GENERATION OF SEMI-GROUPS OF NONLINEAR TRANSFORMATIONS ON GENERAL BANACH SPACES.

By M. G. Crandall and T. M. Liggett.

Introduction. In [8], Hille proved that if A is a closed and densely defined linear operator on a Banach space X with the properties that $(I+\lambda A)^{-1}$ is defined on X and

$$\|(I+\lambda A)^{-1}\| \leqq 1 \tag{1}$$

for $\lambda > 0$, then

$$u(t) = \lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x \tag{2}$$

exists for $t \geqq 0$ and $x \in X$. Moreover, the function defined in (2) is a solution of the Cauchy problem

$$\frac{du}{dt} + Au = 0, \quad u(0) = x \tag{3}$$

if $x \in D(A)$. Roughly speaking, the main result of this work states that the limit (2) exists even if A is nonlinear (and multivalued) provided the nonlinear analogue of (1) holds for A. In addition, we prove that the (multivalued) analogue of (3) has a solution in a strong sense if and only if the limit in (2) is strongly differentiable, and in this case it is the solution to (3). These results extend earlier ones which require additional restrictive conditions, including that X^* be uniformly convex ([2], [3], [7], [9], [10]); that A be continuous ([19]); that (3) have a strong solution ([2]); or that A be linear ([8], [20]).

Of course, the "exponential formula" (2) is properly viewed in the context of the theory of semi-groups of transformations. To state our results more precisely and put them in such a context, it is convenient to introduce some notation and definitions. If $C \subset X$, a semi-group on C is a function S on $[0,\infty)$ such that $S(t)$ maps C into C for each $t \geqq 0$ and satisfies

$$\begin{aligned} &S(t+\tau) = S(t)S(\tau) \quad \text{for } t,\tau \geqq 0, \text{ and} \\ &\lim_{t\downarrow 0} S(t)x = S(0)x = x \quad \text{for } x \in C. \end{aligned} \tag{4}$$

If S is a semi-group on C and there is a real number ω so that

$$\| S(t)x - S(t)y \| \leqq e^{\omega t} \| x - y \| \tag{5}$$

for $t \geqq 0$ and $x, y \in C$, we will write $S \in Q_\omega(C)$.

The Hille-Yosida Theorem precisely describes the infinitesimal generators of those $S \in Q_\omega(X)$ whose values are linear operators. In the nonlinear case, examples show that one should consider semi-groups defined on more general subsets C of X, and that their generators cannot be adequately described by functions. Hence one introduces "multivalued" functions A, which we view as subsets of $X \times X$. If $A \subset X \times X$, we define

$$(6)\quad \begin{array}{ll} \text{(i)} & Ax = \{y : [x, y] \in A\} \\ \text{(ii)} & D(A) = \{x : Ax \neq \emptyset\} \\ \text{(iii)} & R(A) = \cup \{Ax : x \in D(A)\}. \end{array}$$

If $A, B \subset X \times X$, and λ is real, one sets

$$(7)\quad \begin{aligned} A + B &= \{[x, y + z] : y \in Ax \text{ and } z \in Bx\} \\ \lambda A &= \{[x, \lambda y] : y \in Ax\} \\ A^{-1} &= \{[y, x] : [x, y] \in A\}. \end{aligned}$$

Functions are not distinguished from their graphs—for example, I is the diagonal of $X \times X$. If $A \subset X \times X$ is a function, Ax will denote either the value of A at x or the set defined in (6(i)), depending on the context. Finally, a subset B of $X \times X$, is called *accretive* if $(I + \lambda B)^{-1}$ is a function for $\lambda > 0$ and

$$\| (I + \lambda B)^{-1}x - (I + \lambda B)^{-1}y \| \leqq \| x - y \| \tag{8}$$

for $x, y \in D((I + \lambda B)^{-1})$.

A complete extension of the Hille-Yosida Theorem to cover all of $Q_\omega(C)$ is known only in the case in which C is a closed convex set and X is a Hilbert space. In this case, $Q_\omega(C)$ can be put in one to one correspondence with the set of those subsets A of $X \times X$ which satisfy: $A + \omega I$ is accretive, $R(I + \lambda A) = X$ for $\lambda > 0$ and $\omega\lambda < 1$, and $\overline{D(A)} = C$. This is proved in the appendix of [6] in the case $\omega = 0$. The extension to general ω is unpublished, but straightforward. Partial results of the same nature are known if X^* is uniformly convex (see [1] and [10]). However, there have been no results of any generality concerning the case in which X^* is not uniformly convex. In this direction, we will prove:

THEOREM I. *Let $A \subset X \times X$ and ω be a real number such that $A + \omega I$ is accretive. If $R(I + \lambda A) \supset \overline{D(A)}$ for all sufficiently small positive λ, then*

$$\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x \tag{9}$$

exists for $x\in\overline{D(A)}$ *and* $t>0$. *Moreover, if* $S(t)x$ *is defined as the limit in* (9), *then* $S\in Q_\omega(\overline{D(A)})$.

The proof of Theorem I is given in Section 1. The emphasis throughout this work is on semi-groups, but our general approach extends easily to the time dependent case. An example of this is sketched in an appendix to Section 1.

The relationship of the limit (9) and the corresponding Cauchy problem is discussed in Section 2. Basic facts concerning this relationship are given by:

THEOREM II. *In addition to the conditions of Theorem I, suppose that*

(a) $$R(I+\lambda A)\supset \operatorname{clco} D(A)$$

for all sufficiently small positive λ, *and*

(b) A *is a closed subset of* $X\times X$. (*Here* clco *denotes the closure of the convex hull.*) *If* $x\in D(A)$ *and* $0<T\leqq\infty$, *conditions* (i) *and* (ii) *below on a function* $u[0,T)\to X$ *are equivalent:*

(i) u *is a strong solution of*

$$0\in\frac{du}{dt}+Au,\qquad u(0)=x \tag{10}$$

on $[0,T)$.

(ii) $u(t)=\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x$ *for* $t\in[0,T)$ *and* $u(t)$ *is strongly differentiable almost everywhere.*

The assertion (i) $\Rightarrow$ (ii) of Theorem II follows from Theorem 2.1 of Brezis and Pazy [2] (if $\omega=0$). The definition of a strong solution of (10) is given in Section 2, where the significance of Theorem II is also discussed.

Another question which arises in this context is the following. If $S\in Q_\omega(C)$, is there an A such that

$$S(t)=\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}?$$

The results in this direction are rather weak and are summarized in Section 3.

Finally, one might ask whether there is at most one A which leads to a given $S\in Q_\omega(C)$ through the exponential formula. We will see that this is not the case, even in relatively simple situations where $C=X$. The relevant examples will be found in Section 4.

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1. The exponential formula. In the following discussion, a set $A \subset X \times X$ and a real number ω such that $A + \omega I$ is accretive are fixed. If λ is real, J_λ will denote the set $(I + \lambda A)^{-1}$ and $D_\lambda = D(J_\lambda)$ its domain. If $B \subset X \times X$, we define

$$|Bx| = \inf\{\|y\| : y \in Bx\} \tag{1.1}$$

for $x \in D(B)$. The first lemma collects some elementary facts about the sets J_λ.

LEMMA 1.2. *Take $\lambda \geqq 0$ so that $\lambda\omega < 1$. Then the following four statements hold.*

(i) *J_λ is a function, and for $x, y \in D_\lambda$,*

$$\|J_\lambda x - J_\lambda y\| \leqq (1 - \lambda\omega)^{-1} \|x - y\|.$$

(ii) $$\|J_\lambda x - x\| \leqq \lambda(1 - \lambda\omega)^{-1} |Ax|$$

for $x \in D_\lambda \cap D(A)$.

(iii) *If n is a positive integer, $x \in D(J_\lambda{}^n)$ and $\lambda|\omega| < 1$, then*

$$\|J_\lambda{}^n x - x\| \leqq n(1 - \lambda|\omega|)^{-n+1} \|J_\lambda x - x\|.$$

(iv) *If $x \in D_\lambda$, $\lambda > 0$, and μ is real, then*

$$\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x \in D_\mu, \text{ and}$$

$$J_\lambda x \in J_\mu\left(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x\right).$$

Proof. By assumption, $A + \omega I$ is accretive, so

$$(I + t(A + \omega I))^{-1}$$

is a contraction for $t \geqq 0$. Rewriting this, we see that

$$\left(I + \frac{t}{1 + t\omega} A\right)^{-1}$$

has $|1 + t\omega|$ as a Lipschitz constant if $t \geqq 0$ and $1 + t\omega \neq 0$. Setting $\lambda = t(1 + t\omega)^{-1}$ and using the restrictions $\lambda \geqq 0$ and $\lambda\omega < 1$, we find that

$(I+\lambda A)^{-1}$ has $(1-\lambda\omega)^{-1}=|1+t\omega|$ as a Lipschitz constant, so (i) is proved.

To prove (ii), take $[x_1,y_1]\in A$ and $[x,y]\in A$ such that $x_1+\lambda y_1=x$. Then $J_\lambda x=x_1$ and according to (i),

$$\begin{aligned}\|x_1-x\|&=\|J_\lambda(x_1+\lambda y_1)-J_\lambda(x+\lambda y)\|\\&\leqq(1-\lambda\omega)^{-1}\|(x_1+\lambda y_1)-(x+\lambda y)\|\\&=\lambda(1-\lambda\omega)^{-1}\|y\|.\end{aligned}$$

Since $y\in Ax$ was arbitrary, the result follows.

In order to obtain (iii), observe that (i) implies

$$\begin{aligned}\|J_\lambda{}^n x-x\|&=\|\sum_{i=0}^{n-1}(J_\lambda{}^{n-i}x-J_\lambda{}^{n-(i+1)}x)\|\\&\leqq\sum_{i=0}^{n-1}(1-\lambda\omega)^{-n+(i+1)}\|J_\lambda x-x\|\\&\leqq n(1-\lambda|\omega|)^{-n+1}\|J_\lambda x-x\|.\end{aligned}$$

Finally, we show (iv), which is simply the nonlinear "resolvent formula." If $x\in D_\lambda$ then there is an element $[x_0,y_0]$ of A such that $x_0+\lambda y_0=x$. We note that

$$\begin{aligned}\frac{\mu}{\lambda}x+\frac{\lambda-\mu}{\lambda}J_\lambda x&=\frac{\mu}{\lambda}(x_0+\lambda y_0)+\frac{\lambda-\mu}{\lambda}x_0\\&=x_0+\mu y_0.\end{aligned}$$

Now, $[x_0+\mu y_0,x_0]\in J_\mu$ by the way J_μ is defined, and

$$x_0=J_\lambda x\in J_\mu(x_0+\mu y_0).$$

The proof of the lemma is complete.

The most important steps in the proof of Theorem I are given in the next two lemmas, which obtain rather sharp estimates of the difference $J_\mu{}^n x-J_\lambda{}^m x$ in terms of the size of $|n\mu-m\lambda|$, λ and μ. As usual, $B(k,l)$ will denote the binomial coefficient.

LEMMA 1.3. *Let $\lambda\geqq\mu>0$, $\omega\lambda<1$, and $x\in D(J_\lambda{}^m)\cap D(J_\mu{}^n)$, where m and n are positive integers satisfying $n\geqq m$. Then*

$$\|J_\mu{}^n x-J_\lambda{}^m x\|\leqq(1-\omega\mu)^{-n}\sum_{j=0}^{m-1}\alpha^j\beta^{n-j}B(n,j)\|J_\lambda{}^{m-j}x-x\|$$

$$+\sum_{j=m}^{n}(1-\omega\mu)^{-j}\alpha^m\beta^{j-m}B(j-1,m-1)\|J_\mu{}^{n-j}x-x\|,$$

where $\alpha=\dfrac{\mu}{\lambda}$ and $\beta=\dfrac{\lambda-\mu}{\lambda}$.

Proof. For integers j and k satisfying $0 \leqq j \leqq n$ and $0 \leqq k \leqq m$, put

$$a_{k,j} = \| J_\mu{}^j x - J_\lambda{}^k x \|.$$

If j and k are positive, we have by Lemma 1.2, parts (i) and (iv), that

$$\begin{aligned} a_{k,j} &= \| J_\mu{}^j x - J_\mu (\frac{\mu}{\lambda} J_\lambda{}^{k-1} x + \frac{\lambda - \mu}{\lambda} J_\lambda{}^k x) \| \\ &\leqq (1-\mu\omega)^{-1} \| J_\mu{}^{j-1} x - (\frac{\mu}{\lambda} J_\lambda{}^{k-1} x + \frac{\lambda-\mu}{\lambda} J_\lambda{}^k x) \| \\ &\leqq (1-\mu\omega)^{-1} \{ \frac{\mu}{\lambda} \| J_\mu{}^{j-1} x - J_\lambda{}^{k-1} x \| + \frac{\lambda-\mu}{\lambda} \| J_\mu{}^{j-1} x - J_\lambda{}^k x \| \} \\ &= \alpha_1 a_{k-1,j-1} + \beta_1 a_{k,j-1} \end{aligned}$$

with

$$\alpha_1 = (1-\mu\omega)^{-1} \frac{\mu}{\lambda} \text{ and } \beta_1 = (1-\mu\omega)^{-1} \frac{\lambda-\mu}{\lambda}.$$

The inequalities

$$a_{k,j} \leqq \alpha_1 a_{k-1,j-1} + \beta_1 a_{k,j-1}$$

can be solved to estimate $a_{m,n}$ in terms of $a_{k,0}$ and $a_{0,j}$ in precisely the form indicated in the statement of the lemma if $n \geqq m$. If $n \leqq m$ the estimate becomes

$$\| J_\mu{}^n x - J_\lambda{}^m x \| \leqq (1-\omega\mu)^{-n} \sum_{j=0}^{n} \alpha^j \beta^{n-j} B(n,j) \| J_\lambda{}^{m-j} x - x \|.$$

The (somewhat awkward) induction is left to the reader.

LEMMA 1.4. *Let $n \geqq m > 0$ be integers, and α, β be positive numbers satisfying $\alpha + \beta = 1$. Then*

$$\text{(i)} \qquad \sum_{j=0}^{m} B(n,j) \alpha^j \beta^{n-j} (m-j) \leqq \sqrt{(n\alpha - m)^2 + n\alpha\beta},$$

and

$$\text{(ii)} \qquad \sum_{j=m}^{n} B(j-1, m-1) \alpha^m \beta^{j-m} (n-j) \leqq \sqrt{\frac{m\beta}{\alpha^2} + (\frac{m\beta}{\alpha} + m - n)^2}.$$

Proof. To obtain (i), note first that

$$\begin{aligned} \sum_{j=0}^{m} B(n,j) \alpha^j \beta^{n-j} (m-j) &\leqq \sum_{j=0}^{n} B(n,j) \alpha^j \beta^{n-j} | m-j | \\ &\leqq \Big(\sum_{j=0}^{n} B(n,j) \alpha^j \beta^{n-j} \Big)^{\frac{1}{2}} \Big(\sum_{j=0}^{n} B(n,j) \alpha^j \beta^{n-j} (m-j)^2 \Big)^{\frac{1}{2}} \end{aligned} \tag{1.5}$$

by the Schwartz inequality. Now the relations

$$\sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j} = (\alpha+\beta)^n,$$

$$\sum_{j=0}^{n} jB(n,j)\alpha^j\beta^{n-j} = \alpha n(\alpha+\beta)^{n-1},$$

and

$$\sum_{j=0}^{n} j^2 B(n,j)\alpha^j\beta^{n-j} = \alpha^2 n(n-1)(\alpha+\beta)^{n-2} + \alpha n(\alpha+\beta)^{n-1}$$

together with $\alpha+\beta=1$ permit explicit calculation of the right hand side of (1.5), thus verifying the first estimate. In a similar manner, to obtain (ii) one writes

$$(1.6)\quad \begin{aligned} \sum_{j=m}^{n} B(j-1,m-1)\alpha^m\beta^{j-m}(n-j) &\leqq \sum_{j=m}^{\infty} B(j-1,m-1)\alpha^m\beta^{j-m}\,|\,n-j\,| \\ &\leqq \left(\sum_{j=m}^{\infty} B(j-1,m-1)\alpha^m\beta^{j-m}\right)^{\frac{1}{2}} \left(\sum_{j=m}^{\infty} B(j-1,m-1)\alpha^m\beta^{j-m}(n-j)^2\right)^{\frac{1}{2}}. \end{aligned}$$

This last expression can now be evaluated explicitly by using

$$(1.7)\qquad \sum_{j=m}^{\infty} B(j-1,m-1)\beta^{j-m} = \frac{1}{(1-\beta)^m}, \qquad |\,\beta\,| < 1,$$

and the identities obtained by differentiating (1.7) with respect to β. This completes the proof.

Proof of Theorem I. Let $x \in D(A)$, and assume that $\lambda \geqq \mu > 0$, $n \geqq m$ and $\lambda\,|\,\omega\,| < 1$. By assumption, $x \in D(J_\lambda{}^m) \cap D(J_\mu{}^n)$, so Lemma 1.2(ii) and (iii) and Lemma 1.3 combine to yield

$$(1.8)\quad \begin{aligned} \|\, J_\mu{}^n x - J_\lambda{}^m x \,\| \leqq \{&(1-\mu\,|\,\omega\,|)^{-n}(1-\lambda\,|\,\omega\,|)^{-m}\lambda \sum_{j=0}^{m} B(n,j)\alpha^j\beta^{n-j}(m-j) \\ &+ (1-\mu\,|\,\omega\,|)^{-2n}\mu \sum_{j=m}^{n} B(j-1,m-1)\alpha^m\beta^{j-m}(n-j)\}\,|\,Ax\,| \end{aligned}$$

where $\alpha=\mu/\lambda$ and $\beta=(\lambda-\mu)/\lambda$. For $t\in[0,\frac{1}{2}]$, it is easy to verify that

$$(1-t)^{-n} \leqq e^{2nt}.$$

Thus, if $\lambda\,|\,\omega\,| \leqq \frac{1}{2}$, (1.8) and Lemma 1.4 yield

$$(1.9)\quad \begin{aligned} \|\, J_\mu{}^n x - J_\lambda{}^m x \,\| \leqq \{&[(n\mu-\lambda m)^2 + n\mu(\lambda-\mu)]^{\frac{1}{2}} \exp[2\,|\,\omega\,|(n\mu+m\lambda)] \\ &+ [m\lambda(\lambda-\mu) + (m\lambda-n\mu)^2]^{\frac{1}{2}} \exp[4\,|\,\omega\,|\,n\mu]\}\,|\,Ax\,|. \end{aligned}$$

The theorem follows rather quickly now. Taking $\mu = t/n$ and $\lambda = t/m$ in (1.9), the estimate becomes

$$(1.10) \qquad \| J_{t/n}{}^n x - J_{t/m}{}^m x \| \leqq 2t \exp(4 \mid \omega \mid t) \left(\frac{1}{m} - \frac{1}{n}\right)^{\frac{1}{2}} \mid Ax \mid.$$

So, $\lim_{n\to\infty} J_{t/n}{}^n x$ exists.

Since $J_{t/n}{}^n$ has $(1-\omega t/n)^{-n}$ as a Lipschitz constant and

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\omega\right)^{-n} = e^{\omega t},$$

$S(t)x = \lim_{n\to\infty} J_{t/n}{}^n x$ exists for $x \in \overline{D(A)}$ if it exists for $x \in D(A)$, and $S(t)$ has $e^{\omega t}$ as a Lipschitz constant. Clearly $S(t)$ leaves $\overline{D(A)}$ invariant. Now, if $x \in D(A)$ and $\tau > t \geqq 0$, take the limit in (1.9) with $n = m$, $\mu = t/n$ and $\lambda = \tau/n$ to obtain

$$(1.11) \quad \| S(\tau)x - S(t)x \| \leqq \{\exp(2 \mid \omega \mid (t+\tau)) + \exp(4 \mid \omega \mid t)\} \mid Ax \mid (\tau - t).$$

So $S(t)x$ is Lipschitz continuous in t on bounded t-sets. It follows that $S(t)x$ is continuous in t for $x \in \overline{D(A)}$.

Finally, we will verify the semi-group property: $S(t+\tau) = S(t)S(\tau)$. For this, the strong convergence

$$S(t) = \lim_{n\to\infty} J_{t/n}{}^n,$$

together with the uniform Lipschitz continuity of $\{J_{t/n}{}^n\}_{n>N}$ for N sufficiently large, gives

$$[S(t)]^m = \lim_{n\to\infty} [J_{t/n}{}^n]^m = \lim_{n\to\infty} [J_{t/n}{}^m]^n.$$

Therefore,

$$S(mt) = \lim_{n\to\infty} J_{mt/n}{}^n = \lim_{k\to\infty} J_{mt/mk}{}^{mk}$$
$$= \lim_{k\to\infty} [J_{t/k}{}^m]^k = [S(t)]^m.$$

Now let l, k, r, s be positive integers. Then

$$S\left(\frac{l}{k} + \frac{r}{s}\right) = S\left(\frac{ls+rk}{ks}\right) = \left[S\left(\frac{1}{ks}\right)\right]^{ls+rk}$$
$$= \left[S\left(\frac{1}{ks}\right)\right]^{ls} \left[S\left(\frac{1}{ks}\right)\right]^{rk} = S\left(\frac{l}{k}\right) S\left(\frac{r}{s}\right).$$

Hence $S(t+\tau) = S(t)S(\tau)$ holds if t, τ are rational. In view of the continuity in t and Lipschitz continuity in x of S, the proof is complete.

Remark. The estimates of this section actually show that if $\{\epsilon(n)\}$ is a sequence of nonnegative numbers converging to zero and $\{k(n)\}$ is a sequence of nonnegative integers such that $\lim_{n\to\infty} k(n)\epsilon(n) = t$ exists, then $\lim_{n\to\infty}(I+\epsilon(n)A)^{-k(n)}x$ exists whenever each term is defined and $x \in D(A)$. Also, the limit depends only on t. Using this observation, we can define S by

$$S(t)x = \lim_{n\to\infty}(I+\epsilon(n)A)^{-k(n)}x$$

for those t, x such that appropriate sequences $\{\epsilon(n)\}$ and $\{k(n)\}$ exist. It may be, for example, that $S(t)x$ is only defined for $0 \leqq t \leqq T$.

If A is linear, the proof of Theorem I extends to the more general case in which the assumption on A is of the form

$$\|(I+\lambda A)^{-n}\| \leqq M(1-\lambda\omega)^{-n} \tag{1.12}$$

for some fixed M and all integers n, in an obvious way, since then J_μ commutes with the operation of taking linear combinations. The nonlinear generalization of (1.12) does not yield to our proof, however, since in the proof of Lemma 1.3 one must then remove factors of J_μ or J_λ one at a time, and powers of M build up in the estimates.

Appendix to Section 1. The time dependent case. One nice feature of the proofs given in this section is the case with which the arguments extend to the time dependent case. We will state and sketch the proof of a theorem which implies the existence of a nonlinear evolution operator. This theorem extends a main result of [9] concerning singlevalued equations in the case X^* is uniformly convex. Compare also with [10]. However, we have not aimed for generality here, and attempt only to illustrate the power of our approach.

THEOREM A. *Let ω be a real number. For each t in $[0,T]$, let $A(t)$ be a subset of $X \times X$ such that $A(t)+\omega I$ is accretive. Assume in addition that the following conditions hold.*

(i) *$D(A(t))$ is independent of t.*

(ii) *$R(I+\lambda A(t))$ contains the closure of $D(A(0))$ for $0<\lambda\leqq\lambda_0$ and $0\leqq t\leqq T$.*

(iii) *$|A(t)x| \leqq |A(\tau)x| + |t-\tau| L(\|x\|)(1+|A(\tau)x|)$ for $0\leqq t,\tau\leqq T$ and $x\in D(A(0))$.*

(iv) *$\|(I+\lambda A(t))^{-1}x-(I+\lambda A(\tau))^{-1}x\| \leqq \lambda |t-\tau| L(\|x\|+|A(\tau)x|)$ $0\leqq t,\tau\leqq T$, $0<\lambda\leqq\lambda_0$ and $x\in D(A(0))$.*

Here $L:[0,\infty)\to[0,\infty)$ *is an increasing function. Then*

$$\lim_{n\to\infty}\prod_{i=1}^{n}\left(I+\frac{t}{u}A\left(\frac{it}{n}\right)\right)^{-1}x$$

exists for $x\in\overline{D(A(0))}$ *and* $0\leqq t\leqq T$.

In the statement of the above theorem and in the discussion which follows, we use the conventions

$$\begin{aligned}&\prod_{i=j}^{j}T_i x=T_j x\\ &\prod_{i=j}^{k+1}T_i x=T_{k+1}\Big(\prod_{i=j}^{k}T_i x\Big)\quad \text{if } k\geqq j\\ &\prod_{i=j}^{k}T_i x=x\qquad \text{if } k<j\end{aligned}\tag{A.1}$$

for any collection $\{T_i\}$ of functions. The proof of Theorem A is outlined below. First, if $x\in D(A(0))$, then assumptions (i) and (iii), together with the accretive property of $A(t)+\omega I$, imply that there is a constant $M(x)$ such that

$$\Big\|\prod_{i=1}^{l}\left(I+\frac{t}{n}A\left(\frac{it}{n}\right)\right)^{-1}x-x\Big\|\leqq\frac{l}{n}M(x),\tag{A.2}$$

and

$$\Big|A(\mu)\prod_{i=1}^{l}\left(I+\frac{t}{n}A\left(\frac{it}{n}\right)\right)^{-1}x\Big|\leqq M(x)\tag{A.3}$$

whenever $0\leqq l\leqq n$, $0\leqq t\leqq T$, $0\leqq\mu\leqq T$ and n is sufficiently large.

If $0<m\leqq n$, set

$$a_{k,l}=\Big\|\prod_{i=1}^{l}\left(I+\frac{t}{n}A\left(\frac{it}{n}\right)\right)^{-1}x-\prod_{i=1}^{k}\left(I+\frac{t}{m}A\left(\frac{it}{m}\right)\right)^{-1}x\Big\|,$$

and use the resolvent equation as before to find

$$\begin{aligned}a_{k,l}\leqq{}&\Big\|\left(I+\frac{t}{n}A\left(\frac{kt}{m}\right)\right)^{-1}\prod_{i=1}^{l-1}\left(I+\frac{t}{n}A\left(\frac{it}{n}\right)\right)^{-1}x\\ &-\left(I+\frac{t}{m}A\left(\frac{kt}{m}\right)\right)^{-1}\prod_{i=1}^{k-1}\left(I+\frac{t}{m}A\left(\frac{it}{m}\right)\right)^{-1}x\Big\|\\ &+\Big\|\left(I+\frac{t}{n}A\left(\frac{lt}{n}\right)\right)^{-1}\prod_{i=1}^{l-1}\left(I+\frac{t}{n}A\left(\frac{it}{n}\right)\right)^{-1}x\\ &-\left(I+\frac{t}{n}A\left(\frac{kt}{m}\right)\right)^{-1}\prod_{i=1}^{l-1}\left(I+\frac{t}{n}A\left(\frac{it}{u}\right)\right)^{-1}x\Big\|\\ \leqq{}&\left(1-\frac{t}{n}|\omega|\right)^{-1}\left(\frac{m}{n}a_{k-1,l-1}+\frac{n-m}{n}a_{k,l-1}\right)+g(l,k)\end{aligned}\tag{A.4}$$

for $l > 0$ and $k > 0$, where

$$(A.5) \qquad g(l,k) = \| (I + \frac{t}{n} A(\frac{lt}{n}))^{-1} \prod_{i=1}^{l-1} (I + \frac{t}{n} A(\frac{it}{n}))^{-1} x - (I + \frac{t}{n} A(\frac{kt}{m}))^{-1} \prod_{i=1}^{l-1} (I + \frac{t}{n} A(\frac{it}{n}))^{-1} x \|.$$

Assumption (iv) of Theorem A and (A.2)-(A.5) imply that

$$(A.6) \qquad a_{k,l} \leqq (1 - \frac{t}{n} |\omega|)^{-1} (\alpha a_{k-1,l-1} + \beta a_{k,l-1}) + \frac{Ht^2}{mn} | k - l\alpha |$$

for some constant H, where $\alpha = m/n$ and $\beta = (n-m)/n$. Solving (A.6) yields

$$(A.7) \qquad a_{m,n} \leqq (1 - \frac{t}{n} |\omega|)^{-n} \{ \sum_{j=0}^{m} \alpha^j \beta^{n-j} a_{m-j,0} + \sum_{j=m}^{n} \alpha^m \beta^{j-m} B(j-1, m-1) a_{0,n-j} \} + \frac{Ht^2}{mn} \sum_{j=0}^{n-1} \sum_{i=0}^{j} \alpha^i \beta^{j-i} B(j,i) | i - j\alpha |.$$

In view of (A.2), we have already shown that the first two terms in (A.7) are bounded by a multiple of $t/\sqrt{m}$. In a similar way, one sees that the last term is bounded by a constant times $t^2/\sqrt{m}$, which completes the proof. The construction and properties of the evolution operator

$$U(t,s) = \lim_{n \to \infty} \prod_{i=1}^{n} (I + \frac{t-s}{n} A(s + \frac{i(t-s)}{n}))^{-1}$$

will be studied more systematically and generally elsewhere.

2. The Cauchy problem. In this section we study the Cauchy problem

$$(2.1) \qquad 0 \in \frac{du}{dt} + Au, \quad u(0) = x.$$

DEFINITION 2.2. *Let $0 < T \leqq \infty$ and $u: [0,T) \to X$. .Then u is a strong solution of* (2.1) *on $[0,T)$ if* (i) *u is continuous,* (ii) *u is the indefinite integral of a function which is strongly integrable on compact subsets of $(0,T)$,* (iii) *$u(0) = x$, and* (iv) *$u'(t) \in -Au(t)$ for almost all t in the interval $[0,T)$.*

From now on we assume that $A + \omega I$ is accretive for some fixed ω. In this case, we have the following equivalent restatement of Definition 2.2 when $x \in D(A)$.

LEMMA 2.3. *Let $x \in D(A)$. Then a function $u: [0,T) \to X$ is a strong solution of* (2.1) *on* $[0,T)$ *if and only if u is Lipschitz continuous on compact subsets of* $[0,T)$, *u is differentiable almost everywhere on* $[0,T)$ *and* (2.1) *is satisfied a. e.*

The proof is routine and will not be given here. (See [10, Lemma 6.2], and recall that a strongly absolutely continuous function which is differentiable almost everywhere is an indefinite integral of its (strongly integrable) derivative.) See also [2].

Before proving Theorem II, we pause for some remarks. It is an unfortunate fact that even if A satisfies the conditions of Theorem II and $x \in D(A)$, the problem (2.1) may not have a strong solution on $[0,T)$ for any $T > 0$. The situation is even worse than this. In Section 4 we give an example of an accretive set A satisfying $\overline{D(A)} = X$ and $R(I + \lambda A) = X$ for $\lambda > 0$ such that there are *no* Lipschitz continuous X-valued functions u on *any* interval (a,b) for which $0 \in du/dt + Au$, a. e. $t \in (a,b)$, holds. This phenomenon cannot be avoided without severe restrictions on X or on A. The basic problem is that if X is not reflexive, then Lipschitz continuous functions of a real variable with values in X need not be differentiable anywhere. Still, if A satisfies mild conditions we have proved that $u(t) = \lim_{n\to\infty} (I + (t/n)A)^{-n}x$ exists and this function should be a solution of (2.1). To interpret it as a solution in some reasonably clean weak sense seems desirable, but has proven difficult. One possibility is simply to observe that the exponential formula represents the function as a limit of solutions of approximate problems,

$$(2.5) \qquad \begin{cases} 0 \in \dfrac{u_\epsilon(t) - u_\epsilon(t-\epsilon)}{\epsilon} + Au_\epsilon(t) & \text{for } t \geqq 0 \\ u_\epsilon(t) = x & \text{for } -\epsilon \leqq t < 0 \end{cases}$$

in which the derivative is replaced by the backward difference quotient. We have shown that if u_ϵ is defined for small $\epsilon > 0$, then $\lim_{\epsilon\downarrow 0} u_\epsilon(t)$ exists. (In fact, it is $\lim_{n\to\infty}(I + (t/n)A)^{-n}x$.) One could define "generalized" solutions of (2.1) as limits of solutions of (2.5) as $\epsilon \downarrow 0$. A problem with this approach is that it is not obvious that strong solutions of (2.1) are "generalized" solutions in this sense, nor that "regular" generalized solutions are actually strong solutions, two properties a good definition of a generalized solution should possess. Theorem II asserts that these two criteria are satisfied in this case.

The assertion (i) $\Rightarrow$ (ii) of Theorem II is a special case of Theorem

2.1 of [2] if $\omega = 0$, and the extension to general ω is left to the reader. We begin the proof of (ii) $\Rightarrow$ (i), i.e. that $S(t)x = \lim_{n\to\infty}(I + (t/n)A)^{-n}x$ is a strong solution of (2.1) on any interval $[0, T)$ on which $S(t)x$ is differentiable almost everywhere. We assume, without loss of generality, that X is a real Banach space. With each $x \in X$ is associated the set $F(x) = \{x^* \in X^*: (x, x^*) = \| x \|^2 = \| x^* \|^2\}$, where (x, x^*) denotes the value of $x^* \in X^*$ at $x \in X$. By the Hahn-Banach Theorem, $F(x)$ is nonempty for each $x \in X$ and $F(x)$ is also bounded. Thus the functions defined below are finite-valued.

DEFINITION 2.6. *$\langle\ ,\ \rangle_i: X \times X \to R$ and $\langle\ ,\ \rangle_s: X \times X \to R$ are the functions defined by*

$$\langle u, v\rangle_i = \inf\{(u, v^*) : v^* \in F(v)\}, \tag{2.7}$$

and

$$\langle u, v\rangle_s = \sup\{(u, v^*) : v^* \in F(v)\} = -\langle -u, v\rangle_i. \tag{2.8}$$

The main step in the proof of Theorem II is the following extension of a result of Brezis [1].

LEMMA 2.9. *Let the hypotheses of Theorem II be satisfied, and put*

$$S(t)x = \lim_{n\to\infty}(I + \frac{t}{n}A)^{-n}x$$

for $t \geqq 0$ and $x \in \overline{D(A)}$. If $x \in \overline{D(A)}$ and $[x_0, y_0] \in A$, then

$$\sup_{\zeta^* \in F(x-x_0)} \limsup_{t\downarrow 0} (\frac{S(t)x - x}{t} + \omega(x_0 - x), \zeta^*) \leqq \langle y_0, x_0 - x\rangle_s. \tag{2.10}$$

Before proving this lemma, we will use it to complete the proof of Theorem II. Take $z \in \overline{D(A)}$ and assume that $S(t)z$ is strongly differentiable at $t_0 > 0$, so that

$$S(t_0 + h)z = S(t_0)z + hy + o(h) \quad \text{as } h \to 0, \tag{2.11}$$

where $y = \frac{d}{dt}S(t)z$ evaluated at $t = t_0$. Our aim is to show that $[S(t_0)z, -y] \in A$. Since it was already shown in the proof of Theorem I that $S(t)z$ is Lipschitz continuous on bounded t-sets for $z \in D(A)$, this will conclude the proof of (ii) $\Rightarrow$ (i). By assumption, if $0 < \lambda < t_0$, there is a point $[x_\lambda, y_\lambda] \in A$ such that

(2.12) $$x_\lambda + \lambda y_\lambda = S(t_0 - \lambda) z = S(t_0) z - \lambda y + o(\lambda).$$

We will prove that

$$x_\lambda \to S(t_0) z \text{ and } y_\lambda \to -y$$

as $\lambda \downarrow 0$, so that $[S(t_0)z, -y] \in A$ since A is closed. To show this, put $[x_0, y_0] = [x_\lambda, y_\lambda]$ and $x = S(t_0)z$ in (2.10). Since $F(x_\lambda - S(t_0)z)$ is compact in the weak-star topology of X^*, we can replace the supremum on the right hand side of (2.10) by a maximum. This yields the existence of an $\eta^* \in F(x_\lambda - S(t_0)z)$ such that

(2.13) $$(y + \omega(x_\lambda - S(t_0)z), \zeta^*) \leqq (y_\lambda, \eta^*)$$

for all $\zeta^* \in F(S(t_0)z - x_\lambda)$. Letting $\zeta^* = -\eta^*$ in (2.13), using (2.12), and rewriting yields

(2.14) $$(S(t_0)z - x_\lambda + \lambda\omega(x_\lambda - S(t_0)z) + o(\lambda), \eta^*) \geqq 0.$$

Since $\eta^* \in F(x_\lambda - S(t_0)z)$, it follows that

(2.15) $$(1 - \lambda\omega) \| S(t_0)z - x_\lambda \|^2 \leqq o(\lambda) \| x_\lambda - S(t_0)z \|$$

Thus,

$$\lim_{\lambda \downarrow 0} \frac{S(t_0)z - x_\lambda}{\lambda} = 0.$$

Returning to (2.12), we see that $\lim_{\lambda \downarrow 0} \| y_\lambda + y \| = 0$. The proof of Theorem II is complete.

Finally, we turn to Lemma 2.9, whose proof requires some preliminary work. We begin with a lemma which lists several properties of the functions defined in (2.6).

LEMMA 2.16. *Let $x, y, z \in X$. Then:*

(a) $\langle \alpha y + x, y \rangle_j = \alpha \| y \|^2 + \langle x, y \rangle_j$ *for* $\alpha \in R$ *and* $j = i$ *or* s.

(b) $\langle \beta x, \gamma y \rangle_j = \gamma\beta \langle x, y \rangle_j$ *for* $\gamma\beta \geqq 0$ *and* $j = i$ *or* s.

(c) $\langle z + x, y \rangle_j \leqq \| z \| \| y \| + \langle x, y \rangle_j$ *for* $j = i$ *or* s.

(d) $\langle \ , \ \rangle_s : X \times X \to R$ *is upper semicontinuous.*

(e) $B \subset X \times X$ *is accretive if and only if*

$$\langle y_1 - y_2, x_2 - x_1 \rangle_s \geqq 0 \text{ for } [x_i, y_i] \in B \text{ and } i = 1, 2.$$

(f) *If* $u : (a, b) \to X$, *then*

$$\frac{d}{dt}\|u(t)\|^2\,|_{t=t_0}=2\langle u'(t_0),u(t_0)\rangle_s=2\langle u'(t_0),u(t_0)\rangle_i$$

at each point t_0 at which $\|u(t)\|^2$ is differentiable and $u(t)$ is weakly differentiable ($u'(t_0)$ denotes the weak derivative evaluated at $t=t_0$).

Proof. Assertions (a), (b) and (c) are elementary, so we omit their proof. Assertions (e) and (f) are proved in [9]. We will now prove (d). It is necessary to show that $x_n\to x_0$ and $y_n\to y_0$ in X imply that

$$\limsup_{n\to\infty}\langle x_n,y_n\rangle_s\leqq\langle x_0,y_0\rangle_s. \tag{2.17}$$

Since $F(y_n)$ is weak-star compact, we can pick $\zeta_n{}^*\in F(y_n)$ such that

$$\langle x_n,y_n\rangle_s=(x_n,\zeta_n{}^*).$$

We may assume, without loss of generality, that

$$\lim_{n\to\infty}\langle x_n,y_n\rangle_s$$

exists. Let $\zeta_0{}^*$ be any weak-star limit point of $\{\zeta_n{}^*\}$. Then

$$\|\zeta_0{}^*\|\leqq\liminf_{n\to\infty}\|\zeta_n{}^*\|=\liminf_{n\to\infty}\|y_n\|=\|y_0\|, \tag{2.18}$$

and

$$\|y_0\|^2=\lim_{n\to\infty}(y_n,\zeta_n{}^*)=\lim_{n\to\infty}(y_0,\zeta_n{}^*)=(y_0,\zeta_0{}^*), \tag{2.19}$$

since $y_n\to y_0$ strongly. Together, (2.18) and (2.19) imply that $\zeta_0{}^*\in F(y_0)$. Now, since $x_n\to x_0$ strongly, and $\{\langle x_n,y_n\rangle_s\}$ converges

$$\lim_{n\to\infty}\langle x_n,y_n\rangle_s=\lim_{n\to\infty}(x_n,\zeta_n{}^*)=(x_0,\zeta_0{}^*)\leqq\langle x_0,y_0\rangle_s,$$

and we have (2.17). The proof is complete.

To continue with our proof of Lemma 2.9, we introduce the semigroups S_λ on $\operatorname{clco}D(A)$ defined by

$$\frac{d}{dt}S_\lambda(t)x+A_\lambda S_\lambda(t)x=0 \tag{2.20}$$

for $t\geqq 0$ and $0<\lambda\leqq\lambda_0$, where

$$A_\lambda x=\lambda^{-1}(x-J_\lambda x)\qquad x\in D_\lambda. \tag{2.21}$$

The existence of S_λ satisfying (2.20) is a convenient consequence of the technical assumption (a) of Theorem II. (See the remarks at the end of this section, or [4] for more general results of this nature.) We will need to know how well $S_\lambda(t)$ approximates $S(t)$.

LEMMA 2.22. *Let the assumptions of Theorem II be satisfied and take* $x \in D(A)$, $0 < \lambda \leqq \lambda_0$, *and* $\lambda|\omega| < 1/2$. *Then*

$$\begin{aligned} \| S_\lambda(t)x - S(t)x \| \leqq \{ & e^{2t|\omega|}\lambda(1-\lambda\omega)^{-1} + 2e^{4t|\omega|}\sqrt{t\lambda} \\ & + e^{4t|\omega|}(4(t|\omega|)^2\sqrt{\lambda} + 2t|\omega|\sqrt{\lambda} + t)^{\frac{1}{2}}\sqrt{\lambda}(1-\lambda\omega)^{-1} + e^{t|\omega|}\lambda\} | Ax |. \end{aligned} \tag{2.23}$$

Proof. The proof is given for $\omega = 0$ only. This simplifies the presentation, and the general case is not essentially more difficult. Choosing the integer m so that $t = m\lambda + \delta$ and $0 \leqq \delta < \lambda$, we estimate

$$\begin{aligned} \| S_\lambda(t)x - S(t)x \| \leqq & \| S_\lambda(t)x - S_\lambda(m\lambda)x \| + \| S_\lambda(m\lambda)x - J_\lambda{}^m x \| \\ & + \| J_\lambda{}^m x - S(m\lambda)x \| + \| S(m\lambda)x - S(t)x \|. \end{aligned} \tag{2.24}$$

Since $S \in Q_0(\overline{D(A)})$, the last term above admits the estimate

$$\begin{aligned} \| S(m\lambda)x - S(t)x \| & \leqq \| x - S(\delta)x \| \\ & = \lim_{n\to\infty} \| x - J_{\delta/n}{}^n x \| \leqq \delta | Ax | \leqq \lambda | Ax | \end{aligned} \tag{2.25}$$

by Lemma 1.2. The first term also admits such an estimate. Indeed, we have

$$\begin{aligned} \frac{d}{dt} \| S_\lambda(t)z - S_\lambda(t)y \|^2 \leqq 2\lambda^{-1}\{ & \| J_\lambda S_\lambda(t)z - J_\lambda S_\lambda(t)y \| \, \| S_\lambda(t)z - S_\lambda(t)y \| \\ & - \| S_\lambda(t)z - S_\lambda(t)y \|^2 \} \leqq 0 \end{aligned}$$

by Lemma 2.16(c) and (f), and the fact that J_λ is a contraction (recall that $\omega = 0$). Thus $S_\lambda \in Q_0(\mathrm{clco} D(A))$ and

$$\left\| \frac{d}{dt} S_\lambda(t)x \,|_{t=0} \right\| = \| A_\lambda x \| \leqq | Ax |$$

is a Lipschitz constant for $S_\lambda(t)x$. Therefore

$$\| S_\lambda(m\lambda)x - S_\lambda(t)x \| \leqq \delta | Ax | \leqq \lambda | Ax |. \tag{2.26}$$

The third term in (2.24) is estimated via (1.10),

$$\| J_\lambda{}^m x - S(m\lambda)x \| \leqq 2m\lambda(1/\sqrt{m}) | Ax | \leqq 2\sqrt{\lambda t} | Ax |. \tag{2.27}$$

The second term requires a different estimate. We use Lemma 2.4 of [2], which states that if J is a contraction and u satisfies

$$\frac{du}{dt}(t) = Ju(t) - u(t),$$

then

$$\| u(n) - J^n u(0) \| \leqq \sqrt{n} \, \| Ju(0) - u(0) \|, \tag{2.28}$$

to obtain

$$\begin{aligned} \| S_\lambda(m\lambda)x - J_\lambda{}^m x \| &\leqq \sqrt{m} \, \| x - J_\lambda x \| \\ &\leqq \sqrt{m}\,\lambda \, | Ax | \leqq \sqrt{\lambda t} \, | Ax |. \end{aligned} \tag{2.29}$$

Using (2.25)-(2.29), (2.24) becomes

$$\| S_\lambda(t)x - S(t)x \| \leqq \{2\lambda + 3\sqrt{\lambda t}\} \, | Ax |, \tag{2.30}$$

which is (2.23) in the case $\omega = 0$. For $\omega \neq 0$, the estimates (2.25)-(2.27) are generalized in an obvious way, while (2.29) is replaced by an estimate obtained from Lemma 4′ in the appendix of [15]. The proof of Lemma 2.22 is complete.

Next, having chosen $[x_0, y_0] \in A$ as in the statement of Lemma 2.9, set $x_\lambda = x_0 + \lambda y_0$ and compute

$$\begin{aligned} \frac{d}{dt} \| S_\lambda(t)x - x_\lambda \|^2 &= 2\langle -A_\lambda S_\lambda(t)x, S_\lambda(t)x - x_\lambda \rangle_i \\ &\leqq -2\langle A_\lambda S_\lambda(t)x - A_\lambda x_\lambda, S_\lambda(t)x - x_\lambda \rangle_s \\ &\quad + 2\langle A_\lambda x_\lambda, x_\lambda - S_\lambda(t)x \rangle_s \end{aligned} \tag{2.31}$$

Recalling that $A_\lambda = \lambda^{-1}(I - J_\lambda)$, and that J_λ has $(1 - \lambda\omega)^{-1}$ as a Lipschitz constant, we have

$$\begin{aligned} \langle A_\lambda u - A_\lambda v, u - v \rangle_s &\geqq \langle A_\lambda u - A_\lambda v, u - v \rangle_i \\ &\geqq \lambda^{-1}\{ \| u - v \|^2 - (1 - \lambda\omega)^{-1} \| u - v \|^2 \} \\ &\geqq -\omega(1 - \lambda\omega)^{-1} \| u - v \|^2 \end{aligned} \tag{2.32}$$

Using (2.32) and the fact that $A_\lambda x_\lambda = y_0$ in (2.31), we have

$$\begin{aligned} \frac{d}{dt} \| S_\lambda(t)x - x_\lambda \|^2 &\leqq 2\omega(1 - \lambda\omega)^{-1} \| S_\lambda(t)x - x_\lambda \|^2 \\ &\quad + 2\langle y_0, x_\lambda - S_\lambda(t)x \rangle_s. \end{aligned} \tag{2.33}$$

This inequality implies immediately that

$$\begin{aligned} &\exp(-2\omega t/(1 - \lambda\omega)) \| S_\lambda(t)x - x_\lambda \|^2 - \| x - x_\lambda \|^2 \\ &\leqq 2 \int_0^t \langle y_0, x_\lambda - S_\lambda(\tau)x \rangle_s \exp(-2\omega\tau/(1 - \lambda\omega)) \, d\tau. \end{aligned} \tag{2.34}$$

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Note that the right hand side of (2.34) is well defined since the integrand is semicontinuous by Lemma 2.16(d). Now, $x_\lambda = x_0 + \lambda y_0 \to x_0$ as $\lambda \downarrow 0$, and Lemma 2.22 implies that

$$S_\lambda(t)x \to S(t)x$$

as $\lambda \downarrow 0$. Using Lemma 2.16(d) again, we can pass to the limit as $\lambda \downarrow 0$ in (2.34) to see that

$$\begin{aligned} & e^{-2\omega t} \| S(t)x - x_0 \|^2 - \| x - x_0 \|^2 \\ & \leqq 2t \int_0^1 \langle y_0, x_0 - S(t\tau)x \rangle_s e^{-2\omega\tau t} d\tau \end{aligned} \tag{2.35}$$

Finally, observe that if $\zeta^* \in F(x - x_0)$, then

$$\| S(t)x - x_0 \|^2 \geqq \| x - x_0 \|^2 + 2(S(t)x - x, \zeta^*),$$

so that (2.35) yields

$$\begin{aligned} & \left(\frac{e^{-2\omega t} - 1}{t} (x - x_0) + 2e^{-2\omega t} \frac{(S(t)x - x)}{t}, \zeta^* \right) \\ & \leqq 2 \int_0^1 \langle y_0, x_0 - S(t\tau)x \rangle_s e^{-2\omega\tau t} \, d\tau. \end{aligned} \tag{2.36}$$

Taking the lim sup of (2.36) as $t \downarrow 0$ and applying Lemma 2.16(d) one final time completes the proof of Lemma 2.9.

It is of interest to point out various circumstances in which Theorem II is applicable. First of all, if $x \in D(A)$

$$u(t) = \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n} x$$

has $e^{\omega T} |Ax|$ as a Lipschitz constant on $[0, T]$. If X has the property that each Lipschitz continuous X-valued function is differentiable almost everywhere (e.g., if X is reflexive), then condition (ii) of Theorem II is satisfied.

One can also put restrictions on A to achieve the same end. For example, if A is singlevalued and continuous from the strong to the weak topology of X, we have for $x \in D(A)$,

$$\begin{aligned} \frac{(S(t)x - x)}{t} & = \lim_{n \to \infty} \frac{J_{t/n}{}^n x - x}{t} \\ & = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{J_{t/n}(J_{t/n}{}^i x) - J_{t/n}{}^i x}{t/n}. \end{aligned} \tag{2.37}$$

Since $J_\lambda = (I+\lambda A)^{-1}$, $AJ_\lambda = \lambda^{-1}(I-J_\lambda)$, so (2.37) can be rewritten in the form

$$(2.38) \qquad \frac{(S(t)x-x)}{t} = -\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} A(J_{t/n}{}^{i}x).$$

Now, if $0 \leqq i \leqq n$,

$$\| x - J_{t/n}{}^{i}x \| \leqq t(1-\frac{t}{n}|\omega|)^{-i} \| Ax \|.$$

Thus, given any convex weak neighborhood U of 0,

$$Ax - A(J_{t/n}{}^{i}x) \in U$$

if $0 \leqq i \leqq n$ and t is sufficiently small. Equation (2.38) then implies that

$$(2.39) \qquad \frac{(S(t)x-x)}{t} + Ax \in U$$

if t is sufficiently small, so we conclude that

$$(2.40) \qquad w-\lim_{t\downarrow 0} \frac{(S(t)x-x)}{t} + Ax = 0.$$

Restricting A by one of a number of conditions, such as that $D(A)$ be closed (or the weaker condition: $A^{-1}(\{y \colon \| y \| \leqq R\})$ is closed for each R), implies at once that $S(t)$ leaves $D(A)$ invariant. So (2.40) shows that the weak right derivative of $S(t)x$ is $AS(t)x$ which is weakly continuous, so $S(t)x$ is weakly continuously differentiable. This implies that $S(t)x$ is a strong solution of

$$\frac{du}{dt} + Au = 0, \qquad u(0) = x.$$

See [19] for the first treatment of the case of continuous A. The existence of the semi-groups S_λ of (2.20) follows from the above remarks and Theorem I, since under the assumptions of Theorem II A_λ restricted to $\mathrm{clco}D(A)$ satisfies the conditions of Theorem I with ω replaced by $\omega(1-\lambda\omega)^{-1}$.

If A is linear, closed, and $D(A)$ is dense, the equation

$$AJ_{t/n}{}^{n}x = J_{t/n}{}^{n}Ax$$

implies that $S(t)$ leaves $D(A)$ invariant and that A commutes with $S(t)$. Moreover, (2.38) can be rewritten in the form

$$(2.41) \qquad \frac{S(t)x-x}{t} = -\lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{n} J_{t/n}{}^{i})Ax,$$

and we already showed above that

$$\lim_{t \downarrow 0} J_{t/n}{}^{i} x = x$$

holds uniformly in n and $0 \leqq i \leqq n$ for $x \in D(A)$, and hence also for $x \in \overline{D(A)} = X$. Thus (2.41) gives

$$\lim_{t \downarrow 0} \frac{S(t)x - x}{t} + Ax = 0 \tag{2.42}$$

for $x \in D(A)$. The right derivative of $S(t)x$ is therefore

$$AS(t)x = S(t)Ax$$

which is continuous, and we have the standard linear result.

Remark. After the preparation of this manuscript, I. Miyadera [14] showed that in assumption (a) of Theorem II, the closed convex hull of $D(A)$ can be replaced by the closure of $D(A)$.

3. Existence of accretive sets. In this section, we look briefly at the problem which is the converse of that discussed in the preceding sections. One asks, given $S \in Q_{\omega}(C)$, is there an A such that

$$S(t) = \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n} ?$$

In this section C will always be assumed to be closed and convex. The problem stated here has been solved only in two cases, namely if X is a Hilbert space or if S has linear operator values and C is a linear space. We will show that this question also has an affirmative answer if X is finite dimensional, and will obtain several results in the direction of answering the question for a general Banach space. For simplicity, we will assume that $\omega = 0$ throughout this section.

We now fix C and $S \in Q_0(C)$. For $t > 0$, define

$$A^t x = t^{-1}(x - S(t)x).$$

If $\lambda > 0$, $I + \lambda A^t$ satisfies

$$\begin{aligned} \|(I + \lambda A^t)x - (I + \lambda A^t)y\| &\geqq (1 + \lambda/t)\|x - y\| - (\lambda/t)\|S(t)x - S(t)y\| \\ &\geqq \|x - y\|, \end{aligned}$$

so $J_{\lambda,t} = (I + \lambda A^t)^{-1}$ is singlevalued and a contraction. In fact, $J_{\lambda,t}$ is defined on C and leaves C invariant, since if $x \in C$, $J_{\lambda,t}x$ is the unique fixed point of the strictly contractive mapping

$$y \to \frac{t}{\lambda+t}x + \frac{\lambda}{\lambda+t}S(t)y$$

of C into C. In other words, for $x \in C$

(3.1) $$J_{\lambda,t}x = \frac{t}{\lambda+t}x + \frac{\lambda}{\lambda+t}S(t)J_{\lambda,t}x.$$

The maps $J_{\lambda,t}$ have been much used, see [12], [16], [17], [18]. Komura [13] studied $\lim_{t\downarrow 0} J_{\lambda,t}$ in a Hilbert space context. His results were substantially sharpened by Kato [11]. The idea in what follows is that, since

$$J_{\lambda,t} = (I + \lambda \frac{I - S(t)}{t})^{-1},$$

one would expect that $\lim_{t\downarrow 0} J_{\lambda,t} = (I + \lambda A)^{-1}$ would hold in some sense for the generator $-A$ of S. Assuming that $\lim_{n\to\infty} J_{\lambda,t_n} = J_\lambda$ exists for $\lambda > 0$ and some sequence $t_n \downarrow 0$, we will prove that

$$A = \bigcup_{\lambda>0} \{[J_\lambda x, \lambda^{-1}(x - J_\lambda x)] : x \in C\}$$

is accretive and $S(t) = \lim_{n\to\infty}(I + (t/n)A)^{-n}$. Moreover, we will show that such a limit exists if X is finite dimensional

The simplest estimate based on (3.1) is

$$\begin{aligned}
\| J_{\lambda,t}x - x \| &= \| \frac{t}{t+\lambda}x + \frac{\lambda}{t+\lambda}S(t)J_{\lambda,t}x - x \| \\
&= \frac{\lambda}{t+\lambda} \| S(t)J_{\lambda,t}x - x \| \\
&\leqq \frac{\lambda}{t+\lambda}(\| S(t)J_{\lambda,t}x - S(t)x \| + \| S(t)x - x \|) \\
&\leqq \frac{\lambda}{t+\lambda}(\| J_{\lambda,t}x - x \| + \| S(t)x - x \|)
\end{aligned}$$

for $x \in C$, which implies that

(3.2) $$\| J_{\lambda,t}x - x \| \leqq (\lambda/t) \| S(t)x - x \|.$$

However, we will need more information concerning $J_{\lambda,t}x$. The key fact is stated next.

LEMMA 3.3. *For $x \in C$,*

$$\lim_{(t,\lambda)\to(0,0)} \| J_{\lambda,t}x - x \| = 0.$$

This implies, in particular, that $J_{\lambda,t}x$ remains bounded as $t \downarrow 0$. For the proof of Lemma 3.3, we mimic Kato [11] as follows.

LEMMA 3.4. *Fix $x \in C$; $\lambda, t > 0$; n a positive integer. For $\tau > 0$, let $y_\tau = J_{\lambda,\tau}x$. Then there is an integer i, with $1 \leqq i \leqq n$, so that*

$$0 \geqq \langle y_{nt} - x, y_{nt} - y_t \rangle_s + \langle y_t - x, y_t - S(it) y_{nt} \rangle_s.$$

Proof. If (3.1) is solved for $S(t)y_t$, we find that

$$S(t) y_t = y_t + \frac{t}{\lambda}(y_t - x). \tag{3.5}$$

If $z \in C$, this implies that

$$\begin{aligned} \| y_t - z \|^2 &\geqq \| S(t) y_t - S(t) z \|^2 \\ &= \| y_t - S(t) z + \frac{t}{\lambda}(y_t - x) \|^2 \\ &\geqq \| y_t - S(t) z \|^2 + \frac{2t}{\lambda} \langle y_t - x, y_t - S(t) z \rangle_s. \end{aligned} \tag{3.6}$$

Here we have have used the elementary fact that

$$\| u + v \|^2 \geqq \| u \|^2 + 2\langle v, u \rangle_s.$$

Repeated use of (3.6) yields

$$\| y_t - z \|^2 \geqq \| y_t - S(nt) z \|^2 + \frac{2t}{\lambda} \sum_{i=1}^{n} \langle y_t - x, y_t - S(it) z \rangle_s \tag{3.7}$$

for each integer n. Using $z = y_{nt}$ in (3.7), and then (3.5) with nt in place of t, gives

$$\begin{aligned} \| y_t - y_{nt} \|^2 &\geqq \| (y_t - y_{nt}) - \frac{nt}{\lambda}(y_{nt} - x) \|^2 \\ &\quad + \frac{2t}{\lambda} \sum_{i=1}^{n} \langle y_t - x, y_t - S(it) z \rangle_s \\ &\geqq \| y_t - y_{nt} \|^2 + \frac{2nt}{\lambda} \langle y_{nt} - x, y_{nt} - y_t \rangle_s \\ &\quad + \frac{2t}{\lambda} \sum_{i=1}^{n} \langle y_t - x, y_t - S(it) y_{nt} \rangle_s. \end{aligned}$$

Simplifying and dividing by $(2t/\lambda)$ yields

$$0 \geqq \sum_{i=1}^{n} \{ \langle y_{nt} - x, y_{nt} - y_t \rangle_s + \langle y_t - x, y_t - S(it) y_{nt} \rangle_s \}.$$

Since at least one term on the right above must be nonpositive, the lemma is proved.

In order to deduce some consequences from this lemma, we use the following estimates:

$$\langle y_{nt}-x, y_{nt}-y_t\rangle_s \geqq -\| y_{nt}-x\|(\| y_t-x\| + \| y_{nt}-x\|),$$

and

$$\begin{aligned}\langle y_t-x, y_t-S(it)y_{nt}\rangle_s &\geqq \| y_t-S(it)y_{nt}\|^2 - \| S(it)y_{nt}-x\|\,\| y_t-S(it)y_{nt}\| \\ &\geqq (\| y_t-x\| - \| x-S(it)y_{nt}\|)^2 \\ &\qquad - \| S(it)y_{nt}-x\|(\| y_t-x\| + \| x-S(it)y_{nt}\|).\end{aligned}$$

So, by Lemma 3.4, there is an i, $1\leqq i\leqq n$, such that

$$\begin{aligned}0 &\geqq \| y_t-x\|^2 - \| y_t-x\|(3\| x-S(it)y_{nt}\| + \| y_{nt}-x\|) - \| y_{nt}-x\|^2 \\ &\geqq \| y_t-x\|^2 - \| y_t-x\|(4\| y_{nt}-x\| + 3\| x-S(it)x\|) - \| y_{nt}-x\|^2.\end{aligned}$$

This implies that

$$\| y_t-x\| \leqq \frac{7+\sqrt{53}}{2}\max(\| y_{nt}-x\|, \| x-S(it)x\|).$$

Recalling (3.2) and the definition of y_τ, we conclude that there is a universal constant $\eta>0$ so that for each integer n, there is an i, $1\leqq i\leqq n$, satisfying

$$\eta\,\| J_{\lambda,t}x-x\| \leqq \max\left(\frac{\lambda}{nt}\| S(nt)x-x\|, \| x-S(it)x\|\right). \tag{3.8}$$

We can now conclude the proof of Lemma 3.3. For this purpose, define

$$f(t)=\sup\{\| x-S(\tau)x\| : 0\leqq\tau\leqq t\}. \tag{3.9}$$

Given $\epsilon>0$, pick $\delta>0$ so that $f(\delta)\leqq\epsilon$. If $\delta/2\leqq nt\leqq\delta$, (3.8) implies that

$$\eta\,\| J_{\lambda,t}x-x\| \leqq \max(2\lambda\epsilon/\delta, \epsilon). \tag{3.10}$$

Since $2t\leqq\delta$ implies there is an integer n so that $\delta/2\leqq nt\leqq\delta$, (3.10) in turn yields

$$\| J_{\lambda,t}x-x\| \leqq \epsilon/\eta \tag{3.11}$$

for $0<t<\delta/2$ and $\lambda<\delta/2$. The proof is complete.

Lemma 3.12. *If there is a sequence $t_n\downarrow 0$ and a positive number λ such that*

$$J_\lambda x = \lim_{n\to\infty} J_{\lambda, t_n} x$$

exists for each $x \in C$, *then*

$$J_\mu x = \lim_{n\to\infty} J_{\mu, t_n} x$$

exists for $\mu > \lambda$. *Furthermore,*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x\right).$$

Proof. We recall that

$$J_{\lambda, t} = (I + \lambda A^t)^{-1} \text{ where } A^t = \frac{I - S(t)}{t}.$$

Thus the resolvent formula (Lemma 1.2(iv)) holds for $J_{\lambda,t}$ and we have

$$J_{\lambda, t_n} x = J_{\mu, t_n}\left(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_{\lambda, t_n} x\right). \tag{3.13}$$

Since each J_{μ, t_n} is a contraction,

$$\lim_{n\to\infty} J_{\mu, t_n} x_n = z \text{ and } \lim_{n\to\infty} x_n = x$$

imply that $\lim_{n\to\infty} J_{\mu, t_n} x = z$. Thus (3.13) and the hypothesis imply that

$$R\left(\frac{\mu}{\lambda} I + \frac{\lambda - \mu}{\lambda} J_\lambda\right) \subset \{x : \lim_{n\to\infty} J_{\mu, t_n} x \text{ exists}\}.$$

Since J_λ is a limit of contractions, it is also a contraction. To see that

$$C \subset R\left(\frac{\mu}{\lambda} I + \frac{\lambda - \mu}{\lambda} J_\lambda\right),$$

note that if $z \in C$,

$$z = \frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x$$

is equivalent to $x = T_z x$ where

$$T_z x = \frac{\lambda}{\mu} z + \frac{\mu - \lambda}{\mu} J_\lambda x$$

is a strictly contractive mapping of C into C.

As a consequence of Lemmas 3.3 and 3.12, we can now prove

THEOREM III. *Suppose there are sequences* $\{\lambda_m\}$ *and* $\{t_n\}$ *with* $\lambda_m \downarrow 0$ *and* $t_n \downarrow 0$ *such that*

$$\lim_{n\to\infty} J_{\lambda_m, t_n} x$$

exists for each m *and each* $x \in C$. *Then*

(a) $$J_\lambda x = \lim_{n\to\infty} J_{\lambda, t_n} x$$

exists for each $\lambda > 0$ *and* $x \in C$;

(b) $$A = \bigcup_{\lambda>0} \{[J_\lambda x, \lambda^{-1}(x - J_\lambda x)] : x \in C\}$$

is accretive with $R(I + \lambda A) = C$ *for* $\lambda > 0$ *and* $\overline{D(A)} = C$;

(c) $$D(A) \subset \{x \in C : S(t)x \text{ is Lipschitz continuous in } t\};$$

(d) $$S(t)x = \lim_{n\to\infty} (I + \frac{t}{n} A)^{-n} x$$

for each $x \in C$.

Proof. Conclusion (a) is an immediate consequence of Lemma 3.12. To obtain (b), we note that for each $t > 0$, $J_{\lambda,t}$ is a contraction which satisfies the resolvent equation (Lemma 1.2(iv)). Hence J_λ has the same properties. From this it follows easily that the A defined in (b) above is accretive. The range condition is also automatic. $\overline{D(A)} = C$ follows from Lemma 3.3 since for each $x \in C$,

$$\lim_{\lambda\downarrow 0} \| J_\lambda x - x \| = \lim_{\lambda\downarrow 0} \lim_{t\downarrow 0} \| J_{\lambda,t} x - x \| = 0$$

Assertion (c) follows from (3.1), which implies that for $x \in C$,

$$J_{\lambda,t_n} x + \lambda t_n^{-1}(J_{\lambda,t_n} x - S(t_n) J_{\lambda,t_n} x) = x,$$

so that

(3.14) $$L = \lim_{n\to\infty} \frac{\| J_{\lambda,t_n} x - S(t_n) J_{\lambda,t_n} x \|}{t_n} < \infty.$$

Since $S \in Q_0(C)$, it follows that L is a Lipschitz constant for $S(t)J_\lambda x$ as a function of t.

Since $\overline{D(A)} = C$, it suffices to verify (d) for $x \in D(A)$. We define approximating semi-groups S_λ on C by the equation

(3.15) $$\frac{d}{dt} S_\lambda(t) x = \frac{S(\lambda) S_\lambda(t) z - S_\lambda(t) x}{\lambda} = -A^\lambda S_\lambda(t) x.$$

The existence of the S_λ with

$$S_\lambda(t) = \lim_{n\to\infty} (I + \frac{t}{n} A^\lambda)^{-n}$$

follows from the remarks at the end of Section 2 and Theorem I, since A^λ is continuous, accretive, and $R(I+\eta A^\lambda) \supset C$ for all $\eta > 0$. By (2.28) we have

$$\| S_\lambda(\lambda n)x - S(n\lambda)x \| \leqq \sqrt{n} \, \| S(\lambda)x - x \| \leqq K\sqrt{n\lambda}\sqrt{\lambda}$$

where K is a Lipschitz constant for $S(t)x$. Hence if $t > 0$, $t = n\lambda + \delta$ and $0 \leqq \delta < \lambda$,

$$\begin{aligned} \| S_\lambda(t)x - S(t)x \| &\leqq \| S_\lambda(n\lambda)x - S(n\lambda)x \| \\ &\quad + \| S_\lambda(t)x - S_\lambda(n\lambda)x \| + \| S(t)x - S(n\lambda)x \| \\ &\leqq K(\sqrt{\lambda t} + 2\lambda). \end{aligned} \tag{3.16}$$

Note that we have used the fact that

$$\lambda^{-1} \| S(\lambda)x - x \| \leqq K$$

is also a Lipschitz constant for $S_\lambda(t)x$. Now,

$$\begin{aligned} \| S(t)x - (I + \frac{t}{n} A)^{-n}x \| &\leqq \| S(t)x - S_\lambda(t)x \| \\ &+ \| S_\lambda(t)x - (I + \frac{t}{n} A^\lambda)^{-n}x \| + \| (I + \frac{t}{n} A)^{-n}x - (I + \frac{t}{n} A^\lambda)^{-n}x \|. \end{aligned} \tag{3.17}$$

Taking $\lambda = t_k$, (3.16), (3.17), and (1.10) yield

$$\begin{aligned} \| S(t)x - (I + \frac{t}{n} A)^{-n}x \| &\leqq K(\sqrt{tt_k} + 2t_k) \\ &+ 2Kt/\sqrt{n} + \| (I + \frac{t}{n} A)^{-n}x - (I + \frac{t}{n} A^{t_k})^{-n}x \|. \end{aligned} \tag{3.18}$$

By part (a), $(I + (t/n)A^{t_k})^{-1}$ converges strongly to $(I + (t/n)A)^{-1}$ as $k \to \infty$. Thus, (3.18) becomes

$$\| S(t)x - (I + \frac{t}{n} A)^{-n}x \| \leqq 2Kt/\sqrt{n}, \tag{3.19}$$

and the proof is complete.

As an application, we show that if X is finite dimensional then the hypotheses of Theorem III are satisfied. If $S \in Q_0(C)$, and

$$J_{\lambda,t} = (I + \frac{\lambda}{t}(I - S(t)))^{-1},$$

then each $J_{\lambda,t}$ is a contraction on C. Moreover, if $x \in C$ and $\lambda \leqq 1$, (3.10) implies that $J_{\lambda,t}x$ is bounded uniformly in λ as $t \downarrow 0$. So $J_{\lambda,t}$ is bounded on each bounded subset of C as $t \downarrow 0$. By the Arzela-Ascoli Theorem and a diagonal argument, there is a sequence $\{t_n\}$ with $t_n \downarrow 0$ so that

$$\lim_{n\to\infty} J_{1/m,t_n}$$

exists uniformly on bounded subsets of C for each m, which completes the verification. This shows, in particular, that every contraction semi-group on a finite dimensional Banach space can be obtained from an accretive set via the exponential formula. As will be seen in the next section, however, there may be many accretive sets which yield the same semi-group in this way, even if X is two dimensional. This latter phenomenon can only occur, though, if X^* fails to be strictly convex.

In fact, with the tools we have developed one can prove, for example, that if $\tilde{A}$ is defined by

$$(I + \lambda\tilde{A})^{-1} = \lim_{t\downarrow 0}(I + \lambda A^t)^{-1}$$

then

$$S(t) = \lim_{n\to\infty}(I + (t/n)A)^{-n},$$

and

$$(I + \lambda A)^{-1}x = (I + \lambda\tilde{A})^{-1}x$$

for $x \in C$ if X^* is strictly convex. Moreover, if X^* is uniformly convex and $S(t)x$ is uniformly continuous on bounded (t, x) sets, then

$$\lim_{t\downarrow 0}(I + \lambda A^t)^{-1}$$

exists. These results, for example, imply the following theorem, which we state without proof.

THEOREM IV. *Let X be finite dimensional and X^* strictly convex. If $S \in Q_0(C)$ then there is exactly one set $A \subset X \times X$ such that*

(i) *A is accretive,*

(ii) *$\overline{D(A)} = C$,*

(iii) *$\bigcap_{\lambda>0} R(I + \lambda A) \supset C$ and A is the closure of $\bigcup_{\lambda>0} \{[J_\lambda x, \lambda^{-1}(x - J_\lambda)]$: $x \in C\}$, where $J_\lambda = (I + \lambda A)^{-1}$,*

(iv) *$S(t)x = \lim_{n\to\infty}(I + \frac{t}{n}A)^{-n}x$ for $x \in C$.*

It is interesting to note that with Theorem III and the remarks at the end of Section 2 we have completely recovered the Hille-Yosida criteria as special cases of nonlinear theorems. To see this we need only note that if S is linear, then $\lim_{t\downarrow 0}(I+(\lambda/t)(I-S(t)))^{-1}x$ exists for $x\in D(A)$. Indeed, in this case,

$$(I+\frac{\lambda}{t}(I-S(t)))^{-1}x=\frac{t}{\lambda+t}\sum_{i=0}^{\infty}(\frac{\lambda}{\lambda+t})^{i}S(it)x$$

and it is an exercise to show that the limit as $t\downarrow 0$ of the right hand side exists and is given by

$$\int_0^\infty e^{-t}S(\lambda t)x\,dt$$

(even if S is not linear).

Remark. Subsequent to the preparation of this manuscript it has been shown that $\lim_{t\downarrow 0} J_{\lambda,t}$ does not always exist, even if X is finite dimensional. See [5].

4. Examples. In this section we give examples illustrating various phenomena which show, among other things, that the results of Sections 2 and 3 are sharper than it might at first appear. These examples illustrate some of the variety of behavior which semi-groups can display.

The first example is of a semi-group $S\in Q_0(X)$, where $X=C([-1,1])$ which has the property that

$$S(t)=\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}$$

holds for an accretive set A with dense domain, but for which the quotients $(S(t)z-z)/t$ do not converge (in any sense) as $t\downarrow 0$ to an element of X for any $z\in X$. In other words, any of the usual X-valued generators one would assign to S are emplty. The construction is of the following type: For each $x\in[-1,1]$ a semi-group $S_x\in Q_0(R)$ is chosen. Then S is defined by

$$(S(t)f)(x)=S_x(t)f(x),\qquad f\in C([-1,1]). \tag{4.1}$$

This S will be in $Q_0(X)$ if S_x depends on x in such a way that the right hand side of (4.1) is in X.

If $g: R\to R$ is monotone decreasing, let

$$g_{\pm}(x) = \lim_{h \downarrow 0} g(x \pm h),$$

and

$$(4.2) \qquad g^0(x) = \begin{cases} g_-(x) & \text{if } g_-(x) < 0 \\ 0 & \text{if } g_-(x) \geqq 0 \text{ and } g_+(x) \leqq 0 \\ g_+(x) & \text{if } g_+(x) > 0. \end{cases}$$

It is easy to deduce from [6, Theorem I] that there is a semi-group $T \in Q_0(R)$ such that

$$(4.3) \qquad D^+T(t)x = g^0(T(t)x)$$

for $t \geqq 0$ and $x \in R$, where D^+ denotes the right derivative. Choose g to have the following properties:

(4.4)
(i) g is monotone decreasing, and
(ii) the discontinuities of g are dense in R.

If one sets $S_x = T$ for $-1 \leqq x \leqq 1$ in (4.1), where T satisfies (4.3) and (4.4), then clearly $S \in Q_0(C[-1,1])$ and

$$(4.5) \qquad \lim_{t \downarrow 0} \frac{(S(t)f)(x) - f(x)}{t} = g^0(f(x))$$

for $x \in [-1,1]$. In view of (4.4), the only sets K for which $g^0(K)$ is connected are singletons, so the only $f \in C([-1,1])$ for which $g^0 \circ f$ is continuous are the constants. Thus (4.5) implies that $(S(t)f - f)/t$ has no limit as $t \downarrow 0$ in $C([-1,1])$ unless f is constant.

To eliminate the differentiability of $S(t)f$ for all constant functions f, we must take S_x in (4.1) to depend on x in a suitable way. Let g satisfy (4.4) and

$$(4.6) \qquad g \text{ is continuous at } 0.$$

Define a family g_x, $-1 \leqq x \leqq 1$, of monotone functions by

$$(4.7) \qquad g_x(y) = g(x^2 y + x)$$

for $y \in R$ and $-1 \leqq x \leqq 1$, and a corresponding family of semi-groups $S_x \in Q_0(R)$ by

$$(4.8) \qquad D^+S_x(t)y = g_x{}^0(S_x(t)y)$$

for $t \geqq 0$, $y \in R$ and $x \in [-1,1]$. One finds that if T satisfies (4.3), then

$$(4.9) \qquad S_x(t)y = \begin{cases} x^{-2}(T(tx^2)(x^2y+x)-x) & \text{if } x \neq 0 \\ y + tg(0) & \text{if } x = 0. \end{cases}$$

The S defined by (4.1) then lies in $Q_0(C[-1,1])$. The semi-group and contraction properties are evident. The continuity of $S(t)f(x)$ in x for $f \in C([-1,1])$ is obvious except at $x=0$, and continuity at $x=0$ follows from (4.6). Moreover,

$$\lim_{t \downarrow 0} \frac{(S(t)f)(x) - f(x)}{t} = g^0(x^2f(x)+x).$$

But, as above, $g^0(x^2f(x)+x)$ is not continuous in x unless $x^2f(x)+x$ is constant, which is impossible for continuous f on $[-1,1]$.

If we take $S \in Q_0(C[-1,1])$ as above, it is true that

$$(4.10) \qquad \lim_{t \downarrow 0} (I + \frac{\lambda}{t}(I - S(t)))^{-1}$$

exists strongly for $\lambda > 0$. So, by the results of Section 3, it defines an accretive set A such that

$$S(t) = \lim_{n \to \infty} (I + \frac{t}{n}A)^{-n}.$$

In view of Theorem II, there are *no* Lipschitz continuous functions u with values in $C([-1,1])$ such that

$$0 \in \frac{du}{dt} + Au \qquad \text{a.e.}$$

For this A, problem (2.1) has no strong solution on any interval for any initial data. The existence of the limit (4.10) follows, for example, from the estimates of [11]. We do not give details.

Remarks. Making very simple choices of g in (4.3) and setting $S_x = T$ for $x \in [-1,1]$ in (4.1) yields semi-groups S with bad behavior. For example, if g is 2 on $(-\infty, 0)$ and 1 on $[0,\infty)$, the corresponding semi group is given by $\lim_{n \to \infty} (I + (t/n)A)^{-n}$ for an appropriate A, but $S(t)$ does not leave $D(A)$ invariant. Indeed, in this case, if

$$f(x) = x^2 - 1,$$

then $f \in D(A)$ but $S(t)f \notin D(A)$ for $0 < t < 1/2$.

An example of nondifferentiable behavior which is easy to write down explicitly is the following. Let K be the closed convex set of functions f in $C([0,1])$ which satisfy

$$0 \leqq f(x) \leqq x \text{ for } 0 \leqq x \leqq 1.$$

The semi-group $S \in Q_0(K)$ is defined by

$$(4.11) \qquad (S(t)f)(x) = (t + f(x)) \wedge x$$

for $t \geqq 0$ and $0 \leqq x \leqq 1$. It is easy to verify that $S(t)f$ is differentiable at $t = 0$ iff $f(x) \equiv x$. However,

$$S(t) = \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n}$$

where

$$A = \bigcup_{\lambda > 0} \{[(\lambda + f) \wedge x, \frac{f - (\lambda + f) \wedge x}{\lambda}] : f \in K\}$$

is found by computing $\lim_{t \downarrow 0} (I + (\lambda/t)(I - S(t)))^{-1}$.

We now turn to examples of a different kind. Having shown that one cannot hope to describe semi-groups of nonlinear transformations by any usual notion of infinitesimal generator, one might hope to do so by means of the exponential formula. There are essential difficulties here. One problem is that if $S \in Q_\omega(C)$, we do not know if

$$S(t) = \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n}$$

for some A. Section 3 presented some results in this direction. However, even in finite dimensional spaces, where one has the existence of an A, it turns out that A is not unique. This is shown by our next example.

Let $X = R^2$ equipped with the norm

$$(4.12) \qquad \|(a, b)\| = \max(|a|, |b|).$$

Let $g \colon [-1, 1] \to [-1, 1]$ satisfy

(4.13) $g(-1) = +1$, $g(+1) = -1$, g is continuous, and g is monotone nonincreasing.

To such a g, we assign a subset A_g of $R^2 \times R^2$ as follows. Take $D(A) = R^2$ and

$$(4.14) \qquad A_g(a, b) = \begin{cases} \{(-1, 1)\} & \text{if } b > a \\ \{(x, g(x)) : -1 \leqq x \leqq 1\} & \text{if } a = b \\ \{(-1, 1)\} & \text{if } b < a. \end{cases}$$

The set A_g is accretive and $R(I + \lambda A_g) = X$ for $\lambda > 0$. The semi-group S_g, which is defined by

$$S_g(t) = \lim_{n\to\infty} (I + \frac{t}{n} A_g)^{-n},$$

depends on g only through the value $x_g = (I - g)^{-1}(0)$. Note that $(I - g)(-1) = -2$ and $(I - g)(1) = 2$, so since $I - g$ is strictly monotone and continuous, it takes on the value zero once and x_g is well defined. For example, if $g(x) = -x$ and $h(x) = -x^3$, $A_g \neq A_h$ while $S_g = S_h$ (since $x_g = x_h = 0$).

We verify some of the details in this example, since the ideas are relatively unfamiliar. To show that A_g is accretive, we need to verify that

$$(4.15) \qquad \| (a_1, b_1) - (a_2, b_2) + \lambda((c_1, d_1) - (c_2, d_2)) \| \geqq \| (a_1, b_1) - (a_2, b_2) \|$$

for $\lambda \geqq 0$ and $(c_i, d_i) \in A_g(a_i, b_i)$. Consider the following four cases:

$$(4.16) \quad \begin{array}{ll} \text{(i)} & b_1 > a_1 \text{ and } b_2 < a_2 \\ \text{(ii)} & b_1 > a_1 \text{ and } b_2 = a_2 \text{ or } b_1 < a_1 \text{ and } b_2 = a_2 \\ \text{(iii)} & b_1 = a_1 \text{ and } b_2 = a_2 \\ \text{(iv)} & b_1 > a_1 \text{ and } b_2 > a_2 \text{ or } b_1 < a_1 \text{ and } b_2 < a_2. \end{array}$$

In case (iv), (4.15) reduces at once to an equality. For (4.16)(i), (4.15) becomes

$$\max(| a_1 - a_2 - 2\lambda |, | b_1 - b_2 + 2\lambda |) \geqq \max(| a_1 - a_2 |, | b_1 - b_2 |)$$

where $b_1 - b_2 > a_1 - a_2$. If $| a_1 - a_2 | \geqq | b_1 - b_2 |$, it follows that $0 > a_1 - a_2$, so $|(a_1 - a_2) - 2\lambda| \geqq | a_1 - a_2 |$ for $\lambda \geqq 0$. If $| b_1 - b_2 | \geqq | a_1 - a_2 |$, then $b_1 - b_2 > 0$ and $|(b_1 - b_2) + 2\lambda| \geqq | b_1 - b_2 |$, which was to be shown. Case (4.16)(ii) is similar and is omitted. In the remaining case (iii), (4.15) becomes

$$\max(|(a_1 - a_2) + \lambda(x - y)|, |(b_1 - b_2) + \lambda(g(x) - g(y)|) \geqq \max(| a_1 - a_2 |, | b_1 - b_2 |)$$

for $-1 \leqq x, y \leqq 1$. Since $a_1 - a_2 = b_1 - b_2$, the above inequality follows from the monotonicity of g (i.e., $x - y$ and $g(x) - g(y)$ cannot both have the same strict sign).

One easily sees that $R(I + \lambda A_g) = X$ for $\lambda > 0$ and that S_g is determined by the invariance of line $L = \{(a, a) : a \in R\}$ under each $S_g(t)$ and velocities $(1, -1)$ above L, $(-1, 1)$, below L and $(-x_g, -x_g)$ on L.

This example illustrates other phenomena as well. Recalling Lemma 2.16(e), it is worth noting that if g is strictly monotone,

$$\langle (x, g(x)) - (y, g(y)), (a, a) - (b, b) \rangle_i < 0$$

unless $x = y$ or $a = b$. Thus Kato's notion of accretiveness, which we have been using, is superior to that defined, e. g., in [6, Sec. 6]. Moreover, the set $A_g(0, 0)$ is not convex unless g is the identity. It is especially interesting in this situation that

$$\lim_{t \downarrow 0} (I + \frac{\lambda}{t} (I - S_g(t)))^{-1}$$

exists and defines A_h where the graph of h consists of the two line segments joining $(-1, 1)$ to (x_g, x_g) and (x_g, x_g) to $(1, -1)$. Indeed, if $X = R^2$ with the norm (4.12) and $C \subseteq R^2$ is closed and convex, then $S \in Q_0(C)$ implies that

$$\lim_{t \downarrow 0} (I + \frac{\lambda}{t} (I - S(t)))^{-1} x$$

exists for $x \in C$ and $\lambda \geqq 0$. See [5].

UNIVERSITY OF CALIFORNIA, LOS ANGELES.

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