5. Aliasing

In the last lecture we derived the discrete Fourier transform (DFT) which can be defined as

$$Y_k = \sum_{j=0}^{N-1} y_j \exp\left(-\frac{i2\pi jk}{N}\right)$$
$$y_j = \frac{1}{N}\sum_{k=0}^{N-1} Y_k \exp\left(\frac{i2\pi jk}{N}\right) \tag{5-1}$$

We have shown that for the DFT $Y_{k+N} = Y_k$ and we also know that if y_j is a real series $Y_{-k} = Y_k^*$ (i.e., the coefficients for negative and positive k are complex conjugates). From this we can infer that if we specify Y between 0 and $N/2$ we have specified it everywhere. Between $N/2$ and N the amplitudes are reflected with the phase of opposite sign and above N and below 0 the series repeat.

An N sample time series with a sample interval Δt will have a length or period of $N\Delta t$. The spacing of frequency samples is $1/T = 1/N\Delta t$ and the Fourier Transform is periodic (i.e., will repeat itself) at a frequency interval of $N \times \left(1/N\Delta t\right)$ or $1/\Delta t$.

$N/2$ corresponds to a frequency, termed the Nyquist frequency (or folding frequency)

$$f_{Ny} = \frac{1}{2\Delta t}$$

If we know that our digitized signal y_j is composed entirely of frequencies less than f_{Ny} then there is no ambiguity in discrete Fourier transform. The DFT repeats outside the frequency band $-f_{Ny} < f < f_{Ny}$ but we know that these repetitions do not correspond to the continuous signal $y(t)$.

However if the continuous signal contained frequencies above f_{Ny}, these higher frequencies fold into frequencies below f_{Ny} when the signal is sampled. If we take a DFT and an inverse DFT we will reconstruct the sampled time series exactly, but we would interpret the frequency content of the signal incorrectly. The spectra of adequately sampled and aliased signals are illustrated in Figure 5.1.

It is important to understand that aliasing is a problem that occurs when the signal is digitized and not a problem in the DFT itself. If a time series is sampled at too low a frequency then the information at higher frequencies is mapped into lower frequencies (Figure 5.2) leading to a loss of information and an ambiguity that cannot be resolved from the digitized signal. From a practical standpoint, geoscientists recording digital signals must either chose a sampling rate that is higher than the frequencies in the natural signal or they must filter the signal before digitizing so that it only contains frequencies below the Nyquist

You will spend some time in the class and more in Exercise 3 understanding aliasing in the time domain. Once you understand aliasing in the time domain, it will hopefully be easier to see how it is manifested in the frequency domain.

Angular Frequencies

The angular Nyquist frequency is the Nyquist frequency multiplied by 2π

$$\omega_{Ny} = \frac{\pi}{\Delta t} \tag{5-2}$$

If $\Delta t = 1$, $\omega_{Ny} = \pi$ and the angular frequency of a discrete series extends from $-\pi$ to π and repeats outside this interval. Quite a few functions is *MATLAB,* notably filter design functions, specify frequencies in terms of this normalization.

Even and Odd Number of Samples

Since there are N samples in the time series, we know that these must be represented by N independent quantities in the frequency domain.

For N odd, $N/2$ is not an integer and therefore does not correspond to a frequency sample. The N independent frequency values are the real coefficient Y_0 (the DC amplitude of a signal has no phase and so must be real) and complex coefficients Y_1 to $Y_{(N-1)/2}$.

For N even, $N/2$ is an integer and therefore corresponds to a frequency sample. The N independent frequency values are the real coefficient Y_0, the complex coefficients Y_1 to $Y_{(N-1)/2}$ and a real coefficient at $Y_{N/2}$. To understand why the coefficient $Y_{N/2}$ is real, consider sine and cosine waves that are sampled at exactly twice their frequency starting with a sample at time 0 - the sine wave (the imaginary component) is always sampled where it is zero.

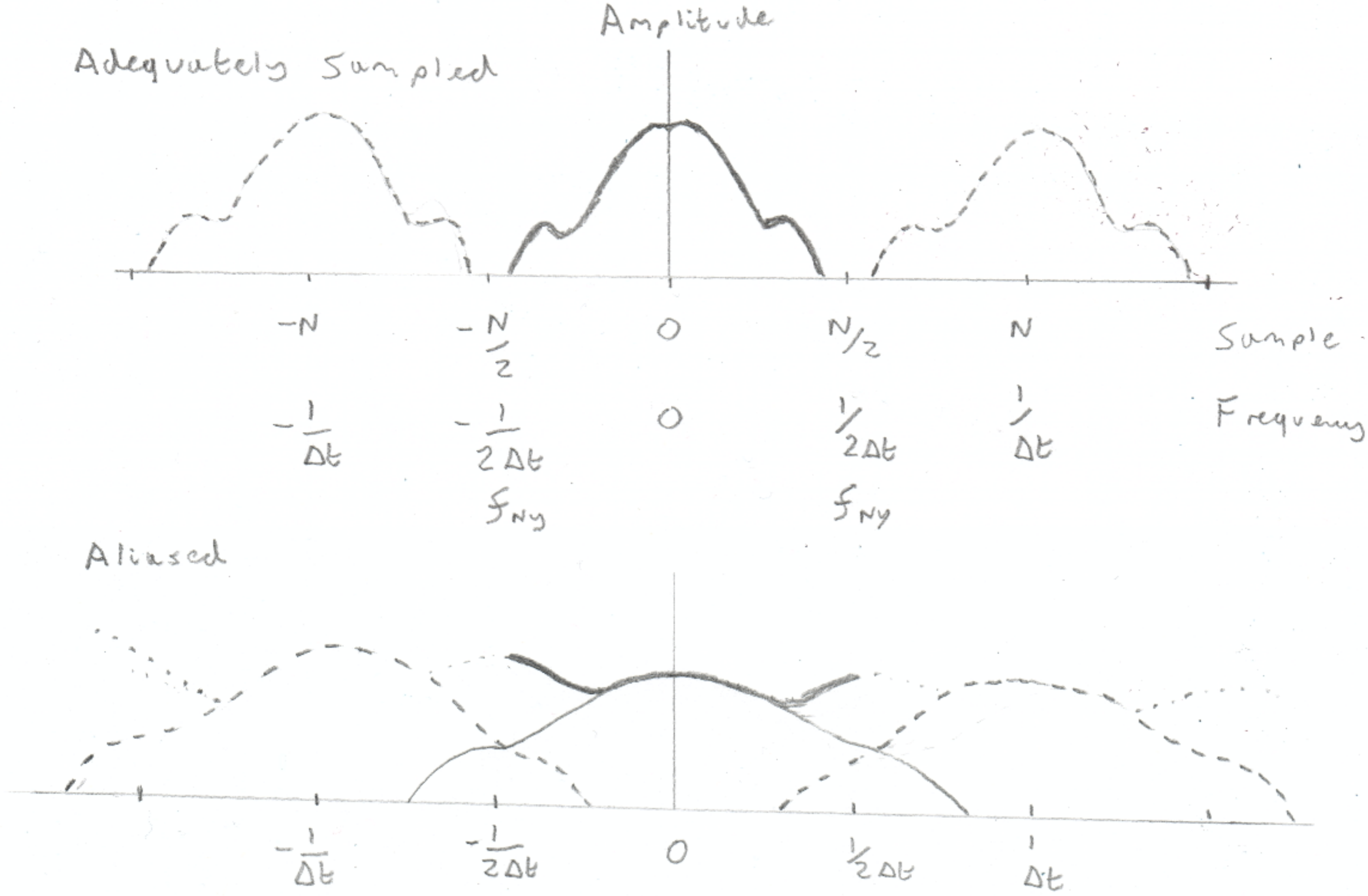

Figure 5.1 Spectra of adequately sampled and aliased time series.

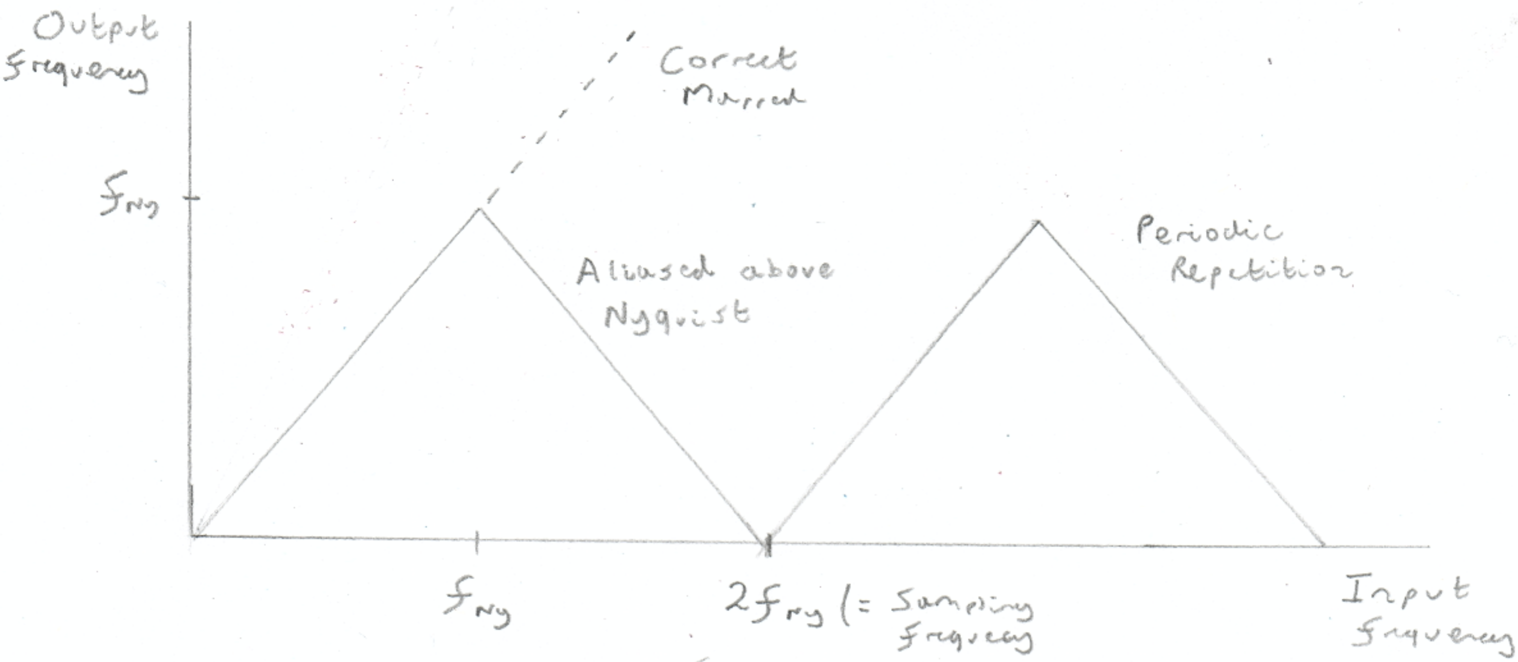

Figure 5.2 Mapping of frequencies by sampling.