Homework 2. Fourier Transform of a Gaussian and Convolution

Note that your written answers can be brief but include plots wherever meaningful/necessary.

1. In class we have looked at the Fourier transform of continuous functions and we have shown that the Fourier transform of a delta function (an impulse) is equally weighted in all frequencies. A sharp function in the time domain yields a very broad function in the frequency domain, and vice versa. This is a form of the uncertainty principle and a very important concept in signal analysis – it is not possible to have a very narrowband signal of short duration. We can demonstrate this by considering the Gaussian or normal distribution that we will encounter later on in the class

The Gaussian distribution is given by

$$g(t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\tau} \exp\left(-\frac{t^2}{2\tau^2}\right), \quad -\infty < t < \infty \tag{1}$$

This function has a mean of zero and a root mean squared (RMS) deviation (a measure of its width) of τ, which is defined by

$$\sqrt{\int_{-\infty}^{\infty} t^2 g(t)dt \Big/ \int_{-\infty}^{\infty} g(t)dt} \,. \tag{2}$$

It turns out that the Fourier transform of a Gaussian is another Gaussian (showing so requires the use of complex variable theory).

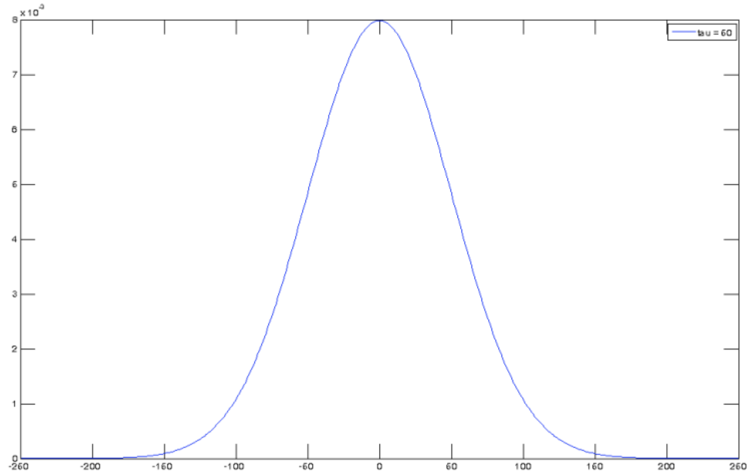
(i) Write a program that returns the Gaussian distribution given input arguments of a vector of times `t` and a value of RMS deviation `tau`. Plot an example. (10pts)

This one's easy, the program should return a bell curve.

```
tau = 50;
t = -5*tau:5*tau;

g = gaussian(t,tau);

% % function g = gaussian(t,tau)
% % % ex05_leakageCC, Problem 1 (i)
% %
% % g = exp(-t.^2./(2*tau^2))./sqrt(2*pi)./tau;
```



(ii) Write a second function to calculate equation (2) and verify that the input Gaussian distribution has an RMS deviation of τ. (Note: since you are doing the integration numerically you will not show this exactly but you should be able to get pretty close provided your input times are fairly finely sampled and extend from -5τ to 5τ) (10pts)

They should write a second function that calculates tau, like so

```
Tau = gaussian_RMS(t,g);

% % function tau = gaussian_RMS(t,g)
% % % ex05_leakageCC, Problem 1 (ii)
% %
% % dt = diff(t);
% % dt = dt(1);
% %
% % tau = sqrt( sum( t.^2.*g*dt) ./ sum( g*dt) );
```

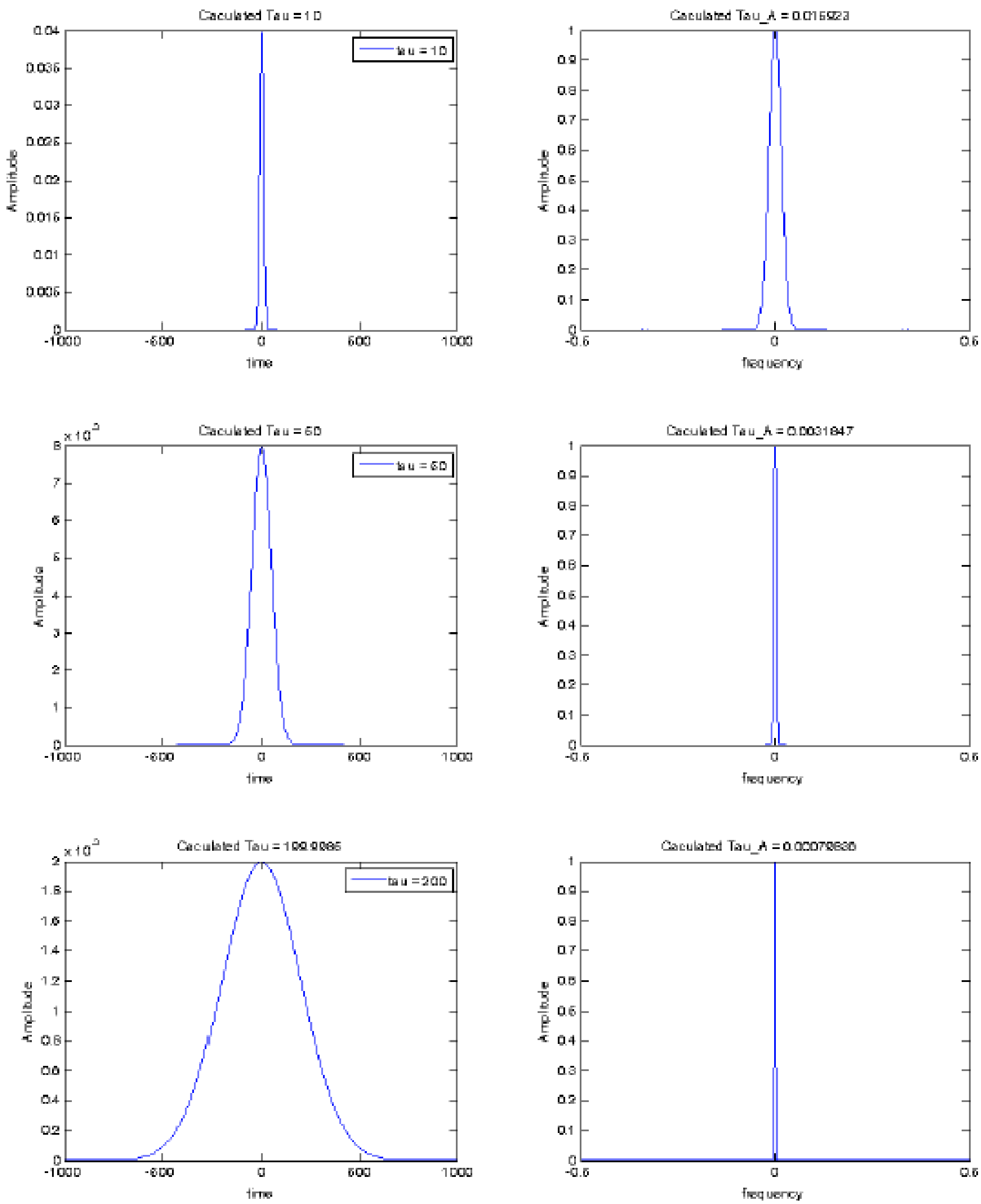
And then check that for a gaussian defined by (1) they get the right answer as output with this second function

(iii) The Fourier transform of (1) is:

$$G(\omega) = \exp\left(-\frac{\omega^2\tau^2}{2}\right). \qquad (3)$$

Write a function to calculate the amplitude spectrum and explore qualitatively with the help of plots how changes in the width of the Gaussian function you create in the time domain (i.e., variations in parameter τ) affect the width of the amplitude spectrum. (10pts)

The function should be used to generate plots like these (time domain is left column, freq. domain is right column)



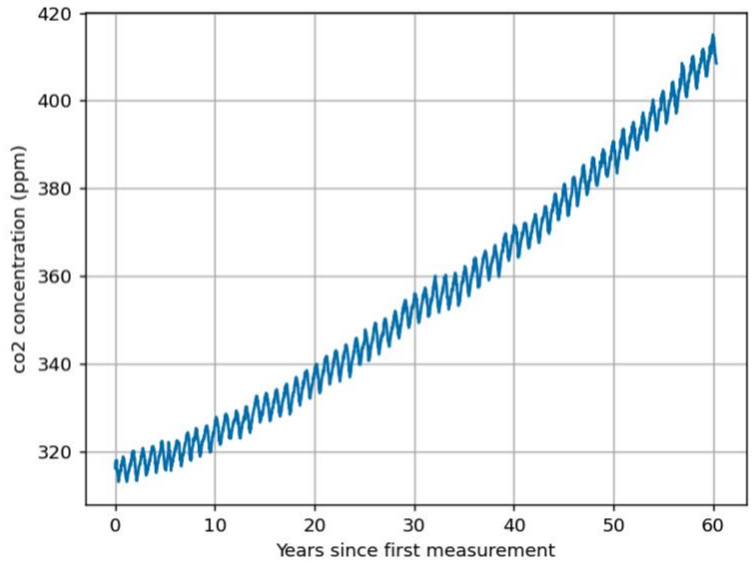
(iv) By applying the function you wrote in (2) to the amplitude spectra of the Gaussian function with different values of τ, deduce a relationship between the width of a Gaussian in the time domain and the width of its amplitude spectrum. (5pts)

Large tau in time domain -> narrow tau in frequency domain and vice versa. More formally time domain tau = 1/freq domain tau

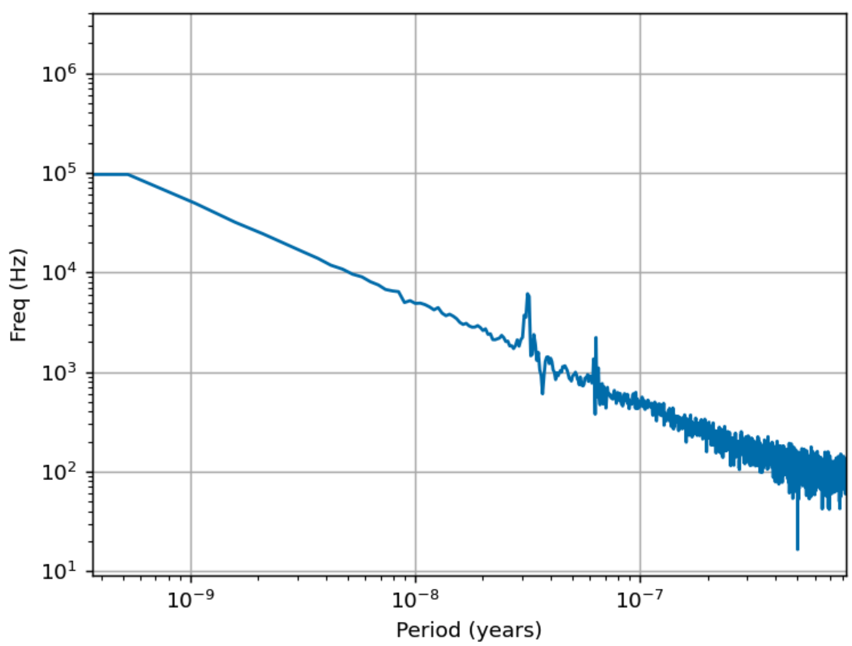
2. On Canvas under the week 3 module find the file “maunaloa_weekly.csv”. This contains CO2 measurements at Mauna Loa (the famous “Keeling Curve”) at weekly (dt=7 days) intervals.

(i) Plot the data. Qualitatively answer: Are there any periodicities in the data? If so at what periods? Sketch what you would expect the Fourier amplitude spectrum of the data would look like. (10pts)

Data, should have labels!



The sketch of the amplitude spectrum can be whatever but MSUT have a peak at periods of 1 year. If they are using units of seconds for time this would be at f~=3e-8 Hz. Here’s the actual spectra, it’s ok if they miss the second peak or the linear trend, or add other peaks. Mostly I want to see the peak at 1 year periods



(ii) A common means to smooth data is to compute an N-point running mean. That is, each point is taken as an average of N points centered upon the point of interest – it is simplest to chose N odd with $N = 2M + 1$. Explain why this is equivalent to convolving the data with a time series of N points with values of $1/N$. (5pts)

Smooth the data with an N-point running mean: N is odd = 2M+1 points. This is like convolving the data with a box car that has a total area of 1. Or summing the N data points around the point of interest and then dividing by N to get the mean.

(iii) Smooth the co2 data with this method using the Python function `numpy.convolve()` or the *MATLAB* `conv` command for different choices of N. What is the best choice of N to get rid of yearly fluctuations? Why is this value best? What happens at either end of the time series after convolving? (10pts)

The best N is anything that is longer (though not crazy long) than the yearly fluctuations you want to remove. So if the sampling is weekly data then N>104ish (2 years) is probably good. The time series drops to zero at each end because there aren't enough points to convolve with

(vi) We can also smooth the temperature data by convolving it with a Gaussian. Create a Gaussian function using the command you wrote for the last exercise whose RMS deviation τ is equivalent to the half-width of the boxcar. Multiply the Gaussian by a constant so that the sum of all the points in unity. Perform convolution in the time domain. Can you see any significant differences to the result from smoothing with a boxcar? (10pts)

Smoothing with the Gaussian should be, well, smoother, than with the box car (simple running mean) because the Fourier transform of the boxcar has more ripples. This should manifest as more "high frequency" wiggles left over after boxcar smoothing.