Data Processing and Analysis

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GEOP 505/Math 587 Primer on Complex Numbers and Arithmetic

These notes summarize some important facts about complex numbers and their arithmetic.

Rectangular Form

If we define $i = \sqrt{-1}$, then we can construct a system of complex numbers of the form $z = a + bi$. In this system,

$$(a+bi)+(c+di) = (a+c)+(b+d)i \tag{1}$$

$$(a+bi)-(c+di) = (a-c)+(b-d)i \tag{2}$$

$$(a+bi)(c+di) = ac+bci+adi+bdi^2 = (ac-bd)+(bc+ad)i \tag{3}$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \tag{4}$$

The *complex conjugate* of a complex number is given by

$$(a+bi)^* = \overline{a+bi} = a-bi. \tag{5}$$

Euler's Formula

Using the power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{6}$$

we can derive a formula for e raised to an imaginary power.

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots$$

Using the facts that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$, we find that

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Rearranging the terms, we get

$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots)$$

Finally, using the Taylor's series for sin and cos, we get

$$e^{ix} = \cos(x) + i\sin(x) \tag{7}$$

For general complex numbers $a + bi$, we find that

$$e^{a+bi} = e^a(\cos(b) + i\sin(b)) \tag{8}$$

Polar Form

Using Euler's formula, we can take any complex number

$$z = a + bi$$

and rewrite it as

$$z = Re^{i\theta} \tag{9}$$

where

$$R = |z| = \sqrt{z^* z} = \sqrt{a^2 + b^2} \tag{10}$$

is variously called the *amplitude*, *modulus*, or *complex norm* and

$$\theta = \angle z = \tan^{-1}\frac{b}{a}. \tag{11}$$

is variously called the *complex angle*, *phase* or *argument* of z. Because sin and cos are 2π periodic, we can add any multiple of 2π to the phase of a complex number without changing its value.

We can also go the other way. If

$$z = Re^{i\theta}$$

then $z = a + bi$, where

$$a = R\cos(\theta) \tag{12}$$

and

$$b = R\sin(\theta). \tag{13}$$

Polar form is very useful for multiplication, division, and exponentiation, but hopeless for addition and subtraction.

$$Ae^{i\theta}Be^{i\phi} = ABe^{\theta+\phi} \tag{14}$$

$$\frac{Ae^{i\theta}}{Be^{i\phi}} = \frac{A}{B}e^{\theta-\phi} \tag{15}$$

$$(Ae^{i\theta})^x = (A^x)e^{ix\theta}. \tag{16}$$

It's also easy to find the complex conjugate of a number in polar form.

$$(Ae^{i\theta})^* = Ae^{-i\theta}. \tag{17}$$

Cosine and Sine in terms of complex exponentials

Using Euler's formula, it's easy to derive formulas for sin and cos in terms of complex exponentials.

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{18}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{19}$$

MATLAB and Complex Numbers

When you first start MATLAB, The variable i is set equal to $\sqrt{-1}$. However, if you change the value of i (for example by using it as the index in a for loop!), then it will no longer have this value. Thus it is a good idea to avoid using i as a loop index.

Nearly all of the functions that are built into MATLAB operate correctly on complex numbers. Thus you can add, subtract, multiply, and divide complex numbers. You can also compute exponentials, logs, sines, cosines, and other functions of complex numbers. MATLAB has several useful functions for manipulating complex numbers. The conj function computes the complex conjugate of a number. The abs function computes the absolute value of a complex number. The angle function computes the phase angle of a complex number.

How LTI's operate on complex exponentials, sines, and cosines

A linear time invariant (LTI) system operates in a simple fashion when a complex exponential, sine, or cosine is input to the system. Recall that if $\phi(t)$ is the impulse response of the system (that is, the response of the system when the input $\delta(t)$), then response to an input function, $x(t)$, is given by the convolution of $x(t)$ with the impulse response.

$$y(t) = \phi(t) * x(t) \tag{20}$$

or

$$y(t) = \int_{-\infty}^{\infty} \phi(\tau)x(t-\tau)d\tau. \tag{21}$$

Consider the special case where

$$x(t) = e^{i2\pi ft}. \tag{22}$$

In this case,

$$y(t) = \int_{-\infty}^{\infty} \phi(\tau) e^{i2\pi x(t-\tau)} d\tau. \tag{23}$$

$$y(t) = e^{i2\pi ft} \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi f\tau} d\tau. \tag{24}$$

Notice that this integral depends only on f, and not t. We can define

$$\Phi(f) = \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi f\tau} d\tau. \tag{25}$$

Where $\Phi(f)$ is the *Fourier Transform* of $\phi(t)$. Then

$$y(t) = e^{i2\pi ft}\Phi(f). \tag{26}$$

Since $\Phi(f)$ is just a complex number, we can write it in exponential form as $\Phi(f) = A(f)e^{i\theta(f)}$, where $A(f)$ and $\theta(f)$ are real numbers. Now, we can write $y(t)$ as

$$y(t) = e^{i2\pi ft} A(f) e^{i\theta(f)}. \tag{27}$$

This says that if we use a complex exponential signal as the input to our LTI system, we'll get a complex exponential signal as the output. The factor $A(f)$ amplifies or attenuates the signal, and the factor $e^{i\theta(f)}$ shifts the phase of the signal.

What if $x(t) = \cos(2\pi ft)$ where f and t are a real frequency and time? We can use (**??**) to write the cosine in terms of complex exponentials.

$$x(t) = \frac{e^{i2\pi ft} + e^{-i2\pi ft}}{2}. \tag{28}$$

Then using the principles of scaling and superposition and (**??**), we get that

$$y(t) = \frac{e^{i2\pi ft}\Phi(f)}{2} + \frac{e^{-i2\pi ft}\Phi(-f)}{2}. \tag{29}$$

It can be shown that if $\phi(t)$ is real, then

$$\Phi(-f) = \Phi(f)^* \tag{30}$$

To see this, simply take the complex conjugate of $\Phi(f)$.

$$\Phi(f)^* = \left(\int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi f\tau} d\tau. \right)^* \tag{31}$$

$$\Phi(f)^* = \int_{-\infty}^{\infty} \left(\phi(\tau) e^{-i2\pi f\tau} \right)^* d\tau. \tag{32}$$

$$\Phi(f)^* = \int_{-\infty}^{\infty} \phi(\tau) e^{+i2\pi f \tau} d\tau. \tag{33}$$

$$\Phi(f)^* = \int_{-\infty}^{\infty} \phi(\tau) e^{-i2\pi(-f)\tau} d\tau. \tag{34}$$

$$\Phi(f)^* = \Phi(-f). \tag{35}$$

Equivalently, this says that $\Phi(f)$ for a real-valued impulse response $\phi(t)$ has even symmetry for its real part, and odd symmetry for its imaginary part (relative to $f = 0$.)

Since $\Phi(f) = A(f)e^{i\theta(f)}$, and $\Phi(-f) = \Phi(f)^*$, $\Phi(-f) = A(f)e^{-i\theta(f)}$. Thus

$$y(t) = \frac{e^{i2\pi ft} A(f) e^{i\theta(f)}}{2} + \frac{e^{-i2\pi ft} A(f) e^{-i\theta(f)}}{2}. \tag{36}$$

$$y(t) = \frac{e^{i(2\pi ft + \theta(f))} + e^{-i(2\pi ft + \theta(f))}}{2} A(f). \tag{37}$$

$$y(t) = \cos(2\pi ft + \theta(f)) A(f). \tag{38}$$

This shows that the output of the LTI system with a real-valued $\phi(t)$ and a cosine input is also a cosine, but with its magnitude scaled by the amplitude, $A(f)$, and shifted in phase by the angle $\theta(f)$. Similarly, it's easy to show that if the input is a sine wave with frequency f, then the output will be a sine wave scaled by the amplitude $A(f)$ and shifted in phase by the angle $\theta(f)$.