3. Convolution and Linear Filters

I have drawn the content for this lecture mostly from Chapter 1 of Bob Crosson's notes on *Data Analysis.*

We will start this class with a thought experiment, which is illustrated in Figure 3.1. We know that if we strike a gong with a hammer and make a ringing that will progressively decay in amplitude. If we strike it twice at a large time interval it will make two discrete sounds. If we strike it twice in quick succession the sounds will merge. But what happens if we hit it many times in quick succession? Convolution is the process that allows us to predict how the gong will respond to an arbitrarily complex set of blows provided we know how it responds to a single blow.

Linear Filters (Systems)

Before discussing convolution, we will first introduce the concept of a linear filter or a linear system (two alternate names for the same thing)

A filter or a system is physical or conceptual objects that has an input and output. For our gong example the input is the sequence of hammer blows and the output is the resulting oscillation that generates a sound. We can write this mathematically as

$$x(t) \rightarrow L \rightarrow y(t) \quad or \quad y(t) = L\left[x(t)\right] \tag{3-1}$$

where $x(t)$ is the input, $y(t)$ the output and L is a the filter.

A filter is described as "linear shift invariant" or just "linear" if it has the following 3 properties:

1. Shift invariance (the filter is not dependent on the time origin)
$$L\left[x(t+a)\right] = y(t+a) \tag{3-2}$$
2. Scale invariant
$$L\left[ax(t)\right] = ay(t) \tag{3-3}$$
3. Superposition invariant
$$L\left[x_1(t) + x_2(t)\right] = L\left[x_1(t)\right] + L\left[x_2(t)\right] \tag{3-4}$$

An example of a filter that fails the 2nd and 3rd criteria is the squaring filter

$$L\left[x(t)\right] = x^2(t) \tag{3-5}$$

A filter that fails the first criterion is one that depends upon time such as

$$L\left[x(t)\right] = x + t \tag{3-6}$$

Two important properties of filters are causality and stability.

- A causal filter gives no output until there is an input. In the real physical world, all natural filters are causal but when we process recorded digital data we can also apply acausal filters that generate an output before the input.
- A stable filter is one in which the output is bounded, that is it decays with time to zero (an example of an unstable filter occurs when the microphone gets placed near the speaker).

Convolution

Convolution is denoted by the "*" symbol and is defined mathematically by

$$c(t) = a(t) * b(t) = \int_{-\infty}^{\infty} a(\tau) b(t-\tau) d\tau \tag{3-7}$$

Physically convolution involves reversing one time series, lagging it (shifting it in time), multiplying the two functions and integrating. It is a very important concept to understand and there are some nice applets on line that do this. For example

http://maxwell.me.gu.edu.au/spl/Excalibar/Jtg/Conv.html

To see how things work check the animate box and use functions that are asymmetric or better still chose the "PW Linear" option and input your own with the mouse.

Convolution has some important mathematical properties. It is commutative

$$a(t) * b(t) = b(t) * a(t) \tag{3-8}$$

which means that it does not matter which of the functions is reversed during the convolution operation.

Convolution is associative

$$\left[a(t) * b(t)\right] * c(t) = a(t) * \left[b(t) * c(t)\right] \tag{3-9}$$

Very often we can think of our recordings of natural signals as the result of applying a sequence of filters. For example a seismic wave passing through the Earth to a seismometer is first filtered by the Earth itself, then filtered by the seismic sensor (i.e., the input signal is the ground motion and an output signal is the voltage generated by the seismometer) and finally by the recording device (the input signal is the voltage from the seismometer and the output digitized time series). The associative property tells us that the order of the filters is immaterial.

It is distributive

$$a(t) * \left[b(t) + c(t)\right] = \left[a(t) * b(t)\right] + \left[a(t) * c(t)\right] \tag{3-10}$$

which means that the sum of two convolved signals is the same as the convolution of the sum.

The Fourier transform of a convolution is

$$\begin{aligned}
FT\left[a(t) * b(t)\right] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} a(\tau) b(t-\tau) d\tau\right] \exp(-i2\pi f t) dt \\
&= \int_{-\infty}^{\infty} a(\tau) \exp(-i2\pi f \tau) \left[\int_{-\infty}^{\infty} b(t-\tau) \exp\left[-i2\pi f (t-\tau)\right] d(t-\tau)\right] d\tau \\
&= \int_{-\infty}^{\infty} a(\tau) \exp(-i2\pi f \tau) \left[\int_{-\infty}^{\infty} b(s) \exp\left[-i2\pi f (s)\right] ds\right] d\tau, \quad \text{(where } s = t - \tau\text{)} \\
&= B(f) \int_{-\infty}^{\infty} a(\tau) \exp(-i2\pi f \tau) d\tau = A(f) B(f)
\end{aligned} \tag{3-11}$$

Thus, convolution in the time domain is a product in the frequency domain. This is a very important result to understand conceptually. It also has practical implications. Since convolution involves multiplication and integration (addition) for each time offset, in the digital domain it is quite computationally intensive. It is often computationally quicker to first obtain the Fourier transform of two signals to be convolved, multiply the Fourier transforms in the frequency domain, and then perform the inverse Fourier transform.

The symmetry of the Fourier transform pair means that convolution in the frequency domain is a product in the time domain

$$FT^{-1}\left[A(f)*B(f)\right]=a(t)b(t) \tag{3-12}$$

Impulse Response of a Linear System

If we plug a delta-function $\delta(t)$ into the convolution integral of equation (3-7) we get

$$\delta(t)*f(t)=\int_{-\infty}^{\infty}\delta(\tau)f(t-\tau)d\tau=f(t) \tag{3-13}$$

For convolution $\delta(t)$ acts like 1 in algebra – it is the identity operator.

If we now return to considering the three properties of a linear shift invariant system, it is possible to construct a formulation for the output of such a system if the response of the system to an impulse is known. Starting with shift invariance of equation (3-2) we can write

$$x(t-\tau)\rightarrow\left[L\right]\rightarrow y(t-\tau) \tag{3-14}$$

If we scale the process by an amplitude $a(\tau)$ which does not depend on t we get from equation (3-3)

$$a(\tau)x(t-\tau)\rightarrow\left[L\right]\rightarrow a(\tau)y(t-\tau) \tag{3-15}$$

Extending the superposition property of equation (3-4) to an integral we can get

$$\int_{-\infty}^{\infty}a(\tau)x(t-\tau)d\tau\rightarrow\left[L\right]\rightarrow\int_{-\infty}^{\infty}a(\tau)y(t-\tau)d\tau=\int_{-\infty}^{\infty}a(\tau)L\left[x(t-\tau)\right]d\tau \tag{3-16}$$

Suppose we input the δ-function into a linear system to get an output g(t)

$$\delta(t)\rightarrow\left[L\right]\rightarrow g(t) \tag{3-17}$$

Replacing $x(t)$ with $\delta(t)$ and $L[x(t)]$ with $g(t)$ in equation (3-16) yields

$$\int_{-\infty}^{\infty}a(\tau)\delta(t-\tau)d\tau\rightarrow\left[L\right]\rightarrow\int_{-\infty}^{\infty}a(\tau)g(t-\tau)d\tau \tag{3-18}$$

However left side is just $a(t)$ because of the properties of the delta-function and so we have

$$a(t)\rightarrow\left[L\right]\rightarrow a(t)*g(t) \tag{3-19}$$

If we know the response of a linear system to a delta-function, we know its response to any arbitrary function. If we go back to our example of the gong, we can think of its response to a single hammer blow as its impulse response – if we know this we can reconstruct the response to any series of blows.

Response of a Linear System to a Sinusoidal Input

Another way to get the response of a linear system is to input sine waves of different frequencies. We can write

$$y_f(t)=L\left[\exp(i2\pi ft)\right] \tag{3-20}$$

Using the shift invariance and scale invariance properties of equations (3-2 and 3-3) yields

$$
\begin{aligned}
y_f(t+\tau) &= L\left[\exp\left(i2\pi f(t+\tau)\right)\right] = \exp\left(i2\pi f\tau\right)L\left[\exp\left(i2\pi ft\right)\right] \\
&= \exp\left(i2\pi f\tau\right)y_f(t)
\end{aligned}
\tag{3-21}
$$

Setting t = 0 yields

$$
y_f(\tau) = \exp\left(i2\pi f\tau\right)y_f(0) \tag{3-22}
$$

The output of the system at an arbitrary time τ is just a scaled version of the input. Since $y_f(0)$ may be complex it can change the amplitude and phase of the output but it remains a sinusoidal function. Writing this schematically, we have

$$
\exp(i2\pi ft) \rightarrow [L] \rightarrow y_f(0)\exp(i2\pi ft) \tag{3-23}
$$

Now if we replace $y_f(0)$ by $G(f)$ integrate the outputs at all frequencies we get

$$
\int_{-\infty}^{\infty}\exp(i2\pi ft)df \rightarrow [L] \rightarrow \int_{-\infty}^{\infty}G(f)\exp(i2\pi ft)df = g(t) \tag{3-24}
$$

This expression includes two inverse Fourier transforms. The left hand side is the inverse Fourier transform of 1 which is equal to the delta-function and so $g(t)$ is just the impulse response of the filter defined in equation (3-17). The frequency response of a filter is just the Fourier transform of the impulse response. We have two alternative but entirely equivalent means to characterize and describe the response of a system.

We are going to come back to considering linear filters in the discrete domain after we have learned about the discrete Fourier transform but in preparation for the exercise, we will initially just consider convolution of digital time series.

Convolution in the discrete domain

In the discrete domain we can write the time domain convolution of two finite series

$$
\begin{aligned}
&a(i), \quad i = 1, M \\
&b(i), \quad i = 1, N \\
&c(j) = [a * b)](j) = \sum_{i=\max(1,j+1-n)}^{\min(j,m)} a(i)b(j+1-i), \quad j = 1, N+M-1
\end{aligned}
\tag{3-25}
$$

where the limits for the sum define all legal indexes of a and b. This equation is a little bit hard envision but just as in the continuous domain, we reverse one time series and slide them past each other and sum the products of aligned sample pairs – the 1st and last terms in the convolution are the result of the overlap of a single sample pairs. In Matlab the function `conv(a,b)` calculates this convolution and will return N+M-1 samples (note that there is an optional 3rd argument that returns just a subsection of the convolution – see the documentation with `help conv` or `doc conv`)

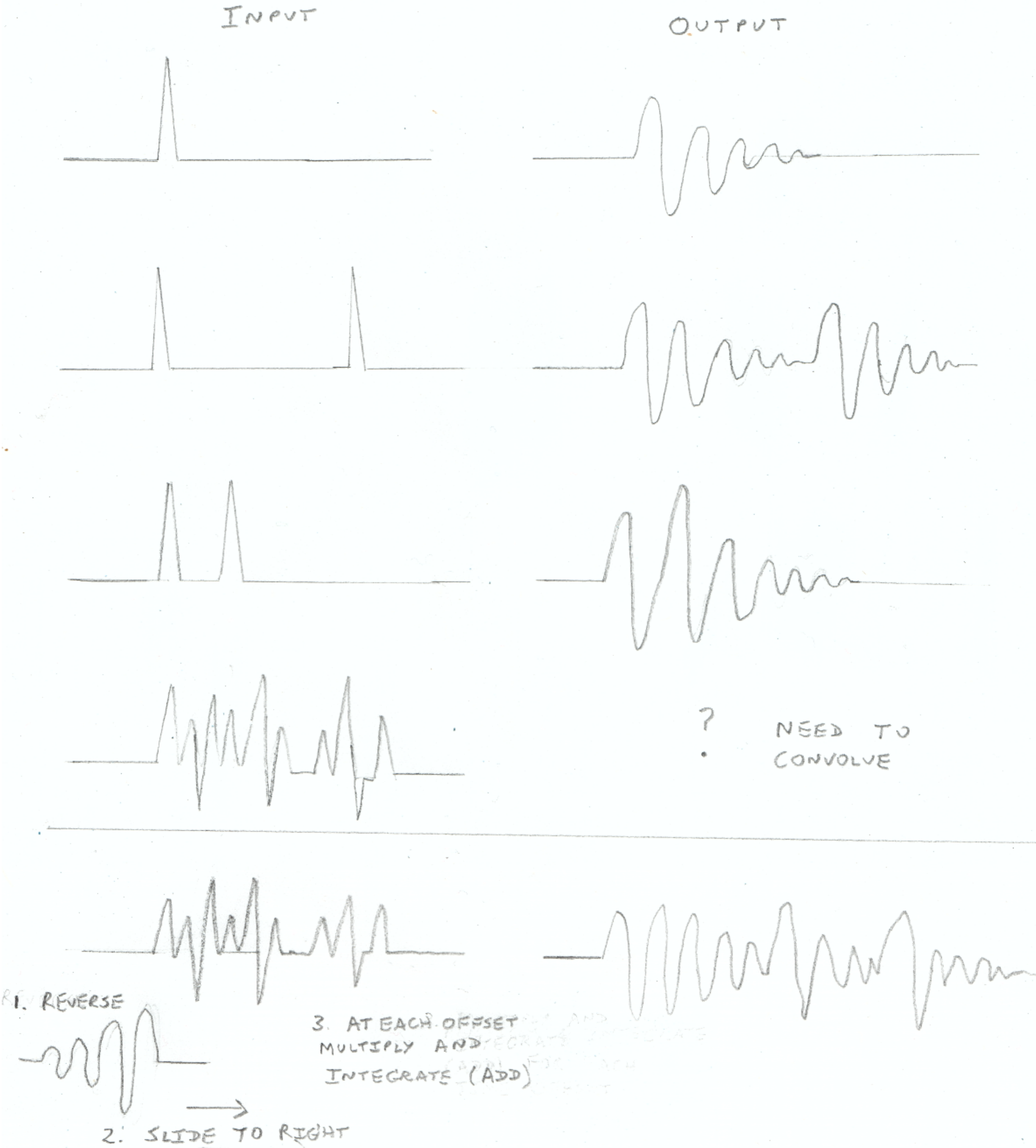

Fig 3.1 Hitting a gong with a hammer