1. Fourier Series

I have drawn the content for this lecture from the book *Mathematical Methods for the Physical Sciences* by K. F. Riley

Whatever your background, you are probably aware that harmonic functions (i.e., sine and cosine functions) are a useful means to describe many physical phenomena. If we are describing how things vary in time, the conventional notation is of the form $\cos(\omega t)$ (or $\sin(\omega t)$) where t is time and ω is known as the angular frequency and is related to the frequency f by $\omega = 2\pi f$. For spatial variations, the notation is $\cos(kx)$ where x is the spatial coordinate and k is the wavenumber and is related to the wavelength, λ, by $k=2\pi/\lambda$. When considering problems that involve harmonic waves (e.g., waves on the ocean surface), descriptions in terms of time and spatial variable are interchangeable.

As Riley notes, harmonic functions can arise from at least 3 different but interrelated ways:
1. As mathematically simple and manageable functions.
2. As solutions to a certain broad class of differential equations (Sturm Liouville equations).
3. As functions that arise naturally in describing a wide variety of physical systems.

Consider the set of functions given by

$$h_n(t) = \cos\left(2\pi nt/T\right), \quad 0 \le n < \infty \text{, and } n \text{ an integer.} \tag{1.1}$$

We can show that

$$\int_{-T/2}^{T/2} \cos\left(2\pi nt/T\right)\cos\left(2\pi mt/T\right)dt = \begin{cases} 0, & n \ne m \\ \dfrac{T}{2}, & n = m \ne 0 \\ T, & n = m = 0 \end{cases} \tag{1.2}$$

You should be able to show the above yourself with the aid of the trigonometric formula

$$2\cos A \cos B = \cos\left(A+B\right) + \cos\left(A-B\right) \tag{1.3}$$

Mathematically, we can say that the series of harmonic functions $h_n(t)$ are **orthogonal** over the range $-T/2 < t < T/2$. This means that we cannot represent one harmonic function in terms of a sum of the other functions. This is a useful property because it means that any sum of the orthogonal functions produces a unique periodic function that cannot be produced by summing the orthogonal functions in a different way (i.e., with different coefficients).

For the series of orthogonal functions to be very useful they must also be **complete** – that is any arbitrary periodic function must be expressible as a sum of the orthogonal functions. It turns out that the functions $h_n(t)$ are incomplete because they can only describe arbitrary periodic functions that are even (symmetric about $t = 0$). Likewise the set of functions given by

$$g_n(t) = \sin\left(2\pi nt/T\right), \quad 1 \le n < \infty \text{, and } n \text{ is an integer}$$

is also incomplete because they can only describe periodic functions that are odd (antisymmetric about $t = 0$).

However if we combine the cosine and sine series, $h_n(t)$ and $g_n(t)$ they are complete. You should be able to show that this set of harmonic functions are also orthogonal. That is in addition to equation (1.1), we can also write

$$\int_{-T/2}^{T/2} \cos(2\pi nt/T)\sin(2\pi mt/T)dt = 0 \tag{1.4}$$

$$\int_{-T/2}^{T/2} \sin(2\pi nt/T)\sin(2\pi mt/T)dt = \begin{cases} 0, & n \neq m \\ \frac{T}{2}, & n = m \end{cases} \tag{1.5}$$

To show the above you need to remember the trigonometric relationships

$$\begin{aligned} 2\cos A\sin B &= \sin(A+B)+\sin(B-A) \\ 2\sin A\sin B &= \cos(A-B)-\cos(A+B) \end{aligned} \tag{1.6}$$

The complete set of harmonic functions are often referred to as **basis** functions since they can be summed to construct all other functions.

An alternate and mathematically very convenient orthogonal and complete set of harmonic basis functions is given by

$$\exp(i2\pi nt/T) = \cos(2\pi nt/T) + i\sin(2\pi nt/T), \quad -\infty < n < \infty \text{, and } n \text{ an integer.} \tag{1.7}$$

This form facilitates compact expressions and as we will see is more convenient for the Fourier transforms.

If you have not familiar with equation (1.7) you can derive it by considering the Maclaurin (Taylor) Series expansions for the 3 terms.

$$\begin{aligned} f(\alpha) &= f(0) + \alpha f'(0) + \frac{\alpha^2}{2!}f''(0) + \frac{\alpha^3}{3!}f'''(0) + \ldots \\ \exp(\alpha) &= 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!}\ldots \\ \sin(\alpha) &= \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!}\ldots \\ \cos(\alpha) &= 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!}\ldots \end{aligned} \tag{1.8}$$

Also note that complex functions $g(x)$ and $h(x)$ are defined as orthogonal if

$$\int \bar{g}(x)h(x)dx = 0 \tag{1.9}$$

where $\bar{g}$ is the complex conjugate that is also often denoted by g^*. Remember that the complex conjugate of a complex number is given by $(a + ib)^* = a - ib$.

We can write a Fourier series for a periodic function

$$y(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(2\pi nt/T\right) + \sum_{n=1}^{\infty} B_n \sin\left(2\pi nt/T\right) = \sum_{n=-\infty}^{\infty} C_n \exp\left(i2\pi nt/T\right) \tag{1.10}$$

where the ½ in front of A_0 is a convention that comes from the way the coefficients are defined (see equation 1.13 below and compare with equation 1.2).

By considering the terms for a particular value of $\pm n$, you can show that the coefficients are related by

$$\begin{aligned} C_{-n} &= \tfrac{1}{2}\left(A_n + iB_n\right) \\ C_n &= \tfrac{1}{2}\left(A_n - iB_n\right) \\ C_0 &= \tfrac{1}{2}A_0 \end{aligned} \tag{1.11}$$

To calculate the coefficients for a particular periodic function, we integrate the product of that function with each of the basis functions. For example

$$\begin{aligned} &\int_{-T/2}^{T/2} y(t)\cos(2\pi mt/T)\,dt \\ &= \int_{-T/2}^{T/2} \frac{A_0}{2}\cos(2\pi mt/T)\,dt + \int_{-T/2}^{T/2} \sum_{n=1}^{\infty} A_n \cos(2\pi nt/T)\cos(2\pi mt/T)\,dt \\ &\qquad + \int_{-T/2}^{T/2} \sum_{n=1}^{\infty} B_n \cos(2\pi nt/T)\sin(2\pi mt/T)\,dt \\ &= \frac{1}{2}A_m T \end{aligned} \tag{1.12}$$

In the above, we have expanded $y(t)$ in terms of the Fourier Series and then taken advantage of the fact that all but one of the integrals is zero because the functions are orthogonal. We can rearrange equation (1.12) to get

$$A_m = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(2\pi mt/T)\,dt \tag{1.13}$$