Homework 1. Fourier Series

(1) (i) Consider a periodic square wave defined in the interval $-T/2 \leq t < T/2$ by

$$y(t) = \begin{cases} -1, & -T/2 < t < 0 \\ +1, & 0 < t < T/2 \\ 0, & t = -T/2, 0 \end{cases}$$

Show that the Fourier Series expansion of y (x) is given by

$$y(t) = \frac{4}{\pi}\left[\sin \omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \ldots\right]$$

where $\omega = 2\pi/\mathrm{T}$.

(ii) Write a program (Python, Matlab, whatever you prefer) to create a time series of a square wave with $T = 2\pi$ in the interval 0 to 10π and with a sampling rate (`nsamp`) of 1000 samples per unit. Additionally write code to calculate the Fourier series equivalent of this time series.

(ii) Make plots comparing the square wave versus the approximation using, 5,10,20 and 100 Fourier series terms (`nterm`). How well does the Fourier series represent a square wave? Where along the time series is the fit worst? How does the character of the misfit change with the number of terms? Can you measure the size of the maximum misfit and does it vary with the number of terms used? You are looking at an effect called the Gibbs phenomenon, which relates to how the Fourier series handles a discontinuity.

(iii) Set `nsamp` = 100 and `nterm` = 100. What does the misfit look like? Set `nsamp` = 10,000 and keep `nterm` = 100. Again what does the misfit look like? The difference can be ascribed to an effect known as aliasing.

(2) We can write both real and complex functions in terms of an exponential Fourier Series given by

$$y(t) = \sum_{n=-\infty}^{\infty} C_n \exp\left(i2\pi nt/T\right)$$

where C_n are <u>complex</u> coefficients.

(i) Show that the functions $\exp\left(i2\pi nt/T\right)$, $-\infty < n < \infty$, are orthogonal over the range $-T/2$ to $T/2$

(ii) If $y(t)$ is a real function then show that $C_{-n} = C_n^*$ (i.e., C_{-n} is the complex conjugate of C_n)

(iii) Derive expressions for C_n and C_{-n} in terms of A_n and B_n in the alternate Fourier Series

$$y(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(2\pi nt/T\right) + \sum_{n=1}^{\infty} B_n \sin\left(2\pi nt/T\right) = \sum_{n=-\infty}^{\infty} C_n \exp\left(i2\pi nt/T\right)$$