Trig Function Evaluation

One of the problems with most trig classes is that they tend to concentrate on right triangle trig and do everything in terms of degrees. Then you get to a calculus course where almost everything is done in radians and the unit circle is a very useful tool.

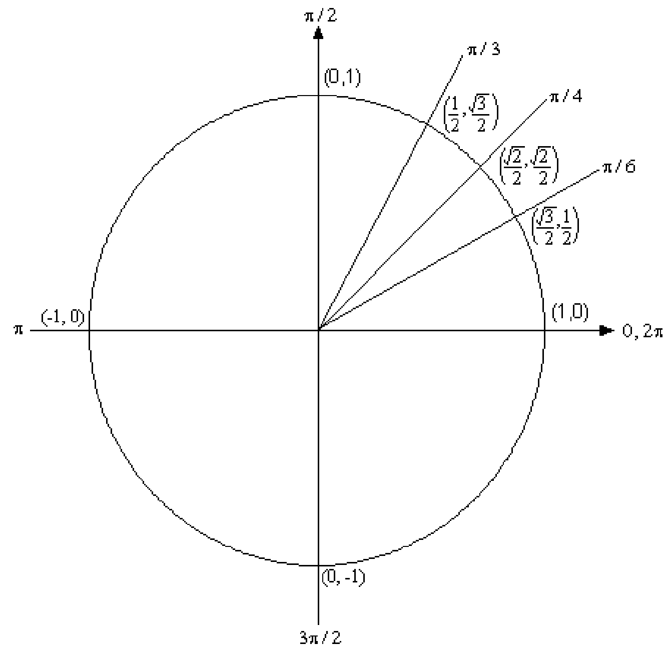
So first off let's look at the following table to relate degrees and radians.

Degree	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Know this table! There are, of course, many other angles in radians that we'll see during this class, but most will relate back to these few angles. So, if you can deal with these angles you will be able to deal with most of the others.

Be forewarned, everything in most calculus classes will be done in radians!

Now, let's look at the unit circle. Below is the unit circle with just the first quadrant filled in. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate is the cosine of that angle and the second coordinate is the sine of that angle. There are a couple of *basic* angles that are commonly used. These are $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π and are shown below along with the coordinates of the intersections. So, from the unit circle below we can see that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.



Remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is 2π, so the positive *x*-axis can correspond to either an angle of 0 or 2π (or 4π, or 6π, or -2π, or -4π, *etc*. depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$\frac{\pi}{6}+2\pi=\frac{13\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around counter clockwise)

$\frac{\pi}{6}+4\pi=\frac{25\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice counter clockwise)

$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around clockwise)

$\frac{\pi}{6}-4\pi=-\frac{23\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice clockwise)

etc.

In fact $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6}+2\pi\, n$, $n=0,\pm 1,\pm 2,\pm 3,\ldots$ In this case n is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of n correspond to counter clockwise rotations and negative values of n correspond to clockwise rotations.

So, why did I only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first. You'll see this in the following examples.

Find the exact value of each of the following. In other words, don't use a calculator.

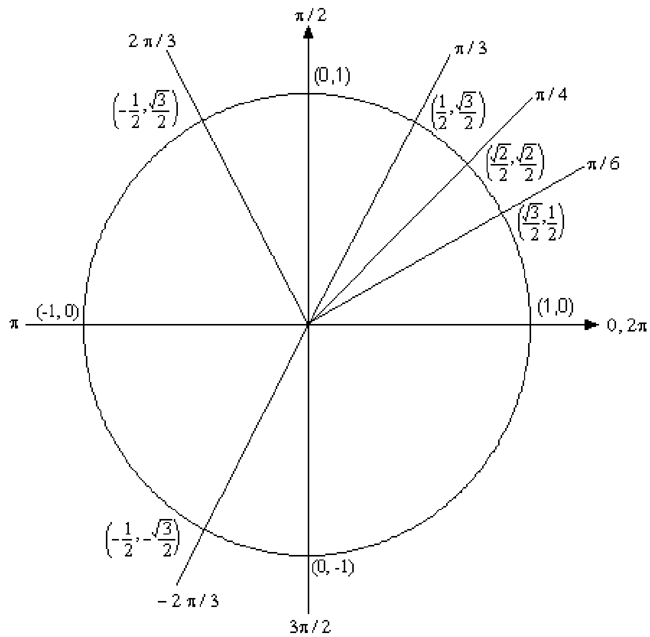
10. $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$

Solution

The first evaluation here uses the angle $\frac{2\pi}{3}$. Notice that $\frac{2\pi}{3}=\pi-\frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative x-axis. This means that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the x coordinate will be negative.

Likewise for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3}=-\pi+\frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative x-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Both of these angles along with their coordinates are shown on the following unit circle.



From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right)=-\frac{\sqrt{3}}{2}$.

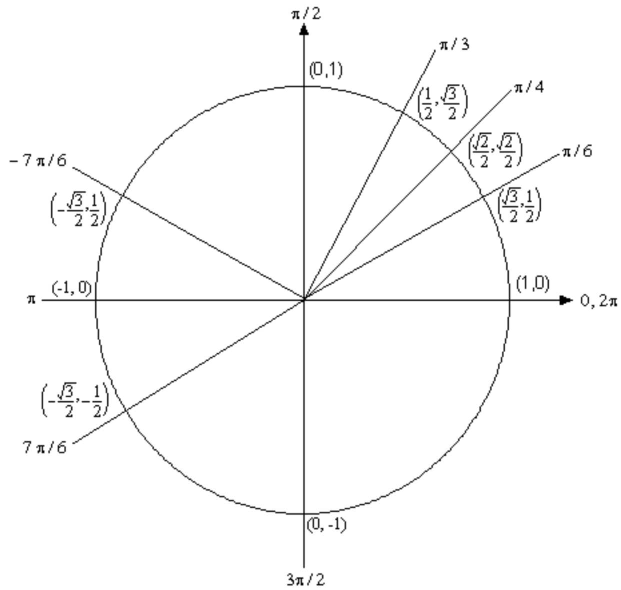
This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta)=-\sin(\theta)$$

11. $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

Solution

For this example notice that $\frac{7\pi}{6}=\pi+\frac{\pi}{6}$ so this means we would rotate down $\frac{\pi}{6}$ from the negative x-axis to get to this angle. Also $-\frac{7\pi}{6}=-\pi-\frac{\pi}{6}$ so this means we would rotate up $\frac{\pi}{6}$ from the negative x-axis to get to this angle. These are both shown on the following unit circle along with appropriate coordinates for the intersection points.



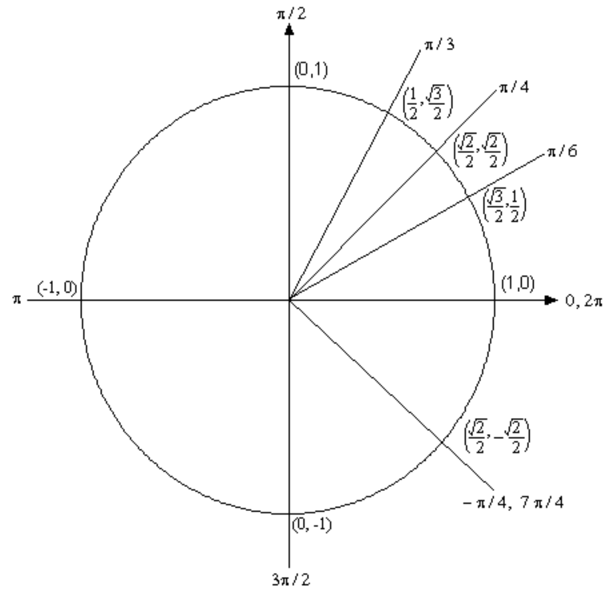
From this unit circle we can see that $\cos\left(\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta)=\cos(\theta).$$

12. $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$

Solution

Here we should note that $\frac{7\pi}{4}=2\pi-\frac{\pi}{4}$ so $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ are in fact the same angle! The unit circle for this angle is



Now, if we remember that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ we can use the unit circle to find the values the tangent function. So,

$$\tan\left(\frac{7\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = \frac{\sin(-\pi/4)}{\cos(-\pi/4)} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1.$$

On a side note, notice that $\tan\left(\dfrac{\pi}{4}\right) = 1$ and we se can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan(-\theta) = -\tan(\theta).$$

13. $\sin\left(\dfrac{9\pi}{4}\right)$

Solution

For this problem let's notice that $\dfrac{9\pi}{4} = 2\pi + \dfrac{\pi}{4}$. Now, recall that one complete revolution is 2π. So, this means that $\dfrac{9\pi}{4}$ and $\dfrac{\pi}{4}$ are at the same point on the unit circle. Therefore,

$$\sin\left(\frac{9\pi}{4}\right) = \sin\left(2\pi + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

This leads us to a very nice fact about the sine function. The sine function is an example of a *periodic* function. Periodic function are functions that will repeat themselves over and over. The "distance" that you need to move to the right or left before the function starts repeating itself is called the **period** of the function.

In the case of sine the period is 2π. This means the sine function will repeat itself every 2π. This leads to a nice formula for the sine function.

$$\sin\left(x+2\pi n\right)=\sin\left(x\right) \qquad n=0,\pm 1,\ \pm 2,\ldots$$

Notice as well that since

$$\csc\left(x\right)=\frac{1}{\sin\left(x\right)}$$

we can say the same thing about cosecant.

$$\csc\left(x+2\pi n\right)=\csc\left(x\right) \qquad n=0,\pm 1,\ \pm 2,\ldots$$

Well, actually we should be careful here. We can say this provided $x\neq n\pi$ since sine will be zero at these points and so cosecant won't exist there!

14. $\sec\left(\frac{25\pi}{6}\right)$

Solution

Here we need to notice that $\frac{25\pi}{6}=4\pi+\frac{\pi}{6}$. In other words, we've started at $\frac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(4\pi+\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec\left(x\right)=\frac{1}{\cos\left(x\right)}$$

and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}$$

We should also note that cosine and secant are periodic functions with a period of 2π. So,

$$\begin{aligned}\cos\left(x+2\pi n\right)&=\cos\left(x\right)\\ \sec\left(x+2\pi n\right)&=\sec\left(x\right)\end{aligned} \qquad n=0,\pm 1,\ \pm 2,\ldots$$

15. $\tan\left(\frac{4\pi}{3}\right)$

Solution

To do this problem it will help to know that tangent (and hence cotangent) is also a periodic function, but unlike sine and cosine it has a period of π.

$$\begin{aligned} \tan(x+\pi n) &= \tan(x) \\ \cot(x+\pi n) &= \cot(x) \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

So, to do this problem let's note that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$. Therefore,

$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Trig Evaluation Final Thoughts
As we saw in the previous examples if you know the first quadrant of the unit circle you can find the value of ANY trig function (not just sine and cosine) for ANY angle that can be related back to one of those shown in the first quadrant. This is a nice idea to remember as it means that you only need to memorize the first quadrant and how to get the angles in the remaining three quadrants!

In these problems I used only "basic" angles, but many of the ideas here can also be applied to angles other than these "basic" angles as we'll see in Solving Trig Equations.

Graphs of Trig Functions

There is not a whole lot to this section. It is here just to remind you of the graphs of the six trig functions as well as a couple of nice properties about trig functions.

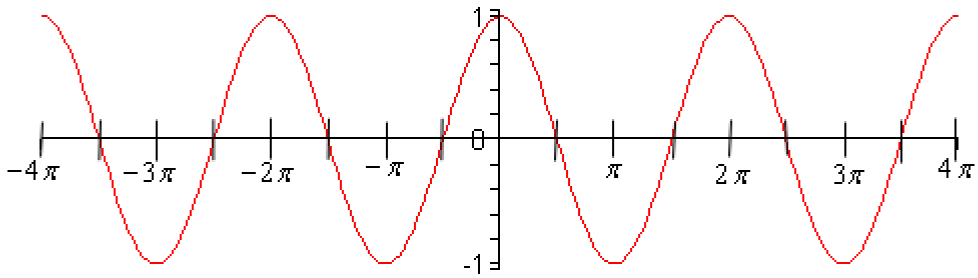
Before jumping into the problems remember we saw in the Trig Function Evaluation section that trig functions are examples of *periodic* functions. This means that all we really need to do is graph the function for one periods length of values then repeat the graph.

Graph the following function.

1. $y = \cos(x)$

Solution
There really isn't a whole lot to this one other than plotting a few points between 0 and 2π, then repeat. Remember cosine has a period of 2π (see Problem 6 in Trig Function Evaluation).

Here's the graph for $-4\pi \le x \le 4\pi$.



Notice that graph does repeat itself 4 times in this range of x's as it should.

Let's also note here that we can put all values of x into cosine (which won't be the case for most of the trig functions) and let's also note that

$$-1 \le \cos(x) \le 1$$

It is important to notice that cosine will never be larger than 1 or smaller than -1. This will be useful on occasion in a calculus class.

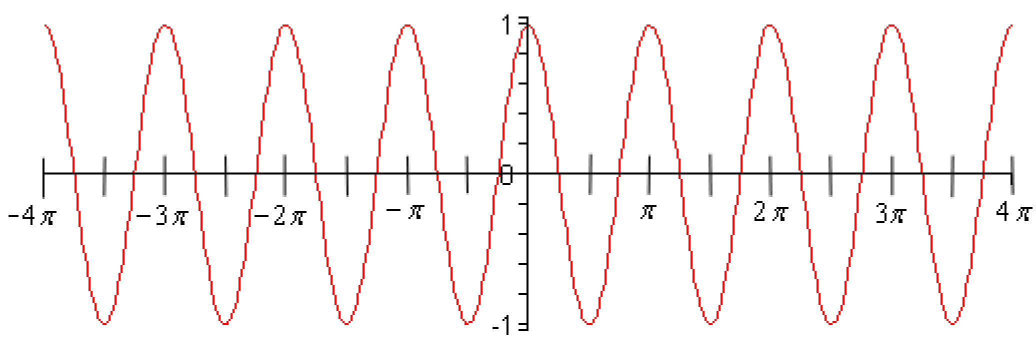
2. $y = \cos(2x)$

Solution

We need to be a little careful with this graph. $\cos(x)$ has a period of 2π, but we're not dealing with $\cos(x)$ here. We are dealing with $\cos(2x)$. In this case notice that if we plug in $x = \pi$ we will get

$$\cos(2(\pi)) = \cos(2\pi) = \cos(0) = 1$$

In this case the function starts to repeat itself after π instead of 2π! So, this function has a period of π. So, we can expect the graph to repeat itself 8 times in the range $-4\pi \le x \le 4\pi$. Here is that graph.



Sure enough, there are twice as many cycles in this graph.

In general we can get the period of $\cos(\omega x)$ using the following.

$$\text{Period} = \frac{2\pi}{\omega}$$

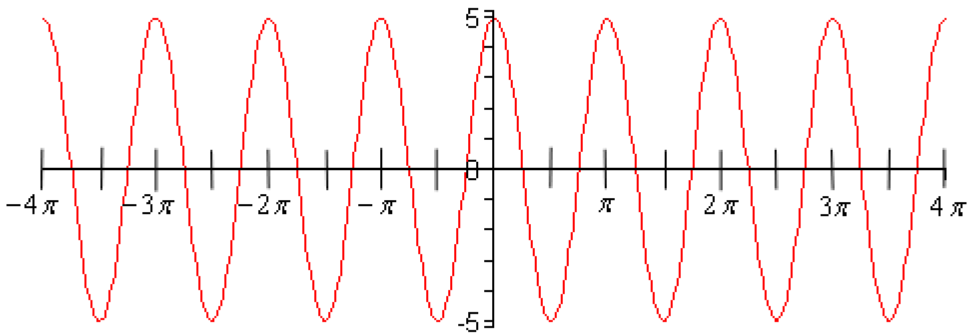
If $\omega > 1$ we can expect a period smaller than 2π and so the graph will oscillate faster. Likewise, if $\omega < 1$ we can expect a period larger than 2π and so the graph will oscillate slower.

Note that the period does not affect how large cosine will get. We still have

$$-1 \le \cos(2x) \le 1$$

3. $y = 5\cos(2x)$

Solution
In this case I added a 5 in front of the cosine. All that this will do is increase how big cosine will get. The number in front of the cosine or sine is called the **amplitude**. Here's the graph of this function.



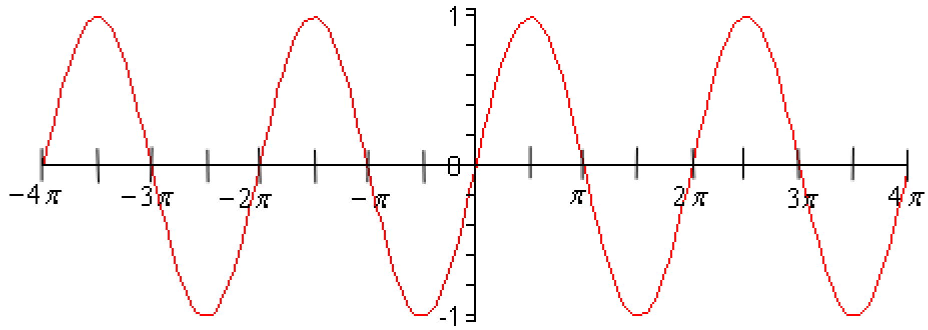
Note the scale on the *y*-axis for this problem and do not confuse it with the previous graph. The *y*-axis scales are different!

In general,

$$-R \le R\cos(\omega x) \le R$$

4. $y = \sin(x)$

Solution
As with the first problem in this section there really isn't a lot to do other than graph it. Here is the graph on the range $-4\pi \le x \le 4\pi$.



From this graph we can see that sine has the same range that cosine does. In general

$$-R \le R\sin(\omega x) \le R$$

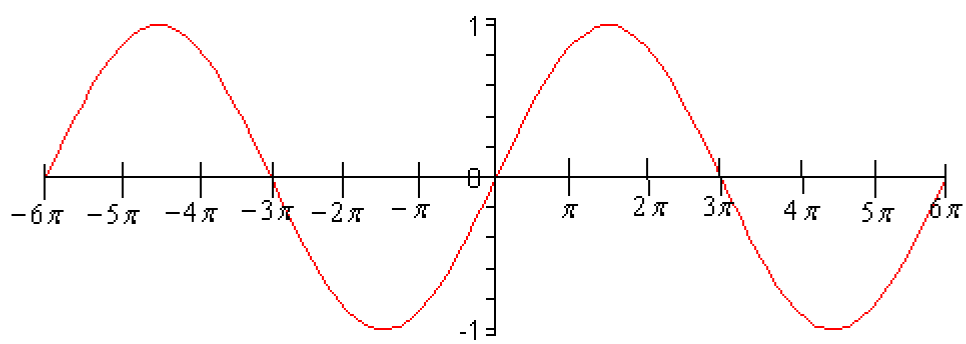
As with cosine, sine itself will never be larger than 1 and never smaller than -1.

5. $y = \sin\left(\frac{x}{3}\right)$

Solution
So, in this case we don't have just an x inside the parenthesis. Just as in the case of cosine we can get the period of $\sin(\omega x)$ by using

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{1/3} = 6\pi$$

In this case the curve will repeat every 6π. So, for this graph I'll change the range to $-6\pi \le x \le 6\pi$ so we can get at least two traces of the curve showing. Here is the graph.



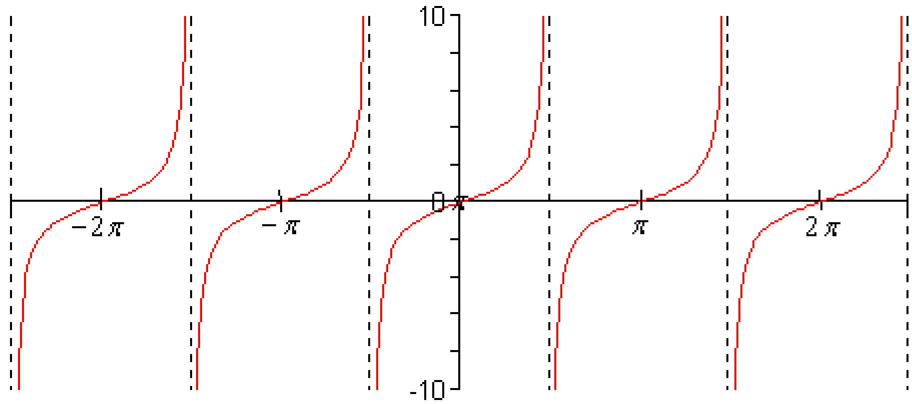
6. $y = \tan(x)$

Solution

In the case of tangent we have to be careful when plugging x's in since tangent doesn't exist wherever cosine is zero (remember that $\tan x = \frac{\sin x}{\cos x}$). Tangent will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of tangent on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



Finally, a couple of quick properties about $R\tan(\omega x)$.

$$-\infty < R\tan(\omega x) < \infty$$

$$\text{Period} = \frac{\pi}{\omega}$$

For the period remember that $\tan(x)$ has a period of π unlike sine and cosine and that accounts for the absence of the 2 in the numerator that was there for sine and cosine.

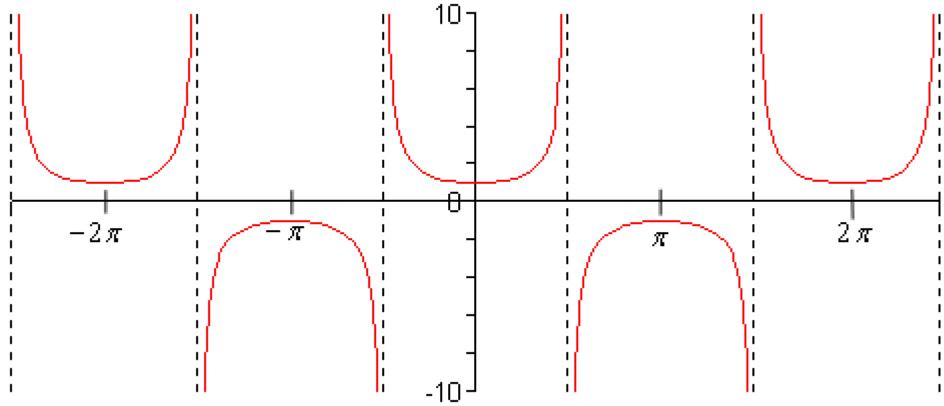
7. $y = \sec(x)$

Solution

As with tangent we will have to avoid x's for which cosine is zero (remember that $\sec x = \frac{1}{\cos x}$). Secant will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of secant on the range $-\frac{5\pi}{2} < x < \frac{5\pi}{2}$.



Notice that the graph is always greater than 1 and less than -1. This should not be terribly surprising. Recall that $-1 \le \cos(x) \le 1$. So, 1 divided by something less than 1 will be greater than 1. Also, $1/\pm 1 = \pm 1$ and so we get the following ranges out of secant.

$$R\sec(\omega x) \ge R \qquad \text{and} \qquad R\sec(\omega x) \le -R$$

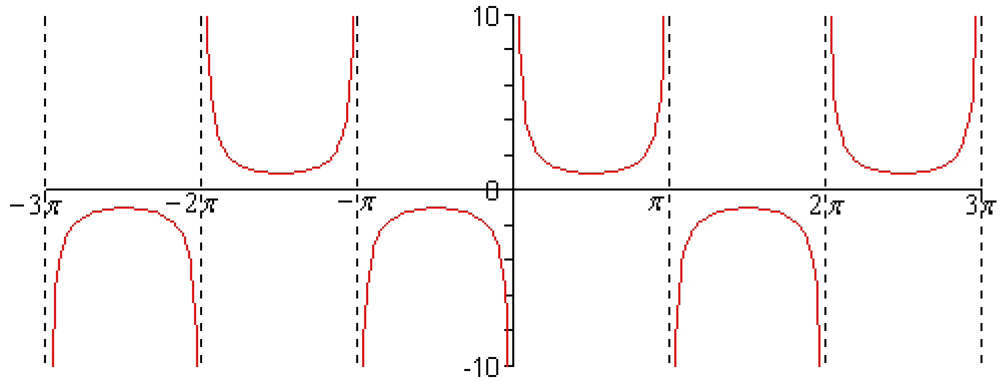
8. $y = \csc(x)$

Solution

For this graph we will have to avoid x's where sine is zero $\left(\csc x = \dfrac{1}{\sin x}\right)$. So, the graph of cosecant will not exist for

$$x = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$$

Here is the graph of cosecant.

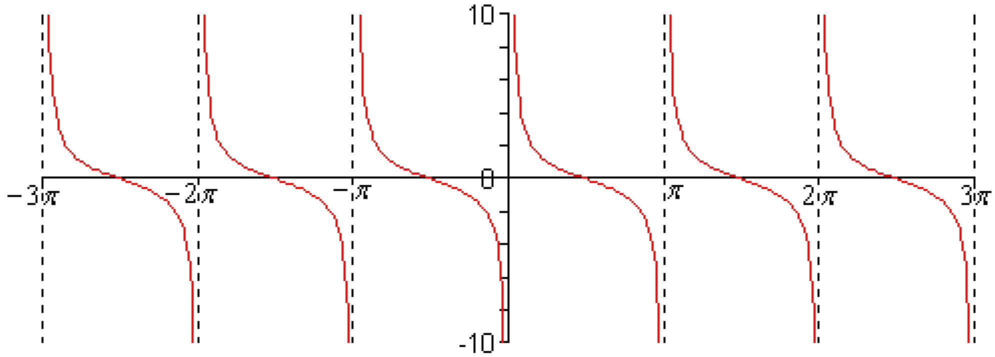


Cosecant will have the same range as secant.

$$R\csc(\omega x) \ge R \qquad \text{and} \qquad R\csc(\omega x) \le -R$$

9. $y = \cot(x)$

Solution
Cotangent must avoid
$$x = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$$
since we will have division by zero at these points. Here is the graph.



Cotangent has the following range.
$$-\infty < R\cot(\omega x) < \infty$$

Trig Formulas

This is not a complete list of trig formulas. This is just a list of formulas that I've found to be the most useful in a Calculus class. For a complete listing of trig formulas you can download my Trig Cheat Sheet.

Complete the following formulas.

1. $\sin^2(\theta) + \cos^2(\theta) =$

Solution
$\sin^2(\theta) + \cos^2(\theta) = 1$

Note that this is true for ANY argument as long as it is the same in both the sine and the cosine. So, for example :
$$\sin^2(3x^4 - 5x^2 + 87) + \cos^2(3x^4 - 5x^2 + 87) = 1$$

2. $\tan^2(\theta) + 1 =$

Solution
$\tan^2(\theta) + 1 = \sec^2(\theta)$

If you know the formula from Problem 1 in this section you can get this one for free.

$$\sin^2(\theta)+\cos^2(\theta)=1$$
$$\frac{\sin^2(\theta)}{\cos^2(\theta)}+\frac{\cos^2(\theta)}{\cos^2(\theta)}=\frac{1}{\cos^2(\theta)}$$
$$\tan^2(\theta)+1=\sec^2(\theta)$$

Can you come up with a similar formula relating $\cot^2(\theta)$ and $\csc^2(\theta)$?

3. $\sin(2t)=$

Solution
$\sin(2t)=2\sin(t)\cos(t)$

This formula is often used in reverse so that a product of a sine and cosine (with the same argument of course) can be written as a single sine. For example,

$$\begin{aligned}\sin^3(3x^2)\cos^3(3x^2)&=\left(\sin(3x^2)\cos(3x^2)\right)^3\\&=\left(\frac{1}{2}\sin\left(2(3x^2)\right)\right)^3\\&=\frac{1}{8}\sin^3(6x^2)\end{aligned}$$

You will find that using this formula in reverse can significantly reduce the complexity of some of the problems that you'll face in a Calculus class.

4. $\cos(2x)=$ (Three possible formulas)

Solution
As noted there are three possible formulas to use here.

$$\cos(2x)=\cos^2(x)-\sin^2(x)$$
$$\cos(2x)=2\cos^2(x)-1$$
$$\cos(2x)=1-2\sin^2(x)$$

You can get the second formula by substituting $\sin^2(x)=1-\cos^2(x)$ (see Problem 1 from this section) into the first. Likewise, you can substitute $\cos^2(x)=1-\sin^2(x)$ into the first formula to get the third formula.

5. $\cos^2(x)=$ (In terms of cosine to the first power)

Solution

$\cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$

This is really the second formula from Problem 4 in this section rearranged and is VERY useful for eliminating even powers of cosines. For example,

$$\begin{aligned} 5\cos^2(3x) &= 5\left(\frac{1}{2}\left(1+\cos\left(2(3x)\right)\right)\right) \\ &= \frac{5}{2}\left(1+\cos(6x)\right) \end{aligned}$$

Note that you probably saw this formula written as

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1+\cos(x)\right)}$$

in a trig class and called a half-angle formula.

6. $\sin^2(x) =$ (In terms of cosine to the first power)

Solution

$\sin^2(x) = \frac{1}{2}\left(1-\cos(2x)\right)$

As with the previous problem this is really the third formula from Problem 4 in this section rearranged and is very useful for eliminating even powers of sine. For example,

$$\begin{aligned} 4\sin^4(2t) &= 4\left(\sin^2(2t)\right)^2 \\ &= 4\left(\frac{1}{2}\left(1-\cos(4t)\right)\right)^2 \\ &= 4\left(\frac{1}{4}\right)\left(1-2\cos(4t)+\cos^2(4t)\right) \\ &= 1-2\cos(4t)+\frac{1}{2}\left(1+\cos(8t)\right) \\ &= \frac{3}{2}-2\cos(4t)+\frac{1}{2}\cos(8t) \end{aligned}$$

As shown in this example you may have to use both formulas and more than once if the power is larger than 2 and the answer will often have multiple cosines with different arguments.

Again, in a trig class this was probably called half-angle formula and written as,

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1-\cos(x)\right)}$$

Solving Trig Equations

Solve the following trig equations. For those without intervals listed find ALL possible solutions. For those with intervals listed find only the solutions that fall in those intervals.

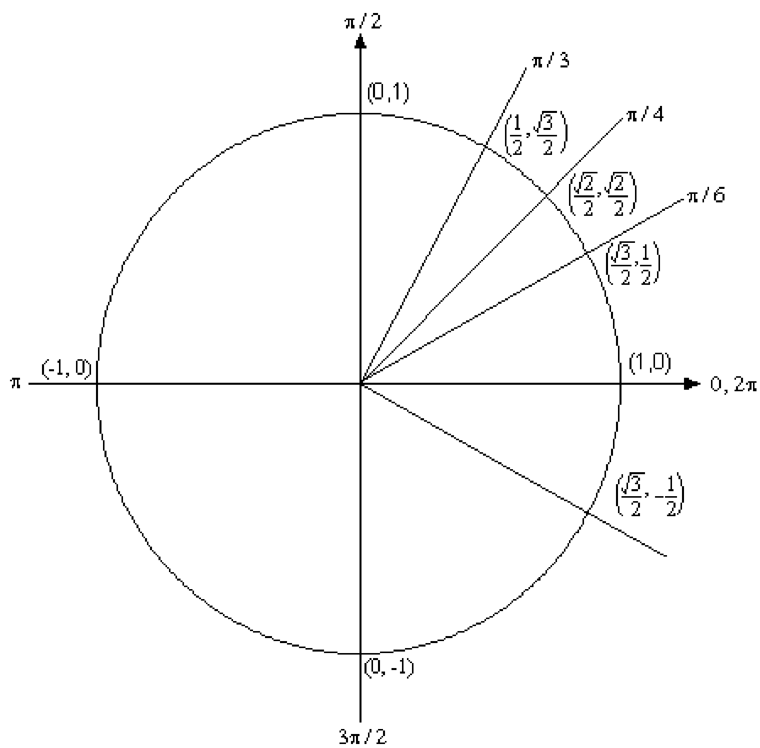
1. $2\cos(t) = \sqrt{3}$

Solution
There's not much to do with this one. Just divide both sides by 2 and then go to the unit circle.

$$2\cos(t) = \sqrt{3}$$
$$\cos(t) = \frac{\sqrt{3}}{2}$$

So, we are looking for all the values of t for which cosine will have the value of $\frac{\sqrt{3}}{2}$.

So, let's take a look at the following unit circle.



From quick inspection we can see that $t = \frac{\pi}{6}$ is a solution. However, as I have shown on the unit circle there is another angle which will also be a solution. We need to determine what this angle is. When we look for these angles we typically want *positive* angles that lie between 0 and 2π . This angle will not be the only possibility of course, but by convention we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive *x*-axis, then so must the angle in the fourth quadrant. So we could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles so, we'll use $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Go back to my introduction in the Trig Function Evaluation section and you'll see there that I used

$$\frac{\pi}{6} + 2\pi\, n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

to represent all the possible angles that can end at the same location on the unit circle, *i.e.* angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction (*n* is positive) or clockwise direction (*n* is negative) for *n* complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

$$\frac{\pi}{6} + 2\pi\, n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi\, n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

2. $2\cos(t) = \sqrt{3}$ on $[-2\pi, 2\pi]$

Solution

This problem is almost identical to the previous except now I want all the solutions that fall in the interval $[-2\pi, 2\pi]$. So we will start out with the list of all possible solutions from the previous problem.

$$\frac{\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
$$\frac{11\pi}{6} + 2\pi\, n\,, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Then start picking values of *n* until we get all possible solutions in the interval.

First notice that since both the angles are positive *adding* on any multiples of 2π (*n* positive) will get us bigger than 2π and hence out of the interval. So, all positive values of *n* are immediately out. Let's take a look at the negatives values of *n*.

$$n = -1$$
$$\frac{\pi}{6} + 2\pi\left(-1\right) = -\frac{11\pi}{6} > -2\pi$$
$$\frac{11\pi}{6} + 2\pi\left(-1\right) = -\frac{\pi}{6} > -2\pi$$

These are both greater than -2π and so are solutions, but if we subtract another 2π off (*i.e* use $n = -2$) we will once again be outside of the interval.

So, the solutions are : $\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{11\pi}{6}$.

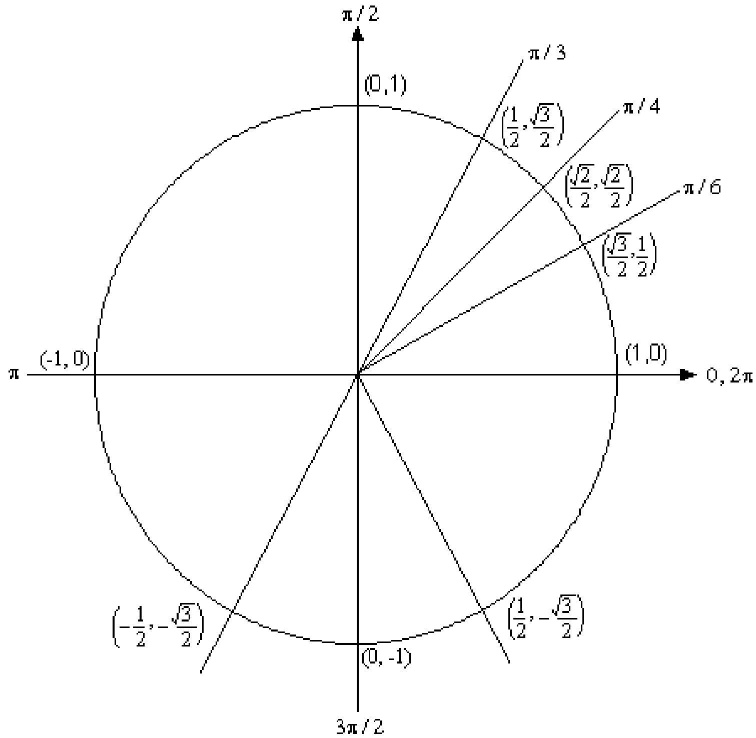
3. $2\sin\left(5x\right) = -\sqrt{3}$

Solution

This one is very similar to Problem 1, although there is a very important difference. We'll start this problem in exactly the same way as we did in Problem 1.

$$2\sin(5x) = -\sqrt{3}$$
$$\sin(5x) = \frac{-\sqrt{3}}{2}$$

So, we are looking for angles that will give $-\frac{\sqrt{3}}{2}$ out of the sine function. Let's again go to our trusty unit circle.



Now, there are no angles in the first quadrant for which sine has a value of $-\frac{\sqrt{3}}{2}$. However, there are two angles in the lower half of the unit circle for which sine will have a value of $-\frac{\sqrt{3}}{2}$. So, what are these angles? A quick way of doing this is to, for a second, ignore the "-" in the problem and solve $\sin(x)=\frac{\sqrt{3}}{2}$ in the first quadrant only. Doing this give a solution of $x=\frac{\pi}{3}$. Now, again using some geometry, this tells us that the angle in the third quadrant will be $\frac{\pi}{3}$ below the ***negative*** x-axis or $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$. Likewise, the angle in the fourth quadrant will $\frac{\pi}{3}$ below the ***positive*** x-axis or $2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$. Remember that we're looking for positive angles between 0 and 2π.

Now we come to the very important difference between this problem and Problem 1. The solution is **NOT**

$$x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$
$$x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$

This is not the set of solutions because we are NOT looking for values of x for which $\sin(x) = -\frac{\sqrt{3}}{2}$, but instead we are looking for values of x for which $\sin(5x) = -\frac{\sqrt{3}}{2}$. Note the difference in the arguments of the sine function! One is x and the other is $5x$. This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

$$5x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$
$$5x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$

Well, actually, that's not quite the solution. We are looking for values of x so divide everything by 5 to get.

$$x = \frac{4\pi}{15} + \frac{2\pi n}{5}, \quad n = 0, \pm1, \pm2, \ldots$$
$$x = \frac{\pi}{3} + \frac{2\pi n}{5}, \quad n = 0, \pm1, \pm2, \ldots$$

Notice that I also divided the $2\pi n$ by 5 as well! This is important! If you don't do that you **WILL** miss solutions. For instance, take $n = 1$.

$$x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{11\pi}{15}\right)\right) = \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

I'll leave it to you to verify my work showing they are solutions. However it makes the point. If you didn't divided the $2\pi n$ by 5 you would have missed these solutions!

4. $2\sin(5x + 4) = -\sqrt{3}$

Solution

This problem is almost identical to the previous problem except this time we have an argument of $5x + 4$ instead of $5x$. However, most of the problem is identical. In this case the solutions we get will be

$$5x + 4 = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$
$$5x + 4 = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$

Notice the difference in the left hand sides between this solution and the corresponding solution in the previous problem.

Now we need to solve for x. We'll first subtract 4 from both sides then divide by 5 to get the following solution.

$$x = \frac{4\pi}{15} + \frac{2\pi n - 4}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{15} + \frac{2\pi n - 4}{5}, \quad n = 0, \pm 1, \pm 2, \ldots$$

It's somewhat messy, but it is the solution. Don't get excited when solutions get messy. They will on occasion and you need to get used to seeing them.

5. $2\sin\left(3x\right) = 1$ on $\left[-\pi, \pi\right]$

Solution

I'm going to leave most of the explanation that was in the previous three out of this one to see if you have caught on how to do these.

$$2\sin\left(3x\right) = 1$$
$$\sin\left(3x\right) = \frac{1}{2}$$

By examining a unit circle we see that

$$3x = \frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$3x = \frac{5\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by 3,

$$x = \frac{\pi}{18} + \frac{2\pi n}{3}, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{18} + \frac{2\pi n}{3}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now, we are looking for solutions in the range $\left[-\pi, \pi\right]$. So, let's start trying some values of n.

$n = 0$ **:** $\qquad x = \frac{\pi}{18} \quad \& \quad x = \frac{5\pi}{18}$

$n = 1$ **:**

$$x = \frac{\pi}{18} + \frac{2\pi}{3} = \frac{13\pi}{18} < \pi \text{ so a solution}$$
$$x = \frac{5\pi}{18} + \frac{2\pi}{3} = \frac{17\pi}{18} < \pi \text{ so a solution}$$

$n = 2$ **:**

$$x = \frac{\pi}{18} + \frac{4\pi}{3} = \frac{25\pi}{18} > \pi \text{ so NOT a solution}$$
$$x = \frac{5\pi}{18} + \frac{4\pi}{3} = \frac{29\pi}{18} > \pi \text{ so NOT a solution}$$

Once, we've hit the limit in one direction there's no reason to continue on. In, other words if using $n = 2$ gets values larger than π then so will all values of n larger than 2. Note as well that it is possible to have one of these be a solution and the other to not be a solution. It all depends on the interval being used.

Let's not forget the negative values of n.

$n = -1$ **:**

$$x = \frac{\pi}{18} - \frac{2\pi}{3} = -\frac{11\pi}{18} > -\pi \text{ so a solution}$$
$$x = \frac{5\pi}{18} - \frac{2\pi}{3} = -\frac{7\pi}{18} > -\pi \text{ so a solution}$$

$n = -2$ **:**

$$x = \frac{\pi}{18} - \frac{4\pi}{3} = -\frac{23\pi}{18} < -\pi \text{ so NOT a solution}$$
$$x = \frac{5\pi}{18} - \frac{4\pi}{3} = -\frac{19\pi}{18} < -\pi \text{ so NOT a solution}$$

Again, now that we've started getting less than $-\pi$ all other values of $n < -2$ will also give values that are less than $-\pi$.

So putting all this together gives the following six solutions.

$$x = -\frac{11\pi}{18}, \frac{\pi}{18}, \frac{13\pi}{18}$$
$$x = -\frac{7\pi}{18}, \frac{5\pi}{18}, \frac{17\pi}{18}$$

Finally, note again that if we hadn't divided the $2\pi n$ by 3 the only solutions we would have gotten would be $\frac{\pi}{18}$ and $\frac{5\pi}{18}$. We would have completely missed four of the solutions!

6. $\sin(4t) = 1$ on $[0, 2\pi]$

Solution

This one doesn't actually have a lot of work involved. We're looking for values of t for which $\sin(4t) = 1$. This is one of the few trig equations for which there is only a single angle in all of $[0, 2\pi]$ which will work. So our solutions are

$$4t = \frac{\pi}{2} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, by dividing by 4,

$$t = \frac{\pi}{8} + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Since we want the solutions on $[0, 2\pi]$ negative values of *n* aren't needed for this problem. So, plugging in values of *n* will give the following four solutions

$$t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

7. $\cos(3x) = 2$

Solution
This is a trick question that is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \cos(\theta) \le 1$ and $-1 \le \sin(\theta) \le 1$. Therefore, since cosine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

8. $\sin(2x) = -\cos(2x)$

Solution
This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

$$\sin(2x) = -\cos(2x)$$
$$\frac{\sin(2x)}{\cos(2x)} = -1$$
$$\tan(2x) = -1$$

So, solving $\sin(2x) = -\cos(2x)$ is the same as solving $\tan(2x) = -1$. At some level we didn't need to do this for this problem as all we're looking for is angles in which sine and cosine have the same value, but opposite signs. However, for other problems this won't be the case and we'll want to convert to tangent.

Looking at our trusty unit circle it appears that the solutions will be,

$$2x = \frac{3\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$2x = \frac{7\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by the 2 we get the solutions

$$x = \frac{3\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$x = \frac{7\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

No interval was given so we'll stop here.

9. $2\sin(\theta)\cos(\theta) = 1$

Solution

Again, we need to do a little work to get this equation into a form we can handle. The easiest way to do this one is to recall one of the trig formulas from the Trig Formulas section (in particular Problem 3).

$$2\sin(\theta)\cos(\theta) = 1$$
$$\sin(2\theta) = 1$$

At this point, proceed as we did in the previous problems..

$$2\theta = \frac{\pi}{2} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, by dividing by 2,

$$\theta = \frac{\pi}{4} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Again, there is no interval so we stop here.

10. $\sin(w)\cos(w) + \cos(w) = 0$

Solution

This problem is very different from the previous problems.

DO NOT DIVIDE BOTH SIDES BY A COSINE!!!!!!

If you divide both sides by a cosine you ***WILL*** lose solutions! The best way to deal with this one is to "factor" the equations as follows.

$$\sin(w)\cos(w) + \cos(w) = 0$$
$$\cos(w)(\sin(w) + 1) = 0$$

So, solutions will be values of w for which

$$\cos(w) = 0$$

or,

$$\sin(w) + 1 = 0 \quad \Rightarrow \quad \sin(w) = -1$$

In the first case we will have $\cos(w) = 0$ at the following values.

$$w = \frac{\pi}{2} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$w = \frac{3\pi}{2} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

In the second case we will have $\sin(w) = -1$ at the following values.

$$w = \frac{3\pi}{2} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Note that in this case we got a repeat answer. Sometimes this will happen and sometimes it won't so don't expect this to always happen. So, all together we get the following solutions,

$$w = \frac{\pi}{2} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$w = \frac{3\pi}{2} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

As with the previous couple of problems this has no interval so we'll stop here. Notice as well that if we'd divided out a cosine we would have lost half the solutions.

11. $2\cos^2(3x) + 5\cos(3x) - 3 = 0$

Solution

This problem appears very difficult at first glance, but only the first step is different for the previous problems. First notice that

$$2t^2 + 5t - 3 = 0$$
$$(2t - 1)(t + 3) = 0$$

The solutions to this are $t = \frac{1}{2}$ and $t = -3$. So, why cover this? Well, if you think about it there is very little difference between this and the problem you are asked to do. First, we factor the equation

$$2\cos^2(3x) + 5\cos(3x) - 3 = 0$$
$$(2\cos(3x) - 1)(\cos(3x) + 3) = 0$$

The solutions to this are

$$\cos(3x) = \frac{1}{2} \qquad \text{and} \qquad \cos(3x) = -3$$

The solutions to the first are

$$3x = \frac{\pi}{3} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$3x = \frac{5\pi}{3} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by 3,

$$x = \frac{\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

The second has no solutions because cosine can't be less that -1. Don't get used to this. Often both will yield solutions!

Therefore, the solutions to this are (again no interval so we're done at this point).

$$x = \frac{\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{5\pi}{9} + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

12. $5\sin\left(2x\right) = 1$

Solution

This problem, in some ways, is VERY different from the previous problems and yet will work in essentially the same manner. To this point all the problems came down to a few "basic" angles that most people know and/or have used on a regular basis. This problem won't, but the solution process is pretty much the same. First, get the sine on one side by itself.

$$\sin\left(2x\right) = \frac{1}{5}$$

Now, at this point we know that we don't have one of the "basic" angles since those all pretty much come down to having 0, 1, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$ or $\frac{\sqrt{3}}{2}$ on the right side of the equal sign. So, in order to solve this we'll need to use our calculator. Every calculator is different, but most will have an inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) and inverse tangent ($\tan^{-1}$) button on them these days. If you aren't familiar with inverse trig functions see the next section in this review. Also, make sure that your calculator is set to do radians and not degrees for this problem.

It is also very important to understand the answer that your calculator will give. First, note that I said answer (*i.e.* a single answer) because that is all your calculator will ever give and we know from our work above that there are infinitely many answers. Next, when using your calculator to solve $\sin\left(x\right) = a$, *i.e.* using $\sin^{-1}\left(a\right)$, we will get the following ranges for x.

$$a \geq 0 \qquad \Rightarrow \qquad 0 \leq x \leq 1.570796327 = \frac{\pi}{2} \qquad \left(\text{Quad I}\right)$$

$$a \leq 0 \qquad \Rightarrow \qquad -\frac{\pi}{2} = -1.570796327 \leq x \leq 0 \qquad \left(\text{Quad IV}\right)$$

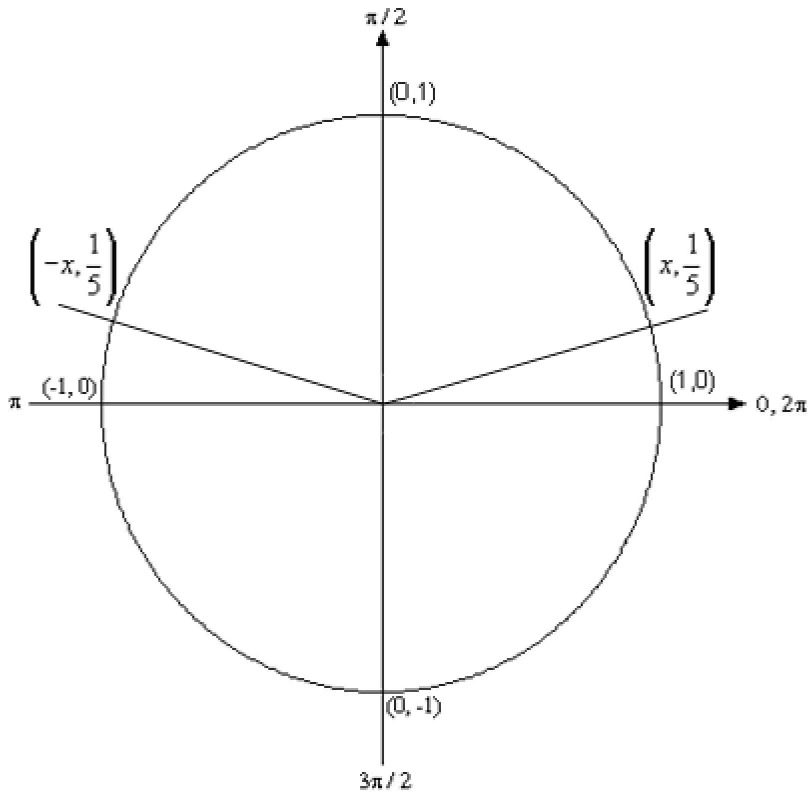
So, when using the inverse sine button on your calculator it will ONLY return answers in the first or fourth quadrant depending upon the sign of a.

Using our calculator in this problem yields,

$$2x = \sin^{-1}\left(\frac{1}{5}\right) = 0.2013579$$

Don't forget the 2 that is in the argument! We'll take care of that in a bit.

Now, we know from our work above that if there is a solution in the first quadrant to this equation then there will also be a solution in the second quadrant and that it will be at an angle of 0.2013579 above the x-axis as shown below.



I didn't put in the x, or cosine value, in the unit circle since it's not needed for the problem. I did however not that they will be the same value, except for the negative sign. The angle in the second quadrant will then be,

$$\pi - 0.2013579 = 2.9402348$$

So, let's put all this together.

$$2x = 0.2013579 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$2x = 2.9402348 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Note that I added the $2\pi n$ onto our angles as well since we know that will be needed in order to get all the solutions. The final step is to then divide both sides by the 2 in order to get all possible solutions. Doing this gives,

$$x = 0.10067895 + \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = 1.4701174 + \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

The answers won't be as "nice" as the answers in the previous problems but there they are. Note as well that if we'd been given an interval we could plug in values of *n* to determine the solutions that actually fall in the interval that we're interested in.

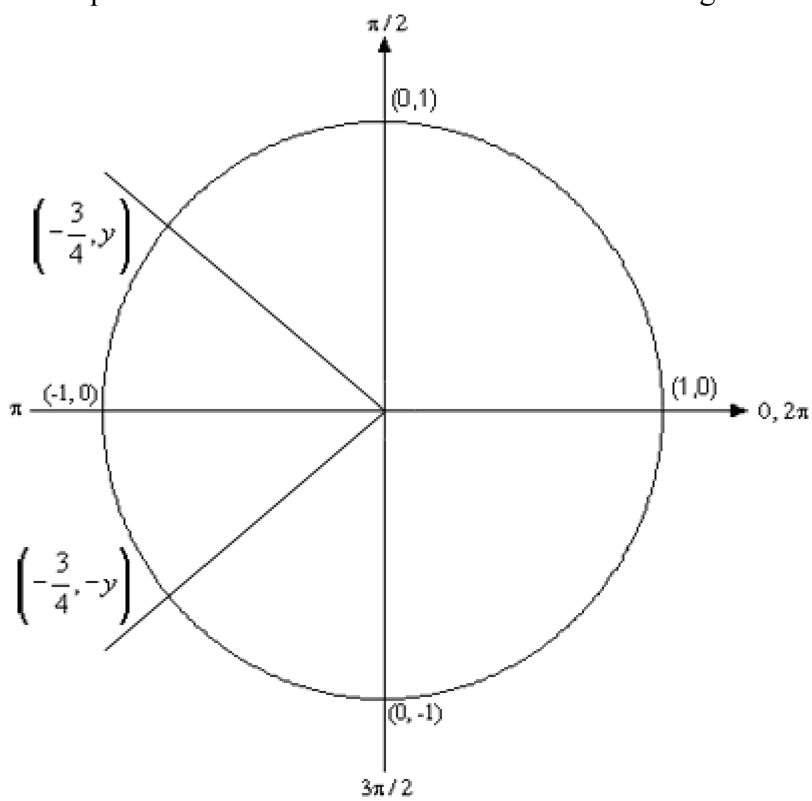
13. $4\cos\left(\frac{x}{5}\right) = -3$

Solution

This problem is again very similar to previous problems and yet has some differences. First get the cosine on one side by itself.

$$\cos\left(\frac{x}{5}\right) = -\frac{3}{4}$$

Now, let's take a quick look at a unit circle so we can see what angles we're after.



I didn't put the *y* values in since they aren't needed for this problem. Note however, that they will be the same except have opposite signs. Now, if this were a problem involving a "basic" angle we'd drop the "-" to determine the angle each of the lines above makes with the *x*-axis and then use that to find the actual angles. However, in this case since we're using a calculator we'll get the angle in the second quadrant for free so we may as well jump straight to that one.

However, prior to doing that let's acknowledge how the calculator will work when working with inverse cosines. If we're going to solve $\cos(x) = a$, using $\cos^{-1}(a)$, then our calculator will give one answer in one of the following ranges, depending upon the sign of *a*.

$$a \geq 0 \qquad \Rightarrow \qquad 0 \leq x \leq 1.570796 = \frac{\pi}{2} \qquad (\text{Quad I})$$

$$a \leq 0 \qquad \Rightarrow \qquad \pi = 3.141593 \leq x \leq 1.570796 = \frac{\pi}{2} \qquad (\text{Quad II})$$

So, using our calculator we get the following angle in the second quadrant.

$$\frac{x}{5} = \cos^{-1}\left(-\frac{3}{4}\right) = 2.418858$$

Now, we need to get the second angle that lies in the third quadrant. To find this angle note that the line in the second quadrant and the line in the third quadrant both make the same angle with the negative *x*-axis. Since we know what the angle in the second quadrant is we can find the angle that this line makes with the negative *x*-axis as follows,

$$\pi - 2.418858 = 0.722735$$

This means that the angle in the third quadrant is,

$$\pi + 0.722735 = 3.864328$$

Putting all this together gives,

$$\frac{x}{5} = 2.418858 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\frac{x}{5} = 3.864328 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Finally, we just need to multiply both sides by 5 to determine all possible solutions.

$$x = 12.09429 + 10\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$x = 19.32164 + 10\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

14. $10\sin(x-2) = -7$

Solution

We'll do this one much quicker than the previous two. First get the sine on one side by itself.

$$\sin(x-2) = -\frac{7}{10}$$

From a unit circle we can see that the two angles we'll be looking for are in the third and fourth quadrants. Our calculator will give us the angle that is in the fourth quadrant and this angle is,

$$x - 2 = \sin^{-1}\left(-\frac{7}{10}\right) = -0.775395$$

Note that in all the previous examples we generally wouldn't have used this answer because it is negative. There is nothing wrong with the answer, but as I mentioned several times in earlier problems we generally try to use positive angles between 0 and 2π . However, in this case since we are doing calculator work we won't worry about that fact that it's negative. If we wanted the positive angle we could always get it as,

$$2\pi - 0.775395 = 5.5077903$$

Now, the line corresponding to this solution makes an angle with the positive x-axis of 0.775395. The angle in the third quadrant will be 0.775395 radians below the negative x-axis and so is,

$$x - 2 = \pi + 0.775395 = 3.916988$$

Putting all this together gives,

$$\begin{aligned} x - 2 &= -0.775395 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \\ x - 2 &= 3.916988 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

To get the final solution all we need to do is add 2 to both sides. All possible solutions are then,

$$\begin{aligned} x &= 1.224605 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \\ x &= 5.916988 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

As the last three examples have shown, solving a trig equation that doesn't give any of the "basic" angles is not much different from those that do give "basic" angles. In fact, in some ways there are a little easier to do since our calculator will always give us one for free and all we need to do is find the second. The main idea here is to always remember that we need to be careful with our calculator and understand the results that it gives us.

Note as well that even for those problems that have "basic" angles as solutions we could have used a calculator as well. The only difference would have been that our answers would have been decimals instead of the exact answers we got.

Inverse Trig Functions

One of the more common notations for inverse trig functions can be very confusing. First, regardless of how you are used to dealing with exponentiation we tend to denote an inverse trig function with an "exponent" of "-1". In other words, the inverse cosine is denoted as $\cos^{-1}(x)$. It is important here to note that in this case the "-1" is NOT an exponent and so,

$$\cos^{-1}(x) \neq \frac{1}{\cos(x)}.$$

In inverse trig functions the "-1" looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trig function. It is a notation that we use in this case to denote inverse trig functions. If I had really wanted exponentiation to denote 1 over cosine I would use the following.

$$\left(\cos(x)\right)^{-1} = \frac{1}{\cos(x)}$$

There's another notation for inverse trig functions that avoids this ambiguity. It is the following.

$$\cos^{-1}(x) = \arccos(x)$$
$$\sin^{-1}(x) = \arcsin(x)$$
$$\tan^{-1}(x) = \arctan(x)$$

So, be careful with the notation for inverse trig functions!

There are, of course, similar inverse functions for the remaining three trig functions, but these are the main three that you'll see in a calculus class so I'm going to concentrate on them.

To evaluate inverse trig functions remember that the following statements are equivalent.

$$\theta = \cos^{-1}(x) \qquad \Leftrightarrow \qquad x = \cos(\theta)$$
$$\theta = \sin^{-1}(x) \qquad \Leftrightarrow \qquad x = \sin(\theta)$$
$$\theta = \tan^{-1}(x) \qquad \Leftrightarrow \qquad x = \tan(\theta)$$

In other words, when we evaluate an inverse trig function we are asking what angle, θ, did we plug into the trig function (regular, not inverse!) to get x.

So, let's do some problems to see how these work. Evaluate each of the following.

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

In Problem 1 of the Solving Trig Equations section we solved the following equation.

$$\cos(t) = \frac{\sqrt{3}}{2}$$

In other words, we asked what angles, t, do we need to plug into cosine to get $\frac{\sqrt{3}}{2}$?

This is essentially what we are asking here when we are asked to compute the inverse trig function.

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

There is one very large difference however. In Problem 1 we were solving an equation which yielded an infinite number of solutions. These were,

$$\frac{\pi}{6}+2\pi\, n,\ \ n=0,\pm 1,\pm 2,\pm 3,\ldots$$
$$\frac{11\pi}{6}+2\pi\, n,\ \ n=0,\pm 1,\pm 2,\pm 3,\ldots$$

In the case of inverse trig functions we are after a single value. We don't want to have to guess at which one of the infinite possible answers we want. So, to make sure we get a single value out of the inverse trig cosine function we use the following restrictions on inverse cosine.

$$\theta=\cos^{-1}(x) \qquad -1\le x\le 1 \qquad \text{and} \qquad 0\le\theta\le\pi$$

The restriction on the θ guarantees that we will only get a single value angle and since we can't get values of x out of cosine that are larger than 1 or smaller than -1 we also can't plug these values into an inverse trig function.

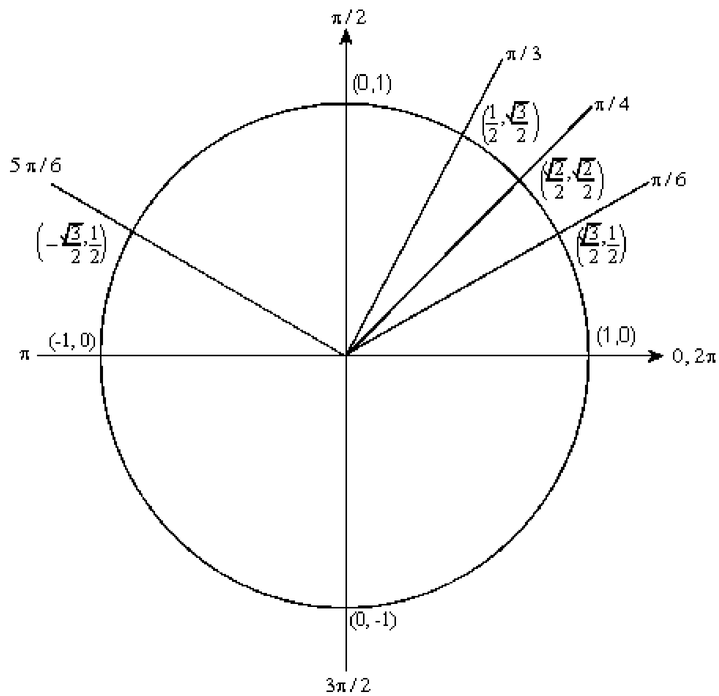
So, using these restrictions on the solution to Problem 1 we can see that the answer in this case is

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$$

2. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution

In general we don't need to actually solve an equation to determine the value of an inverse trig function. All we need to do is look at a unit circle. So in this case we're after an angle between 0 and π for which cosine will take on the value $-\frac{\sqrt{3}}{2}$. So, check out the following unit circle



From this we can see that

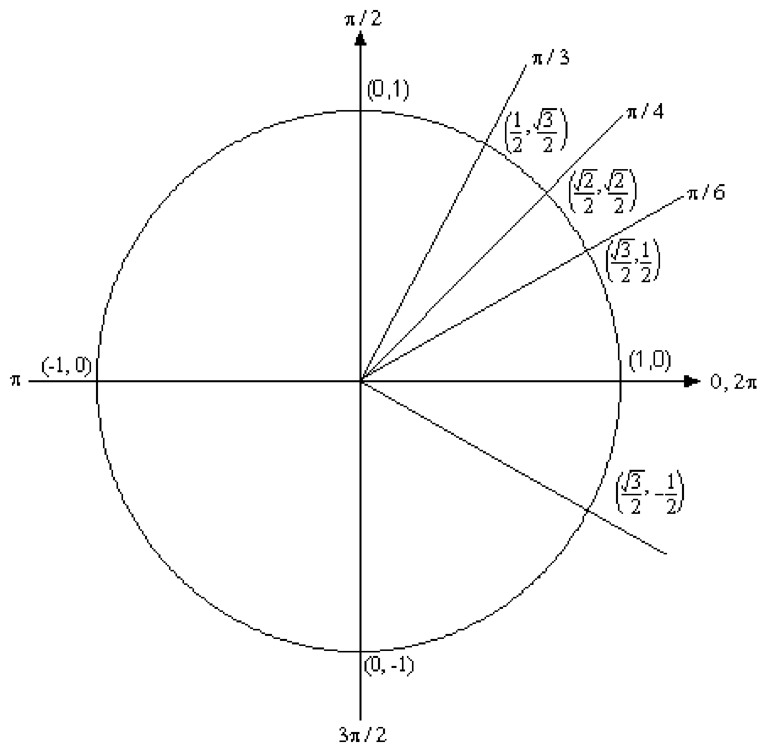
$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$$

3. $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution
The restrictions that we put on θ for the inverse cosine function will not work for the inverse sine function. Just look at the unit circle above and you will see that between 0 and π there are in fact two angles for which sine would be $\frac{1}{2}$ and this is not what we want. As with the inverse cosine function we only want a single value. Therefore, for the inverse sine function we use the following restrictions.

$$\theta=\sin^{-1}(x) \qquad -1\le x\le 1 \qquad \text{and} \qquad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$

By checking out the unit circle



we see

$$\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$$

4. $\tan^{-1}(1)$

Solution
The restriction for inverse tangent is

$$\theta=\tan^{-1}(x) \qquad\qquad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$$

Notice that there is no restriction on x this time. This is because $\tan(\theta)$ can take any value from negative infinity to positive infinity. If this is true then we can also plug any value into the inverse tangent function. Also note that we don't include the two endpoints on the restriction on θ. Tangent is not defined at these two points, so we can't plug them into the inverse tangent function.

In this problem we're looking for the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for which $\tan(\theta)=1$, or $\sin(\theta)=\cos(\theta)$. This can only occur at $\theta=\frac{\pi}{4}$ so,

$$\tan^{-1}(1)=\frac{\pi}{4}$$

5. $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

Solution
Recalling the answer to Problem 1 in this section the solution to this problem is much easier than it look's like on the surface.

$$\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

This problem leads to a couple of nice facts about inverse cosine

$$\cos\left(\cos^{-1}(x)\right)=x \qquad \text{AND} \qquad \cos^{-1}\left(\cos(\theta)\right)=\theta$$

6. $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$

Solution
This problem is also not too difficult (hopefully…).

$$\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)=\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$$

As with inverse cosine we also have the following facts about inverse sine.

$$\sin\left(\sin^{-1}(x)\right)=x \qquad \text{AND} \qquad \sin^{-1}\left(\sin(\theta)\right)=\theta$$

7. $\tan\left(\tan^{-1}(-4)\right)$.

Solution
Just as inverse cosine and inverse sine had a couple of nice facts about them so does inverse tangent. Here is the fact

$$\tan\left(\tan^{-1}(x)\right)=x \qquad \text{AND} \qquad \tan^{-1}\left(\tan(\theta)\right)=\theta$$

Using this fact makes this a very easy problem as I couldn't do $\tan^{-1}(4)$ by hand! A calculator could easily do it but I couldn't get an exact answer from a unit circle.

$$\tan\left(\tan^{-1}(-4)\right)=-4$$

Exponentials / Logarithms

Basic Exponential Functions