MAP562 Optimal design of structures (École Polytechnique)
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Session 4: Jan 30th, 2019 – Parametric Optimization

Exercise 1

Let Ω be an open, bounded and regular domain in $\mathbb{R}^2$. We consider an elastic membrane in Ω with variable thickness The boundary is split into two parts $\partial\Omega = \Gamma_N \cup \Gamma_D$. The vertical displacement of the membrane is the solution $u \in H^1(\Omega)$ of the following problem

$$\begin{cases} -\operatorname{div}(h\nabla u) = f & \text{in } \Omega, \\ h\frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_N \\ u = 0 & \text{sur } \Gamma_D, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ are the applied forces. The thickness $h(x)$ belongs to the admissible set

$$\mathcal{U}_{ad} = \{h \in L^\infty(\Omega)\ , \quad h_{max} \geq h(x) \geq h_{min} > 0 \text{ in } \Omega\}\,.$$

1. Let $u_0 \in L^2(\Gamma_N)$ be a target displacement that we want to match. In this case, the objective functional is given by
$$\inf_{h\in\mathcal{U}_{ad}} \left\{ J(h) = \int_{\Gamma_N} |u-u_0|^2 ds \right\}. \tag{2}$$
   a) Formulate the Lagrangian and derive the adjoint state.
   b) Compute the derivative of $J(h)$ with respect to h.
   c) Construct a `FreeFem++` script which solves (2).
   You may start from the given script `conductivity.edp`.
2. We now change the goal and consider to match $\sigma_0 \in L^2(\Omega)^2$ as goal. The corresponding objectif functional reads:
$$\inf_{h\in\mathcal{U}_{ad}} \left\{ J(h) = \frac{1}{2}\int_{\Omega} |\sigma-\sigma_0|^2 dx \right\}, \tag{3}$$
   where $\sigma = h\nabla u$ is the vector of the constraints.
   a) Formulate the Lagrangian and derive the adjoint state.
   b) Calculate the derivative of $J(h)$ with respect to h.
   c) Construct a `FreeFem++` script which solves (3).

3. **(Optional)** We modify slightly the functional considered in (2). The new objective functional is
$$\inf_{h\in\mathcal{U}_{ad}} \left\{ J(h) = \frac{1}{2}\int_{\Omega} h|u-u_0|^2 dx \right\}. \tag{4}$$
   a) Formulate the Lagrangian and derive the adjoint state.
   b) Calculate the derivative of $J(h)$ with respect to h.
   c) Construct a `FreeFem++` script which solves (3).

Exercise 2

In this exercise we optimize the thickness h of a two-dimensional elastic body in Ω. As in the other exercises, we have two boundary parts Γ_D and Γ_N. For a homogeneous elastic body the governing equations are: Find $u \in V$ such that

$$\begin{cases} -\nabla \cdot (\sigma(u)) &=& f \quad \text{in } \Omega, \\ u &=& (0,0) \quad \text{on } \Gamma_D, \\ \sigma(u) \cdot n &=& g_N \quad \text{on } \Gamma_N, \\ \sigma(u) \cdot n &=& 0 \quad \text{on the rest of } \partial\Omega, \end{cases}$$

where the stress is given by the classical law:

$$\sigma(u) = h(2\mu e(u) + \lambda tr(e(u))I)$$

with the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, I is the identity matrix, and μ and λ are the Lamé coefficients.

We have two goals in mind, each corresponding to the minimization of one of the two following functionals:

$$J_C(h) = \int_\Omega f \cdot u(h)\, dx + \int_{\Gamma_N} g \cdot u(h)\, ds \text{ (Compliance minimization)},$$

or

$$J_D(h) = \int_\omega |u - u_0|^2\, dx \text{ (Displacement matching)},$$

where ω is a small region around the point where the forces are applied. The admissible set for both problem configurations is given by

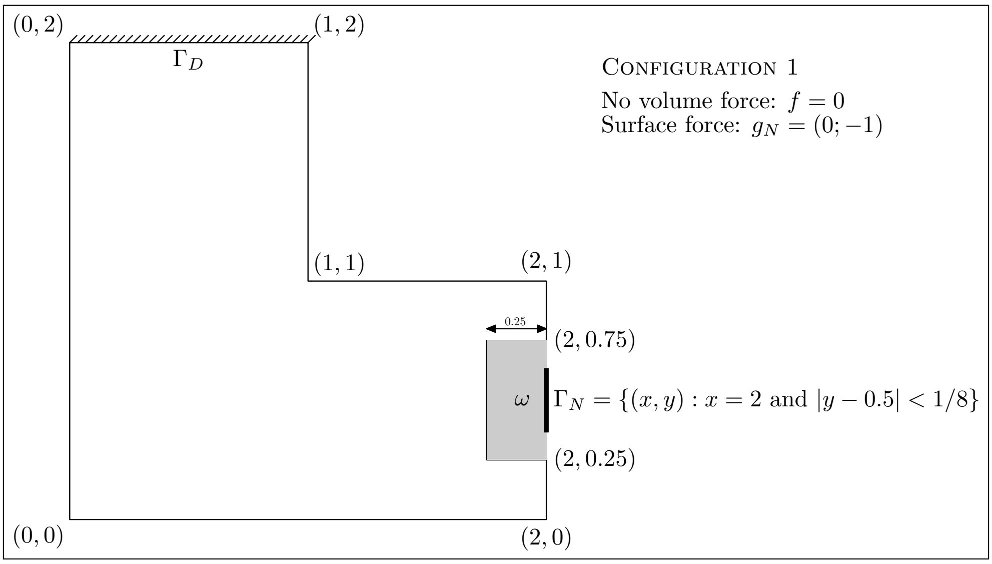
$$U = \{h \in L^\infty(\Omega) : \, h_{min} \leq h \leq h_{max}\},$$

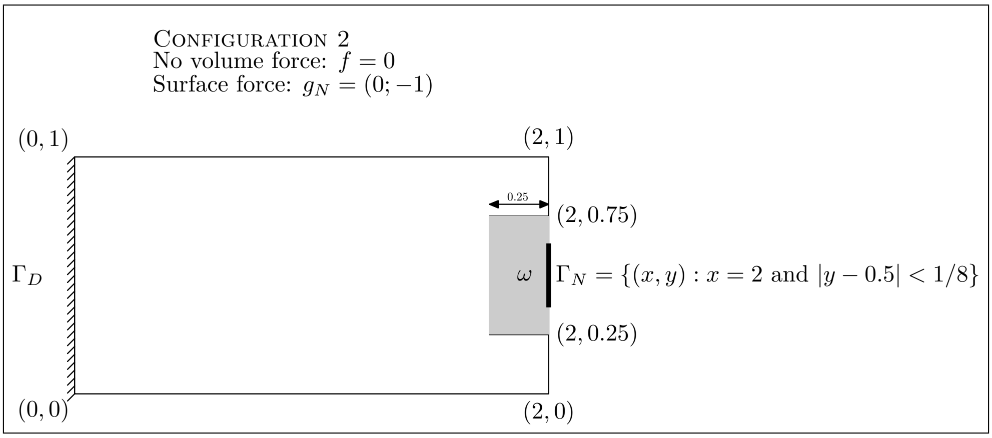
and with the additional *volume* constraint

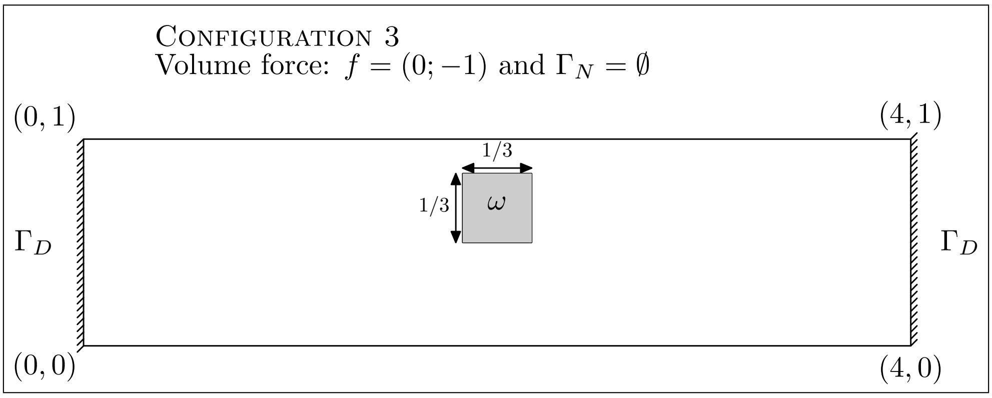
$$\int_\Omega h(x)\, dx = h_{avg}|\Omega|.$$

In all cases the following parameters will be used: $h_{min} = 0.1$, $h_{max} = 1$ and $h_{avg} = 0.3$.

1. For both functionals described above, introduce the Lagrangian, derive the state and adjoint equations.
2. Implement $\min_{h \in U} J_C(h)$ in FreeFem using a gradient algorithm.
3. Implement $\min_{h \in U} J_D(h)$ in FreeFem using a gradient algorithm for the target $u_0 = (0,0)$.
4. Observe how does the mesh refinement and the parameter h_{avg} influence the results.




Configuration 2
No volume force: $f = 0$
Surface force: $g_N = (0; -1)$
$(0,1)$
$(2,1)$
0.25
$(2, 0.75)$
Γ_D
ω
$\Gamma_N = \{(x,y) : x = 2 \text{ and } |y - 0.5| < 1/8\}$
$(2, 0.25)$
$(0,0)$
$(2,0)$


Configuration 3
Volume force: $f = (0; -1)$ and $\Gamma_N = \emptyset$
$(0,1)$
$(4,1)$
$1/3$
$1/3$
ω
Γ_D
Γ_D
$(0,0)$
$(4,0)$