MAP562 Optimal design of structures

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Exercise and Homework Sheet 8, March 6th, 2019

Instructions Upload your solutions as separate files to the course Moodle by March 13th.

Exercise 1

The purpose of this exercise is to compute numerically the homogenized tensor in the case of two-phase mixtures. You may start from the code `composite.edp` which can be found on Moodle. The Hooke law $A(y)$ is given by

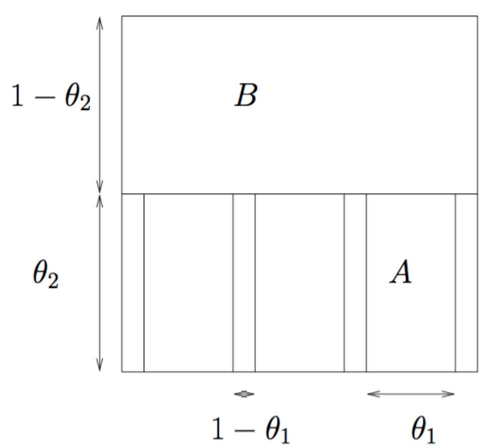
$$A(y) = \alpha\chi(y) + \beta(1 - \chi(y)),$$

where $\chi(y)$ is a characteristic function $\chi : Y \to \{0, 1\}$ of a domain $\omega \subset Y$.

1. Prove that if ω is a disk then the tensor A^* is a multiple of the identity. Verify this fact numerically. For different values of the radius of the disk, verify that the homogeneized tensor verifies the Hashin-Shtrikman bounds. (See the course: slide 43)
2. Prove that if ω is an ellipse with axes parallel to the edges of Y then A^* is diagonal. Verify this fact numerically.
3. Consider a shape which is not symmetric with respect to the vertical and horizontal directions (for example a rotated ellipse) and check that in this case the tensor A^* is not diagonal.

Exercise 2

In this exercise we numerically compute some properties of a composite material which is approximately a rank-two laminate as displayed in the following figure.



The cell equations are those given in the slides. They are implemented in the FreeFem++ script `composite.edp`. Notice in particular that the periodic boundary conditions are defined in the finite element space. The same code could be used to study composite materials in solid mechanics.

This exercise has the following objectives:

- **Task 1:** Implement the new geometry in the given program `composite.edp`. This program will then yield the values A^*_{ij} of the homogenized tensor A^*.
- **Task 2:** Secondly take the formula from Lemma 7.14 (see the lecture notes) using the specific values for m_1 and m_2. With that formula compute as well the values of A^* (possibly see Lemma 7.11) in order to see how we approximately obtain A^*_{ij}.

1. For task 2 from above, show that the proportion of material A is $\theta = \theta_1\theta_2$. We need to find the proportions of the lamination, which yield a rank-two lamination with two orthogonal directions. Here the two phases are assumed to be isotropic. Specifically, show that
$$m_1 = \frac{1-\theta_1}{1-\theta}, \quad m_2 = \frac{\theta_1(1-\theta_2)}{1-\theta}.$$
What are the values of θ_1 and θ_2 that yield an isotropic composite material for a given volume fraction θ ?
2. **(Homework)** Implement task 1 in `FreeFem++` with the following numerical values:
$$\theta_1 = 0.8; \quad \theta_2 = 0.4; \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$
3. **(Homework)** Implement Task 2 using the formula from Lemma 7.14 and compare the result with the one obtained in the first question.
4. **(Homework)** Given A and B, vary the proportions m_1 and m_2 and plot the values of the diagonal entries of A^* ($i.e$, its eigenvalues). Do the same after inverting the values of A and B. What do you recover?

Exercise 3

We consider the following heat optimization problem. Let $\Omega = (0,1)^2$ and let T the temperature. The associated Hooke law is given by $A(y) = \alpha\chi(y) + \beta(1-\chi(y))$ where $\chi(y)$ is a characteristic function. In the numerical examples you can take $\alpha = 1, \beta = 10$ and the volume fraction $\theta = \int_Y \chi(y)dy$ equal to $\theta = 0.7$. We wish to find T such that
$$\begin{cases} -\operatorname{div}(A^*\nabla T) &= 0 & \text{in } \Omega, \\ A^*\nabla T\cdot n &= 1 & \text{on } \Gamma_N, \\ A^*\nabla T\cdot n &= 0 & \text{on } \Gamma, \\ T &= 0 & \text{on } \Gamma_D. \end{cases}$$
In order to optimize the heater, we want to minimize the temperature on the top boundary:
$$\min_{\theta, A^*}\Big\{ J(\theta, A^*) = \int_{\Gamma_N} T\, ds. \Big\} \tag{1}$$
Notice that this problem is self-adjoint and thus for the optimization, we can work with an optimality criterion method.

1. Show that, by the optimality condition, an optimal A^* for (1) is a rank-one laminate such that
$$A^*\nabla T = \lambda_\theta^+ \nabla T,$$
where $\lambda_\theta^+ = \alpha\theta + \beta(1-\theta)$.
2. We now change the cost functional to
$$\min_{\theta, A^*}\Big\{ J(\theta, A^*) = \int_{\Gamma_N} T^2\, ds. \Big\} \tag{2}$$
In this case, the problem is no more self-adjoint. This means that we must adopt the more general approach that is the gradient algorithm. This requires the solution of the adjoint state and there are two variables to be optimized: θ and ϕ.
Find the adjoint equation and compute the derivative of the objective function with respect to its parameters.

In the numerical simulations, when minimizing (1) using the optimality criterion method, we start by performing a fixed number of iterations. Then, in a second step, we can penalize the density θ to force it to take only the values 0 and 1. Rather than working with the true θ_{opt} we introduce
$$\theta_{pen} = \frac{1-\cos(\pi\theta_{opt})}{2}.$$
This will yield $\theta_{pen} < \theta_{opt}$ for $0 < \theta_{opt} < 0.5$ and $\theta_{opt} < \theta_{pen}$ for $0.5 < \theta_{opt} < 1$. For the numerical simulations start from the given script `radiator.edp`

3. **(Homework)** Implement the optimality criterion method for the minimization of (1) in `FreeFem++`. Plot the global evolution of the cost function and interpret the result. Refine the mesh twice and observe how the results change.

When minimizing the functional (2), which is not self-adjoint, a gradient algorithm can be used. There are multiple ways to penalize θ so that it gets close to a characteristic function:

- add a penalization term in the functional: instead of minimizing $J(\theta, A^*)$ we minimize

$$J(\theta, A^*) + c_{pen} \int_\Omega \theta(1-\theta)$$

  where $c_{pen} > 0$. In practice you may choose $c_{pen} = 1$ in the beginning of the optimization procedure, then after a fixed number of iterations further increase c_{pen} such that the penalization is enforced. Note that when increasing c_{pen} the objective function increases, so these iterations should be **accepted** regardless of the slight increase in the objective function.

- a second way to enforce the result to be a characteristic function is to use the so-called **SIMP method**. In this method simply replace the homogenized tensor A^* by $\theta^p A$ for some power $p > 1$ (typically $p = 3$). Note that the material properties are taken into account with weight θ^p, while the volume is computed using the density θ. Again, we do not start with $p = 3$ from the beginning. In practice you can run 10 iterations with $p = 1$, another 10 iterations with $p = 2$ and the rest with $p = 3$. Again, the objective function may increase when increasing p, so these iterations should be **accepted** regardless of the potential increase in the objective function.

4. **(Homework)** Implement the gradient algorithm for the minimization of the functional (2) in `FreeFem++` using the techniques above to enforce the penalization. Plot the global evolution of the cost function and interpret the results. Refine the mesh twice and observe how the results change.
   You may implement the optimization algorithm on one of the geometries shown below. Note that the leftmost geometry is the one used in `radiator.edp`.