MAP562 Optimal design of structures

by Grégoire Allaire, Beniamin Bogosel

École Polytechnique

Homework Sheet 2, Jan 16th, 2018

Instructions Upload your solutions as separate files to the course Moodle. Do not forget to add your own comments to explain what you are doing.

You must choose ONE of the Exercises 4 and 5 below (taken from Sheet 2) and solve it following what was done in class for the p-Laplacian. The deadline is Jan 23rd 2018.

The deadline for Exercise 6 is also Jan 23rd 2018.

Exercise 4 A second nonlinear problem

Let Ω be a smooth bounded open set of $\mathbb{R}^n$. Let $f \in L^2(\Omega)$. Let $a(v)$ be a smooth function from $\mathbb{R}^n$ into $\mathbb{R}^n$ which is bounded uniformly, as well as all its derivatives on $\mathbb{R}^n$, and that satisfies

$$0 < C^- \leq a(v) \leq C^+ < +\infty \quad \forall\, v \in \mathbb{R}^n.$$

We consider the minimization problem

$$E(u) = \min_{v \in H_0^1(\Omega)} E(v); \quad E(v) = \frac{1}{2}\int_\Omega a(v(x))|\nabla v(x)|^2\,dx - \int_\Omega f(x)\,v(x)\,dx\,. \tag{1}$$

Throughout the exercise $u \in H_0^1(\Omega)$ is assumed to be a smooth function.

1. Compute the first order directional derivative of E at u in the direction of v, that we shall denote by $\langle E'(u), v\rangle$.
2. Show that the first-order optimality condition for (1) is a variational formulation for a non-linear partial differential equation, which should be explicitly exhibited.
3. Compute the second order derivative of E at u, in the directions δu and ϕ, that we shall denote by $E''(u)\big(\delta u, \phi\big)$.
4. **(Optional)** Prove that the bilinear form $(v, w) \to E''(u)\big(v, w\big)$ is symmetric and continuous on $H_0^1(\Omega)$. From now on we assume that the first and second order derivatives of $v \to a(v)$ are uniformly small on $\mathbb{R}^n$. Deduce that the bilinear form is coercive on $H_0^1(\Omega)$. Hint: use integration by parts and the Poincaré inequality in Ω.

   In the following, for the numerical implementation, use the domain $\Omega = (0, 10)^2$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D is prescribed on the right and top boundaries and Γ_N on the remaining two parts. Moreover, let $a(v) = (v^2+1)/(v^2+2)$ and $f = 1$. Specifically, we use.

5. Construct and implement a gradient algorithm for minimizing (1).
6. Construct and implement a Newton algorithm for minimizing (1).
7. Prove that, at each iteration of the Newton algorithm, there exists a unique descent direction. Hint: use question 4 and assume that the previous iterate is a smooth function.
8. Compare with the fixed point method where at each iteration v_n minimizes

$$E_{n-1}(v) = \frac{1}{2}\int_\Omega a(v_{n-1}(x))|\nabla v(x)|^2\,dx - \int_\Omega f(x)\,v(x)\,dx\,.$$

Exercise 5 Radiative transfer

Let Ω be a given two dimensional domain and Γ_0 a part of its boundary. Denote by $\Gamma_N = \partial\Omega \setminus \Gamma_0$. Consider the following non-linear problem modeling the radiative heat transfer:

$$\left\{\begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial n} + k|u|^3 u &=& 0 & \text{on } \Gamma_N \end{array}\right. \tag{2}$$

Remark. In the physical context the temperature is considered positive and the absolute value can be eliminated.

1. Write a variational formulation associated to problem (2) defined on the functional space $V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$.
2. Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{dx} + \frac{k}{5}\int_{\Gamma_N} |u|^5 \,\mathrm{d}\sigma - \int_\Omega f u \,\mathrm{dx}\,.$$

   Show that:

   - The derivative of J gives the variational formulation found at the previous question.
   - The functional is strictly convex and if a minimizer exists it is unique. Furthermore, the minimizer verifies the variational formulation.

3. Formulate a gradient algorithm to find the solution u that minimizes $J(u)$.
4. Formulate a Newton algorithm to find the solution u that minimizes $J(u)$.
5. Solve problem (2) in `FreeFem++` using a fixed point algorithm. Compare this approach to the Newton algorithm (what problem is solved at each iteration, number of iterations)

Exercise 6 Optimal control

1. **(Optional)** Solve question 4 of Exercise 6 in the Sheet 2: formulate a gradient algorithm to solve the minimization problem.