MAP562 Optimal design of structures (École Polytechnique)

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Exercise 1

Optimization of a heater with a constant localized control and no convection

In this exercise we consider a simplified heat optimal control problem. In some subregion ω of the domain Ω we have a heat source $u \in \mathbb{R}$ (the control variable) which is a constant. The goal is to match a certain, given, temperature field $x \mapsto T_0(x)$. This leads to the cost functional:

$$J(u) = \int_\Omega |T(u) - T_0|^2 \, dx.$$

We are interested in minimizing $J(u)$. The heat distribution is modeled by a diffusion equation

$$\begin{aligned} -\Delta T &= 1_\omega u \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and we denote by $T(u) = T$ the solution. The resulting problem statement is a so-called **PDE-constrained optimization problem.**

1. Show that
$$T(u) = T(1)u.$$
2. Derive the variational formulation satisfied by T.
3. Compute the derivatives of $T(u)$ and $J(u)$ with respect to u.
   Hint: For the derivative of $J(u)$ we need to employ twice the relation $T(u) = T(1)u$. In most cases such an explicit relation does not exist and then a crucial aspect in derivative-based optimization is the evaluation of the derivatives.
4. Formulate a gradient algorithm to solve the minimization problem (including all necessary steps!) and implement it in `FreeFem++`.

Exercise 2

A heat optimization problem.

In this exercise we consider a simplified heat optimization problem. Here the optimization variable is a vector and it is not strictly necessary to use all the theory developed in the course. For a given smooth bounded domain $\Omega \subset \mathbb{R}^d$ and for any $z \in \mathbb{R}^d$, we investigate the minimization of the cost functional:

$$J(z) = \int_\Omega |T - T_0|^2 \, dx$$

where $x \mapsto T_0(x)$ is a given smooth temperature field and $T \equiv T(x, z)$ is the solution of the following boundary value problem for the x variable

$$\begin{aligned} -\Delta T &= f_z \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f_z(x) = f(x - z)$ and $f(x)$ is a smooth non-negative function with compact support. In the numerical applications, take $d = 2$, $T_0(x) \equiv 0.1$, $\Omega = (0, 10)^2$ and

$$f(x) = \begin{cases} 1 - |x|^2 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

1. For a given direction $e \in \mathbb{R}^d$, find the problem solved by the directional derivative $v \equiv \nabla_z T \cdot e$.
2. Compute the derivative of $J(z)$ in the direction e in terms of v.
3. Introduce the Lagrangian corresponding to the minimization of $J(z)$.

4. Deduce the adjoint problem for this minimization problem and find an explicit formula for the gradient $\nabla_z J(z)$.
5. Implement a gradient algorithm for this problem in `FreeFem++`.
6. Implement a gradient algorithm for this problem in `FreeFem++` with the additional constraint that f_z be supported in Ω.

Exercise 3

Optimization of a heater with localized control and convection

We continue our investigation of an optimal heat source with two differences compared to exercise 1.

The first one is that the control is no longer constant. More precisely the control is a heat flux $v \in L^2(\omega)$. The second difference is the presence of an air stream in the room. It's velocity is given by the field of vectors $x \mapsto u(x) \in \mathbb{R}^2$ in the room. We assume that the air is incompressible, which can be written as

$$\nabla \cdot u(x) = 0 \text{ } (i.e. \text{ } u \text{ is divergence free}).$$

We also assume that u is a smooth function. The temperature in the room satisfies the so-called convection-diffusion equation

$$\begin{aligned} -\Delta T + u \cdot \nabla T &= 1_\omega v \text{ in } \Omega. \\ T &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Our objective is still to minimize the cost function

$$J(v) = \int_\Omega |T(v) - T_0|^2 dx,$$

for a given $T_0 \in \mathbb{R}$.

1. Determine the variational problem satisfied by the solution $T \equiv T(v)$ of the convection-diffusion equation. We admit that this problem has a unique solution that depends continuously on the data.
2. Compute the derivatives of $v \mapsto T(v)$ and $v \mapsto J(v)$ with respect to v. Can these expressions be used to implement a gradient type algorithm applied to the minimization of J ?
3. The gradient of J can be explicitly expressed by introducing an adjoint state. To this end, we first introduce the Lagrangian $\mathcal{L}(v, T, p)$, defined for $T, p \in H_0^1(\Omega)$ and $v \in L^2(\omega)$. Recall and verify that finding the minimizer of J is equivalent to solving the following min-max problem

$$\min_{v, T \in H_0^1(\Omega)} \sup_{p \in H_0^1(\Omega)} \mathcal{L}(v, T, p).$$

4. Determine the derivatives of $\mathcal{L}$ with respect to T and v.
5. By noticing that $J(v) = \mathcal{L}(v, T(v), p)$ for all $p \in H_0^1(\Omega)$, find a new expression of the differential of J depending on the derivatives of $\mathcal{L}$. Prove that a particular choice for p enables us to get rid of the term that depends on $\partial T / \partial v$. Deduce a new expression for the gradient of J and use it to write a gradient type algorithm for minimizing J.

Exercise 4

Optimal control of ODEs. (Optional)

We consider the following linear system of ordinary differential equations, the solution of which (called the state) is a function $y(t)$ with values in $\mathbb{R}^N$

$$\begin{cases} \dfrac{dy}{dt} = Ay + Bv + f \text{ for } 0 \le t \le T \\ y(0) = y_0 \end{cases} \tag{1}$$

where $y_0 \in \mathbb{R}^N$ is the initial state of the system, $f(t) \in \mathbb{R}^N$ is a source term, $v(t) \in \mathbb{R}^M$ is the control which allows us to act on the system, A and B are two constant matrices of respective dimensions $N \times N$ and $N \times M$. We shall denote by y_v the solution of (1).

We look for the optimal control v which minimizes the quadratic functional

$$\begin{aligned} J(v) = &\int_0^T Rv(t) \cdot v(t)dt + \int_0^T Q(y_v - z)(t) \cdot (y_v - z)(t)dt \\ &+ D\left(y_v(T) - z_T\right) \cdot \left(y_v(T) - z_T\right), \end{aligned}$$

where $z(t)$ is a target trajectory, z_T is a target final position, and R, Q, D are three symmetric non-negative matrices, from which only R is assumed to be positive definite. Let K be a closed non-empty convex set of $\mathbb{R}^M$: we restrict the control to the admissible set $L^2(]0,T[;K)$. The minimization problem is thus

$$\inf_{v(t) \in L^2(]0,T[;K)} J(v). \tag{2}$$

We assume that, if $f(t) \in L^2(]0,T[;\mathbb{R}^N)$, there exists a unique solution of (1) $y_v(t) \in H^1(]0,T[;\mathbb{R}^N)$, which is furthermore continuous in time.

A **simplified version** of this exercise is to consider the case: $R = I_M$ ($M \times M$ identity matrix), $Q = I_N$ ($N \times N$ identity matrix) and $D = 0$ ($N \times N$ zero matrix).

1. Prove that there exists a unique optimal control which minimizes (2).
2. Compute the derivative of the map $v \to y_v$ in the direction $w \in L^2(]0,T[;\mathbb{R}^N)$, that shall be denoted by y'_w.
3. Compute the derivative of $J(v)$ in the direction w in terms of y'_w. Explain why this formula is not useful in practice.
4. To get a simpler formula, we introduce the Lagrangian

$$\begin{aligned} \mathcal{L}(v,y,p) \;\; = &\int_0^T Rv(t) \cdot v(t)dt + \int_0^T Q(y - z)(t) \cdot (y - z)(t)dt \\ &+ D\left(y(T) - z_T\right) \cdot \left(y(T) - z_T\right) + \int_0^T p \cdot \left(-\frac{dy}{dt} + Ay + Bv + f\right) dt \\ &- p(0) \cdot \left(y(0) - y_0\right), \end{aligned}$$

   defined for any $v \in L^2(]0,T[;\mathbb{R}^N)$, any $y(t) \in H^1(]0,T[;\mathbb{R}^N)$ and any $p(t) \in H^1(]0,T[;\mathbb{R}^N)$. Check that

$$\sup_{p(t) \in H^1(]0,T[;\mathbb{R}^N)} \mathcal{L}(v,y,p) = \begin{cases} J(v) & \text{if (1) is satisfied,} \\ +\infty & \text{otherwise.} \end{cases}$$

5. Compute the partial derivative of $\mathcal{L}(v,y,p)$ with respect to y in the direction ϕ. By definition the adjoint system is obtained by writing that this partial derivative is zero for any $\phi \in H^1(]0,T[;\mathbb{R}^N)$ when $y = y_v$. We denoted by p_v the adjoint state. Check that it is a solution of the following ODE system (called the adjoint system)

$$\begin{cases} \dfrac{dp_v}{dt} = -A^* p_v - Q(y_v - z) \text{ for } 0 \le t \le T \\ p_v(T) = D(y_v(T) - z_T) \end{cases} \tag{3}$$

   where A^* is the adjoint matrix.

6. Compute the partial derivative with respect to v, in the direction w, of $\mathcal{L}(v, y_v, p)$ where p is a function independent of v. In this partial derivative, replace p by the adjoint state p_v and deduce a formula for the derivative of $J(v)$ in the direction w. Check that

$$J'(v) = B^* p_v + Rv \ .$$