MAP562 Optimal design of structures (École Polytechnique)
by Grégoire Allaire, Beniamin Bogosel

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Exercise 1 Quadratic optimization

(already discussed in class)
In this exercise we summarize basic properties of quadratic optimization. Consider the problem

$$\min_{x \in \mathbb{R}^n,\ Bx=c} J(x); \quad J(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle,$$

where A is a $n \times n$ symmetric, positive definite matrix; $b \in \mathbb{R}^n$ is a vector; B is a $m \times n$ matrix ($m \leq n$) whose rank equals m; and $c \in \mathbb{R}^m$ is a vector.

1. State the optimality condition.
2. Deduce the optimal solution.

Exercise 2 Basic example of optimization

In order to illustrate the concepts treated in the course we consider the following problem. The first part of the problem is simple, its objective being to see and work with initial examples of `FreeFem++` optimization algorithms. The second part adds some supplementary difficulties concerning some basic constraints.

Consider $\Omega \subset \mathbb{R}^2$ a bounded and connected domain and let $f \in L^2(\Omega)$ be a function which is not the null function. The objective, in the first part, will be to find the solution of the equation

$$u^3 = f \text{ in } \Omega. \tag{1}$$

The solution is the obvious $u = f^{1/3}$, but in what follows we will recover this solution using optimization algorithms. In order to do this, define the energy functional

$$J(u) = \frac{1}{4}\int_\Omega u^4 - \int_\Omega fu. \tag{2}$$

Part 1. Unconstrained optimization

1. Prove that J is strictly convex. Deduce that the minimizer verifies (1).
2. Propose and implement a gradient algorithm in order to find the unique minimizer of $J(u)$.
3. Compute the second derivative of J. Propose and implement a Newton algorithm in order to minimize $J(u)$. Study the behavior of the algorithm with respect to the initialization.
4. Propose and implement a fixed point algorithm in order to find the solution of (1). Compare the Newton algorithm to the fixed point algorithm.

Part 2. Constrained optimization

5. Solve the problem $\min J(u)$ where the function u verifies $\frac{1}{|\Omega|}\int_\Omega u = u_0$, where u_0 is given such that the equality above is compatible with the choice of f. (experiment... and give concrete values).
6. Solve the problem $\min J(u)$ where the function u verifies $0 \leq u \leq 1$.
7. Solve the problem $\min J(u)$ under the two constraints above: $0 \leq u \leq 1$ and $\frac{1}{|\Omega|}\int_\Omega u = u_0 \in [0, 1]$.

   Hint: The function $h : t \mapsto \frac{1}{|\Omega|}\int_\Omega \min(\max(u + t, 0), 1)$ is increasing and

   $$h(-\max_\Omega u) = 0,\ h(1 - \min_\Omega u) = 1.$$

   It is enough to make a dichotomy search for a t_0 such that $h(t_0) = u_0 \in [0, 1]$.

Exercise 3 Minimization of the p-Laplace problem

Linear PDEs correctly describe the behavior of physical systems close to their equilibrium state. However, non linear phenomena can appear in more general situations. The aim of this exercise is to extend the analysis performed on a linear system to some non-linear cases.

Let Ω be a bounded open set of $\mathbb{R}^N$, $p \in \mathbb{N}$ such that $p > 2$ and $f \in L^\infty(\Omega)$ the source term (or control). We consider the PDE (whose unknown is u)

$$\begin{aligned} -\operatorname{div}((1+|\nabla u|^{p-2})\nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3}$$

Note that for $|\nabla u| \ll 1$ (small perturbations), we recover the standard Poisson equation.

We introduce the Banach space $W_0^{1,p}(\Omega)$ defined by

$$W_0^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega), \text{ et } u = 0 \text{ on } \partial\Omega\},$$

1. State the variational formulation of (3) in the form $a(u,\phi) = 0$, $\forall\, \phi$. Note that $u \mapsto a(u,\phi)$ is not linear while $\phi \mapsto a(u,\phi)$ is a linear map.
2. Let J be the so called energy functional defined by:

$$J(u) = \int \frac{|\nabla u|^2}{2} + \frac{|\nabla u|^p}{p} - fu$$

   - Justify that the first derivative of $J(u)$ gives the variational formulation, *i.e.*, $J'(u)(\phi) = a(u,\phi)$.
   - Prove that the energy J is strictly convex. Deduce that if a minimizer exists, it is unique.

     For this reason it is equivalent to minimize J and to find a solution of the PDE. We admit that the minimization problem has a minimizer.
3. Formulate a gradient algorithm to find the solution u that minimizes $J(u)$.
4. Formulate a Newton algorithm to find the solution u that minimizes $J(u)$. Establish a justification that there indeed exists a unique descent direction.
   Hint: Prove that the second derivative is coercive and assume that the previous solution u^n is smooth enough.
5. What happens in the case $p = 2$?
6. Propose a fixed point algorithm in order to solve (3) in `FreeFem++`.

Exercise 4 A second nonlinear problem

(**Homework**: follow the instructions given in the Homework Sheet 2)

Let Ω be a smooth bounded open set of $\mathbb{R}^n$. Let $f \in L^2(\Omega)$. Let $a(v)$ be a smooth function from $\mathbb{R}$ into $\mathbb{R}$ which is bounded uniformly, as well as all its derivatives on $\mathbb{R}$, and that satisfies

$$0 < C^- \leq a(v) \leq C^+ < +\infty \quad \forall\, v \in \mathbb{R}.$$

We consider the minimization problem

$$E(u) = \min_{v \in H_0^1(\Omega)} E(v); \quad E(v) = \frac{1}{2}\int_\Omega a(v(x))|\nabla v(x)|^2\, dx - \int_\Omega f(x)\, v(x)\, dx\,. \tag{4}$$

Throughout the exercise $u \in H_0^1(\Omega)$ is assumed to be a smooth function.

1. Compute the first order directional derivative of E at u in the direction of v, that we shall denote by $\langle E'(u), v\rangle$.
2. Show that the first-order optimality condition for (4) is a variational formulation for a non-linear partial differential equation, which should be explicitly exhibited.
3. Compute the second order derivative of E at u, in the directions δu and ϕ, that we shall denote by $E''(u)\big(\delta u, \phi\big)$.

4. **(Optional)** Prove that the bilinear form $(v,w) \to E''(u)\big(v,w\big)$ is symmetric and continuous on $H_0^1(\Omega)$. From now on we assume that the first and second order derivatives of $v \to a(v)$ are uniformly small on $\mathbb{R}$. Deduce that the bilinear form is coercive on $H_0^1(\Omega)$. Hint: use integration by parts and the Poincaré inequality in Ω.
   In the following, for the numerical implementation, use $a(v) = (v^2+1)/(v^2+2)$ and $f = 1$.
5. Construct and implement a gradient algorithm for minimizing (4).
6. Construct and implement a Newton algorithm for minimizing (4).
7. Prove that, at each iteration of the Newton algorithm, there exists a unique descent direction. Hint: use question 4 and assume that the previous iterate is a smooth function.

Exercise 5 Radiative transfer

(**Homework**: follow the instructions given in the Homework Sheet 2)

Let Ω be a given two dimensional domain and Γ_0 a part of its boundary. Denote by $\Gamma_N = \partial\Omega \setminus \Gamma_0$. Consider the following non-linear problem modeling the radiative heat transfer:

$$\left\{ \begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial n} + k|u|^3 u &=& 0 & \text{on } \Gamma_N \end{array} \right. \tag{5}$$

Remark 1. In the physical context the temperature is considered positive and the absolute value can be eliminated.

1. Write a variational formulation associated to problem (5) defined on the functional space $V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$.
2. Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{dx} + \frac{k}{5}\int_{\Gamma_N} |u|^5 \,\mathrm{d}\sigma - \int_\Omega f u \,\mathrm{dx}\,.$$

   Show that:

   - The derivative of J gives the variational formulation found at the previous question.
   - The functional is strictly convex and if a minimizer exists it is unique. Furthermore, the minimizer verifies the variational formulation.

3. Formulate a gradient algorithm to find the solution u that minimizes $J(u)$.
4. Formulate a Newton algorithm to find the solution u that minimizes $J(u)$.
5. Solve problem (5) in `FreeFem++` using a fixed point algorithm. Compare this approach to the Newton algorithm (what problem is solved at each iteration, number of iterations)

Exercise 6 Optimal control

In this exercise we consider a simplified heat optimal control problem. In some subregion ω of the domain Ω we have a heat source $u \in \mathbb{R}$ (the control variable) which is a constant. The goal is to match a certain, given, temperature $T_0(x)$. This leads to the cost functional:

$$J(u) = \int_\Omega |T(u) - T_0|^2 \, dx$$

We are interested in minimizing $J(u)$. The heat distribution is modeled by a diffusion equation

$$\begin{aligned} -\Delta T &= 1_\omega u \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $T = T(u)$ is the solution, $\Omega = (0,1)^2$ and ω a ball with radius 0.1. The resulting problem statement is a so-called **PDE-constrained optimization problem.**

1. Show that

$$T(u) = T(1)u$$

2. Derive the variational formulation for the forward problem.
3. Compute the derivatives of $T(u)$ and $J(u)$ w.r.t. u.
   Hint: For the derivative of $J(u)$ we need to employ twice the relation $T(u) = T(1)u$. In most cases such an explicit relation does not exist and then a crucial aspect in derivative-based optimization is the evaluation of the 'inner' derivatives.
4. **(Homework)** Formulate a gradient algorithm to solve the minimization problem (including all necessary steps!).