MAP562 Optimal design of structures (École Polytechnique)

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Exercise 1 Homogenization method - dimension two

We consider the example given in the course for the functionals

$$J(\theta, A^*) = \int_\Omega f u \, dx \text{ or } J(\theta, A^*) = \int_\Omega |u - u_0|^2 \, dx,$$

for the homogenized state equation

$$\begin{cases} -\operatorname{div}(A^* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

It is possible to prove that there exist optimal solutions where A^* is a rank-1 simple laminate. In dimension 2 a rank one laminate can be defined by

$$A^*(\theta, \phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda_\theta^+ & 0 \\ 0 & \lambda_\theta^- \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \qquad \phi \in [0, \pi]$$

The admissible set becomes

$$\mathcal{U}_{ad}^L = \{(\theta, \phi) \int L^\infty(\Omega; [0,1] \times [0, \pi]) : \int_\Omega \theta(x) dx = V_\alpha\}$$

1. Prove that the partial derivatives of the objective function $J(\theta, A^*) = J(\theta, \phi)$ are given by

$$\nabla_\phi J(\theta, \phi) = \frac{\partial A^*}{\partial \phi} \nabla u \cdot \nabla p \text{ and } \nabla_\theta J(\theta, \phi) = \frac{\partial A^*}{\partial \theta} \nabla u \cdot \nabla p$$

   where u is the solution of the state equation and p is the adjoint state.
2. Compute explicitly the corresponding expressions and use them for the numerical implementation.

Exercise 2 Maximization of compliance

In this exercise we consider the maximization of the compliance treated in the course

$$\min_{(\theta, A^*) \in U_{ad}^L} \left\{ J(\theta, A^*) = -\int_\Omega u(x) dx \right\}$$

where u is the solution of

$$\begin{cases} -\operatorname{div}(A^* \nabla u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

1. Prove that the problem is self-adjoint and the adjoint state is $p = u$.
2. Show that $\nabla_{A^*} J(\theta, A^*) = \nabla u \otimes \nabla u$. Following the results shown on slide 54 show that any optimal A^* satisfies $A^* \nabla u = \lambda_\theta^- \nabla u$.
3. Show that the opposite of the compliance can be written as a minimum thanks to the primal energy:

$$-\int_\Omega u dx = -\int_\Omega \lambda_\theta^- |\nabla u|^2 dx = \min_{v \in H_0^1(\Omega)} \left(\int_\Omega \lambda_\theta^- |\nabla v|^2 dx - 2 \int_\Omega v dx \right)$$

Exercise 3 The SIMP method

The Solid Isotropic Material with Penalization (SIMP) method consists of convexifying the admissible set by allowing intermediate material properties. Hooke's law becomes $\theta(x)A$ with $\theta(x) \in [0,1]$, which gives rise to **fictitious materials**.

In this method simply replace the homogenized tensor A^* by $\theta^p A$ for some power $p \geq 1$ (typically $p = 3$). Note that the material properties are taken into account with weight θ^p, while the volume is computed using the density θ. We do not start with $p = 3$ from the beginning. In practice you can run 10 iterations with $p = 1$, another 10 iterations with $p = 2$ and the rest with $p = 3$. Again, the objective function may increase when increasing p, so these iterations should be **accepted** regardless of the potential increase in the objective function.

1. Consider T which solves the homogenized equation

$$\begin{cases} -\operatorname{div}(A^*\nabla T) &= 0 & \text{in } \Omega, \\ A^*\nabla T \cdot n &= 1 & \text{on } \Gamma_N, \\ A^*\nabla T \cdot n &= 0 & \text{on } \Gamma, \\ T &= 0 & \text{on } \Gamma_D. \end{cases}$$

   We wish to minimize the functional $J(\theta, A^*) = \int_{\Gamma_N} T^2 ds$. Replace A^* by $\alpha(1-\theta^p) + \beta\theta^p$ where α and β are the material parameters and θ is the density.

2. Observe the `FreeFem++` implementation given in the file `simp_heat.edp` found on Moodle and justify the formulas found therein: notably the **choice of the gradient**. In practice the parameters are as follows: $\alpha = 1, \beta = 10$ and the density θ verifies $\int_\Omega \theta = 0.3|\Omega|$.

3. Adapt these ideas to the case of linearized elasticity (see the code `simp_elas.edp` for an example):

$$\begin{cases} -\operatorname{div}(\sigma) &= 0 & \text{in } \Omega, \\ \sigma n &= g & \text{on } \Gamma_N, \\ \sigma n &= 0 & \text{on } \Gamma, \\ u &= 0 & \text{on } \Gamma_D. \end{cases}$$

   where $\sigma = \theta^p A_1 e(u) + (1-\theta^p)A_2 e(u)$, $e(u) = 0.5(\nabla u + \nabla^T u)$. The material A_1 is isotropic defined by its Lamé coefficients and $A_2 = 10^{-3}A_1$.
   The functional to be minimized is $J(\theta) = \int_{\Gamma_N} g_0 \cdot u ds$ where g_0 is a force which is not collinear with g.