MAP562 Optimal design of structures (École Polytechnique)
by Grégoire Allaire, Beniamin Bogosel

Session 5: Feb 6th, 2019 – Parametric Optimization
Exercise Sheet

Exercise 1 Optimize the compliance for the linearized elasticity

In this exercise we optimize the thickness h of a two-dimensional elastic body in Ω. The boundary is partitioned into multiple parts named Γ_D, Γ_N and $\Gamma = \partial\Omega \setminus (\Gamma_D \cup \Gamma_N)$. We consider the following equation governing an elastic body

$$\begin{cases} -\operatorname{div}(\sigma(u)) &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_D, \\ \sigma(u)n &= g & \text{on } \Gamma_N, \\ \sigma(u)n &= 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where the stress is given by the classical law:

$$\sigma(u) = h(2\mu e(u) + \lambda tr(e(u))I)$$

with the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, I is the identity matrix, and μ and λ are the Lamé coefficients. We write below the equations satisfied by the symmetric stress tensor $\sigma \in \mathcal{M}_s$

$$\begin{cases} -\operatorname{div}\tau &= f & \text{in } \Omega \\ \tau n &= g & \text{on } \Gamma_N \\ \tau n &= 0 & \text{on } \Gamma \end{cases} \tag{2}$$

where we denote by $\mathcal{M}_s$ the space of $N \times N$ matrices with real entries which are symmetric.

Preliminary question. Recall what is the variational formulation for problem (2) (you can think of the connection with the linearized elasticity system).

We want to minimize the compliance defined by

$$J(h) = \int_\Omega f \cdot u(h)\, dx + \int_{\Gamma_N} g \cdot u(h)\, ds \tag{3}$$

where h belongs to the set of admissible designs $\mathcal{U}_{ad}$ defined by

$$\mathcal{U}_{ad} = \{h \in L^\infty(\Omega) : 0.1 \leq h \leq 1, \ \text{ and } \int_\Omega h = 0.5|\Omega|\}.$$

1. Write the Lagrangian and check that in the case of the compliance minimization the adjoint is related to the state function. Deduce what is the advantage of working with the compliance from a numerical point of view.
2. Hooke's law governing an elastic material is often written as a linear transformation (tensor) A given by
$$\tau = Ae(u) = 2\mu e(u) + \lambda tr(e(u))I.$$
Prove that the inverse transformation is
$$e(u) = A^{-1}\tau = \frac{1}{2\mu}\tau - \frac{\lambda}{2\mu(2\mu + N\lambda)} tr(\tau)I.$$
Recall that classical conditions regarding the Lamé parameters include $\mu > 0$ and $2\mu + N\lambda > 0$.
3. **(Optional)** Prove that the mapping $\mathcal{U}_{ad} \times L^2(\Omega, \mathcal{M}_s) \ni (h, \tau) \mapsto \int_\Omega h^{-1}A^{-1}\tau \cdot \tau dx$ is strictly convex.

4. Let $h \in \mathcal{U}_{ad}$ be given. Prove that the compliance can be computed in a different way:

$$J(h) = \min_{\substack{\tau \in L^2(\Omega, \mathcal{M}_s) \\ \tau \text{ verifies (2)}}} \int_\Omega h^{-1} A^{-1} \tau \cdot \tau dx.$$

(**Hint:** One inequality should be immediate. For the other, write the Lagrangian associated to the minimization problem and use the associated optimality conditions)

⋆ Remark that the boundary condition for the dual problem (2) are the Neumann boundary condition for the state equation (1).

⋆ Note that using this equivalent computation of the compliance, the problem $\inf_{h \in \mathcal{U}_{ad}} J(h)$ can be written in the following form:

$$\inf_{h \in \mathcal{U}_{ad}} \min_{\substack{\tau \in L^2(\Omega, \mathcal{M}_s)^{N \times N} \\ \tau \text{ verifies (2)}}} \int_\Omega h^{-1} A^{-1} \tau \cdot \tau dx$$

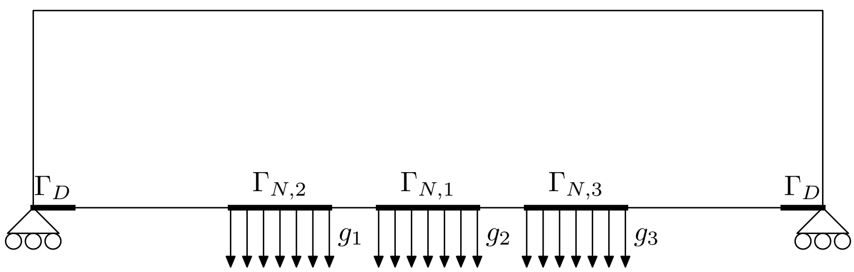
This new problem consists of minimizing a convex function and since we are dealing with a double minimization, the order of the minimization can be inverted. It turns out that if we consider the minimization in h first, then it is possible to find h explicitly as a function of τ.

5. Consider the minimization problem

$$\min_{h \in \mathcal{U}_{ad}} \int_\Omega h^{-1} A^{-1} \tau \cdot \tau dx$$

for $\tau \in L^2(\Omega, \mathcal{M}_s)$ fixed. Write the associated Lagrangian and find the optimality condition satisfied by h. Deduce the expression of h in terms of τ and the Lagrange multiplier ℓ for the volume constraint.

6. Implement in `FreeFem++` the optimality criterion algorithm presented in the course for the load case of the bridge presented in the Figure below, where $g = g_i$ on $\Gamma_{N,i}$ and $\Gamma_N = \cup_{i=1}^3 \Gamma_{N,i}$.



Exercise 2 Optimality criterion - multiple loads case

In this exercise we optimize the thickness h of a two-dimensional elastic body in Ω subject to multiple loads. The Dirichlet part Γ_D will be the same for each load case. However, we have multiple subsets of $\partial\Omega$ denoted $\Gamma_{N,i}$ where the loads g_i are applied. Each loading case $(\Gamma_{N,i}, g_i)$, $(i = 1, ..., n)$ gives a displacement which is the solution of the equation

$$\left\{ \begin{array}{rcll} -\operatorname{div}(\sigma(u_i)) & = & 0 & \text{in } \Omega, \\ u_i & = & 0 & \text{on } \Gamma_D, \\ \sigma(u_i)n & = & g_i & \text{on } \Gamma_{N,i}, \\ \sigma(u_i)n & = & 0 & \text{on the rest of } \partial\Omega, \end{array} \right. \tag{Elas,i}$$

where the stress is given by the classical law

$$\sigma(u_i) = h(2\mu e(u_i) + \lambda tr(e(u_i))I) \quad (= \sigma_i).$$

As usual μ and λ are the Lamé coefficients.

The equation satisfied by the symmetric tensor $\sigma_i = \sigma(u_i)$ is

$$\left\{ \begin{array}{rcll} -\operatorname{div} \sigma_i & = & 0 & \text{in } \Omega \\ \sigma_i n & = & g_i & \text{on } \Gamma_{N,i} \\ \sigma_i n & = & 0 & \text{on } \Gamma = \partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \end{array} \right. \tag{Tensor,i}$$

The objective to be minimized is the total compliance for all the load cases given by

$$J(h) = \sum_{i=1}^{n} \int_{\Gamma_{N,i}} g_i \cdot u_i \, ds \tag{4}$$

1. Let $h \in \mathcal{U}_{ad}$ be given. Prove that the compliance can be computed in a different way:

$$J(h) = \min_{\substack{\tau_i \in L^2(\Omega, \mathcal{M}_s) \\ \tau_i \text{ verifies (Tensor,i)}}} \sum_{i=1}^{n} \int_{\Omega} h^{-1} A^{-1} \tau_i \cdot \tau_i dx.$$

2. Write the associated optimality condition when we minimize

$$\mathcal{U}_{ad} \ni h \mapsto \sum_{i=1}^{n} \int_{\Omega} h^{-1} A^{-1} \tau_i \cdot \tau_i dx$$

   for τ_i fixed.
3. (**Homework**) Implement an algorithm in `FreeFem++` which solves the elasticity problem in the multi-load case. You can use again the configuration of the bridge given above or one of the configurations given below.
4. Compare the results obtained in the single-load case and the multi load case.