OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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CHAPTER VII (the end)

TOPOLOGY OPTIMIZATION

BY THE HOMOGENIZATION METHOD

7.5 Shape optimization in the elasticity setting

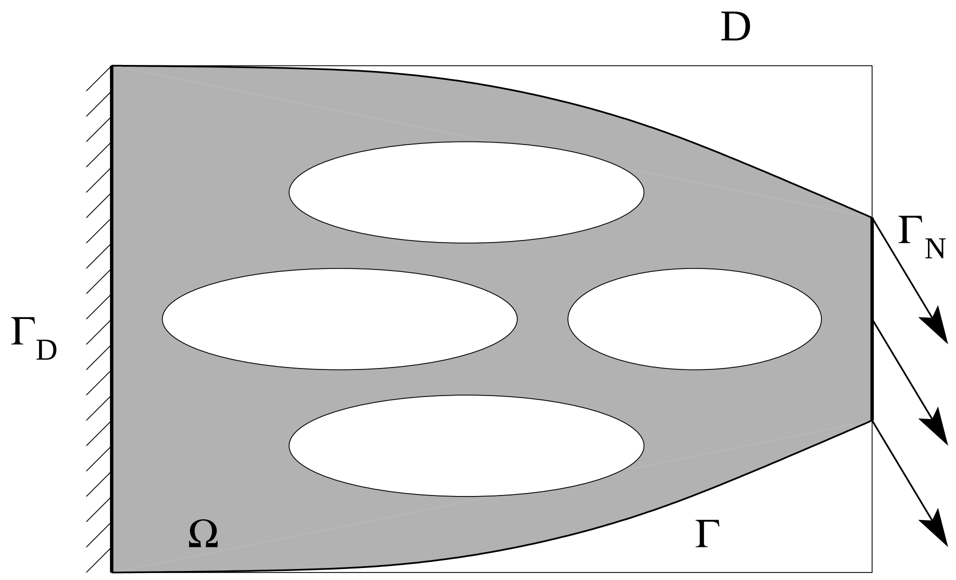
Very similar to the conductivity setting but there are some additional hurdles.

We shall review the results without proofs.

The basic ingredients of the homogenization method are the sames:

- ☞ introduction of composite designs characterized by (θ, A^*),
- ☞ Hashin-Shtrikman bounds for composites,
- ☞ sequential laminates are optimal microstructures.

Model problem: compliance minimization



Bounded working domain $D \in \mathbb{R}^N$ $(N = 2, 3)$.

Linear isotropic elastic material, with Hooke's law A

$$A = (\kappa - \frac{2\mu}{N}) I_2 \otimes I_2 + 2\mu I_4, \quad 0 < \kappa, \mu < +\infty$$

Admissible shape = subset $\Omega \subset D$.

Boundary $\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$ with Γ_N and Γ_D fixed.

$$\begin{cases} -\,\mathrm{div}\sigma = 0 & \text{in } \Omega \\ \sigma = 2\mu e(u) + \lambda\,\mathrm{tr}(e(u))\,\mathrm{Id} & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \Gamma, \end{cases}$$

Weight is minimized and rigidity is maximized. Let $\ell > 0$ be a Lagrange multiplier, the objective function is

$$\inf_{\Omega\subset D} \left\{ J(\Omega) = \int_{\Gamma_N} g \cdot u\, ds + \ell \int_{\Omega} dx \right\}.$$

This shape optimization problem can be approximated by a two-phase optimization problem: the original material A and the holes of rigidity $B \approx 0$.

The Hooke's law of the mixture in D is

$$\chi_\Omega(x)A + \big(1 - \chi_\Omega(x)\big)B \approx \chi_\Omega(x)A$$

The admissible set is

$$\mathcal{U}_{ad} = \Big\{\chi \in L^\infty\left(D; \{0,1\}\right)\Big\}.$$

As in the conductivity/membrane case, one can apply the relaxation approach based on homogenization theory.

The homogenization method can be generalized to the elasticity setting.

Homogenized formulation of shape optimization

We introduce composite materials characterized by a local volume fraction $\theta(x)$ of the phase A (taking any values in the range $[0,1]$) and an homogenized tensor $A^*(x)$, corresponding to its microstructure.

The set of admissible homogenized designs is

$$\mathcal{U}^*_{ad} = \left\{ (\theta, A^*) \in L^\infty \left(D; [0,1] \times \mathbb{R}^{N^4} \right), A^*(x) \in G_{\theta(x)} \text{ in } D \right\}.$$

The homogenized state equation is

$$\begin{cases} \sigma = A^* e(u) & \text{with } e(u) = \frac{1}{2} \left(\nabla u + (\nabla u)^t \right), \\ \mathrm{div}\sigma = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N). \end{cases}$$

The homogenized compliance is defined by

$$c(\theta, A^*) = \int_{\Gamma_N} g \cdot u \, ds.$$

The relaxed or homogenized optimization problem is

$$\min_{(\theta, A^*) \in \mathcal{U}^*_{ad}} \left\{ J(\theta, A^*) = c(\theta, A^*) + \ell \int_D \theta(x) \, dx \right\}.$$

Major inconvenient: in the elasticity setting an explicit characterization of G_θ is still lacking !

Fortunately, for compliance one can replace G_θ by its explicit subset of laminated composites.

The key argument to avoid the knowledge of G_θ is that, thanks to the complementary energy minimization, compliance can be rewritten as

$$c(\theta, A^*) = \int_{\Gamma_N} g \cdot u \, ds = \min_{\substack{div\sigma=0 \text{ in } D \\ \sigma n = g \text{ on } \Gamma_N \\ \sigma n = 0 \text{ on } \partial D \setminus \Gamma_N \cup \Gamma_D}} \int_D A^{*-1} \sigma \cdot \sigma \, dx.$$

The shape optimization problem thus becomes a double minimization (we already used this argument in chapter 5).

Exchanging the order of minimizations

The shape optimization problem is

$$\min_{(\theta,A^\star)\in\mathcal{U}^\star_{ad}} \left\{ \min_{\substack{div\sigma=0 \text{ in } D\\ \sigma n=g \text{ on } \Gamma_N\\ \sigma n=0 \text{ on } \partial D\setminus\Gamma_N\cup\Gamma_D}} \int_D A^{*-1}\sigma\cdot\sigma\,dx + \ell\int_D \theta(x)\,dx \right\}.$$

Since the order of minimization is irrelevant, and the minimization with respect to the design parameters (θ, A^*) is local, it can be rewritten

$$\min_{\substack{div\sigma=0 \text{ in } D\\ \sigma n=g \text{ on } \Gamma_N\\ \sigma n=0 \text{ on } \partial D\setminus\Gamma_N\cup\Gamma_D}} \int_D \min_{\substack{0\leq\theta\leq 1\\ A^*\in G_\theta}} \Big(A^{*-1}\sigma\cdot\sigma+\ell\theta\Big)(x)\,dx.$$

For a given stress tensor σ, the minimization of complementary energy

$$\min_{A^* \in G_\theta} A^{*-1}\sigma \cdot \sigma$$

is a classical problem in homogenization, of finding optimal bounds on the effective properties of composite materials.

It turns out that we can restrict ourselves to sequential laminates which form an explicit subset L_θ of G_θ.

Such a simplification is made possible because compliance is the objective function.

7.5.2 Sequential laminates in elasticity

$$A\xi = 2\mu_A \xi + \lambda_A (tr\xi) I, \quad B\xi = 2\mu_B \xi + \lambda_B (tr\xi) I,$$

with the identity matrix I_2, and $\kappa_{A,B} = \lambda_{A,B} + 2\mu_{A,B}/N$. We assume B to be weaker than A

$$0 \leq \mu_B < \mu_A, \quad 0 \leq \kappa_B < \kappa_A.$$

We work with stresses rather than strains, thus we use inverse elasticity tensors.

Lemma 7.24. The Hooke's law of a simple laminate of A and B in proportions θ and $(1-\theta)$, respectively, in the direction e, is

$$(1-\theta)\left(A^{*-1} - A^{-1}\right)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} + \theta f_A^c(e)$$

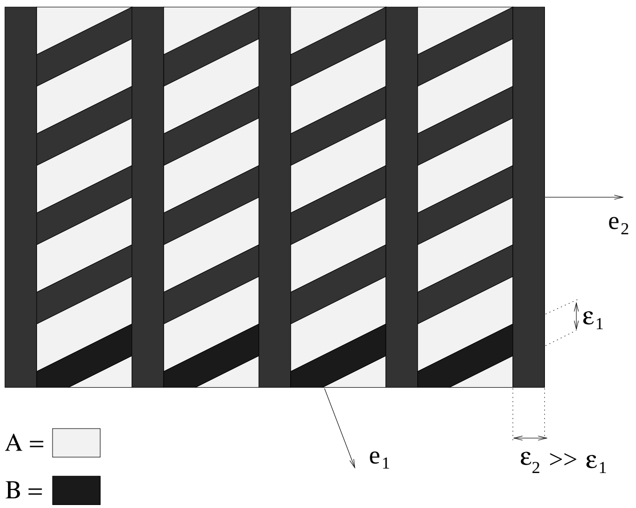
with $f_A^c(e)$ the tensor defined, for any symmetric matrix ξ, by

$$f_A^c(e_i)\xi \cdot \xi = A\xi \cdot \xi - \frac{1}{\mu_A}|A\xi e_i|^2 + \frac{\mu_A + \lambda_A}{\mu_A(2\mu_A + \lambda_A)}((A\xi)e_i \cdot e_i)^2.$$

Reiterated lamination formula

Proposition 7.25. A rank-p sequential laminate with matrix A and inclusions B, in proportions θ and $(1-\theta)$, respectively, in the directions $(e_i)_{1\leq i\leq p}$ with parameters $(m_i)_{1\leq i\leq p}$ such that $0 \leq m_i \leq 1$ and $\sum_{i=1}^p m_i = 1$, is given by

$$(1-\theta)\left({A^*}^{-1} - A^{-1}\right)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} + \theta \sum_{i=1}^{p} m_i f_A^c(e_i)$$



Proposition 7.27. Let G_θ be the set of all homogenized elasticity tensors obtained by mixing the two phases A and B in proportions θ and $(1-\theta)$. Let L_θ be the subset of G_θ made of sequential laminated composites. For any stress σ,

$$HS(\sigma) = \min_{A^* \in G_\theta} {A^*}^{-1}\sigma \cdot \sigma = \min_{A^* \in L_\theta} {A^*}^{-1}\sigma \cdot \sigma.$$

Furthermore, the minimum is attained by a rank-N sequential laminate with lamination directions given by the eigendirections of σ.

Remark.

- An optimal tensor A^* can be interpreted as the most rigid composite material in G_θ able to sustain the stress σ.
- $HS(\sigma)$ is called Hashin-Shtrikman optimal energy bound.
- In the conductivity setting, a rank-1 laminate was enough...
- Practical conclusion: G_θ can be replaced by L_θ.

7.5.4 Homogenized formulation of shape optimization

$$\min_{\substack{div\sigma=0 \text{ in } D \\ \sigma n=g \text{ on } \Gamma_N \\ \sigma n=0 \text{ on } \partial D\setminus\Gamma_N\cup\Gamma_D}} \int_D \min_{\substack{0\leq\theta\leq 1 \\ A^*\in G_\theta}} \left(A^{*-1}\sigma\cdot\sigma+\ell\theta\right)dx.$$

Optimality condition. If (θ, A^*, σ) is a minimizer, then A^* is a rank-N sequential laminate aligned with σ and with explicit proportions

$$A^{*-1} = A^{-1} + \frac{1-\theta}{\theta}\left(\sum_{i=1}^{N} m_i f_A^c(e_i)\right)^{-1},$$

and θ is given in 2-D (similar formula in 3-D)

$$\theta_{opt} = \min\left(1, \sqrt{\frac{\kappa+\mu}{4\mu\kappa\ell}}\left(|\sigma_1|+|\sigma_2|\right)\right),$$

where σ is the solution of the homogenized equation.

7.5.5 Numerical algorithm

Double "alternating" minimization in σ and in (θ, A^*).

- intialization of the shape (θ_0, A_0^*)
- iterations $n \geq 1$ until convergence
  - given a shape $(\theta_{n-1}, A_{n-1}^*)$, we compute the stress σ_n by solving a linear elasticity problem (by a finite element method)
  - given a stress field σ_n, we update the new design parameters (θ_n, A_n^*) with the explicit optimality formula in terms of σ_n.

Remarks.

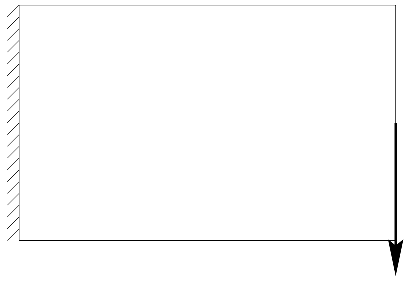
☞ For compliance, the problem is self-adjoint.

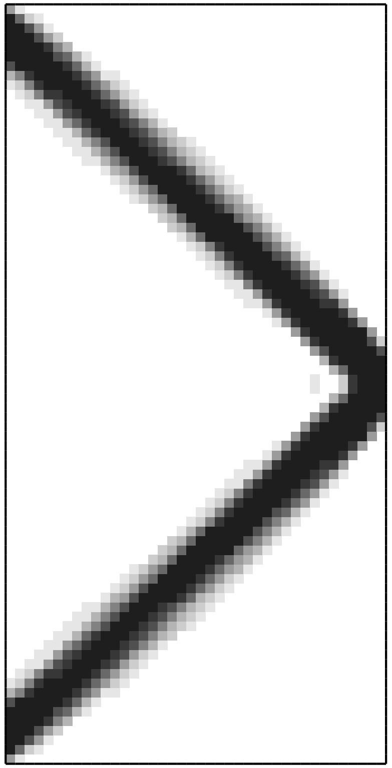
☞ Micro-macro method (local microstructure / global density).

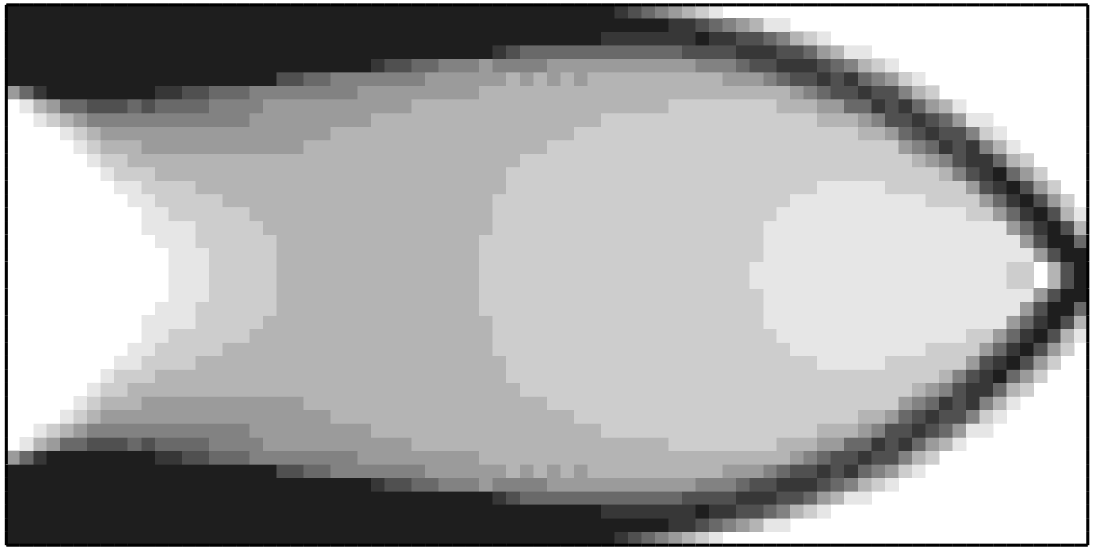
Remarks

- ☞ The objective function always decreases.
- ☞ Algorithm of the type "optimality criteria".
- ☞ Algorithme of "shape capturing" on a fixed mesh of Ω.
- ☞ We replace void by a weak "ersatz" material, or we impose $\theta \geq 10^{-3}$ to get an invertible rigidity matrix.
- ☞ A few tens of iterations are sufficient to converge.

Example: optimal cantilever







Penalization

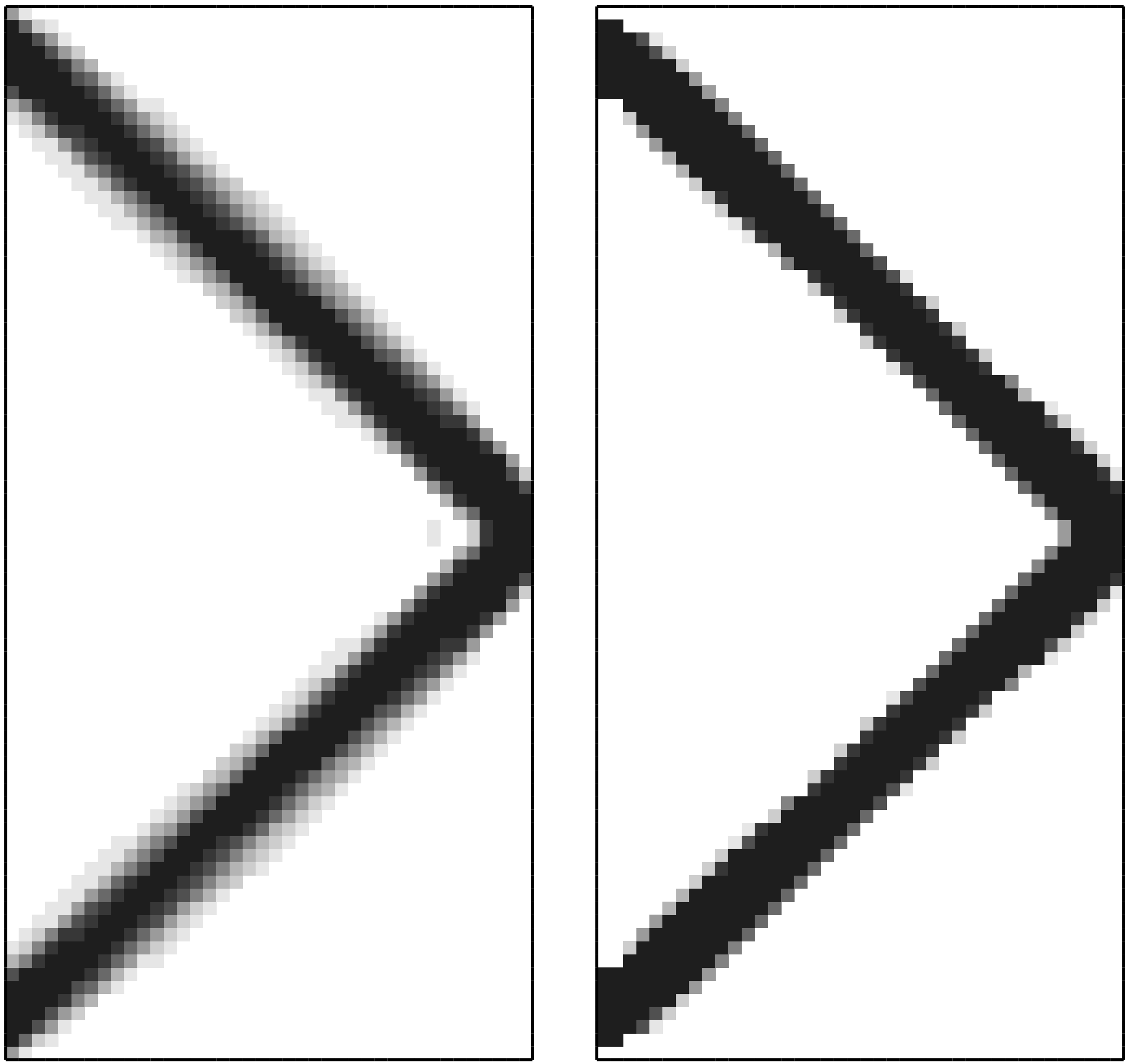
The previous algorithm compute **composite** shapes instead of **classical** shapes.

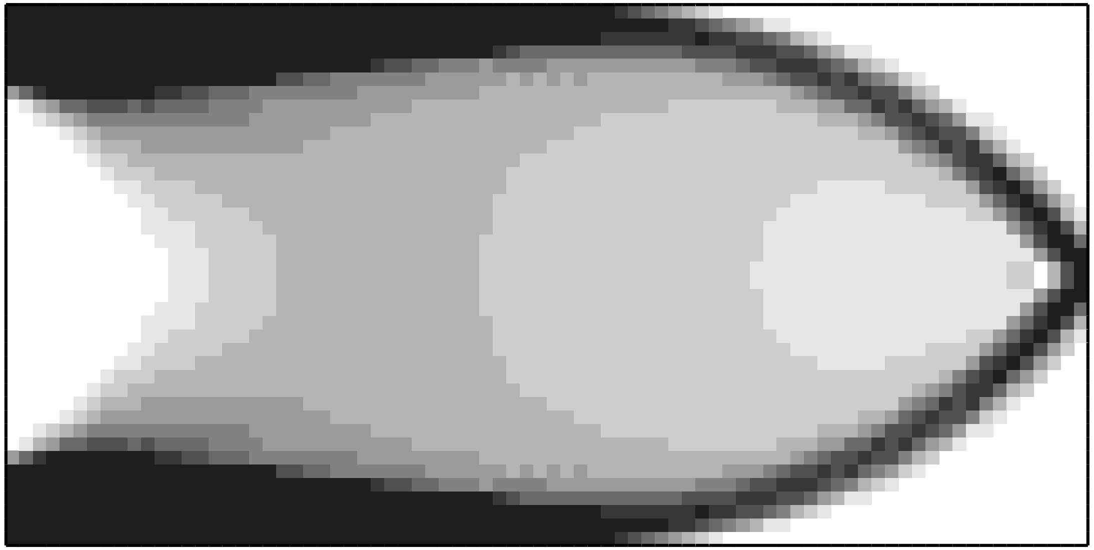
We thus use a penalization technique to force the density in taking values close to 0 or 1.

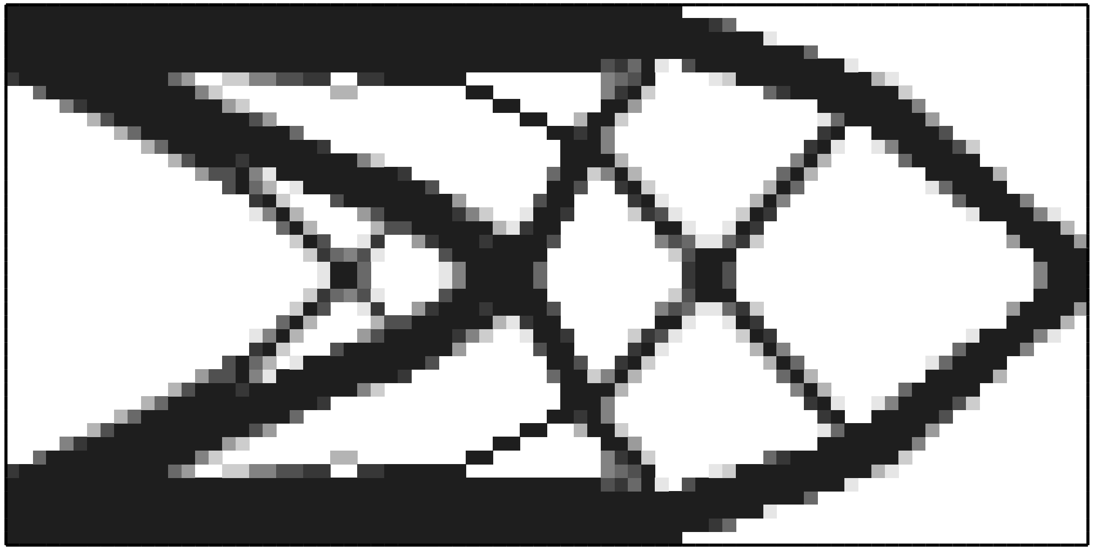
Algorithm: after convergence to a composite shape, we perform a few more iterations with a penalized density

$$\theta_{pen} = \frac{1 - \cos(\pi\theta_{opt})}{2}.$$

If $0 < \theta_{opt} < 1/2$, then $\theta_{pen} < \theta_{opt}$, while, if $1/2 < \theta_{opt} < 1$, then $\theta_{pen} > \theta_{opt}$.

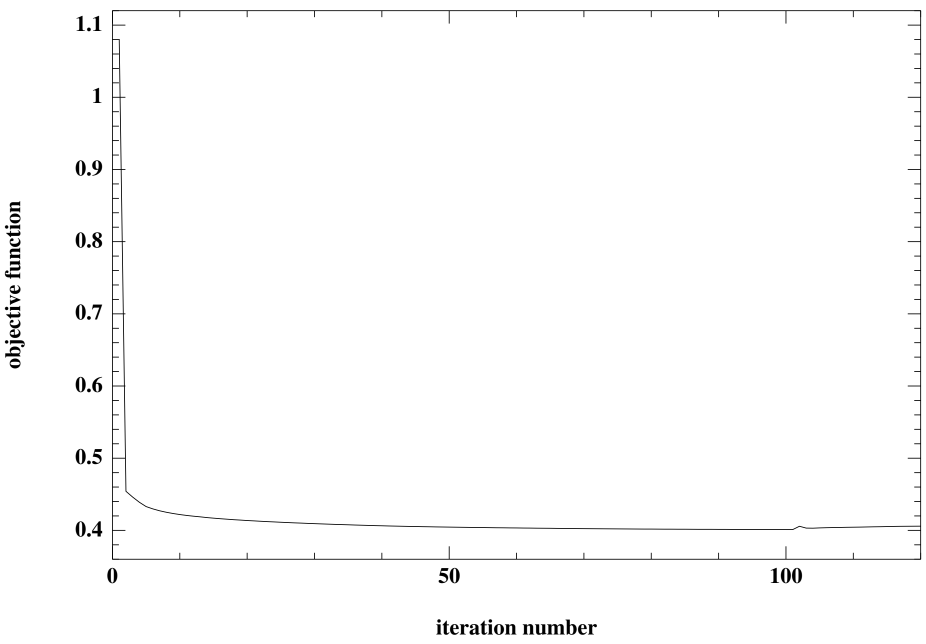


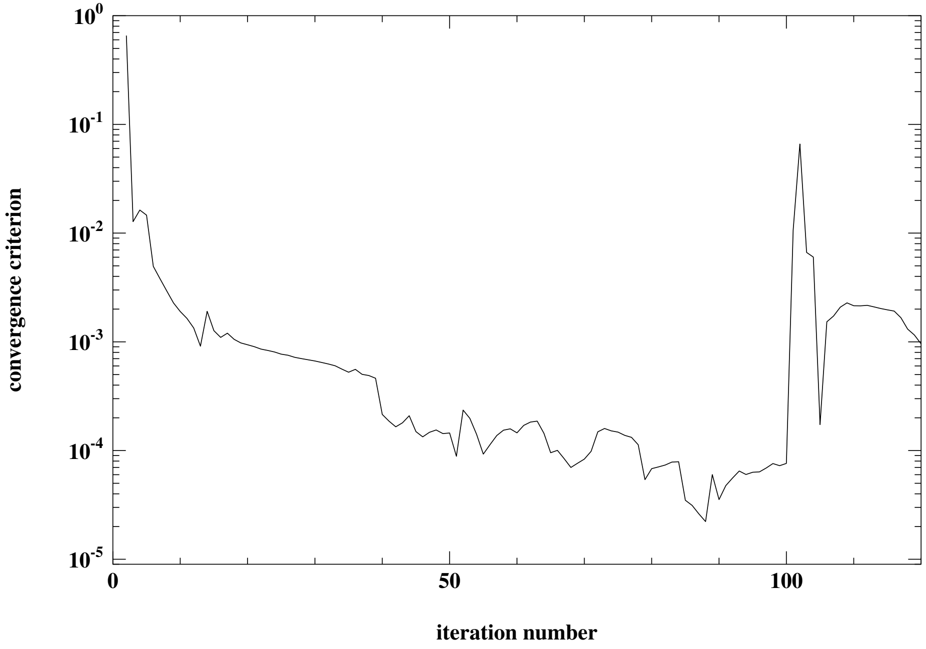




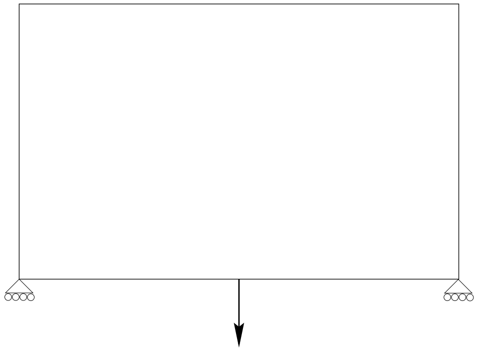
Convergence history:

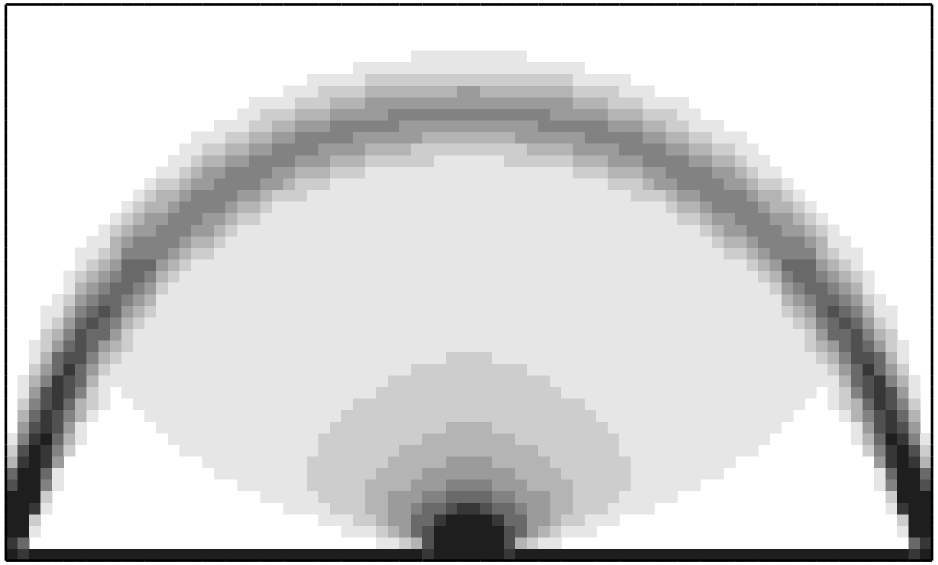
objective function (left), and residual (right),
in terms of the iteration number.

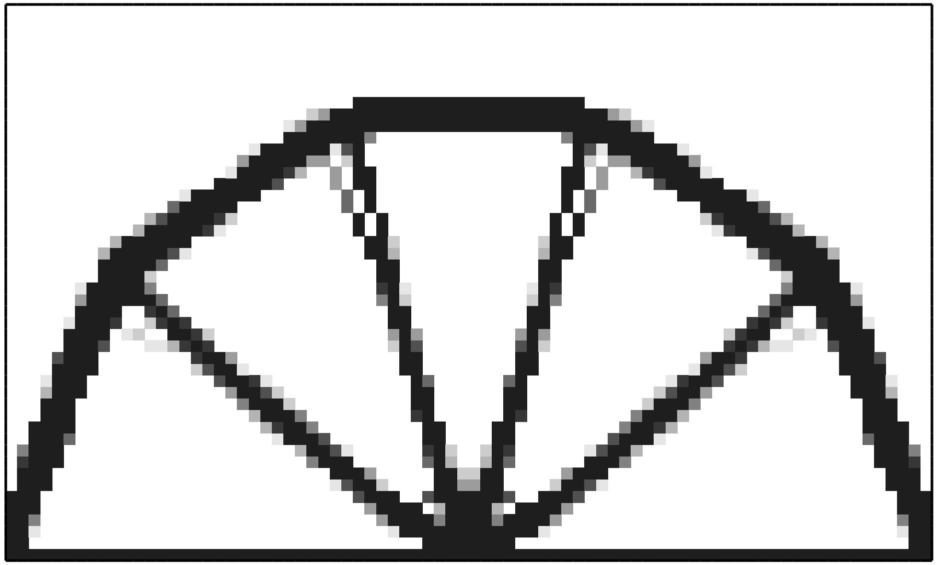




Example: optimal bridge







7.5.6. Convexification and “fictitious materials”

Idea. In the homogenization method composite materials are introduced but discarded at the end by penalization. Can we simplify the approach by introducing merely a density θ ?

A classical shape is parametrized by $\chi(x) \in \{0, 1\}$.

If we **convexify** this admissible set, we obtain $\theta(x) \in [0, 1]$.

The Hooke's law, which was $\chi(x)A$, becomes $\theta(x)A$. We also call this **fictitious materials** because one can not realize them by a true homogenization process (in general). Combined with a penalization scheme, this methode is called **SIMP** (Solid Isotropic Material with Penalization).

Convexified formulation with $0 \leq \theta(x) \leq 1$

$$\begin{cases} \sigma = \theta(x) A e(u) & \text{with } e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right), \\ \operatorname{div}\sigma = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_D \\ \sigma n = g & \text{on } \Gamma_N \\ \sigma n = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N). \end{cases}$$

Compliance minimization

$$\min_{0 \leq \theta(x) \leq 1} \left(c(\theta) + \ell \int_D \theta(x) \right).$$

with

$$c(\theta) = \int_{\Gamma_N} g \cdot u = \int_D (\theta(x) A)^{-1} \sigma \cdot \sigma = \min_{\substack{div\tau=0 \text{ in } D \\ \tau n = g \text{ on } \Gamma_N \\ \tau n = 0 \text{ on } \partial D \setminus \Gamma_N \cup \Gamma_D}} \int_D (\theta(x) A)^{-1} \tau \cdot \tau \, dx.$$

Now, there is **only one single** design parameter: the material density θ (the microstructure A^* has disappeared).

Existence of solutions

Theorem 7.33. The convexified formulation

$$\min_{0\leq\theta(x)\leq 1} \quad \min_{\substack{div\tau=0 \text{ in } D\\ \tau n=g \text{ on } \Gamma_N\\ \tau n=0 \text{ on } \partial D\setminus\Gamma_N\cup\Gamma_D}} \int_D (\theta(x)A)^{-1}\tau\cdot\tau\,dx + \ell\int_D \theta\,dx$$

admits at least one solution.

Proof. The function, defined on $\mathbb{R}^+ \times \mathcal{M}_n^s$,

$$\phi(a,\sigma) = a^{-1}A^{-1}\sigma\cdot\sigma,$$

is convex because

$$\phi(a,\sigma) = \phi(a_0,\sigma_0) + D\phi(a_0,\sigma_0)\cdot(a-a_0,\sigma-\sigma_0) + \phi(a,\sigma - aa_0^{-1}\sigma_0),$$

where the derivative $D\phi$ is given by

$$D\phi(a_0,\sigma_0)\cdot(b,\tau) = -\frac{b}{a_0^2}A^{-1}\sigma_0\cdot\sigma_0 + 2a_0^{-1}A^{-1}\sigma_0\cdot\tau.$$

Optimality condition

If we exchange the minimizations in τ and in θ, we can compute the optimal θ which is

$$\theta(x) = \begin{cases} 1 & \text{if } A^{-1}\tau \cdot \tau \geq \ell \\ \sqrt{\ell^{-1} A^{-1}\tau \cdot \tau} & \text{if } A^{-1}\tau \cdot \tau \leq \ell \end{cases}$$

Again we can use an "alternating" double minimization algorithm.

Numerical algorithm

- intialization of the shape θ_0
- iterations $k \geq 1$ until convergence
  - given a shape θ_{k-1}, we compute the stress σ_k by solving an elasticity problem (by a finite element method)
  - given a stress field σ_k, we update the new material density θ_k with the explicit optimality formula in terms of σ_k.

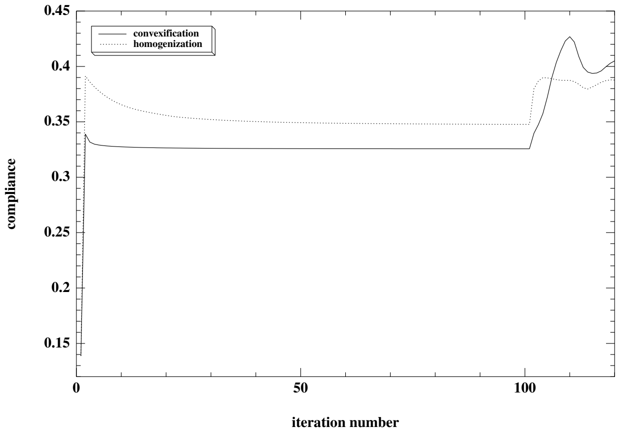
Penalization: we use a penalized density

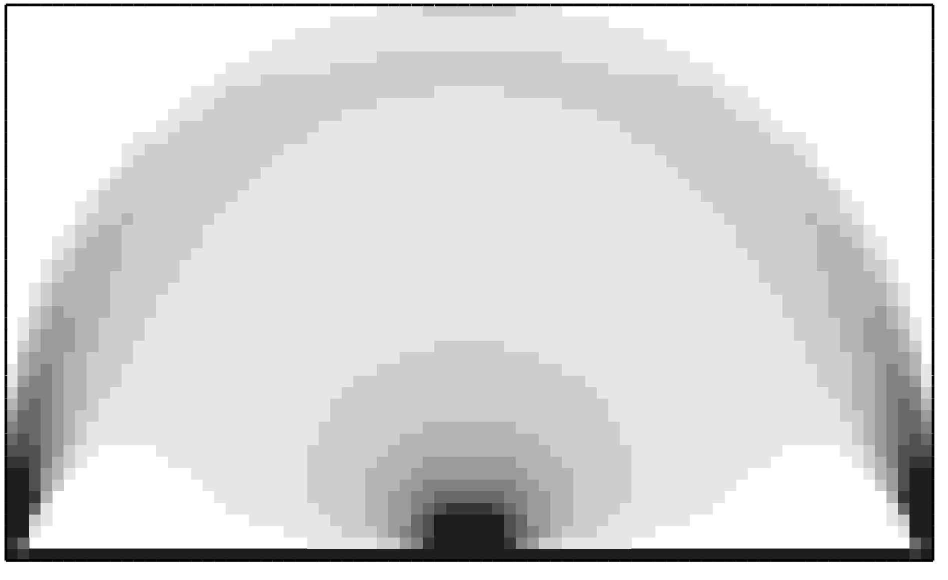
$$\theta_{pen} = \frac{1 - \cos(\pi \theta_{opt})}{2} \quad \text{or (SIMP)} \quad \theta_{pen} = \theta^p \quad p > 1.$$

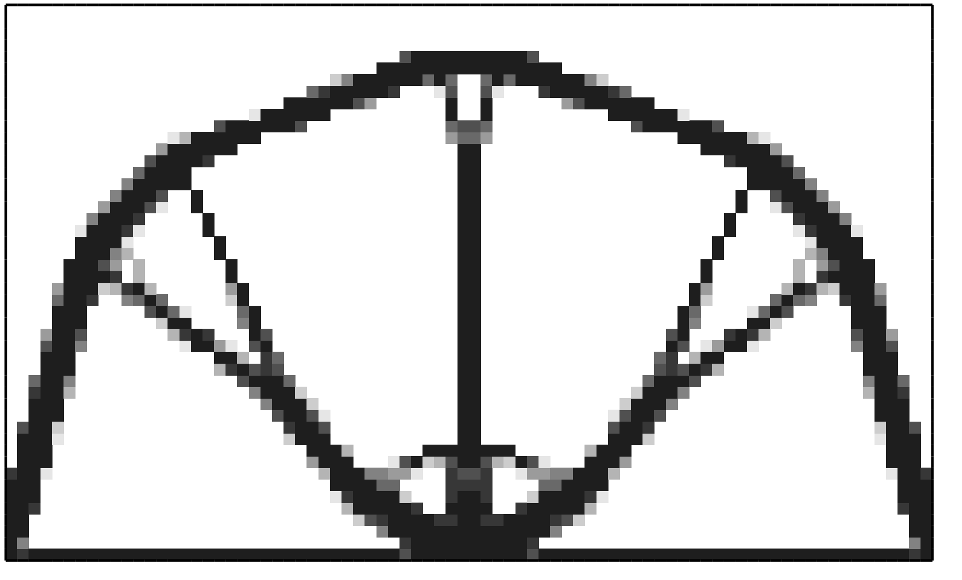
In practice: it is extremely simple ! But the numerical results are not as good ! An explanation is the lack of a relaxation theorem.

Be careful: very delicate monitoring of the penalization...

Optimal bridge by the convexification method







Conclusion

- ☞ SIMP (or convexification, or "fictitious materials") is very simple and **very popular** (many commercial codes are using it).
- ☞ SIMP uses very few informations on composites !
- ☞ On the contrary to the homogenization method, SIMP **is not a relaxation method**: it changes the problem !
- ☞ There is a gap between the true minimal value of the objective function and that of SIMP.
- ☞ SIMP can be delicate to monitor: how to increase the penalization parameter ?

Generalizations of the homogenization method

- ☞ multiple loads
- ☞ vibration eigenfrequency
- ☞ general criterion of the least square type

The two first cases are self-adjoint and we have a complete understanding and justification of the relaxation process. However, the third case is not self-adjoint and only a partial relaxation is known.

Other methods of topology optimization

- ☞ Discrete 0/1 optimization: genetic algorithms.
- ☞ Level set methods based on geometric optimization.
- ☞ Topological derivative: sensitivity to the nucleation of a small hole.
- ☞ Phase-field methods.