MAP562 Optimal design of structures

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Homework Sheet 5, Feb 6th, 2019

Instructions: Upload your solutions as separate `FreeFem++` files to the course Moodle by February 13th. Each group will choose one of the multi-load cases proposed.

Note that the theoretical part of the exercise was already treated in class, so start from that in your numerical investigations.

Exercise 2 Optimality criterion - multiple loads case

In this exercise we optimize the thickness h of a two-dimensional elastic body in Ω subject to multiple loads. The Dirichlet part Γ_D will be the same for each load case. However, we have multiple subsets of $\partial\Omega$ denoted $\Gamma_{N,i}$ where the loads g_i are applied. Each loading case $(\Gamma_{N,i}, g_i)$, $(i = 1, ..., n)$ gives a displacement which is the solution of the equation

$$\begin{cases} -\operatorname{div}(\sigma(u_i)) &= 0 \quad \text{in } \Omega, \\ u_i &= 0 \quad \text{on } \Gamma_D, \\ \sigma(u_i)n &= g_i \quad \text{on } \Gamma_{N,i}, \\ \sigma(u_i)n &= 0 \quad \text{on the rest of } \partial\Omega, \end{cases} \tag{Elas,i}$$

where the stress is given by the classical law

$$\sigma(u_i) = h(2\mu e(u_i) + \lambda tr(e(u_i))I) \quad (= \sigma_i).$$

As usual μ and λ are the Lamé coefficients.

The equation satisfied by the symmetric tensor $\sigma_i = \sigma(u_i)$ is

$$\begin{cases} -\operatorname{div}\sigma_i &= 0 \quad \text{in } \Omega \\ \sigma_i n &= g_i \quad \text{on } \Gamma_{N,i} \\ \sigma_i n &= 0 \quad \text{on } \Gamma = \partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \end{cases} \tag{Tensor,i}$$

The objective to be minimized is the total compliance for all the load cases given by

$$J(h) = \sum_{i=1}^{n} \int_{\Gamma_{N,i}} g_i \cdot u_i \, ds \tag{1}$$

1. Let $h \in \mathcal{U}_{ad}$ be given. Prove that the compliance can be computed in a different way:

$$J(h) = \min_{\substack{\tau_i \in L^2(\Omega, \mathcal{M}_s) \\ \tau_i \text{ verifies (Tensor,i)}}} \sum_{i=1}^{n} \int_\Omega h^{-1}A^{-1}\tau_i \cdot \tau_i dx.$$

2. Write the associated optimality condition when we minimize

$$\mathcal{U}_{ad} \ni h \mapsto \sum_{i=1}^{n} \int_\Omega h^{-1}A^{-1}\tau_i \cdot \tau_i dx$$

for τ_i fixed.

3. **(Homework)** Implement an algorithm in `FreeFem++` which solves the elasticity problem in the multi-load case. You can use again the configuration of the bridge given above or one of the configurations given below.
4. **(Homework)** Compare the results obtained in the single-load case and the multi load case. You may vary the magnitude of the forces in order to observe the behavior of the solution.