MAP562 Optimal design of structures

by Grégoire Allaire, Beniamin Bogosel

École Polytechnique

Homework Sheet 3, Jan 23rd, 2019

Instructions Choose ONE of the exercises below. The chosen exercise is due on January 30th, 2019. Upload your solutions as separate `FreeFem++` files, including detailed comments, to the course Moodle.

Exercise 1

We denote by (x_1, x_2) the coordinates of a point $x \in \mathbb{R}^2$ and by $B(c, r)$ the ball centered at $c \in \mathbb{R}^2$ of radius $r \in \mathbb{R}$.

1. For a **constant heat source** $v \in \mathbb{R}$, using a gradient method, implement in `FreeFem++` the minimization of the functional:

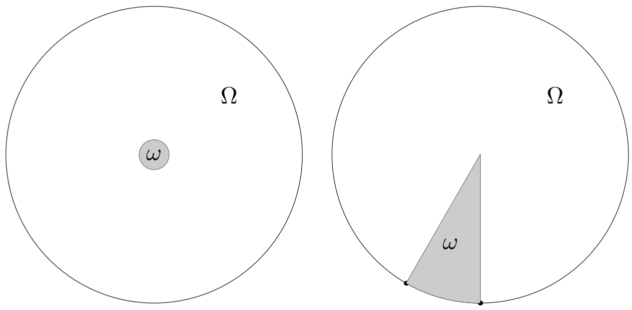
$$J(v) = \int_\Omega |T - T_0|^2 \, dx$$

where

$$\begin{cases} -\Delta T + u \cdot \nabla T &= 1_\omega v \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

in which the desired temperature is the constant function $T_0 = 10$.
The following geometries and velocity fields will be considered:

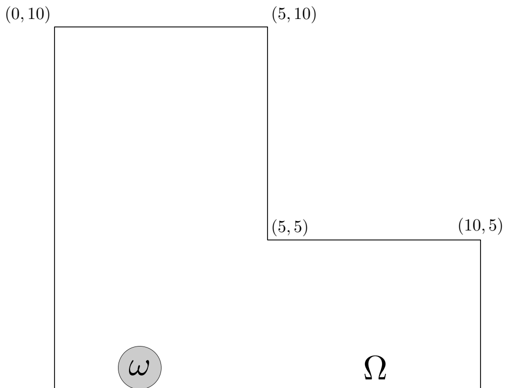
- $\Omega = B((0,0), 1)$, $u(x) = (-x_2, x_1)^\top$
  - $\omega = B((0,0), 0.1)$,
  - ω is a *slice* of Ω with angle $\pi/6$ (see the drawing below).



- Ω is the L shaped domain (see the drawing below), $u = \nabla\Phi$ where Φ is a solution of

$$\Delta\Phi = 0 \text{ in } \Omega, \text{ and } \frac{\partial\Phi}{\partial n} = \begin{cases} -1 & \text{if } x_1 = 0 \text{ and } 0.5 < x_2 < 1; \\ 1 & \text{if } x_1 = 10; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

  - $\omega = B((2,2), 1)$
  - vary the position of ω and observe the behavior of the solution.



2. Compare your result to the case without convection ($u = [0, 0]$): run the simulation and write a short comment on the influence of the velocity field.

Exercise 2 Optimization of the position of a heater

For a given smooth bounded domain $\Omega \subset \mathbb{R}^d$ and for any $z \in \mathbb{R}^d$, we investigate the minimization of the cost functional:

$$J(z) = \int_\Omega |T - T_0|^2 \, dx$$

where $x \mapsto T_0(x)$ is a given smooth temperature field and $T \equiv T(x, z)$ is the solution of the following boundary value problem for the x variable

$$\begin{aligned} -\Delta T &= f_z \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f_z(x) = f(x - z)$ and $f(x)$ is a smooth non-negative function with compact support. In the numerical applications, take $d = 2$, $T_0(x) \equiv 0.1$, $\Omega = (0, 10)^2$ and

$$f(x) = \begin{cases} 1 - |x|^2 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

In a second application use the L-beam geometry described above.

This problem corresponds to an exercise on Exercise Sheet 3. Solving the theoretical questions should help you with the numerical application.

1. Implement a gradient algorithm for this problem in `FreeFem++`.
2. Implement a gradient algorithm for this problem in `FreeFem++` with the additional constraint that f_z be supported in Ω.
3. **(Verification)** Implement an algorithm which computes $J(z)$ for every point z in the mesh and plot the result for one of the given geometries (as a $\mathbb{P}_1$ function, for example). (Since one PDE should be solved for each z, use a small mesh for this part: less than 1000 nodes). Once the result is plotted, verify that the result produced by the gradient algorithm coincides with a local minimum.
   Indication: The number of vertices in a mesh `Th` can be found with the command `Th.nv`. The coordinates of the node `i` of the mesh can be found with `Th(i).x` and `Th(i).y`. If `g` is a $\mathbb{P}_1$ function defined on `Th` then you can set the value of `g` at node `i` with the command `g[][i]=...`. A similar example can be found among the codes on Moodle.