Mean-Field Theory in the General Case

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Contents

1 Introductory Remarks

- I have put full calculations in appendices and tried present the calculations more succinctly in the main text (with some comments).

2 Master Equation

$$
\begin{aligned}
\frac{\partial P(y)}{\partial t} &= u\Big[P(y-1)\mathrm{I}_{y>0} - P(y)\Big] \\
&+ (1-u)[1-P(0)]^L\Big[P(y+1) - P(y)\mathrm{I}_{y>0}\Big] \\
&+ \frac{p}{4}\Big[P(y+2) - P(y)\mathrm{I}_{y>1}\Big] \\
&+ \frac{(1-p)}{4}\Big[P(y-2)\mathrm{I}_{y>1} - P(y)\Big] \,.
\end{aligned} \tag{1}
$$

The lines of the right-hand side of the equation represent:

1. membrane moves up (probability u)
2. membrane moves down (probability $1-u$)
3. particle moves forwards, interface grows up (probability p)
4. particle moves backwards, interface grows down (probability $1-p$)

The factor 1/4 comes from the TASEP maximal current $\rho(1-\rho)$ when the density is 1/2. The factor $[1-P(0)]^L$ describes the probability that all sites have $y > 0$. I_X is an indicator function, defined as:

$$\mathrm{I}_X = \begin{cases} 1\ , & X \text{ is true.} \\ 0\ , & X \text{ is false.} \end{cases} \tag{2}$$

3 Generating Function

Define the generating function

$$G(z) = \sum_{y=0}^{\infty} z^y P(y)\ . \tag{3}$$

In the steady state, $\partial P(y)/\partial t = 0$. Using this, the generating function (3) and the master equation (1) we find

$$G(z) = \frac{-\left[pP(1)z^2 + \{pP(1) + (b+p)P(0)\}z + pP(0)\right]}{[(1-p)z^3 + (a+1-p)z^2 - (b+p)z - p]}\ , \tag{4}$$

where

$$a = 4u\ , \quad b = 4(1-u)(1-P(0))^L\ , \tag{5}$$

and in the rough phase $b \to (1-u)$ as $L \to \infty$. (And in the smooth phase $b \to 0$ as $L \to \infty$?)

By setting $p = 1$ we find

$$G(z) = \frac{-\left[P(1)z^2 + \{P(1) + (b+1)P(0)\}z + P(0)\right]}{[az^2 - (b+1)z - 1]}\ , \tag{6}$$

as was found for the master equation for the $p = 1$ case studied previously.

3.1 Finding the Probability Distribution

We have the following expression for the generating function:

$$G(z) = \frac{-\left[pP(1)z^2 + (pP(1) + (b+p)P(0))z + pP(0)\right]}{[(1-p)z^3 + (a+1-p)z^2(b+p)z - p]}\ . \tag{7}$$

The cubic in the denominator makes it difficult to solve. We assign a function to the denominator:

$$h(z) = (1-p)z^3 + (a+1-p)z^2(b+p)z - p\ . \tag{8}$$

This cubic function $h(z)$ has three real roots[1] z_+, z_- and z_p, such that

$$z_+ > 0\ , \tag{9}$$

$$z_p < z_- < 0\ , \tag{10}$$

and

$$|z_p| > |z_+| > |z_-|\ . \tag{11}$$

[1]Do these roots become complex for certain parameter values?

z_+ and z_- correspond to the two roots of the same name of the quadratic in the $p = 1$ case. In the limit $p \to 1$ the root z_p must disappear as $h(z)$ become a cubic. We can see then that $z_p \sim -a/(1-p)$, such that the order $1/(1-p)^2$ terms in $h(z)$ cancel.

Importantly, because we still have $|z_-| < |z_+|$, the pole at z_- is still closer to the origin and dominates the integral of $G(z)$ which describes $P(y)$. Thus, as in the $p = 1$ case, we must cancel a factor $(z - z_-)$ from top and bottom. Conversely, the pole z_p is further from the origin that z_+, because $|z_p| > |z_+|$, and so this pole (with negative real part) does not dominate the same integral, so does not need to be cancelled.

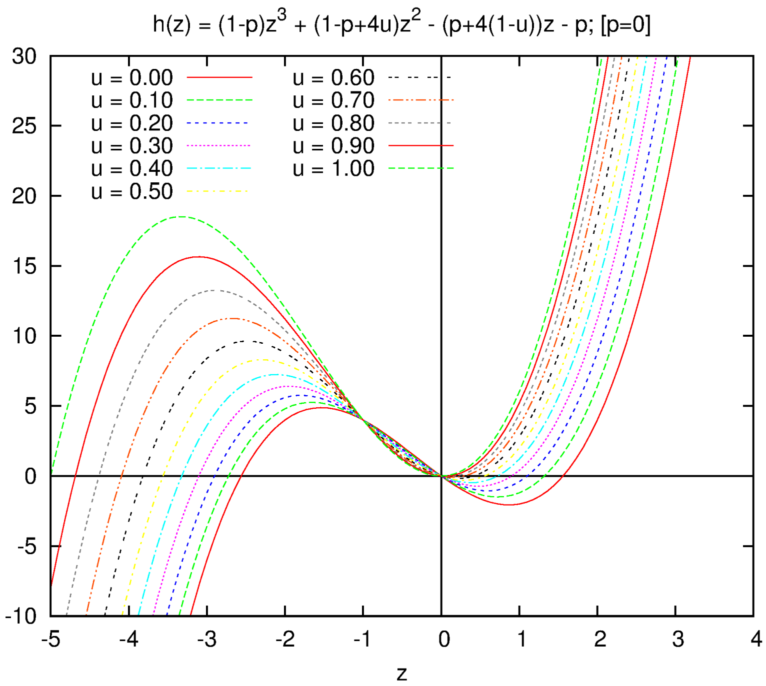

Figure 1: When $p = 0$, $|z_p| > |z_+|$ across the range $u = 0$ to 1. Also, $z_- = 0$ (or at least $z_- \simeq 0$).

So now we can write

$$h(z) = (1-p)(z - z_-)(z - z_+)(z - z_p) \,, \tag{12}$$

and the numerator of $G(z)$ can be written as

$$-\left[pP(0)z^2 + (pP(1) + (b+p)P(0))\, z + pP(0)\right] = -(Az + B)(z - z_-) \,. \tag{13}$$

Immediately from this we can write

$$A = pP(1) \,, \tag{14}$$

and

$$B = -\frac{pP(0)}{z_-} \,, \tag{15}$$

which will be useful later.

Now, coming back to the generating function, we can write $G(z)$ as

$$G(z) = -\frac{(Az + B)}{(1-p)(z - z_+)(z - z_p)} \,. \tag{16}$$

To find an expression for $P(y)$ we will try to rewrite $G(z)$ as a sum of powers of z. To begin, we factorise out $-z_+$, $-z_p$, to find

$$\begin{aligned} G(z) &= -\frac{(Az+B)}{(1-p)z_+z_p(1-z/z_+)(1-z/z_p)} \\ &= -\frac{(Az+B)}{(1-p)z_+z_p}\sum_{l=0}^{\infty}\left(\frac{z}{z_+}\right)^l\sum_{m=0}^{\infty}\left(\frac{z}{z_p}\right)^m \ . \end{aligned} \tag{17}$$

Using the substitution $n = l + m$ and by rearranging the sums we find

$$G(z) = -\frac{(Az+B)}{(1-p)z_+z_p}\sum_{n=0}^{\infty}\frac{z^n}{z_p^n}\sum_{l=0}^{n}\left(\frac{z_p}{z_p}\right)^l \ . \tag{18}$$

We then evaluate the geometric sum over l to find

$$G(z) = -\frac{(Az+B)}{(1-p)z_+z_p}\sum_{n=0}^{\infty}\frac{z^n}{z_p^n}\left[\frac{(z_p/z_+)^{n+1}-1}{(z_p/z_+)-1}\right] \ . \tag{19}$$

We want to find the coefficients of z^n to find the values of $P(n)$. To do this we first multiply through by $(Az + B)$:

$$G(z) = -\frac{1}{(1-p)z_+z_p}\sum_{n=0}^{\infty}\frac{Az^{n+1}+Bz^n}{z_p^n}\left[\frac{(z_p/z_+)^{n+1}-1}{(z_p/z_+)-1}\right] \ , \tag{20}$$

and then relabel the $n \to n - 1$ in the "A" sum:

$$\begin{aligned} G(z) = -\frac{1}{(1-p)z_+z_p} \quad & \left\{\sum_{n=1}^{\infty}\frac{Az^n}{z_p^{n-1}}\left[\frac{(z_p/z_+)^{n}-1}{(z_p/z_+)-1}\right]\right. \\ & \left. +\sum_{n=0}^{\infty}\frac{Bz^n}{z_p^n}\left[\frac{(z_p/z_+)^{n+1}-1}{(z_p/z_+)-1}\right]\right\} \ . \end{aligned} \tag{21}$$

Next, pull out the $n = 0$ term and combine the sums, to get

$$G(z) = -\frac{1}{(1-p)z_+z_p}Bz^0 - \sum_{n=1}^{\infty}\frac{z^n}{(1-p)(z_p-z_+)}\left(\frac{Az_++B}{z_+^{n+1}} - \frac{Az_p+B}{z_p^{n+1}}\right) \ . \tag{22}$$

From this we see that

$$P(0) = -\frac{B}{(1-p)z_+z_p} \ , \tag{23}$$

and, for $n > 0$,

$$P(n) = \frac{1}{(1-p)(z_p-z_+)}\left(\frac{Az_++B}{z_+^{n+1}} - \frac{Az_p+B}{z_p^{n+1}}\right) \ . \tag{24}$$

From the expression for $P(0)$ we have

$$B = -(1-p)z_+z_pP(0) \ , \tag{25}$$

and from earlier we have

$$A = pP(1) \ . \tag{26}$$

We can use these to calculate

$$Az_{+,p} + B \quad = \quad z_{+,p}(pP(1) - (1-p)z_{p,+}P(0)) \ . \tag{27}$$

Substituting back in to the expression for $P(n)$, $n > 0$ we find

$$P(n) \quad = \quad -\frac{1}{(1-p)(z_p - z_+)}\left(\frac{pP(1)(z_p^n - z_+^n) - (1-p)P(0)(z_p^{n+1} - z_+^{n+1})}{z_+^n z_p^n}\right) \ . \tag{28}$$

By setting $n = 1$ we can solve self-consistently for $P(1)$:

$$P(1) \quad = \quad -\frac{P(1)}{(1-p)z_p z_+} + \frac{P(0)(z_p + z_+)}{z_p z_+} \ , \tag{29}$$

and so by rearranging we find

$$P(1) = \frac{(1-p)(z_p + z_+)}{z_+ z_p(1-p) - p}P(0) \ . \tag{30}$$

Now we substitute the expression for $P(1)$ back in to the expression for $P(n)$ to find

$$P(n) = -\frac{P(0)}{z_+^n z_p^n(z_p - z_+)}\left(\frac{p(z_p^n - z_+^n)(z_p + z_+)}{z_+ z_p(1-p) + p} - (z_p^{n+1} - z_+^{n+1})\right) \ . \tag{31}$$

We can simplify a bit, using

$$p(z_p^n - z_+^n)(z_p + z_+) = pz_p^{n+1} - pz_+^{n+1} - pz_p z_+^n + pz_+ z_p^n \ , \tag{32}$$

and

$$-(z_p^{n+1} - z_+^{n+1})(z_+ z_p(1-p) + p) = -pz_p^{n+1} + pz_+^{n+1} - (1-p)z_+ z_p^{n+2} + (1-p)z_p z_+^{n+2} \ , \tag{33}$$

to find

$$P(n) = -\frac{P(0)}{z_+^n z_p^n(z_p - z_+)}\left(\frac{-pz_p z_+^n + pz_+ z_p^n - (1-p)z_+ z_p^{n+2} + (1-p)z_p z_+^{n+2}}{(z_+ z_p(1-p) + p)}\right) \ . \tag{34}$$

Finally, rearrange to find

$$P(n) = \frac{P(0)}{z_+^{n-1} z_p^{n-1}(z_p - z_+)}\left(\frac{(1-p)[z_p^{n+1} - z_+^{n+1}] - p[z_p^{n-1} - z_+^{n-1}]}{(z_+ z_p(1-p) + p)}\right) \ . \tag{35}$$

3.2 Recovering $p = 1$ equation

We now outline how to recover the $p = 1$ solution:

$$P(n) = \frac{P(0)}{1 + z_-}z_+^{-n} \ , \quad n > 0 \ . \tag{36}$$

To begin, we use the two expressions for B:

$$B = -(1-p)z_p z_+ P(0) \ , \quad B = -\frac{p}{z_-}P(0) \ , \tag{37}$$

to define

$$\alpha_p = (1-p)z_p = \frac{p}{z_- z_+} \ . \tag{38}$$

Importantly, α_p remains finite as $p \to 1$, because $z_p \sim (1-p)^{-1}$. For convenience, we define

$$\alpha_1 = \frac{1}{z_- z_+} \ . \tag{39}$$

Using this we rewrite (35) in terms of α_p and powers of z_p^{-1}:

$$P(n) = \frac{P(0)}{z_+^{n-1}} \left\{ \frac{\left(\alpha_p - (1-p) z_+^{n+1} z_p^{-n}\right) - p\left(z_p^{-1} - z_+^{n-1} z_p^{-(n-1)}\right)}{\left(z_+\alpha_p + p\right)\left(1 - z_+ z_p^{-1}\right)} \right\} \tag{40}$$

Next, we informally take the limit $p \to 1$ by setting all terms with powers of z_p^{-1} to zero to find

$$P(n) = \frac{P(0)}{z_+^{n-1}} \frac{\alpha_1}{(z_+\alpha_1 + 1)} \ . \tag{41}$$

Finally, we substitute in the expression for α_1 in terms of $z_\pm$ to find

$$P(n) = \frac{P(0)}{(1+z_-)} \frac{1}{z_+^n} \ , \tag{42}$$

as required.

3.3 Further Simplification

The root z_p is problematic because it diverges as $p \to 1$ as $(1-p)^{-1}$. Using the two definitions of B from (123) and (133) we can define

$$\alpha_p = (1-p)z_p = \frac{p}{z_+ z_-} \ . \tag{43}$$

Unlike z_p, as $p \to 1$, α_p remains finite. We can also define

$$x_p = \frac{z_+}{z_p} \ , \quad x_p < 1 \ , \quad |x_p| < 1 \ , \tag{44}$$

which has the useful property that $x_p \to 0$ as $p \to 1$. **(NOTE: I think I've made assumption about what z_+ does as $p \to 1$, which I haven't justified...)** The aim is to replace all instances of z_p with α_p or x_p. We can now write $P(n)$ in terms of these new variables:

$$P(n) = \frac{P(0)}{z_+^{n-1}} \left(\frac{\alpha_p^2[1 - x_p^{n+1}] - p(1-p)[1 - x_p^{n-1}]}{\alpha_p(1-x_p)(z_+\alpha_p + p)} \right) \ . \tag{45}$$

When n is large, the terms $x_p^n \to 0$, and so we see that

$$P(n) \simeq P(0) z_+ \left(\frac{\alpha_p^2 - p(1-p)}{\alpha_p(z_+\alpha_p + p)} \right) \frac{1}{z_+^n} \ . \tag{46}$$

This is useful because it shows that for large n, the transfer matrix has approximately the same structure as the $p = 1$ case, albeit with a different-but-related multiplying factor. The structure of the transfer matrix is the important part for getting the width exponent 1/3.

We can reintroduce z_p to (45) to find an expression for $P(n)$ as a sum of exponentials. We rearrange the x_p terms and multiply through the $z_+^{-(n-1)}$ term to find

$$P(n) = \frac{P(0)}{\alpha_p(1-x_p)(z_+\alpha_p+p)} \left(\frac{[\alpha_p^2 - p(1-p)]}{z_+^{n-1}} + \frac{[p(1-p) - \alpha_p^2 x_p^2]}{z_+^{n-1}} \right) . \tag{47}$$

4 Transfer Matrix

4.1 Definition

We want to use the solution for $P(n)$ above to give us equations for the statistical weights of heights n in the interface, and then from this build a transfer matrix which selects only interface configurations where the heights between adjacent neighbours differ by exactly 1.

We take (47) and rewrite with weights, as

$$w(n) = w(0)\theta_p \left(\frac{Q_p}{z_+^n} + \frac{R_p}{z_p^n} \right) , \quad n > 0 , \tag{48}$$

where

$$\theta_p = \frac{1}{\alpha_p(1-x_p)(z_+\alpha_p+p)} , \tag{49}$$

$$Q_p = z_+[\alpha_p^2 - p(1-p)] , \tag{50}$$

and

$$R_p = z_p[p(1-p) - \alpha_p^2 x_p^2] . \tag{51}$$

We also introduce the definitions

$$q = \frac{1}{z_+} , \quad q > 0 , \quad |q| < 1 , \tag{52}$$

and

$$r = \frac{1}{z_p} , \quad r < 0 , \quad |r| < |q| < 1 , \tag{53}$$

to write

$$w(n) = w(0)\theta_p Q_p \left(q^n + \frac{R_p}{Q_p} r^n \right) , \quad n > 0 . \tag{54}$$

using this we define the transfer matrix:

$$T = \begin{pmatrix} 0 & w(0) & 0 & 0 & 0 & \cdots \\ w(1) & 0 & w(1) & 0 & 0 & \\ 0 & w(2) & 0 & w(2) & 0 & \\ 0 & 0 & w(3) & 0 & w(3) & \\ \vdots & & & & & \ddots \end{pmatrix} , \tag{55}$$

which can be written as

$$T = w(0)\theta_p Q_p \left[T_q + \frac{R_p}{Q_p} T_r\right] \ , \tag{56}$$

where

$$T_q = \begin{pmatrix} 0 & (\theta_p Q_p)^{-1} & 0 & 0 & 0 & \cdots \\ q & 0 & q & 0 & 0 & \\ 0 & q^2 & 0 & q^2 & 0 & \\ 0 & 0 & q^3 & 0 & q^3 & \\ \vdots & & & & & \ddots \end{pmatrix} \ , \tag{57}$$

and

$$T_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ r & 0 & r & 0 & 0 & \\ 0 & r^2 & 0 & r^2 & 0 & \\ 0 & 0 & r^3 & 0 & r^3 & \\ \vdots & & & & & \ddots \end{pmatrix} \ . \tag{58}$$

As a quick aside: we could also write T_r as

$$T_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ -|r| & 0 & -|r| & 0 & 0 & \\ 0 & |r|^2 & 0 & |r|^2 & 0 & \\ 0 & 0 & -|r|^3 & 0 & -|r|^3 & \\ \vdots & & & & & \ddots \end{pmatrix} \ , \tag{59}$$

which may give us some useful physical insight. (T_r represents an increase in probabilities of occupying even heights (except 0), and a decrease in probability of occupying odd heights?)

For convenience we can redefine T without the multiplying factor $w(0)\theta_p Q_p$:

$$T = \left[T_q + \frac{R_p}{Q_p} T_r\right] \ . \tag{60}$$

4.1.1 Recovering $p = 1$ Transfer Matrix

To make the connection back to our previous work, we can show that it is straightforward to recover the $p = 1$ transfer matrix:

$$T = \begin{pmatrix} 0 & (1 + z_-) & 0 & 0 & 0 & \cdots \\ q & 0 & q & 0 & 0 & \\ 0 & q^2 & 0 & q^2 & 0 & \\ 0 & 0 & q^3 & 0 & q^3 & \\ \vdots & & & & & \ddots \end{pmatrix} \ . \tag{61}$$

To begin notice that for $p = 1$, $R_1 = 0$. Next, using the definition of α_p in (144), we see that

$$\theta_1 = \frac{z_+ z_-^2}{z_- + 1} \tag{62}$$

and

$$Q_1 = \frac{1}{z_+ z_-^2} \ . \tag{63}$$

Thus, the element $(T_q)_{0,1} = (\theta_p Q_p)^{-1}$ becomes

$$(\theta_1 Q_1)^{-1} = \frac{z_- + 1}{z_+ z_-^2} \frac{z_+ z_-^2}{1} = (z_- + 1) \ , \tag{64}$$

and $T_q = T$ in (165).

4.2 Eigenvectors

By defining the basis vectors

$$\langle n| \ , \quad |n\rangle \ , \quad n = 0, 1, 2, 3, \ldots \tag{65}$$

and the eigenvectors

$$T|\phi\rangle = \mu|\phi\rangle \ , \tag{66}$$

$$\langle\psi|T = \mu\langle\psi| \ , \tag{67}$$

where μ is the largest eigenvalue (largest real part), we can write a partition sum

$$Z = \sum_{n=0}^{\infty} \langle n|T^L|n\rangle \simeq \mu^L \sum_{n=0}^{\infty} \langle n|\phi\rangle\langle\psi|n\rangle \tag{68}$$

and similarly the height distribution

$$P(n) = \frac{\langle n|T^L|n\rangle}{Z} \simeq \frac{\langle n|\phi\rangle\langle\psi|n\rangle}{\sum_{n'=0}^{\infty} \langle n'|\phi\rangle\langle\psi|n'\rangle \ .} \ , \tag{69}$$

for large L. We also define

$$\begin{aligned} \langle\phi| &= \sum_{n=0}^{\infty} \phi_n \langle n| \ , \\ |\psi\rangle &= \sum_{n=0}^{\infty} \psi_n |n\rangle \ , \end{aligned} \tag{70}$$

and see that

$$P(n) = \frac{\phi_n \psi_n}{\sum_{n'=0}^{\infty} \phi'_n \psi'_n \ .} \tag{71}$$

4.2.1 Recursion relations for coefficients ϕ_n, ψ_n

To calculate the distribution $P(n)$ we need to find the eigenfunctions ϕ_n, ψ_n. Using the equations

$$T|\phi\rangle = \left[T_q + \frac{R_p}{Q_p} T_r \right] |\phi\rangle = \mu|\phi\rangle \tag{72}$$

and

$$\langle\psi|T = \langle\psi|\left[T_q + \frac{R_p}{Q_p}T_r\right] = \mu\langle\phi| \tag{73}$$

we can find recursion relations for both.

For the right eigenvector we have a boundary equation

$$\frac{1}{\theta_p Q_p}\phi_1 = \mu\phi_0 \ , \tag{74}$$

and for $n > 0$

$$\left(q^n + \frac{R_p}{Q_p}r^n\right)(\phi_{n-1} + \phi_{n+1}) = \mu\phi_n \ . \tag{75}$$

For the left eigenvector we have two boundary terms

$$\left(q + \frac{R_p}{Q_p}r\right)\psi_1 = \mu\psi_0 \tag{76}$$

and

$$\frac{1}{\theta_p Q_p}\psi_0 + \left(q^2 + \frac{R_p}{Q_p}r^2\right)\psi_2 = \mu\psi_1 \ , \tag{77}$$

and for $n > 1$ we have

$$\left(q^{n-1} + \frac{R_p}{Q_p}r^{n-1}\right)\psi_{n-1} + \left(q^{n+1} + \frac{R_p}{Q_p}r^{n+1}\right)\psi_{n+1} = \mu\psi_n \ . \tag{78}$$

Actually, one can show that

$$\frac{1}{\theta_p} = Q_p + R_p \ . \tag{79}$$

This means that for ψ_n the second boundary equation (77) can be rewritten as

$$\left(1 + \frac{R_p}{Q_p}\right)\psi_0 + \left(q^2 + \frac{R_p}{Q_p}r^2\right)\psi_2 = \mu\psi_1 \ , \tag{80}$$

which is actually consistent with the general recursion relation (78), and so it is not a boundary term after all. Thus, the left eigenfunction satisfies

$$\left(q^{n-1} + \frac{R_p}{Q_p}r^{n-1}\right)\psi_{n-1} + \left(q^{n+1} + \frac{R_p}{Q_p}r^{n+1}\right)\psi_{n+1} = \mu\psi_n \ , \tag{81}$$

for $n > 0$, with the boundary condition

$$\left(1 + \frac{R_p}{Q_p}\right)\psi_1 = \mu\psi_0 \ . \tag{82}$$

Also, the boundary condition for the right eigenfunction can be expressed as

$$\left(1 + \frac{R_p}{Q_p}\right)\phi_1 = \mu\phi_0 \ . \tag{83}$$

4.2.2 SUMMARY: Recursion relations

To be clear, we have:

$$\left(q^n + \frac{R_p}{Q_p} r^n\right)(\phi_{n-1} + \phi_{n+1}) = \mu\phi_n \ , \tag{84}$$

for $n > 0$, with the boundary condition

$$\left(1 + \frac{R_p}{Q_p}\right)\phi_1 = \mu\phi_0 \ , \tag{85}$$

for the right eigenfunction, and for the left eigenfunction we have

$$\left(q^{n-1} + \frac{R_p}{Q_p} r^{n-1}\right)\psi_{n-1} + \left(q^{n+1} + \frac{R_p}{Q_p} r^{n+1}\right)\psi_{n+1} = \mu\psi_n \ , \tag{86}$$

for $n > 0$, with the boundary condition

$$\left(1 + \frac{R_p}{Q_p}\right)\psi_1 = \mu\psi_0 \ . \tag{87}$$

(NOTE TO SELF: maybe define $f_n = q^n + (R_p/Q_p)r^n$ for convenience?)

[WORK IN PROGRESS: next steps: 1. continuum approximation, 2. Airy function solution? 3. L scaling?]

4.2.3 Consistency with $p = 1$

How the general p recursion relations above are consistent with those found for $p = 1$ is not obvious at a glance. To see that they are, we need to consider R_p/Q_p and r.

First, when $p = 1$, $R_p/Q_p = z_-$, which means that

$$1 + \frac{R_1}{Q_1} = 1 + z_- \ . \tag{88}$$

Second: $r = r_p = z_p^{-1}$. As $p \to 1$, z_p diverges, and so $r \to 0$. Now we use these results with the recursion relations above. First, the right eigenfunction:

$$\begin{aligned}
\left(q^n + \frac{R_1}{Q_1} r_1^n\right)(\phi_{n-1} + \phi_{n+1}) &= \mu\phi_n \\
(q^n + z_-(0)^n)(\phi_{n-1} + \phi_{n+1}) &= \mu\phi_n \\
q^n(\phi_{n-1} + \phi_{n+1}) &= \mu\phi_n \ ,
\end{aligned} \tag{89}$$

for $n > 0$ with the boundary condition

$$\begin{aligned}
\left(1 + \frac{R_1}{Q_1}\right)\phi_1 &= \mu\phi_0 \\
(1 + z_-)\phi_1 &= \mu\phi_0 \ .
\end{aligned} \tag{90}$$

Second, the left eigenfunction:

$$\left(q^{n-1} + \frac{R_1}{Q_1} r - 1^{n-1}\right) \psi_{n-1} + \left(q^{n+1} + \frac{R_1}{Q_1} r_1^{n+1}\right) \psi_{n+1} = \mu\psi_n$$
$$\left(q^{n-1} + z_- r_1^{n-1}\right) \psi_{n-1} + \left(q^{n+1} + z_- r_1^{n+1}\right) \psi_{n+1} = \mu\psi_n \ , \tag{91}$$

which gives

$$q^{n-1} + \psi_{n-1} + q^{n+1}\psi_{n+1} = \mu\psi_n \tag{92}$$

for $n > 1$, but for $n = 1$ becomes

$$(1 + z_-)\,\psi_0 + q^2\psi_2 = \mu\psi_1 \ , \tag{93}$$

4.3 Continuum Approximation

We can make a continuum approximation

$$\phi_{n\pm1} = \phi(n) \pm \frac{\mathrm{d}\phi}{\mathrm{d}n} + \frac{1}{2}\frac{\mathrm{d}^2\phi}{\mathrm{d}n} + \text{h.o.t.} \ , \tag{94}$$
$$\psi_{n\pm1} = \psi(n) \pm \frac{\mathrm{d}\psi}{\mathrm{d}n} + \frac{1}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}n} + \text{h.o.t.} \ . \tag{95}$$

We also define

$$q = 1 - \epsilon \ , \quad \epsilon \ll 1 \ , \tag{96}$$
$$r = -(1 - \eta) \ , \quad \eta \ll 1 \ . \tag{97}$$

(NOTE: In the $p = 1$ case, $q = 1-\epsilon$ was based on the observation that $z_+^{-1} \simeq 1 - \mathcal{O}(L^{-1})$. Is this still the case in general? And is something similar true for r too?) These expressions allow us to write

$$q^n \simeq 1 - n\epsilon \ , \tag{98}$$
$$r^n \simeq \mathrm{e}^{i\pi n}(1 - n\eta) \ . \tag{99}$$
$$\tag{100}$$

The r terms complicate matters, because they introduce a complex compenent to the coefficents in the recursion relations.

4.3.1 ASIDE: L Scaling and ϵ

It would actually be more useful if we could relate the small parameter scalings of q and r. In an earlier work ("mf.pdf") I showed that

$$z_+ = 1 + \mathcal{O}\left(\frac{1}{L}\right) \ , \quad p = 1 \ ,$$

and thus

$$q = z_+^{-1} = 1 - \mathcal{O}\left(\frac{1}{L}\right) \ , \quad p = 1 \ ,$$

which is why we define

$$q = 1 - \epsilon \ , \quad \epsilon \ll 1 \ .$$

It was possible to find this because:

1. we had expressions for the roots z_+ and z_- in terms of a, b,
2. we knew a relationship between a,b and $P(0)$ (or $w(0)$),
3. numerically we find, and analytically we can argue, that $P(0) \sim 1/L$.

Now, for general p, we have three roots z_-, z_+ and z_p, and we don't know the expressions for any of them[2]. It's likely that $z_+ = 1 + \mathcal{O}(1/L)$ still, because that's what we see in simulation, but we have no idea about how z_p scales with L (and thus how r scales with ϵ).

4.3.2 Right Eigenvector Equations

[This (sub-sub-)section is now a bit out of date.]

In the continuum limit, the right eigenvector equation becomes

$$\left[2 - \mu - 2n\epsilon + 2(1 - n\eta)\frac{R_p}{Q_p}\mathrm{e}^{i\pi n}\right]\phi(n) + \left[1 + \frac{R_p}{Q_p}\mathrm{e}^{i\pi n}\right]\frac{\mathrm{d}^2\phi}{\mathrm{d}n^2} = 0 \ , \tag{101}$$

with the boundary condition

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}n}\right|_{n=0} = (\mu\theta_p Q_p - 1)\phi(0) \ . \tag{102}$$

We have made the assumptions that

$$\epsilon\frac{\mathrm{d}^2\phi}{\mathrm{d}n^2} \ , \quad \eta\frac{\mathrm{d}^2\phi}{\mathrm{d}n^2}$$

are both negligible. **(NOTE: why? "smoothly varying"?)**

The general solution for $\phi(n)$ is complex, so we can write it as

$$\phi(n) = u_\phi(n) + iv_\phi(n) \ , \tag{103}$$

where the functions $u_\phi(n)$ and $v_\phi(n)$ are both real. Then, we split (101) into an equation each for the real part

$$\begin{aligned} 0 \ = \ & \left[2 - \mu - 2n\epsilon + 2(1 - n\eta)\frac{R_p}{Q_p}\cos(\pi n)\right]u_\phi(n) \\ - \ & \left[2(1 - n\eta)\frac{R_p}{Q_p}\sin(\pi n)\right]v_\phi(n) \\ + \ & \left[1 + \frac{R_p}{Q_p}\cos(\pi n)\right]\frac{\mathrm{d}^2u_\phi}{\mathrm{d}n^2} \\ - \ & \left[\frac{R_p}{Q_p}\sin(\pi n)\right]\frac{\mathrm{d}^2v_\phi}{\mathrm{d}n^2} \ , \end{aligned} \tag{104}$$

[2]We can get them from Mathematica, but they are extremely complicated.

and the imaginary part

$$
\begin{aligned}
0 &= \left[2(1-n\eta)\frac{R_p}{Q_p}\sin(\pi n)\right]u_\phi(n) \\
&+ \left[2-\mu-2n\epsilon+2(1-n\eta)\frac{R_p}{Q_p}\cos(\pi n)\right]v_\phi(n) \\
&+ \left[\frac{R_p}{Q_p}\sin(\pi n)\right]\frac{\mathrm{d}^2 u_\phi}{\mathrm{d}n^2} \\
&+ \left[1+\frac{R_p}{Q_p}\cos(\pi n)\right]\frac{\mathrm{d}^2 v_\phi}{\mathrm{d}n^2}\ .
\end{aligned}
\tag{105}
$$

4.3.3 Left Eigenvector Equations

[Work in progress]

A Calculation of Generating Function in Steady State

In the steady state

$$\begin{aligned}
0 &= u\Big[P(y-1)\mathrm{I}_{y>0} - P(y)\Big] \\
&+ (1-u)[1-P(0)]^L\Big[P(y+1) - P(y)\mathrm{I}_{y>0}\Big] \\
&+ \frac{p}{4}\Big[P(y+2) - P(y)\mathrm{I}_{y>1}\Big] \\
&+ \frac{(1-p)}{4}\Big[P(y-2)\mathrm{I}_{y>1} - P(y)\Big] . \qquad (106)
\end{aligned}$$

Using the definitions for a and b from (5), multiply both sides by z^y and sum from $y = 0$ to infinity:

$$\begin{aligned}
0 &= a\left[\sum_{y=1}^{\infty} z^y P(y-1) - \sum_{y=0}^{\infty} z^y P(y)\right] \\
&+ b\left[\sum_{y=0}^{\infty} z^y P(y+1) - \sum_{y=1}^{\infty} z^y P(y)\right] \\
&+ p\left[\sum_{y=0}^{\infty} z^y P(y+2) - \sum_{y=2}^{\infty} z^y P(y)\right] \\
&+ (1-p)\left[\sum_{y=2}^{\infty} z^y P(y-2) - \sum_{y=0}^{\infty} z^y P(y)\right] . \qquad (107)
\end{aligned}$$

Change of variables to make all sums over $P(y)$:

$$\begin{aligned}
0 &= a\left[z\sum_{y=0}^{\infty} z^y P(y) - \sum_{y=0}^{\infty} z^y P(y)\right] \\
&+ b\left[z^{-1}\sum_{y=1}^{\infty} z^y P(y) - \sum_{y=1}^{\infty} z^y P(y)\right] \\
&+ p\left[z^{-2}\sum_{y=2}^{\infty} z^y P(y) - \sum_{y=2}^{\infty} z^y P(y)\right] \\
&+ (1-p)\left[z^2\sum_{y=0}^{\infty} z^y P(y) - \sum_{y=0}^{\infty} z^y P(y)\right] . \qquad (108)
\end{aligned}$$

Rewrite all sums in terms of $G(z)$, $P(0)$ and $P(1)$:

$$\begin{aligned}
0 &= a\,[z-1]\,G(z) \\
&+ b\left[z^{-1}-1\right][G(z) - P(0)] \\
&+ p\left[z^{-2}-1\right][G(z) - zP(1) - P(0)] \\
&+ (1-p)\left[z^2-1\right]G(z) . \qquad (109)
\end{aligned}$$

Group together terms with $G(z)$, $P(0)$ and $P(1)$:

$$\begin{aligned}
0 &= \left[a(z-1)+b(z^{-1}-1)+p(z^{-2}-1)+(1-p)(z^2-1)\right]G(z) \\
&- p\left[z^{-1}-z\right]P(1) \\
&- \left[b(z^{-1}-1)+p(z^{-2}-1)\right]P(0)\ .
\end{aligned} \tag{110}$$

Multiply through by z^2:

$$\begin{aligned}
0 &= \left[az^2(z-1)+bz(1-z)+p(1-z^2)+(1-p)z^2(z^2-1)\right]G(z) \\
&- pz\left[1-z^2\right]P(1) \\
&- \left[bz(1-z)+p(1-z^2)\right]P(0)\ .
\end{aligned} \tag{111}$$

Now notice that every term contains a factor of $z-1$. Divide this out:

$$\begin{aligned}
0 &= \left[az^2-bz-p(1+z)+(1-p)z^2(z+1)\right]G(z) \\
&+ pz\left[1+z\right]P(1) \\
&+ \left[bz+p(1+z)\right]P(0)\ .
\end{aligned} \tag{112}$$

Rearrange by grouping powers of z:

$$\begin{aligned}
0 &= \left[(1-p)z^3+(a+1-p)z^2(b+p)z-p\right]G(z) \\
&+ pP(1)z^2+(pP(1)+(b+p)P(0))z+pP(0)\ .
\end{aligned} \tag{113}$$

Now rearrange to get the expression (3) for $G(z)$:

$$G(z) = \frac{-\left[pP(1)z^2+(pP(1)+(b+p)P(0))z+pP(0)\right]}{\left[(1-p)z^3+(a+1-p)z^2(b+p)z-p\right]} \tag{114}$$

B Finding the Probability Distribution

We have the following expression for the generating function:

$$G(z) = \frac{-\left[pP(1)z^2+(pP(1)+(b+p)P(0))z+pP(0)\right]}{\left[(1-p)z^3+(a+1-p)z^2(b+p)z-p\right]}\ . \tag{115}$$

The cubic in the denominator makes it difficult to solve. We assign a function to the denominator:

$$h(z) = (1-p)z^3+(a+1-p)z^2(b+p)z-p\ . \tag{116}$$

This cubic function $h(z)$ has three real roots[3] z_+, z_- and z_p, such that

$$z_+ > 0\ , \tag{117}$$

$$z_p < z_- < 0\ , \tag{118}$$

and

$$|z_p| > |z_+| > |z_-|\ . \tag{119}$$

[3] Do these roots become complex for certain parameter values?

z_+ and z_- correspond to the two roots of the same name of the quadratic in the $p = 1$ case. In the limit $p \to 1$ the root z_p must disappear as $h(z)$ become a cubic. We can see then that $z_p \sim -a/(1-p)$, such that the order $1/(1-p)^2$ terms in $h(z)$ cancel. **(NOTE: I need to make this analysis more concrete.)(NOTE: include sketch of roots?)**

Importantly, because we still have $|z_-| < |z_+|$, the pole at z_- is still closer to the origin and dominates the integral of $G(z)$ which describes $P(y)$. Thus, as in the $p = 1$ case, we must cancel a factor $(z - z_-)$ from top and bottom. Conversely, the pole z_p is further from the origin that z_+, because $|z_p| > |z_+|$, and so this pole (with negative real part) does not dominate the same integral, so does not need to be cancelled. **(NOTE: are we sure $|z_p| > |z_+|$? I'm pretty confident - see graph.)**

So now we can write

$$h(z) = (1-p)(z - z_-)(z - z_+)(z - z_p) \ , \tag{120}$$

and the numerator of $G(z)$ can be written as

$$-\left[pP(0)z^2 + \left(pP(1) + (b+p)P(0)\right)z + pP(0)\right] = -(Az + B)(z - z_-) \ . \tag{121}$$

Immediately from this we can write

$$A = pP(1) \ , \tag{122}$$

and

$$B = -\frac{pP(0)}{z_-} \ , \tag{123}$$

which will be useful later.

Now, coming back to the generating function, we can write $G(z)$ as

$$G(z) = -\frac{(Az + B)}{(1-p)(z - z_+)(z - z_p)} \ . \tag{124}$$

To find an expression for $P(y)$ we will try to rewrite $G(z)$ as a sum of powers of z. To begin, we factorise out $-z_+$, $-z_p$, to find

$$\begin{aligned} G(z) &= -\frac{(Az + B)}{(1-p)z_+ z_p (1 - z/z_+)(1 - z/z_p)} \\ &= -\frac{(Az + B)}{(1-p)z_+ z_p} \sum_{l=0}^{\infty} \left(\frac{z}{z_+}\right)^l \sum_{m=0}^{\infty} \left(\frac{z}{z_p}\right)^m \ . \end{aligned} \tag{125}$$

Using the substitution $n = l + m$ and by rearranging the sums we find

$$\begin{aligned} G(z) &= -\frac{(Az + B)}{(1-p)z_+ z_p} \sum_{n=0}^{\infty} z^n \sum_{l=0}^{n} \frac{1}{z_p^{n-l}} \frac{1}{z_+^l} \\ &= -\frac{(Az + B)}{(1-p)z_+ z_p} \sum_{n=0}^{\infty} \frac{z^n}{z_p^n} \sum_{l=0}^{n} \left(\frac{z_p}{z_p}\right)^l \ . \end{aligned} \tag{126}$$

We then evaluate the geometric sum over l to find

$$G(z) = -\frac{(Az + B)}{(1-p)z_+ z_p} \sum_{n=0}^{\infty} \frac{z^n}{z_p^n} \left[\frac{(z_p/z_+)^{n+1} - 1}{(z_p/z_+) - 1}\right] \ . \tag{127}$$

We want to find the coefficients of z^n to find the values of $P(n)$. To do this we first multiply through by $(Az+B)$:

$$G(z) = -\frac{1}{(1-p)z_+z_p}\sum_{n=0}^{\infty}\frac{Az^{n+1}+Bz^n}{z_p^n}\left[\frac{(z_p/z_+)^{n+1}-1}{(z_p/z_+)-1}\right] , \tag{128}$$

and then relabel the $n \to n-1$ in the "A" sum:

$$\begin{aligned} G(z) = -\frac{1}{(1-p)z_+z_p} \quad & \left\{\sum_{n=1}^{\infty}\frac{Az^n}{z_p^{n-1}}\left[\frac{(z_p/z_+)^n-1}{(z_p/z_+)-1}\right]\right. \\ & \left. + \sum_{n=0}^{\infty}\frac{Bz^n}{z_p^n}\left[\frac{(z_p/z_+)^{n+1}-1}{(z_p/z_+)-1}\right]\right\} . \end{aligned} \tag{129}$$

Next, pull out the $n=0$ term and combine the sums, to get

$$\begin{aligned} G(z) &= -\frac{1}{(1-p)z_+z_p}Bz^0 - \sum_{n=1}^{\infty}\frac{z^n}{(1-p)z_p^{n+1}(z_p-z_+)}\left(Az_p\left[\left(\frac{z_p}{z_+}\right)^n-1\right]+B\left[\left(\frac{z_p}{z_+}\right)^{n+1}-1\right]\right) \\ &= -\frac{1}{(1-p)z_+z_p}Bz^0 - \sum_{n=1}^{\infty}\frac{z^n}{(1-p)(z_p-z_+)}\left(A\left[\left(\frac{1}{z_+}\right)^n-\left(\frac{1}{z_p}\right)^n\right]+B\left[\left(\frac{1}{z_+}\right)^{n+1}-\left(\frac{1}{z_p}\right)^{n+1}\right]\right) \\ &= -\frac{1}{(1-p)z_+z_p}Bz^0 - \sum_{n=1}^{\infty}\frac{z^n}{(1-p)(z_p-z_+)}\left(\frac{Az_++B}{z_+^{n+1}}-\frac{Az_p+B}{z_p^{n+1}}\right) . \end{aligned} \tag{130}$$

From this we see that

$$P(0) = -\frac{B}{(1-p)z_+z_p} , \tag{131}$$

and, for $n>0$,

$$P(n) = \frac{1}{(1-p)(z_p-z_+)}\left(\frac{Az_++B}{z_+^{n+1}}-\frac{Az_p+B}{z_p^{n+1}}\right) . \tag{132}$$

From the expression for $P(0)$ we have

$$B = -(1-p)z_+z_pP(0) , \tag{133}$$

and from earlier we have

$$A = pP(1) . \tag{134}$$

We can use these to calculate

$$\begin{aligned} Az_{+,p}+B &= pP(1)z_{+,p} - (1-p)z_+z_pP(0) \\ &= z_{+,p}(pP(1)-(1-p)z_{p,+}P(0)) . \end{aligned} \tag{135}$$

Substituting back in to the expression for $P(n)$, $n>0$ we find

$$\begin{aligned} P(n) &= -\frac{1}{(1-p)(z_p-z_+)}\left(\frac{pP(1)-(1-p)P(0)z_p}{z_+^n}-\frac{pP(1)-(1-p)P(0)z_+}{z_p^n}\right) \\ &= -\frac{1}{(1-p)(z_p-z_+)}\left(\frac{pP(1)(z_p^n-z_+^n)-(1-p)P(0)(z_p^{n+1}-z_+^{n+1})}{z_+^nz_p^n}\right) . \end{aligned} \tag{136}$$

By setting $n = 1$ we can solve self-consistently for $P(1)$:

$$
\begin{aligned}
P(1) &= -\frac{1}{(1-p)(z_p - z_+)} \left(\frac{pP(1)(z_p - z_+) - (1-p)P(0)(z_p^2 - z_+^2)}{z_+ z_p} \right) \\
&= -\frac{1}{(1-p)z_p z_+} \left[pP(1) - (1-p)P(0)(z_p + z_+) \right] \\
&= -\frac{P(1)}{(1-p)z_p z_+} + \frac{P(0)(z_p + z_+)}{z_p z_+} \ , \qquad (137)
\end{aligned}
$$

and so by rearranging we find

$$P(1) = \frac{(1-p)(z_p + z_+)}{z_+ z_p (1-p) - p} P(0) \ . \qquad (138)$$

Now we substitute the expression for $P(1)$ back in to the expression for $P(n)$ to find

$$P(n) = -\frac{P(0)}{z_+^n z_p^n (z_p - z_+)} \left(\frac{p(z_p^n - z_+^n)(z_p + z_+)}{z_+ z_p (1-p) + p} - (z_p^{n+1} - z_+^{n+1}) \right) \ . \qquad (139)$$

We can simplify a bit, using

$$p(z_p^n - z_+^n)(z_p + z_+) = pz_p^{n+1} - pz_+^{n+1} - pz_p z_+^n + pz_+ z_p^n \ , \qquad (140)$$

and

$$-(z_p^{n+1} - z_+^{n+1})(z_+ z_p (1-p) + p) = -pz_p^{n+1} + pz_+^{n+1} - (1-p)z_+ z_p^{n+2} + (1-p)z_p z_+^{n+2} \ , \qquad (141)$$

to find

$$P(n) = -\frac{P(0)}{z_+^n z_p^n (z_p - z_+)} \left(\frac{-pz_p z_+^n + pz_+ z_p^n - (1-p)z_+ z_p^{n+2} + (1-p)z_p z_+^{n+2}}{(z_+ z_p (1-p) + p)} \right) \ . \qquad (142)$$

Finally, rearrange to find

$$P(n) = \frac{P(0)}{z_+^{n-1} z_p^{n-1} (z_p - z_+)} \left(\frac{(1-p)[z_p^{n+1} - z_+^{n+1}] - p[z_p^{n-1} - z_+^{n-1}]}{(z_+ z_p (1-p) + p)} \right) \ . \qquad (143)$$

B.1 Further Simplification

The root z_p is problematic because it diverges as $p \to 1$ as $(1-p)^{-1}$. Using the two definitions of B from (123) and (133) we can define

$$\alpha_p = (1-p)z_p = \frac{p}{z_+ z_-} \ . \qquad (144)$$

Unlike z_p, as $p \to 1$, α_p remains finite. We can also define

$$x_p = \frac{z_+}{z_p} \ , \quad x_p < 1 \ , \quad |x_p| < 1 \ , \qquad (145)$$

which has the useful property that $x_p \to 0$ as $p \to 1$. **(NOTE: I think I've made assumption about what z_+ does as $p \to 1$, which I haven't justified...)** The aim is to replace all instances of z_p with α_p or x_p.

First, we factorise out some powers of z_p from (143) to find

$$P(n) = \frac{P(0)}{z_+^{n-1} z_p^{n-1} z_p(1-x_p)} \left(\frac{(1-p)z_p^{n+1}[1-x_p^{n+1}] - p z_p^{n-1}[1-x_p^{n-1}]}{(z_+ z_p(1-p)+p)} \right) . \tag{146}$$

Then we cancel some powers of z_p:

$$P(n) = \frac{P(0)}{z_+^{n-1} z_p(1-x_p)} \left(\frac{(1-p)z_p^2[1-x_p^{n+1}] - p[1-x_p^{n-1}]}{(z_+ z_p(1-p)+p)} \right) . \tag{147}$$

We now multiply top and bottom by $(1-p)$, and substitute in α_p to find

$$P(n) = \frac{P(0)}{z_+^{n-1}} \left(\frac{\alpha_p^2[1-x_p^{n+1}] - p(1-p)[1-x_p^{n-1}]}{\alpha_p(1-x_p)(z_+\alpha_p+p)} \right) . \tag{148}$$

When n is large, the terms $x_p^n \to 0$, and so we see that

$$P(n) \simeq P(0) z_+ \left(\frac{\alpha_p^2 - p(1-p)}{\alpha_p(z_+\alpha_p+p)} \right) \frac{1}{z_+^n} . \tag{149}$$

B.1.1 Sum of Exponentials

We can introduce z_p back in to (148) to find an expression for $P(n)$ as a sum of exponentials. We rearrange the x_p terms and multiply through the $z_+^{-(n-1)}$ term to find

$$P(n) = \frac{P(0)}{\alpha_p(1-x_p)(z_+\alpha_p+p)} \left(\frac{\alpha_p^2 - p(1-p) + x_p^{n-1}[p(1-p) - \alpha_p^2 x_p^2]}{z_p^{n-1}} \right) . \tag{150}$$

This can be rewritten as

$$P(n) = \frac{P(0)}{\alpha_p(1-x_p)(z_+\alpha_p+p)} \left(\frac{[\alpha_p^2 - p(1-p)]}{z_+^{n-1}} + \frac{[p(1-p) - \alpha_p^2 x_p^2]}{z_p^{n-1}} \right) . \tag{151}$$

C Transfer Matrix

C.1 Definition

Take (151) and rewrite with weights, as

$$w(n) = w(0)\theta_p \left(\frac{Q_p}{z_+^n} + \frac{R_p}{z_p^n} \right) , \quad n > 0 , \tag{152}$$

where

$$\theta_p = \frac{1}{\alpha_p(1-x_p)(z_+\alpha_p+p)} , \tag{153}$$

$$Q_p = z_+[\alpha_p^2 - p(1-p)] \ , \tag{154}$$

and

$$R_p = z_p[p(1-p) - \alpha_p^2 x_p^2] \ . \tag{155}$$

We also introduce the definitions

$$q = \frac{1}{z_+} \ , \quad q > 0 \ , \quad |q| < 1 \ , \tag{156}$$

and

$$r = \frac{1}{z_p} \ , \quad r < 0 \ , \quad |r| < |q| < 1 \ , \tag{157}$$

to write

$$w(n) = w(0)\theta_p \left(Q_p q^n + R_p r^n\right) \ , \quad n > 0 \ . \tag{158}$$

Define the transfer matrix:

$$T = \begin{pmatrix} 0 & w(0) & 0 & 0 & 0 & \cdots \\ w(1) & 0 & w(1) & 0 & 0 & \\ 0 & w(2) & 0 & w(2) & 0 & \\ 0 & 0 & w(3) & 0 & w(3) & \\ \vdots & & & & & \ddots \end{pmatrix} . \tag{159}$$

This can be written as

$$T = w(0)\theta_p \left[Q_p T_q + R_p T_r\right] \ , \tag{160}$$

where

$$T_q = \begin{pmatrix} 0 & (\theta_p Q_p)^{-1} & 0 & 0 & 0 & \cdots \\ q & 0 & q & 0 & 0 & \\ 0 & q^2 & 0 & q^2 & 0 & \\ 0 & 0 & q^3 & 0 & q^3 & \\ \vdots & & & & & \ddots \end{pmatrix} , \tag{161}$$

and

$$T_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ r & 0 & r & 0 & 0 & \\ 0 & r^2 & 0 & r^2 & 0 & \\ 0 & 0 & r^3 & 0 & r^3 & \\ \vdots & & & & & \ddots \end{pmatrix} . \tag{162}$$

As a quick aside: we could also write T_r as

$$T_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ -|r| & 0 & -|r| & 0 & 0 & \\ 0 & |r|^2 & 0 & |r|^2 & 0 & \\ 0 & 0 & -|r|^3 & 0 & -|r|^3 & \\ \vdots & & & & & \ddots \end{pmatrix} , \tag{163}$$

which may give us some useful physical insight. (T_r represents an increase in probabilities of occupying even heights (except 0), and a decrease in probability of occupying odd heights?)

For convenience we can redefine T without the multiplying factor $w(0)\theta_p Q_p$:

$$T = \left[T_q + \frac{R_p}{Q_p} T_r\right] \ . \tag{164}$$

C.1.1 Recovering $p = 1$ Transfer Matrix

It is straightforward to recover the $p = 1$ transfer matrix:

$$T = \begin{pmatrix} 0 & (1+z_-) & 0 & 0 & 0 & \cdots \\ q & 0 & q & 0 & 0 & \\ 0 & q^2 & 0 & q^2 & 0 & \\ 0 & 0 & q^3 & 0 & q^3 & \\ \vdots & & & & & \ddots \end{pmatrix} , \tag{165}$$

from our earlier work.

To begin notice that for $p = 1$, $R_1 = 0$. Next, using the definition of α_p in (144), we see that

$$\theta_1 = \frac{z_+ z_-^2}{z_- + 1} \tag{166}$$

and

$$Q_1 = \frac{1}{z_+ z_-^2} . \tag{167}$$

Thus, the element $(T_q)_{0,1} = (\theta_p Q_p)^{-1}$ becomes

$$(\theta_1 Q_1)^{-1} = \frac{z_- + 1}{z_+ z_-^2} \frac{z_+ z_-^2}{1} = (z_- + 1) , \tag{168}$$

and $T_q = T$ in (165).

C.2 Eigenvectors

Some definitions:

$$T|\phi^{(\mu)}\rangle = \mu|\phi^{(\mu)}\rangle , \tag{169}$$

$$\langle\psi^{(\mu)}|T = \mu\langle\psi^{(\mu)}| , \tag{170}$$

$$|\phi^{(\mu)}\rangle = \sum_{n=0}^{\infty} \phi_n^{(\mu)}|n\rangle , \tag{171}$$

$$\langle\psi^{(\mu)}| = \sum_{n=0}^{\infty} \psi_n^{(\mu)}\langle n| , \tag{172}$$

with basis vectors:

$$\langle n| , \quad |n\rangle , \quad n = 0, 1, 2, 3, \ldots \tag{173}$$

Partition function

$$Z = \sum_{n=0}^{\infty} \langle n|T^L|n\rangle , \tag{174}$$

and probability distribution

$$P(n) = \frac{\langle n|T^L|n\rangle}{Z} . \tag{175}$$

We can write

$$
\begin{aligned}
Z &= \sum_{n=0}^{\infty} \langle n | T^L \sum_{\mu} |\phi^{(\mu)}\rangle\langle\psi^{(\mu)}| |n\rangle \\
&= \sum_{\mu} \mu^L \sum_{n=0}^{\infty} \langle n|\phi^{(\mu)}\rangle\langle\psi^{(\mu)}|n\rangle \ . \qquad (176)
\end{aligned}
$$

Assume that for large L ($L \to \infty$) the sum is dominated by the largest eigenvalue, μ (abuse of notation!), to write

$$
Z \simeq \mu^L \sum_{n=0}^{\infty} \langle n|\phi\rangle\langle\psi|n\rangle \ , \qquad (177)
$$

where

$$
\begin{aligned}
|\phi\rangle &= |\phi^{(\mu)}\rangle \\
\langle\psi| &= \langle\psi^{(\mu)}| \ . \qquad (178)
\end{aligned}
$$

Similarly, for $P(n)$ we can write

$$
\begin{aligned}
P(n) &= \frac{\langle n|T^L \sum_\mu |\phi^{(\mu)}\rangle\langle\psi^{(\mu)}||n\rangle}{Z} \\
&= \frac{\sum_\mu \mu^L \langle n|\phi^{(\mu)}\rangle\langle\psi^{(\mu)}|n\rangle}{Z} \ , \qquad (179)
\end{aligned}
$$

and again assuming the sum is dominated by the largest eigenvalue μ (again abusing notation) as $L \to \infty$ we find

$$
P(n) \simeq \frac{\mu^L \langle n|\phi\rangle\langle\psi|n\rangle}{Z} \ . \qquad (180)
$$

A factor of μ^L can be cancelled from both top and bottom to obtain

$$
P(n) = \frac{\langle n|\phi\rangle\langle\psi|n\rangle}{\sum_{n'=0}^{\infty} \langle n'|\phi\rangle\langle\psi|n'\rangle} \ . \qquad (181)
$$

We define

$$
\begin{aligned}
\langle\phi| &= \sum_{n=0}^{\infty} \phi_n \langle n| \ , \\
|\psi\rangle &= \sum_{n=0}^{\infty} \psi_n |n\rangle \ , \qquad (182)
\end{aligned}
$$

and see that

$$
P(n) = \frac{\phi_n \psi_n}{\sum_{n'=0}^{\infty} \phi'_n \psi'_n} \ . \qquad (183)
$$

C.2.1 Recursion relations for coefficients ϕ_n, ψ_n

Using the equations

$$T|\phi\rangle = \left[T_q + \frac{R_p}{Q_p}T_r\right]|\phi\rangle = \mu|\phi\rangle \tag{184}$$

and

$$\langle\psi|T = \langle\psi|\left[T_q + \frac{R_p}{Q_p}T_r\right] = \mu\langle\phi| \tag{185}$$

we can find recursion relations for ϕ_n, ψ_n.

For the right eigenvector we have a boundary equation

$$\frac{1}{\theta_p Q_p}\phi_1 = \mu\phi_0 \ , \tag{186}$$

and for $n > 0$

$$\left(q^n + \frac{R_p}{Q_p}r^n\right)(\phi_{n-1} + \phi_{n+1}) = \mu\phi_n \ . \tag{187}$$

For the left eigenvector we have two boundary terms

$$\left(q + \frac{R_p}{Q_p}r\right)\psi_1 = \mu\psi_0 \tag{188}$$

and

$$\frac{1}{\theta_p Q_p}\psi_0 + \left(q^2 + \frac{R_p}{Q_p}r^2\right)\psi_2 = \mu\psi_1 \ , \tag{189}$$

and for $n > 1$ we have

$$\left(q^{n-1} + \frac{R_p}{Q_p}r^{n-1}\right)\psi_{n-1} + \left(q^{n+1} + \frac{R_p}{Q_p}r^{n+1}\right)\psi_{n+1} = \mu\psi_n \ . \tag{190}$$

C.2.2 Rewriting θ_p

We have

$$(\theta_p)^{-1} = \alpha_p(1-x_p)(z_+\alpha_p + p) \ ,$$

where

$$\alpha_p = z_p(1-p) \ , \quad x_p = \frac{z_+}{z_p} \ .$$

We also have

$$\begin{aligned} Q_p &= z_+[\alpha_p^2 - p(1-p)] \\ R_p &= z_p[p(1-p) - \alpha_p^2 x_p^2] \ . \end{aligned}$$

Now we compute $Q_p + R_p$:

$$\begin{aligned} Q_p + R_p &= z_+[\alpha_p^2 - p(1-p)] + z_p[p(1-p) - \alpha_p^2 x_p^2] \\ &= \alpha_p[z_+\alpha_p - z_p x_p^2 \alpha_p] + p(1-p)[z_p - z_+] \\ &= \alpha_p[z_+\alpha_p - z_+ x_p \alpha_p] + \alpha_p \frac{p}{z_p}[z_p - z_+] \\ &= \alpha_p[z_+\alpha_p(1-x_p)] + \alpha_p p[1-x_p] \\ &= \alpha_p(z_+\alpha_p + p)(1-x_p) \ , \end{aligned} \tag{191}$$

so we can see that

$$Q_p + R_p = \frac{1}{\theta_p} \; , \tag{192}$$

and thus

$$\frac{1}{\theta_p Q_p} = 1 + \frac{R_p}{Q_p} \; . \tag{193}$$

This means that: for the right eigenfunction we have

$$\left(q^n + \frac{R_p}{Q_p} r^n \right) (\phi_{n-1} + \phi_{n+1}) = \mu \phi_n \; . \tag{194}$$

for $n > 0$, with the boundary condition

$$1 + \frac{R_p}{Q_p} \phi_1 = \mu \phi_0 \; ; \tag{195}$$

for the left eigenfunction we have

$$\left(q^{n-1} + \frac{R_p}{Q_p} r^{n-1} \right) \psi_{n-1} + \left(q^{n+1} + \frac{R_p}{Q_p} r^{n+1} \right) \psi_{n+1} = \mu \psi_n \; , \tag{196}$$

for $n > 0$ with the boundary condition

$$\left(q + \frac{R_p}{Q_p} r \right) \psi_1 = \mu \psi_0 \; . \tag{197}$$