Update From Weizmann #2

Justin Whitehouse

June 21, 2016

Contents

1 Introduction

In the last few days I have been (numerically) looking at the heights distribution, in particular at $p = 0.5$. Richard recently presented an exactly solvable case for the model, which I tried to use to fit to the measured height distribution.

1.1 Summary

From Richard's solution I have derived a solution for the probability, $P(y)$, of a point on the interface having height y in terms of the Airy function $\mathrm{Ai}(x)$ in essentially exactly these same way as we did previously for our mean-field theory.

The solution I've found can be fit quite well to the measured distribution (with enough free paramters...). I think this corroborates our numerical evidence from the width measurements for a width exponent of 1/3 near/at $p = 0.5$.

Maybe in (the near?) future I will try to analyse the general case mean-field theory as perturbation/deviation from Richard's equilibrium solution (because the structure is very similar).

I've quickly recapped a couple of Richard's results which I use in my analysis before giving a brief overview of what I've done.

1.2 Note about notation

Richard has used N for the system size. I will instead be using L, to be consistent with my own notation.

2 Quick Overview of Richard's Solution

Richard wrote down a master equation for which a stationary solution can be found for the special parameter combination

$$\left(\frac{p}{1-p}\right)^{\frac{L}{2}} = \frac{1-u}{u} \, . \tag{1}$$

To hold for arbitrary u in the $L \to \infty$ limit, we must take

$$p = \frac{1}{2}\left(1 + \frac{a}{N}\right) \, , \tag{2}$$

and so (1) becomes

$$\mathrm{e}^{a} = \frac{1-u}{u} \, . \tag{3}$$

The stationary distribution $P(\mathcal{C}, y_0)$ can be written as a matrix product

$$P(\mathcal{C}, y_0) = \frac{\langle y_0 | \prod_{i=1}^{L} [\tau_i D + (1-\tau_i) E] | y_0 \rangle}{Z} \tag{4}$$

where

$$Z = \sum_{y_0=0}^{\infty} \langle y_0 | (D+E)^L | y_0 \rangle \tag{5}$$

and $\langle y| \in \{\langle 0|, \langle 1|, \ldots\}$ is the height basis ($\langle y|y'\rangle = \delta_{y,y'}$). The representation of the matrices D and E in this basis is

$$\langle y'|D|y\rangle = \begin{cases} \left(\frac{1-p}{p}\right)^{\frac{y-1}{2}} & y - y' = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$\langle y'|E|y\rangle = \begin{cases} \left(\frac{1-p}{p}\right)^{\frac{y}{2}} & y - y' = -1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

3 Defining $P(y)$, Transfer Matrix, Eigenfunctions

I have defined the probability that a site picked at random has height y as

$$P(y) = \sum_{\mathcal{C}} P(\mathcal{C}, y) = \frac{\langle y|(D+E)^L|y\rangle}{Z} \ . \tag{8}$$

Defining

$$\rho = \left(\frac{1-p}{p}\right) \ , \tag{9}$$

the matrix $T = (D+E)$ has the structure

$$T = \begin{pmatrix} 0 & \rho^0 & 0 & 0 & 0 & \cdots \\ \rho^0 & 0 & \rho^1 & 0 & 0 & \\ 0 & \rho^1 & 0 & \rho^2 & 0 & \\ 0 & 0 & \rho^2 & 0 & \rho^3 & \\ \vdots & & & & & \ddots \end{pmatrix} \ , \tag{10}$$

which as Richard has already pointed out, is very similar to the transfer matrix T' we have in the $p = 1$ mean-field theory:

$$T' = \begin{pmatrix} 0 & c & 0 & 0 & 0 & \cdots \\ q^1 & 0 & q^1 & 0 & 0 & \\ 0 & q^2 & 0 & q^2 & 0 & \\ 0 & 0 & q^3 & 0 & q^3 & \\ \vdots & & & & & \ddots \end{pmatrix} \ . \tag{11}$$

Similarly to how we did it for the mean-field theory, we can define the eigenvectors of T as

$$T|\phi\rangle = \mu|\phi\rangle \tag{12}$$

where

$$|\phi\rangle = \sum_{y=0}^{\infty} \phi_y |y\rangle \ . \tag{13}$$

3.1 Eigenfunction

Using the expressions above we can write down the recursion relations

$$\phi_2 = \mu\phi_0 \ , \tag{14}$$

$$\rho^{y-1}\phi_{y-1} + \rho^y \phi_{y+1} = \mu\phi_y \ , \quad y > 0 \ . \tag{15}$$

Maybe we could find a solution from this for ϕ_y (perhaps in terms of Chebyshev polynomials, but I'm going to make a continuum approximation, as we did previously.

With $p = (1/2)(1 + a/L)$, we find that $\rho \simeq 1 - 2a/L$, so we can write $\rho = 1 - \epsilon$, with $\epsilon \ll 1$ (same as $q \simeq 1 - \epsilon$ in earlier work). We make the expansion

$$\phi_{y\pm1} \simeq \phi(y) \pm \phi'(y) + \frac{1}{2}\phi''(y) + \text{ h.o.t} , \tag{16}$$

to find

$$(3 - \mu - 2y\epsilon)\phi(y) + \phi''(y) = 0 \ (+ \text{h.o.t}) , \tag{17}$$

This is essentially the same differential equation we found in our earlier mean-field theory. The boundary condition can only be satisfied with $\mu - 3 = \delta$, with $\delta \ll 1$. We map the equation onto Airy's equation, and find

$$\phi(y) = c \ \mathrm{Ai}[(y + b)(2\epsilon)^{1/3} + z_0] , \quad \mathrm{Ai}[z_0] = 0 , \tag{18}$$

where b and c are constants.

3.2 Probability Distribution

From (18) we can write

$$P(y) = \frac{\mathrm{Ai}[(y + b)(2\epsilon)^{1/3} + z_0]^2}{Z} , \tag{19}$$

where

$$Z = \int_0^\infty \mathrm{d}x \ \mathrm{Ai}[(x + b)(2\epsilon)^{1/3} + z_0]^2 . \tag{20}$$

4 Fit to data

We can rewrite $P(y)$ as

$$P(y) = \frac{(2\epsilon)^{1/3}\mathrm{Ai}[(y + b)(2\epsilon)^{1/3} + z_0]^2}{Z} , \tag{21}$$

where

$$Z = \int_{b(2\epsilon)^{1/3}+z_0}^\infty \mathrm{d}z \ \mathrm{Ai}[z]^2 . \tag{22}$$

To see how this solution compares to data, I will (not very precisely or rigorously) fit this function to some height distribution data obtained for $p = 0.5$, through the parameters Z and b. By choosing u and knowing the system size L I can set $\epsilon = 2a/L$ and set a by using $a = \ln((1 - u)/u)$.

Plotted in Figure 1 we can see that the distribution in (21) has roughly the right shape, but a good fit cannot be obtained using Z and b alone. Fitting using ϵ as well allows us to obtain quite a good fit (Figure 2).

A quick observation: in Figure 1 I used $\epsilon = 2a/L$, with $a = \ln((1 - u)/u)$, $u = 0.4$ and $L = 1024$. This gives $\epsilon = 7.92 \times 10^{-4}$, which is approximately twice the value for which the fit was obtained. (it would be nice if it turns out I got a factor of two wrong somewhere and that we can fit this with just the two parameters Z and b.)

A second quick observation: with $u = 0.4$, $L = 1024$, the corresponding value of p according to (1) is $p = 0.50020$.

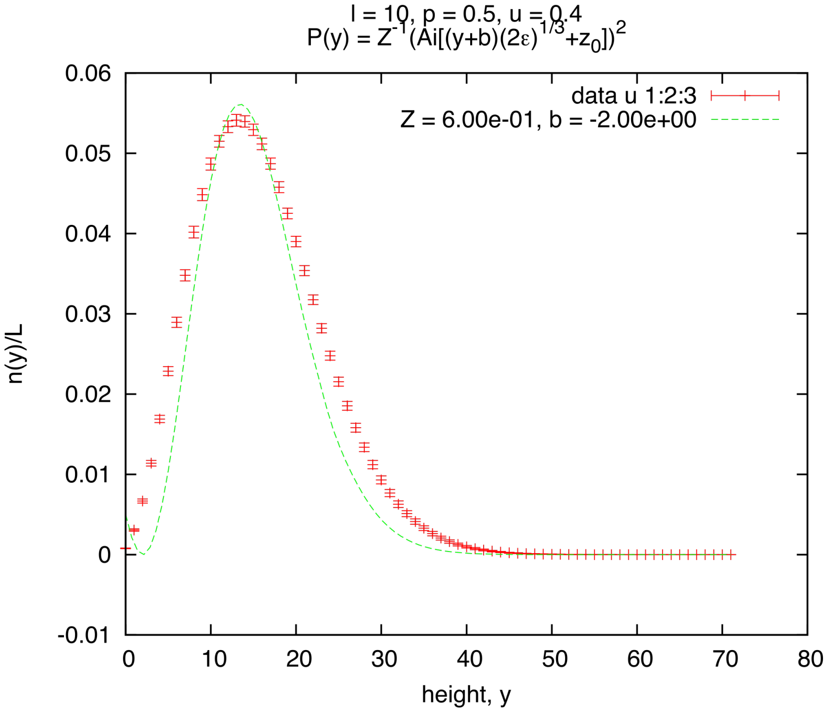

Figure 1: Attempt at a fit of the height distribution for $p = 0.5$, $u = 0.4$ using Z and b (values chosen by eye).

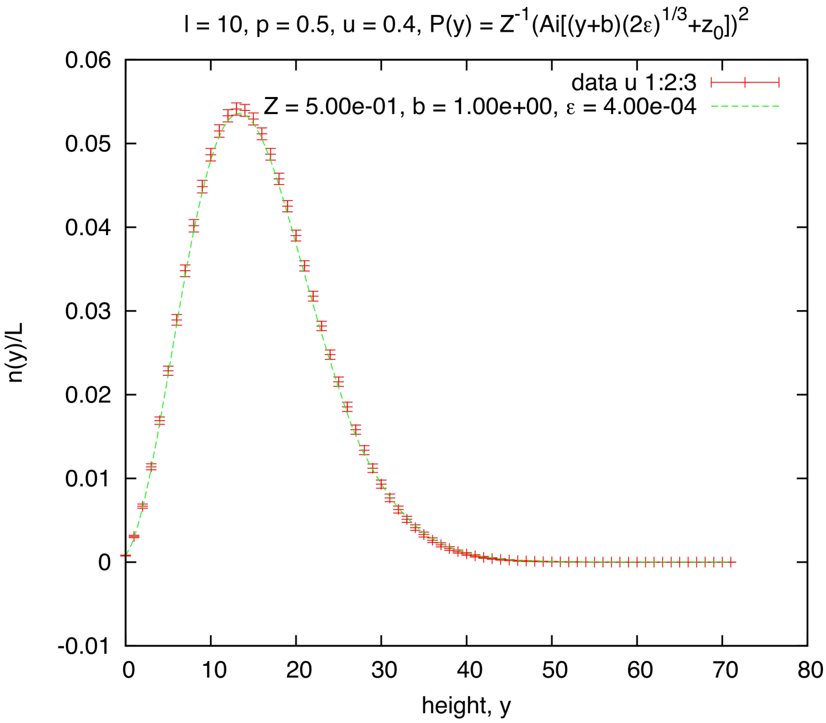

Figure 2: Attempt at a fit of the height distribution for $p = 0.5$, $u = 0.4$ using Z, b and ϵ (values chosen by eye).