Membrane-Interface: Martin's Siple Mean Field Model

1 Table of Definitions

u	probability that the membrane hops away from the interface.
$1-u$	probability that the membrane hops towards the interface.
$P(y)$	Probability that interface is distance y away from membrane.
$\mathbb{1}_{y>Y}$	indicator function, 1 if condition met, 0 otherwise.
a	$a = 4u$
b	$b = 4(1-u)[1-P(0)]^L$
$G(z)$	Generating function

2 Equations: Results, Definitions

2.1 Simple Definitions

$$a = 4u \tag{1}$$

$$b = 4(1-u)[1-P(0)]^L \tag{2}$$

In the bound phase, we require b to be finite (and non-zero). Thus, $P(0)$ must scale as $\sim 1/L$. Thus we define

$$P(0) = -\frac{\ln c}{L}\ , \quad 0 < c < 1\ , \tag{3}$$

and thus

$$b \simeq 4(1-u)c\left(1 - \frac{(\ln c)^2}{2}\frac{1}{L} + \mathcal{O}\left(\frac{1}{L}\right)\right) \tag{4}$$

2.2 Partition Function

$$G(z) = \sum_{y=1}^{\infty} z^y P(y) \tag{5}$$

$$G(z) = \frac{P(1)(1+z)z + P(0)(1+(1+b)z)}{1+(b+1)z-az^2} \tag{6}$$

$$G(z) = \frac{P(1)z^2 + (P(1)+(1+b)P(0))z + P(0)}{-a(z-z_-)(z-z_+)} \tag{7}$$

$$G(z) = \frac{Az+B}{z-z_+} \tag{8}$$

$$G(z) = -\frac{B}{z_+} - \left(A + \frac{B}{z_+}\right)\sum_{y=1}^{\infty}\left(\frac{z}{z_+}\right)^y \tag{9}$$

2.3 Roots $z_\pm$

$$z_\pm = \frac{(1+b) \pm \sqrt{(1+b)^2 + 4a}}{2a} \tag{10}$$

$$z_+ z_- = -\frac{1}{a} \tag{11}$$

$$z_+ + z_- = \frac{1+b}{a} \tag{12}$$

$$z_+ - z_- = \frac{\sqrt{(1+b)^2 + 4a}}{a} \tag{13}$$

2.4 $P(1)$ and $P(0)$

Normalisation:

$$2P(1) + (1+b)P(0) = 2 + b - a \tag{14}$$

Positivity:

$$P(1)(1+z_-)z_- + P(0)(1+(1+b)z_-) = 0 \tag{15}$$

Factorisation of the partition function:

$$-a(B - Az_-) = P(1) + (1+b)P(0)\ . \tag{16}$$

From self-consistency in expression for $P(y)$:

$$z_+(1+z_-)P(1) = P(0)\ . \tag{17}$$

2.5 Expressions for $P(y)$

2.5.1 From (9)

$y = 0$

$$P(0) = -\frac{B}{z_+} \tag{18}$$

$y > 0$:

$$P(y) = -\left(A + \frac{B}{z_+}\right)\left(\frac{z}{z_+}\right)^y \tag{19}$$

2.5.2 In terms of $P(0)$, $P(1)$

$$P(y) = \frac{P(1)}{z_+^{n-1}} \tag{20}$$

$$P(y) = \frac{1}{1+z_-}\frac{P(0)}{z_+^n}\ . \tag{21}$$

3 Mean Field Equation

$$\begin{aligned}\frac{\partial P(y)}{\partial t} &= uP(y-1)\mathbb{1}_{y>0} + \frac{1}{4}P(y+2) + (1-u)[1-P(0)]^L P(y+1) \\ &- uP(y) - \frac{1}{4}P(y)\mathbb{1}_{y>1} - (1-u)[1-P(0)]^L P(y)\mathbb{1}_{y>0}\end{aligned} \tag{22}$$

The factor $1/4$ comes from the TASEP maximal current $\rho(1-\rho)$ when the density is $1/2$. The factor $[1-P(0)]^L$ describes the probability that all sites have $y > 0$.

3.1 Steady State

$$\begin{aligned} 0 &= aP(y-1)\mathbb{1}_{y>0} + P(y+2) + bP(y+1) \\ &- aP(y) - P(y)\mathbb{1}_{y>1} - bP(y)\mathbb{1}_{y>0} \end{aligned} \tag{23}$$

where a and b are defined in (1) and (2) respectively. This gives the equation for the generating function

$$\begin{aligned} 0 &= zaG(z) + \frac{G(z)}{z^2} - \frac{P(1)}{z} - \frac{P(0)}{z^2} + \frac{bG(z)}{z} - \frac{bP(0)}{z} \\ &- (a+b+1)G(z) + P(0) + bP(0) + zP(1)\ . \end{aligned} \tag{24}$$

This can be rearranged to give

$$G(z)[az^3 - (a+b+1)z^2 + bz + 1] = (z - z^3)P(1) + (1 - bz - z^2 - bz^2)P(0)\ , \tag{25}$$

from which one can factorise out $(1-z)$ to get

$$G(z)[1 + (b+1)z - az^2] = z(1+z)P(1) + (1 + (1+b)z)P(0)\ , \tag{26}$$

which can be solved to give (6).

4 Calculations

4.1 Finding Conditions to Fix $P(0)$, $P(1)$

Starting from the generating function (6):

$$G(z) = \frac{P(1)(1+z)z + P(0)(1+(1+b)z)}{1+(b+1)z - az^2}\ ,$$

by requiring the distribution to be normalised, i.e.

$$G(1) = \sum_{y=0}^{\infty} P(y) = 1 \tag{27}$$

we find the equation (given in (14)) relating $P(0)$ and $P(1)$:

$$2P(1) + (1+b)P(0) = 2 + b - a\ .$$

Then, we notice that the denominator of $G(z)$ is quadratic in z, and thus $G(z)$ can be written as

$$G(z) = \frac{P(1)z^2 + (P(1) + (1+b)P(0))z + P(0)}{-a(z-z_-)(z-z_+)}\ .$$

Because

$$|z_-| < |z_+| \tag{28}$$

the pole at z_- is nearer the origin, and will dominate the integral

$$P(n) = \oint \frac{\mathrm{d}z}{2\pi i}\frac{G(z)}{z^n} \tag{29}$$

at large n, but because

$$z_- < 0 \tag{30}$$

this means that the distribution would oscillate between positive and negative values as n is increased. Negative values are obviously unphysical, so the term $(z - z_-)$ in the denominator must be cancelled by the numerator. Thus,

$$P(1)(1+z)z + P(0)(1+(1+b)z) = -a(Az+B)(z-z_-) \ , \tag{31}$$

and we obtain another condition (15) which relates $P(0)$ and $P(1)$:

$$P(1)(1+z_-)z_- + P(0)(1+(1+b)z_-) = 0 \ .$$

$G(z)$ can now be written as

$$G(z) = \frac{-a(z-z_-)(Az+B)}{(z-z_-)(z-z_+)} \ . \tag{32}$$

Comparing the numerator of this equation with that of (7), we find

$$P(1)z^2 + (P(1) + (1+b)P(0))z + P(0) = -aAz^2 + (az_-A - aB)z + az_-B \ . \tag{33}$$

We can then read off

$$P(0) = -aA \ , \quad A = -\frac{1}{a}P(0) = z_+z_-P(0) \ , \tag{34}$$

$$P(1) = -\frac{B}{z_+} \ , \quad B = -z_+P(0) \ , \tag{35}$$

and another condition (16) for $P(0)$ and $P(1)$:

$$-a(B - Az_-) = P(1) + (1+b)P(0) \ .$$

4.2 Finding $P(0)$, $P(1)$ in terms of a and b (and z_+ and z_-)

Using (34), (35) and (16) we find that

$$P(1) = \frac{az_+ - (1+b)}{1+z_-}P(0) \ . \tag{36}$$

Now, substitute this in to (15):

$$\begin{aligned}
\frac{2(az_+ - (1+b))}{(1+z_-)}P(0) &= (2+b)(1-P(0)) - a \\
2(az_+ - (1+b))P(0) &= (2+b-a)(1+z_-) - (2+b)(1+z_-)P(0) \\
(2az_+ - 2(1+b) + (2+b)(1+z_-))P(0) &= (2+b-a)(1+z_-) \\
(2az_+ + (2+b)z_- - b)P(0) &= (2+b-a)(1+z_-) \\
\left(-\frac{2}{z_-} + (2+b)z_- - b\right)P(0) &= (2+b-a)(1+z_-) \\
P(0) = -\frac{B}{z_+}\frac{(2+b)z_-^2 - bz_- - 2}{z_-}P(0) &= (2+b-a)(1+z_-) \\
\frac{(z_- - 1)((2+b)z_- + 2)}{z_-}P(0) &= (2+b-a)(1+z_-)
\end{aligned} \tag{37}$$

and finally you get

$$P(0) = \frac{-z_-(1+z_-)(2+b-a)}{(1-z_-)((2+b)z_- + 2)} \tag{38}$$

4.3 Finding an Expression for $P(y)$ in terms of $P(0)$ and/or $P(1)$

Going back to (8),

$$G(z) = \frac{Az + B}{z - z_+} \ ,$$

we can see that this can be rewritten as

$$\begin{aligned} G(z) &= -\frac{Az+B}{z_+}\frac{1}{1-z/z_+} \\ &= -\frac{Az+B}{z_+}\sum_{y=0}^{\infty}\left(\frac{z}{z_+}\right)^n \\ &= -\frac{B}{z_+} - \left(A + \frac{B}{z_+}\right)\sum_{y=1}^{\infty}\left(\frac{z}{z_+}\right)^y \end{aligned} \tag{39}$$

Substituting in from (34) and (35) we find

$$G(z) = P(0) + [P(0) - z_+ z_- P(1)]\sum_{y=1}^{\infty}\left(\frac{z}{z_+}\right)^y \ . \tag{40}$$

Thus, for $y > 0$, we can read off

$$P(y) = \frac{P(0) - z_+ z_- P(1)}{z_+^y} \ . \tag{41}$$

Self-consistently,

$$P(1) = \frac{P(0) - z_+ z_- P(1)}{z_+} \tag{42}$$

gives

$$z_+(1 + z_-)P(1) = P(0) \ . \tag{43}$$

Substituting back into (41) we can write $P(y)$ as either

$$P(y) = \frac{P(1)}{z_+^{n-1}} \tag{44}$$

or

$$P(y) = \frac{1}{1+z_-}\frac{P(0)}{z_+^n} \ . \tag{45}$$

Thus

$$G(z) = P(0)\left[1 + \left(\frac{1}{1+z_-}\right)\sum_{y=1}^{\infty}\left(\frac{z}{z_+}\right)^y\right] \ . \tag{46}$$

4.4 Mean separation, $\bar{y}$

The means separation $\bar{y}$ can be found from

$$\bar{y} = \left.\frac{\mathrm{d}G}{\mathrm{d}z}\right|_{z=1} = \sum_{y=1}^{\infty} yP(y) \ . \tag{47}$$

From (46)

$$\frac{\mathrm{d}G}{\mathrm{d}z} = \frac{P(0)}{1+z_-}\sum_{y=1}^{\infty} y\frac{z^{y-1}}{z_+^y} \ , \tag{48}$$

thus,

$$
\begin{aligned}
\bar{y} &= \frac{P(0)}{1+z_-}\sum_{y=1}^{\infty} y\frac{1}{z_+^y} \\
&= \frac{P(0)}{1+z_-}\sum_{y=1}^{\infty} y\zeta^y \ , \quad \zeta = z_+^{-1} \\
&= \frac{P(0)\zeta}{1+z_-}\frac{\mathrm{d}}{\mathrm{d}\zeta}\sum_{y=1}^{\infty}\zeta^y \\
&= \frac{P(0)\zeta}{1+z_-}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\sum_{y=0}^{\infty}\zeta^y - 1\right] \\
&= \frac{P(0)\zeta}{1+z_-}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{1}{1-\zeta} - 1\right] \ , \quad |\zeta| < 1 \\
&= \frac{P(0)\zeta}{1+z_-}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\zeta}{1-\zeta}\right] \\
&= \frac{P(0)\zeta}{1+z_-}\frac{(1-\zeta)+\zeta}{(1-\zeta)^2} \\
&= \frac{P(0)}{1+z_-}\frac{\zeta}{(1-\zeta)^2} \\
\bar{y} &= \frac{P(0)}{1+z_-}\frac{z_+}{(z_+-1)^2}
\end{aligned}
\tag{49}
$$

4.5 Width, W

The width W is defined as

$$W = \sqrt{\overline{y^2} - \overline{y}^2} \ . \tag{50}$$

Using

$$\frac{\mathrm{d}^2 G}{\mathrm{d}z^2} = \sum_{y=1}^{\infty} y(y-1)z^{y-2}P(y) \ , \tag{51}$$

we see that

$$\left.\frac{\mathrm{d}^2 G}{\mathrm{d}z^2}\right|_{z=1} = \overline{y^2} - \overline{y} \ . \tag{52}$$

Thus

$$W^2 = \left.\frac{\mathrm{d}^2 G}{\mathrm{d}z^2}\right|_{z=1} + \overline{y} - \overline{y}^2 \tag{53}$$

Differentiating (48) again we find

$$\frac{\mathrm{d}^2 G}{\mathrm{d}z^2} = \frac{P(0)}{1+z_-}\sum_{y=1}^{\infty} y(y-1)\frac{z^{y-2}}{z_+^y} \ . \tag{54}$$

$$
\begin{aligned}
\left.\frac{\mathrm{d}^2 G}{\mathrm{d}z^2}\right|_{z=1} &= \frac{P(0)}{1+z_-}\sum_{y=1}^{\infty} y(y-1)\zeta^y \ , \quad \zeta = z_+^{-1} \\
&= \frac{P(0)\zeta^2}{1+z_-}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\sum_{y=1}^{\infty}\zeta^y \\
&= \frac{P(0)\zeta^2}{1+z_-}\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{1}{(1-\zeta)^2} \ , \quad |\zeta| < 1 \\
&= \frac{P(0)\zeta^2}{1+z_-}\frac{2}{(1-\zeta)^3} \\
&= \frac{P(0)}{1+z_-}\frac{2z_+}{(z_+-1)^3} \\
&= \frac{2\overline{y}}{(z_+-1)} \ , \qquad (55)
\end{aligned}
$$

so

$$
\begin{aligned}
W^2 &= \overline{y}\left(\frac{2}{(z_+-1)} + 1 - \overline{y}\right) \\
&= \overline{y}\left(\frac{z_+ + 1}{z_+ - 1} - \overline{y}\right) \qquad (56)
\end{aligned}
$$

4.6 Scaling

From (43), $P(1) \propto P(0)$ and thus, from (14), we see that

$$2 + b - a \propto P(0) \propto \frac{1}{L} \ . \qquad (57)$$

This means that we can write

$$2 + b - a = \frac{d}{L} \ . \qquad (58)$$

Now, to leading order in L (which is L^0),

$$2 + b - a = 0 \ . \qquad (59)$$

We can substitute (4) in for b, giving

$$2 + 4(1-u)c - 4u = 0 \ , \qquad (60)$$

to find

$$c = \frac{2u-1}{2(1-u)} \ . \qquad (61)$$

We can now compute $z_\pm$ in terms of u, d and L:

$$
\begin{aligned}
z_\pm &= \frac{(1+b) \pm [(1+b)^2+4a]^{1/2}}{2a} \\
z_\pm &= \frac{(4u-1)+\frac{d}{L} \pm \left[(4u-1)^2+2(4u-1)\frac{d}{L}+16u+\mathcal{O}\left(\frac{1}{L^2}\right)\right]^{1/2}}{8u} \\
z_\pm &= \frac{(4u-1)+\frac{d}{L} \pm \left[(4u+1)^2+2(4u-1)\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right]^{1/2}}{8u} \\
z_\pm &= \frac{(4u-1)+\frac{d}{L} \pm (4u+1)\left[1+2\frac{(4u-1)}{(4u+1)^2}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right]^{1/2}}{8u} \\
z_\pm &= \frac{(4u-1)+\frac{d}{L} \pm (4u+1)\left[1+\frac{(4u-1)}{(4u+1)^2}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right]}{8u} \qquad (62)
\end{aligned}
$$

Now,

$$
\begin{aligned}
z_+ &= \frac{8u}{8u}+\frac{1}{8u}\left[1+\frac{(4u-1)}{(4u+1)}\right]\frac{d}{L} \\
z_+ &= 1+\frac{1}{(4u+1)}\frac{d}{L}\ , \qquad (63)
\end{aligned}
$$

and

$$
\begin{aligned}
z_- &= -\frac{2}{8u}+\frac{1}{8u}\left[1-\frac{(4u-1)}{(4u+1)}\right]\frac{d}{L} \\
z_- &= -\frac{2}{8u}+\frac{2}{8u}\left[\frac{1}{(4u+1)}\right]\frac{d}{L} \\
z_- &= -\frac{1}{4u}\left[1-\frac{1}{(4u+1)}\frac{d}{L}\right] \qquad (64)
\end{aligned}
$$

Using our expressions (38) and (3) for $P(0)$, as well as (58), we can relate c to d:

$$
-\frac{\ln c}{L} = -\frac{z_-(1+z_-)}{(1-z_-)((2+b)z_-+2)}\frac{d}{L}\ . \qquad (65)
$$

Using (63) and (64),

$$
\begin{aligned}
-z_-(1+z_-) &\simeq \frac{1}{4u}\left(1-\frac{1}{(4u+1)}\frac{d}{L}\right)\left(1-\frac{1}{4u}+\frac{1}{4u(4u+1)}\frac{d}{L}\right) \\
-z_-(1+z_-) &\simeq \frac{1}{(4u)^2}\left(1-\frac{1}{(4u+1)}\frac{d}{L}\right)\left(4u-1+\frac{1}{(4u+1)}\frac{d}{L}\right) \\
-z_-(1+z_-) &\simeq \frac{1}{(4u)^2}\left(4u-1-\frac{(4u-1)}{(4u+1)}\frac{d}{L}+\frac{1}{(4u+1)}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right) \\
-z_-(1+z_-) &\simeq \frac{1}{(4u)^2}\left(4u-1+\frac{(2-4u)}{(4u+1)}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right)\ , \qquad (66)
\end{aligned}
$$

and

$$
\begin{aligned}
(1-z_-)^{-1} &\simeq \left(1+\frac{1}{4u}-\frac{1}{4u}\frac{1}{(4u+1)}\frac{d}{L}\right)^{-1} \\
(1-z_-)^{-1} &\simeq \left(\frac{(4u+1)}{4u}-\frac{(4u+1)}{4u}\frac{1}{(4u+1)^2}\frac{d}{L}\right)^{-1} \\
(1-z_-)^{-1} &\simeq \frac{4u}{(4u+1)}\left(1-\frac{1}{(4u+1)^2}\frac{d}{L}\right)^{-1} \\
(1-z_-)^{-1} &\simeq \frac{4u}{(4u+1)}\left(1+\frac{1}{(4u+1)^2}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right) \ . \qquad (67)
\end{aligned}
$$

Also,

$$
\begin{aligned}
(2+b)z_- &\simeq \left[2+4u-2+\frac{d}{L}\right]\left[1-\frac{1}{4u+1}\frac{d}{L}\right]\left(-\frac{1}{4u}\right) \\
(2+b)z_- &\simeq \left(-\frac{1}{4u}\right)\left[4u+\left(1-\frac{4u}{(4u+1)}\right)\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right] \\
(2+b)z_- &\simeq \left(-\frac{1}{4u}\right)\left[4u+\frac{1}{(4u+1)}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right] \\
(2+b)z_- &\simeq -1-\frac{1}{4u(4u+1)}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right) \ , \qquad (68)
\end{aligned}
$$

$$
[(2+b)z_-+2] \simeq 1-\frac{1}{4u(4u+1)}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right) \ , \qquad (69)
$$

and

$$
[(2+b)z_-+2]^{-1} \simeq 1+\frac{1}{4u(4u+1)}\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right) \ . \qquad (70)
$$

Now we can substitute these in to (65):

$$
\begin{aligned}
-\frac{\ln c}{d} &\simeq \frac{1}{(4u)^2}\frac{4u}{(4u+1)}\left[4u-1+\frac{(2-4u)}{(4u+1)}\frac{d}{L}\right]\left[1+\frac{1}{(4u+1)^2}\frac{d}{L}\right]\left[1+\frac{1}{4u(4u+1)}\frac{d}{L}\right] \\
&\simeq \frac{1}{4u(4u+1)}\left[4u-1+\left(\frac{(2-4u)}{(4u+1)}+\frac{(4u-1)}{(4u+1)^2}+\frac{(4u-1)}{4u(4u+1)}\right)\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right)\right] \\
&\simeq \frac{(4u-1)}{4u(4u+1)}+k\frac{d}{L}+\mathcal{O}\left(\frac{1}{L^2}\right) \\
&\simeq \frac{(4u-1)}{4u(4u+1)}+\mathcal{O}\left(\frac{1}{L}\right) \ , \qquad (71)
\end{aligned}
$$

where

$$
k=\frac{1}{(4u+1)}\left((2-4u)+\frac{(4u-1)}{(4u+1)}+\frac{(4u-1)}{4u}\right) \ . \qquad (72)
$$

Thus

$$
\begin{aligned}
d &= -\frac{4u(4u+1)}{(4u-1)}\ln c \\
d &= \frac{4u(4u+1)}{(4u-1)}\ln\left[\frac{2(1-u)}{(2u-1)}\right] \qquad (73)
\end{aligned}
$$

4.6.1 Density of Contacts

The density of contacts is the same as $P(0)$. Using (3) and (61),

$$P(0) = \frac{1}{L} \ln \left[\frac{2(1-u)}{2u-1} \right] . \tag{74}$$

The mean number of contacts

$$\bar{C} = P(0)L = \ln \left[\frac{2(1-u)}{2u-1} \right] \tag{75}$$

4.6.2 Mean separation, $\bar{y}$

Using (49) we can calculate how $\bar{y}$ scales with L. For convenience, we define

$$D = \frac{d}{4u+1} . \tag{76}$$

We can then write

$$z_+ \simeq 1 + \frac{D}{L} , \tag{77}$$

$$1 + z_- \simeq \frac{(4u-1)}{4u} \left(1 + \frac{1}{(4u-1)} \frac{D}{L} \right) , \tag{78}$$

and

$$z_+ - 1 \simeq \frac{D}{L} \left(1 + \frac{\kappa}{D} \frac{1}{L} \right) , \tag{79}$$

where κ is the coefficient of the order L^{-2} term in the expansion of z_+ (which we may not have calculated explicitly already). Now we substitute these in to the expression for $\bar{y}$:

$$\begin{aligned}
\bar{y} &= \left(\frac{-\ln c}{L} \right) \left(1 + \frac{D}{L} \right) \left(\frac{4u}{4u-1} \right) \left(1 + \frac{1}{4u-1} \frac{D}{L} \right)^{-1} \left(\frac{L}{D} \right)^2 \left(1 + \frac{\kappa}{D} \frac{1}{L} \right)^{-2} \\
&= (-\ln c) \frac{4u}{4u-1} \frac{L}{D^2} \left(1 + \frac{D}{L} \right) \left(1 - \frac{1}{4u-1} \frac{D}{L} \right) \left(1 - \frac{2\kappa}{D} \frac{1}{L} \right) \\
\frac{\bar{y}}{L} &= (-\ln c) \frac{4u}{4u-1} \frac{1}{D^2} \left(1 + \frac{4u-2}{4u-1} \frac{D}{L} - \frac{2\kappa}{D} \frac{1}{L} + \mathcal{O}\left(\frac{1}{L^2} \right) \right) \\
&= \frac{(-\ln c)}{D^2} \frac{4u}{4u-1} \left(1 + \left[\frac{4u-2}{4u-1} D - \frac{2\kappa}{D} \right] \frac{1}{L} + \mathcal{O}\left(\frac{1}{L^2} \right) \right) ,
\end{aligned} \tag{80}$$

so to leading order in L,

$$\bar{y} = \frac{(-\ln c)}{D^2} \frac{4u}{4u-1} L ; \tag{81}$$

$\bar{y}$ scales *linearly* with L.

4.6.3 Width, W

W^2 is defined in (56) as

$$W^2 = \bar{y} \left(\frac{z_+ + 1}{z_+ - 1} - \bar{y} \right) .$$

To simplify the calculation, we write

$$\bar{y} = Y(L + \gamma) , \tag{82}$$

where

$$Y = \frac{(-\ln c)}{D^2}\frac{4u}{4u-1} \tag{83}$$

and

$$\gamma = \frac{4u-2}{4u-1}D - \frac{2\kappa}{D}\ . \tag{84}$$

Also,

$$z_+ + 1 \simeq 2 + \frac{D}{L}\ , \tag{85}$$

and $z_+ - 1$ is given up to order L^{-2} in (79). Thus

$$\begin{aligned}
W^2 &= Y(L+\gamma)\left[\left(2+\frac{D}{L}\right)\frac{L}{D}\left(1-\frac{\kappa}{D}\frac{1}{L}\right) - Y(L+\gamma)\right] \\
&= Y(L+\gamma)\left[\frac{L}{D}\left(2+\frac{D}{L}-\frac{2\kappa}{D}\frac{1}{L}\right) - YL - Y\gamma + \mathcal{O}\left(\frac{1}{L}\right)\right] \\
&= Y(L+\gamma)\left[\frac{2L}{D}+1-\frac{2\kappa}{D^2} - YL - Y\gamma + \mathcal{O}\left(\frac{1}{L}\right)\right] \\
&= Y(L+\gamma)\left[\left(\frac{2}{D}-Y\right)L + \left(1-\frac{2\kappa}{D^2}-Y\gamma\right) + \mathcal{O}\left(\frac{1}{L}\right)\right] \\
&= Y\left[\left(\frac{2}{D}-Y\right)L^2 + \left(\gamma\left(\frac{2}{D}-Y\right) + \left(1-\frac{2\kappa}{D^2}-Y\gamma\right)\right)L + \mathcal{O}\left(\frac{1}{L}\right)\right] \\
\frac{W^2}{L^2} &= Y\left[\frac{2-YD}{D} + \left(\gamma\left(\frac{2-DY}{D}\right) + \left(\frac{D^2(1-Y\gamma)-2\kappa}{D^2}\right)\right)\frac{1}{L} + \mathcal{O}\left(\frac{1}{L^2}\right)\right] \\
&= \frac{Y}{D^2}\left[D(2-YD) + \left(\gamma D(2-DY) + D^2(1-Y\gamma) - 2\kappa\right)\frac{1}{L} + \mathcal{O}\left(\frac{1}{L^2}\right)\right] \\
&= \frac{Y}{D^2}\left[D(2-YD) + \left((1-2\gamma Y)D^2 + 2\gamma D - 2\kappa\right)\frac{1}{L} + \mathcal{O}\left(\frac{1}{L^2}\right)\right]\ ;.
\end{aligned} \tag{86}$$

To leading order in L,

$$W^2 \simeq \frac{Y}{D^2}D(2-YD)L^2\ , \tag{87}$$

and

$$\begin{aligned}
W &\simeq \frac{\sqrt{YD(2-YD)}}{D}L + \frac{1}{2}\frac{(1-2\gamma Y)D^2+2\gamma D-2\kappa}{D(2-YD)} + \mathcal{O}\left(\frac{1}{L}\right) \\
&\simeq \sqrt{\frac{Y(2-YD)}{D}}L + \frac{(1-2\gamma Y)D^2+2\gamma D-2\kappa}{2D(2-YD)} + \mathcal{O}\left(\frac{1}{L}\right) \\
&\simeq \sqrt{\frac{Y(2-YD)}{D}}L\ .
\end{aligned} \tag{88}$$

From the earlier definitions,

$$\frac{Y(2-YD)}{D} = \frac{(-\ln c)}{D^3}\left(\frac{4u}{4u-1}\right)\left[2+\frac{(\ln c)}{D}\left(\frac{4u}{4u-1}\right)\right]\ . \tag{89}$$

4.6.4 Scaling Plots

We can make plots of the numerical measurements of the quantities $C(u)$, $\bar{y}$ and W to see if they scale with system size in the bound phase is the same way that the mean field theory predicts.

From Figure 1 we see that $C(u)$ is independent of system size, in agreement with the prediciton in (75).

Figure 2 shows that numerically the mean separation $\bar{y}$ seems to scale like $L^{1/2}$, whereas (81) predicts that $\bar{Y}$ scales like L.

It's difficult to see, but by looking at the data points for $L = 2^{12}$,2^{13}, Figure **??** indicates that in the bound phase the width W scales like $L^{1/2}$, whereas our result (88) predicts that W scales linearly with L.

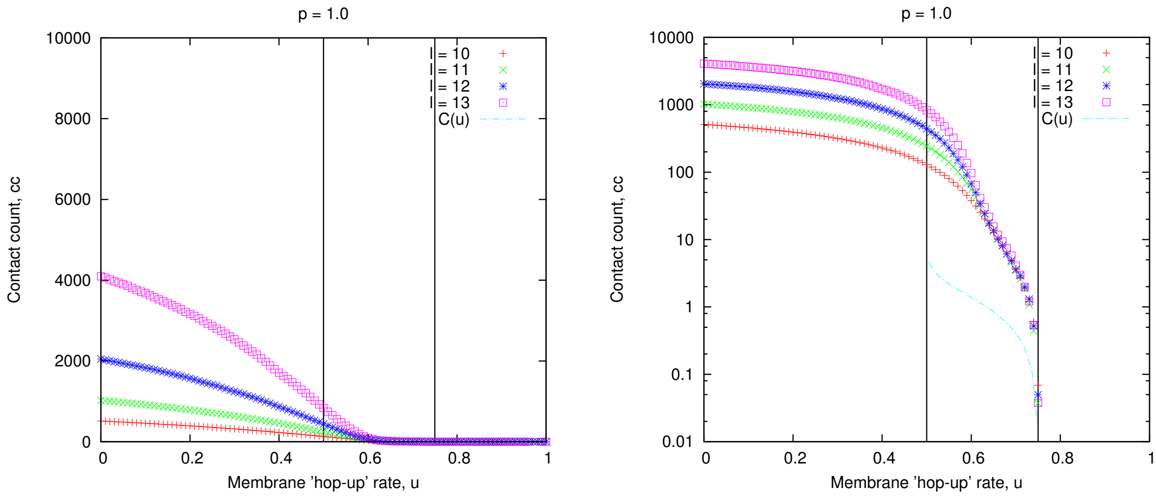

Figure 1: Contact count from simulations, "*cc*", and the analytic leading order contact count $C(u)$ from (75) plotted against u. *Left*: linear C-axis; *Right*: logarithmic C-axis. Vertical lines at $u_1 = 0.5$ and $u_2 = 0.75$.

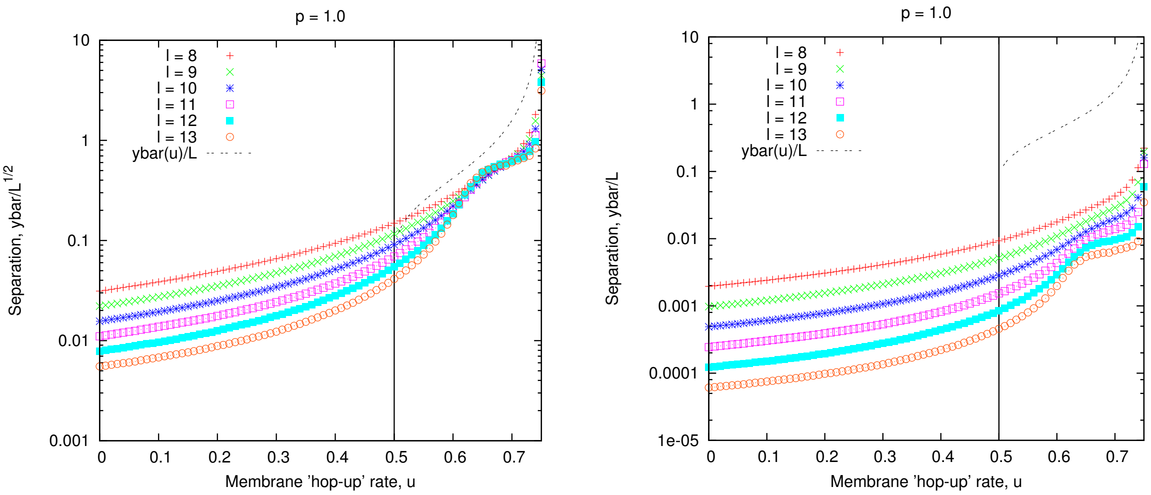

Figure 2: Plot of numerical data for $\bar{y}$ scaled by $L^{-1/2}$ (*left*) and L^{-1} (*right*). Also shown is the coefficient $Y(u)$ of the leading order (linear) dependence of $\bar{y}$ given in (83).

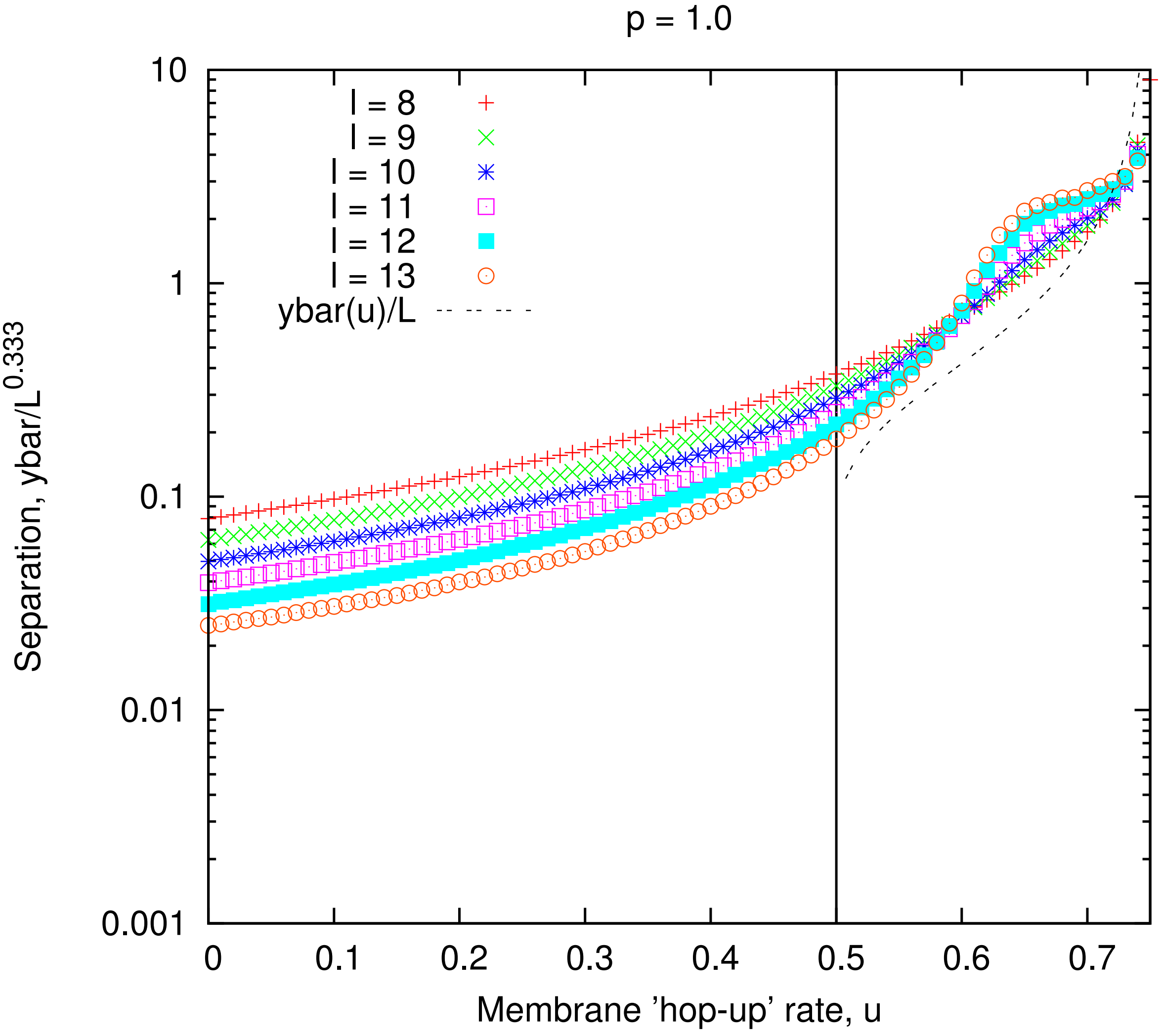

Figure 3: As Figure 2, but rescaled by $L^{0.333}$.

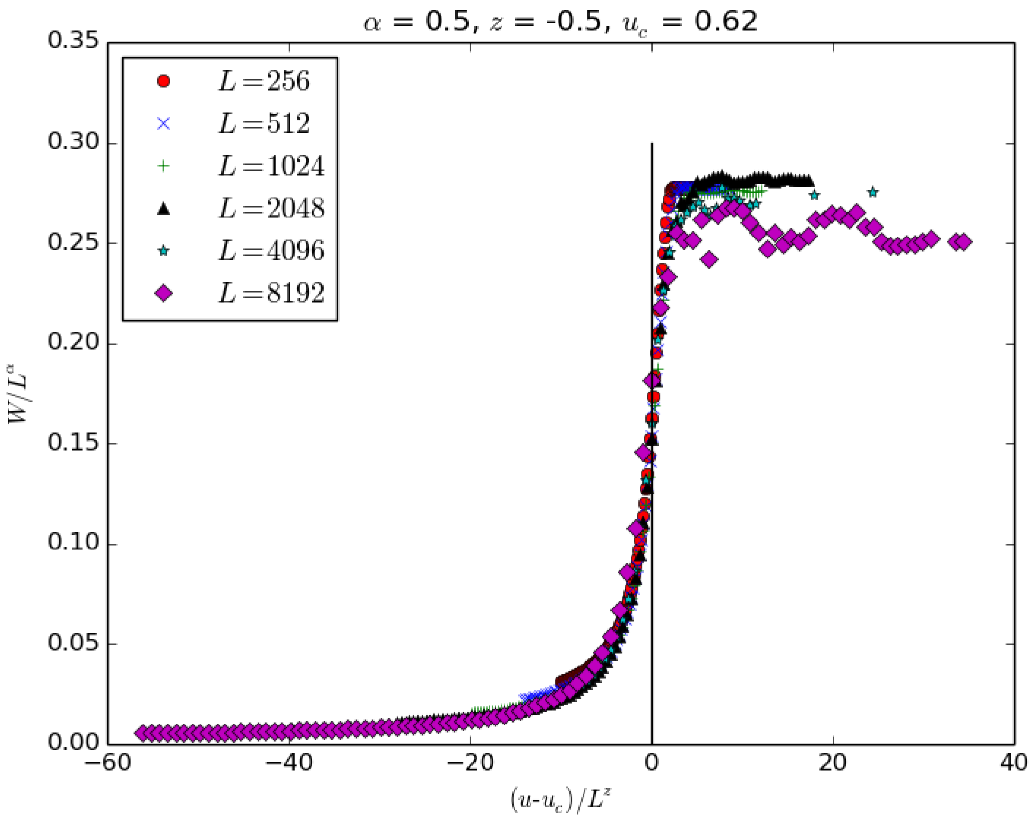

Figure 4: Plot of a collapse of the width data around an estimated value for the lower critical value $u_1 \simeq 0.62$. The width is scaled by L^α, $\alpha = 1/2$, and to obtain the collapse, $u - u_1$ (here $u_c \equiv u_1$) is scaled by L^z, $z = -1/2$.