Population Dynamics

Population Dynamics

Population dynamics is the branch of life sciences that

- studies the size (and age composition) of (human or animal) populations as dynamical systems,
- investigate the biological and environmental processes driving them (birth and death rates, and by immigration and emigration, predation, etc.).

Bug Dynamics

Problem:

The flour beetle Tribolium has been studied in a laboratory in which the biologists experimentally adjusted the adult mortality rate (number dying per unit time).

- for some values of the mortality rates, an equilibrium population resulted.
- when the mortality rate is increased beyond some value, the population was found to undergo periodic oscillations in time.
- under some conditions, the variation in population level became chaotic, i.e. with no discernible regularity or repeating pattern.

Bug Dynamics cont'd

Questions:

- Why do we get such different-looking patterns of population dynamics?
- What general mathematical model could produce these patterns?

Bug Dynamics cont'd

- To model the bugs dynamics, we shall start with a certain number of bugs that are currently alive, the present generation
- In the next generation, the change in the number of bugs due to births and deaths depends on the number of bugs in the present generation
- Births clearly arise from breeding. Deaths arise from aging, disease, starvation, and predation
- Death rates tend to increase as populations grow, due to increased competition for food and increased rates for disease
- Net change = births - deaths

Bug Dynamics cont'd

- For a given environment, the population tends not to grow beyond some maximum value, known as the carrying capacity. It's the maximum equilibrium population size of a biological species that can be sustained by the environment.

- If we can calculate the change in population, we can add it to the present population, and thus obtain the population for the next generation

- Then using the next generation's population, we can repeat the process to obtain the population of the next next generation

- If we can have a mathematical expression relating the populations of the present and the next generation, we can use the computer to predict the population at all later times

Logistic Map

Mathematical Representation

Some notations: We shall use

- n to denote the generation,
- X_n to denote the population in the nth generation, and
- ΔX_n to denote the total change that takes generation n to generation $n+1$.
- X_0 will be the starting population.

We then have

$$X_{n+1} = X_n + \Delta X_n$$

Mathematical Representation cont'd

What could be ΔX_n?

First, a crude guess: the change ΔX_n is just proportional to the present population i.e.

$$\Delta X_n = \text{constant} \times X_n$$

and we shall denote the constant ("growth rate") as $(\lambda - 1)$

This however gives rise a relatively trivial evolution:

$$X_{n+1} = X_n + (\lambda - 1)X_n = \lambda X_n,$$

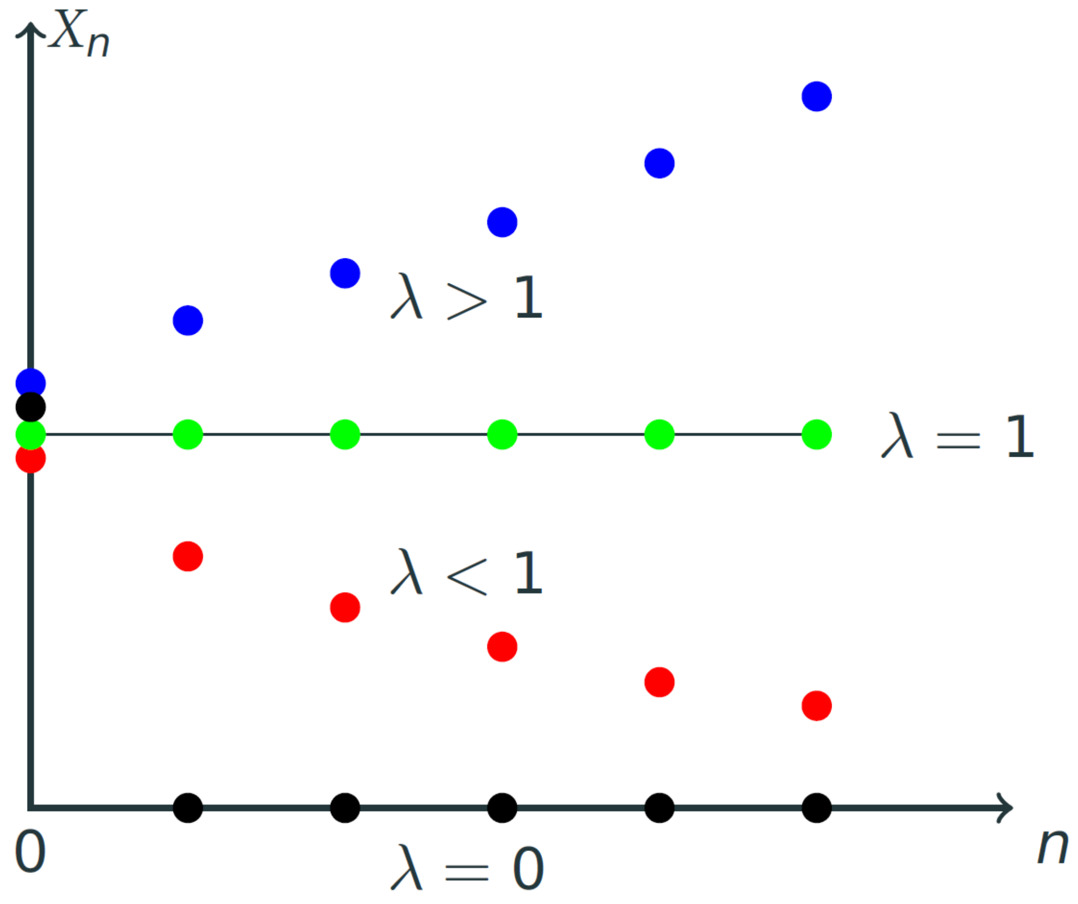
from which we can solve exactly:

$$X_{n+1} = \lambda X_n = \lambda \times (\lambda X_{n-1}) = \cdots = \lambda^{n+1} X_0$$

The Crude Guess result

$$X_{n+1} = \lambda^{n+1} X_0$$

- Clearly, if $\lambda = 0$, the population becomes, and stays at, 0
- If $\lambda = 1$, the population stays constant
- If $\lambda > 1$, the population grows indefinitely
- If $\lambda < 1$, the population eventually shrinks to zero
- There is no other possible behavior – not so good



Growth rate that depends on the present population

So we need a growth rate that slows down when the population approaches the maximum K (carrying capacity) that can be sustained

This can be achieved if we replace λ with $\Lambda(X_n)$ in the form

$$\lambda \to \Lambda(X_n) = \alpha\left(1 - \frac{X_n}{K}\right),$$

where α is kept constant, leading to

$$X_{n+1} = \alpha\left(1 - \frac{X_n}{K}\right) X_n$$

Rescaling ...

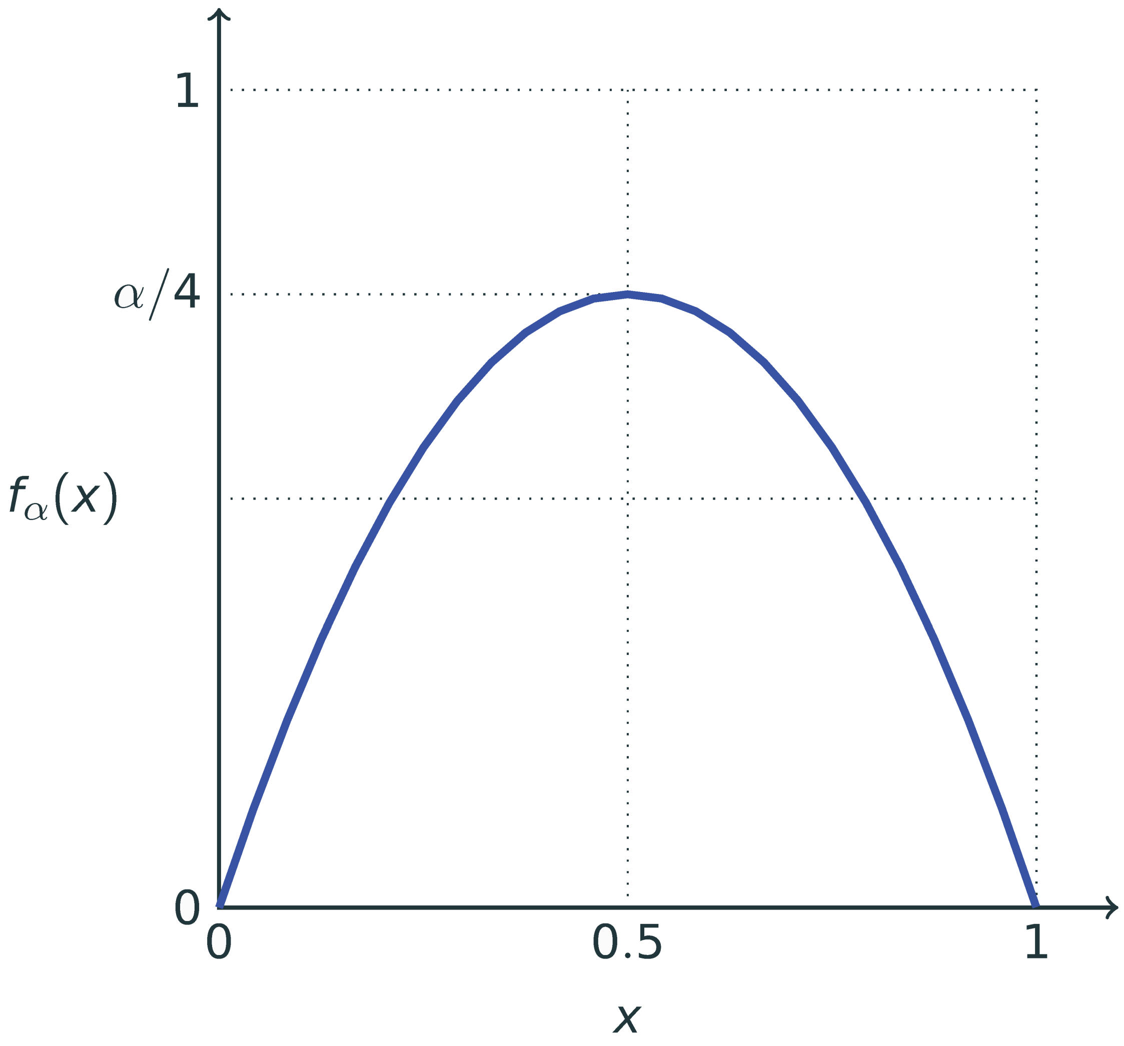
It is useful to work with the new (scaled) variable $x_n = X_n/K$

Dividing the previous equation by K, we are led to

$$x_{n+1} = \alpha(1 - x_n)x_n$$

This is called the Logistic map, and the right hand side is usually referred to as the Logistic function: $f_\alpha(x) = \alpha x(1 - x)$

https://en.wikipedia.org/wiki/Logistic_map



Does this work?

With this formulation, we would like to know if the model can give rise to (some or all of) the observational characteristics of the bug population:

- First, will there be an equilibrium population? In other words, for a given value of the system parameter α, is there a stable population that stays constant?
- Will there be oscillations in the population?
- What else can happen to the population for various values of the system parameter α?

What can be deduced already?

Equilibrium population: this corresponds to the case when there is no change in the population as time passes.

Let us denote the equilibrium population by $x^\star$. Then when the population x_n is exactly $x^\star$, then x_{n+1} is also $x^\star$. From the logistic map

$$x_{n+1} = \alpha(1 - x_n)x_n$$

we have

$$\begin{aligned} x^\star &= \alpha(1 - x^\star)x^\star \\ \Rightarrow x^\star\left(1 - \alpha + \alpha x^\star\right) &= 0 \end{aligned}$$

What possible value(s) of $x^\star$ will allow this equation to be satisfied?

Equilibrium population

$$x^\star\left(1-\alpha+\alpha x^\star\right)=0$$

- Clearly, $x^\star = 0$ (Extinction!) satisfies the equation
- The other solution gives

$$1-\alpha+\alpha x^\star = 0 \Rightarrow x^\star = \frac{\alpha-1}{\alpha}$$

- We thus have 2 possible values of $x^\star$:

$$x^\star = 0, \quad \text{and} \quad x^\star = \frac{\alpha-1}{\alpha}$$

Clearly, the second possibility exists only if $\alpha > 1$.

- The existence of a non-zero equilibrium population (at least for $\alpha > 1$) is encouraging.
- But will oscillations be possible with this model? At what values of α?
- Is there constraint on the values of the system parameter α?

Constraint on the parameter α

Logistic map: $x_{n+1} = \alpha(1 - x_n)x_n$

- $\alpha \neq 0$: Otherwise no story to tell anymore
- $\alpha > 0$: negative α leads to negative population, which is not acceptable
- $\alpha \leq 4$: Maximum value of the logistic function is $\frac{\alpha}{4}$. Thus

$$\frac{\alpha}{4} \leq 1 \Rightarrow \alpha \leq 4$$

So, we have the following constraint:

$$0 < \alpha \leq 4$$

Brute-force iteration

To find out what other possible behavior this model will lead to, we can do the following "experiment":

- Pick a value of α between 0 and 4.
- Starting with a value of x_0 (chosen arbitrarily), we can use the logistic map $x_{n+1} = \alpha(1 - x_n)x_n$ to compute the subsequent populations $x_1, x_2, x_3, \cdots$ and see if the population settles to some pattern.
- Change the α value, and repeat the process of iteration

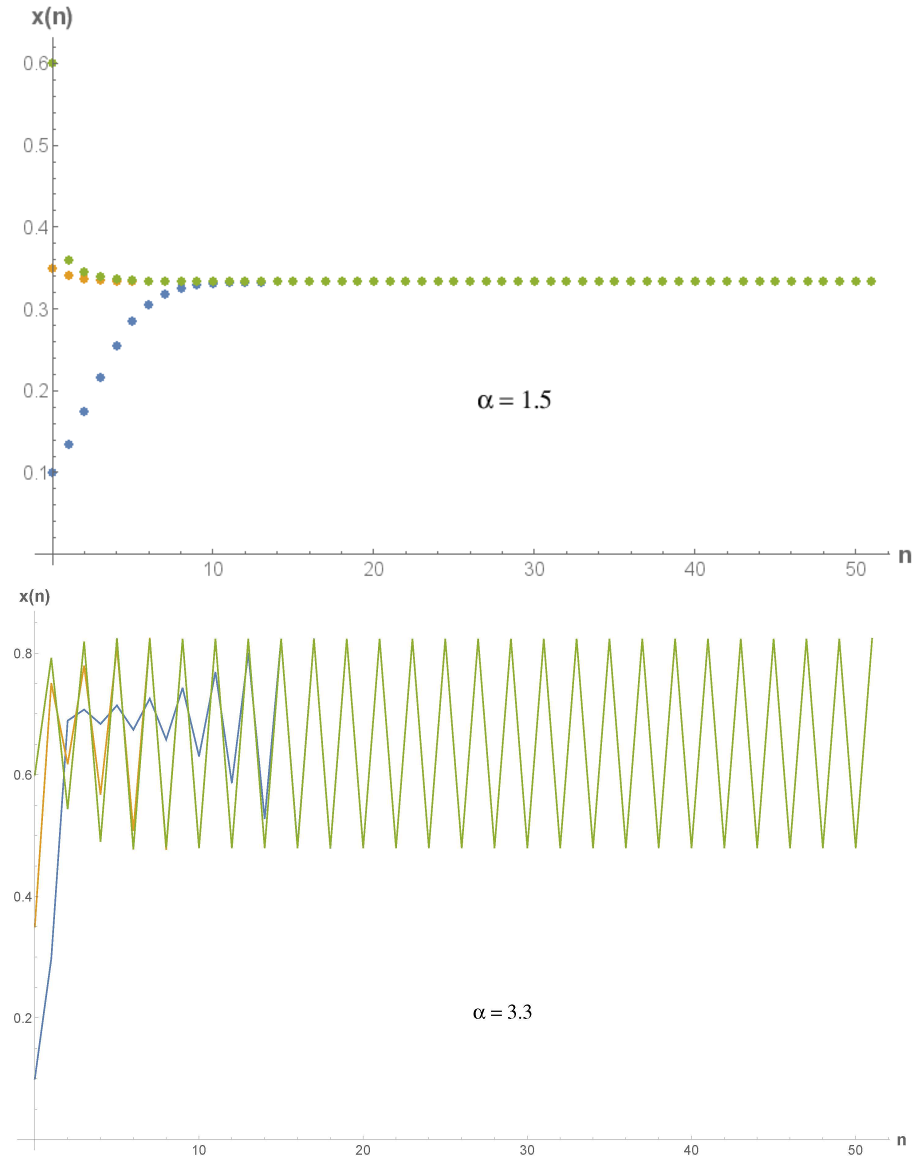
Clearly, this experiment will be so much easier if we can get the computer to carry it out.

Transients ...

- Starting from an arbitrary x_0, it takes a number of iterations before we reach equilibrium (steady state) or the periodic points – unless of course x_0 is already the equilibrium or one of the periodic points.

- So the system goes from the initial state to the steady state eventually, separated by a transient stage.

- To find out what the steady state is for a particular α and x_0, we need to ignore the transients.

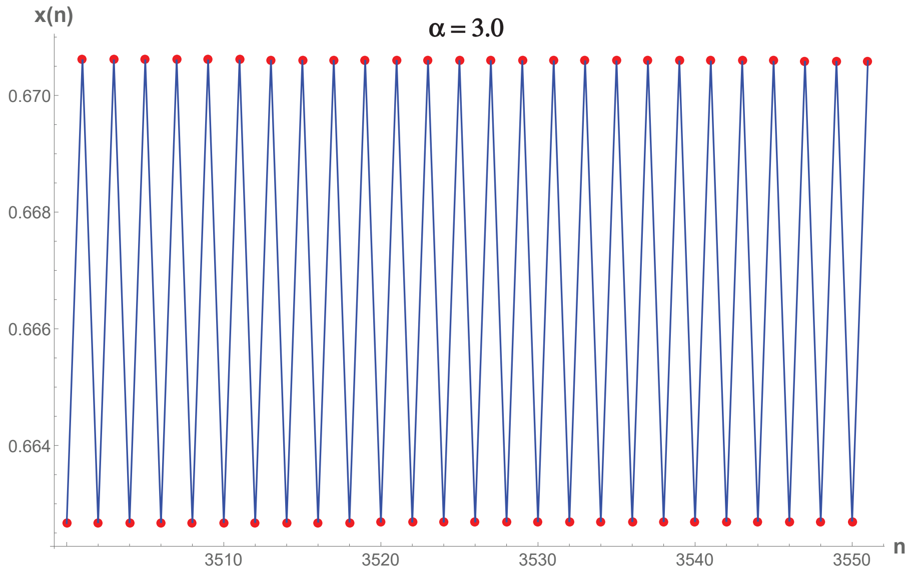


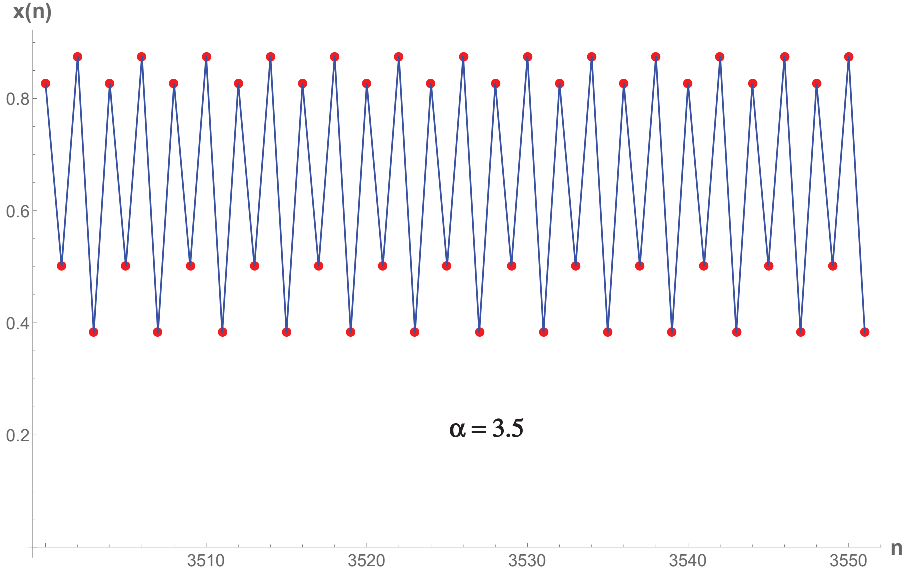
It appears that for most (but not all) values of α, the steady states/periodic points of the logistic map are independent of the initial values x_0.

Let us explore these states by making the plot of x_n vs. α:

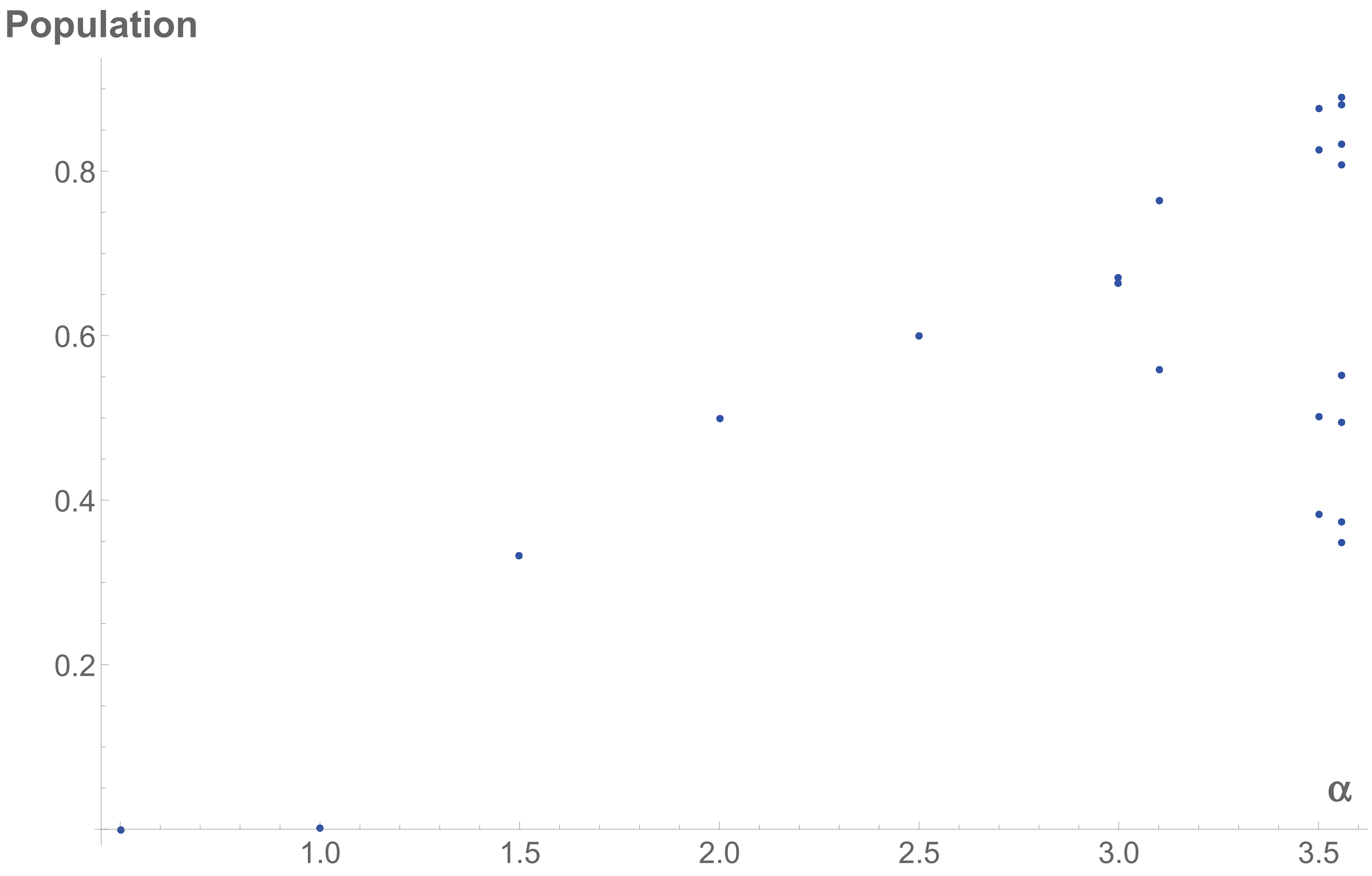
- the value of α on the horizontal axis;
- the steady state/periodic point values of x_n (for $n \gg 1$) on the vertical axis.

That is, fix a value of α, iterate the logistic map and retain only the steady states/periodic points

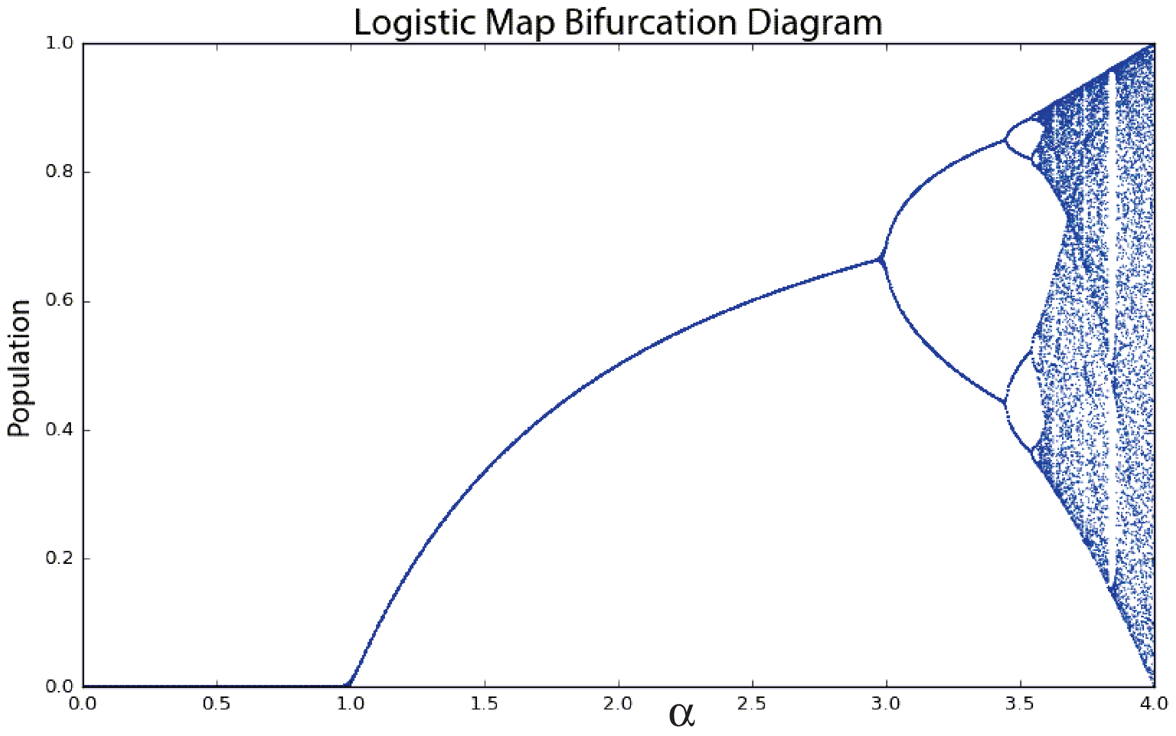


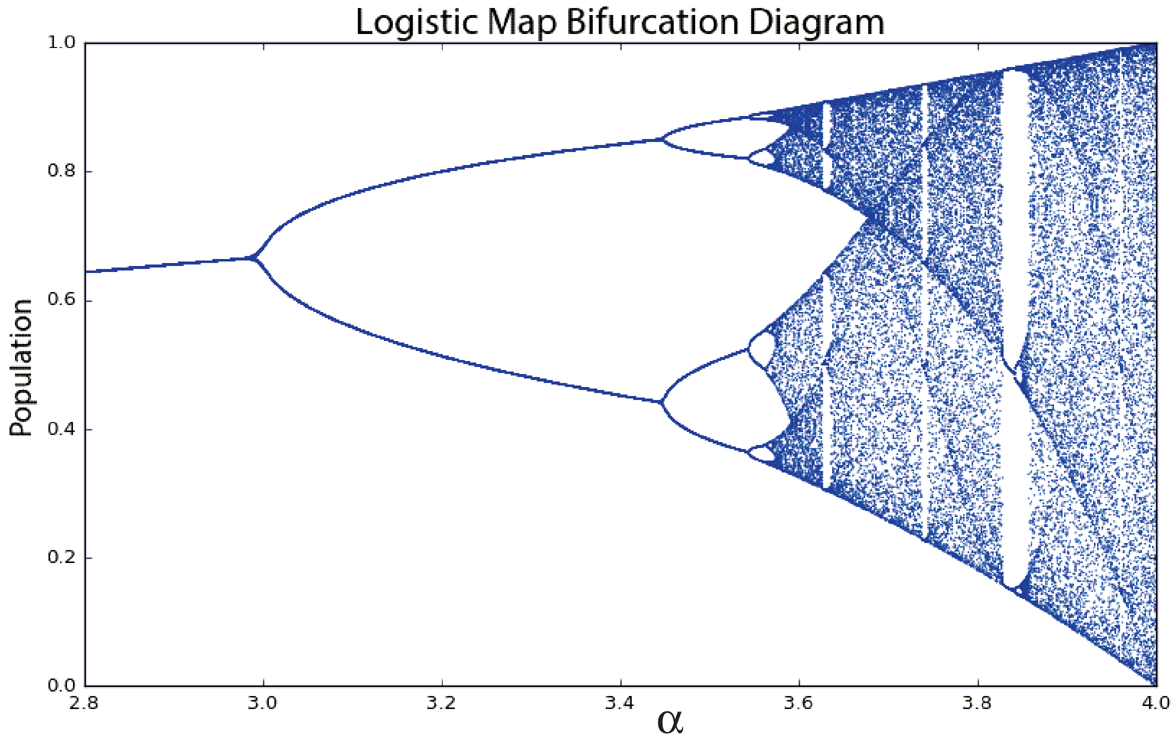


Scanning α to see behavior ...

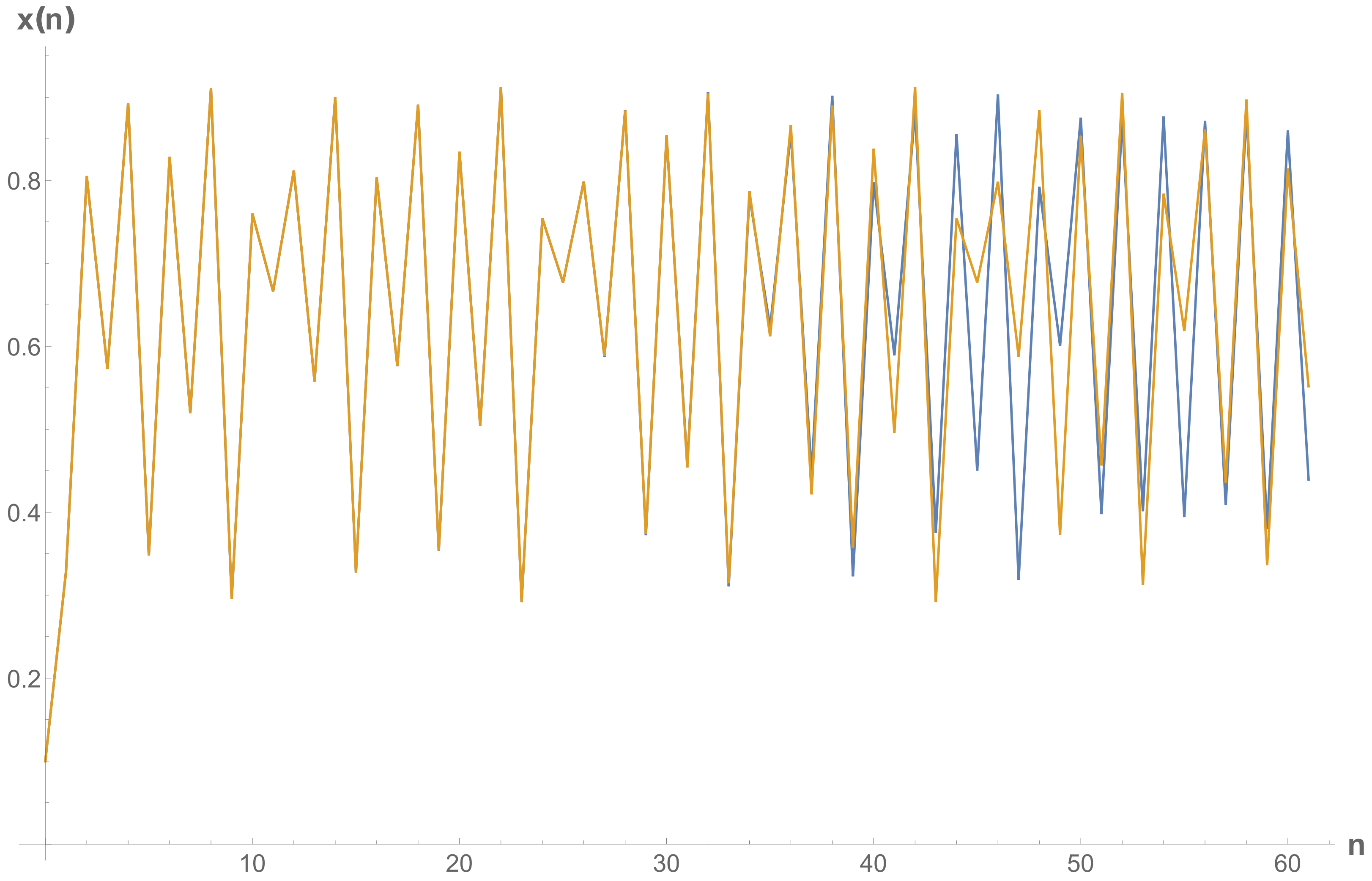


Bifurcation Diagram



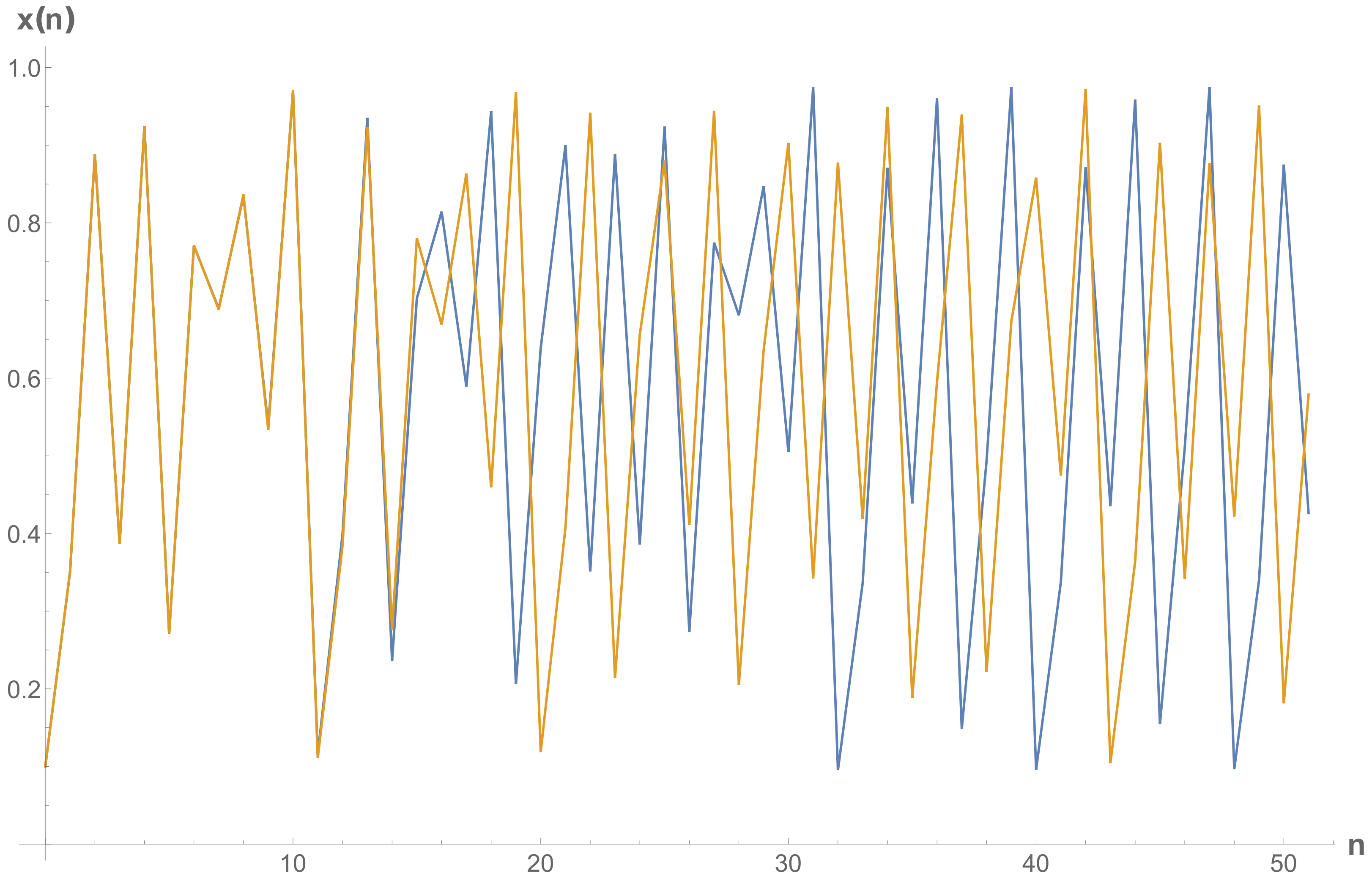


Sensitivity to system parameter α



$\alpha = 3.65$ and $\alpha = 3.65001$, $x(0) = 0.1$

Chaotic regime: Sensitivity to initial conditions



$\alpha = 3.9$, $x(0) = 0.1$ and $x(0) = 0.10001$

Behavior of the Logistic Map for varying α: A Summary

α interval	Long-term behavior
0 and 1	$x_n \to 0$ as $n \to \infty$, independent of x_0.
1 and 2	$x_n \to (\alpha - 1)/\alpha$ as $n \to \infty$, independent of x_0.
2 and 3	Still have $x_n \to (\alpha - 1)/\alpha$ as $n \to \infty$, but first will fluctuate around that value for some time that depend on the value of x_0.
3 and 3.449489...	for almost all initial conditions, x_n will approach permanent oscillations between 2 values
3.449489... and 3.54409...	for almost all initial conditions, x_n will approach permanent oscillations among 4 values.
3.54409... to $\approx$ 3.569946...	for almost all initial conditions, x_n will approach oscillations among 8 values, then 16, 32, etc.

Lessons from the bifurcation diagram

- How the system behaves in the long-time limit (i.e. for large enough n) depends on the value of the system parameter α.
- As α is varied, we have fixed point $\rightarrow$ periodic points $\rightarrow$ chaotic oscillations $\rightarrow$ periodic points $\rightarrow$ chaotic oscillations ...
- In some domain of α values, the behaviors of two systems vary drastically even when their difference in α is very small $\Rightarrow$ sensitive dependence on the system parameter.
- For some (domains of) α values, the behaviors of two systems, with the same α value, vary drastically even when their difference in the initial values is very small $\Rightarrow$ sensitive dependence on the initial values

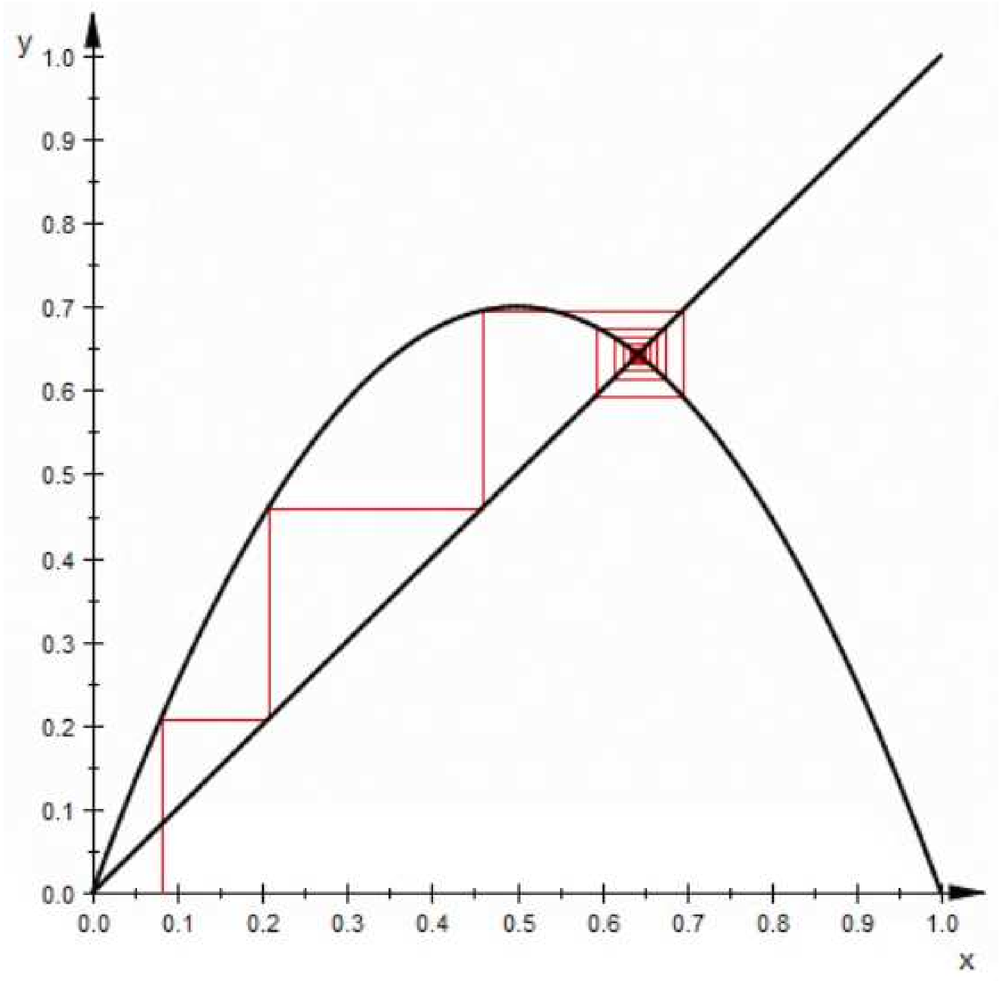
Cobweb Diagram

Visualizing function iteration via cobwebbing

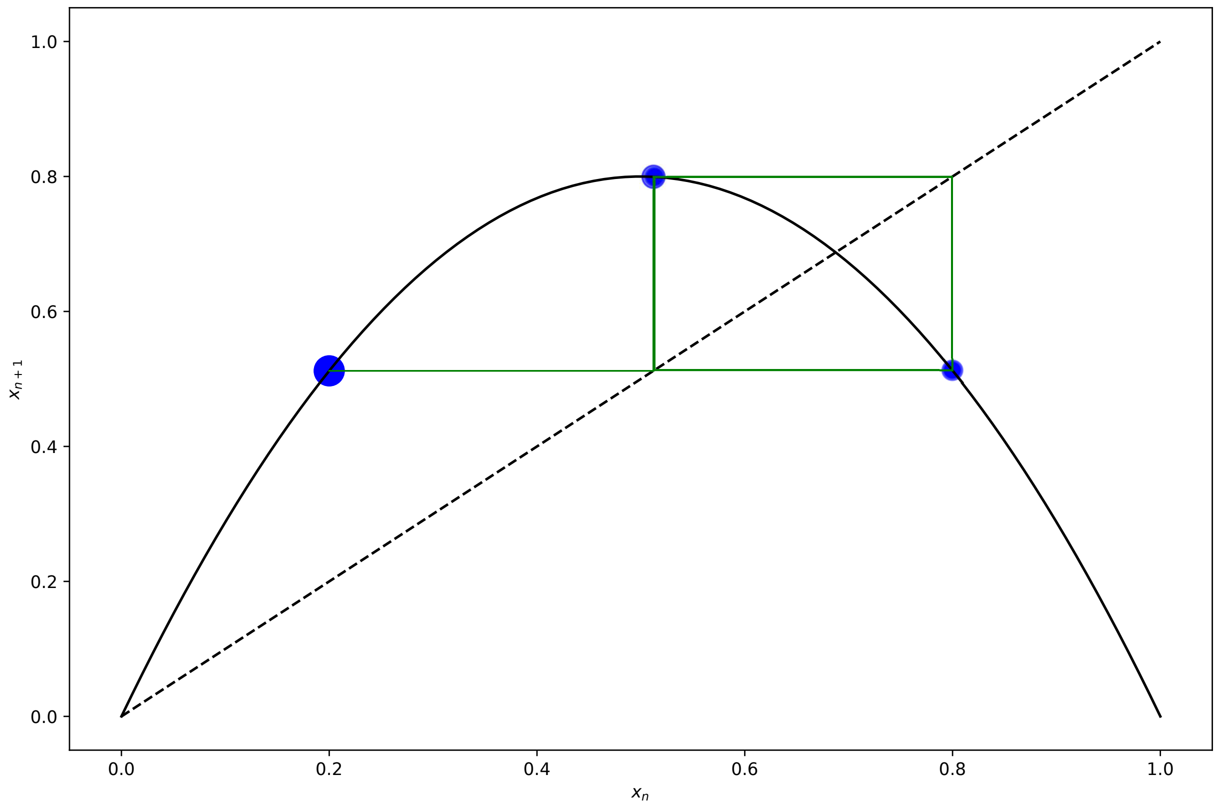
Cobwebbing is a graphical method of exploring the behavior of repeatedly applying a function $f(x)$ to an initial value $x0$.

To draw a cobweb diagram for the recursive formula $x_{n+1} = f(x_n)$, proceed as follows:

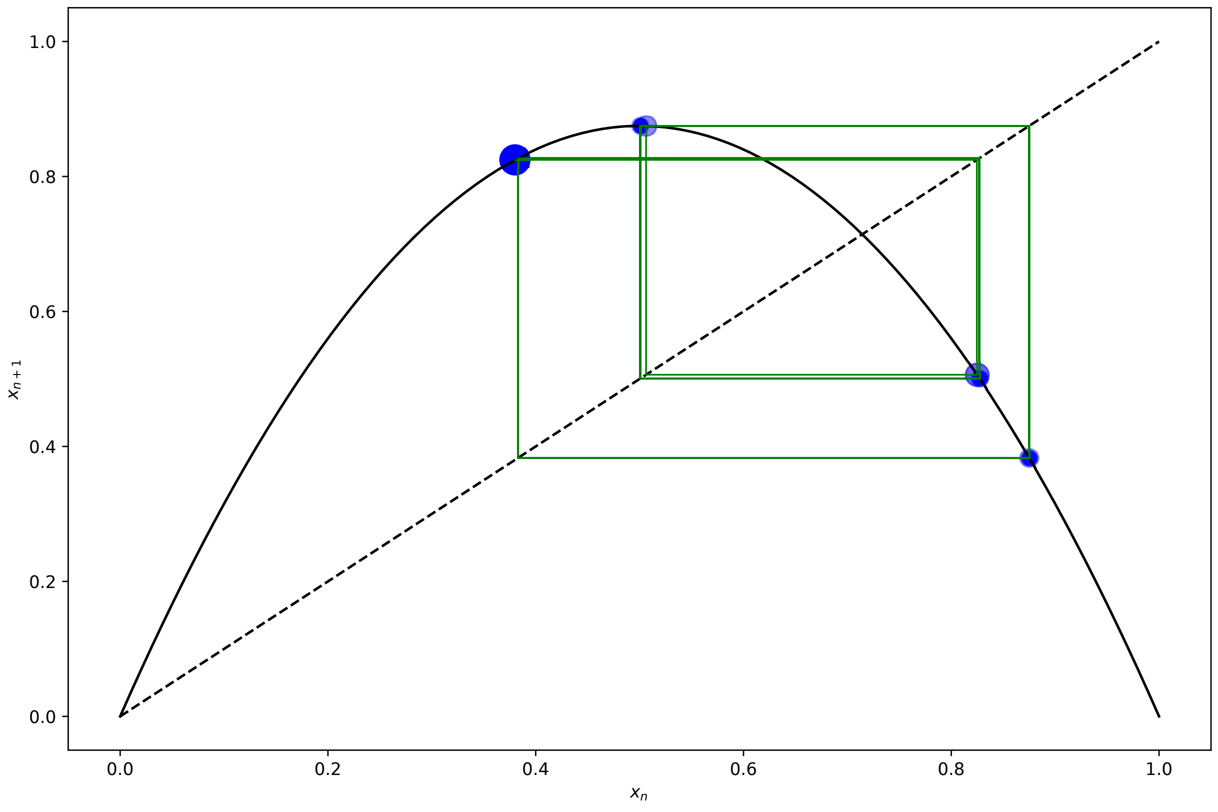
1. Plot the equations $y = f(x)$ and $y = x$
2. Start at the point $(x_0, 0)$
3. Draw a line vertically to meet the graph of $y = f(x)$
4. Draw a line horizontally to meet the graph of $y = x$
5. Repeat steps 3 to 4



Visualizing function iteration via cobwebbing cont'd



$\alpha = 3.2,\ x_0 = 0.2$



$\alpha = 3.5,\ x_0 = 0.38$