Name:______________________________ **Tutorial Group:**__________________ **(day/time)**

1 Determine whether $((p \vee q) \wedge (q \rightarrow r)) \rightarrow r$ is a tautology. [2 marks]

Solution:

It is not a tautology. Consider the following row in the truth table:

p	q	r	$p \vee q$	$q \rightarrow r$	$(p \vee q) \wedge (q \rightarrow r)$	$((p \vee q) \wedge (q \rightarrow r)) \rightarrow r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	F	F	T	T	T	F
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

2 Let $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 4\}$ and $C = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. Which of the following are true? Justify your answer.

(i) $\forall x \in C\ (x \in A) \leftrightarrow (x^2 \in B)$. [2 marks]

(ii) $\forall x \in C\ (\forall y \in B\ xy \in B) \rightarrow (x^2 = x)$. [2 marks]

(iii) $\exists x \in A\ \forall y \in A\ (x \neq 0) \wedge (xy \in B)$. [2 marks]

(iv) $\sim (\forall x \in A\ \exists y \in A\ (x \neq 0) \wedge (xy \in B))$. [2 marks]

Solution:

(i) *Case* $x = -4, -3, 3, 4$: $x \in A$ is false and $x^2 \in B$ is false, so $x \in A \leftrightarrow x^2 \in B$ is true.
Case $x = -2, -1, 0, 1, 2$: $x \in A$ is true and $x^2 \in B$ is true, so $x \in A \leftrightarrow x^2 \in B$ is true.
Therefore (i) is **true**.

(ii) *Case* $x = 0, 1$: $\forall y \in B\ xy \in B$ is true and $x^2 = x$ is true, so $(\forall y \in B\ xy \in B) \rightarrow x^2 = x$ is true.
Case $x = -4, -3, -2, -1, 2, 3, 4$: $\forall y \in B\ xy \in B$ is false (counterexample: $y = 4$),
so $(\forall y \in B\ xy \in B) \rightarrow x^2 = x$ is true.
Therefore (ii) is **true**.

(iii) *Case* $x = -2$: counterexample $y = 2$
Case $x = -1$: counterexample $y = 2$
Case $x = 0$: $x \neq 0$ is false
Case $x = 1$: counterexample $y = 2$
Case $x = 2$: counterexample $y = -1$
Thus, $\forall y \in A\ (x \neq 0) \wedge (xy \in B)$ is false for every x in A, so (iii) is **false**.

(iv) $\sim (\forall x \in A\ \exists y \in A\ (x \neq 0) \wedge (xy \in B)) \equiv \exists x \in A\ \forall y \in A\ (x = 0) \vee (xy \notin B)$ is **true**
(example: $x = 0$)

3 Consider the claim:

"For any integers m and n, if $m+n$ is even, then either both m and n are even or both are odd."

(i) State the claim symbolically, using predicates $Even(x)$ and $Odd(y)$. [2 marks]

The following is a proof:

"We prove by contradiction. Suppose one of them is odd and the other is even. Without loss of generality, we may assume m is even and n is odd. Then, $m = 2h$ and $n = 2k+1$ for some integers h and k, so $m+n = 2(h+k)+1$. Since $h+k$ is an integer, $m+n$ is therefore odd, so we get a contradiction."

(ii) Let p be the claim in (i). Why does the proof for p start by assuming that one integer is odd and the other is even? [2 marks]

(iii) Point out one example of universal instantiation in this proof. [1 mark]

(iv) Point out one example of modus ponens in this proof. [1 mark]

(v) Explain what is meant by "Without loss of generality" in this proof. [1 mark]

Solution:

(i) $\forall m \in \mathbf{Z}\ \forall n \in \mathbf{Z}\ Even(m+n) \to (Even(m) \wedge Even(n)) \vee (Odd(m) \wedge Odd(n))$

(ii) Let p be the claim in (i). Then $\sim p$ is
$\exists m \in \mathbf{Z}\ \exists n \in \mathbf{Z}\ Even(m+n) \wedge \sim ((Even(m) \wedge Even(n)) \vee (Odd(m) \wedge Odd(n)))$
where $\sim ((Even(m) \wedge Even(n)) \vee (Odd(m) \wedge Odd(n)))$
$\equiv (Odd(m) \vee Odd(n)) \wedge (Even(m) \vee Even(n))$
$\equiv ((Odd(m) \wedge (Even(m) \vee Even(n)) \vee (Odd(n) \wedge (Even(m) \vee Even(n))$
$\equiv (Odd(m) \wedge Even(n)) \vee (Odd(n) \wedge Even(m))$

(iii) Many possibilities; example:

$$\begin{array}{ll} & \forall x \in \mathbf{Z}\ Even(x) \to\ \exists h \in \mathbf{Z}\ x = 2h \\ & m \in \mathbf{Z} \\ \hline \text{U.I.} & Even(m) \to\ \exists h \in \mathbf{Z}\ m = 2h \end{array}$$

(iv) Many possibilities; example:

$$\begin{array}{ll} & Even(m) \to\ \exists h \in \mathbf{Z}\ m = 2h \\ & Even(m) \\ \hline \text{M.P.} & \exists h \in \mathbf{Z}\ m = 2h \end{array}$$

(v) There are two cases to consider: (a) m is odd and n is even; (b) n is odd and m is even. The proof for one case is the same as the proof for the other case, except for switching the roles of m and n.

4 Two sequences β and γ are said to **span** a space S over field F if and only if

"every sequence α in S can be expressed as $\alpha = b\beta + c\gamma$ for some b and c in F".

(i) State the condition (in "...") symbolically. [1 mark]

A student writes: "ψ and η span S because $\omega \in S$, $0 \in F$ and $\omega = 0\psi + 0\eta$."
(Note: $\omega \in S$, $0 \in F$ and $\omega = 0\psi + 0\eta$ are all correct.)

(ii) Explain why this argument might be wrong. [1 mark]

(iii) Explain why this argument might be correct. [1 mark]

Solution:

(i) $\forall \alpha \in S\ \exists b \in F\ \exists c \in F\ \ \alpha = b\beta + c\gamma$

(ii) If $S \neq \{w\}$, then the student must also check cases where $\alpha \in S$ and $\alpha \neq w$.

(iii) If $S = \{w\}$, then the student is correct.