National University of Singapore
Department of Computer Science
2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 5**

1. Consider the relation R from Tutorial 4 Problem 5. Let $S = R^{-1} \circ R$ and $T = S \circ S$.

   (a) Determine whether S is a total order.

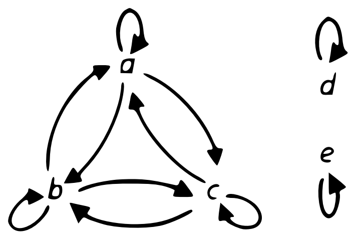
   (b) Draw an arrow diagram for T.

   (c) Why is T an equivalence relation? Determine the equivalence classes with respect to T.

Solution: Recall $S = R^{-1} \circ R = \{(a,a),(a,c),(b,b),(b,c),(c,a),(c,c),(d,d),(e,e),(c,b)\}$.
So $T = S \circ S = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\}$.

(a) S is *not* antisymmetric since $(a,c) \in S$ and $(c,a) \in S$ but $a \neq c$.
Thus S is not a partial order and, consequently, not a total order.

(b)



(c) *reflexive*: Every node in the arrow diagram above has a loop.
symmetric: Every $x \to y$ edge has an $x \leftarrow y$ counterpart.
transitive: Every $x \to y$ and $y \to z$ pair has a corresponding $x \to z$.
$[a]_T = \{a,b,c\} = [b]_T = [c]_T$; $[d]_T = \{d\}$; $[e]_T = \{e\}$.

2.* Let A and B be sets and R a relation from A to B. Prove that $R^{-1} \circ R$ is symmetric.

Solution:
Let $R \subseteq A \times B$. Suppose $(x,y) \in R^{-1} \circ R$.
Then there is $b \in B$ such that $(x,b) \in R$ and $(b,y) \in R^{-1}$.
Therefore $(b,x) \in R^{-1}$ and $(y,b) \in R$. So $(y,x) \in R^{-1} \circ R$. These prove $R^{-1} \circ R$ is symmetric.

Alternative:
Proposition 5.2.7 tells us $(R^{-1} \circ R)^{-1} = R^{-1} \circ (R^{-1})^{-1} = R^{-1} \circ R$.
So if $(x,y) \in R^{-1} \circ R$, then $(x,y) \in (R^{-1} \circ R)^{-1}$, and so $(y,x) \in R^{-1} \circ R$.
This shows $R^{-1} \circ R$ is symmetric.

3. For each relation below, determine if it is reflexive, symmetric, antisymmetric, and transitive:

(a) $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x^2 \leqslant y^2\}$, as a relation on $\mathbb{Z}$;

(b)* $\{(x,y) \in \mathbb{R} \times \mathbb{R} : xy \geqslant 0\}$, as a relation on $\mathbb{R}$;

(c)* $\{(A,B) \in \mathcal{P}(U) \times \mathcal{P}(U) : A \cap B \neq \varnothing\}$, as a relation on $\mathcal{P}(U)$, where U is a set with at least 2 elements;

(d) $\{\big((a,b),(c,d)\big) \in \mathbb{R}^2 \times \mathbb{R}^2 : (a \leqslant c) \wedge (b \leqslant d)\big\}$, as a relation on $\mathbb{R}^2$.

If a relation R above is not transitive, then give an example to show $R \circ R \not\subseteq R$. Which of the above is a partial order? Is it a total order?

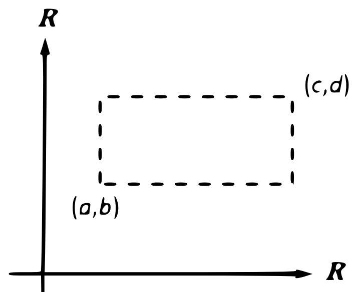
Solution:

(a) $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x^2 \leqslant y^2\}$
reflexive: $\forall x \in \mathbb{Z}\ x^2 \leqslant x^2$.
not *symmetric*: $0^2 \leqslant 1^2$ but $1^2 \nleqslant 0^2$.
not *antisymmetric*: $(-1)^2 \leqslant 1^2$ and $1^2 \leqslant (-1)^2$ but $1 \neq -1$ (so **not** a partial order).
transitive: $x^2 \leqslant y^2$ and $y^2 \leqslant z^2$ implies $x^2 \leqslant z^2$.

(b)* $S = \{(x,y) \in \mathbb{R}^2 : xy \geqslant 0\}$
reflexive: $\forall x \in \mathbb{R}\ x^2 \geqslant 0$.
symmetric: $xy \geqslant 0$ implies $yx \geqslant 0$.
not *antisymmetric*: $(-1)(-2) \geqslant 0$ and $(-2)(-1) \geqslant 0$ but $-1 \neq -2$ (so **not** a partial order).
not *transitive*: $(-1)0 \geqslant 0$ and $0(2) \geqslant 0$ but $(-1)2 \ngeqslant 0$.
$(-1,0) \in S \wedge (0,2) \in S$, implying $(-1,2) \in S \circ S$.
But $(-1,2) \notin S$. So $S \circ S \not\subseteq S$.

(c)* $R = \{(A,B) \in \mathcal{P}(U) \times \mathcal{P}(U) : A \cap B \neq \varnothing\}$
not *reflexive* (even for $U = \varnothing$): $\varnothing \cap \varnothing = \varnothing$.
symmetric: $A \cap B \neq \varnothing$ implies $B \cap A \neq \varnothing$.
not *antisymmetric*: $\{b\} \cap \{b,c\} \neq \varnothing$ and $\{b,c\} \cap \{b\} \neq \varnothing$ but $\{b\} \neq \{b,c\}$.
not *transitive*: $\{b\} \cap \{b,c\} \neq \varnothing$ and $\{b,c\} \cap \{c\} \neq \varnothing$ but $\{b\} \cap \{c\} = \varnothing$.
$(\{b\},\{b,c\}) \in R \wedge (\{b,c\},\{c\}) \in R$, implying $(\{b\},\{c\}) \in R \circ R$.
But $(\{b\},\{c\}) \notin R$. So $R \circ R \not\subseteq R$.

(d) $T = \big\{\big((a,b),(c,d)\big) \in (\mathbb{R}^2)^2 : (a \leqslant c) \wedge (b \leqslant d)\big\}$
reflexive: $\forall (x,y) \in \mathbb{R}^2\ (x \leqslant x \wedge y \leqslant y)$
not *symmetric*: $((0,0),(1,1)) \in T$ but $((1,1),(0,0)) \notin T$.
antisymmetric: $((a,b),(c,d)) \in T$ and $((c,d),(a,b)) \in T$
imply $a \leqslant c \wedge b \leqslant d$ and $c \leqslant a \wedge d \leqslant b$.
So $a = c$ and $b = d$. This means $(a,b) = (c,d)$.
transitive: $((a,b),(c,d)) \in T$ and $((c,d),(e,f)) \in T$
imply $a \leqslant c \wedge b \leqslant d$ and $c \leqslant e \wedge d \leqslant f$.
So $a \leqslant e$ and $b \leqslant f$.
This means $((a,b),(e,f)) \in T$.



T is a partial order, but **not** a total order: $((0,1),(1,0)) \notin T$ and $((1,0),(0,1)) \notin T$.

4.* Prove that the relation $S = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m^3 + n^3 \text{ is even}\}$ on $\mathbb{Z}$ from Tutorial 4 Question 6 is an equivalence relation. What are the equivalence classes?

Solution: $S = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m^3 + n^3 \text{ is even}\}$

reflexive: For any $n \in \mathbb{Z}$, $n^3 + n^3 = 2n^3$ is even.

symmetric: If $m^3 + n^3$ is even, then $n^3 + m^3 (= m^3 + n^3)$ is even.

transitive: $m^3 + n^3$ is even and $n^3 + k^3$ is even
imply $\exists r \in \mathbb{Z}\ m^3 + n^3 = 2r$ and $\exists s \in \mathbb{Z}\ n^3 + k^3 = 2s$.
Given such $r, s \in \mathbb{Z}$, we have $m^3+k^3 = 2r+2s-2n^3 = 2(r+s-n^3)$, where $r+s-n^3 \in \mathbb{Z}$.
So $m^3 + k^3$ is even.

$[0]_S = \{m \in \mathbb{Z} : m^3 + 0^3 \text{ is even}\}$, which equals the set of all even integers.
$[1]_S = \{m \in \mathbb{Z} : m^3 + 1^3 \text{ is even}\}$, which equals the set of all odd integers.

5.* Let $k \in \mathbb{Z}^+$. Define the relation $\equiv_k$ on $\mathbb{Z}$ by setting, for all $m, n \in \mathbb{Z}$,

$$m \equiv_k n \qquad \text{if and only if} \qquad k \text{ divides } m - n.$$

Prove that $\equiv_k$ is an equivalence relation. What are the equivalence classes?

Solution: $k \in \mathbb{Z}^+$ fixed, and $\forall m \in \mathbb{Z}\ \forall n \in \mathbb{Z}\ (m \equiv_k n \Leftrightarrow k \text{ divides } m - n)$.

reflexive: For any $n \in \mathbb{Z}$, $n - n = 0$, which is divisible by k because $0 = k \cdot 0$; so $n \equiv_k n$.

symmetric: Suppose $m \equiv_k n$. Then $m - n = kq$ for some $q \in \mathbb{Z}$.
For such $q \in \mathbb{Z}$, we have $n - m = -(m - n) = k(-q)$ where $-q \in \mathbb{Z}$. Thus $n \equiv_k m$.

transitive: Suppose $m \equiv_k n$ and $n \equiv_k h$. Then $m - n = kq$ and $n - h = kq'$ for some $q, q' \in \mathbb{Z}$.
For such q and q', we have $m - h = (m - n) + (n - h) = k(q + q')$, where $q + q' \in \mathbb{Z}$.
So $m \equiv_k h$.

$$[0] = \{\ldots, -2k, -k, 0, k, 2k, 3k, \ldots\}.$$
$$[1] = \{\ldots, -2k+1, -k+1, 1, k+1, 2k+1, 3k+1, \ldots\}.$$
$$[2] = \{\ldots, -2k+2, -k+2, 2, k+2, 2k+2, 3k+2, \ldots\}.$$
$$\vdots$$
$$[k-1] = \{\ldots, -2k+(k-1), -k+(k-1), (k-1), k+(k-1), 2k+(k-1), 3k+(k-1), \ldots\}.$$

6.* Let R be a binary relation on a set X, and $Y \subseteq X$. The **restriction** of R to Y, denoted $R\big|_Y$, is the relation on Y defined by $R\big|_Y = R \cap (Y \times Y)$. If R is an equivalence relation on X, then we call the partition X/R given by Theorem 6.3.10 the **partition (of X) induced by** R.

(a) Prove that, if R is an equivalence relation, then $R\big|_Y$ is an equivalence relation on Y.

(b) Let $B = \{-2, -1, 0, 1, 2, 3, 4\}$ and let S be the equivalence relation in Problem 4. How would you draw an undirected graph to represent $S\big|_B$? Determine the equivalence classes and the partition induced by $S\big|_B$.

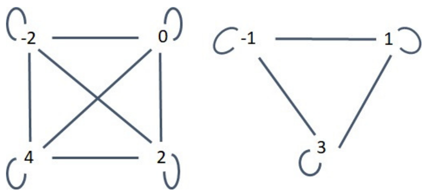
(c) Let $C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$ and let $\equiv_6$ be as in Problem 5. How would you draw an undirected graph to represent $\equiv_6 \big|_C$? Determine the equivalence classes and the partition induced by $\equiv_6\big|_C$.

[Without (a), we would have to prove all over again for (b) that $\{(m, n) \in B \times B : m^3 + n^3 \text{ is even}\}$ is an equivalence relation on B.]

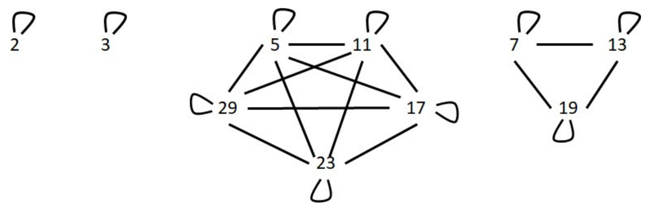
Solution: $R \subseteq X^2, \qquad Y \subseteq X, \qquad R\big|_Y = R \cap Y^2.$

(a) Suppose R is an equivalence relation.
$R\big|_Y$ *is reflexive*: Let $y \in Y$. Then $(y, y) \in R$ since $y \in X$ and R is reflexive.
So $(y, y) \in R \cap (Y \times Y) = R\big|_Y$.
$R\big|_Y$ *is symmetric*: Let $(a, b) \in R\big|_Y$. Then $(a, b) \in R$ and $(a, b) \in Y^2$.
As R is symmetric, this implies $(b, a) \in R$.
Since $(b, a) \in Y^2$, we deduce that $(b, a) \in R \cap Y^2 = R\big|_Y$.
$R\big|_Y$ *is transitive*: Suppose $(a, b) \in R\big|_Y$ and $(b, c) \in R\big|_Y$.
Then $(a, b) \in R$ and $(b, c) \in R$, and $a, b, c \in Y$.
As R is transitive, this implies $(a, c) \in R$. So $(a, c) \in R \cap Y^2 = R\big|_Y$.

(b) Relation $S = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m^3 + n^3 \text{ is even}\}$ on $\mathbb{Z}$,
$B = \{-2, -1, 0, 1, 2, 3, 4\}$.
One can represent $S\big|_B$ by the right diagram.
Equivalence classes: $\{-2, 0, 2, 4\}$ and $\{-1, 1, 3\}$
Partition: $\{\{-2, 0, 2, 4\}, \{-1, 1, 3\}\}$



(c) Relation $\equiv_6$ on $\mathbb{Z}$ defined by $m \equiv_6 n$ if and only if 6 divides $m - n$, for all $m, n \in \mathbb{Z}$, and $C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$.
One can represent the equivalence relation $\equiv_6 \big|_C$ by



Equivalence classes: $\{2\}$, $\{3\}$, $\{5, 11, 17, 23, 29\}$, $\{7, 13, 19\}$
Partition: $\{\{2\}, \{3\}, \{5, 11, 17, 23, 29\}, \{7, 13, 19\}\}$

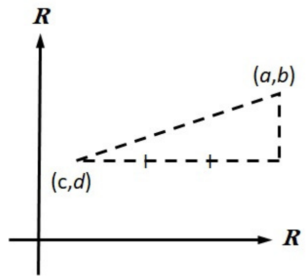
7. Consider the following relation on the set of all points in the plane:

$$\mathcal{L} = \{((a,b),(c,d)) \in \mathbb{R}^2 \times \mathbb{R}^2 : a - c = 3(b-d)\}.$$

(a) Prove that $\mathcal{L}$ is an equivalence relation.

(b) For a point (u,v) in the plane, determine the equivalence class $[(u,v)]_{\mathcal{L}}$, and represent it geometrically.

(c) Determine the partition of $\mathbb{R}^2$ induced by $\mathcal{L}$.

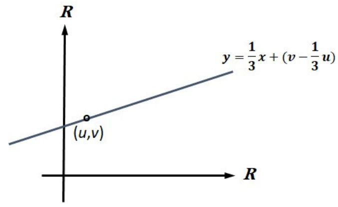


Solution: $\mathcal{L} = \{((a,b),(c,d)) \in \mathbb{R}^2 \times \mathbb{R}^2 : a - c = 3(b-d)\}$

(a) *reflexive*: For any $(a,b) \in \mathbb{R}^2$, $a - a = 3(b-b)$, so $(a,b)\ \mathcal{L}\ (a,b)$.
symmetric: If $(a,b)\ \mathcal{L}\ (c,d)$, then $a-c = 3(b-d)$, so $c - a = 3(d-b)$, making $(c,d)\ \mathcal{L}\ (a,b)$.
transitive: If $(a,b)\ \mathcal{L}\ (c,d)$ and $(c,d)\ \mathcal{L}\ (e,f)$, then $a-c = 3(b-d)$ and $c-e = 3(d-f)$, so $a-e = (a-c)+(c-e) = 3(b-d)+3(d-f) = 3(b-f)$, making $(a,b)\ \mathcal{L}\ (e,f)$.

(b)
$$\begin{aligned}[(u,v)]_{\mathcal{L}} &= \{(x,y) \in \mathbb{R}^2 : (x,y)\ \mathcal{L}\ (u,v)\} \\ &= \{(x,y) \in \mathbb{R}^2 : x - u = 3(y-v)\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : y = \frac{1}{3}x + \left(v - \frac{1}{3}u\right)\right\}.\end{aligned}$$

So $[(u,v)]_{\mathcal{L}}$ is the set of all points on the straight line passing through (u,v) with gradient $1/3$.



(c) Each equivalence class is a straight line, and it cuts the y-axis at some point, say $(0,c)$; we can use this point to represent the equivalence class.
Therefore, the partition of $\mathbb{R}^2$ induced by $\mathcal{L}$ is $\Pi_{\mathcal{L}} = \{[(0,c)]_{\mathcal{L}} : c \in \mathbb{R}\}$.
The fact that $\Pi_{\mathcal{L}}$ satisfies the definition of a partition translates to:
(a) each line in $\Pi_{\mathcal{L}}$ is a nonempty set of points, (b) the union of all the lines in $\Pi_{\mathcal{L}}$ is $\mathbb{R}^2$, and (c) any two distinct lines in $\Pi_{\mathcal{L}}$ have empty intersection.

8. Let R be an equivalence relation on set X. Prove that, for any $b, c \in X$,

$$b\ R\ c \qquad \text{if and only if} \qquad [b]_R = [c]_R.$$

Solution: $[a]_R = \{y \in X : a\ R\ y\}$

($\Rightarrow$) Assume $b\ R\ c$. Then $c \in [b]_R$.
Consider any $y \in [c]_R$. Then $c\ R\ y$. Since $b\ R\ c$, we deduce that $b\ R\ y$ by transitivity. So $y \in [c]_R$.
Thus $[b]_R \subseteq [c]_R$.
Symmetry implies $c\ R\ b$. So $[c]_R \subseteq [b]_R$ by a similar argument.
All these show $[b]_R = [c]_R$.

($\Leftarrow$) Assume $[b]_R = [c]_R$. Reflexivity implies $c \in [c]_R$ and so $c \in [b]_R$. This means $b\ R\ c$.

9. Prove or disprove:

(a) A relation that is symmetric cannot be antisymmetric.

(b) A relation that is not symmetric must be antisymmetric.

Solution:

(a) A symmetric relation cannot be antisymmetric: **false**
Example: the relation $\{(a, a), (b, b)\}$ on $\{a, b\}$ is both symmetric and antisymmetric.
Similarly for $\{(x, y) \in \mathbb{Z}^2 : x = y\}$ as a relation on $\mathbb{Z}$.

(b) A relation that is not symmetric must be antisymmetric: **false**
Example: the relation $\{(m, n) \in \mathbb{Z}^2 : m \text{ divides } n\}$ on $\mathbb{Z}$
It is not symmetric as 2 divides 6 but 6 does not divide 2.
It is not antisymmetric as 2 divides -2 and -2 divides 2, but $2 \neq -2$.
Another example: problem 3(a) above

10. (a) The following is a "proof" that every relation that is symmetric and transitive must be reflexive:

"Suppose R is symmetric and transitive. Then $x\ R\ y$ and $y\ R\ x$ for any x and y in A, because R is symmetric. Thus $x\ R\ x$ by transitivity. So R is reflexive."

What is wrong with this "proof"?

(b) Give an example of a symmetric, transitive relation that is not reflexive.

Solution:

(a) Symmetric means $\forall x \in A\ \forall y \in A\ (x\ R\ y \Rightarrow y\ R\ x)$.
Suppose $b \in A$ such that $\forall y \in A\ (b \not R\ y)$.
Then $\forall y \in A\ (b\ R\ y \Rightarrow y\ R\ b)$ is (vacuously) true, but $b\ R\ y \wedge y\ R\ b$ is false for every $y \in A$.
This falsehood means we cannot deduce $b\ R\ b$.
So R may not be reflexive (despite being symmetric and transitive).
This "proof" confuses $\forall x \in A\ \forall y \in A\ (x\ R\ y \Rightarrow y\ R\ x)$ with $\forall x \in A\ \forall y \in A\ (x\ R\ y \wedge y\ R\ x)$.

(b) Example: the relation $R = \{(0, 0)\}$ on the set $A = \{0, 1\}$
It is symmetric and transitive, but it is not reflexive since $(1, 1) \notin R$.
Another example: $R = \{(m, n) \in \mathbb{Z}^2 : mn \text{ is odd}\}$
It is symmetric and transitive, but it is not reflexive since $(2, 2) \notin R$.

11.* For a positive integer n, define $S_n = \{q \in \mathbb{Z} : \exists k \in \mathbb{Z}_{\geqslant 0}\ n = 2^k q\}$.

(a) Determine S_{7680}.

(b) Use S_n and the Well-Ordering Principle to prove that, for any $n \in \mathbb{Z}^+$, there exists an integer h and an odd integer r such that $n = 2^h r$.

Solution: $S_n = \{q \in \mathbb{Z} : \exists k \in \mathbb{Z}_{\geqslant 0}\ n = 2^k q\}$ for all $n \in \mathbb{Z}^+$.

(a) $7680 = 2^9 \times 15$. So $S_{7680} = \{15, 30, 60, 120, 240, 480, 960, 1920, 3840, 7680\}$.

(b) By definition, $S_n \subseteq \mathbb{Z}$. Since $n = 2^0 n$, we have $n \in S_n$. So $S_n \neq \varnothing$.
As 2^k is positive, if $n = 2^k q$, then q must be positive.
So $\forall q \in S_n\ q \geqslant 1$. So 1 is a lower bound for S_n.
Apply the Well-Ordering Principle to find $r \in S_n$ such that $\forall q \in S_n\ q \geqslant r$.
Since $r \in S_n$, we get $h \in \mathbb{Z}_{\geqslant 0}$ satisfying $n = 2^h r$.
It remains to prove that r is odd.
For a contradiction, suppose r is even, say $r = 2t$ where $t \in \mathbb{Z}$.
Then $n = 2^h r = 2^{h+1} t$ with $h + 1 \in \mathbb{Z}_{\geqslant 0}$. So $t \in S_n$. Therefore $r \leqslant t$ as $\forall q \in S_n\ q \geqslant r$.
Note $2^{h+1} t = n > 0$ implies $t > 0$. So $r = 2t > t \geqslant r$, which is a contradiction.

12.* Explain why the definitions in (a) and (b) below are not valid.

(a) For any real number x, define $\hat{x}$ to be the largest integer n such that $n \geqslant x$.

(b) For any real number x, define $\langle x \rangle$ to be the integer n such that $|x - n| < 1$.

(c) One can define the ceiling $\lceil x \rceil$ of a real number x to be the smallest integer in $\{n \in \mathbb{Z} : n \geqslant x\}$. Explain why this is a valid definition, i.e., why this integer always exists and is always unique.

Solution:

(a) $\hat{x}$ does not exist: there is no largest integer n such that $n \geqslant x$.

(b) $\langle x \rangle$ is ambiguous: e.g., there are two integers n (namely, 12 and 13) such that $|12.31 - n| < 1$.

(c) Let $B_x = \{n \in \mathbb{Z} : n \geqslant x\}$. Note that $B_x \subseteq \mathbb{Z}$.
It is a fact that above any real number there is an integer.
(This fact is sometimes called the Archimedean property of the real numbers.) So $B_x \neq \varnothing$.
It also tells us that $\forall n \in B_x \; n \geqslant b$ for some $b \in \mathbb{Z}$:
if $x \geqslant 0$, then we can set $b = 0$, else we can set $b = -m$ for any integer $m \geqslant -x$.
By the Well-Ordering Principle, the set B_x has a smallest element, say c.
This c is an integer because all elements of B_x are integers.
This smallest element is unique because if d is also a smallest element,
then $c \leqslant d$ and $d \leqslant c$, implying $c = d$.
Check to see that c has the properties of (the intuitively defined) $\lceil x \rceil$:
  $c \in B_x$, so $c \in \mathbb{Z}$.
  $\forall n \in B_x \; n \geqslant c$, i.e., c is the smallest integer bigger than or equal to x.
(This exercise shows that the concepts $\lfloor x \rfloor$ and $\lceil x \rceil$, which are commonly used in Computer Science, are fundamentally based on the Well-Ordering Principle.)

13.* Recall that, for all $x \in \mathbb{R}$, if $x \geqslant 0$, then $|x| = x$, else $|x| = -x$. Consider the claim:

"$|a_1 + a_2 + \cdots + a_n| \leqslant |a_1| + |a_2| + \cdots + |a_n|$ for all real numbers $a_1, \ldots, a_n$."

(This is called the *Triangle Inequality*, which is often used in Calculus, as well as in Complexity Analysis, e.g., the *Travelling Salesman Problem* remains NP-Complete even if the distances satisfy the Triangle Inequality.)

(a) The following is a "proof" of the claim.

"We will use the Second Induction Principle. Since $|a_1| \leqslant |a_1|$ for any $a_1 \in \mathbb{R}$, the claim is trivially true for $n = 1$. Suppose the claim is true for all $n < k + 1$, where $k \geqslant 1$. For any $a_1, \ldots, a_{k-1}, a_k, a_{k+1} \in \mathbb{R}$, letting $a'_k = a_k + a_{k+1}$,

$$\begin{aligned}
&|a_1 + \cdots + a_{k-1} + a_k + a_{k+1}| \\
&= |a_1 + \cdots + a_{k-1} + a'_k| \\
&\leqslant |a_1| + \cdots + |a_{k-1}| + |a'_k| && \text{by the induction hypothesis;} \\
&= |a_1| + \cdots + |a_{k-1}| + |a_k + a_{k+1}| \\
&\leqslant |a_1| + \cdots + |a_{k-1}| + |a_k| + |a_{k+1}| && \text{as } |b + c| \leqslant |b| + |c| \text{ by the induction hypothesis.}
\end{aligned}$$

So the claim is true for $n = k + 1$. By induction, the claim is true for all integers $n \geqslant 1$."

What is wrong with the "proof" above? (Note that the same "proof" can be used to show "$|a_1 + a_2 + \cdots + a_n| \geqslant |a_1| + |a_2| + \cdots + |a_n|$ for any real numbers $a_1, \ldots, a_n$", which is false.)

(b) Either fix the error in (a), or give your own proof of the claim.

Solution:

(a) The error lies in the claim $|a_k + a_{k+1}| \leqslant |a_k| + |a_{k+1}|$.
To claim that "$|b + c| \leqslant |b| + |c|$ by the induction hypothesis",
i.e., the inequality holds when there are 2 terms,
the induction hypothesis "Suppose the claim is true for all $n < k + 1$, where $k \geqslant 1$"
must cover $n = 2$, so we require $k \geqslant 2$.
The basis must therefore settle $n = 1$ and $n = 2$, instead of just $n = 1$.

(b) **Basis**: $n = 1$: $|a| \leqslant |a|$ for any $a \in \mathbb{R}$.
$n = 2$: For any $b, c \in \mathbb{R}$,

$$\begin{aligned}
&& bc &\leqslant |bc| \\
\therefore && b^2 + 2bc + c^2 &\leqslant b^2 + 2|bc| + c^2 \\
\therefore && (b + c)^2 &\leqslant (|b| + |c|)^2 \\
\therefore && |b + c|^2 &\leqslant (|b| + |c|)^2 \\
\therefore && |b + c| &\leqslant |b| + |c| && \text{since } |b + c| \geqslant 0.
\end{aligned}$$

Induction Hypothesis: Suppose the claim is true for all $n < k + 1$, where $k \geqslant 2$.
Induction Step: Consider the case $n = k + 1$.
For any $a_1, \ldots, a_{k-1}, a_k, a_{k+1} \in \mathbb{R}$, letting $a'_k = a_k + a_{k+1}$,

$$\begin{aligned}
&|a_1 + \cdots + a_{k-1} + a_k + a_{k+1}| \\
&= |a_1 + \cdots + a_{k-1} + a'_k| \\
&\leqslant |a_1| + \cdots + |a_{k-1}| + |a'_k| && \text{by the induction hypothesis;} \\
&= |a_1| + \cdots + |a_{k-1}| + |a_k + a_{k+1}| \\
&\leqslant |a_1| + \cdots + |a_{k-1}| + |a_k| + |a_{k+1}| && \text{by the induction hypothesis.}
\end{aligned}$$

So the claim is true for $n = k + 1$.

By the Second Induction Principle, the claim is true for all $n \geqslant 1$.

14. Continued from Tutorial 4 Problem 13. Prove that, when $C = 2^n$ where $n \in \mathbb{Z}^+$, there is always a solution, i.e., no matter which unit square is singled out on a $2^n \times 2^n$ chessboard, the rest can be covered by non-overlapping L-tiles.

Solution:

Proof (by induction on n):

Basis: $n = 1$.

The singled out unit square on a $2^1 \times 2^1$ chessboard corresponds to the missing square on an L-tile.

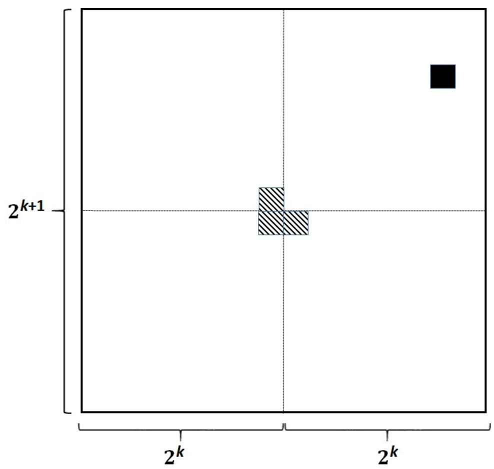
Induction Hypothesis: Assume the claim is true when $n = k$, where $k \geqslant 1$.

Induction Step:

Consider a $2^{k+1} \times 2^{k+1}$ chessboard with a unit square singled out in one quadrant.

Single out another smaller square in each of the other 3 quadrants,

so that the 3 smaller squares (together) can be covered by an L-tile, as shown in the figure below.



Now, each quadrant is a $2^k \times 2^k$ chessboard with a unit square singled out.

Applying the Induction Hypothesis to each of the 4 quadrants,

we see that the rest of the chessboard can be covered by L-tiles.