CS1231 Chapter 7

Functions

7.1 Basics

Definition 7.1.1. Let A, B be sets. A *function* or a *map* from A to B is a relation f from A to B such that any element of A is f-related to a unique element of B, i.e.,

(F1) every element of A is f-related to at least one element of B, or in symbols,

$$\forall x \in A \ \ \exists y \in B \ \ (x, y) \in f;$$

(F2) every element of A is f-related to at most one element of B, or in symbols,

$$\forall x \in A \ \ \forall y_1, y_2 \in B \ \ \big((x, y_1) \in f \wedge (x, y_2) \in f \ \Rightarrow \ y_1 = y_2\big).$$

We write $f\colon A \to B$ for "f is a function from A to B". Here A is called the *domain* of f, and B is called the *codomain* of f.

Remark 7.1.2. The negations of (F1) and (F2) can be expressed respectively as

(∼F1) $\exists x \in A \ \ \forall y \in B \ \ (x, y) \notin f$; and

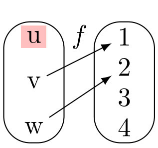
(∼F2) $\exists x \in A \ \ \exists y_1, y_2 \in B \ \ \big((x, y_1) \in f \wedge (x, y_2) \in f \wedge y_1 \neq y_2\big)$.

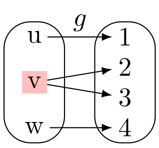
Example 7.1.3. Let $A = \{\mathrm{u}, \mathrm{v}, \mathrm{w}\}$ and $B = \{1, 2, 3, 4\}$.

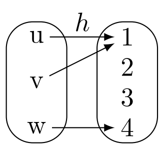
(1) $f = \{(\mathrm{v}, 1), (\mathrm{w}, 2)\}$ is *not* a function $A \to B$ because $\mathrm{u} \in A$ such that no $y \in B$ makes $(\mathrm{u}, y) \in f$, violating (F1).

(2) $g = \{(\mathrm{u}, 1), (\mathrm{v}, 2), (\mathrm{v}, 3), (\mathrm{w}, 4)\}$ is *not* a function $A \to B$ because $\mathrm{v} \in A$ and $2, 3 \in B$ such that $(\mathrm{v}, 2), (\mathrm{v}, 3) \in g$ but $2 \neq 3$, violating (F2).

(3) $h = \{(\mathrm{u}, 1), (\mathrm{v}, 1), (\mathrm{w}, 4)\}$ *is* a function $A \to B$ because both (F1) and (F2) are satisfied.



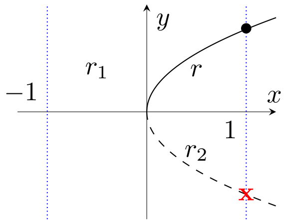




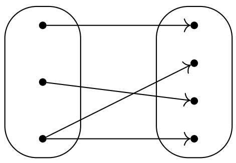
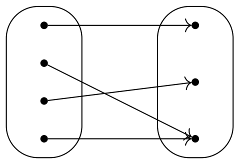
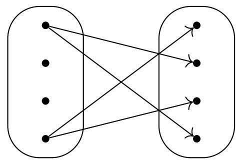
Example 7.1.4. (1) $r = \{(x, y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} : x = y^2\}$ *is* a function $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ because for every $x \in \mathbb{R}_{\geqslant 0}$, there is a unique $y \in \mathbb{R}_{\geqslant 0}$ such that $(x, y) \in r$, namely $y = \sqrt{x}$.

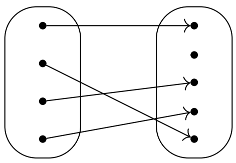
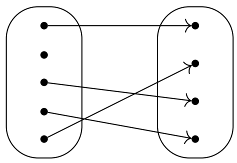
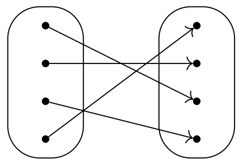
(2) $r_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_{\geqslant 0} : x = y^2\}$ is *not* a function $\mathbb{R} \to \mathbb{R}_{\geqslant 0}$ because $-1 \in \mathbb{R}$ that is not equal to y^2 for any $y \in \mathbb{R}_{\geqslant 0}$, violating (F1).

(3) $r_2 = \{(x,y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R} : x = y^2\}$ is *not* a function $\mathbb{R}_{\geqslant 0} \to \mathbb{R}$ because $1 \in \mathbb{R}_{\geqslant 0}$ and $-1, 1 \in \mathbb{R}$ such that $1 = (-1)^2$ and $1 = 1^2$ but $-1 \neq 1$, violating (F2).



Question 7.1.5. Which of the arrow diagrams below represent a function from the LHS set to the RHS set? 7a

(a)

(b)

(c)


(d)

(e)

(f)


7.2 Images

Definition 7.2.1. Let $f\colon A \to B$.

(1) If $x \in A$, then $f(x)$ denotes the unique element $y \in B$ such that $(x,y) \in f$. We call $f(x)$ the *image* of x under f.

(2) The *range* of f, denoted range(f), is defined by

$$\operatorname{range}(f) = \{f(x) : x \in A\}.$$

Remark 7.2.2. It follows from the definition of images that if $f\colon A \to B$ and $x \in A$, then for all $y \in B$,

$$(x,y) \in f \quad \Leftrightarrow \quad y = f(x).$$

Example 7.2.3. The function $r\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ in Example 7.1.4(1) satisfies

$$\forall x, y \in \mathbb{R}_{\geqslant 0} \ \ \big(y = r(x) \Leftrightarrow x = y^2\big).$$

Note that range$(r) = \mathbb{R}_{\geqslant 0}$, because for every $y \in \mathbb{R}_{\geqslant 0}$, there is $x \in \mathbb{R}_{\geqslant 0}$ such that $y = r(x)$, namely $x = y^2$.

Definition 7.2.4. A *Boolean function* is a function $\{\mathrm{T},\mathrm{F}\}^n \to \{\mathrm{T},\mathrm{F}\}$ where $n \in \mathbb{Z}^+$.

Example 7.2.5. We can view the inclusive or $\vee$ as the Boolean function $d\colon \{\mathrm{T},\mathrm{F}\}^2 \to \{\mathrm{T},\mathrm{F}\}$ satisfying, for all $p, q \in \{\mathrm{T},\mathrm{F}\}$,

$$d(p,q) = \begin{cases} \mathrm{F}, & \text{if } p = \mathrm{F} = q; \\ \mathrm{T}, & \text{otherwise.} \end{cases}$$

Note that range$(d) = \{\mathrm{T},\mathrm{F}\}$, because $d(\mathrm{T},\mathrm{T}) = \mathrm{T}$ and $d(\mathrm{F},\mathrm{F}) = \mathrm{F}$.

Proposition 7.2.6. Let $f, g\colon A \to B$. Then $f = g$ if and only if $f(x) = g(x)$ for all $x \in A$.

Proof. ($\Rightarrow$) Assume $f = g$. Let $x \in A$. Then

$$\begin{array}{lll} & (x, f(x)) \in f & \text{by the } \Leftarrow \text{ part of Remark 7.2.2.} \\ \therefore & (x, f(x)) \in g & \text{as } f = g. \\ \therefore & f(x) = g(x) & \text{by the } \Rightarrow \text{ part of Remark 7.2.2.} \end{array}$$

($\Leftarrow$) Assume $f(x) = g(x)$ for all $x \in A$. For each $x \in A$ and each $y \in B$,

$$\begin{array}{rcll} (x, y) \in f & \Leftrightarrow & y = f(x) & \text{by Remark 7.2.2;} \\ & \Leftrightarrow & y = g(x) & \text{by our assumption;} \\ & \Leftrightarrow & (x, y) \in g & \text{by Remark 7.2.2.} \end{array}$$

So $f = g$. $\square$

Example 7.2.7. The descriptions of r and d in Examples 7.2.3 and 7.2.5 in terms of $r(x)$ and $d(p, q)$ uniquely characterize these functions by Proposition 7.2.6, and can thus serve as definitions of r and d.

Example 7.2.8. Let $f\colon \{0, 2\} \to \mathbb{Z}$ and $g\colon \{0, 2\} \to \mathbb{Z}$ defined by setting, for all $x \in \{0, 2\}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2.$$

Then $f = g$ by Proposition 7.2.6, because $f(x) = g(x)$ for every $x \in \{0, 2\}$.

Example 7.2.9. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Q} \to \mathbb{Q}$ defined by

$$\forall x \in \mathbb{Z} \ \ (f(x) = x^3) \quad \text{and} \quad \forall x \in \mathbb{Q} \ \ (g(x) = x^3).$$

Then $f \neq g$ because $(1/2, 1/8)$ is an element of g but not of f.

7.3 Composition

Proposition 7.3.1. Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f$ is a function $A \to C$. Moreover, for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$

Proof. (F1) Let $x \in A$. Use (F1) for f to find $y \in B$ such that $(x, y) \in f$. Use (F1) for g to find $z \in C$ such that $(y, z) \in g$. Then $(x, z) \in g \circ f$ by the definition of $g \circ f$.

(F2) Let $x \in A$ and $z_1, z_2 \in C$ such that $(x, z_1), (x, z_2) \in g \circ f$. Use the definition of $g \circ f$ to find $y_1, y_2 \in B$ such that $(x, y_1), (x, y_2) \in f$ and $(y_1, z_1), (y_2, z_2) \in g$. Then (F2) for f implies $y_1 = y_2$. So $z_1 = z_2$ as g satisfies (F2).

These show $g \circ f$ is a function $A \to C$. Now, for every $x \in A$,

$$\begin{array}{lll} & (x, f(x)) \in f \quad \text{and} \quad \big(f(x), g(f(x))\big) \in g & \text{by the } \Leftarrow \text{ part of Remark 7.2.2;} \\ \therefore & (x, g(f(x))) \in g \circ f & \text{by the definition of } g \circ f; \\ \therefore & g(f(x)) = (g \circ f)(x) & \text{by the } \Rightarrow \text{ part of Remark 7.2.2.} \end{array}$$ $\square$

Example 7.3.2. Let $f, g\colon \mathbb{Z} \to \mathbb{Z}$ such that for every $x \in \mathbb{Z}$,

$$f(x) = 3x \quad \text{and} \quad g(x) = x + 1.$$

By Proposition 7.3.1, for every $x \in \mathbb{Z}$,

$$(g \circ f)(x) = g(f(x)) = g(3x) = 3x + 1 \quad \text{and} \quad (f \circ g)(x) = f(g(x)) = f(x + 1) = 3(x + 1).$$

Note $(g \circ f)(0) = 1 \neq 3 = (f \circ g)(0)$. So $g \circ f \neq f \circ g$ by Proposition 7.2.6.

$$\begin{tikzcd}
& x \arrow[rr, mapsto, "f"] & & 3x \arrow[dd, mapsto, "g"] \\
x \arrow[dd, mapsto, "g"'] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} \arrow[d, "g"] & \\
& \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} & 3x+1 \\
x+1 \arrow[rr, mapsto, "f"'] & & 3(x+1) &
\end{tikzcd}$$

Definition 7.3.3. Let A be a set. Then the *identity function* on A, denoted id_A, is the function $A \to A$ which satisfies, for all $x \in A$,

$$\mathrm{id}_A(x) = x.$$

Example 7.3.4. Let $f\colon A \to B$.

(1) $f \circ \mathrm{id}_A = f$ by Proposition 7.2.6, because Proposition 7.3.1 implies

- $f \circ \mathrm{id}_A$ is a function $A \to B$; and
- $(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x)$ for all $x \in A$.

(2) $\mathrm{id}_B \circ f = f$ by Proposition 7.2.6, because Proposition 7.3.1 implies

- $\mathrm{id}_B \circ f$ is a function $A \to B$; and
- $(\mathrm{id}_B \circ f)(x) = \mathrm{id}_B(f(x)) = f(x)$ for all $x \in A$.

Question 7.3.5. Which of the following define a function $f\colon \mathbb{Z} \to \mathbb{Z}$ that satisfies $f \circ f = f$? ✎ 7b

(1) $f(x) = 1231$ for all $x \in \mathbb{Z}$.

(2) $f(x) = x$ for all $x \in \mathbb{Z}$.

(3) $f(x) = -x$ for all $x \in \mathbb{Z}$.

(4) $f(x) = 3x + 1$ for all $x \in \mathbb{Z}$.

(5) $f(x) = x^2$ for all $x \in \mathbb{Z}$.

7.4 Inverse and bijectivity

Definition 7.4.1. Let $f\colon A \to B$.

(1) f is *surjective* or *onto* if

$$\forall y \in B \;\; \exists x \in A \;\; y = f(x). \tag{F^{-1}1}$$

A *surjection* is a surjective function.

(2) f is *injective* or *one-to-one* if

$$\forall x_1, x_2 \in A \;\; (f(x_1) = f(x_2) \Rightarrow x_1 = x_2). \tag{F^{-1}2}$$

An *injection* is an injective function.

Figure 7.1: Surjectivity (left) and injectivity (right)

(3) f is *bijective* if it is both surjective and injective. A *bijection* is a bijective function.

Remark 7.4.2. In view of Remark 7.2.2, one can formulate ($\mathrm{F}^{-1}1$) and ($\mathrm{F}^{-1}2$) for a general relation f from A to B as follows:

($\mathrm{F}^{-1}1$) $\forall y \in B \;\; \exists x \in A \;\; (x, y) \in f$;

($\mathrm{F}^{-1}2$) $\forall x_1, x_2 \in A \;\; \forall y \in B \;\; \big((x_1, y) \in f \wedge (x_2, y) \in f \Rightarrow x_1 = x_2\big)$.

By the definition of f^{-1}, these are equivalent respectively to (F1) and (F2) for f^{-1}, i.e.,

- $\forall y \in B \;\; \exists x \in A \;\; (y, x) \in f^{-1}$; and
- $\forall x_1, x_2 \in A \;\; \forall y \in B \;\; \big((y, x_1) \in f^{-1} \wedge (y, x_2) \in f^{-1} \Rightarrow x_1 = x_2\big)$.

So f^{-1} is a function $B \to A$ if and only if f satisfies the relational version of ($\mathrm{F}^{-1}1$) and ($\mathrm{F}^{-1}2$). Similarly, the conditions (F1) and (F2) are equivalent to ($\mathrm{F}^{-1}1$) and ($\mathrm{F}^{-1}2$) for f^{-1}.

Proposition 7.4.3. If f is a bijection $A \to B$, then f^{-1} is a bijection $B \to A$.

Proof. In view of the discussion in Remark 7.4.2, conditions (F1), (F2), ($\mathrm{F}^{-1}1$), and ($\mathrm{F}^{-1}2$) for f are equivalent respectively to conditions ($\mathrm{F}^{-1}1$), ($\mathrm{F}^{-1}2$), (F1), and (F2) for f^{-1}. □

Example 7.4.4. The function $f : \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is surjective.

Proof. Take any $y \in \mathbb{Q}$. Let $x = (y - 1)/3$. Then $x \in \mathbb{Q}$ and $f(x) = 3x + 1 = y$. □

Remark 7.4.5. A function $f : A \to B$ is *not* surjective if and only if

$$\exists y \in B \;\; \forall x \in A \;\; (y \neq f(x)).$$

Example 7.4.6. Define $g : \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not surjective.

Proof. Note $g(x) = x^2 \geqslant 0 > -1$ for all $x \in \mathbb{Z}$. So $g(x) \neq -1$ for all $x \in \mathbb{Z}$, although $-1 \in \mathbb{Z}$. □

Example 7.4.7. As in Example 7.4.4, define $f : \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Then f is injective.

Proof. Let $x_1, x_2 \in \mathbb{Q}$ such that $f(x_1) = f(x_2)$. Then $3x_1 + 1 = 3x_2 + 1$. So $x_1 = x_2$. □

Remark 7.4.8. A function $f : A \to B$ is *not* injective if and only if

$$\exists x_1, x_2 \in A \;\; (f(x_1) = f(x_2) \wedge x_1 \neq x_2).$$

Example 7.4.9. As in Example 7.4.6, define $g : \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not injective.

Proof. Note $g(1) = 1^2 = 1 = (-1)^2 = g(-1)$, although $1 \neq -1$. □

Question 7.4.10. Amongst the arrow diagrams in Question 7.1.5 that represent a function, which ones represent injections, which ones represent surjections, and which ones represent bijections? ✐ 7c

Proposition 7.4.11. Let $f\colon A \to B$ and $g\colon B \to A$. Then

$$g = f^{-1} \quad \Leftrightarrow \quad \forall x \in A \;\; \forall y \in B \;\; \big(g(y) = x \Leftrightarrow y = f(x)\big).$$

Proof.

$$\begin{aligned}
g = f^{-1} \quad &\Leftrightarrow \quad \forall y \in B \;\; \forall x \in A \;\; \big((y,x) \in g \Leftrightarrow (y,x) \in f^{-1}\big) && \text{as } g, f^{-1} \subseteq B \times A;\\
&\Leftrightarrow \quad \forall x \in A \;\; \forall y \in B \;\; \big((y,x) \in g \Leftrightarrow (x,y) \in f\big) && \text{by the definition of } f^{-1};\\
&\Leftrightarrow \quad \forall x \in A \;\; \forall y \in B \;\; \big(g(y) = x \Leftrightarrow y = f(x)\big) && \text{by Remark 7.2.2.}
\end{aligned}$$

□

Example 7.4.12. As in Example 7.4.7, define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Note that for all $x, y \in \mathbb{Q}$,

$$y = 3x + 1 \quad \Leftrightarrow \quad x = (y-1)/3.$$

Let $g\colon \mathbb{Q} \to \mathbb{Q}$ such that $g(y) = (y-1)/3$ for all $y \in \mathbb{Q}$. The equivalence above implies

$$\forall x, y \in \mathbb{Q} \;\; \big(y = f(x) \Leftrightarrow x = g(y)\big).$$

So Proposition 7.4.11 tells us $g = f^{-1}$.

Note 7.4.13. We have no guarantee of a description of the inverse of a general bijection that is much different from what is given by the definition.

Proposition 7.4.14. Let f be a bijection $A \to B$. Then $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

Proof. We know f^{-1} is a function by Proposition 7.4.3, because f is bijection.

For the first part, let $x \in A$. Define $y = f(x)$. Then

$$\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}(f(x)) && \text{by Proposition 7.3.1;}\\
&= f^{-1}(y) && \text{by the definition of } y;\\
&= x && \text{by Proposition 7.4.11, as } y = f(x);\\
&= \mathrm{id}_A(x) && \text{by the definition of } \mathrm{id}_A.
\end{aligned}$$

So $f^{-1} \circ f = \mathrm{id}_A$ by Proposition 7.2.6.

For the second part, let $y \in B$. Define $x = f^{-1}(y)$. Then

$$\begin{aligned}
(f \circ f^{-1})(y) &= f(f^{-1}(y)) && \text{by Proposition 7.3.1;}\\
&= f(x) && \text{by the definition of } x;\\
&= y && \text{by Proposition 7.4.11, as } f^{-1}(y) = x;\\
&= \mathrm{id}_B(y) && \text{by the definition of } \mathrm{id}_B.
\end{aligned}$$

So $f \circ f^{-1} = \mathrm{id}_B$ by Proposition 7.2.6. □