Answers to selected exercises

4a, page 2

(1) We want to prove that $E = \mathbb{Z}^+$, where $E = \{x + 1 : x \in \mathbb{Z}_{\geqslant 0}\}$.

Proof. ($\Rightarrow$) Let $z \in E$. Use the definition of E to find $x \in \mathbb{Z}_{\geqslant 0}$ such that $x + 1 = z$. Then $x \in \mathbb{Z}$ and $x \geqslant 0$ by the definition of $\mathbb{Z}_{\geqslant 0}$. As $x \in \mathbb{Z}$, we know $x + 1 \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under $+$. As $x \geqslant 0$, we know $x + 1 \geqslant 0 + 1 = 1 > 0$. So $z = x + 1 \in \mathbb{Z}^+$ by the definition of $\mathbb{Z}^+$.

($\Leftarrow$) Let $z \in \mathbb{Z}^+$. Then $z \in \mathbb{Z}$ and $z > 0$. Define $x = z - 1$. As $z \in \mathbb{Z}$, we know $x \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under $-$. As $z > 0$, we know $x = z - 1 > 0 - 1 = -1$, and thus $x \geqslant 0$ as $x \in \mathbb{Z}$. So $x \in \mathbb{Z}_{\geqslant 0}$ by the definition of $\mathbb{Z}_{\geqslant 0}$. Hence the definition of E tells us $z = x + 1 \in E$. $\square$

(2) We want to prove that $F = \mathbb{Z}$, where $F = \{x - y : x, y \in \mathbb{Z}_{\geqslant 0}\}$.

Proof. ($\Rightarrow$) Let $z \in F$. Use the definition of F to find $x, y \in \mathbb{Z}_{\geqslant 0}$ such that $x - y = z$. Then $x, y \in \mathbb{Z}$ by the definition of $\mathbb{Z}_{\geqslant 0}$. So $z = x - y \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under $-$.

($\Leftarrow$) Let $z \in \mathbb{Z}$.

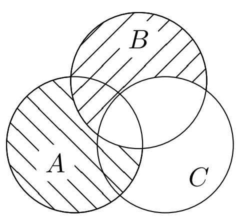
- Case 1: suppose $z \geqslant 0$. Let $x = z$ and $y = 0$. Then $x, y \in \mathbb{Z}_{\geqslant 0}$. So $z = z - 0 = x - y \in F$ by the definition of F.
- Case 2: suppose $z < 0$. Let $x = 0$ and $y = -z$. Then $x, y \in \mathbb{Z}_{\geqslant 0}$ as $z < 0$. So $z = 0 - (-z) = x - y \in F$ by the definition of F.

So $z \in F$ in all the cases. $\square$

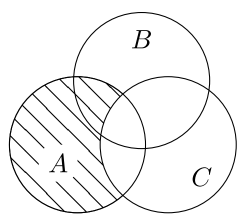
4b, page 3

- $\{1\} \in C$ but $\{1\} \not\subseteq C$;
- $\{2\} \notin C$ but $\{2\} \subseteq C$;
- $\{3\} \in C$ and $\{3\} \subseteq C$; and
- $\{4\} \notin C$ and $\{4\} \not\subseteq C$.

4c, page 6



$(A \setminus B) \cup (B \setminus C)$



$A \setminus C$

No. For a counterexample, let $A = C = \varnothing$ and $B = \{1\}$. Then

$$(A \setminus B) \cup (B \setminus C) = \varnothing \cup \{1\} = \{1\} \neq \varnothing = A \setminus C.$$

✎ 4d, page 6

Ideas. (1) The set of all sets?

(2) $\left\{\left\{\left\{\{\{\ldots\ \ldots\}\}\right\}\right\}\right\}$?

✎ 4e, page 7

Maybe, but is it better?

Proof. Take any set R. Split into two cases.

- Case 1: assume $R \in R$. Then $\sim(R \notin R)$. So $\sim(R \in R \Rightarrow R \notin R)$. Hence

$$\exists x \sim(x \in R \quad \Leftrightarrow \quad x \notin x).$$

- Case 2: assume $R \notin R$. Then $\sim(R \in R)$. So $\sim(R \notin R \Rightarrow R \in R)$. Hence

$$\exists x \sim(x \in R \quad \Leftrightarrow \quad x \notin x).$$

In either case, we showed $\sim\forall x\ (x \in R \Leftrightarrow x \notin x)$. So the proof is finished. $\square$