CS1231 Chapter 4

Sets

4.1 Basics

Definition 4.1.1. (1) A *set* is an unordered collection of objects.

(2) These objects are called the *members* or *elements* of the set.

(3) Write

$$\begin{array}{rll} x \in A & \text{for} & x \text{ is an element of } A; \\ x \notin A & \text{for} & x \text{ is not an element of } A; \\ x, y \in A & \text{for} & x, y \text{ are elements of } A; \\ x, y \notin A & \text{for} & x, y \text{ are not elements of } A; \qquad \text{etc.} \end{array}$$

(4) We may read $x \in A$ also as "x is in A" or "A *contains* x (as an element)".

Warning 4.1.2. Some use "contains" for the subset relation, but in this module we do *not*.

Symbol	Meaning	Examples	Non-examples
$\mathbb{N}$	the set of all natural numbers	$0, 1, 2, 3, 31 \in \mathbb{N}$	$-1, \frac{1}{2} \notin \mathbb{N}$
$\mathbb{Z}$	the set of all integers	$0, 1, -1, 2, -10 \in \mathbb{Z}$	$\frac{1}{2}, \sqrt{2} \notin \mathbb{Z}$
$\mathbb{Q}$	the set of all rational numbers	$-1, 10, \frac{1}{2}, -\frac{7}{5} \in \mathbb{Q}$	$\sqrt{2}, \pi, \sqrt{-1} \notin \mathbb{Q}$
$\mathbb{R}$	the set of all real numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi \in \mathbb{R}$	$\sqrt{-1}, \sqrt{-10} \notin \mathbb{R}$
$\mathbb{C}$	the set of all complex numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi, \sqrt{-1}, \sqrt{-10} \in \mathbb{C}$	
$\mathbb{Z}^+$	the set of all positive integers	$1, 2, 3, 31 \in \mathbb{Z}^+$	$0, -1, -12 \notin \mathbb{Z}^+$
$\mathbb{Z}^-$	the set of all negative integers	$-1, -2, -3, -31 \in \mathbb{Z}^-$	$0, 1, 12 \notin \mathbb{Z}^-$
$\mathbb{Z}_{\geqslant 0}$	the set of all non-negative integers	$0, 1, 2, 3, 31 \in \mathbb{Z}_{\geqslant 0}$	$-1, -12 \notin \mathbb{Z}_{\geqslant 0}$

$\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geqslant m}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_{\geqslant m}$, etc. are defined similarly.

Table 4.1: Common sets

Note 4.1.3. Some define $0 \notin \mathbb{N}$, but in this module we do *not*.

Definition 4.1.4 (roster notation)**.** (1) The set whose only elements are $x_1, x_2, \ldots, x_n$ is denoted $\{x_1, x_2, \ldots, x_n\}$.

(2) The set whose only elements are $x_1, x_2, x_3, \ldots$ is denoted $\{x_1, x_2, x_3, \ldots\}$.

Example 4.1.5. (1) The only elements of $A = \{1, 5, 6, 3, 3, 3\}$ are 1, 5, 6 and 3. So $6 \in A$ but $7 \notin A$.

(2) The only elements of $B = \{0, 2, 4, 6, 8, \dots\}$ are the non-negative even integers. So $4 \in B$ but $5 \notin B$.

To check whether an object z is an element of a set $S = \{x_1, x_2, \dots, x_n\}$. If z is in the list $x_1, x_2, \dots, x_n$, then $z \in S$, else $z \notin S$.

Definition 4.1.6 (set-builder notation)**.** Let U be a set and $P(x)$ be a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted

$$\{x \in U : P(x)\} \quad \text{or} \quad \{x \in U \mid P(x)\}.$$

This is read as "the set of all x in U such that $P(x)$".

Example 4.1.7. (1) The elements of $C = \{x \in \mathbb{Z}_{\geqslant 0} : x \text{ is even}\}$ are precisely the elements of $\mathbb{Z}_{\geqslant 0}$ that are even, i.e., the non-negative even integers. So $6 \in C$ but $7 \notin C$.

(2) The elements of $D = \{x \in \mathbb{Z} : x \text{ is a prime number}\}$ are precisely the elements of $\mathbb{Z}$ that are prime numbers, i.e., the prime integers. So $7 \in D$ but $9 \notin D$.

To check whether an object z is an element of $S = \{x \in U : P(x)\}$. If $z \in U$ and $P(z)$ is true, then $z \in S$, else $z \notin S$. Hence $z \notin U$ implies $z \notin S$, and $P(z)$ is false implies $z \notin S$.

Definition 4.1.8 (replacement notation)**.** Let A be a set and $t(x)$ be a term in a variable x. Then the set of all objects of the form $t(x)$ where x ranges over the elements of A is denoted

$$\{t(x) : x \in A\} \quad \text{or} \quad \{t(x) \mid x \in A\}.$$

This is read as "the set of all $t(x)$ where $x \in A$".

Example 4.1.9. (1) The elements of $E = \{x + 1 : x \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x + 1$ where $x \in \mathbb{Z}_{\geqslant 0}$, i.e., the positive integers. So $1 = 0 + 1 \in E$ but $0 \notin E$.

(2) The elements of $F = \{x - y : x, y \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x - y$ where $x, y \in \mathbb{Z}_{\geqslant 0}$, i.e., the integers. So $-1 = 1 - 2 \in F$ but $\sqrt{2} \notin F$.

To check whether an object z is an element of $S = \{t(x) : x \in A\}$. If there is an element $x \in A$ such that $t(x) = z$, then $z \in S$, else $z \notin S$.

Definition 4.1.10. Two sets are *equal* if they have the same elements, i.e., for all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B).$$

Convention 4.1.11. In mathematical definitions, people often use "if" between the term being defined and the phrase being used to define the term. This is the *only* situation in mathematics when "if" should be understood as a (special) "if and only if".

Example 4.1.12. $\{1, 5, 6, 3, 3, 3\} = \{1, 5, 6, 3\} = \{1, 3, 5, 6\}$.

Slogan 4.1.13. Order and repetition do not matter.

Example 4.1.14.
$$\begin{aligned} \{y^2 : y \text{ is an odd integer}\} &= \{x \in \mathbb{Z} : x = y^2 \text{ for some odd integer } y\} \\ &= \{1^2, 3^2, 5^2, \dots\}. \end{aligned}$$

Example 4.1.15. $\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Proof. ($\Rightarrow$) Take any $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. Then $z \in \mathbb{Z}$ and $z^2 = 1$. So

$$\begin{aligned} & z^2 - 1 = (z-1)(z+1) = 0. \\ \therefore \quad & z - 1 = 0 \quad \text{or} \quad z + 1 = 0. \\ \therefore \quad & z = 1 \quad \text{or} \quad z = -1. \end{aligned}$$

This means $z \in \{1, -1\}$.

($\Leftarrow$) Take any $z \in \{1, -1\}$. Then $z = 1$ or $z = -1$. In either case, we have $z \in \mathbb{Z}$ and $z^2 = 1$. So $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. $\square$

Exercise 4.1.16. Write down proofs of the claims made in Example 4.1.9. In other words, prove that $E = \mathbb{Z}^+$ and $F = \mathbb{Z}$, where E and F are as defined in Example 4.1.9. ✎ 4a

Theorem 4.1.17. There exists a unique set with no element, i.e.,

- there is a set with no element; and (existence part)
- for all sets A, B, if both A and B have no element, then $A = B$. (uniqueness part)

Proof.
- (existence part) The set $\{\}$ has no element.
- (uniqueness part) Let A, B be sets with no element. Then vacuously,

$$\forall z\ (z \in A \Rightarrow z \in B) \quad \text{and} \quad \forall z\ (z \in B \Rightarrow z \in A)$$

because the hypotheses in the implications are never true. So $A = B$. □

Definition 4.1.18. The set with no element is called the *empty set*. It is denoted by $\varnothing$.

4.2 Subsets

Definition 4.2.1. Let A, B be sets. Call A a *subset* of B, and write $A \subseteq B$, if

$$\forall z\ (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that B *includes* A, and write $B \supseteq A$ in this case.

Note 4.2.2. We avoid using the symbol $\subset$ because it may have different meanings to different people.

Example 4.2.3. (1) $\{1, 5, 2\} \subseteq \{5, 2, 1, 4\}$ but $\{1, 5, 2\} \not\subseteq \{2, 1, 4\}$.

(2) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Remark 4.2.4. Let A, B be sets.

(1) $$A \not\subseteq B \quad \Leftrightarrow \quad \exists z\ (z \in A \text{ and } z \notin B).$$

(2) $$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

(3) $$A \subseteq A.$$

Definition 4.2.5. Let A, B be sets. Call A a *proper subset* of B, and write $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is *proper* or *strict*.

Example 4.2.6. All the inclusions in Example 4.2.3 are strict.

Proposition 4.2.7. The empty set is a subset of any set, i.e., for any set A,

$$\varnothing \subseteq A.$$

Proof. Vacuously,

$$\forall z\ (z \in \varnothing \Rightarrow z \in A)$$

because the hypothesis in the implication is never true. So $\varnothing \subseteq A$ by the definition of $\subseteq$. □

Note 4.2.8. Sets can be elements of sets.

Example 4.2.9. (1) The set $A = \{\varnothing\}$ has exactly 1 element, namely the empty set. So A is not empty.

(2) The set $B = \{\{1\}, \{2, 3\}\}$ has exactly 2 elements, namely $\{1\}$, $\{2, 3\}$. So $\{1\} \in B$, but $1 \notin B$.

Note 4.2.10. Membership and inclusion can be different.

Question 4.2.11. Let $C = \{\{1\}, 2, \{3\}, 3, \{\{4\}\}\}$. Which of the following are true? ✎ 4b

- $\{1\} \in C$.
- $\{2\} \in C$.
- $\{3\} \in C$.
- $\{4\} \in C$.
- $\{1\} \subseteq C$.
- $\{2\} \subseteq C$.
- $\{3\} \subseteq C$.
- $\{4\} \subseteq C$.

Definition 4.2.12. Let A be a set. The set of all subsets of A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example 4.2.13. (1) $\mathcal{P}(\varnothing) = \{\varnothing\}$.

(2) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.

(3) $\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.

(4) The following are subsets of $\mathbb{Z}_{\geqslant 0}$ and thus are elements of $\mathcal{P}(\mathbb{Z}_{\geqslant 0})$.

$$\begin{aligned}
&\varnothing, \{0\}, \{1\}, \{2\}, \dots \{0,1\}, \{0,2\}, \{0,3\} \dots \{1,2\}, \{1,3\}, \{1,4\} \dots \\
&\{2,3\}, \{2,4\}, \{2,5\} \dots \{0,1,2\}, \{0,1,3\}, \{0,1,4\}, \dots \\
&\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \dots \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \dots \\
&\mathbb{Z}_{\geqslant 0}, \mathbb{Z}_{\geqslant 1}, \mathbb{Z}_{\geqslant 2}, \dots \{0,2,4,\dots\}, \{1,3,5,\dots\}, \{2,4,6,\dots\}, \{3,5,7,\dots\}, \dots \\
&\{x \in \mathbb{Z}_{\geqslant 0} : (x-1)(x-2) < 0\}, \{x \in \mathbb{Z}_{\geqslant 0} : (x-2)(x-3) < 0\}, \dots \\
&\{3x+2 : x \in \mathbb{Z}_{\geqslant 0}\}, \{4x+3 : x \in \mathbb{Z}_{\geqslant 0}\}, \{5x+4 : x \in \mathbb{Z}_{\geqslant 0}\}, \dots
\end{aligned}$$

4.3 Boolean operations

Definition 4.3.1. Let A, B be sets.

(1) The *union* of A and B, denoted $A \cup B$, is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Read $A \cup B$ as "A union B".

(2) The *intersection* of A and B, denoted $A \cap B$, is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Read $A \cap B$ as "A intersect B".

(3) The *complement* of B in A, denoted $A - B$ or $A \setminus B$, is defined by

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Read $A \setminus B$ as "A minus B".

Convention and terminology 4.3.2. When working in a particular context, one usually works within a fixed set U which includes all the sets one may talk about, so that one only needs to consider the elements of U when proving set equality and inclusion (because no other object can be the element of a set). This U is called a *universal set*.

Definition 4.3.3. Let B be a set. In a context where U is the universal set (so that implicitly $U \supseteq B$), the *complement* of B, denoted $\overline{B}$ or B^{c}, is defined by

$$\overline{B} = U \setminus B.$$

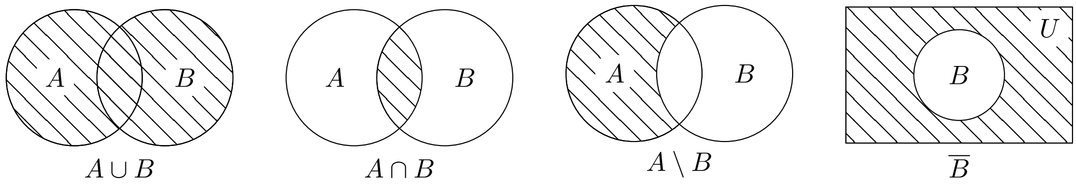

Figure 4.2: Boolean operations on sets

Example 4.3.4. Let $A = \{x \in \mathbb{Z} : x \leqslant 10\}$ and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 15\}$. Then

$$\begin{aligned} A \cup B &= \{x \in \mathbb{Z} : (x \leqslant 10) \vee (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x \leqslant 15\}; \\ A \cap B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 10\}; \\ A \setminus B &= \{x \in \mathbb{Z} : (x \leqslant 10) \wedge {\sim}(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x < 5\}; \\ \overline{B} &= \{x \in \mathbb{Z} : {\sim}(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : (x < 5) \vee (x > 15)\}, \end{aligned}$$

in a context where $\mathbb{Z}$ is the universal set. To show the first equation, one shows

$$\forall x \in \mathbb{Z} \ \big((x \leqslant 10) \vee (5 \leqslant x \leqslant 15) \Leftrightarrow (x \leqslant 15)\big), \qquad \text{etc.}$$

Theorem 4.3.5 (Set Identities)**.** For all set A, B, C in a context where U is the universal set, the following hold.

Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotence	$A \cup A = A$	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Identities	$A \cup \varnothing = A$	$A \cap U = A$
Annihilators	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Complement	$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$
Double Complement Law	$\overline{(\overline{A})} = A$	
Top and bottom	$\overline{\varnothing} = U$	$\overline{U} = \varnothing$
Set difference	$A \setminus B = A \cap \overline{B}$	

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Venn Diagrams. In the left diagram below, hatch the regions representing A and B with and respectively. In the right diagram below, hatch the regions representing $\overline{A}$ and $\overline{B}$ with and respectively.

Then the region represents $\overline{A \cup B}$ in the left diagram, and the region represents $\overline{A} \cap \overline{B}$ in the right diagram. Since these regions occupy the same region in the respective diagrams, we infer that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Note 4.3.6. This argument depends on the fact that each possibility for membership in A and B is represented by a region in the diagram.

Proof using a truth table. The rows in the following table list all the possibilities for an element $x \in U$:

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in \overline{A \cup B}$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in \overline{A} \cap \overline{B}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns under "$x \in \overline{A \cup B}$" and "$x \in \overline{A} \cap \overline{B}$" are the same, for any $x \in U$,

$$x \in \overline{A \cup B} \quad \Leftrightarrow \quad x \in \overline{A} \cap \overline{B}$$

no matter in which case we are. So $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$

Direct proof. Let $z \in U$. Then

$$\begin{array}{cll} & z \in \overline{A \cup B} & \\ \Leftrightarrow & z \notin A \cup B & \text{by the definition of } \overline{\cdot}; \\ \Leftrightarrow & \sim\big((z \in A) \vee (z \in B)\big) & \text{by the definition of } \cup; \\ \Leftrightarrow & (z \notin A) \wedge (z \notin B) & \text{by De Morgan's Laws for propositions;} \\ \Leftrightarrow & (z \in \overline{A}) \wedge (z \in \overline{B}) & \text{by the definition of } \overline{\cdot}; \\ \Leftrightarrow & z \in \overline{A} \cap \overline{B} & \text{by the definition of } \cap. \end{array}$$ $\square$

Example 4.3.7. Under the universal set U, show that $(A \cap B) \cup (A \setminus B) = A$ for all sets A, B.

Proof.
$$\begin{aligned} (A \cap B) \cup (A \setminus B) &= (A \cap B) \cup (A \cap \overline{B}) && \text{by the properties of set difference;} \\ &= A \cap (B \cup \overline{B}) && \text{by distributivity;} \\ &= A \cap U && \text{by the properties of set complement;} \\ &= A && \text{as } U \text{ is the identity for } \cap. \end{aligned}$$ $\square$

Example 4.3.8. Show that $A \cap B \subseteq A$ for all sets A, B.

Proof. By the definition of $\subseteq$, we need to show that

$$\forall z\ (z \in A \cap B \Rightarrow z \in A).$$

Let $z \in A \cap B$. Then $z \in A$ and $z \in B$ by the definition of $\cap$. In particular, we know $z \in A$, as required. $\square$

Question 4.3.9. Is the following true? ✎ 4c

$$(A \setminus B) \cup (B \setminus C) = A \setminus C \quad \text{for all sets } A, B, C.$$

4.4 Russell's Paradox

Example 4.4.1. (1) $\varnothing \notin \varnothing$.

(2) $\mathbb{Z} \notin \mathbb{Z}$.

(3) $\{\varnothing\} \notin \{\varnothing\}$.

Question 4.4.2. Is there a set x such that $x \in x$? ✎ 4d

Theorem 4.4.3 (Russell 1901)**.** There is no set R such that

$$\forall x \; (x \in R \quad \Leftrightarrow \quad x \notin x). \tag{$*$}$$

In words, there is no set R whose elements are precisely the sets x that are not elements of themselves.

Proof. We prove this by contradiction. Suppose R is a set satisfying ($*$). Applying ($*$) to $x = R$ gives

$$R \in R \quad \Leftrightarrow \quad R \notin R. \tag{$\dagger$}$$

Split into two cases.

- Case 1: assume $R \in R$. Then $R \notin R$ by the $\Rightarrow$ part of ($\dagger$). This contradicts our assumption that $R \in R$.
- Case 2: assume $R \notin R$. Then $R \in R$ by the $\Leftarrow$ part of ($\dagger$). This contradicts our assumption that $R \notin R$.

In either case, we get a contradiction. So the proof is finished. □

Question 4.4.4 (tongue-in-cheek)**.** Can you write a proof of Theorem 4.4.3 that does not mention contradiction? 4e