National University of Singapore
Department of Computer Science
2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 8**

A proof is rarely blind deduction from definitions and theorems. Before writing a proof, you must already see in your mind the skeleton joining various parts of the argument. Writing down the technical details is just fleshing out that skeleton.

1. Let A and B be sets.

   (a) Suppose A and B are disjoint (i.e., $A \cap B = \varnothing$) and countable. Prove that $A \cup B$ is countable.

   (b) Suppose A and B are (not necessarily disjoint but) countable. Prove that $A \cup B$ is countable.

2. Let $A_0, A_1, A_2, \ldots$ be countable sets. Recall from Tutorial 3 Problem 9 that for all $n \in \mathbb{Z}_{\geqslant 0}$,

$$\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \cdots \cup A_n,$$

   and for all x,

$$x \in \bigcup_{i=0}^{\infty} A_i \qquad \text{if and only if} \qquad x \in A_i \text{ for some non-negative integer } i.$$

   (a) Prove by induction that $\bigcup_{i=0}^{n} A_i$ is countable for any integer $n \geqslant 0$.

   (b) Does (a) prove that $\bigcup_{i=0}^{\infty} A_i$ is countable?

   (c) Using the countability of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ from Theorem 8.4.4, or otherwise, prove that $\bigcup_{i=0}^{\infty} A_i$ is countable. (Hint: You may find Tutorial 7 Problem 9 useful.)

3.* The set $\mathbb{Q}$ of rational numbers can be defined by $\mathbb{Q} = \{r \in \mathbb{R} : \exists m \in \mathbb{Z}\ \exists n \in \mathbb{Z}^+\ r = \frac{m}{n}\}$.

   (a) Consider the following "proof" that $\mathbb{Q}$ is countable.

   > "Note that $\mathbb{Z} \subseteq \mathbb{Q}$. Since $\mathbb{Z}$ is countable and every subset of a countable set is countable, we know $\mathbb{Q}$ is countable."

   What is wrong with this "proof"?

   (b) Using the countability of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, or otherwise, prove that $\mathbb{Q}$ is countable.
   (This is even more surprising than the countability of $\mathbb{Z}$, since there are infinitely many rational numbers between any two rational numbers.)

   (c) In essence, a set X is countable means we can write $X = \{x_0, x_1, x_2, \ldots\}$. Write $\mathbb{Q}$ in this form.

4. Let Y and Z be sets that are countable and infinite. Prove that $Y \times Z$ is countable.
   (This is a generalization of the fact that $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is countable.)

5. Prove that:

   (a) if a set X has an uncountable subset, then X is also uncountable;

   (b)* if A is uncountable and B is countable, then $A \setminus B$ is uncountable.

6. It will be shown in the lectures that if X is a finite set, then $\mathcal{P}(X)$ is finite and has cardinality $2^{|X|}$. Use this to prove that a set X has countably many subsets if and only if X is finite.

7.* Prove that

   (a) $\{S \in \mathcal{P}(\{\mathrm{b}\}^*) : S \text{ contains exactly 3 strings}\}$ is countable;

   (b) $\mathcal{P}(\{\mathrm{b}\}^*)$ is uncountable.

   (Part (a) can be generalized to "there are countably many finite subsets of $\{\mathrm{b}\}^*$", while part (b) says $\{\mathrm{b}\}^*$ has uncountably many subsets. Therefore, the point here is: the uncountability of $\mathcal{P}(\{\mathrm{b}\}^*)$ must be from the infinite subsets.)

8.* Let B be a finite subset of an infinite set C. Prove that there are uncountably many countable sets X such that $B \subseteq X \subseteq C$.