Definition
An integer n is **even** if and only if $n = 2k$ for some integer k.
An integer n is **odd** if and only if $n = 2k + 1$ for some integer k.

Direct Proof (one direction)
Claim: If the sum of two integers is even,
then their difference is even.
Proof:
Let $m, n \in \mathbf{Z}$.

Suppose $m + n$ is even.

Then $m + n = 2k$ for some $k \in \mathbf{Z}$,

so $m - n = (2k - n) - n = 2(k - n)$.

But $k - n \in \mathbf{Z}$,

so $m - n$ is even.

Direct Proof (both directions)
Claim : For any positive real number x,
$x > 1$ if and only if $x^2 > 1$.

Proof :
Consider any positive real number x.

$(\Rightarrow)$ Suppose $x > 1$.

Since x is positive,

we have $x \cdot x > x \cdot 1 = x > 1$

i.e. $x^2 > 1$.

$(\Leftarrow)$ Conversely, suppose $x^2 > 1$.

Then $x^2 - 1 > 0$,

i.e. $(x - 1)(x + 1) > 0$,

so $x - 1 > 0$ and $x + 1 > 0$

or $x - 1 < 0$ and $x + 1 < 0$.

But $x > 0$, so we can't have $x + 1 < 0$.

Therefore $x - 1 > 0$ and $x + 1 > 0$.

In particular, $x - 1 > 0$ gives $x > 1$.

Proof by Cases

Claim : $x^2 \geq 0$ for every real number x.

Proof :

Case $x > 0$: Then $x \cdot x > 0 \cdot x$

$= 0,$

so $x^2 > 0$.

Case $x = 0$: Then $x \cdot x = 0 \cdot 0$

so $x^2 = 0$.

Case $x < 0$: Then $-x > 0$,

so $0 \cdot (-x) > x \cdot (-x)$.

Thus $0 > -x^2$

i.e. $x^2 > 0$.

Indirect Proof

Claim : Suppose x and y are real numbers.
If $xy = 0$, then $x = 0$ or $y = 0$.

Proof :
Suppose the claim is false,

so $xy = 0$ and $x \neq 0$ and $y \neq 0$.
for some $x, y \in \mathbf{R}$.

Since $x \neq 0$, we can

divide both sides of $xy = 0$ by x,

so $y = 0$.

This contradicts $y \neq 0$,

so the claim is true.

Constructive (Existence) Proof

Claim : There is a real number
between any two different real numbers.

Proof :
Let b and c be real numbers, $b \neq c$.

Without loss of generality,
we may assume $b < c$.

Let $d = \frac{b+c}{2}$, so $d \in \mathbf{R}$.

Then $d = \frac{b+c}{2} > \frac{b+b}{2} = b$

and $d = \frac{b+c}{2} < \frac{c+c}{2} = c$,

so $b < d$ and $d < c$.

Nonconstructive (Existence) Proof

Theorem 1.2 (*Pigeonhole Principle*):
Let B and C be positive integers, $B < C$.
If C cards are distributed among B boxes,
then there is a box with more than one card.

Proof :
Let n_i be the number of cards in box i,
for $i = 1, \ldots, B$.

Then $n_1 + \cdots + n_B = C \qquad — \quad (*)$

Suppose $n_i \leq 1$ for all i.

Then $n_1 + \cdots + n_B \leq 1 + \cdots + 1$

$$= B < C,$$

contradicting $(*)$.

Therefore it is not true that $n_i \leq 1$ for all i
i.e. $n_k > 1$ for some k.

Inductive Proof

First Principle of Mathematical Induction (**1PI**)
Let $P(n)$ be a predicate, where $n \in \mathbf{Z}$, and $b \in \mathbf{Z}$.
If $P(b)$ is true and $P(k) \to P(k+1)$ for all $k \geq b$,
then $P(n)$ is true for all $n \geq b$.

How to use 1PI:

Basis: Prove the claim is true for $n = b$.

Induction Hypothesis: Assume the claim is true if $n = k$, for some $k \geq b$.

Induction Step: Use the Induction Hypothesis to prove that the claim is true for $n = k + 1$.

It follows from 1PI that the claim is true for all $n \geq b$.

Claim: Let $p_1, p_2, \ldots$ be statements. Then

$$\sim (p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\sim p_1) \vee (\sim p_2) \vee \cdots \vee (\sim p_n)$$

for any $n \geq 2$.

Second Principle of Mathematical Induction (**2PI**)

Basis: Prove the claim is true for $n = b, b+1, \ldots, c$.

Induction Hypothesis: Assume the claim is true
if $b \leq n \leq k$, for some $k \geq c$.

Induction Step: Use the Induction Hypothesis to prove that the claim is true for $n = k+1$.

This proves the claim is true for all $n \geq b$.

Definition
Let p be a statement variable; then p and $\sim p$ are **literals**.
A **clause** is a literal or a Boolean expression of the form $\ell_1 \vee \ell_2 \vee \cdots \vee \ell_r$, where each ℓ_i is a literal.
A Boolean expression is in **Conjunctive Normal Form** (CNF) iff it is a clause or has the form $C_1 \wedge C_2 \wedge \cdots \wedge C_s$, where each C_j is a clause.

Example
$\big((\sim p) \wedge (q \vee r)\big) \vee \Big(\sim \big(s \vee p \vee (\sim q)\big)\Big)$

Claim: Every Boolean expression is logically equivalent to a Boolean expression in CNF.