1. Use a counting argument to prove that, for any $n, r \in \mathbb{Z}^+$ and $1 \le r \le n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

[One can prove this combinatorial identity algebraically by using the formula for $\binom{n}{k}$. However, since $\binom{n}{k}$ is *defined* as number of ways to choose a k-element subset from an n-element set, there should be some way of proving the identity by counting ways of choosing subsets from sets; this is what is meant by a "counting argument".]

2.* For $n \in \mathbb{Z}^+$, determine

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

and

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^r \binom{n}{r} + \cdots + (-1)^n \binom{n}{n}$$

3. The Binomial Theorem states that, for any $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$,

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{r} x^{n-r} y^r + \cdots + \binom{n}{n} x^0 y^n.$$

(i)* Give an inductive proof of the theorem.

(ii) Give a counting argument for the theorem.

4. (i) Give an inductive proof of the following:

$$\binom{0}{r} + \binom{1}{r} + \binom{2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} \quad \text{for any } n, r \in \mathbb{N}.$$

(ii) Give a counting argument for the result.

5.* Let $m, n, r \in \mathbb{N}$. Prove the following (Vandermonde's identity):

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}.$$

[An algebraic proof of this identity would be painfully tedious.]

6. Recall the definition of $\bigcup_{k=1}^{n} A_k$ and $\bigcup_{k=1}^{\infty} A_k$ in Tutorial 3.

(i) Consider the claim:
"Suppose $A_1, A_2, \ldots$ are finite sets. Then $\bigcup_{i=1}^{n} A_i$ is finite for any $n \ge 2$."
An inductive proof was given in class (Theorem 3.10). Here is an alternative "proof":
"We will prove by induction on n. Since A_1 and A_2 are finite, $A_1 \cup A_2$ is finite (by Lemma 3.9), so the claim is true for $n = 2$. Now suppose the claim is true for $n = k$, so $\bigcup_{i=1}^{k} A_i$ is finite. Let $A_{k+1} = \varnothing$. Then $\bigcup_{i=1}^{k+1} A_i = \left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1} = \left(\bigcup_{i=1}^{k} A_i\right) \cup \varnothing = \bigcup_{i=1}^{k} A_i$, which is finite by the induction hypothesis, so the claim is true for $n = k+1$. Therefore, the claim is true for all $n \ge 2$."
What is wrong with this "proof"?

(ii) Prove the following is false: "Suppose $A_1, A_2, \ldots$ are finite sets. Then $\bigcup_{i=1}^{\infty} A_i$ is finite."
[The point here is: induction takes you to any finite n, but not to infinity.]

7. State and prove the Inclusion/Exclusion Rule for four sets.

8. Let $n \in \mathbb{Z}^+$. A **complete graph** for n nodes, denoted K_n, is an undirected graph with an edge between every pair of nodes and a loop at every node. Draw K_n for $n \leq 6$.

9. Suppose A and B are nonempty finite sets, $|A| = n$ and $|B| = k$.

   (i) How many relations are there from A to B?

   (ii)* How many functions are there from A to B?
   (In particular, how many Boolean functions are there for m variables?)

   (iii) How many injective functions are there from A to B?

   (iv)* For $k \leq 4$, how many surjective functions are there from A to B?

   (v) How many bijections are there from A to B?

   (vi)* For $k \leq 3$, how many functions are there from A to B that are not injective and not surjective?

10.* Let U be a nonempty finite set. A 3-partition is a partition of U into three subsets X, Y and Z such that

- $X \neq \varnothing$, $Y \neq \varnothing$ and $Z \neq \varnothing$,
- $X \cap Y = Y \cap Z = X \cap Z = \varnothing$ and
- $X \cup Y \cup Z = U$.

   (i) List all possible 3-partitions of $\{a, b, c, d\}$.
   Suppose U has n elements, where $n \geq 3$. Let P_n be the number of 3-partitions of U. What is P_4?

   (ii) Prove that $P_{n+1} = 3P_n + 2^{n-1} - 1$ for all $n \geq 3$.

   (iii) Prove that $P_n = \frac{1}{2}(3^{n-1} - 2^n + 1)$ for all $n \geq 3$.

11.* Consider a Boolean expression α with statement variables $x_1, \ldots, x_n$. A **truth assignment** is a function $f : \{x_1, \ldots, x_n\} \to \{T, F\}$. We say f **satisfies** α (or f is a **satisfying truth assignment** for α) if and only if α is true when, for all i, $f(x_i)$ is the truth value of x_i.

   (i) Does $f(p) = T$, $f(q) = T$ and $f(r) = F$ satisfy $(p \vee q) \wedge ((r \vee \sim q) \vee \sim (p \vee r))$?

   (ii) What is the maximum number of truth assignments that can satisfy α?

   (iii) For $n = 3$, give an example of α that has the maximum number of satisfying truth assignments.

   (iv) A Boolean expression is **satisfiable** if and only if it has at least one satisfying truth assignment. For $n = 4$, give an example of a Boolean expression that is not satisfiable.

   (v) How many satisfying truth assignments are there for the Boolean expression in (i)?

   [The **Satisfiability Problem** is: Given a Boolean expression α, is α satisfiable? Mathematically, this question has a trivial solution: simply try all possible truth assignments. Computationally, however, this problem is believed to be **intractable**, in the sense that no one has found a fast solution algorithm. The problem does not become much easier even if α is in CNF: if α has two literals per clause, there is a polynomial algorithm to determine satisfiability; but if α has three literals per clause, the satisfiability problem becomes NP-complete.]