National University of Singapore
Department of Computer Science
2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 10**

1. Suppose A is a set with n elements.

   (i) How many binary relations are there on A?

   (ii)* How many of these relations are reflexive?

   (iii) How many of these relations are symmetric?

   (iv) How many of these relations are antisymmetric?

   (v) How many of these relations are antisymmetric and symmetric?

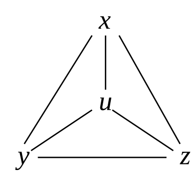
   (vi)* How many of these relations are not reflexive and not symmetric?

   [Hint: Consider the graph representation of the relation.
   You should use small values of n to check the validity of your formulas.]

2. Let $\mathcal{G}_3$ be the set of all undirected graphs whose vertices are a, b, c. Suppose $G = (\{a, b, c\}, E) \in \mathcal{G}_3$. Determine the number of possible G's such that:

   (i)* G has a loop; (ii)* G has a cycle;
   (iii) G is cyclic; (iv)* G is connected;
   (v) G is a tree; (vi) G has exactly two connected components.

3.* The diagram here illustrates an undirected graph K, called a **3-sided wheel**:



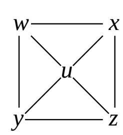
3-sided wheel K
with vertices u, x, y, z

   (i) Let $K = (V_K, E_K)$. List the elements of E_K.

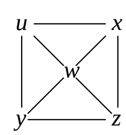
   (ii) How many different 3-sided wheels are there with $V_K = \{u, x, y, z\}$?

   (iii) Determine the number of different 3-sided wheels with $V_K \subseteq \{1, 2, 3, 4, 5, 6\}$ (e.g. $u = 4, x = 6, y = 2, z = 3$)?

   The diagram here shows two **4-sided wheels** H and H':



4-sided wheel H



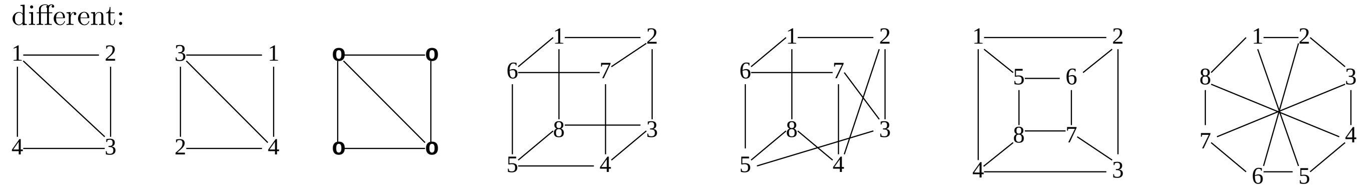
4-sided wheel H'

   (iv) Explain why $H \neq H'$.

   (v) Determine the number of different 4-sided wheels H with vertex set $V_H = \{1, 2, 3, 4, 5\}$.

   (vi) Determine the number of different 4-sided wheels H with vertex set $V_H \subseteq \{1, 2, 3, 4, 5, 6, 7\}$.

4. Our definition for undirected graphs *labels* the vertices. Thus (a) and (b) below are considered different:



(a) (b) (c) (d) (e) (f) (g)

However, they are the same if we ignore the labels, as in (c). We now define what "same" means: Two finite loopless undirected graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are **isomorphic** (denoted $G \simeq H$) iff there is a permutation $\pi : V_G \to V_H$ such that $\{u, v\} \in E_G \leftrightarrow \{\pi(u), \pi(v)\} \in E_H$. (An undirected graph is **loopless** if and only if all its vertices do not have loops.) Thus (a) and (b) are isomorphic — consider $\pi(1) = 3, \pi(2) = 1, \pi(3) = 4, \pi(4) = 2$.

(i) Which of the graphs in (d), (e), (f) and (g) are isomorphic?

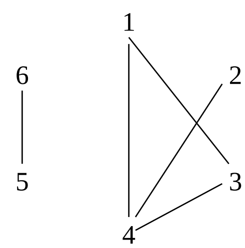
(ii)* Let $\mathcal{G}$ be the set of all loopless undirected graphs whose nodes are $\{1, 2, \ldots, n\}$. Prove that $\simeq$ is an equivalence relation on $\mathcal{G}$. What are in each equivalence class?

(iii) Determine the number of nonisomorphic loopless undirected graphs with n nodes, for $n = 2, 3, 4$.

[The computational complexity for determining whether two given graphs are isomorphic is a 30-year-old open problem that lies at the heart of the P$\neq$NP question.]

5.* Prove that if a loopless undirected graph has n vertices, where $n \geq 2$, and more than $\binom{n-1}{2}$ edges, then it is connected. Is the converse true?

6. Let $G = (V, E)$ be a loopless undirected graph. The **complement** of G is the loopless graph $\overline{G} = (V, F)$, where $\{u, v\} \in F$ if and only if $\{u, v\} \notin E$. Draw the complement of the following graph:



Prove that (for any G) G and $\overline{G}$ cannot both be unconnected.

7.* Let R be a binary relation on a set. Prove that R is transitive if and only if $R_+ \subseteq R$.

8. Recall from Tutorial 9 (Problem 8) the definition of a complete graph. Let R be an equivalence relation on a nonempty set A, and let G be the undirected graph representing R. Prove that every connected component of G is a complete graph.

9. Consider an undirected graph G, whose connected components are $H_1, \ldots, H_k$, where $k \geq 2$. Suppose $G = (V, E)$ and $H_1 = (V_1, E_1), \ldots, H_k = (V_k, E_k)$. Prove that $\{V_1, \ldots, V_k\}$ is a partition of V. Is $\{E_1, \ldots, E_k\}$ a partition of E?