**National University of Singapore**
**Department of Computer Science**

**2021/22 (Sem.1)** **CS1231 Discrete Structures** **Tutorial 11**

1. Consider an undirected graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{6, 8\}, \{7, 8\}, \{8, 9\}\}$.

   (i) Draw $G$.

   (ii) Draw a subgraph $(V', E')$ that is not a tree but satisfies $|E'| = |V'| - 1$.

   (iii) Draw a connected subgraph $(V'', E'')$ that is not a tree, but has an edge $\{x, y\}$ such that $(V'', E'' \setminus \{\{x, y\}\})$ is not connected.

   (iv) Let $T$ be a rooted tree such that $T$'s edges are in $E$ and $T$'s root is 7. Draw all possible $T$s.

   (v) Draw another graph $G'' = (\{1, 2, 3, 4, 5, 6, 7, 8, 9\}, E'')$ such that $E'' \neq E$ but $G''$ is isomorphic to $G$.

   (vi) Determine the number of graphs $G'' = (\{1, 2, 3, 4, 5, 6, 7, 8, 9\}, E'')$ such that $E'' \neq E$ but $G''$ is isomorphic to $G$.

2.* (i) Draw all trees with $n$ nodes for $n = 1, 2, 3, 4$. What is the general formula for the number of trees with $n$ nodes?

   (ii) Determine the number of nonisomorphic trees with $n$ nodes, for $n = 1, 2, 3, 4$.

   What is the relationship between (i) and (ii)?

3. For two rooted trees to be isomorphic under a permutation $\pi$, we require that the image $\pi(u)$ of a root $u$ must also be a root. Determine the number of nonisomorphic rooted trees with $n$ nodes, for $n = 1, 2, 3, 4$.

4.* Let $G = (V, E)$ be an undirected graph. Prove that if $G$ is connected, then $|E| \geq |V| - 1$. Is the converse true?

5. Let $G = (V, E)$ be an undirected graph. Prove that if $G$ is acyclic, then $|E| \leq |V| - 1$. Is the converse true?

6.* Prove that a loopless undirected graph is a tree if and only if there is exactly one path between every pair of nodes.

7. Recall from Tutorial 9 (Problem 8) that a complete graph has all loops and all edges.
   How many spanning trees are there for the complete graph with 4 nodes?

8.* Consider a rooted tree $T$ in which every parent has at most $b$ children ($b \in \mathbb{Z}^+$; a binary tree has $b = 2$).

   State and prove a result relating the number of leaves and number of parents in $T$, if each parent has exactly $b$ children.

9. For $n \geq 2$, a directed graph $(\{v_1, v_2, \ldots, v_n\}, \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\})$ is called a **cycle**. A directed graph $G = (V, D)$ is **cyclic** if it contains a loop or a cycle as a subgraph; otherwise, it is **acyclic**

(i) Prove that if $G$ is acyclic, then $D$ is antisymmetric.

(ii) Prove or disprove the converse of (i).

(iii) Prove that if $D$ is a partial order, then $G$ does not contain any cycles.
[We can hence arrange the edges in the graph for a partial order, so they all point in one direction.]

(iv)* Prove that, for any $n \geq 2$, there is a directed acyclic graph with $n$ nodes and $\frac{1}{2}n(n-1)$ edges.
[Contrast this with the $|E| = |V| - 1$ characterization for undirected acyclic graphs.]

(v)* Prove that any directed graph with $n$ nodes and more than $\frac{1}{2}n(n-1)$ edges must be cyclic.

10. Let $\mathcal{D}_3$ be the set of all directed graphs whose nodes are $a, b, c$. Suppose $G = (\{a, b, c\}, D) \in \mathcal{D}_3$. Determine the number of possible $G$'s such that:

| | |
|---|---|
| (i)* $G$ has a loop; | (ii) $G$ is acyclic.; |
| (iii)* $D$ is reflexive; | (iv) $D$ is symmetric; |
| (v)* $D$ is antisymmetric; | (vi) $D$ is a total order. |

11.* Consider a loopless undirected graph $G = (V, E)$ and $V' \subseteq V$. We say $V'$ **covers** an edge $\{x, y\} \in E$ iff $x \in V'$ or $y \in V'$, and $V'$ is a **vertex cover** iff $V'$ covers every edge. For example, $\{00, 02, 11, 12, 21, 22\}$ is a vertex cover for the graph in (a).

(i) For the graph in (a), find a vertex cover of smallest possible size.

A Boolean expression $\alpha$ is **3CNF** iff $\alpha$ is a conjunction of clauses with exactly 3 literals per clause. A 3CNF expression $\alpha$ with $k$ clauses can be transformed into a loopless undirected graph $G_\alpha$, so that $\alpha$ is satisfiable (Tutorial 9) iff $G_\alpha$ has a vertex cover of size at most $2k$. The following illustrates this transformation: Suppose, for $k = 4$,

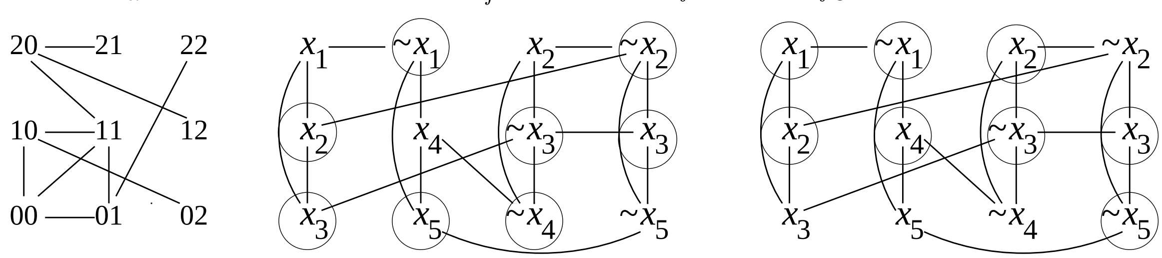
$$\alpha = (x_1 \vee x_2 \vee x_3) \wedge (\sim x_1 \vee x_4 \vee x_5) \wedge (x_2 \vee \sim x_3 \vee \sim x_4) \wedge (\sim x_2 \vee x_3 \vee \sim x_5).$$

Then the graph in (b) and in (c) (without the circles) is $G_\alpha$.

(ii) Consider the truth assignment $f(x_1) = f(x_2) = f(x_4) = T$ and $f(x_3) = f(x_5) = F$. Verify that $f$ satisfies $\alpha$. Let $C_f$ be the set of vertices in (b) that are circled. Verify that $C_f$ is a vertex cover of size 8.

(iii) Consider the set $C$ of vertices in (c) that are circled. Verify that $C$ is a vertex cover of size 8. Define a truth function $f_C$ by $f_C(x_1) = f_C(x_3) = f_C(x_5) = T$ and $f_C(x_2) = f_C(x_4) = F$. Verify that $f_C$ satisfies $\alpha$.

(iv) How is $G_\alpha$ derived from $\alpha$? How is $C_f$ derived from $f$? How is $f_C$ derived from $C$?




(a) (b) (c)

[This problem suggests that finding a vertex cover is "harder" than finding a satisfying truth assignment. Actually, they are "equally hard", since both are NP-Complete.]