National University of Singapore
Department of Computer Science

2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 4**

1. Let $A = \{0, 1\}$, $B = \{a, b, c\}$ and $C = \{01, 10\}$. Determine the following:
(a) $B \times C$; (b) $A \times B \times C$; (c) $\varnothing \times A$; (d) $\mathcal{P}(\{\varnothing\}) \times A$.

Solution:
$A = \{0, 1\}$, $B = \{a, b, c\}$ and $C = \{01, 10\}$.

(a) $B \times C = \{(a, 01), (b, 01), (c, 01), (a, 10), (b, 10), (c, 10)\}$.

(b) $A \times B \times C = \{(0, a, 01), (0, b, 01), (0, c, 01), (0, a, 10), (0, b, 10), (0, c, 10),$
$(1, a, 01), (1, b, 01), (1, c, 01), (1, a, 10), (1, b, 10), (1, c, 10)\}$.

(c) $\varnothing \times A = \{(b, c) : b \in \varnothing \wedge c \in A\} = \varnothing$.

(d) $\mathcal{P}(\{\varnothing\}) \times A = \{\varnothing, \{\varnothing\}\} \times \{0, 1\} = \{(\varnothing, 0), (\varnothing, 1), (\{\varnothing\}, 0), (\{\varnothing\}, 1)\}$.

2.* Consider a probabilistic experiment like the following: first toss a coin; if the toss is head, then pick a ball from a box with 2 black balls and 3 white balls; if the toss is tail, then pick a ball from a box with 4 red balls and 5 white balls. An **outcome** of such an experiment may be, say, a tail followed by a red ball.

(a) Use an ordered pair to represent each possible outcome of the experiment; the set of all such ordered pairs is called the **sample space**.

(b) An **event** for this experiment may be "the toss is tail" or "a white ball is picked". How can these events be represented as subsets of the sample space?

[This is an example of mathematical **modeling**, which is ubiquitous in Computer Science; here, sets and ordered pairs are used to model the experiment.]

Solution:
Notation: Head H; Tail T; black balls B_1, B_2; red balls R_1, R_2, R_3, R_4;
white balls W_1, W_2, W_3 in the first box; white balls $W'_1, W'_2, W'_3, W'_4, W'_5$ in the second box

(a) sample space $= \{(H, B_1), (H, B_2), (H, W_1), (H, W_2), (H, W_3),$
$(T, R_1), (T, R_2), (T, R_3), (T, R_4),$
$(T, W'_1), (T, W'_2), (T, W'_3), (T, W'_4), (T, W'_5)\}$.

(b) "toss is tail": $\{(T, R_1), (T, R_2), (T, R_3), (T, R_4), (T, W'_1), (T, W'_2), (T, W'_3), (T, W'_4), (T, W'_5)\}$
"a white ball is picked": $\{(H, W_1), (H, W_2), (H, W_3),$
$(T, W'_1), (T, W'_2), (T, W'_3), (T, W'_4), (T, W'_5)\}$

Alternative:

(a) sample space $= \{(H, B), (H, W), (T, R), (T, W)\}$.
The outcomes in this sample space are *not* equilikely;
information about the number of balls is not captured in the notation.

(b) "toss is tail": $\{(T, R), (T, W)\}$; "white ball is picked": $\{(H, W), (T, W)\}$

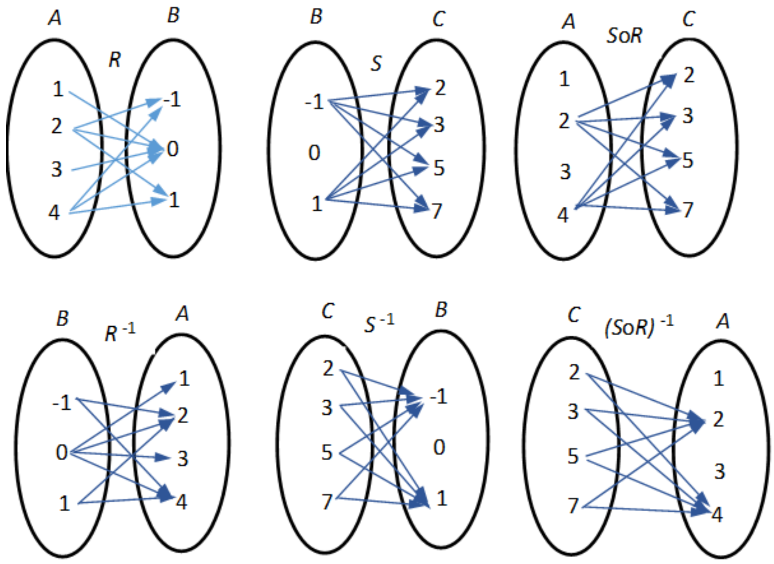
3. Let $A = \{1, 2, 3, 4\}$, $B = \{-1, 0, 1\}$, $C = \{2, 3, 5, 7\}$, $R = \{(a, b) \in A \times B : ab \text{ is even}\}$ and $S = \{(b, c) \in B \times C : b + 2c \text{ is odd}\}$.

   (a) Draw arrow diagrams for R, S, $S \circ R$ and their inverses R^{-1}, S^{-1} and $(S \circ R)^{-1}$.

   (b) Verify that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Solution: $A = \{1, 2, 3, 4\}$, $B = \{-1, 0, 1\}$, $C = \{2, 3, 5, 7\}$,
$R = \{(a, b) \in A \times B : ab \text{ is even}\}$ and $S = \{(b, c) \in B \times C : b + 2c \text{ is odd}\}$.

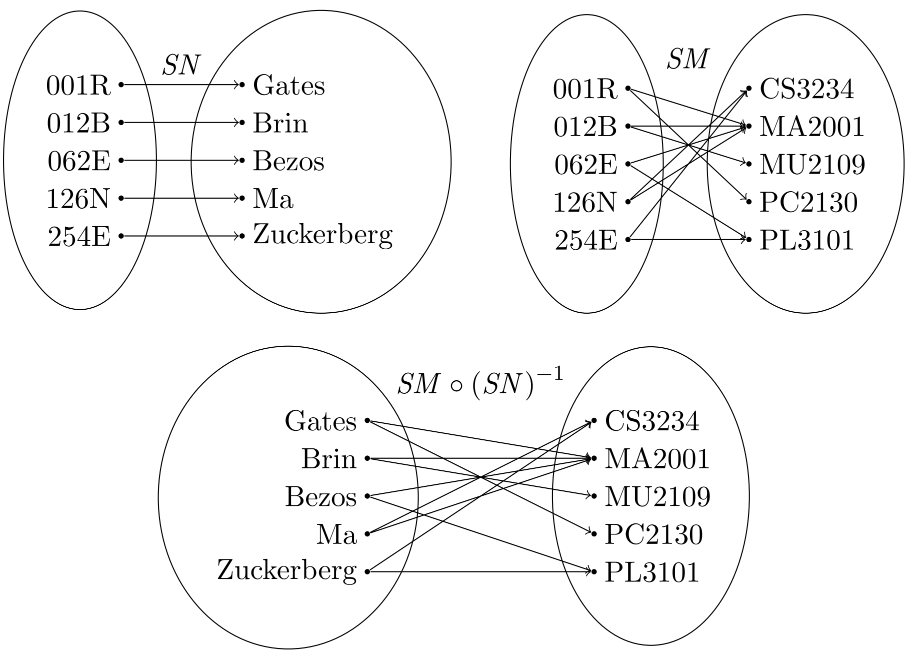
(a)
$$R = \{(1, 0), (2, -1), (2, 1), (3, 0), (4, -1), (4, 1), (2, 0), (4, 0)\}.$$
$$S = \{(-1, 2), (-1, 3), (-1, 5), (-1, 7), (1, 2), (1, 3), (1, 5), (1, 7)\}.$$
$$S \circ R = \{(2, 2), (2, 3), (2, 5), (2, 7), (4, 2), (4, 3), (4, 5), (4, 7)\}.$$
$$R^{-1} = \{(0, 1), (-1, 2), (1, 2), (0, 3), (-1, 4), (1, 4), (0, 2), (0, 4)\}.$$
$$S^{-1} = \{(2, -1), (3, -1), (5, -1), (7, -1), (2, 1), (3, 1), (5, 1), (7, 1)\}.$$
$$(S \circ R)^{-1} = \{(2, 2), (3, 2), (5, 2), (7, 2), (2, 4), (3, 4), (5, 4), (7, 4)\}.$$



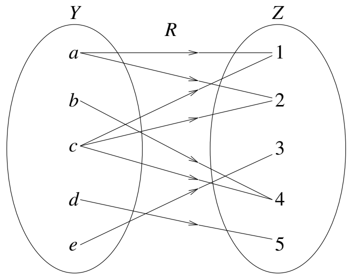
(b) $R^{-1} \circ S^{-1} = \{(2, 2), (2, 4), (3, 2), (3, 4), (5, 2), (5, 4), (7, 2), (7, 4)\} = (S \circ R)^{-1}$.

4. Recall the database relations in Figure 5.1 of the lecture notes. Draw arrow diagrams for SN, SM and $SM \circ (SN)^{-1}$.

Solution:



5. Let $Y = \{a, b, c, d, e\}$ and $Z = \{1, 2, 3, 4, 5\}$, and the arrow diagram of the relation R from Y to Z be as shown on the right.



(a) Determine R^{-1}.

(b) Determine $R^{-1} \circ R$.

Solution:

(a) $R^{-1} = \{(1, a), (1, c), (2, a), (2, c), (3, e), (4, b), (4, c), (5, d)\}$.

(b) $R^{-1} \circ R = \{(a, a), (a, c), (b, b), (b, c), (c, a), (c, c), (d, d), (e, e), (c, b)\}$.

6. Consider the relation $S = \{(m, n) \in \mathbb{Z}^2 : m^3 + n^3 \text{ is even}\}$ on $\mathbb{Z}$. Determine S^{-1}, $S \circ S$ and $S \circ S^{-1}$.

Solution:
$S = \{(m, n) \in \mathbb{Z}^2 : m^3 + n^3 \text{ is even}\}$.
$S^{-1} = \{(x, y) \in \mathbb{Z}^2 : (y, x) \in S\} = \{(x, y) \in \mathbb{Z}^2 : y^3 + x^3 \text{ is even}\}$
$= \{(x, y) \in \mathbb{Z}^2 : x^3 + y^3 \text{ is even}\} = S$.

We claim that $S \circ S = S$.

($\Rightarrow$) Suppose $(x, z) \in S \circ S$. Find $(x, y) \in S$ and $(y, z) \in S$ where $y \in \mathbb{Z}$.
Then $x^3 + y^3$ is even and $y^3 + z^3$ is even. It follows that $(x^3 + y^3) + (y^3 + z^3)$ is even.
This implies $(x^3+2y^3+z^3)$ is even and hence (x^3+z^3) is even. So $(x, z) \in S$. These show $S \circ S \subseteq S$.

($\Leftarrow$) Conversely, suppose $(x, z) \in S$, so that $x^3 + z^3$ is even.
Suppose x^3 is odd. Then z^3 is also odd. So both $x^3 + 1^3$ and $1^3 + z^3$ are even.
Thus $(x, 1) \in S$ and $(1, z) \in S$, making $(x, z) \in S \circ S$.
Similarly, suppose x^3 is even. Then z^3 is also even since $x^3 + z^3$ is even.
Now both $x^3 + 0^3$ and $0^3 + z^3$ are even. Thus $(x, 0) \in S$ and $(0, z) \in S$, making $(x, z) \in S \circ S$.
These show $S \subseteq S \circ S$.

It follows that $S \circ S^{-1} = S \circ S = S$.

7.* A predicate $P(x, y)$ can be represented by a relation R, so that $P(x, y)$ is true if and only if $(x, y) \in R$. For example, if the domains of x and y are $B = \{2, 3, 5, 7, 11, 13\}$ and $C = \{0, 2, 4, 6, 8\}$ respectively, then one can represent the predicate $x = y + 1$ by the relation $\{(3, 2), (5, 4), (7, 6)\}$ over B and C.

(a) For the domains B for x and C for y above, determine the relations that represent the predicates
(i) $x < y$; (ii) x divides y (see Tutorial 2, Problem 2); (iii) $x - y \in C$.

(b) In general, what can you say about the relation R over X and Y that represents a predicate $P(x, y)$ if
(i) $\forall x \in X\ \forall y \in Y\ P(x, y)$ is true? (ii) $\exists x \in X\ \exists y \in Y\ P(x, y)$ is true?

(c) In (b), viewing $X \times Y$ as the universal set, and what is $P(x, y)$ when $(x, y) \in \overline{R}$?

Solution:
$B = \{2, 3, 5, 7, 11, 13\}$ and $C = \{0, 2, 4, 6, 8\}$.

(a) (i) $x < y$: $\{(2, 4), (2, 6), (2, 8), (3, 4), (3, 6), (3, 8), (5, 6), (5, 8), (7, 8)\}$

(ii) x divides y: $\{(2, 0), (2, 2), (2, 4), (2, 6), (2, 8), (3, 6), (3, 0), (5, 0), (7, 0), (11, 0), (13, 0)\}$

(iii) $x - y \in C$: $\{(2, 0), (2, 2)\}$

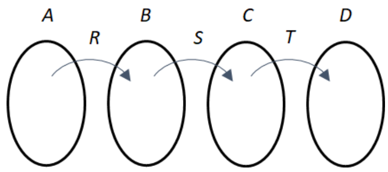
(b) (i) $\forall x \in X\ \forall y \in Y\ P(x, y)$ iff $\forall x \in X\ \forall y \in Y\ (x, y) \in R$ iff $R = X \times Y$.

(ii) $\exists x \in X\ \exists y \in Y\ P(x, y)$ iff $\exists x \in X\ \exists y \in Y\ (x, y) \in R$ iff $R \neq \varnothing$.

(c) If $(x, y) \in \overline{R}$, then $\sim((x, y) \in R)$ and so $\sim P(x, y)$, i.e., $P(x, y)$ is false.

8. Let A, B, C, D be sets and $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq C \times D$. Prove that $T \circ (S \circ R) = (T \circ S) \circ R$ (i.e. composition is associative for relations).

Solution:

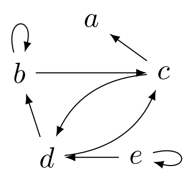


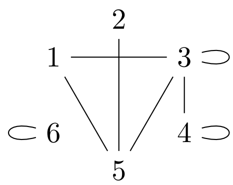
Note that $S \circ R \subseteq A \times C$ and $T \circ S \subseteq B \times D$.
Suppose $(a, d) \in (T \circ S) \circ R$. Find $b \in B$ such that $(a, b) \in R$ and $(b, d) \in T \circ S$.
Note $(b, d) \in T \circ S$ gives $c \in C$ such that $(b, c) \in S$ and $(c, d) \in T$.
Now $(a, b) \in R$ and $(b, c) \in S$ imply $(a, c) \in S \circ R$.
And $(a, c) \in S \circ R$ and $(c, d) \in T$ imply $(a, d) \in T \circ (S \circ R)$. These show $(T \circ S) \circ R \subseteq T \circ (S \circ R)$.

Conversely, suppose $(a, d) \in T \circ (S \circ R)$.
Take $c \in C$ such that $(a, c) \in S \circ R$ and $(c, d) \in T$.
Further, take $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
Therefore, we get $(a, b) \in R$ and $(b, d) \in T \circ S$. So $(a, d) \in (T \circ S) \circ R$.
These show $T \circ (S \circ R) \subseteq (T \circ S) \circ R$.

Hence $(T \circ S) \circ R = T \circ (S \circ R)$.

9. The directed graph (A, D) and the undirected graph (B, E) are shown below:





Determine A, D, B and E.

Solution:
$A = \{a, b, c, d, e\}$, $D = \{(b, b), (b, c), (c, a), (c, d), (d, b), (d, c), (e, d), (e, e)\}$.

$B = \{1, 2, 3, 4, 5, 6\}$, $E = \{\{3, 3\}, \{4, 4\}, \{6, 6\}, \{1, 3\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{3, 4\}\}$.

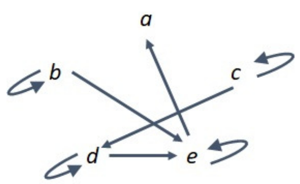
10.* Let $A = \{a, b, c, d, e\}$ and $R = \{(b, b), (b, e), (c, c), (c, d), (d, d), (d, e), (e, a), (e, e)\}$, considered as a relation on A.

(a) Draw an arrow diagram for R.

(b) Determine R^{-1}.

(c) Determine $R \circ R$.

Solution:
$A = \{a, b, c, d, e\}$ and $R = \{(b, b), (b, e), (c, c), (c, d), (d, d), (d, e), (e, a), (e, e)\}$.



(a)

(b) $R^{-1} = \{(b, b), (e, b), (c, c), (d, c), (d, d), (e, d), (a, e), (e, e)\}$.

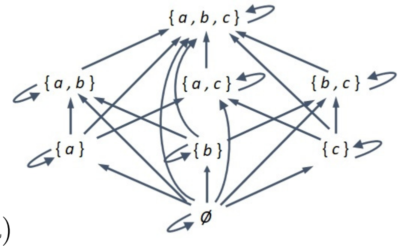
(c) $R \circ R = \{(b, e), (b, a), (c, d), (c, e), (d, e), (d, a), (e, a), (b, b)(d, d), (c, c), (e, e)\}$.

11. Draw the following directed graphs:

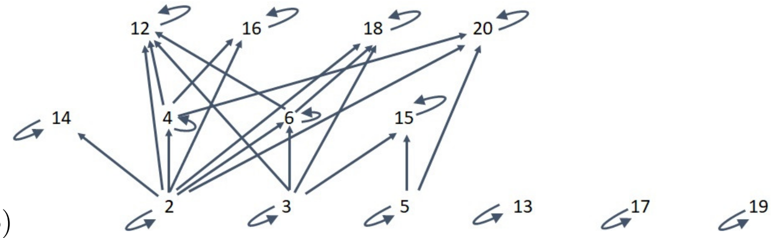
(a) $(\mathcal{P}(\{a,b,c\}), \subseteq)$ where $\subseteq$ is the "subset" relation;

(b) $(\{2,3,4,5,6,12,13,14,15,16,17,18,19,20\}, |)$, where $|$ is the "divides" relation (from Tutorial 2, Problem 2).

Solution:



(a)



(b)

12.* Let $C = \{-4,-3,-2,-1,0,1,2,3,4\}$ and $R = \{(x,y) \in C^2 : y = x^2 - 1\}$, considered as a relation on C.

(a) Draw an arrow diagram for R.

(b) Determine $R \circ R$.

Solution:

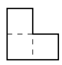
(a)

(b) $R \circ R = \{(1,-1), (-1,-1), (0,0)\}$.

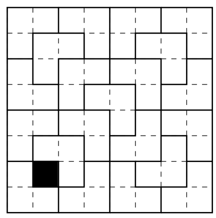
13. A $C \times C$ *chessboard* is a square divided into C rows of C unit squares, where $C \in \mathbb{Z}^+$. For example, the usual chessboard is a $2^3 \times 2^3$ chessboard. An *L-tile* is a $2^1 \times 2^1$ chessboard with one unit square missing (as shown).

Given a $C \times C$ chessboard and any one of its unit squares singled out (like the black one below), can the rest of the chessboard can be covered by non-overlapping L-tiles? (See the example below.)

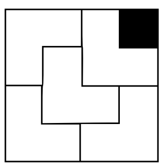
Investigate into the cases $C = 4$, $C = 5$ and $C = 6$. (To be continued in Tutorial 5.)



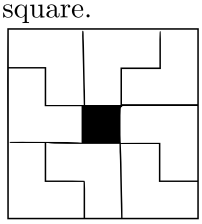
L-tile



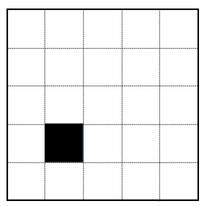
Solution:
Let X denote the position of the black square.



(a)



(b)



(c)

$C = 4$: For any choice of X, a solution exists. For example, see (a) above.
(Informally, this says $\forall X\ \exists$solution.)

$C = 5$: For (b) above, a solution exists; for (c), no solution exists.
(Informally, (b) tells us $\exists X\ \exists$solution, and (c) tells us $\exists X\ {\sim}\exists$solution.)

$C = 6$: Other than X, there are $6^2 - 1 = 35$ squares to be covered by L-tiles.
Each L-tile has 3 squares, and 3 does not divide 35.
Therefore, for any choice of X, no solution exists.
(Informally, this says $\forall X\ {\sim}\exists$solution.)