National University of Singapore
Department of Computer Science

2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 2**

The tutors will discuss the problems that do not have a $*$; if there is sufficient time remaining, they will discuss the problems with a $*$ as well.

1. Let P and Q be predicates. Prove that

   (i) $(\forall x \in D\ P(x)) \wedge (\forall x \in D\ Q(x))$ is true if and only if $\forall x \in D\ (P(x) \wedge Q(x))$ is true;

   (ii) $(\exists x \in D\ P(x)) \wedge (\exists x \in D\ Q(x))$ and $\exists x \in D\ (P(x) \wedge Q(x))$ are not equivalent.

2. An elementary definition in number theory is the following:

   "For integers d and n, $d|n$ if and only if $d \neq 0$ and $n = kd$ for some integer k."

   (Here, $d|n$ is "d *divides* n", and d is called a *divisor* or *factor*.)

   State the above definition symbolically. (Does 2 divide $2\sqrt{2}$?)

3.* A long time ago, you already knew the following:

   (i) There is no biggest number, i.e. no matter how big a number is, there is always another number that is bigger.

   (ii) Between any two given numbers, you can always find another number.

   Formulate these two ideas symbolically, using $\forall$, $\exists$, etc.

4. We will soon introduce the concept of a "relation". For a binary relation R on a set A, there are three important properties:

   R is said to be *reflexive* if and only if "xRx for any x in A".

   R is *symmetric* if and only if "for every x and y in A, if xRy then yRx."

   R is *transitive* if and only if "for all x, y and z in A, if xRy and yRz, then xRz."

   For each property, state the condition ("...") symbolically.

5. Fermat's Last Theorem is a famous claim made more than 300 years ago, and only recently proved. One version of the theorem is:

   "$a^n + b^n \neq c^n$ for all positive integers a, b, c and n, when $n > 2$."

   (i) State the theorem symbolically.

   (ii)* Give a different but equivalent statement of the theorem.

   (iii) Repeat (i), but without the condition $n > 2$.

   (iv) Why is the claim in (iii) false?

6. Another famous claim is the Goldbach Conjecture (about 200 years old, still unproven):

   "Every even integer greater than 2 can be represented as the sum of two prime numbers."

   (i) State the conjecture symbolically.

   (ii) How can you show that the conjecture is wrong (and therefore become instantly famous)?

   (Definitions: An integer n is *even* if and only if there is an integer k such that $n = 2k$; an integer n is *odd* if and only if there is an integer k such that $n = 2k + 1$. We can introduce predicates *Even* and *Odd* and write these symbolically as $Even(n) \leftrightarrow \exists k \in \mathbb{Z}\ n = 2k$ and $Odd(n) \leftrightarrow \exists k \in \mathbb{Z}\ n = 2k + 1$.)

7. * Consider the statement: $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x > y) \rightarrow (x^2 > y^2)$.

   (i) Prove that the statement is false.

   (ii) What is wrong with this argument: "Let $x = -1$ and $y = 2$. Then $x^2 = 1$, $y^2 = 4$, and x^2 is not larger than y^2, so the statement is false."

8. (i) The following is a "proof" that $x^2 \geq 0$ for all real numbers x:
"There are three cases to consider: $x < 0$, $x = 0$ and $x > 0$. If $x < 0$, for example $x = -3$, then $x^2 = 9 > 0$; if $x = 0$, then $x^2 = 0$; if $x > 0$, for example $x = 4$, then $x^2 = 16 > 0$."
What's wrong with this "proof"?

(ii) Use the "method" in (i) to prove that $x^3 = x$ for all real numbers x.

(iii) Here is another "proof" that $x^2 \geq 0$ for all real numbers x:
"We will prove by contradiction. Suppose $x^2 < 0$ for all real numbers x. If we let $x = 3$, then $x^2 = 9$, which is larger than 0, so we get a contradiction. Therefore $x^2 \geq 0$ for all real numbers x."
What's wrong with this "proof"?

(iv) Use the "method" in (iii) to prove that $x^3 = x$ for all real numbers x.

[The point of (ii) and (iv) is: You can "prove" nonsense with bad logic.]

9. ***(i)** The following is a "proof" that $x > 1$ implies $x^2 > 1$ for any real number x:
"Consider any real number x such that $x > 1$. Assume $x^2 > 1$ is true, so $x^2 - 1 > 0$. But $x^2 - 1 = (x-1)(x+1)$, therefore $(x-1)(x+1) > 0$. Since $x > 1$, we have $x - 1 > 0$. Dividing $(x-1)(x+1) > 0$ by the positive number $x - 1$, we get $x + 1 > 0$, which is true since $x > 1$. Therefore $x^2 > 1$ is true." What is wrong with this "proof"?

(ii) Use the "method" in (i) to prove that $1 < 0$.

10.* State symbolically the proverb: "All that glitters is not gold." What does it mean?

[This is an example for why this course avoids non-mathematical statements.]

11. Consider the claim: "If x is a real number and $x^2 > x$, then either $x < 0$ or $x > 1$."

(i) State the claim symbolically.

(ii) Prove the claim.

(iii) Explain the logic behind your proof, i.e. point out where (if any) you have used Universal Instantiation, Modus Ponens, Proof by Cases, Proof by Contradiction, etc.

12. Recall from Problem 6 the definition of odd and even integers. Consider the claim: "There is no integer that is both even and odd."

(i) State the claim symbolically and prove it.

(ii) Explain the logic behind your proof.

(iii) What's wrong with this "proof"?
"Suppose there is an integer n that is both even and odd. Since n is even, there is an integer k such that $n = 2k$. Since n is odd, there is an integer k such that $n = 2k + 1$. Therefore, $2k = 2k + 1$; subtracting $2k$ gives $0 = 1$. This is impossible, so n cannot exist."

13. *(i) Prove the following lemma:
For any $n \in \mathbb{Z}$, if n is odd, then n^2 is odd.

(ii) Prove by contraposition the following statement:
If a, b, c are integers such that $a^2 + b^2 = c^2$, then a and b cannot both be odd.