National University of Singapore
Department of Computer Science

2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 2**

The tutors will discuss the problems that do not have a $*$; if there is sufficient time remaining, they will discuss the problems with a $*$ as well.

1. Let P and Q be predicates. Prove that

   (i) $(\forall x \in D\ P(x)) \wedge (\forall x \in D\ Q(x))$ is true if and only if $\forall x \in D\ (P(x) \wedge Q(x))$ is true;

   (ii) $(\exists x \in D\ P(x)) \wedge (\exists x \in D\ Q(x))$ and $\exists x \in D\ (P(x) \wedge Q(x))$ are not equivalent.

> **Solution:**
>
> (i) ($\Rightarrow$) Suppose $(\forall x \in D\ P(x)) \wedge (\forall x \in D\ Q(x))$ is true. Consider any $a \in D$. Since $\forall x \in D\ P(x)$ is true, we have $P(a)$ is true; similarly, $Q(a)$ is true. Therefore $P(a) \wedge Q(a)$ is true for any $a \in D$, i.e. $\forall x \in D\ (P(x) \wedge Q(x))$ is true.
> ($\Leftarrow$) Suppose $\forall x \in D\ (P(x) \wedge Q(x))$ is true. Consider any $a \in D$. Then $P(a) \wedge Q(a)$ is true, so $P(a)$ is true and $Q(a)$ is true. Since $P(a)$ is true for any $a \in D$, we have $\forall x \in D\ P(x)$ is true; similarly, $\forall x \in D\ Q(x)$ is true. Therefore, $(\forall x \in D\ P(x)) \wedge (\forall x \in D\ Q(x))$ is true.
>
> (ii) To claim that $(\exists x \in D\ P(x)) \wedge (\exists x \in D\ Q(x))$ and $\exists x \in D\ (P(x) \wedge Q(x))$ are equivalent is to claim they have the same truth values for any D, P and Q; i.e. there is an implicit universal quantification over D, P and Q. To prove inequivalence, it therefore suffices to give a counterexample.
> There are many possible counterexamples. Here's one:
> Let $D = \mathbb{N}$, $P(x)$ be "$x^2 = 0$" and $Q(x)$ be "$x^2 = 1$".
> Then $(\exists x \in \mathbb{N}\ x^2 = 0) \wedge (\exists x \in \mathbb{N}\ x^2 = 1)$ is true, but $\exists x \in \mathbb{N}\ (x^2 = 0 \wedge x^2 = 1)$ is false.

2. An elementary definition in number theory is the following:

   "For integers d and n, $d|n$ if and only if $d \neq 0$ and $n = kd$ for some integer k."

   (Here, $d|n$ is "*d divides n*", and d is called a *divisor* or *factor*.)

   State the above definition symbolically. (Does 2 divide $2\sqrt{2}$?)

> **Solution:**
> $\forall d \in \mathbb{Z}\ \forall n \in \mathbb{Z}\ d|n \leftrightarrow (d \neq 0 \wedge \exists k \in \mathbb{Z}\ n = kd)$
> (It does not make sense to ask if 2 divides $2\sqrt{2}$, since the definition for $d|n$ assumes n is an integer and $2\sqrt{2} \notin \mathbb{Z}$, nor is there any $k \in \mathbb{Z}$ such that $2\sqrt{2} = 2k$.)

3.* A long time ago, you already knew the following:

   (i) There is no biggest number, i.e. no matter how big a number is, there is always another number that is bigger.

   (ii) Between any two given numbers, you can always find another number.

   Formulate these two ideas symbolically, using $\forall$, $\exists$, etc.

> **Solution:**
>
> (i) $\sim \exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ y \leq x$ or, equivalently, $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ y > x$.
> Nonnegative integers $\mathbb{N}$ here can be replaced by integers $\mathbb{Z}$, or rational numbers $\mathbb{Q}$, or real numbers $\mathbb{R}$ (but not complex numbers $\mathbb{C}$).
>
> (ii) $\forall x \in \mathbb{Q}\ \forall y \in \mathbb{Q}\ (x < y) \rightarrow \exists z \in \mathbb{Q}\ (x < z) \wedge (z < y)$
> Rational numbers $\mathbb{Q}$ here can be replaced by $\mathbb{R}$, but cannot be replaced by $\mathbb{Z}$, nor $\mathbb{C}$.

4. We will soon introduce the concept of a "relation". For a binary relation R on a set A, there are three important properties:

   R is said to be *reflexive* if and only if "xRx for any x in A".

   R is *symmetric* if and only if "for every x and y in A, if xRy then yRx."

   R is *transitive* if and only if "for all x, y and z in A, if xRy and yRz, then xRz."

   For each property, state the condition ("...") symbolically.

Solution:

Reflexive: $\forall x \in A\ xRx$

Symmetric: $\forall x \in A\ \forall y \in A\ xRy \rightarrow yRx$

Transitive: $\forall x \in A\ \forall y \in A\ \forall z \in Z\ xRy \wedge yRz \rightarrow xRz$

Here, "for any", "for every", "for all" mean the same thing.

5. Fermat's Last Theorem is a famous claim made more than 300 years ago, and only recently proved. One version of the theorem is:

   "$a^n + b^n \neq c^n$ for all positive integers a, b, c and n, when $n > 2$."

   (i) State the theorem symbolically.

   (ii)* Give a different but equivalent statement of the theorem.

   (iii) Repeat (i), but without the condition $n > 2$.

   (iv) Why is the claim in (iii) false?

Solution:

(i) $\forall n \in \mathbb{Z}^+\ n > 2 \rightarrow \forall a \in \mathbb{Z}^+\ \forall b \in \mathbb{Z}^+\ \forall c \in \mathbb{Z}^+\ a^n + b^n \neq c^n$.
(alternative: $\forall a \in \mathbb{Z}^+\ \forall b \in \mathbb{Z}^+\ \forall c \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}^+\ n > 2 \rightarrow\ a^n + b^n \neq c^n$, etc.)

(ii)* $\sim (\sim \forall a \in \mathbb{Z}^+\ \forall b \in \mathbb{Z}^+\ \forall c \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}^+\ n > 2 \rightarrow\ a^n + b^n \neq c^n$
$\equiv \sim (\exists a \in \mathbb{Z}^+ \exists b \in \mathbb{Z}^+ \exists c \in \mathbb{Z}^+ \exists n \in \mathbb{Z}^+\ (n > 2) \wedge (a^n + b^n = c^n))$
i.e. there are no positive integers a, b, c and n such that $n > 2$ and $a^n + b^n = c^n$.

(iii) $\forall n \in \mathbb{Z}^+\ \forall a \in \mathbb{Z}^+\ \forall b \in \mathbb{Z}^+\ \forall c \in \mathbb{Z}^+\ a^n + b^n \neq c^n$.

(iv) The negation of (iii) is $\exists n \in \mathbb{Z}^+\ \exists a \in \mathbb{Z}^+\ \exists b \in \mathbb{Z}^+\ \exists c \in \mathbb{Z}^+\ a^n + b^n = c^n$,
which is true (examples: $1 + 2 = 3$, $3^2 + 4^2 = 5^2$), so the claim in (iii) is false.

Note: $(n > 2 \rightarrow\ a^n + b^n \neq c^n) \equiv (a^n + b^n = c^n \rightarrow\ n \leq 2)$
$\equiv (a^n + b^n = c^n \rightarrow\ (n = 1\ \vee\ n = 2))$.

6. Another famous claim is the Goldbach Conjecture (about 200 years old, still unproven):
   "Every even integer greater than 2 can be represented as the sum of two prime numbers."

   (i) State the conjecture symbolically.

   (ii) How can you show that the conjecture is wrong (and therefore become instantly famous)?

   (Definitions: An integer n is *even* if and only if there is an integer k such that $n = 2k$; an integer n is *odd* if and only if there is an integer k such that $n = 2k + 1$. We can introduce predicates *Even* and *Odd* and write these symbolically as $Even(n) \leftrightarrow \exists k \in \mathbb{Z}\ n = 2k$ and $Odd(n) \leftrightarrow \exists k \in \mathbb{Z}\ n = 2k + 1$.)

Solution:

(i) $\forall n \in \mathbb{Z}\ (n \text{ is even}) \ \wedge \ (n > 2) \rightarrow$
$\exists q_1 \in \mathbb{Z}\ \exists q_2 \in \mathbb{Z}\ (q_1 \text{ is prime}) \ \wedge \ (q_2 \text{ is prime}) \ \wedge \ (n = q_1 + q_2)$.

Alternatives:

- Define predicates $Even(n)$ and $Prime(n)$; then
  $\forall n \in \mathbb{Z}\ Even(n) \wedge \ (n > 2) \rightarrow \ \exists q_1 \in \mathbb{Z}\ \exists q_2 \in \mathbb{Z}\ Prime(q_1) \wedge \ Prime(q_2) \wedge \ n = q_1 + q_2$.
- Define sets $\mathbf{E} = \{\text{even integers}\}$ and $\mathbf{P} = \{\text{primes}\}$; then
  $\forall n \in \mathbb{Z}\ (n \in \mathbf{E}) \ \wedge \ (n > 2) \rightarrow \ \exists q_1 \in \mathbf{P}\ \exists q_2 \in \mathbf{P}\ n = q_1 + q_2$.

(ii) Negating gives $\exists n \in \mathbb{Z}\ (n \in \mathbf{E}) \ \wedge \ (n > 2) \ \wedge \ \forall q_1 \in \mathbf{P}\ \forall q_2 \in \mathbf{P}\ n \neq q_1 + q_2$.
To become famous, find an even n larger than 2 that is not the sum of two primes.

7. * Consider the statement: $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x > y) \rightarrow (x^2 > y^2)$.
   (i) Prove that the statement is false.
   (ii) What is wrong with this argument: "Let $x = -1$ and $y = 2$. Then $x^2 = 1$, $y^2 = 4$, and x^2 is not larger than y^2, so the statement is false."

Solution:

(i) $\sim (\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x > y) \rightarrow (x^2 > y^2)) \equiv \exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x > y) \wedge (x^2 \leq y^2)$,
which is **true** (example: $x = 1, y = -2$) so the original statement is **false**.

(ii) The counterexample $x = -1$ and $y = 2$ is irrelevant since it does not satisfy $x > y$.
Counterexamples have to satisfy the hypothesis ($x > y$ in this case).

Note to tutors:
Similar errors in this and the following exercises were actually committed by students before.

8. (i) The following is a "proof" that $x^2 \geq 0$ for all real numbers x:
   "There are three cases to consider: $x < 0$, $x = 0$ and $x > 0$. If $x < 0$, for example $x = -3$, then $x^2 = 9 > 0$; if $x = 0$, then $x^2 = 0$; if $x > 0$, for example $x = 4$, then $x^2 = 16 > 0$."
   What's wrong with this "proof"?
   (ii) Use the "method" in (i) to prove that $x^3 = x$ for all real numbers x.
   (iii) Here is another "proof" that $x^2 \geq 0$ for all real numbers x:
   "We will prove by contradiction. Suppose $x^2 < 0$ for all real numbers x. If we let $x = 3$, then $x^2 = 9$, which is larger than 0, so we get a contradiction. Therefore $x^2 \geq 0$ for all real numbers x."
   What's wrong with this "proof"?
   (iv) Use the "method" in (iii) to prove that $x^3 = x$ for all real numbers x.

   [The point of (ii) and (iv) is: You can "prove" nonsense with bad logic.]

Solution:

(i) The statement is $\forall x \in \mathbb{R}\ x^2 \geq 0$.
Looks like Proof by Cases (but is really a "Proof by Examples", which is invalid for a $\forall$ statement).
Error: does not consider arbitrary x.

(ii) Claim: $\forall x \in \mathbb{R}\ x^3 = x$.
"Proof": "There are three cases to consider: $x < 0$, $x = 0$ and $x > 0$. If $x < 0$, for example $x = -1$, then $x^3 = (-1)^3 = -1 = x$; if $x = 0$, then $x^3 = 0 = x$; if $x > 0$, for example $x = 1$, then $x^3 = 1^3 = 1 = x$."

(iii) To prove by contradiction, one must consider $\sim \forall x \in \mathbb{R}\ x^2 \geq 0$,
which is $\exists x \in \mathbb{R} \sim (x^2 \geq 0)$.
Error: $\sim \forall x \in \mathbb{R}\ x^2 \geq 0 \not\equiv \forall x \in \mathbb{R}\ \sim (x^2 \geq 0)$.

(iv) Claim: $\forall x \in \mathbb{R}\ x^3 = x$.
"Proof": "We will prove by contradiction. Suppose $x^3 \neq x$ for all real numbers x. If we let $x = 0$, then $x^3 = 0 = x$, so we get $x^3 = x$, a contradiction. Therefore $x^3 = x$ for all real numbers x."

Note to tutor: Please make sure the students understand $\forall x \in \mathbb{R}\ x^3 = x$ is false.

9.* (i) The following is a "proof" that $x > 1$ implies $x^2 > 1$ for any real number x:
"Consider any real number x such that $x > 1$. Assume $x^2 > 1$ is true, so $x^2 - 1 > 0$. But $x^2 - 1 = (x-1)(x+1)$, therefore $(x-1)(x+1) > 0$. Since $x > 1$, we have $x - 1 > 0$. Dividing $(x-1)(x+1) > 0$ by the positive number $x - 1$, we get $x + 1 > 0$, which is true since $x > 1$. Therefore $x^2 > 1$ is true." What is wrong with this "proof"?

(ii) Use the "method" in (i) to prove that $1 < 0$.

Solution:

(i) The claim is: $\forall x \in \mathbb{R}\ x > 1 \rightarrow\ x^2 > 1$.
Error: Assuming what is to be proven ($x^2 > 1$) to be true.

(ii) "Proof": "Assume $1 < 0$ is true. Since $0 < 2$, we get $1 < 2$, which is true. Therefore $1 < 0$ is true."
Many other such "proofs" are possible.

10.* State symbolically the proverb: "All that glitters is not gold." What does it mean?

[This is an example for why this course avoids non-mathematical statements.]

Solution:
$\forall x\ Glitters(x) \rightarrow\ \sim Gold(x)$
The proverb actually means $\sim \forall x\ Glitters(x) \rightarrow\ Gold(x)$.

Note to tutors:

The next two problems require the students to prove some simple statements, then explain their proofs.

The solutions provided here only suggest what explanations are expected. The students are completely free (and likely) to offer proofs that are different from the ones given here. The focus is on the logic in the proofs, not the proofs themselves.

The proofs invoke elementary properties of numbers, which can be found in Appendix A (attached) from the reference book by Epp.

When explaining their proofs, the students must clarify
(i) how they deal with the quantifiers (∀, ∃) and
(ii) the flow of logic.

11. Consider the claim: "If x is a real number and $x^2 > x$, then either $x < 0$ or $x > 1$."

(i) State the claim symbolically.

(ii) Prove the claim.

(iii) Explain the logic behind your proof, i.e. point out where (if any) you have used Universal Instantiation, Modus Ponens, Proof by Cases, Proof by Contradiction, etc.

Solution:

(i) $\forall x \in \mathbb{R}\ (x^2 > x) \rightarrow\ (x < 0) \vee (x > 1)$

(ii) Proof:	(iii) Explanation:
Consider any $x \in \mathbb{R}$.	Pick an arbitrary element from the domain.
	If $x^2 > x$ is false,
	then $(x^2 > x) \rightarrow\ (x < 0) \vee (x > 1)$ is true
	by definition of $p \rightarrow q$
Suppose $x^2 > x$.	If $x^2 > x$ is true, we must prove $(x < 0) \vee (x > 1)$ is true,
	to show $(x^2 > x) \rightarrow\ (x < 0) \vee (x > 1)$ is true.
Then $x^2 - x > 0$.	This shuffling is universal instantiation (U.I.)
	followed by modus ponens (M.P.).
	The Appendix gives
	$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \forall c \in \mathbb{R}\ a < b\ \rightarrow\ a + c < b + c$
	$x \in \mathbb{R},\ x^2 \in \mathbb{R},\ -x \in \mathbb{R}$
	U.I. $\quad x < x^2 \rightarrow\ x + (-x) < x^2 + (-x)$
	Suppose $\quad x < x^2$
	M.P. $\quad 0 < x^2 - x$
so $x(x-1) > 0$. Therefore	etc.
either $x > 0$ and $x - 1 > 0$	etc.
or $x < 0$ and $x - 1 < 0$	etc.
i.e. $x > 0$ and $x > 1$	etc.
or $x < 0$ and $x < 1$.	etc.
This is iff $x > 1$ or $x < 0$.	i.e. $(x < 0) \vee (x > 1)$ is true,
	so $(x^2 > x) \rightarrow\ (x < 0) \vee (x > 1)$ is true.

12. Recall from Problem 6 the definition of odd and even integers. Consider the claim: "There is no integer that is both even and odd."

(i) State the claim symbolically and prove it.

(ii) Explain the logic behind your proof.

(iii) What's wrong with this "proof"?
"Suppose there is an integer n that is both even and odd. Since n is even, there is an integer k such that $n = 2k$. Since n is odd, there is an integer k such that $n = 2k + 1$. Therefore, $2k = 2k + 1$; subtracting $2k$ gives $0 = 1$. This is impossible, so n cannot exist."

Solution:

(i) $\sim \exists n \in \mathbb{Z}\ Even(n) \wedge\ Odd(n)$	$Even(n)$ and $Odd(n)$ are defined in Problem 6
(ii) Proof:	Proof by contradiction:
Suppose there is some integer n that is both even and odd.	
Then $n = 2k_1$ for some $k_1 \in \mathbb{Z}$ and $n = 2k_2 + 1$ for some $k_2 \in \mathbb{Z}$	The k in the definition of $Even(n)$ and $Odd(n)$ is a dummy variable, so k_1 and k_2 can be different.
Therefore $2k_1 = 2k_2 + 1$,	
so $2(k_1 - k_2) = 1$,	Algebra that hides U.I., M.P. etc.
i.e. $2m = 1$ for some $m \in \mathbb{Z}$.	Invoking closure property of subtraction.
This is impossible, so n does not exist.	A rigorous argument for "impossible" requires knowledge of elementary number theory.

(iii) See remark above on dummy variables.

13.* (i) Prove the following lemma:
For any $n \in \mathbb{Z}$, if n is odd, then n^2 is odd.

(ii) Prove by contraposition the following statement:
If a, b, c are integers such that $a^2 + b^2 = c^2$, then a and b cannot both be odd.

Solution:

(i) Suppose n is odd, so $n = 2k + 1$ for some integer k. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so $n^2 = 2m + 1$ for integer $m = 2k^2 + 2k$. Therefore n^2 is odd.

(ii) For a proof by contraposition, we start by assuming a and b are both odd, so $a = 2h + 1$ and $b = 2k + 1$ for some integers h and k. Then $a^2 + b^2 = (2h + 1)^2 + (2k + 1)^2 = 2(2h^2 + 2h + 2k^2 + 2k + 1)$, so $a^2 + b^2$ is even.
Now, suppose $a^2 + b^2 = c^2$. Then c^2 is even, so by (i), c must be even (if c is odd, then (i) says c^2 is also odd, contradicting Problem 12.) Therefore, $c = 2m$ for some integer m, and $a^2 + b^2 = c^2$ gives $4(h^2 + h + k^2 + k) + 2 = 4m^2$. Dividing by 2, we get $2(h^2 + h + k^2 + k) + 1 = 2m^2$, where the left-hand side is odd and the right-hand side is even, contradicting Problem 12. We arrive at this contradiction by supposing $a^2 + b^2 = c^2$, so it must be that $a^2 + b^2 \neq c^2$.
The claim now follows by contraposition.

Appendix A

Properties of the Real Numbers*

In this text we take the real numbers and their basic properties as our starting point. We give a core set of properties, called axioms, which the real numbers are assumed to satisfy, and we state some useful properties that can be deduced from these axioms.

We assume that there are two binary operations defined on the set of real numbers, called **addition** and **multiplication**, such that if a and b are any two real numbers, the **sum** of a and b, denoted $\boldsymbol{a+b}$, and the **product** of a and b, denoted $\boldsymbol{a \cdot b}$ or $\boldsymbol{ab}$, are also real numbers. These operations satisfy properties F1, F2, F3, F4, F5, and F6, which are called the **field axioms**.

F1. *Commutative Laws* For all real numbers a and b,

$$a+b=b+a \quad \text{and} \quad ab=ba.$$

F2. *Associative Laws* For all real numbers a, b, and c,

$$(a+b)+c=a+(b+c) \quad \text{and} \quad (ab)c=a(bc).$$

F3. *Distributive Laws* For all real numbers a, b, and c,

$$a(b+c)=ab+ac \quad \text{and} \quad (b+c)a=ba+ca.$$

F4. *Existence of Identity Elements* There exist two distinct real numbers, denoted 0 and 1, such that for every real number a,

$$0+a=a+0=a \quad \text{and} \quad 1\cdot a=a\cdot 1=a.$$

F5. *Existence of Additive Inverses* For every real number a, there is a real number, denoted $-a$ and called the **additive inverse** of a, such that

$$a+(-a)=(-a)+a=0.$$

F6. *Existence of Reciprocals* For every real number $a \neq 0$, there is a real number, denoted $1/a$ or a^{-1}, called the **reciprocal** of a, such that

$$a\cdot\left(\frac{1}{a}\right)=\left(\frac{1}{a}\right)\cdot a=1.$$

All the usual algebraic properties of the real numbers that do not involve order can be derived from the field axioms. The most important are collected as theorems T1, T2, T3, T4, T5, T6, T7, T8, T9, T10, T11, T12, T13, T14, T15, and T16 as follows. In all these theorems the symbols a, b, c, and d represent arbitrary real numbers.

T1. *Cancellation Law for Addition* If $a+b=a+c$, then $b=c$. (In particular, this shows that the number 0 of Axiom F4 is unique.)

T2. *Possibility of Subtraction* Given a and b, there is exactly one x such that $a+x=b$. This x is denoted by $b-a$. In particular, $0-a$ is the additive inverse of a, $-a$.

T3. $b-a=b+(-a)$.

T4. $-(-a)=a$.

T5. $a(b-c)=ab-ac$.

T6. $0\cdot a=a\cdot 0=0$.

T7. *Cancellation Law for Multiplication* If $ab=ac$ and $a\neq 0$, then $b=c$. (In particular, this shows that the number 1 of Axiom F4 is unique.)

T8. *Possibility of Division* Given a and b with $a\neq 0$, there is exactly one x such that $ax=b$. This x is denoted by b/a and is called the **quotient** of b and a. In particular, $1/a$ is the reciprocal of a.

T9. If $a\neq 0$, then $b/a=b\cdot a^{-1}$.

T10. If $a\neq 0$, then $\left(a^{-1}\right)^{-1}=a$.

T11. *Zero Product Property* If $ab=0$, then $a=0$ or $b=0$.

T12. *Rule for Multiplication with Negative Signs*

$$(-a)b=a(-b)=-(ab), \quad (-a)(-b)=ab,$$

and

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}.$$

T13. *Equivalent Fractions Property*

$$\frac{a}{b}=\frac{ac}{bc}, \quad \text{if } b\neq 0 \text{ and } c\neq 0.$$

T14. *Rule for Addition of Fractions*

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}, \quad \text{if } b\neq 0 \text{ and } d\neq 0.$$

T15. *Rule for Multiplication of Fractions*

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}, \quad \text{if } b\neq 0 \text{ and } d\neq 0.$$

T16. *Rule for Division of Fractions*

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{ad}{bc}, \quad \text{if } b\neq 0, c\neq 0, \text{and } d\neq 0.$$

The real numbers also satisfy the following axioms, called the **order axioms.** It is assumed that among all real numbers there are certain ones, called the **positive real numbers**, that satisfy properties Ord1, Ord2, and Ord3.

Ord1. For any real numbers a and b, if a and b are positive, so are $a + b$ and ab.

Ord2. For every real number $a \neq 0$, either a is positive or $-a$ is positive but not both.

Ord3. The number 0 is not positive.

The symbols $<$, $>$, $\leq$, and $\geq$, and negative numbers are defined in terms of positive numbers.

Definition

Given real numbers a and b,

$a < b$ means $b + (-a)$ is positive.	$b > a$ means $a < b$.
$a \leq b$ means $a < b$ or $a = b$.	$b \geq a$ means $a \leq b$.
If $a < 0$, we say that a is **negative**.	If $a \geq 0$, we say that a is **nonnegative**.

From the order axioms Ord1, Ord2, and Ord3 and the above definition, all the usual rules for calculating with inequalities can be derived. The most important are collected as theorems T17, T18, T19, T20, T21, T22, T23, T24, T25, T26, and T27 as follows. In all these theorems the symbols a, b, c, and d represent arbitrary real numbers.

T17. *Trichotomy Law* For arbitrary real numbers a and b, exactly one of the three relations $a < b$, $b < a$, or $a = b$ holds.

T18. *Transitive Law* If $a < b$ and $b < c$, then $a < c$.

T19. If $a < b$, then $a + c < b + c$.

T20. If $a < b$ and $c > 0$, then $ac < bc$.

T21. If $a \neq 0$, then $a^2 > 0$.

T22. $1 > 0$.

T23. If $a < b$ and $c < 0$, then $ac > bc$.

T24. If $a < b$, then $-a > -b$. In particular, if $a < 0$, then $-a > 0$.

T25. If $ab > 0$, then both a and b are positive or both are negative.

T26. If $a < c$ and $b < d$, then $a + b < c + d$.

T27. If $0 < a < c$ and $0 < b < d$, then $0 < ab < cd$.