1. Consider an undirected graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{6, 8\}, \{7, 8\}, \{8, 9\}\}$.

   (i) Draw G.

   (ii) Draw a subgraph (V', E') that is not a tree but satisfies $|E'| = |V'| - 1$.

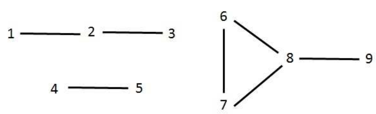
   (iii) Draw a connected subgraph (V'', E'') that is not a tree, but has an edge $\{x, y\}$ such that $(V'', E'' \setminus \{\{x, y\}\})$ is not connected.

   (iv) Let T be a rooted tree such that T's edges are in E and T's root is 7. Draw all possible Ts.

   (v) Draw another graph $G'' = (\{1, 2, 3, 4, 5, 6, 7, 8, 9\}, E'')$ such that $E'' \neq E$ but G'' is isomorphic to G.

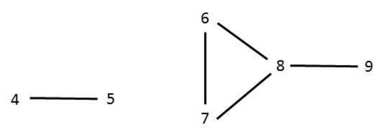
   (vi) Determine the number of graphs $G'' = (\{1, 2, 3, 4, 5, 6, 7, 8, 9\}, E'')$ such that $E'' \neq E$ but G'' is isomorphic to G.

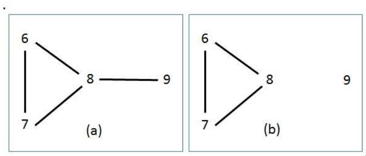
Solution:



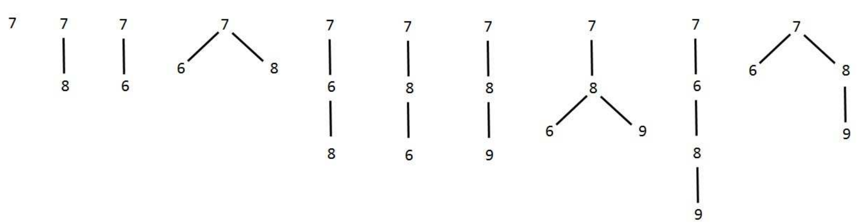
(i)

(ii) Many possibilities. Example:



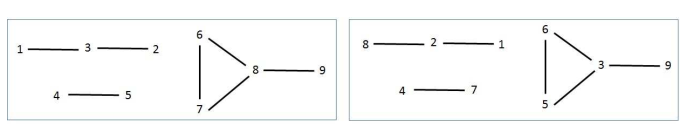


(iii) (a) is connected but not a tree since it has a cycle;
if edge $\{8, 9\}$ is removed, it becomes disconnected.



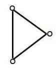
(iv)

(v) Many possibilities. Examples:



(vi) There are:

$\binom{9}{3}$ ways of choosing three nodes for
and $\binom{3}{1}$ ways of choosing the middle node
$\binom{6}{2}$ ways of choosing two nodes for



$\binom{4}{3}$ ways of choosing the cycle
and $\binom{3}{1}$ ways of choosing where to attach the last node.
By Multiplication Rule, number of isomorphic G's is:
$\binom{9}{3}\binom{3}{1}\binom{6}{2}\binom{4}{3}\binom{3}{1} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} 3 \frac{6 \cdot 5}{2 \cdot 1} 4 \cdot 3 = 45360.$

Alternative: $\frac{9!}{2!2!2!}$
Number of isomorphic G'' with $E'' \neq E$ is $45360 - 1 = 45359$.

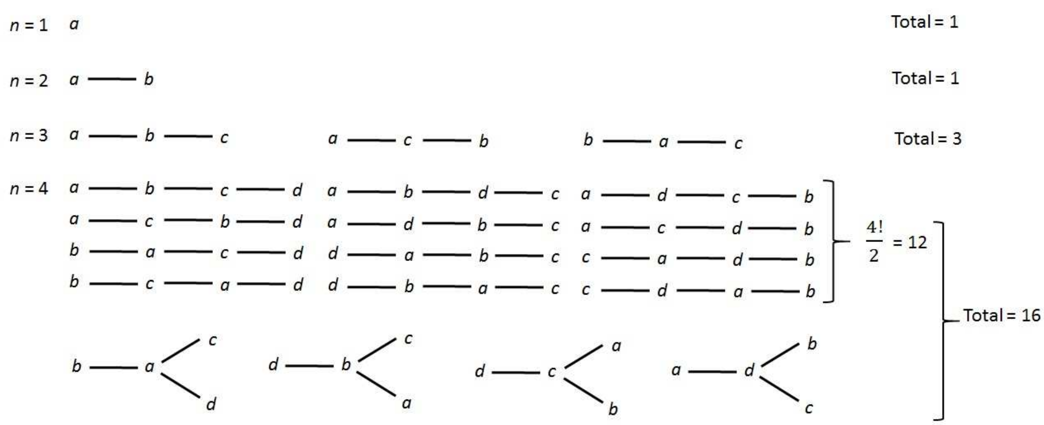
2.* (i) Draw all trees with n nodes for $n = 1, 2, 3, 4$. What is the general formula for the number of trees with n nodes?

(ii) Determine the number of nonisomorphic trees with n nodes, for $n = 1, 2, 3, 4$.

What is the relationship between (i) and (ii)?

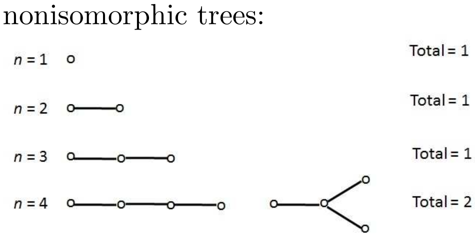
Solution:

(i) trees:



For $n \geq 2$, the general formula is n^{n-2}.
Note to tutors: Proof not expected. (Induction is the obvious choice.)

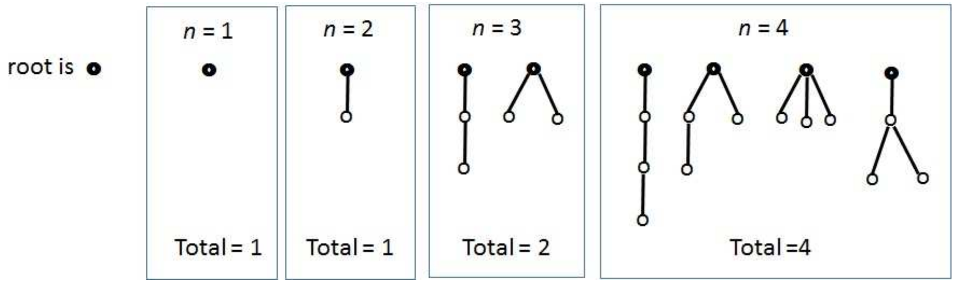
(ii) nonisomorphic trees:



For each tree in (ii), labelling it and permuting the labels gives us the trees in (i).

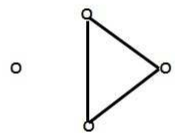
3. For two rooted trees to be isomorphic under a permutation π, we require that the image $\pi(u)$ of a root u must also be a root. Determine the number of nonisomorphic rooted trees with n nodes, for $n = 1, 2, 3, 4$.

Solution:



4.* Let $G = (V, E)$ be an undirected graph. Prove that if G is connected, then $|E| \geq |V| - 1$. Is the converse true?

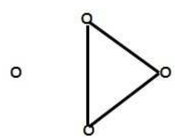
Solution:
Since $G = (V, E)$ is connected, it has a spanning tree (V, F), where $F \subseteq E$.
Therefore $|E| \geq |F| = |V| - 1$, since (V, F) is a tree.



The converse is **false**; example:
$|E| = |V| - 1$ but (V, E) is not connected.

5. Let $G = (V, E)$ be an undirected graph. Prove that if G is acyclic, then $|E| \leq |V| - 1$. Is the converse true?

Solution: Suppose $G = (V, E)$ is an acyclic undirected graph.
Let $H_1 = (V_1, E_1), \ldots, H_k = (V_k, E_k)$ be connected components of G.
Then each H_i is connected and acyclic, i.e. H_i is a tree.
Therefore $|E_i| = |V_i| - 1$,
so $|E| = |E_1| + \cdots + |E_k|$ by Addition Rule
$= (|V_1| - 1) + \cdots + (|V_k| - 1) = |V| - k$.
Thus, $|V| - |E| = k \geq 1$, so $|E| \leq |V| - 1$.



The converse is **false**; example:
$|E| = 3 = |V| - 1$ but (V, E) is not acyclic.

6.* Prove that a loopless undirected graph is a tree if and only if there is exactly one path between every pair of nodes.

Solution:

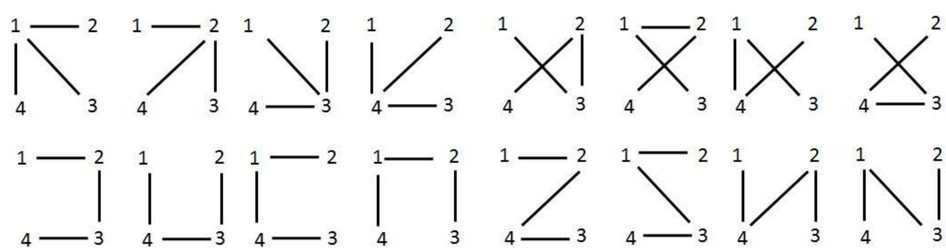
(⇒) Let G be a tree. Then G is connected, so for every pair of nodes x and y, there is a path between x and y.
If some pair has more than one path, then G is cyclic (Theorem 4.3).
Therefore, every pair of nodes has exactly one path between them.

(⇐) Assume G is a loopless undirected graph in which there is exactly one path between every pair of nodes. Then G is connected.
Suppose G is cyclic. Then, since G is loopless and connected, there is a pair of nodes that has more than one path connecting them (Theorem 4.3), contradicting the given assumption. Therefore G is acyclic.
Since G is connected, G is thus a tree.

7. Recall from Tutorial 9 (Problem 7) that a complete graph has all loops and all edges.
How many spanning trees are there for the complete graph with 4 nodes?

Solution:

The complete graph for 4 nodes has 16 $(= \binom{4}{1} + \frac{1}{2}4!)$ spanning trees.



8.* Consider a rooted tree T in which every parent has at most b children ($b \in \mathbb{Z}^+$; a binary tree has $b = 2$).

State and prove a result relating the number of leaves and number of parents in T, if each parent has exactly b children.

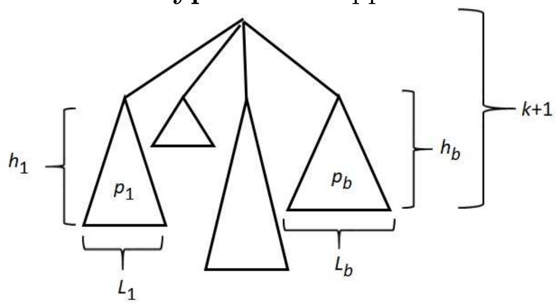
Solution:

Claim: Let T be a rooted tree in which every parent has exactly b children.
If T has L leaves and p parents, then $L = (b-1)p + 1$.

Proof: By 2PI on height h.

Basis $h = 0$: Then there is just 1 leaf (the root) and no parents.
Therefore, $L = 1 = (b-1)0 + 1$, so the claim is true for $h = 0$.

Induction Hypothesis Suppose the claim is true if $0 \le h \le k$, for some $k \ge 0$.



Induction Step Consider a tree with height $h = k + 1$.
The root has b children, which are roots of subtrees.
Let $L_1, \ldots, L_b$ be the number of leaves in these subtrees,
$h_1, \ldots, h_b$ the heights of these subtrees,
and $p_1, \ldots, p_b$ the number of parents in these subtrees.
Since $h_i \le k$, the Induction Hypothesis gives $L_i = (b-1)p_i + 1$.
Hence, $L = L_1 + \cdots + L_b = \sum_{i=1}^{b}((b-1)p_i + 1)$
$= (b-1)(\sum_{i=1}^{b} p_i) + b = (b-1)(p-1) + b$,
where T's root is not among the p_i.
Therefore $L = (b-1)p + 1$ and the claim is true for $h = k + 1$.
By induction, the claim is true for all $h \ge 0$.

9. For $n \geq 2$, a directed graph $(\{v_1, v_2, \ldots, v_n\}, \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\})$ is called a **cycle**. A directed graph $G = (V, D)$ is **cyclic** if it contains a loop or a cycle as a subgraph; otherwise, it is **acyclic**

(i) Prove that if G is acyclic, then D is antisymmetric.

(ii) Prove or disprove the converse of (i).

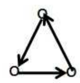
(iii) Prove that if D is a partial order, then G does not contain any cycles.
[We can hence arrange the edges in the graph for a partial order, so they all point in one direction.]

(iv)* Prove that, for any $n \geq 2$, there is a directed acyclic graph with n nodes and $\frac{1}{2}n(n-1)$ edges.
[Contrast this with the $|E| = |V| - 1$ characterization for undirected acyclic graphs.]

(v)* Prove that any directed graph with n nodes and more than $\frac{1}{2}n(n-1)$ edges must be cyclic.

Solution:

(i) Suppose D is not antisymmetric.
Then there exist $x, y \in V$, $x \neq y$, such that $(x, y) \in D$ and $(y, x) \in D$, so G has a cycle.
The contrapositive follows.

(ii) The converse is *false*:  is antisymmetric but not acyclic.

(iii) Let D be a partial order. Suppose G has a cycle
$\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$, where $n \geq 2$ and $v_1, v_2, \ldots, v_n$ are distinct.
The $v_1 D v_2, v_2 D v_3, \ldots, v_{n-1} D v_n, v_n D v_1$.
By transitivity, $v_1 D v_3, v_1 D v_4, \ldots, v_1 D v_n$.
We thus have $v_n D v_1$, $v_1 D v_n$ and $v_1 \neq v_n$,
contradicting the antisymmetric property of a partial order.

(iv)* Construct (V, D) as follows:

$V \leftarrow \{v_1\}$;
$D \leftarrow \emptyset$;
for $k = 2$ to n do
  $V \leftarrow V \cup \{v_k\}$;
  $D \leftarrow D \cup \{(v_i, v_k) \ : \ i = 1, \ldots, k-1\}$;
  $k \leftarrow k + 1$;
end

At the end of the loop, $V = \{v_1, \ldots, v_n\}$ and (V, D) has no loops and no cycles since all edges are of the form (v_i, v_j) where $i < j$, so (V, D) is acyclic.
The number of edges is $1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1)$.

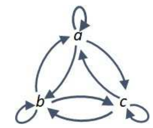
Alternative: Take the complete graph on n nodes, remove all loops,
lay all the nodes down on a line, and point all the edges in one direction.
A complete graph has $\binom{n}{2} = \frac{1}{2}n(n-1)$ edges.

(v)* Suppose $G = (V, D)$ where $|V| = n$ and $|D| > \frac{1}{2}n(n-1)$.
Case G has a loop: Then G is cyclic.
Case G has no loops: Then all the edges in D are of the form (x, y), where x, y are in V.
There are only $\binom{n}{2} = \frac{1}{2}n(n-1)$ different choices of $\{x, y\}$.
Since $|D| > \frac{1}{2}n(n-1)$, by Pigeonhole Principle,
there are two edges with the same choice of $\{x, y\}$,
i.e. (x, y) and (y, x) are both in D.
These form a cycle, so G is cyclic.

10. Let $\mathcal{D}_3$ be the set of all directed graphs whose nodes are a, b, c. Suppose $G = (\{a,b,c\}, D) \in \mathcal{D}_3$. Determine the number of possible G's such that:

(i)* G has a loop;	(ii) G is acyclic.;
(iii)* D is reflexive;	(iv) D is symmetric;
(v)* D is antisymmetric;	(vi) D is a total order.

Solution:



$\mathcal{D}_3 = \{(\{a,b,c\}, D) \ : \ D \subseteq \{a,b,c\}^2\} = \{\{a,b,c\}\} \times \mathcal{P}(\{a,b,c\}^2)$.

$|\mathcal{D}_3| = |\mathcal{P}(\{a,b,c\}^2)| = 2^{|\{a,b,c\}^2|} = 2^{(3^2)} = 2^9 = 512$.

(i)* G has a loop:
There are $2^6 = 64$ possible Ds *without* loops, so the number with loops is $512 - 64 = 448$.

(ii) G is acyclic:
The acyclic Ds have no loops.
There are $\binom{3}{2} = 3$ pairs of nodes;
for each pair, there are 3 choices: $x \to y$, $x \leftarrow y$, or no edge.

Altogether, there are 3^3 possibilities, 2 of which are cyclic:
Number of acyclic Gs is $3^3 - 2 = 25$.

(iii)* D is reflexive:
all loops are included, and each possible edge may be included/excluded,
so there are $1^3 2^6 = 64$ possible D's.

(iv) D is symmetric:
all (x,y) and (y,x) chosen together: $2^3 \times 2^3 = 64$ possibilities.

(v)* D is antisymmetric:
D is antisymmetric iff there is no $x \to y$ and $x \leftarrow y$ pair.
There are $\binom{3}{2}$ pairs of nodes.
For each pair, there are 3 choices: $\circ \to \circ$, $\circ \leftarrow \circ$, or no edge.
For each node, there is a choice of whether to have a loop.
Altogether: $2^3 3^{\binom{3}{2}} = 2^3 3^3 = 216$ possible Ds.

(vi) D is a total order:
all permuations of $a \to b \to c$ (all loops are included)
i.e. $3! = 6$ possibilities.

11.* Consider a loopless undirected graph $G = (V, E)$ and $V' \subseteq V$. We say V' **covers** an edge $\{x, y\} \in E$ iff $x \in V'$ or $y \in V'$, and V' is a **vertex cover** iff V' covers every edge. For example, $\{00, 02, 11, 12, 21, 22\}$ is a vertex cover for the graph in (a).

(i) For the graph in (a), find a vertex cover of smallest possible size.

A Boolean expression α is **3CNF** iff α is a conjunction of clauses with exactly 3 literals per clause. A 3CNF expression α with k clauses can be transformed into a loopless undirected graph G_α, so that α is satisfiable (Tutorial 9) iff G_α has a vertex cover of size at most $2k$. The following illustrates this transformation: Suppose, for $k = 4$,

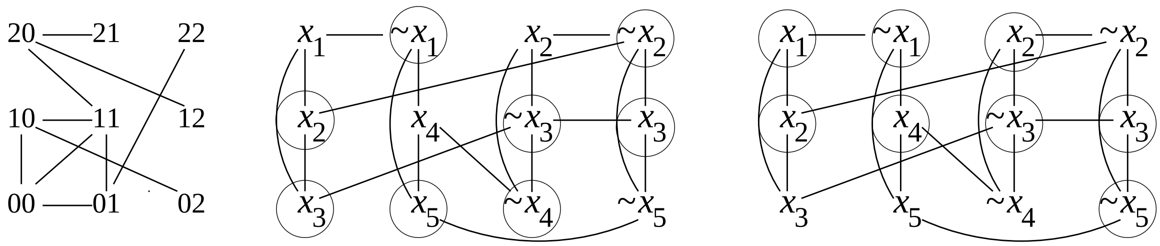
$$\alpha = (x_1 \vee x_2 \vee x_3) \wedge (\sim x_1 \vee x_4 \vee x_5) \wedge (x_2 \vee \sim x_3 \vee \sim x_4) \wedge (\sim x_2 \vee x_3 \vee \sim x_5).$$

Then the graph in (b) and in (c) (without the circles) is G_α.

(ii) Consider the truth assignment $f(x_1) = f(x_2) = f(x_4) = T$ and $f(x_3) = f(x_5) = F$. Verify that f satisfies α. Let C_f be the set of vertices in (b) that are circled. Verify that C_f is a vertex cover of size 8.

(iii) Consider the set C of vertices in (c) that are circled. Verify that C is a vertex cover of size 8. Define a truth function f_C by $f_C(x_1) = f_C(x_3) = f_C(x_5) = T$ and $f_C(x_2) = f_C(x_4) = F$. Verify that f_C satisfies α.

(iv) How is G_α derived from α? How is C_f derived from f? How is f_C derived from C?



(a) (b) (c)

[This problem suggests that finding a vertex cover is "harder" than finding a satisfying truth assignment. Actually, they are "equally hard", since both are NP-Complete.]

Solution:

(i) Covering $\{20, 12\}$, $\{10, 02\}$, $\{00, 11\}$ and $\{01, 22\}$ in (a) requires at least 4 vertices.
Possible size 4 vertex covers: $\{20, 10, 00, 01\}$ or $\{20, 10, 11, 01\}$.

(ii) Each clause in α is true under truth assignment f.
C_f has 8 vertices, and every edge has at least one end covered by C_f.

(iii) C has 8 vertices, and every edge has at least one end covered.
f_C makes every clause in α true.

(iv) **Note to tutors:**
The purpose of this exercise is to demonstrate how
two seemingly different problems are in fact related.
It suffices that the students grasp the connection intuitively, without proving it rigorously.
The following is background information to help you lead the discussion.

This problem shows how 3CNF satisfiability can be "reduced" to vertex cover
by the transformation $\alpha \mapsto G_\alpha$,
so that α is satisfiable iff G_α has a vertex cover of size $2k$.
Since G_α is a graph of a particular form (e.g. it has $3k$ vertices),
finding a vertex cover for G_α is possibly easier than for arbitrary graphs.
In this sense, 3CNF satisfiability is an easier problem than vertex cover:
Any algorithm for finding a vertex cover can be used to solve 3CNF satisfiability
(after first applying $\alpha \mapsto G_\alpha$).

Actually, both problems are (NP-complete and therefore) equally hard.
In principle, there is also a transformation $G \mapsto \alpha_G$ from graphs to 3CNF
such that G has a vertex cover iff α_G is satisfiable.
How is G_α derived from α?

Suppose $\alpha = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k$, where $\alpha_i = \alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$.
Define $G_\alpha = (V, E)$ where $V = \{\alpha_{ij} \ : \ i = 1, \ldots, k \text{ and } j = 1, 2, 3\}$
and $E = E_1 \cup \cdots \cup E_k \cup E_\sim$, where $E_i = \{\{\alpha_{i1}, \alpha_{i2}\}, \{\alpha_{i2}, \alpha_{i3}\}, \{\alpha_{i1}, \alpha_{i3}\}\}$
and $E_\sim = \{\{\beta, \gamma\} \ : \ \beta, \gamma \in V \text{ and } \beta \equiv \sim \gamma\}$.

How is C_f derived from f?

Suppose f satisfies α, denoted $f(\alpha) \equiv T$.
Then for each $i = 1, \ldots, k$, there is j_i such that $f(\alpha_{ij_i}) \equiv T$.
(Fix one j_i in case there are 2 or 3 possibilities.)
Define $C_f = V \setminus \{\alpha_{ij_i} \ : \ i = 1, \ldots, k\}$. Then $|C_f| = 3k - k = 2k$.
C_f is a cover: Consider any edge $\{x, y\} \in E$.

Suppose $\{x, y\} \in E_i$. Then $\{x, y\}$ is covered since 2 vertices from α_{i1}, α_{i2}, α_{i3} are in C_f.
Suppose $\{x, y\} \in E_\sim$. If $\{x, y\}$ is not covered, then $x \notin C_f$ and $y \notin C_f$,
so $f(x) \equiv T$ and $f(y) \equiv T$, which is impossible since $x \equiv \sim y$ and f is a truth assignment.

How is f derived from C?

Suppose C is a cover of size $2k$.
Each triangle E_i has 3 edges, so at least 2 vertices per triangle must be in C.
It follows that each triangle has exactly 2 vertices in C.
Thus, for each i, there is j_i such that $\alpha_{j_i} \notin C$.
Define f_C by $f_C(\alpha_{j_i}) \equiv T$ and,
for any other x such that $x \not\equiv \alpha_{ij_i}$ and $x \not\equiv \sim \alpha_{ij_i}$,
define $f_C(x)$ arbitrarily (e.g. $f_C(x_1)$ can be T or F in (c)).
Then $f_C(x)$ is defined for every x.
Moreover, f_C is a truth assignment: because C covers the edges in $E_\sim$,
we cannot have $f_C(x) = f_C(\sim x)$ for any x.
f_C satisfies α: For each i, $f_C(\alpha_{ij_i}) = T$, so $f_C(\alpha) \equiv T$.

The derivations $f \mapsto C_f$ and $C \mapsto f_C$ show that
α is satisfiable iff G_α has a vertex cover of size $2k$.