National University of Singapore
Department of Computer Science

2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 3**

1. Which of the following are true? ($\varnothing$ denotes the empty set.)
   (a) $\{1, 2, 4\} = \{4, 1, 2\}$. (b) $\{5, \varnothing\} = \{5\}$. (c) $\{5\} \in \{2, 5\}$.
   (d) $\varnothing \in \{1, 2\}$. (e) $\{1, 2\} \in \{1, \{2, 1\}\}$. (f) $1 \in \{\{1, 2\}\}$.

2. List the elements of the following sets:
   (a) $\{x \in \mathbb{N} : x \text{ is odd and } x^2 < 30\}$; (b) $\{x \in \mathbb{Z} : \exists y \in \mathbb{N}\ x^2 + y^2 = 20\}$.

3. Here $\mathbb{R}$ is the universal set. Let $A = \{x \in \mathbb{R} : -2 \leqslant x \leqslant 1\}$ and $B = \{x \in \mathbb{R} : -1 < x < 3\}$. Determine
   (a) $A \cup B$, (b) $A \cap B$, (c) $\overline{A}$, (d) $\overline{A} \cap \overline{B}$, (e) $A \setminus B$.

4. Let U denote the universal set. Prove the set identities that are **not** between double square brackets $[\![\ldots]\!]$ below, for all sets A, B, and C.

(a)*	Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
(b)	Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
(c)*	Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(d)	Idempotence	$A \cup A = A$	$A \cap A = A$
(e)	Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
(f)	De Morgan's Laws	$[\![\overline{A \cup B} = \overline{A} \cap \overline{B}]\!]$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
(g)*	Identities	$[\![A \cup \varnothing = A]\!]$	$A \cap U = A$
(h)*	Annihilators	$A \cup U = U$	$[\![A \cap \varnothing = \varnothing]\!]$
(i)*	Complement	$[\![A \cup \overline{A} = U]\!]$	$A \cap \overline{A} = \varnothing$
(j)*	Double Complement Law	$\overline{(\overline{A})} = A$	
(k)*	Top and bottom	$[\![\overline{\varnothing} = U]\!]$	$\overline{U} = \varnothing$
(l)*	Set difference	$A \setminus B = A \cap \overline{B}$	

5. Let U denote the universal set. Prove the following for all sets A, B, C. You may use what you showed in Question 4 in your proofs.
   (a)* $A \cap \varnothing = \varnothing$ and $A \cup \varnothing = A$. (b) $\overline{\varnothing} = U$ and $A \cup \overline{A} = U$.
   (c) If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$. (d)* $A \subseteq A \cup B$.
   (e) If $A \subseteq B$, then $A \cap C \subseteq B \cap C$. (f) $B \subseteq A$ if and only if $A \cap B = B$.
   (g)* $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$. (h)* If $B = (A \cap \overline{B}) \cup (B \cap \overline{A})$, then $A = \varnothing$.

6. In lexical analysis (CS4212), regular expressions are used to describe how tokens are constructed from strings. The basic construction is **concatenation**: If x and y are strings, then xy is the string formed by the symbols of x followed by the symbols of y; e.g., if $x = \text{CS}$ and $y = 1231$, then $xy = \text{CS1231}$, $yx = \text{1231CS}$ and $yy = 12311231$. If X and Y are sets of strings, define $XY = \{xy : x \in X \wedge y \in Y\}$.

   (a) Let $X = \{1, 01, 11, 011\}$ and $Y = \{00, 100\}$. Determine XY, YX and XX.

   (b) If S is a set of strings, what is $\varnothing S$?

7. Determine $\mathcal{P}(\mathcal{P}(\varnothing))$.

8. For each of the following, determine whether it is true for all sets A, B.
   (a) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. (b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

9. Let $A_1, A_2, \ldots$ be sets. Then the finite unions and the finite intersections can be defined for each positive integer n as follows:

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \text{and} \qquad \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n.$$

(a) Let n be an integer and $n \geqslant 2$. Determine $\bigcup_{k=1}^{n} A_k$ and $\bigcap_{k=1}^{n} A_k$ in each of the following cases.
(i) $A_k = \{k\}$. (ii) $A_k = \{x \in \mathbb{R} : 0 < x < k\}$. (iii) $A_k = \{x \in \mathbb{R} : 0 \leqslant x \leqslant \frac{1}{k}\}$.

Define X and Y by: for all x, y,

$x \in X$ if and only if $x \in \bigcup_{k=1}^{n} A_k$ for some positive integer n,
and $y \in Y$ if and only if $y \in \bigcap_{k=1}^{n} A_k$ for all positive integer n.

(b) State the definitions of X and Y symbolically (using $\exists$, $\forall$, etc.).

(c) Determine X and Y for the three cases in (a).

(d)* In program semantics (CS4214), the meaning of a program is sometimes defined with **fixed points**, which are either an infinite union or an infinite intersection. One way to define them is:

$x \in \bigcup_{k=1}^{\infty} A_k$ if and only if $x \in A_k$ for some positive integer k,
and $y \in \bigcap_{k=1}^{\infty} A_k$ if and only if $y \in A_k$ for all positive integer k.

Prove that $X = \bigcup_{k=1}^{\infty} A_k$ and $Y = \bigcap_{k=1}^{\infty} A_k$, where X and Y are as in (b).
[In other words, part (b) gives equivalent definitions for $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$.]

10. Let B and $E_1, E_2, \ldots$ be sets.

(a)* Suppose E_i and E_j are disjoint (i.e., have empty intersection) for all distinct positive integers i, j. Prove that $E_i \cap B$ and $E_j \cap B$ are disjoint for all distinct positive integers i, j.

(b) Prove that

$$\left(\bigcup_{k=1}^{\infty} E_k\right) \cap B = \bigcup_{k=1}^{\infty} (E_k \cap B).$$

11.* Consider the claim:

For all sets A, B and C, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

The following is a "proof": For all z,

$$\begin{aligned}
& z \in (A \setminus B) \cup (B \setminus A) \\
\Rightarrow \quad & z \in A \setminus B \text{ or } z \in B \setminus A \\
\Rightarrow \quad & z \in A \text{ and } z \notin B \text{ or } z \in B \text{ and } z \notin A \\
\Rightarrow \quad & z \in A \text{ or } z \in B \text{ and } z \notin B \text{ and } z \notin A \\
\Rightarrow \quad & z \in A \cup B \text{ and } z \in \overline{B \cap A} \\
\Rightarrow \quad & z \in (A \cup B) \cap \overline{B \cap A} \\
\Rightarrow \quad & z \in (A \cup B) \setminus (B \cap A).
\end{aligned}$$

Therefore $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(a) Point out the errors in the "proof".

(b) Prove or disprove the claim.