CS1231 Chapter 6

Equivalence relations and partial orders

6.1 Partitions

Definition 6.1.1. Call $\mathscr{C}$ a *partition* of a set A if

(0) $\mathscr{C}$ is a set of *nonempty* subsets of A;

(1) every element of A is in some element of $\mathscr{C}$; and

(2) if two elements of $\mathscr{C}$ have a nonempty intersection, then they are equal.

Elements of a partition are called *components* of the partition.

Remark 6.1.2. One can rewrite the three conditions in the definition of partitions respectively as follows:

(0) $\forall S \in \mathscr{C} \ \ (\varnothing \neq S \subseteq A)$;

(1) $\forall x \in A \ \ \exists S \in \mathscr{C} \ \ (x \in S)$;

(2) $\forall S_1, S_2 \in \mathscr{C} \ \ (S_1 \cap S_2 \neq \varnothing \Rightarrow S_1 = S_2)$.

Yet another way to formulate this is to say that $\mathscr{C}$ is a set of mutually disjoint nonempty subsets of A whose union is A.

Example 6.1.3. One partition of the set $A = \{1, 2, 3\}$ is $\{\{1\}, \{2, 3\}\}$. The others are

$$\{\{1\}, \{2\}, \{3\}\}, \quad \{\{2\}, \{1, 3\}\}, \quad \{\{3\}, \{1, 2\}\}, \quad \{\{1, 2, 3\}\}.$$

Example 6.1.4. One partition of $\mathbb{Z}$ is

$$\{\{2k : k \in \mathbb{Z}\}, \{2k + 1 : k \in \mathbb{Z}\}\}.$$

6.2 Reflexivity, symmetry, and transitivity

Definition 6.2.1. Let A be a set and R be a relation on A.

(1) R is *reflexive* if every element of A is R-related to itself, i.e.,

$$\forall x \in A \ \ (x \mathrel{R} x).$$

(2) R is *symmetric* if x is R-related to y implies y is R-related to x, for all $x, y \in A$, i.e.,

$$\forall x, y \in A \ \ (x \mathrel{R} y \Rightarrow y \mathrel{R} x).$$

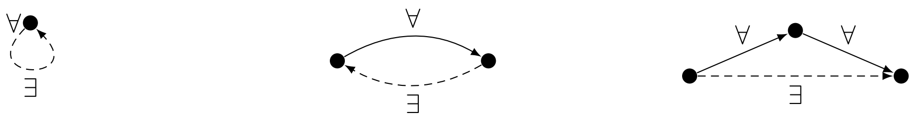
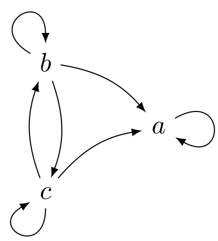


Figure 6.1: Reflexivity, symmetry, and transitivity

(3) R is *transitive* if x is R-related to y and y is R-related to z imply x is R-related to z, for all $x, y, z \in A$, i.e.,

$$\forall x, y, z \in A \ \ (x \; R \; y \wedge y \; R \; z \Rightarrow x \; R \; z).$$

Example 6.2.2. Let R be the relation represented by the following arrow diagram.



Then R is reflexive. It is not symmetric because $b \; R \; a$ but $a \not R \; b$. It is transitive, as one can show by exhaustion:

$$\begin{gathered} a \; R \; a \wedge a \; R \; a \Rightarrow a \; R \; a; \\ a \; R \; a \wedge a \; R \; b \Rightarrow a \; R \; b; \\ a \; R \; a \wedge a \; R \; c \Rightarrow a \; R \; c; \\ a \; R \; b \wedge b \; R \; a \Rightarrow a \; R \; a; \\ \vdots \\ c \; R \; c \wedge c \; R \; b \Rightarrow c \; R \; b; \\ c \; R \; c \wedge c \; R \; c \Rightarrow c \; R \; c. \end{gathered}$$

Example 6.2.3. Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad x = y.$$

Then R is reflexive, symmetric, and transitive.

Example 6.2.4. Let R' denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \; R' \; y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R' is reflexive, may not be symmetric (when U contains x, y such that $x \subsetneq y$), but is transitive.

Example 6.2.5. Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \; R \; y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is reflexive, not symmetric, but transitive.

Example 6.2.6. Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \, R' \, y \quad \Leftrightarrow \quad x < y.$$

Then R' is not reflexive as $0 \not< 0$. It is not symmetric because $0 < 1$ but $1 \not< 0$. It is transitive.

Definition 6.2.7 (recall)**.** Let $n, d \in \mathbb{Z}$. Then d is said to *divide* n if

$$n = dk \quad \text{for some } k \in \mathbb{Z}.$$

We write $d \mid n$ for "d divides n", and $d \nmid n$ for "d does not divide n". We also say

"n is *divisible* by d" or "n is a *multiple* of d" or "d is a *factor/divisor* of n"

for "d divides n".

Example 6.2.8. Let R denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x \, R \, y \quad \Leftrightarrow \quad x \mid y.$$

Then R is reflexive, not symmetric, but transitive.

Proof. (Reflexivity.) For each $a \in \mathbb{Z}^+$, we know $a = a \times 1$ and so $a \mid a$ by the definition of divisibility.

(Non-symmetry.) Note $1 \mid 2$ but $2 \nmid 1$.

(Transitivity.) Let $a, b, c \in \mathbb{Z}^+$ such that $a \mid b$ and $b \mid c$. Use the definition of divisibility to find $k, \ell \in \mathbb{Z}$ such that $b = ak$ and $c = b\ell$. Then $c = b\ell = (ak)\ell = a(k\ell)$, where $k\ell \in \mathbb{Z}$. Thus $a \mid c$ by the definition of divisibility. □

Exercise 6.2.9. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$. View R as a relation on A. Is R reflexive? Is R symmetric? Is R transitive? ✎ 6a

Exercise 6.2.10. Let R be a relation on a set A. Prove that R is transitive if and only if $R \circ R \subseteq R$. ✎ 6b

Definition 6.2.11. An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Convention 6.2.12. People usually use equality-like symbols such as $\sim$, $\approx$, $\simeq$, $\cong$, and $\equiv$ to denote equivalence relations. These symbols are often defined and redefined to mean different equivalence relations in different situations. We may read $\sim$ as "is equivalent to".

Example 6.2.13. The equality relation on a set, as defined in Example 6.2.3, is an equivalence relation.

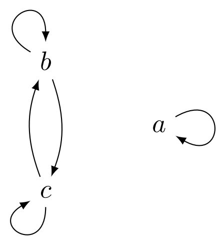
Proposition 6.2.14. Let $\mathscr{C}$ be a partition of a set A. Denote by $\sim_{\mathscr{C}}$ the same-component relation with respect to $\mathscr{C}$, i.e., for all $x, y \in A$,

$$x \sim_{\mathscr{C}} y \quad \Leftrightarrow \quad x, y \in S \text{ for some } S \in \mathscr{C}.$$

Then $\sim_{\mathscr{C}}$ is an equivalence relation on A.

Proof. Reflexivity holds because every element is in the same component as itself. Symmetry holds because if x is in the same component as y, then y is in the same component as x. Transitivity holds because if x is in the same component as y, and y is in the same component as z, then x is in the same component as z. □

Example 6.2.15. Let R be the relation represented by the following arrow diagram.



Then R is reflexive, symmetric, and transitive. So it is an equivalence relation on $\{a, b, c\}$.

6.3 Equivalence classes

Definition 6.3.1. Let $\sim$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x with respect to $\sim$, denoted $[x]_\sim$, is defined to be the set of all elements of A that are $\sim$-related to x, i.e.,

$$[x]_\sim = \{y \in A : x \sim y\}.$$

When there is no risk of confusion, we may drop the subscript and write simply $[x]$.

Example 6.3.2. Let A be a set. The equivalence classes with respect to the equality relation on A are of the form

$$[x] = \{y \in A : x = y\} = \{x\},$$

where $x \in A$.

Example 6.3.3. If R is the equivalence relation represented by the arrow diagram in Example 6.2.15, then

$$[a] = \{a\} \quad \text{and} \quad [b] = \{b, c\} = [c].$$

Lemma 6.3.4. Let $\sim$ be an equivalence relation on a set A.

(1) $x \in [x]$ for all $x \in A$.

(2) Any equivalence class is nonempty.

Proof. (1) Let $x \in A$. Then $x \sim x$ by reflexivity. So $x \in [x]$ by the definition of $[x]$.

(2) Any equivalence class is of the form $[x]$ for some $x \in A$, and so it must be nonempty by (1). □

Lemma 6.3.5. Let $\sim$ be an equivalence relation on a set A. For all $x, y \in A$, if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$.

Proof. Assume $[x] \cap [y] \neq \varnothing$. Say, we have $w \in [x] \cap [y]$. This means $x \sim w$ and $y \sim w$ by the definition of $[x]$ and $[y]$.

To show $[x] = [y]$, we need to prove both $[x] \subseteq [y]$ and $[y] \subseteq [x]$. We will concentrate on the former; the latter is similar.

Take $z \in [x]$. Then $x \sim z$ by the definition of $[x]$. By symmetry, we know from the first paragraph that $w \sim x$. Altogether we have $y \sim w \sim x \sim z$. So transitivity tells us $y \sim z$. Thus $z \in [y]$ by the definition of $[y]$. □

Question 6.3.6. Consider an equivalence relation. Is it true that if x is an element of an equivalence class S, then $S = [x]$? ✐ 6c

Definition 6.3.7. Let A be a set and $\sim$ be an equivalence relation on A. Denote by $A/{\sim}$ the set of all equivalence classes with respect to $\sim$, i.e.,

$$A/{\sim} = \{[x]_\sim : x \in A\}.$$

We may read $A/{\sim}$ as "the quotient of A by $\sim$".

Example 6.3.8. Let A be a set. Then from Example 6.3.2 we know $A/{=}$ is equal to $\{\{x\} : x \in A\}$.

Example 6.3.9. If R is the equivalence relation on the set $A = \{a, b, c\}$ represented by the arrow diagram in Example 6.2.15, then from Example 6.3.3 we know

$$A/{\sim} = \{[a], [b], [c]\} = \big\{\{a\}, \{b, c\}, \{b, c\}\big\} = \big\{\{a\}, \{b, c\}\big\}.$$

Theorem 6.3.10. Let $\sim$ be an equivalence relation on a set A. Then $A/{\sim}$ is a partition of A.

Proof. Conditions (0) and (1) in the definition of partitions are guaranteed by the definition of equivalence classes and Lemma 6.3.4. Condition (2) is given by Lemma 6.3.5. □

6.4 Partial orders

Definition 6.4.1. Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A \ \ (x \mathrel{R} y \wedge y \mathrel{R} x \Rightarrow x = y)$.

(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

(3) Suppose R is a partial order. Let $x, y \in A$. Then x, y are *comparable (under R)* if

$$x \mathrel{R} y \quad \text{or} \quad y \mathrel{R} x.$$

(4) R is a *(non-strict) total order* or a *(non-strict) linear order* if R is a partial order and every pair of elements is comparable, i.e.,

$$\forall x, y \in A \ \ (x \mathrel{R} y \vee y \mathrel{R} x).$$

Note 6.4.2. A total order is always a partial order.

Example 6.4.3. Let R denote the non-strict less-than relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is antisymmetric. In fact, it is a total order.

Example 6.4.4. Let R denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R is antisymmetric. It is always a partial order, but it may not be a total order.

Example 6.4.5. Let R denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \mid y.$$

Is R antisymmetric? Is R a partial order? Is R a total order? ✎ 6d

Here are the arrow diagrams of 8 of the binary relations on the 3-element set $A = \{a, b, c\}$. Set inclusion $\subseteq$ is a partial order on the set consisting of these 8 relations, the Hasse diagram of which is also shown.

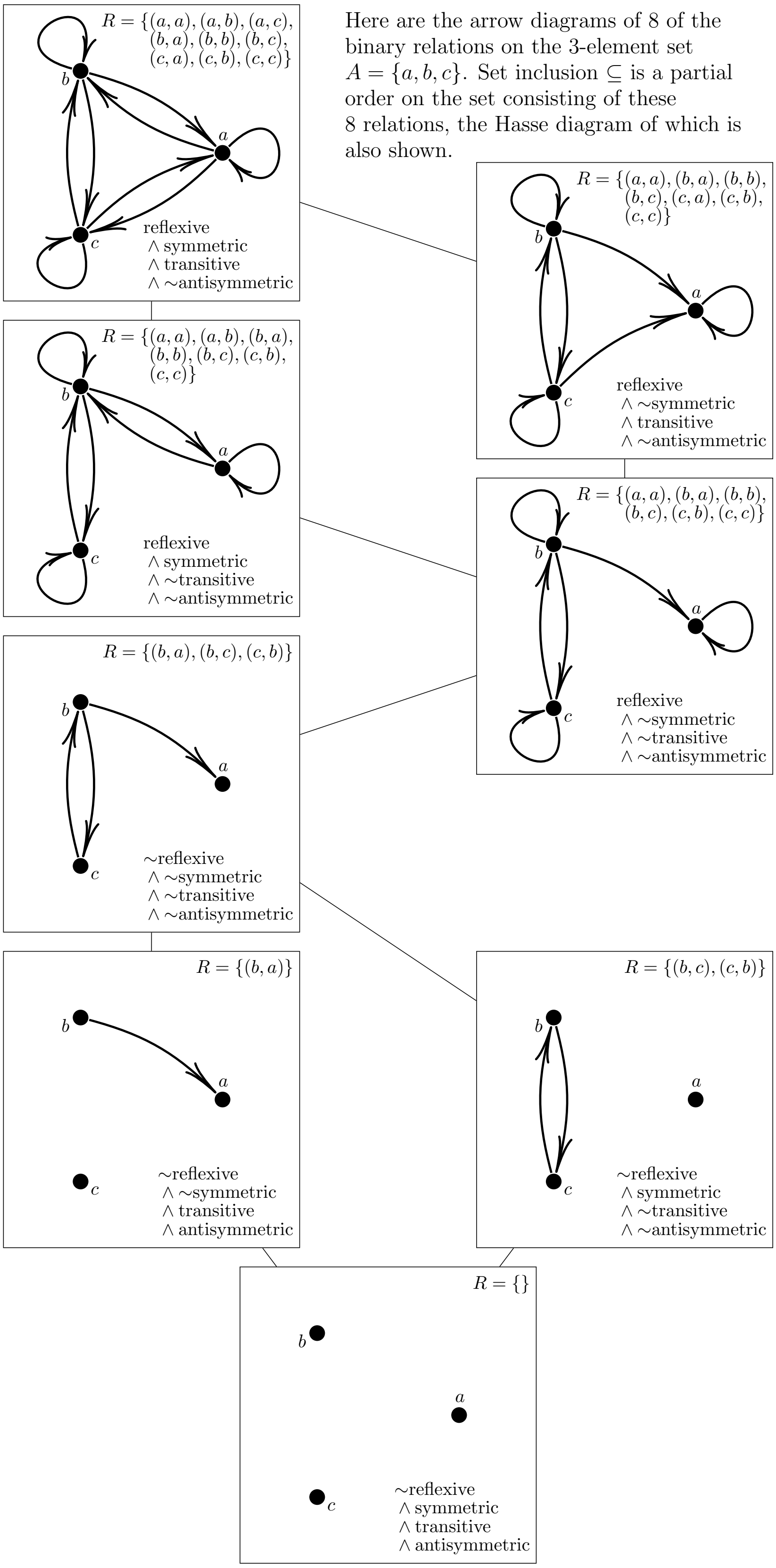


Figure 6.2: A partial order on a set of relations

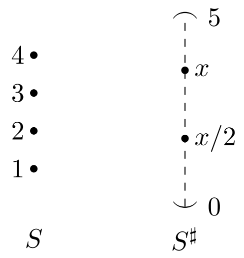


Figure 6.3: A difference between $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Q}_{\geqslant 0}$

Example 6.4.6. Let R' denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x\ R'\ y \quad \Leftrightarrow \quad x \mid y.$$

Is R antisymmetric? Is R a partial order? Is R a total order? 6e

Definition 6.4.7. Let R be a (non-strict) partial order on a set A. A *smallest element* of A (with respect to the partial order R) is an element $m \in A$ such that $m\ R\ x$ for all $x \in A$.

Example 6.4.8. (1) $S = \{x \in \mathbb{Z}_{\geqslant 0} : 0 < x < 5\}$ has smallest element 1.

(2) $S^\sharp = \{x \in \mathbb{Q}_{\geqslant 0} : 0 < x < 5\}$ has no smallest element because if $x \in S^\sharp$, then $x/2 \in S^\sharp$ and $x/2 < x$.

Second Principle of Mathematical Induction (2PI, recall)**.** Let $b, c \in \mathbb{Z}$, and $P(n)$ be a statement for each integer $n \geqslant b$. Here are steps to prove that $P(n)$ is true for all integers $n \geqslant b$ by 2PI.

Establish the **Basis:** Prove that $P(b), P(b+1), \ldots, P(c)$ are true.

Make the **Induction Hypothesis:** Suppose $k \in \mathbb{Z}_{\geqslant c}$ such that $P(b), P(b+1), \ldots, P(k)$ are true.

Complete the **Induction Step:** Use the Induction Hypothesis to prove that $P(k+1)$ is true.

Theorem 6.4.9 (Well-Ordering Principle)**.** Let $b \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}_{\geqslant b}$. If $S \neq \varnothing$, then S has a smallest element.

Proof. We prove the contrapositive. Assume that S has no smallest element. As $S \subseteq \mathbb{Z}_{\geqslant b}$, it suffices to show that $n \notin S$ for all $n \in \mathbb{Z}_{\geqslant b}$. We prove this by 2PI on n.

Basis: If $b \in S$, then b is the smallest element of S because $S \subseteq \mathbb{Z}_{\geqslant b}$, which contradicts our assumption. So $b \notin S$.

Induction Hypothesis: Suppose $k \in \mathbb{Z}_{\geqslant b}$ such that $b, b+1, \ldots, k \notin S$.

Induction Step: If $k+1 \in S$, then $k+1$ is the smallest element of S, because $S \subseteq \mathbb{Z}_{\geqslant b}$, which contradicts our assumption. So $k+1 \notin S$.

This concludes the induction and the proof. $\square$