Addition Rule

If A and B are finite sets that are **disjoint** (i.e. $A \cap B = \emptyset$), then $A \cup B$ is finite and $|A \cup B| = |A| + |B|$.

Lemma 3.9 Let Y be a finite set.

(i) If X is finite, then $X \cup Y$ is finite.

(ii) If $X \subseteq Y$, then $|Y - X| = |Y| - |X|$.

Theorem 3.10 Suppose $A_1, A_2, \ldots$ are finite sets.
Then $A_1 \cup A_2 \cup \cdots \cup A_n$ is finite for any $n \geq 2$.

Theorem 3.11 (Inclusion/Exclusion Rule for 2 and 3 sets)
Let A, B and C be finite sets. Then

$$\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \quad \text{and} \\
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.
\end{aligned}$$

Multiplication Rule

Consider the 2-tuple (x, y).
Suppose there are m possible values for x and,
for each choice of x, there are n possible choices for y.
Then there are mn possible choices for (x, y).

Example For $A = \{a, b, c, d\}$ and $B = \{0, 1\}$,
$|A \times B| =$

Theorem 3.12

Suppose $A_1, A_2, \ldots$ are finite sets.

Then $|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2| \cdots |A_n|$ for any $n \geq 2$.

Theorem 3.13 Let $x \in \mathbb{R}$, $x \neq 1$. Then

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} \quad \text{for any } n \in \mathbb{Z}^+.$$

Corollary 3.14

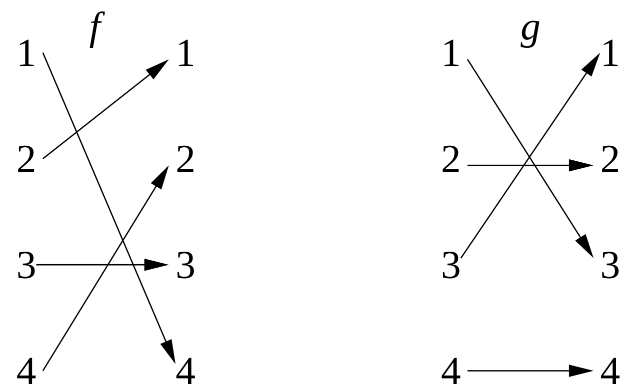
Suppose Γ is an alphabet and $|\Gamma| = s$. If $\ell \in \mathbb{Z}^+$,
then there are $\frac{s^\ell - 1}{s-1}$ strings over Γ with length smaller than ℓ.

Definition For $n \in \mathbb{N}$, define n **factorial** as

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n((n-1)!) & \text{if } n \geq 1 \end{cases}$$

Definition Let $n \in \mathbb{Z}^+$ and $S = \{x_1, \ldots, x_n\}$.
A bijection $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is called a **permutation**,
and the string $x_{f(1)} x_{f(2)} \cdots x_{f(n)}$ is also called a **permutation** of S.
Similarly, the bijection $\emptyset : \emptyset \to \emptyset$ is called a permutation,
and the empty string is a permutation of $\emptyset$.

Example $S = \{A, E, T, R\}$



Theorem 3.15
Let S be a set with n elements, $n \in \mathbb{N}$.
Then there are $n!$ permutations of S.

Definition Let $r, n \in \mathbb{N}$, and let S be a set of n elements.
A subset of r elements is called an r-**combination** of S.
$\binom{n}{r}$ (read "n choose r") denotes the number of r-combinations of S.

Definition

Let $r, n \in \mathbb{Z}^+$, $r \leq n$, and let $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. If $S = \{x_1, \ldots, x_n\}$, then $x_{f(1)} x_{f(2)} \cdots x_{f(r)}$ is an r**-permutation** of S.

${}^nP_r$ denotes the number of r-permutations of S.