National University of Singapore
Department of Computer Science

2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 3**

1. Which of the following are true? ($\varnothing$ denotes the empty set.)

(a)	$\{1,2,4\} = \{4,1,2\}$.	(b)	$\{5,\varnothing\} = \{5\}$.	(c)	$\{5\} \in \{2,5\}$.
(d)	$\varnothing \in \{1,2\}$.	(e)	$\{1,2\} \in \{1,\{2,1\}\}$.	(f)	$1 \in \{\{1,2\}\}$.

Solution:

(a)	$\{1,2,4\} = \{4,1,2\}$: True	(b)	$\{5,\varnothing\} = \{5\}$: False	(c)	$\{5\} \in \{2,5\}$: False
(d)	$\varnothing \in \{1,2\}$: False	(e)	$\{1,2\} \in \{1,\{2,1\}\}$: True	(f)	$1 \in \{\{1,2\}\}$: False

2. List the elements of the following sets:
(a) $\{x \in \mathbb{N} : x \text{ is odd and } x^2 < 30\}$; (b) $\{x \in \mathbb{Z} : \exists y \in \mathbb{N}\ x^2 + y^2 = 20\}$.

Solution:

(a) $\{x \in \mathbb{N} : x \text{ is odd and } x^2 < 30\} = \{1,3,5\}$.

(b) $\{x \in \mathbb{Z} : \exists y \in \mathbb{N}\ x^2 + y^2 = 20\} = \{-4,-2,2,4\}$.

3. Here $\mathbb{R}$ is the universal set. Let $A = \{x \in \mathbb{R} : -2 \leqslant x \leqslant 1\}$ and $B = \{x \in \mathbb{R} : -1 < x < 3\}$. Determine
(a) $A \cup B$, (b) $A \cap B$, (c) $\overline{A}$, (d) $\overline{A} \cap \overline{B}$, (e) $A \setminus B$.

Solution:

(a) $A \cup B = \{x \in \mathbb{R} : -2 \leqslant x < 3\}$.

(b) $A \cap B = \{x \in \mathbb{R} : -1 < x \leqslant 1\}$.

(c) $\overline{A} = \{x \in \mathbb{R} : (x < -2) \vee (x > 1)\}$.

(d) $\overline{A} \cap \overline{B} = \{x \in \mathbb{R} : (x < -2) \vee (x \geqslant 3)\}$.

(e) $A \setminus B = \{x \in \mathbb{R} : -2 \leqslant x \leqslant -1\}$.

4. Let U denote the universal set. Prove the set identities that are **not** between double square brackets $\llbracket \ldots \rrbracket$ below, for all sets A, B, and C.

(a)*	Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
(b)	Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
(c)*	Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(d)	Idempotence	$A \cup A = A$	$A \cap A = A$
(e)	Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
(f)	De Morgan's Laws	$\llbracket \overline{A \cup B} = \overline{A} \cap \overline{B} \rrbracket$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
(g)*	Identities	$\llbracket A \cup \varnothing = A \rrbracket$	$A \cap U = A$
(h)*	Annihilators	$A \cup U = U$	$\llbracket A \cap \varnothing = \varnothing \rrbracket$
(i)*	Complement	$\llbracket A \cup \overline{A} = U \rrbracket$	$A \cap \overline{A} = \varnothing$
(j)*	Double Complement Law	$\overline{(\overline{A})} = A$	
(k)*	Top and bottom	$\llbracket \overline{\varnothing} = U \rrbracket$	$\overline{U} = \varnothing$
(l)*	Set difference	$A \setminus B = A \cap \overline{B}$	

Solution: Each identity can be proved using a truth table or be proved directly. Note that $A, B, C \subseteq U$ as U is the universal set.

(a)*

$x \in A$	$x \in B$	$x \in A \cap B$	$x \in B \cap A$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

So $\forall x\ (x \in A \cap B \Leftrightarrow x \in B \cap A)$.
This means $A \cap B = B \cap A$.

Similarly for $A \cup B$.

(b)

$x \in A$	$x \in B$	$x \in C$	$x \in A \cap B$	$x \in (A \cap B) \cap C$	$x \in B \cap C$	$x \in A \cap (B \cap C)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

So $\forall x \left(x \in (A \cap B) \cap C \Leftrightarrow x \in A \cap (B \cap C)\right)$. This means $(A \cap B) \cap C = A \cap (B \cap C)$.

Alternatively, for every z,

$$\begin{aligned} z \in (A \cap B) \cap C \quad &\Leftrightarrow \quad (z \in A \cap B) \wedge z \in C && \text{by the definition of } \cap; \\ &\Leftrightarrow \quad (z \in A \wedge z \in B) \wedge z \in C && \text{by the definition of } \cap; \\ &\Leftrightarrow \quad z \in A \wedge (z \in B \wedge z \in C) && \text{as } \wedge \text{ is associative;} \\ &\Leftrightarrow \quad z \in A \wedge (z \in B \cap C) && \text{by the definition of } \cap; \\ &\Leftrightarrow \quad z \in A \cap (B \cap C) && \text{by the definition of } \cap. \end{aligned}$$

So $(A \cap B) \cap C = A \cap (B \cap C)$. One can rewrite this as:

$$\begin{aligned} (A \cap B) \cap C &= \{x : (x \in A \cap B) \wedge (x \in C)\} && \text{by the definition of } \cap; \\ &= \{x : ((x \in A) \wedge (x \in B)) \wedge (x \in C)\} && \text{by the definition of } \cap; \\ &= \{x : (x \in A) \wedge ((x \in B) \wedge (x \in C))\} && \text{as } \wedge \text{ is associative;} \\ &= \{x : (x \in A) \wedge (x \in B \cap C)\} && \text{by the definition of } \cap; \\ &= A \cap (B \cap C) && \text{by the definition of } \cap. \end{aligned}$$

Similarly for $A \cup B \cup C$.

(c)* $A \cup (B \cap C)$

$$\begin{aligned} &= \{x : (x \in A) \vee (x \in B \cap C)\} && \text{by the definition of } \cup; \\ &= \{x : (x \in A) \vee (x \in B \wedge x \in C)\} && \text{by the definition of } \cap; \\ &= \{x : ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))\} && \text{as } \vee \text{ distributes over } \wedge; \\ &= \{x : (x \in A \cup B) \wedge (x \in A \cup C)\} && \text{by the definition of } \cup; \\ &= (A \cup B) \cap (A \cup C) && \text{by the definition of } \cap. \end{aligned}$$

Similarly for $A \cap (B \cup C)$.

(d)

$x \in A$	$x \in A \cap A$
T	T
F	F

So $\forall x \ (x \in A \cap A \Leftrightarrow x \in A)$.
This means $A \cap A = A$.

Similarly for $A \cup A$.

(e) For every z,

$$\begin{aligned} z \in A \cup (A \cap B) \quad &\Leftrightarrow \quad z \in A \vee z \in A \cap B && \text{by the definition of } \cup; \\ &\Leftrightarrow \quad z \in A \vee (z \in A \wedge z \in B) && \text{by the definition of } \cap; \\ &\Leftrightarrow \quad z \in A && \text{by absorption in propositional logic.} \end{aligned}$$

So $A \cup (A \cap B) = A$.

Similarly for $A \cap (A \cup B) = A$.

(f) $\overline{A \cap B} = \{x \in U : x \notin A \cap B\}$ by the definition of $\overline{\cdot}$;

$= \big\{x \in U : \sim\big((x \in A) \wedge (x \in B)\big)\big\}$ by the definition of $\cap$;

$= \{x \in U : \sim(x \in A) \vee \sim(x \in B)\}$ by De Morgan's Laws for propositional logic;

$= \{x \in U : x \in \overline{A} \vee x \in \overline{B}\}$ by the definition of $\overline{\cdot}$;

$= \overline{A} \cup \overline{B}$ by the definition of $\cup$.

(g)*

$x \in A$	$x \in U$	$x \in A \cap U$
T	T	T
F	T	F

So $\forall x \in U$ $(x \in A \cap U \Leftrightarrow x \in A)$.
This means $A \cap U = A$.

(h)* For every $z \in U$,

$$\begin{aligned} z \in A \cup U &\Leftrightarrow z \in A \wedge z \in U && \text{by the definition of } U; \\ &\Leftrightarrow z \in U && \text{as } z \in U. \end{aligned}$$

So $A \cup U = U$.

(i)*

$x \in A$	$x \in \overline{A}$	$x \in A \cap \overline{A}$	$x \in \varnothing$
T	F	F	F
F	T	F	F

So $\forall x \in U$ $(x \in A \cap \overline{A} \Leftrightarrow x \in \varnothing)$.
This means $A \cap \overline{A} = \varnothing$.

(j)* $\overline{\overline{A}} = \{x \in U : \sim(x \in \overline{A})\}$ by the definition of $\overline{\cdot}$;

$= \{x \in U : \sim(\sim(x \in A))\}$ by the definition of $\overline{\cdot}$;

$= \{x \in U : x \in A\}$ by the Double Negation Law;

$= A$.

(k)*

$x \in U$	$x \in \overline{U}$	$x \in \varnothing$
T	F	F

So $\forall x \in U$ $(x \in \overline{U} \Leftrightarrow x \in \varnothing)$.
This means $\overline{U} = \varnothing$.

(l)* For every $z \in U$,

$$\begin{aligned} z \in A \setminus B &\Leftrightarrow z \in A \wedge z \notin B && \text{by the definition of } \setminus; \\ &\Leftrightarrow z \in A \wedge (z \in U \wedge z \notin B) && \text{as } z \in U; \\ &\Leftrightarrow z \in A \wedge z \in \overline{B} && \text{by the definition of } \overline{\cdot}; \\ &\Leftrightarrow z \in A \cap \overline{B} && \text{by the definition of } \cap. \end{aligned}$$

So $A \setminus B = A \cap \overline{B}$.

5. Let U denote the universal set. Prove the following for all sets A, B, C. You may use what you showed in Question 4 in your proofs.
(a)* $A \cap \varnothing = \varnothing$ and $A \cup \varnothing = A$.
(b) $\overline{\varnothing} = U$ and $A \cup \overline{A} = U$.
(c) If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.
(d)* $A \subseteq A \cup B$.
(e) If $A \subseteq B$, then $A \cap C \subseteq B \cap C$.
(f) $B \subseteq A$ if and only if $A \cap B = B$.
(g)* $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.
(h)* If $B = (A \cap \overline{B}) \cup (B \cap \overline{A})$, then $A = \varnothing$.

Solution:

(a)* $A \cap \varnothing = A \cap \overline{U} = \overline{\overline{A}} \cap \overline{U} = \overline{\overline{A} \cup U} = \overline{U} = \varnothing$.

$A \cup \varnothing = A \cup \overline{U} = \overline{\overline{A}} \cup \overline{U} = \overline{\overline{A} \cap U} = \overline{\overline{A}} = A$.

(b) $\overline{\varnothing} = \overline{\overline{U}} = U$.

$A \cup \overline{A} = \overline{\overline{A}} \cup \overline{A} = \overline{\overline{A} \cap A} = \overline{A \cap \overline{A}} = \overline{\varnothing} = U$.

(c) Suppose $A \subseteq B$ and $A \subseteq C$. This means that if $x \in A$, then $x \in B$ and $x \in C$, and thus $x \in B \cap C$. So $A \subseteq B \cap C$.

(d)* If $x \in A$, then $x \in A$ or $x \in B$, and thus $x \in A \cup B$. This shows $A \subseteq A \cup B$.

(e) Suppose $A \subseteq B$. Example 4.3.8 in the notes then tells us $A \cap C \subseteq A \subseteq B$ and $A \cap C \subseteq C$. So $A \cap C \subseteq B \cap C$ by (c).

(f) ($\Rightarrow$) Suppose $B \subseteq A$. Then $B \cap B \subseteq A \cap B$ by (e). So $B \subseteq A \cap B$.
Conversely, Example 4.3.8 in the notes tells us $A \cap B \subseteq B$.
Hence $A \cap B = B$.

($\Leftarrow$) Suppose $A \cap B = B$. Then $B = A \cap B \subseteq A$ by Example 4.3.8 in the notes.

(g)* ($\Rightarrow$) Suppose $(A \cap B) \cup C = A \cap (B \cup C)$.
Then (d) and Example 4.3.8 in the notes imply $C \subseteq (A \cap B) \cup C = A \cap (B \cup C) \subseteq A$.

($\Leftarrow$) Suppose $C \subseteq A$. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = (A \cap B) \cup C$ by (f).

Note: The point here is to be careful about parentheses.

(h)* Suppose $B = (A \cap \overline{B}) \cup (B \cap \overline{A})$. Note $A = A \cap U = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$. So it suffices to show both $A \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$.

- Suppose $A \cap B \neq \varnothing$. Let $z \in A \cap B$. Then $z \in A \cap B \subseteq B = (A \cap \overline{B}) \cup (B \cap \overline{A})$. So either $z \in A \cap \overline{B}$ or $z \in B \cap \overline{A}$. However, we know $z \notin A \cap \overline{B}$ because $z \in B$. Similarly, we know $z \notin B \cap \overline{A}$ because $z \in A$. So we have a contradiction.
- Suppose $A \cap \overline{B} \neq \varnothing$. Let $z \in A \cap \overline{B}$. Then $z \in (A \cap \overline{B}) \cup (B \cap \overline{A}) = B$. These contradict each other because the former says $z \in \overline{B}$ and the latter says $z \in B$.

Alternatively, one can proceed algebraically as follows.

- As $B = (A \cap \overline{B}) \cup (B \cap \overline{A})$, we have

$$\begin{aligned} B = B \cap B &= \big((A \cap \overline{B}) \cup (B \cap \overline{A})\big) \cap B = (A \cap \overline{B} \cap B) \cup (B \cap \overline{A} \cap B) \\ &= (A \cap \varnothing) \cup (B \cap \overline{A}) = \varnothing \cup (B \cap \overline{A}) = B \cap \overline{A}. \\ \therefore \qquad A \cap B &= A \cap B \cap \overline{A} = B \cap \varnothing = \varnothing. \end{aligned}$$

- Part (d) implies $A \cap \overline{B} \subseteq (A \cap \overline{B}) \cup (B \cap \overline{A}) = B$. So applying part (e) gives

$$A \cap \overline{B} = A \cap \overline{B} \cap \overline{B} \subseteq B \cap \overline{B} = \varnothing.$$

As $\varnothing$ is a subset of any set, we conclude that $A \cap \overline{B} = \varnothing$.

There are many other proofs.

6. In lexical analysis (CS4212), regular expressions are used to describe how tokens are constructed from strings. The basic construction is **concatenation**: If x and y are strings, then xy is the string formed by the symbols of x followed by the symbols of y; e.g., if x = CS and y = 1231, then xy = CS1231, yx = 1231CS and yy = 12311231. If X and Y are sets of strings, define $XY = \{xy : x \in X \wedge y \in Y\}$.

(a) Let $X = \{1, 01, 11, 011\}$ and $Y = \{00, 100\}$. Determine XY, YX and XX.

(b) If S is a set of strings, what is $\varnothing S$?

Solution:

(a) $X = \{1, 01, 11, 011\}$, $\quad Y = \{00, 100\}$.
$XY = \{100, 1100, 0100, 01100, 11100, 011100\}$. **Note:** XY has 6 elements, not 8.
$YX = \{001, 1001, 0001, 10001, 0011, 10011, 00011, 100011\}$
$XX = \{11, 101, 111, 1011, 011, 0101, 0111, 01011, 1101, 1111, 11011, 01101, 01111, 011011\}$.
Note: XX has 14 elements, not 16.

(b) If $w \in \varnothing S$, then $w = xy$ for some $x \in \varnothing$ and $y \in S$.
But there is no $x \in \varnothing$. So there can be no $w \in \varnothing S$. This means $\varnothing S = \varnothing$.

7. Determine $\mathcal{P}(\mathcal{P}(\varnothing))$.

Solution:
$\mathcal{P}(\mathcal{P}(\varnothing)) = \mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$.

8. For each of the following, determine whether it is true for all sets A, B.
(a) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. (b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Solution:

(a) Let $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
If $S \in \mathcal{P}(A)$, then $S \subseteq A$, so $S \subseteq A \cup B$, i.e. $S \in \mathcal{P}(A \cup B)$. Similarly for $S \in \mathcal{P}(B)$.
Thus $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

(b) Consider $A = \{2\}$ and $B = \{3\}$. Then $\{2, 3\} \in \mathcal{P}(A \cup B)$.
But $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{2\}, \{3\}\}$. So $\{2, 3\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.
Therefore $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ in general.

9. Let $A_1, A_2, \ldots$ be sets. Then the finite unions and the finite intersections can be defined for each positive integer n as follows:

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \text{and} \qquad \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n.$$

(a) Let n be an integer and $n \geqslant 2$. Determine $\bigcup_{k=1}^{n} A_k$ and $\bigcap_{k=1}^{n} A_k$ in each of the following cases.
(i) $A_k = \{k\}$. (ii) $A_k = \{x \in \mathbb{R} : 0 < x < k\}$. (iii) $A_k = \{x \in \mathbb{R} : 0 \leqslant x \leqslant \frac{1}{k}\}$.

Define X and Y by: for all x, y,

$x \in X$ if and only if $x \in \bigcup_{k=1}^{n} A_k$ for some positive integer n,
and $y \in Y$ if and only if $y \in \bigcap_{k=1}^{n} A_k$ for all positive integer n.

(b) State the definitions of X and Y symbolically (using $\exists$, $\forall$, etc.).

(c) Determine X and Y for the three cases in (a).

(d)* In program semantics (CS4214), the meaning of a program is sometimes defined with **fixed points**, which are either an infinite union or an infinite intersection. One way to define them is:

$x \in \bigcup_{k=1}^{\infty} A_k$ if and only if $x \in A_k$ for some positive integer k,
and $y \in \bigcap_{k=1}^{\infty} A_k$ if and only if $y \in A_k$ for all positive integer k.

Prove that $X = \bigcup_{k=1}^{\infty} A_k$ and $Y = \bigcap_{k=1}^{\infty} A_k$, where X and Y are as in (b).
[In other words, part (b) gives equivalent definitions for $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$.]

Solution:

(a) (i) $\bigcup_{k=1}^{n}\{k\} = \{1, \ldots, n\}$.
$\bigcap_{k=1}^{n}\{k\} = \varnothing$ since $n \geqslant 2$.

(ii) $\bigcup_{k=1}^{n}\{x \in \mathbb{R} : 0 < x < k\} = \{x \in \mathbb{R} : 0 < x < n\}$.
$\bigcap_{k=1}^{n}\{x \in \mathbb{R} : 0 < x < k\} = \{x \in \mathbb{R} : 0 < x < 1\}$.

(iii) $\bigcup_{k=1}^{n}\{x \in \mathbb{R} : 0 \leqslant x \leqslant \frac{1}{k}\} = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$.
$\bigcap_{k=1}^{n}\{x \in \mathbb{R} : 0 \leqslant x \leqslant \frac{1}{k}\} = \{x \in \mathbb{R} : 0 \leqslant x \leqslant \frac{1}{n}\}$.

(b) $x \in X \leftrightarrow \exists n \in \mathbb{Z}^+ \; x \in \bigcup\limits_{k=1}^{n} A_k$.

$y \in Y \leftrightarrow \forall n \in \mathbb{Z}^+ \; y \in \bigcap\limits_{k=1}^{n} A_k$.

(c) (i) $X = \mathbb{Z}^+$; $Y = \varnothing$.
(ii) $X = \mathbb{R}^+$; $Y = \{x \in \mathbb{R} : 0 < x < 1\}$.
(iii) $X = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$; $Y = \{0\}$.

(d)* $X = \bigcup_{k=1}^\infty A_k$: Suppose $x \in X$. So $\exists n \in \mathbb{Z}^+ \; x \in \bigcup_{k=1}^n A_k = A_1 \cup \cdots \cup A_n$.
Then $x \in A_k$ for some $k \in \mathbb{Z}^+$. So $x \in \bigcup_{k=1}^\infty A_k$. Therefore $X \subseteq \bigcup_{k=1}^\infty A_k$.
Suppose $x \in \bigcup_{k=1}^\infty A_k$. If $k \in \mathbb{Z}^+$ such that $x \in A_k$, then $x \in \bigcup_{i=1}^k A_i$.
So $x \in X$. Therefore $\bigcup_{k=1}^\infty A_k \subseteq X$.
Equality follows.

$Y = \bigcap_{k=1}^\infty A_k$: Suppose $y \in Y$. So $\forall n \in \mathbb{Z}^+ \; y \in \bigcap_{k=1}^n A_k$. In particular, $\forall n \in \mathbb{Z}^+ \; y \in A_n$.
So $y \in \bigcap_{k=1}^\infty A_k$. Therefore $Y \subseteq \bigcap_{k=1}^\infty A_k$.
Suppose $y \in \bigcap_{k=1}^\infty A_k$. This means $\forall k \in \mathbb{Z}^+ \; y \in A_k$.
Then $y \in \bigcap_{k=1}^n A_k$ for any $n \in \mathbb{Z}^+$. So $y \in Y$. Therefore $\bigcap_{k=1}^\infty A_k \subseteq Y$.
Equality follows.

10. Let B and $E_1, E_2, \ldots$ be sets.

(a)* Suppose E_i and E_j are disjoint (i.e., have empty intersection) for all distinct positive integers i, j. Prove that $E_i \cap B$ and $E_j \cap B$ are disjoint for all distinct positive integers i, j.

(b) Prove that

$$\left(\bigcup_{k=1}^\infty E_k\right) \cap B = \bigcup_{k=1}^\infty (E_k \cap B).$$

Solution:

(a)* Let i, j be distinct positive integers.
Since $E_i \cap E_j = \varnothing$, we have $(E_i \cap B) \cap (E_j \cap B) = E_i \cap E_j \cap B = \varnothing \cap B = \varnothing$.

(b) For all z,

$$\begin{aligned}
z \in \left(\bigcup_{k=1}^\infty E_k\right) \cap B \quad &\Leftrightarrow \quad z \in \bigcup_{k=1}^\infty E_k \wedge z \in B \\
&\Leftrightarrow \quad (\exists k \in \mathbb{Z}^+ \; z \in E_k) \wedge z \in B \\
&\Leftrightarrow \quad \exists k \in \mathbb{Z}^+ \; (z \in E_k \wedge z \in B) && \text{(note)} \\
&\Leftrightarrow \quad \exists k \in \mathbb{Z}^+ \; z \in E_k \cap B \\
&\Leftrightarrow \quad z \in \bigcup_{k=1}^\infty (E_k \cap B).
\end{aligned}$$

So $\left(\bigcup_{k=1}^\infty E_k\right) \cap B = \bigcup_{k=1}^\infty (E_k \cap B)$.

11.* Consider the claim:

For all sets A, B and C, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

The following is a "proof": For all z,

$$\begin{aligned}
& z \in (A \setminus B) \cup (B \setminus A) \\
\Rightarrow \quad & z \in A \setminus B \text{ or } z \in B \setminus A \\
\Rightarrow \quad & z \in A \text{ and } z \notin B \text{ or } z \in B \text{ and } z \notin A \\
\Rightarrow \quad & z \in A \text{ or } z \in B \text{ and } z \notin B \text{ and } z \notin A \\
\Rightarrow \quad & z \in A \cup B \text{ and } z \in \overline{B \cap A} \\
\Rightarrow \quad & z \in (A \cup B) \cap \overline{B \cap A} \\
\Rightarrow \quad & z \in (A \cup B) \setminus (B \cap A).
\end{aligned}$$

Therefore $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(a) Point out the errors in the "proof".

(b) Prove or disprove the claim.

Solution:

(a) For all z,

$$\begin{aligned}
& z \in (A \setminus B) \cup (B \setminus A) \\
\Rightarrow\ & z \in A \setminus B \text{ or } z \in B \setminus A \\
\Rightarrow\ & z \in A \text{ and } z \notin B \text{ or } z \in B \text{ and } z \notin A && \longleftarrow \textbf{ambiguous} \\
\Rightarrow\ & z \in A \text{ or } z \in B \text{ and } z \notin B \text{ and } z \notin A && \longleftarrow \textbf{unclear why one can reorder } \wedge \textbf{ and } \vee \\
\Rightarrow\ & z \in A \cup B \text{ and } z \in \overline{B \cap A} && \longleftarrow \overline{B} \cap \overline{A} \neq \overline{B \cap A} \textbf{ in general} \\
\Rightarrow\ & z \in (A \cup B) \cap \overline{B \cap A} \\
\Rightarrow\ & z \in (A \cup B) \setminus (B \cap A).
\end{aligned}$$

Moreover, the $\Leftarrow$ direction of the proof is missing.

(b)
$$\begin{aligned}
(A \setminus B) \cup (B \setminus A) &= (A \cap \overline{B}) \cup (B \cap \overline{A}) \\
&= \big((A \cap \overline{B}) \cup B\big) \cap \big((A \cap \overline{B}) \cup \overline{A}\big) \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) \\
&= (A \cup B) \cap U \cap U \cap (\overline{B \cap A}) \\
&= (A \cup B) \cap (\overline{B \cap A}) \\
&= (A \cup B) \setminus (B \cap A).
\end{aligned}$$