National University of Singapore
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2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 9**

1. Use a counting argument to prove that, for any $n, r \in \mathbb{Z}^+$ and $1 \le r \le n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

[One can prove this combinatorial identity algebraically by using the formula for $\binom{n}{k}$. However, since $\binom{n}{k}$ is *defined* as number of ways to choose a k-element subset from an n-element set, there should be some way of proving the identity by counting ways of choosing subsets from sets; this is what is meant by a "counting argument".]

Solution:
Consider choosing an r-element subset A from $\{b_1, \ldots, b_n, c\}$.
There are $\binom{n+1}{r}$ choices of A.
On the other hand, if A includes c, there are $\binom{n}{r-1}$ choices;
if A excludes c, there are $\binom{n}{r}$ choices.
Therefore $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.
Note:
With a counting argument, one can solve this problem without knowing the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
Alternative: Algebraically,
$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!} = \frac{n!}{r!(n+1-r)!}(r + (n+1-r)) = \binom{n+1}{r}$.

2.* For $n \in \mathbb{Z}^+$, determine

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

and

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^r \binom{n}{r} + \cdots + (-1)^n \binom{n}{n}$$

Solution:
Note to tutor: I put this problem before the Binomial Theorem on purpose,
so students can better appreciate the Binomial Theorem.

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$ if we use the Binomial Theorem.

Alternative counting argument: Chooose a subset of $\{b_1, \ldots, b_n\}$
left-hand side: $\binom{n}{r}$ for picking a subset of size r.
right-hand side: each b_i is chosen or not chosen (2 possibilities).

$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots (-1)^n \binom{n}{n} = (1-1)^n = 0$ if we use the Binomial Theorem.

3. The Binomial Theorem states that, for any $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}x^0 y^n.$$

(i)* Give an inductive proof of the theorem.

(ii) Give a counting argument for the theorem.

Solution:

(i) *Proof*: By induction on n.
Basis $n = 1$: $(x+y)^n = (x+y)^1 = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1$.
Induction Hypothesis Suppose the theorem is true if $n = k$, for some $k \geq 1$.
Induction Step Consider $n = k+1$.

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^k \\ &= (x+y)(\tbinom{k}{0}x^ky^0 + \tbinom{k}{1}x^{k-1}y^1 + \cdots + \tbinom{k}{r}x^{k-r}y^r + \cdots + \tbinom{k}{k}x^0y^k) \text{ by the Ind. Hyp.} \\ &= \tbinom{k}{0}x^{k+1}y^0 + \tbinom{k}{1}x^ky^1 + \cdots + \tbinom{k}{r}x^{k+1-r}y^r + \cdots + \tbinom{k}{k}x^1y^k) \\ &\quad + \tbinom{k}{0}x^ky^1 + \cdots + \tbinom{k}{r-1}x^{k-(r-1)}y^r + \cdots + \tbinom{k}{k-1}x^1y^k + \tbinom{k}{k}x^0y^{k+1}.\end{aligned}$$

Note the $\binom{k}{0} = 1 = \binom{k+1}{0}$, $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$ by Problem 1, and $\binom{k}{k} = 1 = \binom{k+1}{k+1}$, so the claim is true for $n = k+1$.

(ii) $(x+y)^n = (x_1+y_1)(x_2+y_2)\cdots(x_n+y_n)$ where $x_i = x$ and $y_i = y$ for all i.
In expanding the product, we can view each term in the expansion as picking x_i or y_i from each of the (x_i+y_i)'s.
The coefficient of $x^{n-r}y^r$ is thus the number of ways of picking r y_i's (and picking x_i's from the rest), i.e. $\binom{n}{r}$.

4. (i) Give an inductive proof of the following:

$$\binom{0}{r} + \binom{1}{r} + \binom{2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} \quad \text{for any } n, r \in \mathbb{N}.$$

(ii) Give a counting argument for the result.

Solution:

(i) **Claim:** $\binom{0}{r} + \binom{1}{r} + \binom{2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$ for any $n, r \in \mathbb{N}$.
Proof: By induction on n. (This proof works for any $r \in \mathbb{N}$.)

Basis $n = 0$: $\binom{0}{r} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r > 0 \end{cases}$ $\qquad \binom{n+1}{r+1} = \binom{0+1}{r+1} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r > 0 \end{cases}$

Induction Hypothesis Suppose the claim is true if $n = k$, for some $k \geq 0$.
Induction Step Consider $n = k+1$.

$$\begin{aligned}\tbinom{0}{r} + \tbinom{1}{r} + \tbinom{2}{r} + \cdots + \tbinom{k}{r} + \tbinom{k+1}{r} &= \tbinom{k+1}{r+1} + \tbinom{k+1}{r} \text{ by the Ind. Hyp.} \\ &= \tbinom{(k+1)+1}{r+1} \text{ by Problem 1.}\end{aligned}$$

so the claim is true for $n = k+1$.
By induction, the claim is true for all $n \in \mathbb{N}$.
Alternative non-inductive proof:
By Problem 1, $\binom{n}{r+1} + \binom{n}{r} = \binom{n+1}{r+1}$, so $\binom{n}{r} = \binom{n+1}{r+1} - \binom{n}{r+1}$.

Therefore $\binom{0}{r} + \binom{1}{r} + \binom{2}{r} + \cdots + \binom{n}{r}$

$$\begin{aligned}&= (\tbinom{1}{r+1} - \tbinom{0}{r+1}) + (\tbinom{2}{r+1} - \tbinom{1}{r+1}) + (\tbinom{3}{r+1} - \tbinom{2}{r+1}) + \cdots + (\tbinom{n+1}{r+1} - \tbinom{n}{r+1}) \\ &= -\tbinom{0}{r+1} + \tbinom{n+1}{r+1} = \tbinom{n+1}{r+1}.\end{aligned}$$

(ii) Consider picking an $(r+1)$-element subset A from $\{0, 1, \ldots, n\}$.
If the largest number in A is s, there are $\binom{s}{r}$ choices from $0, 1, \ldots, s-1$.
The total is $\sum_{s=0}^{n} \binom{s}{r}$, but this is also $\binom{n+1}{r+1}$.

5.* Let $m, n, r \in \mathbb{N}$. Prove the following (Vandermonde's identity):

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}.$$

[An algebraic proof of this identity would be painfully tedious.]

Solution:
Consider choosing r persons from m men and n women; there are $\binom{m+n}{r}$ possibilities.
The chosen r can include k men, for $k = 0, 1, \ldots, r$.
For each k, there are $\binom{m}{k}\binom{n}{r-k}$ possibilities by the Multiplication Rule.
By the Addition Rule, the total number of possibilities is
$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$,
so the claim is true.

Note This problem illustrates the power of counting arguments.

6. Recall the definition of $\bigcup_{k=1}^{n} A_k$ and $\bigcup_{k=1}^{\infty} A_k$ in Tutorial 3.

(i) Consider the claim:
"Suppose $A_1, A_2, \ldots$ are finite sets. Then $\bigcup_{i=1}^{n} A_i$ is finite for any $n \geq 2$."
An inductive proof was given in class (Theorem 3.10). Here is an alternative "proof":
"We will prove by induction on n. Since A_1 and A_2 are finite, $A_1 \cup A_2$ is finite (by Lemma 3.9), so the claim is true for $n = 2$. Now suppose the claim is true for $n = k$, so $\bigcup_{i=1}^{k} A_i$ is finite. Let $A_{k+1} = \varnothing$. Then $\bigcup_{i=1}^{k+1} A_i = \left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1} = \left(\bigcup_{i=1}^{k} A_i\right) \cup \varnothing = \bigcup_{i=1}^{k} A_i$, which is finite by the induction hypothesis, so the claim is true for $n = k+1$. Therefore, the claim is true for all $n \geq 2$."
What is wrong with this "proof"?

(ii) Prove the following is false: "Suppose $A_1, A_2, \ldots$ are finite sets. Then $\bigcup_{i=1}^{\infty} A_i$ is finite."
[The point here is: induction takes you to any finite n, but not to infinity.]

Solution:

(i) Error in "proof":
There is an implicit universal quantification on $A_1, A_2, \ldots$,
i.e. we have to prove the claim is true for all possible $A_1, A_2, \ldots$,
so we cannot just consider the special case $A_{k+1} = \varnothing$.
The proof must work for any given $A_1, A_2, \ldots$.

(ii) Let $A_i = \{i\}$. Then $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$, which is infinite.

7. State and prove the Inclusion/Exclusion Rule for four sets.

Solution:

$$
\begin{aligned}
|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\
& -|A \cap B| - |A \cap C| - |B \cap C| - |B \cap D| - |C \cap D| - |A \cap D| \\
& +|A \cap B \cap C| + |A \cap C \cap D| + |B \cap C \cap D| + |A \cap B \cap D| - |A \cap B \cap C \cap D|.
\end{aligned}
$$

Proof:

$|A \cup B \cup C \cup D| = |A| + |B \cup C \cup D| - |A \cap (B \cup C \cup D)|$ by Inclusion/Exclusion for 2 sets

$|B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D|$

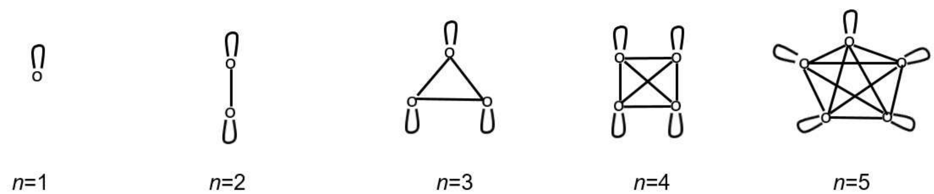
by Inclusion/Exclusion for 3 sets.

$$
\begin{aligned}
|A \cap (B \cup C \cup D)| &= |(A \cap B) \cup (A \cap C) \cup (A \cap D)| \\
&= |A \cap B| + |A \cap C| + |A \cap D| - |A \cap B \cap C| - |A \cap B \cap D| - |A \cap C \cap D| \\
&\quad +|A \cap B \cap C \cap D| \qquad \text{by Inclusion/Exclusion for 3 sets}
\end{aligned}
$$

The claim follows when the second and third equations are substituted into the first equation.

8. Let $n \in \mathbb{Z}^+$. A **complete graph** for n nodes, denoted K_n, is an undirected graph with an edge between every pair of nodes and a loop at every node. Draw K_n for $n \le 5$.

Solution:



9. Suppose A and B are nonempty finite sets, $|A| = n$ and $|B| = k$.

(i) How many relations are there from A to B?

(ii)* How many functions are there from A to B?
(In particular, how many Boolean functions are there for m variables?)

(iii) How many injective functions are there from A to B?

(iv)* For $k \leq 4$, how many surjective functions are there from A to B?

(v) How many bijections are there from A to B?

(vi)* For $k \leq 3$, how many functions are there from A to B that are not injective and not surjective?

Solution:

(i) A relation R is a subset of $A \times B$, so $R \in \mathcal{P}(A \times B)$.
$\mathcal{P}(A \times B)$ has $2^{|A \times B|} = 2^{|A||B|}$ elements, so there are 2^{nk} relations.

(ii)* For a function $f : A \to B$, there are k choices for each $f(a_i)$.
There are n possible a_i's, so the number of functions is $k^n = |B|^{|A|}$.
For Boolean functions of m variables, $A = \{T, F\}^m$ and $B = \{T, F\}$,
so $|A| = 2^m$ and $|B| = 2$, and there are $|B|^{|A|} = 2^{2^m}$ Boolean functions.

(iii) If $|A| > |B|$, there are no injective functions $f : A \to B$ (Theorem 8.1.2).
If $|A| \leq |B|$, there are k choices for $f(a_1)$, $k - 1$ choices for $f(a_2)$, etc.
so the total number of injective functions is $k(k-1)\cdots(k-n+1) = {}^kP_n$.

(iv)* Let $T(n, k)$ be the number of surjective functions from A to B,
and $F = \{f : A \to B \mid f \text{ is not surjective}\}$. (By Theorem 8.1.3, $T(n, k) = 0$ if $n < k$.)
Then $T(n, k) = k^n - |F|$, by (ii) above.
Let $F_i = \{f : A \to B \mid b_i \notin range(f)\}$. Then $F = F_1 \cup F_2 \cup \ldots \cup F_k$.
By Inclusion/Exclusion, $|F| = \sum_i |F_i| - \sum_{i,j} |F_i \cap F_j| + \sum_{h,i,j} |F_h \cap F_i \cap F_j| - \cdots$
where $|F_i| = (k-1)^n, |F_i \cap F_j| = (k-2)^n, \ldots, |F_1 \cap \cdots \cap F_r| = (k-r)^n$.
Therefore $|F| = \binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n - \cdots = \sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}(k-i)^n$,
so $T(n, k) = k^n - |F| = (-1)^0\binom{k}{0}(k-0)^n - |F| = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$.
$\underline{k=1}$: $T(n, 1) = 1^n + (-1)\binom{1}{1}(1-1)^n = 1$ (all of A maps to b_1).
$\underline{k=2}$: $T(n, 2) = 2^n - \binom{2}{1}(2-1)^n + \binom{2}{2}(2-2)^n = 2^n - 2$ (all of A map to b_1 or all map to b_2).
$\underline{k=3}$: $T(n, 3) = 3^n - \binom{3}{1}(3-1)^n + \binom{3}{2}(3-2)^n = 3^n - 3(2^n) + 3$.
$\underline{k=4}$: $T(n, 4) = 4^n - \binom{4}{1}(4-1)^n + \binom{4}{2}(4-2)^n - \binom{4}{3}(4-3)^n = 4^n - 4(3^n) + 6(2^n) - 4$.

(v) There is a bijection $f : A \to B$ if and only if $n = k$ (Theorem 8.1.4).
There are k choices for $f(a_1)$, $k-1$ choices for $f(a_2)$, ..., total $= \begin{cases} k! & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$

(vi)* $\#(\sim\text{injective} \wedge \sim\text{surjective})$
$= \#(\sim(\text{injective} \vee \text{surjective})) = \#\text{functions} - \#(\text{injective} \vee \text{surjective})$
$= \#\text{functions} - (\#\text{injective} + \#\text{surjective} - \#\text{bijections})$.
$\underline{k=1}$: All functions are surjective, so 0.
$\underline{k=2}$: $n > k = 2$: $2^n - (0 + (2^n - 2) - 0) = 2$
$n < k = 2$: $2^n - ({}^2P_n + 0 - 0) = 0$ since $n = 1$.
$n = k = 2$: $2^n - ({}^2P_2 + (2^2 - 2) - 2!) = 2$.

$\underline{k=3}$: $n > k = 3$: $3^n - (0 + (3^n - 3(2^n) + 3) - 0) = 3(2^n) - 3$
$n < k = 3$: $3^n - ({}^3P_n + 0 - 0) = \begin{cases} 0 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \end{cases}$
$n = k = 3$: $3^3 - ({}^3P_3 + (3^3 - 3(2^3) + 3) - 3!) = 21$.

Note: For $n = k$, the general formula is $n^n - n!$.

10.* Let U be a nonempty finite set. A 3-partition is a partition of U into three subsets X, Y and Z such that

- $X \neq \varnothing$, $Y \neq \varnothing$ and $Z \neq \varnothing$,
- $X \cap Y = Y \cap Z = X \cap Z = \varnothing$ and
- $X \cup Y \cup Z = U$.

(i) List all possible 3-partitions of $\{a, b, c, d\}$.
Suppose U has n elements, where $n \geq 3$. Let P_n be the number of 3-partitions of U. What is P_4?

(ii) Prove that $P_{n+1} = 3P_n + 2^{n-1} - 1$ for all $n \geq 3$.

(iii) Prove that $P_n = \frac{1}{2}(3^{n-1} - 2^n + 1)$ for all $n \geq 3$.

Solution:

(i) $\{\{a\}, \{b\}, \{c, d\}\}, \{\{a\}, \{c\}, \{b, d\}\}, \{\{a\}, \{d\}, \{b, c\}\},$
$\{\{b\}, \{c\}, \{a, d\}\}, \{\{b\}, \{d\}, \{a, c\}\}, \{\{c\}, \{d\}, \{a, b\}\} \Rightarrow P_4 = 6.$

Let $U = \{a_1, \ldots, a_n\}$.

(ii) **Case (1)** The 3-partition is $\{X, Y, \{a_{n+1}\}\}$.
The number of possible X is the number of possible $X \in \mathcal{P}(\{a_1, \ldots, a_n\})$, except $X = \varnothing$ and $X = \{a_1, \ldots, a_n\}$ (which makes $Y = \varnothing$), i.e. $|\mathcal{P}(\{a_1, \ldots, a_n\})| - 2 = 2^n - 2$.
Once X is determined, $Y = \{a_1, \ldots, a_n\} \setminus X$.
However, $\{X, Y\} = \{Y, X\}$, so the total number of $\{X, Y\}$ is $\frac{2^n - 2}{2} = 2^{n-1} - 1$.

Case (2) The 3-partition is formed by taking a 3-partition $\{X, Y, Z\}$ of $\{a_1, \ldots, a_n\}$ and adding a_{n+1} to one of them (X or Y or Z).
The number of $\{X, Y, Z\}$ is P_n, so the total number is $3P_n$.

Adding the two cases gives $P_{n+1} = 3P_n + 2^{n-1} - 1$.

(iii) We prove $P_n = \frac{1}{2}(3^{n-1} - 2^n + 1)$ for all $n \geq 3$ by induction on n.
Basis $n = 3$: There is just one partition, namely $\{\{a_1\}, \{a_2\}, \{a_3\}\}$, so $P_3 = 1$.
Also $\frac{1}{2}(3^{3-1} - 2^3 + 1) = \frac{1}{2}(9 - 8 + 1) = 1 = P_3$,
so the claim is true for $n = 3$.

Induction Hypothesis: Suppose the equation is correct if $n = k$, for some $k \geq 3$.

Induction Step: Consider $n = k + 1$.

$$\begin{aligned} \text{By (ii), } P_{k+1} &= 3P_k + 2^{k-1} - 1 \\ &= \tfrac{3}{2}(3^{k-1} - 2^k + 1) + 2^{k-1} - 1 \text{ by the Induction Hypothesis} \\ &= \tfrac{1}{2}(3^k) - 3(2^{k-1}) + \tfrac{3}{2} + 2^{k-1} - 1 \\ &= \tfrac{1}{2}(3^k) - 2^k + \tfrac{1}{2} \\ &= \tfrac{1}{2}(3^k - 2^{k+1} + 1) \text{ so the claim is true for } n = k + 1. \end{aligned}$$

By induction, the claim is true for all $n \geq 3$.

11.* Consider a Boolean expression α with statement variables $x_1, \ldots, x_n$. A **truth assignment** is a function $f : \{x_1, \ldots, x_n\} \to \{T, F\}$. We say f **satisfies** α (or f is a **satisfying truth assignment** for α) if and only if α is true when, for all i, $f(x_i)$ is the truth value of x_i.

(i) Does $f(p) = T$, $f(q) = T$ and $f(r) = F$ satisfy $(p \vee q) \wedge ((r \vee \sim q) \vee \sim (p \vee r))$?

(ii) What is the maximum number of truth assignments that can satisfy α?

(iii) For $n = 3$, give an example of α that has the maximum number of satisfying truth assignments.

(iv) A Boolean expression is **satisfiable** if and only if it has at least one satisfying truth assignment. For $n = 4$, give an example of a Boolean expression that is not satisfiable.

(v) How many satisfying truth assignments are there for the Boolean expression in (i)?

[The **Satisfiability Problem** is: Given a Boolean expression α, is α satisfiable? Mathematically, this question has a trivial solution: simply try all possible truth assignments. Computationally, however, this problem is believed to be **intractable**, in the sense that no one has found a fast solution algorithm. The problem does not become much easier even if α is in CNF: if α has two literals per clause, there is a polynomial algorithm to determine satisfiability; but if α has three literals per clause, the satisfiability problem becomes NP-complete.]

Solution:
n is the number of statement variables.

(i) No. See the truth table below.

(ii) Each truth assignment corresponds to one row of the truth table for α, so the maximum number of truth assignments that can satisfy α is the number of rows in the truth table, i.e. 2^n.

(iii) Any tautology will do; e.g. $((p \to q) \wedge (q \to r)) \to (p \to r)$.

(iv) α is not satisfiable iff it does not have any satisfying truth assignments. This happens iff every row in the truth table makes α false, i.e. α is a contradiction. Any contradiction will do; e.g. $(x_1 \wedge \sim x_1 \wedge x_2) \vee (x_3 \wedge \sim x_3 \wedge x_4)$.

(v) There are 5 satisfying assignments:

p	q	r	$p \vee q$	$r \vee \sim q$	$\sim (p \vee r)$	$(p \vee q) \wedge ((r \vee \sim q) \vee \sim (p \vee r))$
T	T	T	T	T	F	T
T	T	F	T	F	F	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	T	F	T	T
F	F	T	F	F	F	F
F	F	F	F	F	T	F