**Definition**
A **statement** (or **proposition**) is a sentence that is true or false, but not both.

**Examples:**
$1 + 2$

$1 + 2 = 3$

$1 + 2 > 4$

$x + y > 4$

**Definition**
Let $p$ and $q$ be statements. Then $\sim p$, $p \wedge q$ and $p \vee q$ are statements.
$\sim p$ (read as "**not** $p$") is the **negation** of $p$.
$p \wedge q$ (read as "$p$ **and** $q$") is the **conjunction** of $p$ and $q$.
$p \vee q$ (read as "$p$ **or** $q$") is the **disjunction** of $p$ and $q$.

| $x = 0$ | $x \neq 0$ |
|---|---|
| | |

| $x$ is an integer | $x > 0$ | $x$ is a positive integer |
|---|---|---|
| | | |

| $x > 0$ | $x = 0$ | $x \geq 0$ |
|---|---|---|
| | | |

**Definition** (continued) $\sim p$ is $\begin{cases} \text{true} & \text{if } p \text{ is false} \\ \text{false} & \text{otherwise} \end{cases}$

$p \wedge q$ is $\begin{cases} \text{true} & \text{if } p \text{ is true and } q \text{ is true} \\ \text{false} & \text{otherwise} \end{cases}$ $p \vee q$ is $\begin{cases} \text{true} & \text{if } p \text{ is true or } q \text{ is true (or both)} \\ \text{false} & \text{otherwise} \end{cases}$

(This is an example of a **recursive** definition.)

$(p \vee q) \wedge \sim (p \wedge q)$

| $p$ | $q$ | $p \vee q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $(p \vee q) \wedge \sim (p \wedge q)$ |
|---|---|---|---|---|---|
| | | | | | |

**Definition** (continued)
We call $p$ and $q$ **statement variables**.
A **compound statement** or **Boolean expression** is a statement constructed from statement variables by using $\vee$, $\wedge$ and $\sim$.

**Definition**
Two Boolean expressions $P$ and $Q$ with the same statement variables are **logically equivalent** (denoted $P \equiv Q$) if and only if they have the same truth values for all choices of truth values for the variables.

$(p \vee q) \wedge r \equiv p \vee (q \wedge r)$?

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |

$\sim (p \wedge q) \equiv (\sim p) \wedge (\sim q)$?

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |

| $x < 0$ | $x \geq 0$ | $x < 0$ or $x \geq 0$ | $x < 0$ and $x \geq 0$ |
|---|---|---|---|
| | | | |

**Definition**
If a Boolean expression $P$ is always true regardless of the truth values of its statement variables, we call $P$ a **tautology**, denoted $P \equiv T$.
If a Boolean expression $Q$ is always false regardless of the truth values of its statement variables, we call $Q$ a **contradiction**, denoted $Q \equiv F$.

Let $t$ be a tautology and $c$ be a contradiction.

| $p$ | $t$ | $p \wedge t$ |
|---|---|---|
| | | |

| $p$ | $c$ | $p \vee c$ |
|---|---|---|
| | | |

so $t$ is the **conjunctive identity** and $c$ is the **disjunctive identity**.

**Theorem 1.1** For any statement variables $p$, $q$ and $r$, a tautology $t$ and a contradiction $c$,

| | | | |
|---|---|---|---|
| (a) | Commutativity | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| (b) | Associativity | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| (c) | Distributivity | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| (d) | Idempotence | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| (e) | Absorption | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| (f) | De Morgan's Law | $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$ | $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$ |
| (g) | Identities | $p \wedge t \equiv p, \quad p \vee t \equiv t$ | $p \vee c \equiv p, \quad p \wedge c \equiv c$ |
| (h) | Negation | $p \vee \sim p \equiv t, \quad \sim (\sim p) \equiv p$ | $p \wedge \sim p \equiv c, \quad \sim t \equiv c$ |

| if $x > 1$ then $x + y > 2$ | $x > 1$ | $x + y > 2$ |
|---|---|---|
| | | |

| $x > 1$ | $x + y > 2$ | if $x > 1$ then $x + y > 2$ |
|---|---|---|
| | | |

**Definition**
Let $p$ and $q$ be statements.
Then $p \to q$ (read as "if $p$ then $q$" or "$p$ implies $q$" or "$q$ if $p$")
is a **conditional statement**, whose truth table is:

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| | | |

**Note**
Avoid "$p$ only if $q$" (confusing).

**Definition** The **converse** of $p \to q$ is $q \to p$.

$((p \to q) \wedge q) \to p \equiv \mathrm{T}$ ?

| $p$ | $q$ | $p \to q$ | $(p \to q) \wedge q$ | $((p \to q) \wedge q) \to p$ |
|---|---|---|---|---|
| | | | | |

**Definition** The **inverse** of $p \to q$ is $(\sim p) \to (\sim q)$.

**Definition** The **contrapositive** of $p \to q$ is $(\sim q) \to (\sim p)$.

**Definition**
Let $p$ and $q$ be statements.
Then $p \leftrightarrow q$ (read as "$p$ if and only if $q$" and denoted "$p$ iff $q$")
is a **biconditional** statement defined by

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| | | |

**Definition**
Consider the conditional statement $p \to q$.
$p$ is a **sufficient** condition for $q$ and
$q$ is a **necessary** condition for $p$.

**Theorem** (*CS4232 Theory of Computation*) Every regular language is context-free.

"... Since $\{a^n b^n : n \geq 0\}$ is not regular,
therefore it is not context-free ..."

"... Since $\{a^n b^n c^n : n \geq 0\}$ is not context-free,
therefore it is not regular ..."

"... Since $\{a^n b^n : n \geq 0\}$ is context-free,
therefore it is regular ..."

"... Since $\{a^m b^n : m \geq 0, n \geq 0\}$ is regular,
therefore it is context-free ..."