A proof is rarely blind deduction from definitions and theorems. Before writing a proof, you must already see in your mind the skeleton joining various parts of the argument. Writing down the technical details is just fleshing out that skeleton.

1. Let A and B be sets.

   (a) Suppose A and B are disjoint (i.e., $A \cap B = \varnothing$) and countable. Prove that $A \cup B$ is countable.

   (b) Suppose A and B are (not necessarily disjoint but) countable. Prove that $A \cup B$ is countable.

Solution:

(a) A and B are disjoint and countable.

Case A and B are both finite.
Find $m, n \in \mathbb{Z}_{\geqslant 0}$ and bijections $f\colon \{1,2,\ldots,m\} \to A$ and $g\colon \{1,2,\ldots,n\} \to B$.
Define the function $h\colon \{1,2,\ldots,m+n\} \to A \cup B$ by setting

$$h(x) = \begin{cases} f(x), & \text{if } x \leqslant m; \\ g(x-m), & \text{if } x > m, \end{cases}$$

for each $x \in \{1,2,\ldots,m+n\}$.

(h is injective) Let $x_1, x_2 \in \{1,2,\ldots,m+n\}$ such that $h(x_1) = h(x_2)$.
As $A \cap B = \varnothing$, according to the definition of h,
x_1, x_2 are either both less than or equal to m, or both strictly bigger than m.
If x_1, x_2 are both less than or equal to m, then $f(x_1) = h(x_1) = h(x_2) = f(x_2)$, and so $x_1 = x_2$ as f is injective.
If x_1, x_2 are both strictly bigger than m, then $g(x_1 - m) = h(x_1) = h(x_2) = g(x_2 - m)$, and so $x_1 - m = x_2 - m$ as g is injective, and this implies $x_1 = x_2$.

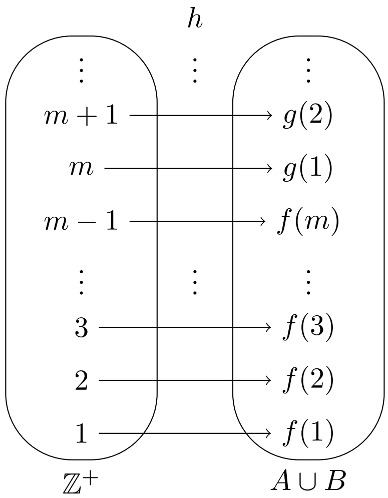
(h is surjective) Let $y \in A \cup B$. Then $y \in A$ or $y \in B$.
If $y \in A$, then the surjectivity of f gives $x \in \{1,2,\ldots,m\}$ such that $y = f(x) = h(x)$.
If $y \in B$, then the surjectivity of g gives $w \in \{1,2,\ldots,n\}$ such that
$y = g(w) = g\big((w+m) - m\big) = h(w+m)$.

These show h is a bijective. So $A \cup B$ is finite and thus countable.

(a) **Case** A is finite and B is infinite.
Find $m \in \mathbb{Z}_{\geqslant 0}$ and bijections $f\colon \{1, 2, \ldots, m\} \to A$ and $g\colon \mathbb{Z}^+ \to B$.
Define the function $h\colon \mathbb{Z}^+ \to A \cup B$ by setting

$$h(x) = \begin{cases} f(x), & \text{if } x \leqslant m; \\ g(x-m), & \text{if } x > m, \end{cases}$$

for each $x \in \mathbb{Z}^+$. As in the previous case, one can show that h is bijective.
So $A \cup B$ has the same cardinality as $\mathbb{Z}^+$, and is thus countable.

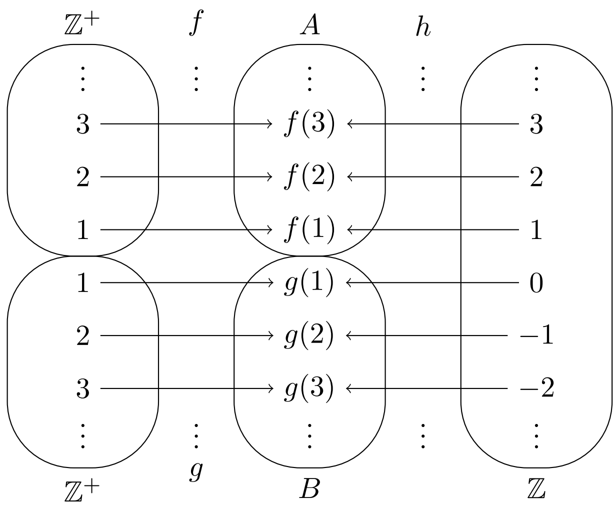


Case A is infinite and B is finite.
Same as in the previous case, except that A and B are interchanged.

(a) **Case** A is infinite and B is infinite.
Find bijections $f\colon \mathbb{Z}^+ \to A$ and $g\colon \mathbb{Z}^+ \to B$.
Define the function $h\colon \mathbb{Z} \to A \cup B$ by setting

$$h(x) = \begin{cases} f(x), & \text{if } x \geqslant 1; \\ g(-x+1), & \text{if } x < 1, \end{cases}$$

for each $x \in \mathbb{Z}^+$.



(h is injective) Let $x_1, x_2 \in \mathbb{Z}$ such that $h(x_1) = h(x_2)$.
As $A \cap B = \varnothing$, according to the definition of h, x_1, x_2 are either both bigger than or equal to 1, or both strictly less than 1.
If x_1, x_2 are both bigger than or equal to 1, then $f(x_1) = h(x_1) = h(x_2) = f(x_2)$, and so $x_1 = x_2$ as f is injective.
If x_1, x_2 are both strictly less than 1, then $g(-x_1+1) = h(x_1) = h(x_2) = g(-x_2+1)$, and so $-x_1 + 1 = -x_2 + 1$ as g is injective, and this implies $x_1 = x_2$.

(h is surjective) Let $y \in A \cup B$. Then $y \in A$ or $y \in B$.
If $y \in A$, then the surjectivity of f gives $x \in \mathbb{Z}^+$ such that $y = f(x) = h(x)$.
If $y \in B$, then the surjectivity of g gives $w \in \mathbb{Z}^+$ such that $y = g(w) = g\bigl(-(1-w)+1\bigr) = h(1-w)$, as $1 - w < 1 - 0 = 1$.

So $A \cup B$ has the same cardinality as $\mathbb{Z}$. But we know that $\mathbb{Z}$ has the same cardinality as $\mathbb{Z}^+$. So $A \cup B$ has the same cardinality as $\mathbb{Z}^+$, and is thus countable.

(b) $A \cup B = (A \setminus B) \cup B$, where $A \setminus B$ and B are disjoint.
$A \setminus B$ and B are subsets of countable sets. So $A \setminus B$ and B are countable by Proposition 8.3.5.
Hence (a) implies $(A \setminus B) \cup B$, and thus $A \cup B$, is countable.

2. Let $A_0, A_1, A_2, \ldots$ be countable sets. Recall from Tutorial 3 Problem 9 that for all $n \in \mathbb{Z}_{\geqslant 0}$,

$$\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \cdots \cup A_n,$$

and for all x,

$$x \in \bigcup_{i=0}^{\infty} A_i \qquad \text{if and only if} \qquad x \in A_i \text{ for some non-negative integer } i.$$

(a) Prove by induction that $\bigcup_{i=0}^{n} A_i$ is countable for any integer $n \geqslant 0$.

(b) Does (a) prove that $\bigcup_{i=0}^{\infty} A_i$ is countable?

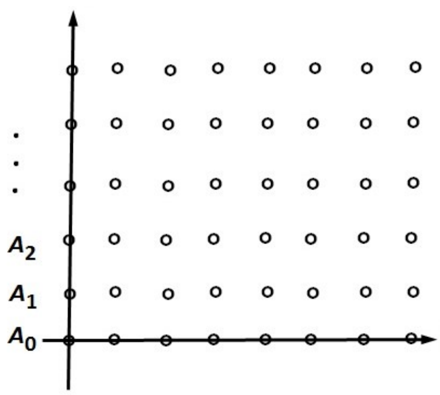
(c) Using the countability of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ from Theorem 8.4.4, or otherwise, prove that $\bigcup_{i=0}^{\infty} A_i$ is countable. (Hint: You may find Tutorial 7 Problem 9 useful.)

Solution: $A_0, A_1, A_2, \ldots$ are countable.

(a) $\bigcup_{i=0}^{n} A_i$ is countable:
Basis $n = 0$. $\bigcup_{i=0}^{0} A_i = A_0$, which is given as countable.
Induction Hypothesis. Suppose the claim is true for $n = k$ where $k \in \mathbb{Z}_{\geqslant 0}$.
Induction Step. Consider $n = k + 1$. Note $\bigcup_{i=0}^{k+1} A_i = (\bigcup_{i=0}^{k} A_i) \cup A_{k+1}$.
Since $\bigcup_{i=0}^{k} A_i$ is countable by the Induction Hypothesis,
and A_{k+1} is given as countable, the claim is true by Problem 1(b).

(b) No, because maybe $\bigcup_{i=0}^{\infty} A_i \neq \bigcup_{i=0}^{n} A_i$ for any $n \in \mathbb{Z}_{\geqslant 0}$: consider the case when each $A_i = \{i\}$.

(c) Intuition: The elements of $A_0, A_1, A_2, \ldots$ can be mapped row by row injectively to $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$.
Complications: finite A_i or $A_i \cap A_j \neq \varnothing$.
Solution: map $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ to $\bigcup_{i=0}^{\infty} A_i$ instead.



Claim. Let $B = \bigcup_{i=0}^{\infty} A_i$. Then B is countable.

Proof. If $A_i = \varnothing$ for all integer $i \geqslant 0$, then $B = \varnothing$, and so B is countable.
Suppose $A_i \neq \varnothing$ for some integer $i \geqslant 0$. Pick any b that is in some A_i.
For each nonempty A_i, fix a surjection $f_i \colon \mathbb{Z}^+ \to A_i$
given by Tutorial 7 Problem 9 and the countability of A_i.
Define $g \colon \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to B$ by setting, for all $i, j \in \mathbb{Z}_{\geqslant 0}$,

$$g(i,j) = \begin{cases} b, & \text{if } A_i = \varnothing; \\ f_i(j+1), & \text{if } A_i \neq \varnothing. \end{cases}$$

Consider any $y \in B$. Then $y \in A_k$ for some integer $k \geqslant 0$. Fix such a k.
Then the surjectivity of f_k gives $x \in \mathbb{Z}^+$
such that $y = f_k(x) = f_k\big((x-1)+1\big) = g(k, x-1)$, as $x - 1 \geqslant 1 - 1 = 0$.
This shows g is a surjection $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to B$.
Use the countability of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ to find a surjection $h \colon \mathbb{Z}^+ \to \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$.
Then $g \circ h$ is a surjection $\mathbb{Z}^+ \to B$ by Proposition 8.1.1(1).
So B is countable by Tutorial 7 Problem 9.

3.* The set $\mathbb{Q}$ of rational numbers can be defined by $\mathbb{Q} = \{r \in \mathbb{R} : \exists m \in \mathbb{Z}\ \exists n \in \mathbb{Z}^+\ r = \frac{m}{n}\}$.

(a) Consider the following "proof" that $\mathbb{Q}$ is countable.

> "Note that $\mathbb{Z} \subseteq \mathbb{Q}$. Since $\mathbb{Z}$ is countable and every subset of a countable set is countable, we know $\mathbb{Q}$ is countable."

What is wrong with this "proof"?

(b) Using the countability of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, or otherwise, prove that $\mathbb{Q}$ is countable.
(This is even more surprising than the countability of $\mathbb{Z}$, since there are infinitely many rational numbers between any two rational numbers.)

(c) In essence, a set X is countable means we can write $X = \{x_0, x_1, x_2, \ldots\}$. Write $\mathbb{Q}$ in this form.

Solution:

(a) It was proved in Proposition 8.3.5 that "$A \subseteq B \rightarrow$ (B is countable $\rightarrow A$ is countable)".
Converse error in "proof", which says "$\mathbb{Z} \subseteq \mathbb{Q} \rightarrow$ ($\mathbb{Z}$ is countable $\rightarrow \mathbb{Q}$ is countable)"

(b) Intuition: The elements of $\mathbb{Q}$ can be mapped injectively to $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, which is countable.
Complications: possible $\frac{m}{n} = \frac{h}{k}$ and negative r.
Solution: Use Problem 2(c).

Define $A_0 = \varnothing$, and $A_i = \{\frac{m}{i} : m \in \mathbb{Z}\}$ for all $i \in \mathbb{Z}^+$.
Note $A_0, A_1, A_2, \ldots$ are all countable since A_i has the same cardinality as $\mathbb{Z}$ for each $i \in \mathbb{Z}^+$.
As $\mathbb{Q} = \bigcup_{i=0}^{\infty} A_i$, we deduce that $\mathbb{Q}$ is countable by Problem 2(c).

(c) In the proof of Theorem 8.4.4, we enumerated the elements of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ as

$$\begin{aligned}\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} = \{&(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(3,0),(2,1),(1,2),(0,3),\\ &(4,0),(3,1),(2,2),(1,3),(0,4),\ldots\}.\end{aligned}$$

For $\mathbb{Q}$, we can follow this enumeration, treating (m,n) as $\frac{m}{n}$,
remove the fractions with zero denominator, then add the negatives:

$$\mathbb{Q} = \left\{0, 1, -1, 0, 2, -2, \frac{1}{2}, -\frac{1}{2}, 0, 3, -3, 1, -1, \frac{1}{3}, -\frac{1}{3}, 0, \ldots\right\}.$$

There are many alternative answers.

4. Let Y and Z be sets that are countable and infinite. Prove that $Y \times Z$ is countable.
(This is a generalization of the fact that $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is countable.)

Solution: Intuition: Map $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ surjectively to $Y \times Z$.
Use the fact that Y and Z are countable and infinite to find bijections $f\colon \mathbb{Z}^+ \to Y$ and $g\colon \mathbb{Z}^+ \to Z$.
Define $h\colon \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to Y \times Z$ by setting $h(i,j) = \big(f(i+1), g(j+1)\big)$ for each $(i,j) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$.

Injectivity: $h(i_1, j_1) = h(i_1, j_2) \quad \Rightarrow \quad \big(f(i_1+1), g(j_1+1)\big) = \big(f(i_2+1), g(j_2+1)\big)$
$\Rightarrow \quad f(i_1+1) = f(i_2+1)$ and $g(j_1+1) = g(j_2+1)$
$\Rightarrow \quad i_1+1 = i_2+1$ and $j_1+1 = j_2+1$ as f and g are injective
$\Rightarrow \quad i_1 = i_2$ and $j_1 = j_2$
$\Rightarrow \quad (i_1, j_1) = (i_2, j_2)$.

Surjectivity: Consider any $(y,z) \in Y \times Z$.
Use the surjectivity of f and g to find $i \in \mathbb{Z}^+$ and $j \in \mathbb{Z}^+$ such that $f(i) = y$ and $g(j) = z$.
Then $(y,z) = \big(f(i), g(j)\big) = \big(f((i-1)+1), g((j-1)+1)\big) = h(i-1, j-1)$.

Since h is bijective, we know $Y \times Z$ has the same cardinality as $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, which is countable.
So $Y \times Z$ is also countable.

5. Prove that:

(a) if a set X has an uncountable subset, then X is also uncountable;

(b)* if A is uncountable and B is countable, then $A \setminus B$ is uncountable.

Solution:

(a) Let $Y \subseteq X$. Proposition 8.3.5 tells us that if X is countable, then Y is countable.
Taking the contrapositive gives: if Y is uncountable, then X is uncountable.

(b)* Note that $A = (A \setminus B) \cup (A \cap B)$.
As $A \cap B \subseteq B$ and B is countable, we know $A \cap B$ is countable by Proposition 8.3.5.
If $A \setminus B$ is countable, then $(A \setminus B) \cup (A \cap B)$ is countable by Problem 1,
contradicting the uncountability of A. So $A \setminus B$ must be uncountable.

6. It will be shown in the lectures that if X is a finite set, then $\mathcal{P}(X)$ is finite and has cardinality $2^{|X|}$.
Use this to prove that a set X has countably many subsets if and only if X is finite.

Solution:
Claim: $\mathcal{P}(X)$ is countable if and only if X is finite.

($\Leftarrow$) If X is finite, then $\mathcal{P}(X)$ is finite and thus countable.
($\Rightarrow$) We prove the contrapositive, that X is infinite implies $\mathcal{P}(X)$ is uncountable.
Suppose X is infinite. Then X has a countable infinite subset, say Y, by Proposition 8.3.4.
Now Corollary 9.2.2 tells us $\mathcal{P}(Y)$ is uncountable.
As $Y \subseteq X$, every subset of Y is also a subset of X. So $\mathcal{P}(Y) \subseteq \mathcal{P}(X)$.
Hence Problem 5(a) implies $\mathcal{P}(X)$ is uncountable.

7.* Prove that

(a) $\{S \in \mathcal{P}(\{\mathrm{b}\}^*) : S \text{ contains exactly 3 strings}\}$ is countable;

(b) $\mathcal{P}(\{\mathrm{b}\}^*)$ is uncountable.

(Part (a) can be generalized to "there are countably many finite subsets of $\{\mathrm{b}\}^*$", while part (b) says $\{\mathrm{b}\}^*$ has uncountably many subsets. Therefore, the point here is: the uncountability of $\mathcal{P}(\{\mathrm{b}\}^*)$ must be from the infinite subsets.)

Solution:

(a) For a symbol c and $n \in \mathbb{Z}_{\geqslant 0}$, let c^n denote the string $\overbrace{cccc\ldots cc}^{n\text{-many } c\text{'s}}$;
e.g., c^0 is the empty string, $c^1 = c$, and $c^3 = ccc$.
Let $U = \{(m, n, k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ : m < n < k\}$.
Since $\mathbb{Z}^+$ is countable, Problem 4 tells us $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ is also countable.
So U is countable too by Proposition 8.3.5.
Let $\mathcal{V} = \{0^m 1 0^n 1 0^k : m < n < k \text{ for some } m, n, k \in \mathbb{Z}_{\geqslant 0}\}$.
As one can directly verify, the function $f \colon U \to \mathcal{V}$
such that $f(m, n, k) = 0^{m-1} 1 0^{n-1} 1 0^{k-1}$ for all $(m, n, k) \in U$ is a bijection.
So $\mathcal{V}$ is countable.

Note that if $S \in \mathcal{P}(\{\mathrm{b}\}^*)$ and S contains exactly 3 strings,
then $S = \{\mathrm{b}^m, \mathrm{b}^n, \mathrm{b}^k\}$ where m, n, k are different non-negative integers, since S is a set.
Define $\mathcal{T}_3 = \{S \in \mathcal{P}(\{\mathrm{b}\}^*) : S \text{ contains exactly 3 strings}\}$.
Define a function $g \colon \mathcal{T}_3 \to \mathcal{V}$ by setting $f(\{\mathrm{b}^m, \mathrm{b}^n, \mathrm{b}^k\}) = 0^m 1 0^n 1 0^k$
whenever $m, n, k \in \mathbb{Z}_{\geqslant 0}$ satisfying $m < n < k$.

(**g is surjective**) For any $0^m 1 0^n 1 0^k \in \mathcal{V}$, we have $m < n < k$,
so b^m, b^n and b^k are three different strings;
therefore $\{\mathrm{b}^m, \mathrm{b}^n, \mathrm{b}^k\} \in \mathcal{T}_3$ and $g(\{\mathrm{b}^m, \mathrm{b}^n, \mathrm{b}^k\}) = 0^m 1 0^n 1 0^k$.

(**g is injective**) Suppose $g(S_1) = 0^m 1 0^n 1 0^k = g(S_2)$. Then $S_1 = \{\mathrm{b}^m, \mathrm{b}^n, \mathrm{b}^k\} = S_2$.

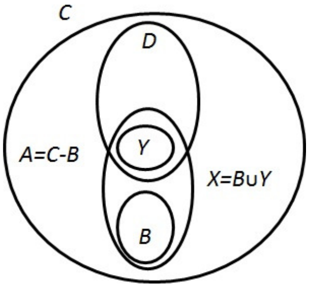
Since g is bijective, the sets $\mathcal{T}_3$ and $\mathcal{V}$ have the same cardinality. So $\mathcal{T}_3$ is countable.

(b) One can verify that the function $h \colon \mathbb{Z}^+ \to \{\mathrm{b}\}^*$
satisfying $h(n) = \mathrm{b}^n$ for each $n \in \mathbb{Z}^+$ is injective.
As $\mathbb{Z}^+$ is infinite, this implies $\{\mathrm{b}\}^*$ is infinite too. So $\mathcal{P}(\{\mathrm{b}\}^*)$ is uncountable by Problem 6.

8.* Let B be a finite subset of an infinite set C. Prove that there are uncountably many countable sets X such that $B \subseteq X \subseteq C$.

Solution: Given: infinite C and finite $B \subseteq C$.
Intuition: Take a countable infinite subset D of C, then add B to the subsets of D.

Proof.



Let $A = C \setminus B$.
As B is finite and C is infinite,
we know A is infinite by the solution to Problem 1(a).
So A has a countably infinite subset, say D, by Proposition 8.3.4.
Note that $\mathcal{P}(D)$ is uncountable by Corollary 9.2.2.
Since D is countable,
Proposition 8.3.5 says every $Y \in \mathcal{P}(D)$ is countable.
Thus, for each $Y \subseteq D$, the union $B \cup Y$ is countable by Problem 1.
As $B \cap D \subseteq B \cap A = B \cap (C \setminus B) = \varnothing$,
one can verify that $\{B \cup Y : Y \in \mathcal{P}(D)\}$ has the same cardinality as $\mathcal{P}(D)$.
So $\{B \cup Y : Y \in \mathcal{P}(D)\}$ is uncountable.
Moreover, for each $Y \in \mathcal{P}(D)$, we have $B \subseteq B \cup Y \subseteq C$ as $B \subseteq C$ and $Y \subseteq D \subseteq A = C \setminus B \subseteq C$.