National University of Singapore
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2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 5**

1. Consider the relation R from Tutorial 4 Problem 5. Let $S = R^{-1} \circ R$ and $T = S \circ S$.
   (a) Determine whether S is a total order.
   (b) Draw an arrow diagram for T.
   (c) Why is T an equivalence relation? Determine the equivalence classes with respect to T.

2.* Let A and B be sets and R a relation from A to B. Prove that $R^{-1} \circ R$ is symmetric.

3. For each relation below, determine if it is reflexive, symmetric, antisymmetric, and transitive:
   (a) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 \leqslant y^2\}$, as a relation on $\mathbb{Z}$;
   (b)* $\{(x, y) \in \mathbb{R} \times \mathbb{R} : xy \geqslant 0\}$, as a relation on $\mathbb{R}$;
   (c)* $\{(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U) : A \cap B \neq \varnothing\}$, as a relation on $\mathcal{P}(U)$, where U is a set with at least 2 elements;
   (d) $\big\{\big((a, b), (c, d)\big) \in \mathbb{R}^2 \times \mathbb{R}^2 : (a \leqslant c) \wedge (b \leqslant d)\big\}$, as a relation on $\mathbb{R}^2$.

   If a relation R above is not transitive, then give an example to show $R \circ R \nsubseteq R$. Which of the above is a partial order? Is it a total order?

4.* Prove that the relation $S = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m^3 + n^3 \text{ is even}\}$ on $\mathbb{Z}$ from Tutorial 4 Question 6 is an equivalence relation. What are the equivalence classes?

5.* Let $k \in \mathbb{Z}^+$. Define the relation $\equiv_k$ on $\mathbb{Z}$ by setting, for all $m, n \in \mathbb{Z}$,
$$m \equiv_k n \qquad \text{if and only if} \qquad k \text{ divides } m - n.$$
   Prove that $\equiv_k$ is an equivalence relation. What are the equivalence classes?

6.* Let R be a binary relation on a set X, and $Y \subseteq X$. The **restriction** of R to Y, denoted $R\big|_Y$, is the relation on Y defined by $R\big|_Y = R \cap (Y \times Y)$. If R is an equivalence relation on X, then we call the partition X/R given by Theorem 6.3.10 the **partition (of X) induced by** R.
   (a) Prove that, if R is an equivalence relation, then $R\big|_Y$ is an equivalence relation on Y.
   (b) Let $B = \{-2, -1, 0, 1, 2, 3, 4\}$ and let S be the equivalence relation in Problem 4. How would you draw an undirected graph to represent $S\big|_B$? Determine the equivalence classes and the partition induced by $S\big|_B$.
   (c) Let $C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$ and let $\equiv_6$ be as in Problem 5. How would you draw an undirected graph to represent $\equiv_6 \big|_C$? Determine the equivalence classes and the partition induced by $\equiv_6\big|_C$.

   [Without (a), we would have to prove all over again for (b) that $\{(m, n) \in B \times B : m^3 + n^3 \text{ is even}\}$ is an equivalence relation on B.]

7. Consider the following relation on the set of all points in the plane:
$$\mathcal{L} = \{((a, b), (c, d)) \in \mathbb{R}^2 \times \mathbb{R}^2 : a - c = 3(b - d)\}.$$
   (a) Prove that $\mathcal{L}$ is an equivalence relation.
   (b) For a point (u, v) in the plane, determine the equivalence class $[(u, v)]_{\mathcal{L}}$, and represent it geometrically.
   (c) Determine the partition of $\mathbb{R}^2$ induced by $\mathcal{L}$.

8. Let R be an equivalence relation on set X. Prove that, for any $b, c \in X$,

$$b \; R \; c \qquad \text{if and only if} \qquad [b]_R = [c]_R.$$

9. Prove or disprove:

   (a) A relation that is symmetric cannot be antisymmetric.

   (b) A relation that is not symmetric must be antisymmetric.

10. (a) The following is a "proof" that every relation that is symmetric and transitive must be reflexive:

    "Suppose R is symmetric and transitive. Then $x \; R \; y$ and $y \; R \; x$ for any x and y in A, because R is symmetric. Thus $x \; R \; x$ by transitivity. So R is reflexive."

    What is wrong with this "proof"?

    (b) Give an example of a symmetric, transitive relation that is not reflexive.

11.* For a positive integer n, define $S_n = \{q \in \mathbb{Z} : \exists k \in \mathbb{Z}_{\geqslant 0} \; n = 2^k q\}$.

   (a) Determine S_{7680}.

   (b) Use S_n and the Well-Ordering Principle to prove that, for any $n \in \mathbb{Z}^+$, there exists an integer h and an odd integer r such that $n = 2^h r$.

12.* Explain why the definitions in (a) and (b) below are not valid.

   (a) For any real number x, define $\hat{x}$ to be the largest integer n such that $n \geqslant x$.

   (b) For any real number x, define $\langle x \rangle$ to be the integer n such that $|x - n| < 1$.

   (c) One can define the ceiling $\lceil x \rceil$ of a real number x to be the smallest integer in $\{n \in \mathbb{Z} : n \geqslant x\}$. Explain why this is a valid definition, i.e., why this integer always exists and is always unique.

13.* Recall that, for all $x \in \mathbb{R}$, if $x \geqslant 0$, then $|x| = x$, else $|x| = -x$. Consider the claim:

   "$|a_1 + a_2 + \cdots + a_n| \leqslant |a_1| + |a_2| + \cdots + |a_n|$ for all real numbers $a_1, \ldots, a_n$."

   (This is called the *Triangle Inequality*, which is often used in Calculus, as well as in Complexity Analysis, e.g., the *Travelling Salesman Problem* remains NP-Complete even if the distances satisfy the Triangle Inequality.)

   (a) The following is a "proof" of the claim.

   "We will use the Second Induction Principle. Since $|a_1| \leqslant |a_1|$ for any $a_1 \in \mathbb{R}$, the claim is trivially true for $n = 1$. Suppose the claim is true for all $n < k + 1$, where $k \geqslant 1$. For any $a_1, \ldots, a_{k-1}, a_k, a_{k+1} \in \mathbb{R}$, letting $a'_k = a_k + a_{k+1}$,

   $$\begin{aligned} &|a_1 + \cdots + a_{k-1} + a_k + a_{k+1}| \\ &= |a_1 + \cdots + a_{k-1} + a'_k| \\ &\leqslant |a_1| + \cdots + |a_{k-1}| + |a'_k| && \text{by the induction hypothesis;} \\ &= |a_1| + \cdots + |a_{k-1}| + |a_k + a_{k+1}| \\ &\leqslant |a_1| + \cdots + |a_{k-1}| + |a_k| + |a_{k+1}| && \text{as } |b + c| \leqslant |b| + |c| \text{ by the induction hypothesis.} \end{aligned}$$

   So the claim is true for $n = k + 1$. By induction, the claim is true for all integers $n \geqslant 1$."

   What is wrong with the "proof" above? (Note that the same "proof" can be used to show "$|a_1 + a_2 + \cdots + a_n| \geqslant |a_1| + |a_2| + \cdots + |a_n|$ for any real numbers $a_1, \ldots, a_n$", which is false.)

   (b) Either fix the error in (a), or give your own proof of the claim.

14. Continued from Tutorial 4 Problem 13. Prove that, when $C = 2^n$ where $n \in \mathbb{Z}^+$, there is always a solution, i.e., no matter which unit square is singled out on a $2^n \times 2^n$ chessboard, the rest can be covered by non-overlapping L-tiles.