CS1231 Chapter 5

Relations

5.1 Basics

Definition 5.1.1. An *alphabet* is a finite set of symbols. A *string* over an alphabet Γ is a finite sequence of symbols from Γ. The set of all strings over an alphabet Γ is denoted Γ^*.

Definition 5.1.2. An *ordered pair* is an expression of the form

$$(x, y).$$

Let (x_1, y_1) and (x_2, y_2) be ordered pairs. Then

$$(x_1, y_1) = (x_2, y_2) \qquad \Leftrightarrow \qquad x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Example 5.1.3. (1) $(1, 2) \neq (2, 1)$, although $\{1, 2\} = \{2, 1\}$.

(2) $(3, 0.5) = (\sqrt{9}, \frac{1}{2})$.

Definition 5.1.4. Let A, B be sets. The *Cartesian product* of A and B, denoted $A \times B$, is defined to be

$$\{(x, y) : x \in A \text{ and } y \in B\}.$$

Read $A \times B$ as "A cross B".

Example 5.1.5. $\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

Definition 5.1.6. Let A, B be sets.

(1) A *relation* from A to B is a subset of $A \times B$.

(2) Let R be a relation from A to B and $(x, y) \in A \times B$. Then we may write

$$x \mathrel{R} y \text{ for } (x, y) \in R \quad \text{and} \quad x \not\mathrel{R} y \text{ for } (x, y) \notin R.$$

We read "$x \mathrel{R} y$" as "x is *R-related* to y" or simply "x is *related* to y".

Example 5.1.7. Let $\Gamma = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, 0, 1, 2, \ldots, 9\}$ and $\Phi = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, \mathrm{a}, \mathrm{b}, \ldots, \mathrm{z}\}$. As in Figure 5.1, define

$$SN = \{(\mathrm{001R}, \mathrm{Gates}), (\mathrm{012B}, \mathrm{Brin}), (\mathrm{062E}, \mathrm{Bezos}), (\mathrm{126N}, \mathrm{Ma}), (\mathrm{254E}, \mathrm{Zuckerberg})\}.$$

Then SN is a relation from Γ^* to Φ^*.

identity	
Student ID	name
001R	Gates
012B	Brin
062E	Bezos
126N	Ma
254E	Zuckerberg

$$SN = \{ (001R, \text{Gates}), (012B, \text{Brin}), (062E, \text{Bezos}), (126N, \text{Ma}), (254E, \text{Zuckerberg}) \}$$

is enrolled in	
Student ID	module
126N	CS3234
254E	CS3234
001R	MA2001
012B	MA2001
062E	MA2001
126N	MA2001
012B	MU2109
001R	PC2130
062E	PL3103
254E	PL3103

$$SM = \{ (126N, CS3234), (254E, CS3234), (001R, MA2001), (012B, MA2001), (062E, MA2001), (126N, MA2001), (012B, MU2109), (001R, PC2130), (062E, PL3101), (254E, PL3101) \}$$

progress		
Student ID	faculty	year
062E	Arts	1
254E	Arts	2
012B	Science	2
001R	Science	1
126N	Science	3

$$SFY = \{ (062E, \text{Arts}, 1), (254E, \text{Arts}, 2), (012B, \text{Science}, 2), (001R, \text{Science}, 1), (126N, \text{Science}, 3) \}$$

teaching			
module	department	faculty	instructor
CS3234	CS	Computing	Turing
MA2001	Mathematics	Science	Gauss
MU2109	Music	Arts	Mozart
PC2130	Physics	Science	Newton
PL3101	Psychology	Arts	Freud

$$MDFI = \{ (CS3234, \text{CS}, \text{Computing}, \text{Turing}), (MA2001, \text{Mathematics}, \text{Science}, \text{Gauss}), (MU2109, \text{Music}, \text{Arts}, \text{Mozart}), (PC2130, \text{Physics}, \text{Science}, \text{Newton}), (PL3101, \text{Psychology}, \text{Arts}, \text{Freud}) \}$$

The set $\{SM, SN, SFY, MDFI\}$ represents the relational database.

Figure 5.1: A fictitious miniature university database and its set-theoretic representation

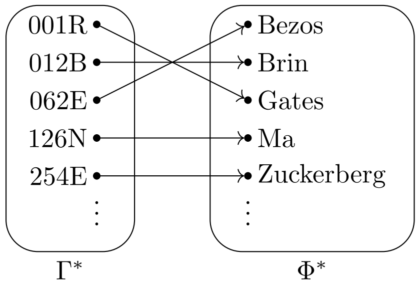
Example 5.1.8. Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4\}$. Define the relation R from A to B by setting

$$x \, R \, y \quad \Leftrightarrow \quad x < y.$$

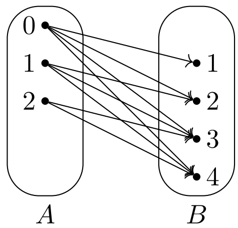
Then $0 \, R \, 1$ and $0 \, R \, 2$, but $2 \not R \, 1$. Thus

$$R = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Arrow diagrams (for relations from a set to another set). One can use the figure below to represent the relation *SN* in Example 5.1.7, where the existence of an arrow from x to y indicates x is related to y.



Similarly, one can use the figure below to represent the relation R in Example 5.1.8.



Definition 5.1.9. Let $n \in \{x \in \mathbb{Z} : x \geqslant 2\}$. An *ordered n-tuple* is an expression of the form

$$(x_1, x_2, \ldots, x_n).$$

Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be ordered n-tuples. Then

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } \ldots \text{ and } x_n = y_n.$$

Example 5.1.10. (1) $(1, 2, 5) \neq (2, 1, 5)$, although $\{1, 2, 5\} = \{2, 1, 5\}$.

(2) $(3, (-2)^2, 0.5, 0) = (\sqrt{9}, 4, \frac{1}{2}, 0)$

Definition 5.1.11. Let $n \in \{x \in \mathbb{Z} : x \geqslant 2\}$ and $A_1, A_2, \ldots, A_n$ be sets. The *Cartesian product* of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is defined to be

$$\{(x_1, x_2, \ldots, x_n) : x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \ldots \text{ and } x_n \in A_n\}.$$

If A is a set, then $A^n = \underbrace{A \times A \times \cdots \times A}_{n\text{-many } A\text{'s}}$.

Example 5.1.12. $\{0, 1\} \times \{0, 1\} \times \{x, y\} = \{(0, 0, x), (0, 0, y), (0, 1, x), (0, 1, y), (1, 0, x), (1, 0, y), (1, 1, x), (1, 1, y)\}$.

Definition 5.1.13. Let $n \in \{x \in \mathbb{Z} : n \geqslant 2\}$ and $A_1, A_2, \ldots, A_n$ be sets. A *n-ary relation over* $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

Example 5.1.14. Following Example 5.1.7, let $\Gamma = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, 0, 1, 2, \ldots, 9\}$ and $\Phi = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}, \mathrm{a}, \mathrm{b}, \ldots, \mathrm{z}\}$. As in Figure 5.1, define

$$\begin{aligned} \mathit{MDFI} = \{&(\mathrm{CS3234}, \mathrm{CS}, \mathrm{Computing}, \mathrm{Turing}), (\mathrm{MA2001}, \mathrm{Mathematics}, \mathrm{Science}, \mathrm{Gauss}),\\ &(\mathrm{MU2109}, \mathrm{Music}, \mathrm{Arts}, \mathrm{Mozart}), (\mathrm{PC2130}, \mathrm{Physics}, \mathrm{Science}, \mathrm{Newton}),\\ &(\mathrm{PL3101}, \mathrm{Psychology}, \mathrm{Arts}, \mathrm{Freud})\}. \end{aligned}$$

Then *MDFI* is a 4-ary relation over $\Gamma^*, \Phi^*, \Phi^*, \Phi^*$.

5.2 Operations on relations

Figure 5.2: Relation composition and inversion

Definition 5.2.1. Let R be a relation from A to B, and S be a relation from B to C. Then $S \circ R$ is the relation from A to C defined by

$$S \circ R = \{(x, z) \in A \times C : (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y \in B\}.$$

We read $S \circ R$ as "S composed with R" or "S circle R".

Note 5.2.2. We compose two binary relations together only when there is a common middle set.

Definition 5.2.3 (recall)**.** The *floor* of a real number x, denoted $\lfloor x \rfloor$, is the greatest integer that is less than or equal to x.

Example 5.2.4. Define a relation R from $\mathbb{Q}_{\geqslant 0}$ to $\mathbb{Z}_{\geqslant 0}$ and a relation S from $\mathbb{Z}_{\geqslant 0}$ to $\mathbb{R}$ by:

$$\begin{aligned} R &= \{(x, y) \in \mathbb{Q}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} : \lfloor x \rfloor = y\}, \quad \text{and} \\ S &= \{(y, z) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{R} : y = z^2\}. \end{aligned}$$

- $(4.8, 2) \in S \circ R$ because $4 \in \mathbb{Z}_{\geqslant 0}$ such that $(4.8, 4) \in R$ and $(4, 2) \in S$.
- $(5/2, -\sqrt{2}) \in S \circ R$ because $2 \in \mathbb{Z}_{\geqslant 0}$ such that $(5/2, 2) \in R$ and $(2, -\sqrt{2}) \in S$.

In general, we have $S \circ R = \{(x, z) \in \mathbb{Q}_{\geqslant 0} \times \mathbb{R} : \lfloor x \rfloor = z^2\}$.

Definition 5.2.5. Let R be a relation from A to B. Then the *inverse of* R is the relation R^{-1} from B to A defined by

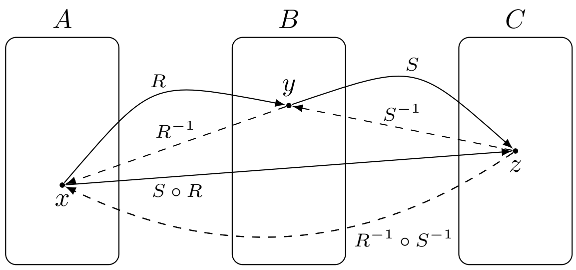
$$R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}.$$

Example 5.2.6. As in Example 5.1.8, let R be the relation from A to B where

$$\begin{aligned} &A = \{0, 1, 2\}, \qquad B = \{1, 2, 3, 4\}, \\ &R = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}. \end{aligned}$$

Then $R^{-1} = \{(1, 0), (2, 0), (3, 0), (4, 0), (2, 1), (3, 1), (4, 1), (3, 2), (4, 2)\}$.

Proposition 5.2.7. Let R be a relation from A to B, and S be a relation from B to C. Then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.



Proof. Since $S \circ R$ is a relation from A to C, we know $(S \circ R)^{-1}$ is a relation from C to A. Since S^{-1} is a relation from C to B, and R^{-1} is a relation from B to A, we know $R^{-1} \circ S^{-1}$ is a relation from C to A as well. Now for all $(z, x) \in C \times A$,

$$\begin{aligned}
&(z,x) \in (S \circ R)^{-1} \quad \Leftrightarrow \quad (x,z) \in S \circ R && \text{by the definition of inverses;}\\
\Leftrightarrow \quad &(x,y) \in R \text{ and } (y,z) \in S \ \text{ for some } y \in B && \text{by the definition of composition;}\\
\Leftrightarrow \quad &(y,x) \in R^{-1} \text{ and } (z,y) \in S^{-1} \ \text{ for some } y \in B && \text{by the definition of inverses;}\\
\Leftrightarrow \quad &(z,x) \in R^{-1} \circ S^{-1} && \text{by the definition of composition.}
\end{aligned}$$

So $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. □

Exercise 5.2.8. Let $A = \{0, 1, 2\}$. Define two relations R, S from A to A by: 5a

$$R = \{(x,y) \in A^2 : x < y\} \quad \text{and} \quad S = \{(0,1),(1,2),(2,0)\}.$$

Is $R \circ S = S \circ R$? Prove that your answer is correct.

5.3 Graphs

Definition 5.3.1. A *(binary) relation on a set* A is a relation from A to A.

Remark 5.3.2. It follows from Definition 5.1.6 and Definition 5.3.1 that the relations on a set A are precisely the subsets of $A \times A$.

Definition 5.3.3. A *directed graph* is an ordered pair (V, D) where V is a set and D is a binary relation on V. In the case when (V, D) is a directed graph,

(1) the *vertices* or the *nodes* are the elements of V;

(2) the *edges* are the elements of D;

(3) an edge *from* x *to* y is the element $(x, y) \in D$;

(4) a *loop* is an edge from a vertex to itself.

Example 5.3.4. Let

$$\begin{aligned}
&V = \{\mathrm{B}, \mathrm{P}, \mathrm{F}, \mathrm{M}, \mathrm{K}, \mathrm{N}\}, \quad \text{and}\\
&D = \{(\mathrm{B},\mathrm{P}),(\mathrm{P},\mathrm{B}),(\mathrm{F},\mathrm{M}),(\mathrm{M},\mathrm{F}),(\mathrm{B},\mathrm{B}),(\mathrm{P},\mathrm{P}),(\mathrm{F},\mathrm{F}),(\mathrm{M},\mathrm{M}),(\mathrm{K},\mathrm{K}),(\mathrm{N},\mathrm{N})\}.
\end{aligned}$$

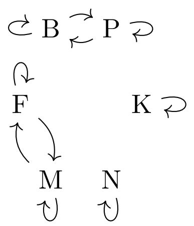
Then (V, D) is a directed graph.

Arrow diagrams (for relations on a set). One can draw an arrow diagram representing a relation R on a set A as follows.

(1) Draw all the elements of A.

(2) For all $x, y \in A$, draw an arrow from x to y if and only if $x \mathrel{R} y$.

Example 5.3.5. The arrow diagram



represents the relation D on the set V from Example 5.3.4.

Definition 5.3.6. A *undirected graph* is an ordered pair (V, E) where V is a set and E is a set all of whose elements are of the form $\{x, y\}$ with $x, y \in V$. In the case when (V, E) is an undirected graph,

(1) the *vertices* or the *nodes* are the elements of V;

(2) the *edges* are the elements of E;

(3) an edge *between x and y* is the element $\{x, y\} \in E$;

(4) a *loop* is an edge between a vertex and itself.

Example 5.3.7. Following Example 5.3.5, define

$$\begin{aligned} V &= \{\mathrm{B}, \mathrm{P}, \mathrm{F}, \mathrm{M}, \mathrm{K}, \mathrm{N}\}, \quad \text{and} \\ E &= \{\{\mathrm{B}, \mathrm{P}\}, \{\mathrm{F}, \mathrm{M}\}, \{\mathrm{B}, \mathrm{B}\}, \{\mathrm{P}, \mathrm{P}\}, \{\mathrm{F}, \mathrm{F}\}, \{\mathrm{M}, \mathrm{M}\}, \{\mathrm{K}, \mathrm{K}\}, \{\mathrm{N}, \mathrm{N}\}\}. \end{aligned}$$

Then (V, E) is an undirected graph.

Drawings of an undirected graph. One can make a drawing representing an undirected graph (V, E) as follows:

(1) Draw all the elements of V.

(2) For all $x, y \in A$, draw a line between x and y if and only if $\{x, y\} \in E$.

Example 5.3.8. Here is a drawing of the undirected graph from Example 5.3.7.

$$\begin{array}{llll} \subset \mathrm{B} & - & \mathrm{P} \supset & \\ \cap & & & \\ \mathrm{F} & & & \mathrm{K} \supset \\ \quad\diagdown & & & \\ \quad\mathrm{M} & & \mathrm{N} & \\ \quad\cup & & \cup & \end{array}$$