

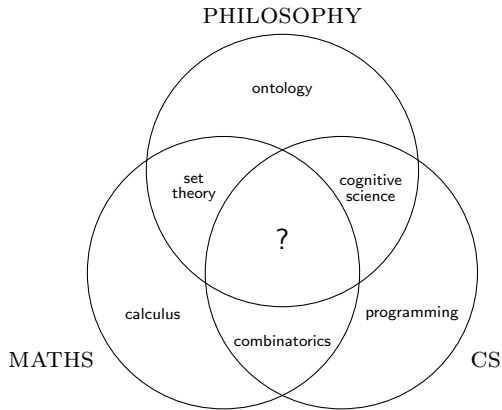
## Chapter 4: Sets

### CS1231 Discrete Structures

Wong Tin Lok

National University of Singapore

2021/21 Semester 1



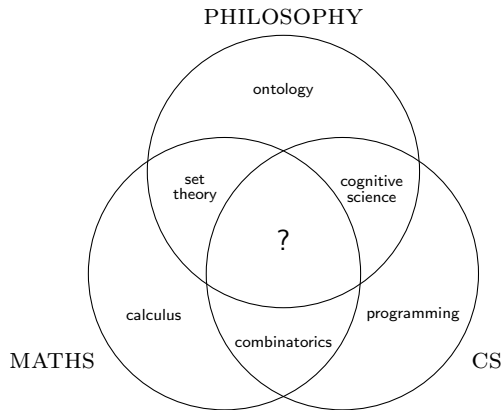
What can one put in the centre?

Answer at <https://pollev.com/wtl>.

- ▶ There is no need to log in to pollev.
- ▶ Use hyphens (-) for spaces in multi-word answers.

## About me

- ▶ WONG Tin Lok Lawrence
- ▶ Department of Mathematics,  
Faculty of Science (S17-05-18)
- ▶ [matwong@nus.edu.sg](mailto:matwong@nus.edu.sg)
- ▶ <https://blog.nus.edu.sg/matwong/>
- ▶ definitions → undefinables
- ▶ proofs → (true) unprovables
- ▶ necessary truth  
→ possible truth




What can one put in the centre?

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## Practicalities

- ▶ Lectures: Zoom (Mute yourself when you are not speaking.)
  - Wednesday 12:00 – ~~2:00pm~~ 1:35pm, with a 5-minute “break” in the middle
  - Friday 12:00 – ~~1:00pm~~ 12:45pm
- ▶ Slides and notes are posted on LumiNUS (<https://luminus.nus.edu.sg>) and in the CS1231 Telegram group.
- ▶ Try out the questions marked with  in the notes. Answers will be provided.
- ▶ There are a pre-lecture and a post-lecture version of slides. Pages that are different are marked by a red line near the top left-hand corner.
- ▶ If you have any questions/comments for me during lectures, then you can unmute yourself and speak, or you can type in the Zoom chat.
- ▶ Consultation: online
  - preferably immediately after the lectures (or by individual/group appointment)
  - CS1231 Telegram group
- ▶ Additional resources: search for “discrete mathematics” on the Internet or in the library (catalogue).
- ▶ Weeks 4.5–9.5: sets, relations, functions, cardinality — *mathematical maturity*

# Sets



## Why sets?

- ▶ The **language** of sets is an important part of modern mathematical discourse.
- ▶ Sets are **interesting** mathematical objects.
- ▶ For this module, they provide a topic on which we practise writing and understanding **proofs**.

Young man, in mathematics you don't understand things.  
You just get used to them. (reportedly) John von Neumann

## Definition 4.1.1

- (1) A **set** is an unordered collection of objects.
- (2) These objects are called the **members** or **elements** of the set.
- (3) Write
$$\begin{aligned}x &\in A && \text{for } x \text{ is an element of } A; \\x &\notin A && \text{for } x \text{ is not an element of } A; \\x, y &\in A && \text{for } x, y \text{ are elements of } A; \\x, y &\notin A && \text{for } x, y \text{ are not elements of } A; \end{aligned}$$

**Warning 4.1.2.** Some use “contains” for the subset relation, but we do **not**.

- (4) We may read  $x \in A$  also as “ $x$  is in  $A$ ” or “ $A$  **contains**  $x$  (as an element)”.

## Common sets (Table 4.1)

Note 4.1.3. Some define  $0 \notin \mathbb{N}$ , but we do *not*.

Symbol	Meaning	Examples	Non-examples
$\mathbb{N}$	the set of all <b>natural numbers</b>	$0, 1, 2, 3, 31 \in \mathbb{N}$	$-1, \frac{1}{2} \notin \mathbb{N}$
$\mathbb{Z}$	the set of all <b>integers</b>	$0, 1, -1, 2, -10 \in \mathbb{Z}$	$\frac{1}{2}, \sqrt{2} \notin \mathbb{Z}$
$\mathbb{Q}$	the set of all <b>rational numbers</b>	$-1, 10, \frac{1}{2}, -\frac{7}{5} \in \mathbb{Q}$	$\sqrt{2}, \pi, \sqrt{-1} \notin \mathbb{Q}$
$\mathbb{R}$	the set of all <b>real numbers</b>	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi \in \mathbb{R}$	$\sqrt{-1}, \sqrt{-10} \notin \mathbb{R}$
$\mathbb{C}$	the set of all <b>complex numbers</b>	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi, \sqrt{-1}, \sqrt{-10} \in \mathbb{C}$	
$\mathbb{Z}^+$	the set of all <b>positive</b> integers	$1, 2, 3, 31 \in \mathbb{Z}^+$	$0, -1, -12 \notin \mathbb{Z}^+$
$\mathbb{Z}^-$	the set of all <b>negative</b> integers	$-1, -2, -3, -31 \in \mathbb{Z}^-$	$0, 1, 12 \notin \mathbb{Z}^-$
$\mathbb{Z}_{\geq 0}$	the set of all <b>non-negative</b> integers	$0, 1, 2, 3, 31 \in \mathbb{Z}_{\geq 0}$	$-1, -12 \notin \mathbb{Z}_{\geq 0}$

$\mathbb{Q}^+, \mathbb{Q}^-, \mathbb{Q}_{\geq m}, \mathbb{R}^+, \mathbb{R}^-, \mathbb{R}_{\geq m}$ , etc. are defined similarly.

$\mathbb{Z}$  is for *Zahlen*.

$\mathbb{Q}$  is for quotients.

“Positive” means  $> 0$ .

“Negative” means  $< 0$ .

“Non-negative” means  $\geq 0$ .

## Specifying a set by listing out all its elements

### Definition 4.1.4 (roster notation)

- (1) The set whose only elements are  $x_1, x_2, \dots, x_n$  is denoted  $\{x_1, x_2, \dots, x_n\}$ .
- (2) The set whose only elements are  $x_1, x_2, x_3, \dots$  is denoted  $\{x_1, x_2, x_3, \dots\}$ .

### Example 4.1.5

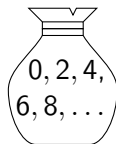
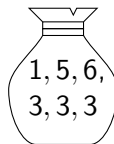
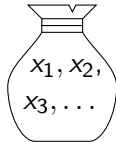
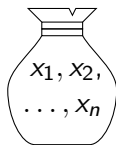
- (1) The only elements of  $A = \{1, 5, 6, 3, 3, 3\}$  are 1, 5, 6 and 3.  
So  $6 \in A$  but  $7 \notin A$ .
- (2) The only elements of  $B = \{0, 2, 4, 6, 8, \dots\}$  are the non-negative even integers.  
So  $4 \in B$  but  $5 \notin B$ .

To check whether an object  $z$  is an element of a set  $S = \{x_1, x_2, \dots, x_n\}$

If  $z$  is in the list  $x_1, x_2, \dots, x_n$ , then  $z \in S$ , else  $z \notin S$ .

### Question

What are the elements of  $\{2, 3, \dots\}$ ? All integers  $x \geq 2$ ?



## Specifying a set by describing its elements

### Definition 4.1.6 (set-builder notation)

Let  $U$  be a set and  $P(x)$  be a predicate over  $U$ . Then the set of all elements  $x \in U$  such that  $P(x)$  is true is denoted

$$\{x \in U : P(x)\} \quad \text{or} \quad \{x \in U \mid P(x)\}.$$

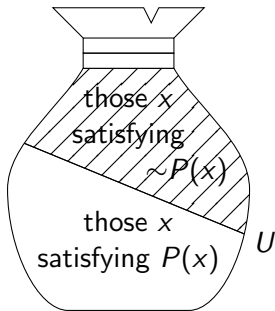
This is read as “the set of all  $x$  in  $U$  such that  $P(x)$ ”.

### Example 4.1.7

- (1) The elements of  $C = \{x \in \mathbb{Z}_{\geq 0} : x \text{ is even}\}$  are precisely the elements of  $\mathbb{Z}_{\geq 0}$  that are even, i.e., the non-negative even integers. So  $6 \in C$  but  $7 \notin C$ .
- (2) The elements of  $D = \{x \in \mathbb{Z} : x \text{ is a prime number}\}$  are precisely the elements of  $\mathbb{Z}$  that are prime numbers, i.e., the prime integers. So  $7 \in D$  but  $9 \notin D$ .

To check whether an object  $z$  is an element of  $S = \{x \in U : P(x)\}$

If  $z \in U$  and  $P(z)$  is true, then  $z \in S$ , else  $z \notin S$ . Hence  $z \notin U$  implies  $z \notin S$ , and  $P(z)$  is false implies  $z \notin S$ .



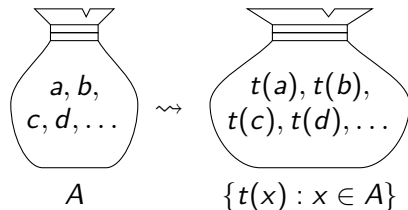
## Specifying a set by replacement

### Definition 4.1.8 (replacement notation)



Let  $A$  be a set and  $t(x)$  be a term in a variable  $x$ . Then the set of all objects of the form  $t(x)$  where  $x$  ranges over the elements of  $A$  is denoted

$$\{t(x) : x \in A\} \quad \text{or} \quad \{t(x) \mid x \in A\}.$$

This is read as “the set of all  $t(x)$  where  $x \in A$ ”.



### Example 4.1.9

- (1) The elements of  $E = \{x + 1 : x \in \mathbb{Z}_{\geq 0}\}$  are precisely those  $x + 1$  where  $x \in \mathbb{Z}_{\geq 0}$ , i.e., the positive integers. So  $1 = 0 + 1 \in E$  but  $0 \notin E$ . 
- (2) The elements of  $F = \{x - y : x, y \in \mathbb{Z}_{\geq 0}\}$  are precisely those  $x - y$  where  $x, y \in \mathbb{Z}_{\geq 0}$ , i.e., the integers. So  $-1 = 1 - 2 \in F$  but  $\sqrt{2} \notin F$ . 

To check whether an object  $z$  is an element of  $S = \{t(x) : x \in A\}$

If there is  $x \in A$  such that  $t(x) = z$ , then  $z \in S$ , else  $z \notin S$ .



# Equality of sets

## Definition 4.1.10

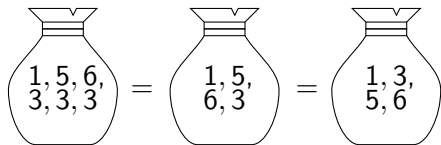
Two sets are *equal* if they have the same elements, i.e., for all sets  $A, B$ ,

$$A = B \iff \forall z (z \in A \iff z \in B).$$

Convention 4.1.11. This is the *only* situation in mathematics when “if” should be understood as a (special) “if and only if”.

## Example 4.1.12

$$\{1, 5, 6, 3, 3, 3\} = \{1, 5, 6, 3\} = \{1, 3, 5, 6\}.$$



Slogan 4.1.13. Order and repetition do not matter.

## Example 4.1.14

$$\begin{aligned} \{y^2 : y \text{ is an odd integer}\} &= \{x \in \mathbb{Z} : x = y^2 \text{ for some odd integer } y\} \\ &= \{1^2, 3^2, 5^2, \dots\}. \end{aligned}$$

## Equality of sets

### Definition 4.1.10

Two sets are *equal* if they have the same elements, i.e., for all sets  $A, B$ ,

$$A = B \iff \forall z (z \in A \iff z \in B).$$

Convention 4.1.11. This is the *only* situation in mathematics when “if” should be understood as a (special) “if and only if”.

### Example 4.1.15

$$\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}.$$

Slogan 4.1.13. Order and repetition do not matter.

### Proof

( $\Rightarrow$ ) Take any  $z \in \{x \in \mathbb{Z} : x^2 = 1\}$ . Then  $z \in \mathbb{Z}$  and  $z^2 = 1$ . So

$$\begin{aligned} z^2 - 1 &= (z - 1)(z + 1) = 0. \\ \therefore z - 1 = 0 \quad \text{or} \quad z + 1 = 0. \\ \therefore z = 1 \quad \text{or} \quad z = -1. \end{aligned}$$

This means  $z \in \{1, -1\}$ .

( $\Leftarrow$ ) Take any  $z \in \{1, -1\}$ . Then  $z = 1$  or  $z = -1$ . In either case, we have  $z \in \mathbb{Z}$  and  $z^2 = 1$ . So  $z \in \{x \in \mathbb{Z} : x^2 = 1\}$ .  $\square$

# The empty set

Definition 4.1.10. For all sets  $A, B$ ,  
 $A = B \Leftrightarrow \forall z (z \in A \Leftrightarrow z \in B).$

## Theorem 4.1.17

There exists a unique set with no element, i.e.,

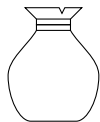
- ▶ there is a set with no element; and (existence part)
- ▶ for all sets  $A, B$ , if both  $A$  and  $B$  have no element, then  $A = B$ . (uniqueness part)

## Proof

- ▶ (existence part) The set  $\{\}$  has no element.
- ▶ (uniqueness part) Let  $A, B$  be sets with no element. Then vacuously,

$$\forall z (z \in A \Rightarrow z \in B) \quad \text{and} \quad \forall z (z \in B \Rightarrow z \in A)$$

because the hypotheses in the implications are never true. So  $A = B$ .



## Definition 4.1.18

The set with no element is called the *empty set*. It is denoted by  $\emptyset$ .

## Checkpoint

### What we saw

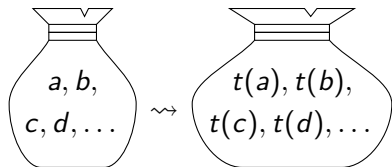
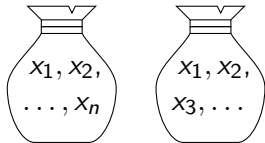
- ▶ membership  $\in$
- ▶ ways to specify sets
  - the set of all ...
  - roster notation
  - set-builder notation
  - replacement notation
  - the unique set that satisfies a property
- ▶ set equality

### Question (to be revisited later)

Are there other ways to specify sets?

### Next

inclusion



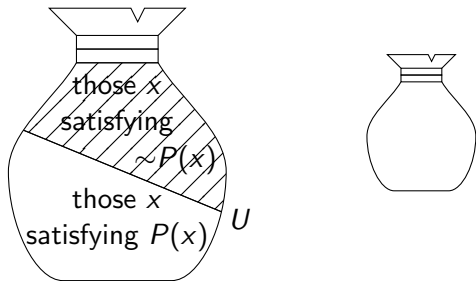
the set of all integers

$\{x_1, x_2, \dots, x_n\}, \{x_1, x_2, x_3, \dots\}$

$\{x \in U : P(x)\}$

$\{t(x) : x \in A\}$

the unique set with no element



## Inclusion of sets

### Definition 4.2.1

Call  $A$  a **subset** of  $B$ , and write  $A \subseteq B$ , if

$$\forall z (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that  $B$  **includes**  $A$ , and write  $B \supseteq A$  in this case.

### Example 4.2.3 and Example 4.2.6

- (1)  $\{1, 5, 2\} \subsetneq \{5, 2, 1, 4\}$  but  $\{1, 5, 2\} \not\subseteq \{2, 1, 4\}$ .  
(2)  $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ . All these inclusions are proper.

### Remark 4.2.4

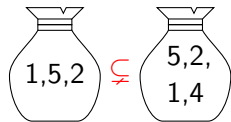
- (1)  $A \not\subseteq B \Leftrightarrow \exists z (z \in A \text{ and } z \notin B)$ .  
(2)  $A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$ .  
(3)  $A \subseteq A$ .

### Definition 4.2.5

Call  $A$  a **proper subset** of  $B$ , write  $A \subsetneq B$ , if  $A \subseteq B$  and  $A \neq B$ . In this case, we may say that the inclusion of  $A$  in  $B$  is **proper** or **strict**.

Let  $A, B$  be sets.

Definition 4.1.10. For all sets  $A, B$ ,  
 $A = B \Leftrightarrow \forall z (z \in A \Leftrightarrow z \in B)$ .



Note 4.2.2. We avoid using the symbol  $\subset$  because it may have different meanings to different people.

## Vacuous inclusion

Let  $A, B$  be sets.

Definition 4.1.10. For all sets  $A, B$ ,  
 $A = B \iff \forall z (z \in A \iff z \in B)$ .

### Definition 4.2.1

Call  $A$  a *subset* of  $B$ , and write  $A \subseteq B$ , if

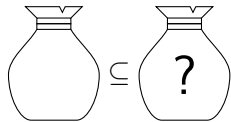
$$\forall z (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that  $B$  *includes*  $A$ , and write  $B \supseteq A$  in this case.

### Proposition 4.2.7

The empty set is a subset of any set, i.e., for any set  $A$ ,

$$\emptyset \subseteq A.$$



### Proof

Vacuously,

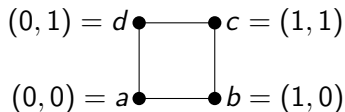
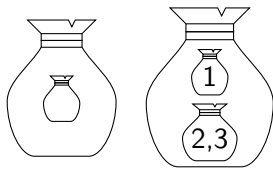
$$\forall z (z \in \emptyset \Rightarrow z \in A)$$

because the hypothesis in the implication is never true. So  $\emptyset \subseteq A$  by the definition of  $\subseteq$ . □

## Sets of sets

### Note 4.2.8

Sets can be elements of sets.



### Example 4.2.9

- (1) The set  $A = \{\emptyset\}$  has exactly 1 element, namely the empty set. So  $A$  is not empty.
- (2) The set  $B = \{\{1\}, \{2, 3\}\}$  has exactly 2 elements, namely  $\{1\}$ ,  $\{2, 3\}$ . So  $\{1\} \in B$ , but  $1 \notin B$ .

How can one use a set to represent the square above?

- If one only wants to represent the connectivity between the points, then use

$$\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}.$$

- If one also wants to represent the positions of the lines, then use

$$\{(x, y) : (x = 0 \text{ and } y \in [0, 1]) \text{ or } (x = 1 \text{ and } y \in [0, 1]) \\ \text{or } (y = 0 \text{ and } x \in [0, 1]) \text{ or } (y = 1 \text{ and } x \in [0, 1])\}.$$

## Power set

### Definition 4.2.12

Let  $A$  be a set. The set of all subsets of  $A$ , denoted  $\mathcal{P}(A)$ , is called the *power set* of  $A$ .

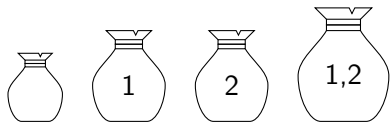
### Example 4.2.13

(1)  $\mathcal{P}(\emptyset) = \{\emptyset\}$

(2)  $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}.$

(3)  $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$

(4) The following are subsets of  $\mathbb{Z}_{\geq 0}$  and thus are elements of  $\mathcal{P}(\mathbb{Z}_{\geq 0})$ .



$\emptyset, \{0\}, \{1\}, \{2\}, \dots \{0, 1\}, \{0, 2\}, \{0, 3\} \dots \{1, 2\}, \{1, 3\}, \{1, 4\} \dots$

$\{2, 3\}, \{2, 4\}, \{2, 5\} \dots \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \dots$

$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \dots \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \dots \dots$

$\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 1}, \mathbb{Z}_{\geq 2}, \dots \{0, 2, 4, \dots\}, \{1, 3, 5, \dots\}, \{2, 4, 6, \dots\}, \{3, 5, 7, \dots\}, \dots$

$\{x \in \mathbb{Z}_{\geq 0} : (x-1)(x-2) < 0\}, \{x \in \mathbb{Z}_{\geq 0} : (x-2)(x-3) < 0\}, \dots$

$\{3x+2 : x \in \mathbb{Z}_{\geq 0}\}, \{4x+3 : x \in \mathbb{Z}_{\geq 0}\}, \{5x+4 : x \in \mathbb{Z}_{\geq 0}\}, \dots \dots$



## Checkpoint

### Note 4.2.10

Membership and inclusion can be different.

### Question 4.2.11

Let  $C = \{\{1\}, 2, \{3\}, 3, \{\{4\}\}\}$ . Which of the following are true?

- |                   |                         |
|-------------------|-------------------------|
| ▶ $\{1\} \in C$ . | ▶ $\{1\} \subseteq C$ . |
| ▶ $\{2\} \in C$ . | ▶ $\{2\} \subseteq C$ . |
| ▶ $\{3\} \in C$ . | ▶ $\{3\} \subseteq C$ . |
| ▶ $\{4\} \in C$ . | ▶ $\{4\} \subseteq C$ . |

### Next

Boolean  
operations

The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; [... ] and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind. Boole 1854

# Boolean operations

Let  $A, B$  be sets.

## Definition 4.3.1

(1) The **union** of  $A$  and  $B$ , denoted  $A \cup B$ , is defined by

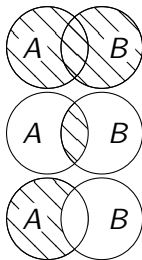
read as 'A union B'  $\longrightarrow A \cup B = \{x : x \in A \text{ or } x \in B\}$ .

(2) The **intersection** of  $A$  and  $B$ , denoted  $A \cap B$ , is defined by

read as 'A intersect B'  $\longrightarrow A \cap B = \{x : x \in A \text{ and } x \in B\}$ .

(3) The **complement** of  $B$  in  $A$ , denoted  $A - B$  or  $A \setminus B$ , is defined by

read as 'A minus B'  $\longrightarrow A \setminus B = \{x : x \in A \text{ and } x \notin B\}$ .

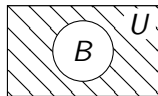


## Convention and terminology 4.3.2

When working in a particular context, one usually works within a fixed set  $U$  which includes all the sets one may talk about, so that one only needs to consider the elements of  $U$  when proving set equality and inclusion. This  $U$  is called a **universal set**.

## Definition 4.3.3

In a context where  $U$  is the universal set (so that implicitly  $U \supseteq B$ ), the **complement** of  $B$ , denoted  $\bar{B}$  or  $B^c$ , is defined by  $\bar{B} = U \setminus B$ .



## Example 4.3.4 on Boolean operations

For all sets  $A, B$ ,

$$A \cup B = \{x : (x \in A) \vee (x \in B)\},$$

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\},$$

$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\},$$

$$\overline{B} = \{x \in U : x \notin B\}, \quad \text{in a context where } U \text{ is the universal set.}$$

Let  $A = \{x \in \mathbb{Z} : x \leq 10\}$  and  $B = \{x \in \mathbb{Z} : 5 \leq x \leq 15\}$ . Then

$$A \cup B = \{x \in \mathbb{Z} : (x \leq 10) \vee (5 \leq x \leq 15)\} = \{x \in \mathbb{Z} : x \leq 15\};$$

$$A \cap B = \{x \in \mathbb{Z} : (x \leq 10) \wedge (5 \leq x \leq 15)\} = \{x \in \mathbb{Z} : 5 \leq x \leq 10\};$$

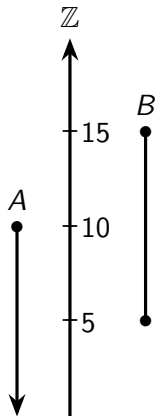
$$A \setminus B = \{x \in \mathbb{Z} : (x \leq 10) \wedge \sim(5 \leq x \leq 15)\} = \{x \in \mathbb{Z} : x < 5\};$$

$$\overline{B} = \{x \in \mathbb{Z} : \sim(5 \leq x \leq 15)\} = \{x \in \mathbb{Z} : (x < 5) \vee (x > 15)\},$$

in a context where  $\mathbb{Z}$  is the universal set. To show the first equation, one shows

$$\forall x \in \mathbb{Z} \ ((x \leq 10) \vee (5 \leq x \leq 15) \Leftrightarrow (x \leq 15)),$$

etc.



## Set identities (Theorem 4.3.5)

For all set  $A, B, C$  in a context where  $U$  is the universal set, the following hold.

Commutativity

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associativity

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Idempotence

$$A \cup A = A$$

$$A \cap A = A$$

Absorption

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

De Morgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Identities

$$A \cup \emptyset = A$$

$$A \cap U = A$$

Annihilators

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

Complement

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \emptyset$$

Double Complement Law

$$\overline{(\overline{A})} = A$$

Top and bottom

$$\overline{\emptyset} = U$$

$$\overline{U} = \emptyset$$





Set difference

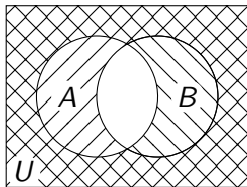
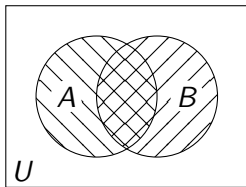
$$A \setminus B = A \cap \overline{B}$$



## Venn diagrams

One of De Morgan's Laws. Work in the universal set  $U$ . For all sets  $A, B$ ,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

In the left diagram, hatch the regions representing  $A$  and  $B$  with  and  respectively. In the right diagram, hatch the regions representing  $\overline{A}$  and  $\overline{B}$  with  and  respectively.



Then the  region represents  $\overline{A \cup B}$  on the left diagram, and the  region represents  $\overline{A} \cap \overline{B}$  on the right diagram. Since these regions occupy the same region in the respective diagrams, we infer that  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ .

**Note 4.3.6.** This argument depends on the fact that each possibility for membership in  $A$  and  $B$  is represented by a region in the diagram.

## Proving set identities using truth tables

One of De Morgan's Laws. Work in the universal set  $U$ . For all sets  $A, B$ ,  
$$\overline{A \cup B} = \bar{A} \cap \bar{B}.$$

### Proof #1

The rows in the following table list all the possibilities for an element  $x \in U$ :

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in \overline{A \cup B}$	$x \in \bar{A}$	$x \in \bar{B}$	$x \in \bar{A} \cap \bar{B}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns under " $x \in \overline{A \cup B}$ " and " $x \in \bar{A} \cap \bar{B}$ " are the same, for any  $x \in U$ ,

$$x \in \overline{A \cup B} \Leftrightarrow x \in \bar{A} \cap \bar{B}$$

no matter in which case we are. So  $\overline{A \cup B} = \bar{A} \cap \bar{B}$ .



## Proving set identities directly

One of De Morgan's Laws. Work in the universal set  $U$ . For all sets  $A, B$ ,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

### Proof #2

Let  $z \in U$ . Then

	$z \in \overline{A \cup B}$	
$\Leftrightarrow$	$z \notin A \cup B$	by the definition of $\overline{\cdot}$ ;
$\Leftrightarrow$	$\sim((z \in A) \vee (z \in B))$	by the definition of $\cup$ ;
$\Leftrightarrow$	$(z \notin A) \wedge (z \notin B)$	by De Morgan's Laws for propositions;
$\Leftrightarrow$	$(z \in \overline{A}) \wedge (z \in \overline{B})$	by the definition of $\overline{\cdot}$ ;
$\Leftrightarrow$	$z \in \overline{A} \cap \overline{B}$	by the definition of $\cap$ .



## Applications of the set identities

Fix a universal set  $U$ . The following are true for all sets  $A, B, C$ .

Identity  $A \cap U = A$ .

Distributivity  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

Complement  $A \cup \bar{A} = U$ .

Set difference  $A \setminus B = A \cap \bar{B}$ .

### Example 4.3.7

Under the universal set  $U$ , show that  $(A \cap B) \cup (A \setminus B) = A$  for all sets  $A, B$ .

#### Proof

$$\begin{aligned}(A \cap B) \cup (A \setminus B) &= (A \cap B) \cup (A \cap \bar{B}) \\ &= A \cap (B \cup \bar{B}) \\ &= A \cap U \\ &= A\end{aligned}$$

by the properties of set difference;

by distributivity;

by the properties of set complement;

as  $U$  is the identity for  $\cap$ .





## Boolean operations and inclusion

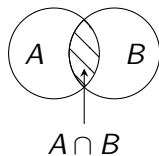
### Example 4.3.8

Show that  $A \cap B \subseteq A$  for all sets  $A, B$ .

Let  $A, B$  be sets.

Definition 4.2.1.  $A \subseteq B \Leftrightarrow \forall z (z \in A \Rightarrow z \in B)$ .

Definition 4.3.1(2).  $A \cap B = \{x : x \in A \text{ and } x \in B\}$ .



### Proof

By the definition of  $\subseteq$ , we need to show that

$$\forall z (z \in A \cap B \Rightarrow z \in A).$$

Let  $z \in A \cap B$ . Then  $z \in A$  and  $z \in B$  by the definition of  $\cap$ . In particular, we know  $z \in A$ , as required. □

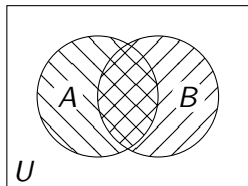
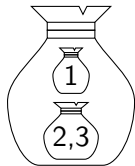
Example 4.3.9: Is the following true?

$$(A \setminus B) \cup (B \setminus C) = A \setminus C \quad \text{for all sets } A, B, C.$$

# Checkpoint

## What we saw

- ▶ membership, inclusion, and equality of sets
- ▶ power sets
- ▶ union, intersections, complements
- ▶ set identities and their proofs
- ▶ Venn diagrams



## Questions

- ▶ Is there a complete list of set operations?
- ▶ Are sets simply predicates in disguise?
- ▶ Why do we work with a universal set?

## Next

Russell's Paradox

Hardly anything more unwelcome can befall a scientific writer than that one of the foundations of his edifice be shaken after the work is finished. Frege (1902)

# A set that is an element of itself?

## Note 4.2.8 (recall)

Sets can be elements of sets.

## Example 4.4.1

- (1)  $\emptyset \notin \emptyset$ .
- (2)  $\mathbb{Z} \notin \mathbb{Z}$ .
- (3)  $\{\emptyset\} \notin \{\emptyset\}$ .

## Question 4.4.2

Is there a set  $x$  such that  $x \in x$ ?



Hogarth (1754)

# Consternation

There is just one point where I have encountered a difficulty. Russell (1902)

## Theorem 4.4.3 (Russell 1901)

There is *no* set  $R$  such that

$\{x : x \notin x\}$ ?

$$\forall x (x \in R \Leftrightarrow x \notin x). \quad (*)$$

## Proof (by contradiction)

Suppose  $R$  is a set satisfying  $(*)$ . Applying  $(*)$  to  $x = R$  gives

$$R \in R \Leftrightarrow R \notin R. \quad (\dagger)$$

Split into two cases.

- **Case 1:** assume  $R \in R$ . Then  $R \notin R$  by the  $\Rightarrow$  part of  $(\dagger)$ . This contradicts our assumption that  $R \in R$ .
- **Case 2:** assume  $R \notin R$ . Then  $R \in R$  by the  $\Leftarrow$  part of  $(\dagger)$ . This contradicts our assumption that  $R \notin R$ .

In either case, we get a contradiction. So the proof is finished.  $\square$

**Question 4.4.4.** Can you write a proof that does not mention contradiction?



# Consternation?

There is just one point where I have encountered a difficulty. Russell (1902)

## Theorem 4.4.3 (Russell 1901)

There is *no* set  $R$  such that

$$\forall x (x \in R \iff x \notin x).$$

$\{x : x \notin x\}$ ?

## Morals

- ▶ Some predicates do not correspond to any set.
- ▶ The set of all sets, if it exists, needs to be handled with *extreme* care.

Suppose a contradiction were to be found in the axioms of set theory. Do you seriously believe that that bridge would fall down? (reportedly) Ramsey

## Next

how sets can represent mathematical objects

