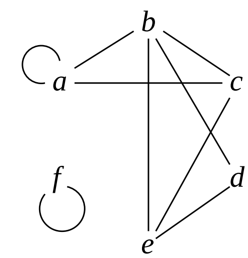
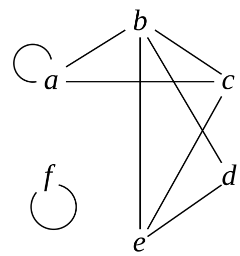
Definition
Consider an undirected graph $G = (V, E)$.
G is **trivial** if and only if $|V| = 1$.
G is **finite** if V is finite; G is **infinite** if V is infinite.

Definition
Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be undirected graphs.
H is a **subgraph** of G (or G **contains** H) if and only if
$V_H \subseteq V_G$ and $E_H \subseteq E_G$.
H is a **proper subgraph** of G if and only if
H is a subgraph of G and $H \neq G$.



Definition
Let $G = (V, E)$ be an undirected graph and $p \geq 2$.
A subgraph of the form $(\{x_1, \ldots, x_p\}, \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{p-1}, x_p\}\})$ is called a **path** between x_1 and x_p in G;
this path has **length** $p - 1$.

Example
$(\{a, b, d, e\}, \{\{a, b\}, \{b, d\}, \{d, e\}\})$



$\{a, b\}, \{b, d\}, \{d, e\}, \{e, b\}, \{b, c\}$

Definition
An undirected graph is **connected** if and only if
it is trivial or there is a path between any two distinct nodes.

Definition

Let R be a binary relation on a nonempty set A. For $n \in \mathbb{Z}^+$, define

$$R_n = \begin{cases} R & \text{if } n = 1 \\ R \circ R_{n-1} & \text{if } n \geq 2 \end{cases}$$

Theorem 4.1

Let $G = (V, E)$ be an undirected graph with $|V| \geq 2$, and
$R = \{(b, c) \in V \times V \mid b \neq c \text{ and } \{b, c\} \in E\}$.
Consider two different nodes x and y in G, and $n \in \mathbb{Z}^+$.
(i) If there is a path of length n between x and y, then $(x, y) \in R_n$.
(ii) If $(x, y) \in R_n$, then there is a path of length at most n between x and y.

Definition
Let R be a binary relation on a set A.
The **transitive closure** of R is $R_+ = \bigcup_{n=1}^{\infty} R_n$.

Corollary 4.2
Let $G = (V, E)$ be an undirected graph and
$R = \{(b, c) \in V \times V \mid b \neq c \text{ and } \{b, c\} \in E\}$.
Then G is connected if and only if
$(x, y) \in R_+$ for any $x, y \in V$ such that $x \neq y$.

Definition

Let $n \in \mathbb{Z}$, $n \geq 3$. An undirected graph of the form
$(\{x_1, x_2, \ldots, x_n\}, \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\})$
is called a **cycle**.

Example

$(\{a, b, d, e, c\}, \{\{a, b\}, \{b, d\}, \{d, e\}, \{e, c\}, \{c, a\}\})$

$$\{a, b\}, \{b, d\}, \{d, e\}, \{e, b\}, \{b, c\}, \{c, a\}$$

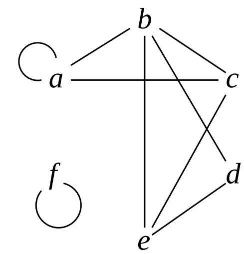
Definition

An undirected graph is **cyclic** if it contains a loop or a cycle; otherwise, G is **acyclic**.

Theorem 4.3
Let G be a connected undirected graph with no loops.
G is cyclic if and only if
there are two distinct nodes with more than one path between them.

Definition

Let G be an undirected graph and H a connected subgraph of G.
If G does not contain another connected subgraph H'
such that H is a proper subgraph of H',
then H is called a **connected component** of G.



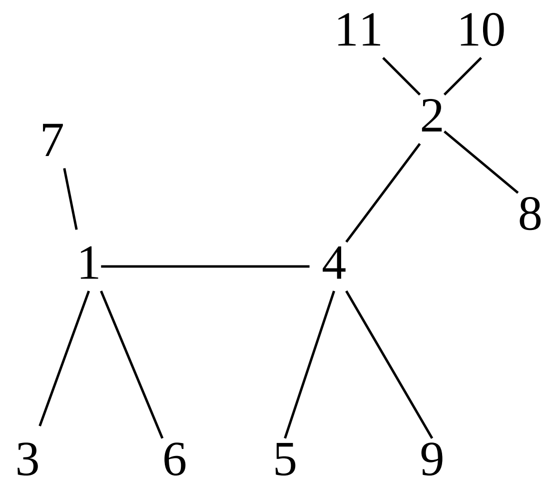
Theorem 4.4

Let x and y be distinct nodes in an undirected graph G.
Then there is a path in G between x and y if and only if
x and y are in the same connected component of G.

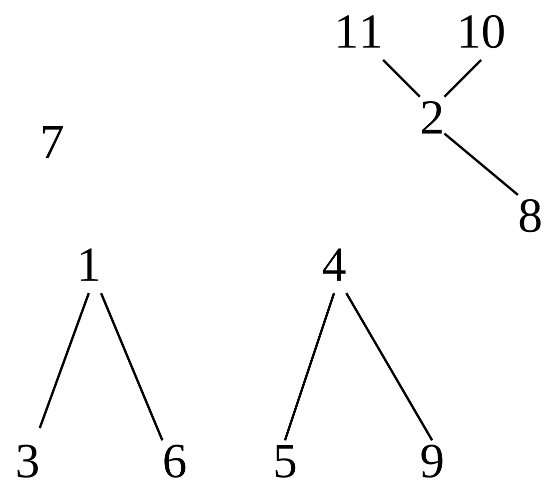
Corollary 4.5

Let A be a nonempty set and R an equivalence relation on A.
Let G be the undirected graph representing R,
and suppose x and y are different nodes in G.
Then x and y are in the same equivalence class under R
if and only if x and y are in the same connected component in G.

Definition
A connected acyclic undirected graph is called a **tree**.

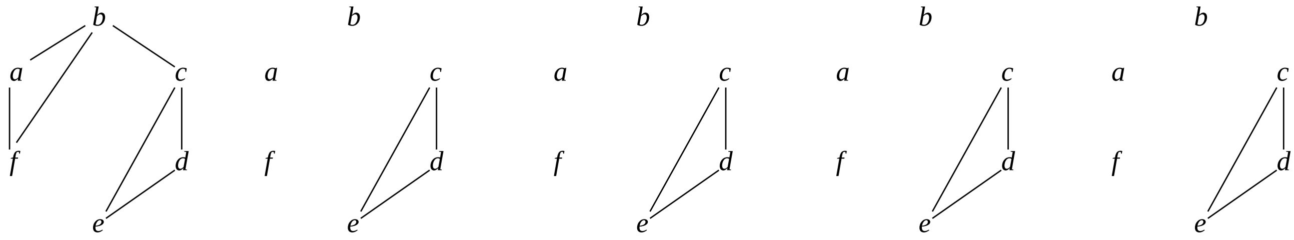


Definition
An acyclic graph is called a **forest**.

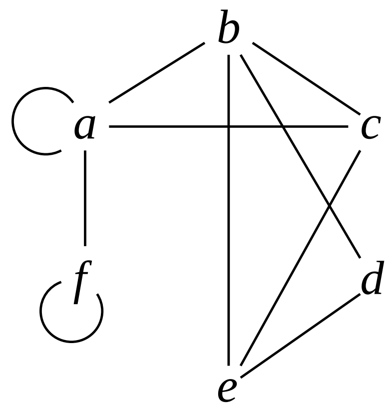


Theorem 4.6 (Tree Theorem)
Let $G = (V, E)$ be a finite connected undirected graph.
Then the following are equivalent:

(1) G is a tree.
(2) removing any edge disconnects G.
(3) $|E| = |V| - 1$.

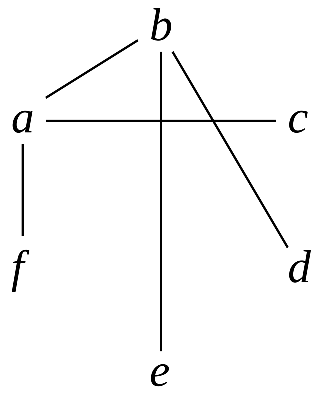

a b c d e f

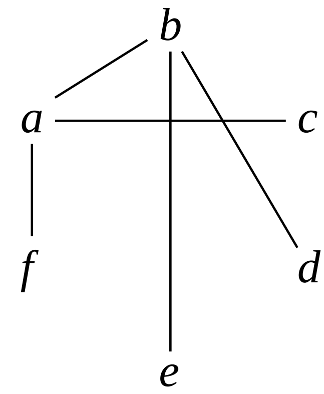
Definition
Let G be an undirected graph.
Any subgraph of G that is a tree and contains all nodes of G
is called a **spanning tree**.



Theorem 4.7
Every finite connected undirected graph has a spanning tree.

Definition
A **rooted** tree is a tree with a distinguished node called the **root**





Definition
Let r be the root of a rooted tree T.
The **level** of r is 0,
and the **level** of any node $x \neq r$ is the number of edges
in the (unique) path from r to x.

Let level(x) denote the level of x.
The **height** of T is the maximum level of any node in T.

Consider any x.
Any node y, $y \neq x$, on the path from r to x (including $y = r$)
is called an **ancestor** of x.
If y is an ancestor of x, then x is a **descendant** of y.

If x is a descendant of p and level(x)=level(p)+1,
p is called the **parent** of x
and x is called a **child** of p.

A node that has a child is an **internal node**;
a node with no children is called a **leaf**.

Definition
A **binary tree** is a rooted tree in which
every node has at most two children.

Theorem 4.8

For any binary tree with m leaves and height h,

$$m \leq 2^h.$$

Theorem 4.9
Consider a binary tree T
in which every parent has exactly two children.
If T has m leaves, then it has $m - 1$ parents.