National University of Singapore
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2021/22 (Sem.1) | **CS1231 Discrete Structures** | **Tutorial 7**

1. Recall from Proposition 8.1.1 that both injections and surjections are closed under composition. In the question, we investigate the converses. Let $f\colon X \to Y$ and $g\colon Y \to Z$.

   (a)* Suppose $g \circ f$ is injective. Prove that f must also be injective.
   Give an example to show that g need not be injective.

   (b) Suppose $g \circ f$ is surjective. Prove that g must also be surjective.
   Give an example to show that f need not be surjective.

   (c) Give an example where $g \circ f$ is bijective but g and f are both not bijective.

2.* Consider a function $f\colon X \to Y$. Recall from Proposition 7.4.14 that if $f^{-1}\colon Y \to X$, then $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. Here are investigate the converse.

   Suppose $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Prove that f must be bijective, and that $g = f^{-1}$.

3. Let X be a nonempty set and $f\colon X \to X$ such that $f \circ f = \mathrm{id}_X$.

   (a) Prove that f is bijective.

   (b) Determine f^{-1}.

   (Hint: You may find Problem 2 helpful.)

4.* Let $k \in \mathbb{Z}^+$. Consider the equivalence relation $\equiv_k$ on $\mathbb{Z}$ from Tutorial 5 Problem 5, where $m \equiv_k n$ if and only if k divides $m - n$ for all $m, n \in \mathbb{Z}$.
   Prove that the equivalence classes have the same cardinality.

5. Consider the equivalence relation

$$\mathcal{L} = \{((a,b),(c,d)) \in (\mathbb{R}^2)^2 : a - c = 3(b - d)\}$$

   on $\mathbb{R}^2$ from Tutorial 5 Problem 7. Prove that all the equivalence classes have the same cardinality.

6. Let U be a finite set. Consider the relation

$$R = \{(A,B) \in \mathcal{P}(U) \times \mathcal{P}(U) : \text{there is a bijection from } A \text{ to } B\}$$

   on $\mathcal{P}(U)$. What are the equivalence classes?

7.* For $a, b \in \mathbb{R}$, define $I_{(a,b)} = \{x \in \mathbb{R} : a < x < b\}$. Give an example of a bijection from $I_{(0,1)}$ to $I_{(-1,1)}$ and another from $I_{(-1,1)}$ to $\mathbb{R}$. Deduce that $I_{(0,1)}$ and $\mathbb{R}$ have the same cardinality.

8. (a) Pick any 7 integers yourself. Find two integers m and n amongst the 7 you chose such that 6 divides $m - n$.

   (b) (We now generalize the observation in (a).) Let $k \in \mathbb{Z}^+$. Use the Pigeonhole Principle and the fact that the partition of $\mathbb{Z}$ induced by the equivalence relation $\equiv_k$ from Problem 4 has exactly k components to deduce that, amongst any $k + 1$ integers, there are two whose difference is divisible by k.

9.* Prove that a set X is countable if and only if it is empty, or there is a surjection $\mathbb{Z}^+ \to X$.
   (Hint: You may find ideas from Tutorial 6 Problem 9 useful.)