National University of Singapore
Department of Computer Science

2021/22 (Sem.1) **CS1231 Discrete Structures** **Tutorial 10**

1. Suppose A is a set with n elements.

   (i) How many binary relations are there on A?

   (ii)* How many of these relations are reflexive?

   (iii) How many of these relations are symmetric?

   (iv) How many of these relations are antisymmetric?

   (v) How many of these relations are antisymmetric and symmetric?

   (vi)* How many of these relations are not reflexive and not symmetric?

   [Hint: Consider the graph representation of the relation.
   You should use small values of n to check the validity of your formulas.]

Solution:
$|A| = n$. In the following, consider the directed/undirected graph representing the relation.

(i) binary relations:
2^{n^2} from Tutorial 9, Problem 9(i).

(ii)* reflexive relations:
Consider all n loops and $\binom{n}{2}$ node pairs in the graph representation.
For reflexive, all loops must be included.
For each node pair, there are 4 choices ($\rightarrow$, $\leftarrow$, $\rightleftarrows$, no edge). Total: $1^n 4^{\binom{n}{2}}$.

(iii) symmetric relations:
In the graph representation, there is a maximum of n loops and $\binom{n}{2}$ node pairs.
From these, pick any subset of node pairs $\{x, y\}$ and add edges $x \rightarrow y$ and $x \leftarrow y$.
Total: $2^{n+\binom{n}{2}} = 2^{n(n+1)/2}$.

(iv) antisymmetric relations:
Like in (c), pick any subset of node pairs $\{x, y\}$, and add either $x \rightarrow y$ or $x \leftarrow y$, but not both.
Total: $2^n 3^{\binom{n}{2}}$, where 2^n is for the loops, and 3 is for no edge, $x \rightarrow y$ or $x \leftarrow y$.

(v) antisymmetric and symmetric:
The only possibility is the graph has no $x \rightarrow y$ edges, only loops.
Each node may or may not have a loop, so the total is $2^n 1^{\binom{n}{2}}$.

(vi)* not reflexive and not symmetric:
$\sim p \wedge \sim q \equiv \sim (p \vee q)$, so the total is
(number of relations $-$ number of reflexive or symmetric relations).
Using Inclusion/Exclusion, number of reflexive or symmetric relations is
$4^{\binom{n}{2}} + 2^{n(n+1)/2} - 2^{\binom{n}{2}}$,
so #(not reflexive and not symmetric) $= 2^{n^2} - (2^{n^2-n} + 2^{n(n+1)/2} - 2^{n(n-1)/2})$.
Alternative:
#(not reflexive and not symmetric) = #(not reflexive) #(not symmetric)
$= (2^n - 1)(4^{\binom{n}{2}} - 2^{\binom{n}{2}}) = (2^n - 1)(2^{n(n-1)} - 2^{n(n-1)/2}) = 2^{n^2} - 2^{n(n-1)} - 2^{n(n+1)/2} + 2^{n(n-1)/2}$.

2. Let $\mathcal{G}_3$ be the set of all undirected graphs whose vertices are a, b, c. Suppose $G = (\{a, b, c\}, E) \in \mathcal{G}_3$. Determine the number of possible G's such that:

(i)* G has a loop; (ii)* G has a cycle;
(iii) G is cyclic; (iv)* G is connected;
(v) G is a tree; (vi) G has exactly two connected components.

Solution:
Consider (labelled) undirected graphs with 3 nodes.
Each node may or may not have a loop (2^3 possibilities),
and each node pair may or may not have an edge ($2^{\binom{3}{2}}$ possibilities).
Total $= 2^3(2^3) = 2^6 = 64$.

(i)* #graphs with no loops $= 1^3 2^{\binom{3}{2}} = 8$
$\Rightarrow$ #graphs with loops $= 64 - 8 = 56$.

(ii)* #graphs with a cycle (choice only in loops) $= 2^3 = 8$.

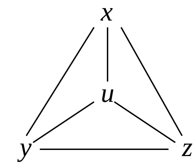
(iii) #($\sim$loop$\wedge$ $\sim$cycle)$= 2^{\binom{3}{2}} - 1 = 7$.
#cyclic $=$ #(loop$\vee$cycle) $= 64 - 7 = 57$.

(iv)* #connected$\Leftrightarrow \wedge, <, >$ or $\triangle$ and any choice of loops: $4(2^3) = 32$ possibilites

(v) tree $\Leftrightarrow \wedge, <,$ or $>$ and no loops: 3 possibilities

(vi) G has exactly 2 components $\Leftrightarrow$ 1 edge only and any choice of loops:
$\binom{3}{1}2^3 = 24$ possibilities.

3.* The diagram here illustrates an undirected graph K, called a **3-sided wheel**:



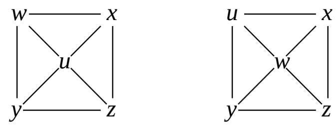
3-sided wheel K
with vertices u, x, y, z

(i) Let $K = (V_K, E_K)$. List the elements of E_K.

(ii) How many different 3-sided wheels are there with $V_K = \{u, x, y, z\}$?

(iii) Determine the number of different 3-sided wheels with $V_K \subseteq \{1, 2, 3, 4, 5, 6\}$ (e.g. $u = 4, x = 6, y = 2, z = 3$)?

The diagram here shows two **4-sided wheels** H and H':



4-sided wheel H 4-sided wheel H'

(iv) Explain why $H \neq H'$.

(v) Determine the number of different 4-sided wheels H with vertex set $V_H = \{1, 2, 3, 4, 5\}$.

(vi) Determine the number of different 4-sided wheels H with vertex set $V_H \subseteq \{1, 2, 3, 4, 5, 6, 7\}$.

Solution:

(i) $E_K = \{\{u, x\}, \{u, y\}, \{u, z\}, \{x, y\}, \{y, z\}, \{x, z\}\}$

(ii) Just 1, since E_K already has all possible edges

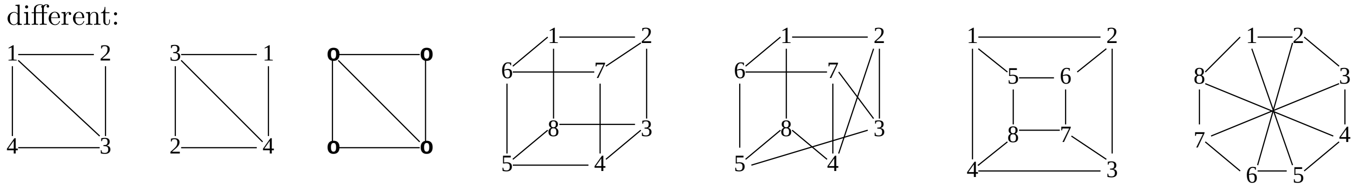
(iii) There are $\binom{6}{4} = 15$ choices for V_K; each choice gives one 3-sided wheel. Therefore, there are 15 possibilites.

(iv) $H \neq H'$ since u in H has 4 edges, but u in H' has 3 edges.

(v) With u at the center, there are just 3 possible wheels, determined by who is not connected to x by one edge. There are 5 possible choices for the center, so there are $5 \times 3 = 15$ possible 4-sided wheels (Multiplication Rule)

(vi) There are $\binom{7}{5} = 21$ choices for V_H; each choice gives 15 4-sided wheels. Therefore there are $21 \times 15 = 315$ possible 4-sided wheels for $V_H \subseteq \{1, 2, 3, 4, 5, 6, 7\}$.

4. Our definition for undirected graphs *labels* the vertices. Thus (a) and (b) below are considered different:



(a) (b) (c) (d) (e) (f) (g)

However, they are the same if we ignore the labels, as in (c). We now define what "same" means: Two finite loopless undirected graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are **isomorphic** (denoted $G \simeq H$) iff there is a permutation $\pi : V_G \to V_H$ such that $\{u, v\} \in E_G \leftrightarrow \{\pi(u), \pi(v)\} \in E_H$. (An undirected graph is **loopless** if and only if all its vertices do not have loops.) Thus (a) and (b) are isomorphic — consider $\pi(1) = 3, \pi(2) = 1, \pi(3) = 4, \pi(4) = 2$.

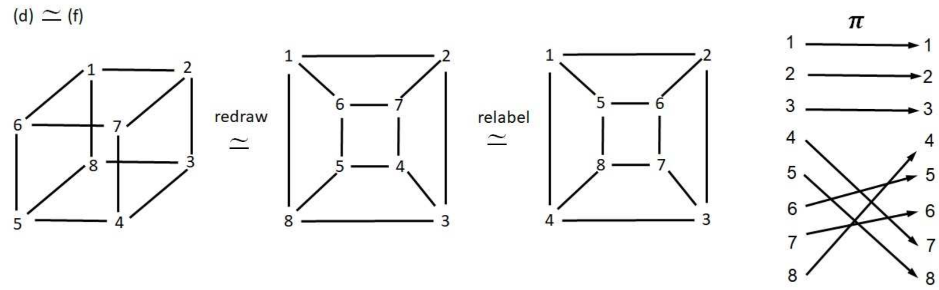
(i) Which of the graphs in (d), (e), (f) and (g) are isomorphic?

(ii)* Let $\mathcal{G}$ be the set of all loopless undirected graphs whose nodes are $\{1, 2, \ldots, n\}$. Prove that $\simeq$ is an equivalence relation on $\mathcal{G}$. What are in each equivalence class?

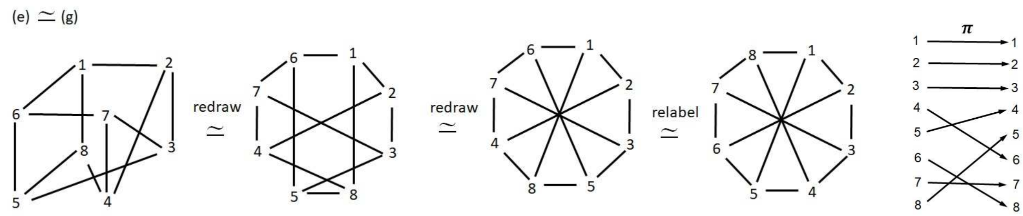
(iii) Determine the number of nonisomorphic loopless undirected graphs with n nodes, for $n = 2, 3, 4$.

[The computational complexity for determining whether two given graphs are isomorphic is a 30-year-old open problem that lies at the heart of the P$\neq$NP question.]

Solution:



(i)



Note to tutor: There are many other possible π.

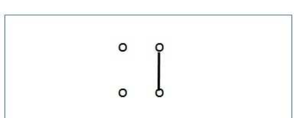
(f)$\not\simeq$(g): (g) has cycles with 5 edges (e.g. 1–2–3–4–8–1) but (f) has no such cycles.
The "structure" of (f) and (g) are thus different,
and no amount of redrawing and relabelling can help.
It follows from transitivity in (ii) that (d)$\not\simeq$(g), (d)$\not\simeq$(e) and (e)$\not\simeq$(f).

(ii)* **reflexive:** For any (V, E), we have $(V, E) \simeq (V, E)$ since we can use
the identity permutation $i_V : V \to V$, $i_V(x) = x$ for all $x \in V$.

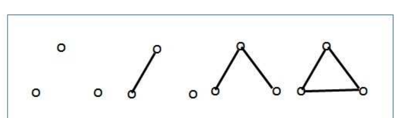
symmetric: Suppose $(V, E_G) \simeq (V, E_H)$, so there is a bijection $\pi : V \to V$
such that $\{u, v\} \in E_G \leftrightarrow \{\pi(u), \pi(v)\} \in E_H$.
Now, $\pi^{-1} : V \to V$ is also a bijection,
and $\{x, y\} \in E_H \leftrightarrow \{\pi(\pi^{-1}(x)), \pi(\pi^{-1}(y))\} \in E_H \leftrightarrow \{(\pi^{-1}(x)), \pi^{-1}(y))\} \in E_G$,
so π^{-1} is an isomorphism and $(V, E_H) \simeq (V, E_G)$.

transitive: Suppose $(V, E_G) \simeq (V, E_H)$ and $(V, E_H) \simeq (V, E_K)$,
so there are bijections $\pi : V \to V$ and $\rho : V \to V$ such that
$\{u, v\} \in E_G \leftrightarrow \{\pi(u), \pi(v)\} \in E_H$ and $\{x, y\} \in E_H \leftrightarrow \{\rho(x), \rho(y)\} \in E_K$.
Now, $\rho \circ \pi : V \to V$ is a bijection and
$\{u, v\} \in E_G \leftrightarrow \{\pi(u), \pi(v)\} \in E_H \leftrightarrow \{\rho(\pi(u)), \rho(\pi(v))\} \in E_K$,
i.e. $\{\rho \circ \pi(u), \rho \circ \pi(v)\} \in E_K$,
so $\rho \circ \pi$ is an isomorphism and $(V, E_G) \simeq (V, E_K)$.

Each equivalence class contains all the graphs that are isomorphic to each other
(i.e. the "same" except for labelling).

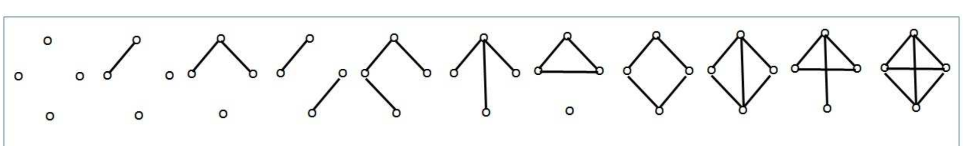


n = 2: 2 non-isomorphic graphs



n = 3: 4 non-isomorphic graphs

(iii)



n = 4: 11 non-isomorphic graphs

Note: "Total" here is actually the number of equivalence classes.

5.* Prove that if a loopless undirected graph has n vertices, where $n \geq 2$, and more than $\binom{n-1}{2}$ edges, then it is connected. Is the converse true?

Solution:

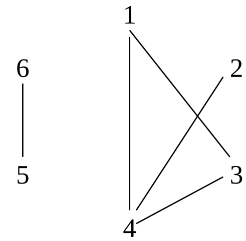
Let $G = (V, E)$ be a loopless undirected graph with n nodes and more than $\binom{n-1}{2}$ edges.
Suppose G is not connected, so it can be divided into a subgraph H with k nodes,
and a subgraph H' with $n-k$ nodes, $1 \leq k \leq n-1$,
such that there is no edge $\{x, x'\}$ in E for any x in H and x' in H'.
Now, H has at most $\binom{k}{2}$ edges and H' has at most $\binom{n-k}{2}$ edges.

$\binom{k}{2} + \binom{n-k}{2} = \frac{1}{2}(k^2 - k) + \frac{1}{2}((n-k)^2 - (n-k))$
$= \frac{1}{2}(n^2 - n - 2nk + 2k^2)$
$= \frac{1}{2}((n-1)(n-2) - 2(k-1)(n-(k+1)))$
$\leq \binom{n-1}{2}$ since $(k-1)((n-k)-1) \geq 0$
contradicting the fact that G has more than $\binom{n-1}{2}$ edges.
Therefore G must be connected.

(Counting argument: The maximum number of edges for an unconnected graph is when there is one isolated node, i.e. $\binom{n-1}{2}$ edges.)

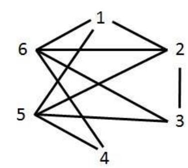
The converse is false: E.g. a–b–c–d–e is connected, but has $4 < \binom{5-1}{2}$ edges.

6. Let $G = (V, E)$ be a loopless undirected graph. The **complement** of G is the loopless graph $\overline{G} = (V, F)$, where $\{u, v\} \in F$ if and only if $\{u, v\} \notin E$. Draw the complement of the following graph:

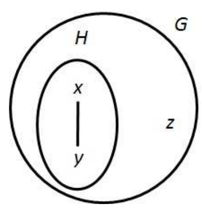


Prove that (for any G) G and $\overline{G}$ cannot both be unconnected.

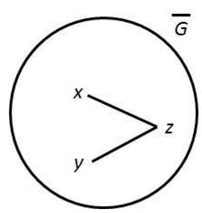
Solution:



Consider any $G = (V, E)$.
Either G is connected, or G is not connected.
If G is not connected, consider any $x, y \in V$, $x \neq y$.
Either $\{x, y\} \in F$ or $\{x, y\} \notin F$.



If $\{x, y\} \notin F$, then $\{x, y\} \in E$,
so x and y are in the same connected component of G;
call this component H.
Since G is unconnected, there is some node z in G such that z is not in H.
The $\{x, z\} \notin E$ and $\{y, z\} \notin E$,
so $\{x, z\} \in F$ and $\{y, z\} \in F$.



Thus, if G is not connected,
then any $x, y \in V$, $x \neq y$, will have a path in F between them
(either $\{x, y\}$, or $\{x, z\}$ and $\{z, y\}$).

In other words, if G is not connected, then $\overline{G}$ is connected.

7.* Let R be a binary relation on a set. Prove that R is transitive if and only if $R_+ \subseteq R$.

Solution:
Recall Exercise 6.2.10: R is transitive if and only if $R \circ R \subseteq R$.
Therefore, it suffices to prove that $R \circ R \subseteq R$ if and only if $R_+ \subseteq R$.

($\Leftarrow$) $R \circ R = R_2 \subseteq \bigcup_{n=1}^{\infty} R_n = R_+$, so $R_+ \subseteq R$ implies $R \circ R \subseteq R$.

($\Rightarrow$) Suppose $R \circ R \subseteq R$.
We prove by induction on n that $R_n \subseteq R$ for $n \geq 2$.
Basis $n = 2$: $R_2 = R \circ R \subseteq R$.
Induction Hpothesis Suppose $R_k \subseteq R$ for some $k \geq 2$.
Induction Step Consider any $(x, z) \in R_{k+1} = R \circ R_k$.
Then there is $y \in A$ such that $(x, y) \in R_k$ and $(y, z) \in R$.
But $R_k \subseteq R$ (Ind. Hyp.), so $(x, y) \in R$ and $(y, z) \in R$.
Thus $(x, z) \in R \circ R \subseteq R$, i.e. $(x, z) \in R$.
We conclude that $R_{k+1} \subseteq R$.
By induction $R_n \subseteq R$ for all $n \geq 2$.

Now, $(x, y) \in R_+ = \bigcup_{i=1}^{\infty} R_i \Rightarrow (x, y) \in R_n$ for some n
$\Rightarrow (x, y) \in R$ since $R_n \subseteq R$,
so $R_+ \subseteq R$.

8. Recall from Tutorial 9 (Problem 8) the definition of a complete graph. Let R be an equivalence relation on a nonempty set A, and let G be the undirected graph representing R. Prove that every connected component of G is a complete graph.

Solution: Let R be an equivalence relation on a set $A(\neq \varnothing)$,
and let the undirected graph $G = (A, E)$ represent R.
Consider any $x, y \in A$, $x \neq y$, and x and y in the same connected component.
Corollary 4.5 says x and y are in the same equivalence class, so $[x]_R = [y]_R$.
By Tutorial 5, Problem 7, we get xRy, so $\{x, y\} \in E$.
Thus, every 2 nodes in a connected component have an edge between them.
Since R is reflexive, G also has a loop at every node.
Therefore, every connected component of G is a complete graph.

9. Consider an undirected graph G, whose connected components are $H_1, \dots, H_k$, where $k \geq 2$. Suppose $G = (V, E)$ and $H_1 = (V_1, E_1), \dots, H_k = (V_k, E_k)$. Prove that $\{V_1, \dots, V_k\}$ is a partition of V. Is $\{E_1, \dots, E_k\}$ a partition of E?

Solution: $G = (V, E)$ is an undirected graph, and
$H_1 = (V_1, E_1), \dots, H_k = (V_k, E_k)$ are connected components, $k \geq 2$.

Claim: $\{V_1, \dots, V_k\}$ is a partition of V.
Proof:
$\underline{V_1 \cup V_2 \cup \cdots \cup V_k = V}$: Since $V_i \subseteq V$ for all i, we have $V_1 \cup \cdots \cup V_k \subseteq V$.
For any $u \in V$, u must belong to H_i for some i, so $V \subseteq V_1 \cup \cdots \cup V_k$.
Thus $V = V_1 \cup \cdots \cup V_k$.

$\underline{V_1, \cdots, V_k \text{ are disjoint}}$: Consider $i \neq j$.
Since $V_i \neq V_j$, we have $\sim (V_i \subseteq V_j \wedge \ V_j \subseteq V_i)$, i.e. $V_i \nsubseteq V_j$ or $V_j \nsubseteq V_i$,
so either $V_i \setminus V_j \neq \varnothing$ or $V_j \setminus V_i \neq \varnothing$.
Without loss of generality, assume $V_i \setminus V_j \neq \varnothing$, so there is some $y \in V_i \setminus V_j$.

Suppose $V_i \cap V_j \neq \varnothing$, so there is some $b \in V_i \cap V_j$.
Since $b \in V_j$ and $y \notin V_j$, we have $y \neq b$.
Since $y, b \in V_i$, there is a path $P = (V_P, E_P)$ in H_i between b and y,
say $V_P = \{x_1, \dots, x_n\}$ and $E_P = \{\{x_1, x_2\}, \dots, \{x_{n-1}, x_n\}\}$, where $b = x_1$ and $y = x_n$.

Let $H_j' = (V_j \cup V_P, E_j \cup E_P)$. Then H_j' is connected
(any two nodes in V_j are connected via edges in E_j,
any two nodes in V_P are connected via edges in E_P,
and a node in V_j and a node in V_P are both connected to b).
Moreover, y is in H_j' but not in H_j,
so H_j is a connected and proper subgraph of H_j'.
This contradicts the fact that H_j is a connected component.
We conclude that $V_i \cap V_j = \varnothing$ if $i \neq j$.

Thus $\{V_1, \dots, V_k\}$ is a partition of V.

$\{E_1, \dots, E_k\}$ may not be a partition since it is possible that $E_i = \varnothing$ for some i.
Suppose $E_i \neq \varnothing$ for all i.
Then, similarly, $E = E_1 \cup \cdots \cup E_k$.
Also, if $i \neq j$, then $E_i \cap E_j = \varnothing$: if $\{b, c\} \in E_i \cap E_j$, then $b, c \in V_i \cap V_j$, contradicting $V_i \cap V_j = \varnothing$.