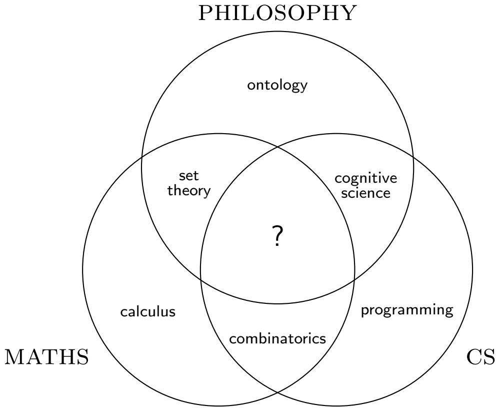
Chapter 4: Sets

CS1231 Discrete Structures

Wong Tin Lok

National University of Singapore

2021/22 Semester 1



What can one put in the centre?

Answer at `https://pollev.com/wtl`.

- There is no need to log in to pollev.
- Use hyphens (-) for spaces in multi-word answers.

About me

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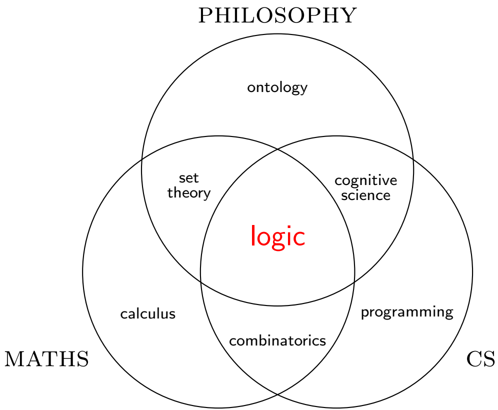
- definitions → undefinables
- proofs → (true) unprovables
- necessary truth → possible truth



What can one put in the centre?

Answer at `https://pollev.com/wtl`.

- There is no need to log in to pollev.
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Practicalities

- Lectures: Zoom (Mute yourself when you are not speaking.)
  - Wednesday 12:00 – ~~2:00pm~~ 1:35pm, with a 5-minute “break” in the middle
  - Friday 12:00 – ~~1:00pm~~ 12:45pm
- Slides and notes are posted on LumiNUS (`https://luminus.nus.edu.sg`) and in the CS1231 Telegram group.
- Try out the questions marked with ✐ in the notes. Answers will be provided.
- There are a pre-lecture and a post-lecture version of slides. Pages that are different are marked by a red line near the top left-hand corner.
- If you have any questions/comments for me during lectures, then you can unmute yourself and speak, or you can type in the Zoom chat.
- Consultation: online
  - preferably immediately after the lectures (or by individual/group appointment)
  - CS1231 Telegram group
- Additional resources: search for “discrete mathematics” on the Internet or in the library (catalogue).
- Weeks 4.5–9.5: sets, relations, functions, cardinality — *mathematical maturity*

Sets

Why sets?

- The language of sets is an important part of modern mathematical discourse.
- Sets are interesting mathematical objects.
- For this module, they provide a topic on which we practise writing and understanding proofs.

Young man, in mathematics you don't understand things. You just get used to them. (reportedly) John von Neumann

Definition 4.1.1

(1) A *set* is an unordered collection of objects.

(2) These objects are called the *members* or *elements* of the set.

(3) Write

$$\begin{array}{rll} x \in A & \text{for} & x \text{ is an element of } A; \\ x \notin A & \text{for} & x \text{ is not an element of } A; \\ x, y \in A & \text{for} & x, y \text{ are elements of } A; \\ x, y \notin A & \text{for} & x, y \text{ are not elements of } A; \qquad \text{etc.} \end{array}$$

(4) We may read $x \in A$ also as "x is in A" or "A *contains* x (as an element)".

Warning 4.1.2. Some use "contains" for the subset relation, but we do *not*.

Common sets (Table 4.1)

Note 4.1.3. Some define $0 \notin \mathbb{N}$, but we do *not*.

Symbol	Meaning	Examples	Non-examples
$\mathbb{N}$	the set of all natural numbers	$0, 1, 2, 3, 31 \in \mathbb{N}$	$-1, \frac{1}{2} \notin \mathbb{N}$
$\mathbb{Z}$	the set of all integers	$0, 1, -1, 2, -10 \in \mathbb{Z}$	$\frac{1}{2}, \sqrt{2} \notin \mathbb{Z}$
$\mathbb{Q}$	the set of all rational numbers	$-1, 10, \frac{1}{2}, -\frac{7}{5} \in \mathbb{Q}$	$\sqrt{2}, \pi, \sqrt{-1} \notin \mathbb{Q}$
$\mathbb{R}$	the set of all real numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi \in \mathbb{R}$	$\sqrt{-1}, \sqrt{-10} \notin \mathbb{R}$
$\mathbb{C}$	the set of all complex numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi, \sqrt{-1}, \sqrt{-10} \in \mathbb{C}$	
$\mathbb{Z}^+$	the set of all positive integers	$1, 2, 3, 31 \in \mathbb{Z}^+$	$0, -1, -12 \notin \mathbb{Z}^+$
$\mathbb{Z}^-$	the set of all negative integers	$-1, -2, -3, -31 \in \mathbb{Z}^-$	$0, 1, 12 \notin \mathbb{Z}^-$
$\mathbb{Z}_{\geqslant 0}$	the set of all non-negative integers	$0, 1, 2, 3, 31 \in \mathbb{Z}_{\geqslant 0}$	$-1, -12 \notin \mathbb{Z}_{\geqslant 0}$

$\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geqslant m}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_{\geqslant m}$, etc. are defined similarly.

$\mathbb{Z}$ is for *Zahlen*.

$\mathbb{Q}$ is for quotients.

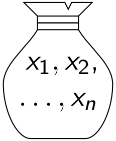
"Positive" means > 0.

"Negative" means < 0.

"Non-negative" means $\geqslant 0$.

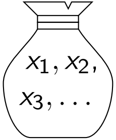
Specifying a set by listing out all its elements

Definition 4.1.4 (roster notation)



(1) The set whose only elements are $x_1, x_2, \dots, x_n$ is denoted $\{x_1, x_2, \dots, x_n\}$.

(2) The set whose only elements are $x_1, x_2, x_3, \dots$ is denoted $\{x_1, x_2, x_3, \dots\}$.



Example 4.1.5

(1) The only elements of $A = \{1, 5, 6, 3, 3, 3\}$ are 1, 5, 6 and 3.
So $6 \in A$ but $7 \notin A$.

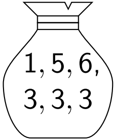
(2) The only elements of $B = \{0, 2, 4, 6, 8, \dots\}$ are the non-negative even integers.
So $4 \in B$ but $5 \notin B$.

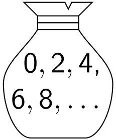
To check whether an object z is an element of a set $S = \{x_1, x_2, \dots, x_n\}$

If z is in the list $x_1, x_2, \dots, x_n$, then $z \in S$, else $z \notin S$.

Question

What are the elements of $\{2, 3, \dots\}$? All integers $x \geqslant 2$?





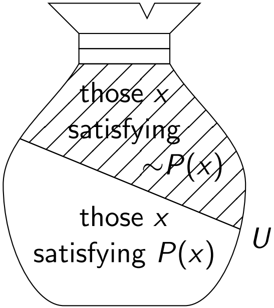
Specifying a set by describing its elements

Definition 4.1.6 (set-builder notation)

Let U be a set and $P(x)$ be a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted

$$\{x \in U : P(x)\} \quad \text{or} \quad \{x \in U \mid P(x)\}.$$

This is read as "the set of all x in U such that $P(x)$".



Example 4.1.7

(1) The elements of $C = \{x \in \mathbb{Z}_{\geqslant 0} : x \text{ is even}\}$ are precisely the elements of $\mathbb{Z}_{\geqslant 0}$ that are even, i.e., the non-negative even integers. So $6 \in C$ but $7 \notin C$.

(2) The elements of $D = \{x \in \mathbb{Z} : x \text{ is a prime number}\}$ are precisely the elements of $\mathbb{Z}$ that are prime numbers, i.e., the prime integers. So $7 \in D$ but $9 \notin D$.

To check whether an object z is an element of $S = \{x \in U : P(x)\}$

If $z \in U$ and $P(z)$ is true, then $z \in S$, else $z \notin S$. Hence $z \notin U$ implies $z \notin S$, and $P(z)$ is false implies $z \notin S$.

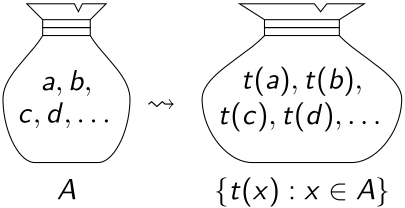
Specifying a set by replacement

Definition 4.1.8 (replacement notation)

Let A be a set and $t(x)$ be a term in a variable x. Then the set of all objects of the form $t(x)$ where x ranges over the elements of A is denoted

$$\{t(x) : x \in A\} \quad \text{or} \quad \{t(x) \mid x \in A\}.$$

This is read as "the set of all $t(x)$ where $x \in A$".



Example 4.1.9

(1) The elements of $E = \{x + 1 : x \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x + 1$ where $x \in \mathbb{Z}_{\geqslant 0}$, i.e., the positive integers. So $1 = 0 + 1 \in E$ but $0 \notin E$.

(2) The elements of $F = \{x - y : x, y \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x - y$ where $x, y \in \mathbb{Z}_{\geqslant 0}$, i.e., the integers. So $-1 = 1 - 2 \in F$ but $\sqrt{2} \notin F$.

To check whether an object z is an element of $S = \{t(x) : x \in A\}$

If there is $x \in A$ such that $t(x) = z$, then $z \in S$, else $z \notin S$.

Equality of sets

Convention 4.1.11. This is the *only* situation in mathematics when "if" should be understood as a (special) "if and only if".

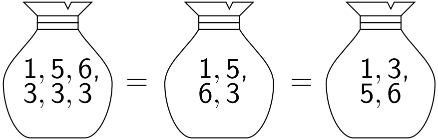
Definition 4.1.10

Two sets are *equal* if they have the same elements, i.e., for all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z \; (z \in A \Leftrightarrow z \in B).$$

Example 4.1.12

$\{1, 5, 6, 3, 3, 3\} = \{1, 5, 6, 3\} = \{1, 3, 5, 6\}$.



Slogan 4.1.13. Order and repetition do not matter.

Example 4.1.14

$$\{y^2 : y \text{ is an odd integer}\} = \{x \in \mathbb{Z} : x = y^2 \text{ for some odd integer } y\}$$
$$= \{1^2, 3^2, 5^2, \dots\}.$$

Equality of sets

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$$A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B).$$

Example 4.1.15

$\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Slogan 4.1.13. Order and repetition do not matter.

Proof

($\Rightarrow$) Take any $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. Then $z \in \mathbb{Z}$ and $z^2 = 1$. So

$$\begin{aligned} & z^2 - 1 = (z-1)(z+1) = 0. \\ \therefore \quad & z - 1 = 0 \quad \text{or} \quad z + 1 = 0. \\ \therefore \quad & z = 1 \quad \text{or} \quad z = -1. \end{aligned}$$

This means $z \in \{1, -1\}$.

($\Leftarrow$) Take any $z \in \{1, -1\}$. Then $z = 1$ or $z = -1$. In either case, we have $z \in \mathbb{Z}$ and $z^2 = 1$. So $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. □

The empty set

Definition 4.1.10. For all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B).$$

Theorem 4.1.17

There exists a unique set with no element, i.e.,

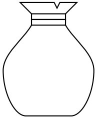
- there is a set with no element; and (existence part)
- for all sets A, B, if both A and B have no element, then $A = B$. (uniqueness part)

Proof

- (existence part) The set $\{\}$ has no element.
- (uniqueness part) Let A, B be sets with no element. Then vacuously,

$$\forall z\ (z \in A \Rightarrow z \in B) \quad \text{and} \quad \forall z\ (z \in B \Rightarrow z \in A)$$

because the hypotheses in the implications are never true. So $A = B$. $\square$



Definition 4.1.18

The set with no element is called the *empty set*. It is denoted by $\varnothing$.

Checkpoint

What we saw

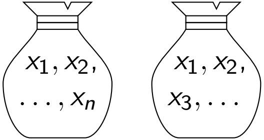
- membership $\in$
- ways to specify sets
  - the set of all . . .
  - roster notation
  - set-builder notation
  - replacement notation
  - the unique set that satisfies a property
- set equality

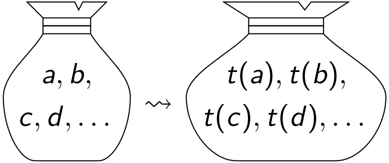
Question (to be revisited later)

Are there other ways to specify sets?

Next

inclusion





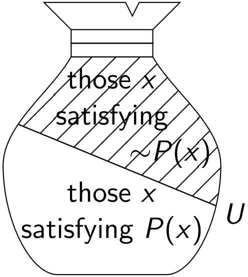
the set of all integers

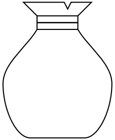
$\{x_1, x_2, \ldots, x_n\}$, $\{x_1, x_2, x_3, \ldots\}$

$\{x \in U : P(x)\}$

$\{t(x) : x \in A\}$

the unique set with no element





Inclusion of sets

Let A, B be sets.

Definition 4.1.10. For all sets A, B,
$$A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B).$$

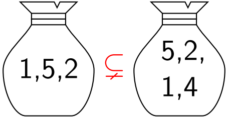
Definition 4.2.1

Call A a *subset* of B, and write $A \subseteq B$, if
$$\forall z\ (z \in A \Rightarrow z \in B).$$
Alternatively, we may say that B *includes* A, and write $B \supseteq A$ in this case.

Example 4.2.3 and Example 4.2.6

(1) $\{1, 5, 2\} \subsetneq \{5, 2, 1, 4\}$ but $\{1, 5, 2\} \not\subseteq \{2, 1, 4\}$.

(2) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. All these inclusions are proper.



Remark 4.2.4

(1) $$A \not\subseteq B \quad \Leftrightarrow \quad \exists z\ (z \in A \text{ and } z \notin B).$$

(2) $$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

(3) $$A \subseteq A.$$

Note 4.2.2. We avoid using the symbol $\subset$ because it may have different meanings to different people.

Definition 4.2.5

Call A a *proper subset* of B, write $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is *proper* or *strict*.

Vacuous inclusion

Let A, B be sets.

Definition 4.1.10. For all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B).$$

Definition 4.2.1

Call A a *subset* of B, and write $A \subseteq B$, if

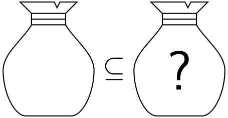
$$\forall z\ (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that B *includes* A, and write $B \supseteq A$ in this case.

Proposition 4.2.7

The empty set is a subset of any set, i.e., for any set A,

$$\varnothing \subseteq A.$$



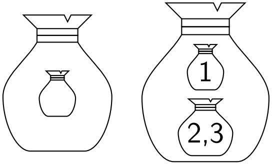
Proof

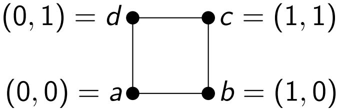
Vacuously,

$$\forall z\ (z \in \varnothing \Rightarrow z \in A)$$

because the hypothesis in the implication is never true. So $\varnothing \subseteq A$ by the definition of $\subseteq$. □

Sets of sets





Note 4.2.8

Sets can be elements of sets.

Example 4.2.9

(1) The set $A = \{\varnothing\}$ has exactly 1 element, namely the empty set. So A is not empty.

(2) The set $B = \{\{1\}, \{2,3\}\}$ has exactly 2 elements, namely $\{1\}$, $\{2,3\}$. So $\{1\} \in B$, but $1 \notin B$.

How can one use a set to represent the square above?

- If one only wants to represent the connectivity between the points, then use

$$\big\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\big\}.$$

- If one also wants to represent the positions of the lines, then use

$$\{(x,y) : (x = 0 \text{ and } y \in [0,1]) \text{ or } (x = 1 \text{ and } y \in [0,1])$$
$$\text{or } (y = 0 \text{ and } x \in [0,1]) \text{ or } (y = 1 \text{ and } x \in [0,1])\}.$$

Power set

Definition 4.2.12

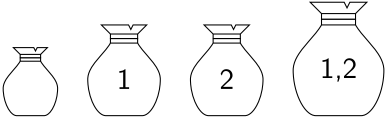
Let A be a set. The set of all subsets of A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example 4.2.13

(1) $\mathcal{P}(\varnothing) = \{\varnothing\}$

(2) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.

(3) $\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.



(4) The following are subsets of $\mathbb{Z}_{\geqslant 0}$ and thus are elements of $\mathcal{P}(\mathbb{Z}_{\geqslant 0})$.

$$\varnothing, \{0\}, \{1\}, \{2\}, \ldots \{0,1\}, \{0,2\}, \{0,3\} \ldots \{1,2\}, \{1,3\}, \{1,4\} \ldots$$

$$\{2,3\}, \{2,4\}, \{2,5\} \ldots \{0,1,2\}, \{0,1,3\}, \{0,1,4\}, \ldots$$

$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \ldots \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \ldots \ldots$$

$$\mathbb{Z}_{\geqslant 0}, \mathbb{Z}_{\geqslant 1}, \mathbb{Z}_{\geqslant 2}, \ldots \{0,2,4,\ldots\}, \{1,3,5,\ldots\}, \{2,4,6,\ldots\}, \{3,5,7,\ldots\}, \ldots$$

$$\{x \in \mathbb{Z}_{\geqslant 0} : (x-1)(x-2) < 0\}, \{x \in \mathbb{Z}_{\geqslant 0} : (x-2)(x-3) < 0\}, \ldots$$

$$\{3x+2 : x \in \mathbb{Z}_{\geqslant 0}\}, \{4x+3 : x \in \mathbb{Z}_{\geqslant 0}\}, \{5x+4 : x \in \mathbb{Z}_{\geqslant 0}\}, \ldots \ldots$$

Checkpoint

Note 4.2.10

Membership and inclusion can be different.

Question 4.2.11

Let $C = \{\{1\}, 2, \{3\}, 3, \{\{4\}\}\}$. Which of the following are true?

- $\{1\} \in C$. ✓
- $\{2\} \in C$.
- $\{3\} \in C$. ✓
- $\{4\} \in C$.
- $\{1\} \subseteq C$.
- $\{2\} \subseteq C$. ✓
- $\{3\} \subseteq C$. ✓
- $\{4\} \subseteq C$.

Next

Boolean operations

> The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; [. . .] and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind. Boole 1854

Zoom poll

Boolean operations

Let A, B be sets.

Definition 4.3.1

(1) The *union* of A and B, denoted $A \cup B$, is defined by

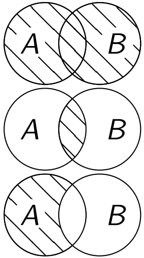
read as 'A union B' $\longrightarrow$ $A \cup B = \{x : x \in A \;\text{ or }\; x \in B\}.$

(2) The *intersection* of A and B, denoted $A \cap B$, is defined by

read as 'A intersect B' $\longrightarrow$ $A \cap B = \{x : x \in A \text{ and } x \in B\}.$

(3) The *complement* of B in A, denoted $A - B$ or $A \setminus B$, is defined by

read as 'A minus B' $\longrightarrow$ $A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$

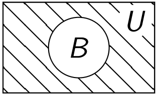


Convention and terminology 4.3.2

When working in a particular context, one usually works within a fixed set U which includes all the sets one may talk about, so that one only needs to consider the elements of U when proving set equality and inclusion. This U is called a *universal set*.

Definition 4.3.3

In a context where U is the universal set (so that implicitly $U \supseteq B$), the *complement* of B, denoted $\overline{B}$ or B^{c}, is defined by $\overline{B} = U \setminus B$.



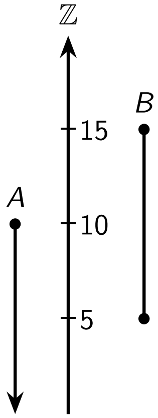
Example 4.3.4 on Boolean operations

For all sets A, B,

$$A \cup B = \{x : (x \in A) \vee (x \in B)\},$$
$$A \cap B = \{x : (x \in A) \wedge (x \in B)\},$$
$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\},$$
$$\overline{B} = \{x \in U : x \notin B\}, \quad \text{in a context where } U \text{ is the universal set.}$$

Let $A = \{x \in \mathbb{Z} : x \leqslant 10\}$ and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 15\}$. Then

$$A \cup B = \{x \in \mathbb{Z} : (x \leqslant 10) \vee (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x \leqslant 15\};$$
$$A \cap B = \{x \in \mathbb{Z} : (x \leqslant 10) \wedge (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 10\};$$
$$A \setminus B = \{x \in \mathbb{Z} : (x \leqslant 10) \wedge \sim(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x < 5\};$$
$$\overline{B} = \{x \in \mathbb{Z} : \sim(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : (x < 5) \vee (x > 15)\},$$



in a context where $\mathbb{Z}$ is the universal set. To show the first equation, one shows

$$\forall x \in \mathbb{Z} \; \big((x \leqslant 10) \vee (5 \leqslant x \leqslant 15) \Leftrightarrow (x \leqslant 15)\big),$$

etc.

Set identities (Theorem 4.3.5)

For all set A, B, C in a context where U is the universal set, the following hold.

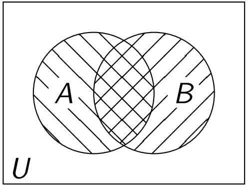
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotence	$A \cup A = A$	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Identities	$A \cup \varnothing = A$	$A \cap U = A$
Annihilators	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Complement	$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$
Double Complement Law	$\overline{(\overline{A})} = A$	
Top and bottom	$\overline{\varnothing} = U$	$\overline{U} = \varnothing$
Set difference	$A \setminus B = A \cap \overline{B}$	

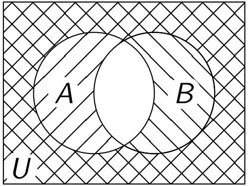
Venn diagrams

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

In the left diagram, hatch the regions representing A and B with and respectively.
In the right diagram, hatch the regions representing $\overline{A}$ and $\overline{B}$ with and respectively.





Then the region represents $\overline{A \cup B}$ on the left diagram, and the region represents $\overline{A} \cap \overline{B}$ on the right diagram. Since these regions occupy the same region in the respective diagrams, we infer that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Note 4.3.6. This argument depends on the fact that each possibility for membership in A and B is represented by a region in the diagram.

Proving set identities using truth tables

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Proof #1

The rows in the following table list all the possibilities for an element $x \in U$:

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in \overline{A \cup B}$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in \overline{A} \cap \overline{B}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns under "$x \in \overline{A \cup B}$" and "$x \in \overline{A} \cap \overline{B}$" are the same, for any $x \in U$,

$$x \in \overline{A \cup B} \quad \Leftrightarrow \quad x \in \overline{A} \cap \overline{B}$$

no matter in which case we are. So $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$

Proving set identities directly

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Proof #2

Let $z \in U$. Then

$$\begin{array}{clr} & z \in \overline{A \cup B} & \\ \Leftrightarrow & z \notin A \cup B & \text{by the definition of } \overline{\cdot}; \\ \Leftrightarrow & \sim\big((z \in A) \vee (z \in B)\big) & \text{by the definition of } \cup; \\ \Leftrightarrow & (z \notin A) \wedge (z \notin B) & \text{by De Morgan's Laws for propositions;} \\ \Leftrightarrow & (z \in \overline{A}) \wedge (z \in \overline{B}) & \text{by the definition of } \overline{\cdot}; \\ \Leftrightarrow & z \in \overline{A} \cap \overline{B} & \text{by the definition of } \cap. \end{array}$$

□

Applications of the set identities

Fix a universal set U. The following are true for all sets A, B, C.

Identity	$A \cap U = A.$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$
Complement	$A \cup \overline{A} = U.$
Set difference	$A \setminus B = A \cap \overline{B}.$

Example 4.3.7

Under the universal set U, show that $(A \cap B) \cup (A \setminus B) = A$ for all sets A, B.

Proof

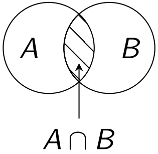
$$
\begin{aligned}
(A \cap B) \cup (A \setminus B) &= (A \cap B) \cup (A \cap \overline{B}) && \text{by the properties of set difference;} \\
&= A \cap (B \cup \overline{B}) && \text{by distributivity;} \\
&= A \cap U && \text{by the properties of set complement;} \\
&= A && \text{as } U \text{ is the identity for } \cap.
\end{aligned}
$$

$\square$

Boolean operations and inclusion

Example 4.3.8

Show that $A \cap B \subseteq A$ for all sets A, B.



Let A, B be sets.

Definition 4.2.1. $A \subseteq B \quad \Leftrightarrow \quad \forall z\ (z \in A \Rightarrow z \in B)$.

Definition 4.3.1(2). $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Proof

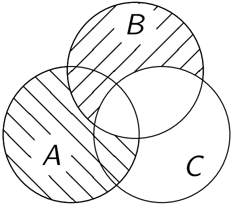
By the definition of $\subseteq$, we need to show that

$$\forall z\ (z \in A \cap B \Rightarrow z \in A).$$

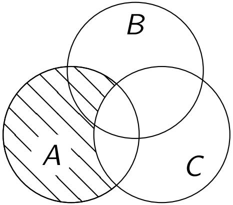
Let $z \in A \cap B$. Then $z \in A$ and $z \in B$ by the definition of $\cap$. In particular, we know $z \in A$, as required. $\square$

Example 4.3.9: Is the following true?

$$\boxed{(A \setminus B) \cup (B \setminus C) = A \setminus C \quad \text{for all sets } A, B, C.}$$



$(A \setminus B) \cup (B \setminus C)$



$A \setminus C$

No. For a counterexample, let $A = C = \varnothing$ and $B = \{1\}$. Then

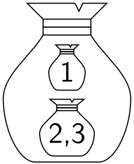
$$(A \setminus B) \cup (B \setminus C) = \varnothing \cup \{1\} = \{1\} \neq \varnothing = A \setminus C.$$

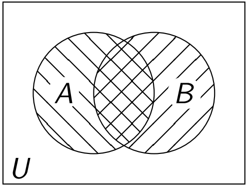
Zoom poll

Checkpoint

What we saw

- membership, inclusion, and equality of sets
- power sets
- union, intersections, complements
- set identities and their proofs
- Venn diagrams





Questions

- Is there a complete list of set operations?
- Are sets simply predicates in disguise?
- Why do we work with a universal set?

Next

Russell's Paradox

> Hardly anything more unwelcome can befall a scientific writer than that one of the foundations of his edifice be shaken after the work is finished. Frege (1902)

A set that is an element of itself?

Note 4.2.8 (recall)

Sets can be elements of sets.

Example 4.4.1

(1) $\varnothing \notin \varnothing$.

(2) $\mathbb{Z} \notin \mathbb{Z}$.

(3) $\{\varnothing\} \notin \{\varnothing\}$.

Question 4.4.2

Is there a set x such that $x \in x$?

Ideas

(1) The set of all sets?

(2) $\left\{ \left\{ \left\{ \{ \{ \dots \quad \dots \} \} \right\} \right\} \right\}$?

Hogarth (1754)

Zoom poll

Consternation

There is just one point where I have encountered a difficulty. Russell (1902)

$\{x : x \notin x\}$?

Theorem 4.4.3 (Russell 1901)

There is *no* set R such that

$$\forall x \, (x \in R \quad \Leftrightarrow \quad x \notin x). \tag{$*$}$$

Proof (by contradiction)

Suppose R is a set satisfying $(*)$. Applying $(*)$ to $x = R$ gives

$$R \in R \quad \Leftrightarrow \quad R \notin R. \tag{$\dagger$}$$

Split into two cases.

- Case 1: assume $R \in R$. Then $R \notin R$ by the $\Rightarrow$ part of $(\dagger)$. This contradicts our assumption that $R \in R$.
- Case 2: assume $R \notin R$. Then $R \in R$ by the $\Leftarrow$ part of $(\dagger)$. This contradicts our assumption that $R \notin R$.

In either case, we get a contradiction. So the proof is finished. $\square$

Question 4.4.4. Can you write a proof that does not mention contradiction? ✐

Consternation?

There is just one point where I have encountered a difficulty. Russell (1902)

Theorem 4.4.3 (Russell 1901)

There is *no* set R such that

$$\forall x \, (x \in R \quad \Leftrightarrow \quad x \notin x).$$

$\{x : x \notin x\}$?

Morals

- Some predicates do not correspond to any set.
- The set of all sets, if it exists, needs to be handled with *extreme* care.

Suppose a contradiction were to be found in the axioms of set theory. Do you seriously believe that that bridge would fall down? (reportedly) Ramsey

Next

how sets can represent mathematical objects