CS2100 COMPUTER ORGANISATION

http://www.comp.nus.edu.sg/~cs2100/

Lecture #13

# Boolean Algebra ?

NUS National University of Singapore | School of Computing

# Lecture #13: Boolean Algebra

1. Digital Circuits
2. Boolean Algebra
3. Truth Table
4. Precedence of Operators
5. Laws of Boolean Algebra
6. Duality
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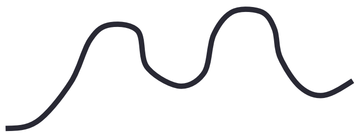
8. Boolean Functions
9. Complement Functions
10. Standard Forms
11. Minterms and Maxterms
12. Canonical Forms:
    Sum-of-Minterms and Product-of-Maxterms

# 1. Digital Circuits (1/2)

- Two voltage levels
  - High/true/1/asserted
  - Low/false/0/deasserted


*Signals in digital circuit*



*Signals in analog circuit*

- Advantages of digital circuits over analog circuits
  - More reliable (simpler circuits, less noise-prone )
  - Specified accuracy (determinable)
  - Abstraction can be applied using simple mathematical model – Boolean Algebra
  - Ease design, analysis and simplification of digital circuit – Digital Logic Design

# 1. Digital Circuits (2/2)

- Combinational: no memory, output depends solely on the input
  - Gates
  - Decoders, multiplexers
  - Adders, multipliers
- Sequential: with memory, output depends on both input and current state
  - Counters, registers
  - Memories

# 2. Boolean Algebra

- Boolean values:
  - True (T or 1)
  - False (F or 0)
- Connectives
  - Conjunction (AND)
    - A · B; A ∧ B
  - Disjunction (OR)
    - A + B; A ∨ B
  - Negation (NOT)
    - A'; $\overline{A}$ ; ¬A;

In CS2100, we use the symbols 1 for true, 0 for false, · for AND, + for OR, and ' for negation (you may use the accent bar). Please follow.

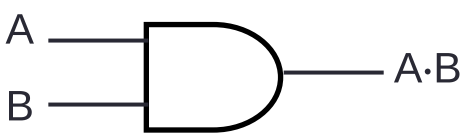
- Truth tables

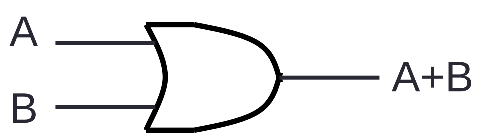
| A | B | A · B |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

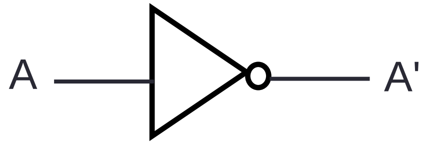
| A | B | A + B |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

| A | A' |
|---|---|
| 0 | 1 |
| 1 | 0 |

- Logic gates



# 2. Boolean Algebra: AND


- Do write the AND operator · (instead of omitting it)
  - Example: Write ***a·b*** instead of ***ab***
  - Why? Writing ***ab*** could mean that it is a 2-bit value.

# 3. Truth Table

- Provide a listing of every possible combination of inputs and its corresponding outputs.
  - Inputs are usually listed in binary sequence.
- Example
  - Truth table with 3 inputs x, y, z and 2 outputs $(y + z)$ and $(x \cdot (y + z))$.

| x | y | z | y + z | x · (y + z) |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

# 3. Proof using Truth Table

- Prove: **x · (y + z) = (x · y) + (x · z)**
  - Construct truth table for LHS and RHS

| x | y | z | y + z | **x · (y + z)** | x · y | x · z | **(x · y) + (x · z)** |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | **0** | 0 | 0 | |
| 0 | 0 | 1 | 1 | **0** | 0 | 0 | |
| 0 | 1 | 0 | 1 | **0** | 0 | 0 | |
| 0 | 1 | 1 | 1 | **0** | 0 | 0 | |
| 1 | 0 | 0 | 0 | **0** | 0 | 0 | |
| 1 | 0 | 1 | 1 | **1** | 0 | 1 | |
| 1 | 1 | 0 | 1 | **1** | 1 | 0 | |
| 1 | 1 | 1 | 1 | **1** | 1 | 1 | |

- Check that column for LHS = column for RHS

- DLD page 59 Quick Review Questions Question 3-1.

# 4. Precedence of Operators

- Precedence from highest to lowest
  - Not (')
  - And (·)
  - Or (+)

Note the difference with CS1231/CS1231S. Here in CS2100, AND has higher precedence than OR.

- Examples:
  - A · B + C = (A · B) + C
  - X + Y' = X + (Y')
  - P + Q' · R = P + ((Q') · R)

Hence, A · B + C is <u>not</u> ambiguous in CS2100.

- Use parenthesis to overwrite precedence. Examples:
  - A · (B + C) [ Without parenthesis, it means A·B+C or (A·B)+C ]
  - (P + Q)' · R [ Without parenthesis, it means P+Q'·R or P+(Q'·R) ]

# 5. Laws of Boolean Algebra

| Identity laws | |
|---|---|
| $A + 0 = 0 + A = A$ | $A \cdot 1 = 1 \cdot A = A$ |
| **Inverse/complement laws** | |
| $A + A' = A' + A = 1$ | $A \cdot A' = A' \cdot A = 0$ |
| **Commutative laws** | |
| $A + B = B + A$ | $A \cdot B = B \cdot A$ |
| **Associative laws *** | |
| $A + (B + C) = (A + B) + C$ | $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ |
| **Distributive laws** | |
| $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$ | $A + (B \cdot C) = (A + B) \cdot (A + C)$ |

* Due to the associative laws, $A + B + C$ is unambiguous. It may be evaluated as $A + (B + C)$ or $(A + B) + C$. Likewise for $A \cdot B \cdot C$.

# 6. Duality

- If the AND/OR operators and identity elements 0/1 in a Boolean equation are interchanged, it remains valid.
- Example:
  - The dual equation of $a+(b\cdot c)=(a+b)\cdot(a+c)$ is $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$.
- Duality gives free theorems – “two for the price of one”, as a Boolean equation is logically equivalent to its dual. So, you prove one theorem and the other comes for free!
- Examples:
  - If $(x+y+z)' = x'\cdot y'\cdot z'$ is valid, then its dual $(x\cdot y\cdot z)' = x'+y'+z'$ is also valid.
  - If $x+1 = 1$ is valid, then its dual $x\cdot 0 = 0$ is also valid.

Do not confuse duality with negation!

# 7. Theorems

| | |
|---|---|
| Idempotency | |
| $X + X = X$ | $X \cdot X = X$ |
| One element / Zero element | |
| $X + 1 = 1 + X = 1$ | $X \cdot 0 = 0 \cdot X = 0$ |
| Involution | |
| $(X')' = X$ | |
| Absorption 1 | |
| $X + X \cdot Y = X$ | $X \cdot (X + Y) = X$ |
| Absorption 2 | |
| $X + X' \cdot Y = X + Y$ | $X \cdot (X' + Y) = X \cdot Y$ |
| DeMorgans' (can be generalised to more than 2 variables) | |
| $(X + Y)' = X' \cdot Y'$ | $(X \cdot Y)' = X' + Y'$ |
| Consensus | |
| $X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$ | $(X+Y) \cdot (X'+Z) \cdot (Y+Z) = (X+Y) \cdot (X'+Z)$ |

# 7. Proving a Theorem

- Theorems can be proved using truth table, or by algebraic manipulation using other theorems/laws.
- Example: Prove absorption theorem $X + X \cdot Y = X$

$$
\begin{aligned}
X + X \cdot Y &= X \cdot 1 + X \cdot Y \text{ (by identity law)} \\
&= X \cdot (1+Y) \text{ (by distributivity)} \\
&= X \cdot (Y+1) \text{ (by commutativity)} \\
&= X \cdot 1 \text{ (by one element law)} \\
&= X \text{ (by identity law)}
\end{aligned}
$$

- By the principle of duality, we may also cite (<u>without proof</u>) that $X \cdot (X+Y) = X$.

# 8. Boolean Functions

- Examples of Boolean functions (logic equations):

  F1(x,y,z) = x·y·z'

  F2(x,y,z) = x + y'·z

  F3(x,y,z) = x'·y'·z + x'·y·z + x·y'

  F4(x,y,z) = x·y' + x'·z

| x | y | z | F1 | F2 | F3 | F4 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | | |
| 0 | 0 | 1 | 0 | 1 | | |
| 0 | 1 | 0 | 0 | 0 | | |
| 0 | 1 | 1 | 0 | 0 | | |
| 1 | 0 | 0 | 0 | 1 | | |
| 1 | 0 | 1 | 0 | 1 | | |
| 1 | 1 | 0 | 1 | 1 | | |
| 1 | 1 | 1 | 0 | 1 | | |

# 9. Complement Functions

- Given a Boolean function F, the complement of F, denoted as F', is obtained by interchanging 1 with 0 in the function's output values.
- Example: F1 = x·y·z'
- What is F1' ?

| x | y | z | F1 | F1' |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 0 | 0 | 1 | 0 | |
| 0 | 1 | 0 | 0 | |
| 0 | 1 | 1 | 0 | |
| 1 | 0 | 0 | 0 | |
| 1 | 0 | 1 | 0 | |
| 1 | 1 | 0 | 1 | |
| 1 | 1 | 1 | 0 | |

# 10. Standard Forms (1/2)

- Certain types of Boolean expressions lead to circuits that are desirable from an implementation viewpoint.
- Two standard forms:
  - Sum-of-Products (SOP)
  - Product-of-Sums (POS)
- Literals
  - A Boolean variable on its own or in its complemented form
  - Examples: (1) $x$, (2) $x'$, (3) $y$, (4) $y'$
- Product term
  - A single literal or a logical product (AND) of several literals
  - Examples: (1) $x$, (2) $x \cdot y \cdot z'$, (3) $A' \cdot B$, (4) $A \cdot B$, (5) $d \cdot g' \cdot v \cdot w$

# 10. Standard Forms (2/2)

- Sum term
  - A single literal or a logical sum (OR) of several literals
  - Examples: (1) $x$, (2) $x+y+z'$, (3) $A'+B$, (4) $A+B$, (5) $c+d+h'+j$
- Sum-of-Products (SOP) expression
  - A product term or a logical sum (OR) of several product terms
  - Examples: (1) $x$, (2) $x + y \cdot z'$, (3) $x \cdot y' + x' \cdot y \cdot z$, (4) $A \cdot B + A' \cdot B'$,
    (5) $A + B' \cdot C + A \cdot C' + C \cdot D$
- Product-of-Sums (POS) expression
  - A sum term or a logical product (AND) of several sum terms
  - Examples: (1) $x$, (2) $x \cdot (y+z')$, (3) $(x+y') \cdot (x'+y+z)$,
    (4) $(A+B) \cdot (A'+B')$, (5) $(A+B+C) \cdot D' \cdot (B'+D+E')$
- Every Boolean expression can be expressed in SOP or POS form.

- DLD page 59 Quick Review Questions Questions 3-2 to 3-5.

# Quiz Time!

**SOP** expr: A product term or a logical sum (OR) of several product terms.
**POS** expr: A sum term or a logical product (AND) of several sum terms.

- Put the right ticks in the following table.

| | *Expression* | *SOP?* | *POS?* |
|---|---|---|---|
| (1) | X'·Y + X·Y' + X·Y·Z | | |
| (2) | (X+Y')·(X'+Y)·(X'+Z') | | |
| (3) | X' + Y + Z | | |
| (4) | X·(W' + Y·Z) | | |
| (5) | X·Y·Z' | | |
| (6) | W·X'·Y + V·(X·Z + W') | | |

# 11. Minterms and Maxterms (1/2)

- A minterm of *n* variables is a product term that contains *n* literals from all the variables.
  - Example: On 2 variables x and y, the minterms are: x'·y', x'·y, x·y' and x·y
- A maxterm of *n* variables is a sum term that contains *n* literals from all the variables.
  - Example: On 2 variables x and y, the maxterms are: x'+y', x'+y, x+y' and x+y
- In general, with *n* variables we have up to $2^n$ minterms and $2^n$ maxterms.

# 11. Minterms and Maxterms (2/2)

- The minterms and maxterms on 2 variables are denoted by m0 to m3 and M0 to M3 respectively.

| x | y | Minterms | | Maxterms | |
|---|---|---|---|---|---|
| | | Term | Notation | Term | Notation |
| 0 | 0 | $x' \cdot y'$ | m0 | $x+y$ | M0 |
| 0 | 1 | $x' \cdot y$ | m1 | $x+y'$ | M1 |
| 1 | 0 | $x \cdot y'$ | m2 | $x'+y$ | M2 |
| 1 | 1 | $x \cdot y$ | m3 | $x'+y'$ | M3 |

- Important fact: Each minterm is the complement of its corresponding maxterm. Likwise, each maxterm is the complement of its corresponding minterm.
  - Example: $m2 = x \cdot y'$

    $m2' = ( x \cdot y' )' = x' + ( y' )' = x' + y = M2$

# Quiz Time Again!

- Ability to convert minterms and maxterms from its Boolean expression to its notation (and vice versa) is important.
- Test yourself with the following quiz, assuming that you are given a Boolean function on 4 variables A, B, C, D.

## Minterm

| | ***Boolean expression*** | ***Minterm notation*** |
|---|---|---|
| (1) | A'·B'·C·D | m3 |
| (2) | | m10 |
| (3) | | m11 |
| (4) | A·B·C·D' | |
| (5) | A·B'·C'·D | |

## Maxterm

| | ***Boolean expression*** | ***Maxterm notation*** |
|---|---|---|
| (1) | A+B+C'+D' | M3 |
| (2) | | M13 |
| (3) | | M0 |
| (4) | A+B+C'+D | |
| (5) | A'+B+C+D' | |

# 12. Canonical Forms

- Canonical/normal form: a unique form of representation.
  - Sum-of-minterms = Canonical sum-of-products
  - Product-of-maxterms = Canonical product-of-sums

# 12.1 Sum-of-Minterms

- Given a truth table, example:

| x | y | z | F1 | F2 | F3 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 |

- Obtain sum-of-minterms expression by gathering the minterms of the function (where output is 1).

F1 = x·y·z' = m6

F2 =

F3 =

# 12.2 Product-of-Maxterms

- Given a truth table, example:
- Obtain product-of-maxterms expression by gathering the maxterms of the function (where output is 0).

| x | y | z | F1 | F2 | F3 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 |

$$F2 = (x+y+z) \cdot (x+y'+z) \cdot (x+y'+z')$$
$$= M0 \cdot M2 \cdot M3 = \Pi M(0,2,3)$$

F3 =

# 12.3 Conversion of Standard Forms

- We can convert between sum-of-minterms and product-of-maxterms easily
- Example: F2 = $\Sigma$m(1,4,5,6,7) = $\Pi$M(0,2,3)
- Why? See F2' in truth table.
- F2' = m0 + m2 + m3
  Therefore,
  F2 = (m0 + m2 + m3)'
  = m0' · m2' · m3' (by DeMorgan's)
  = M0 · M2 · M3 (as m*x*' = M*x*)

| x | y | z | F2 | F2' |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 0 |

- Read up DLD section 3.4, pg 57 – 58.
- Quick Review Questions: pg 60 – 61, Q3-6 to 3-13.

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