NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION FOR

(Semester I : 2018/2019)

Names of examiners: LWC Wong and CS Ng

EE2023 – SIGNALS & SYSTEMS

Nov/Dec 2018 - Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. This paper contains **EIGHT (8)** questions and comprises **TEN (10)** printed pages.
2. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.
3. This is a **CLOSED BOOK** examination. However you are allowed to bring one self-prepared A4-size crib sheet to the examination hall.
4. Programmable and/or graphic calculators are not allowed.
5. Tables of formulas are provided on Pages **7** to **10**.
6. Write only your student number on your answer book(s). Do not write your name.

SECTION A : Answer ALL questions in this section

Q.1 The transfer function of a second order LTI system is given by

$$H(s) = \frac{-s+1}{(s+3)^2+16}.$$

(a) Find the poles and zeros of $H(s)$. Is the system stable, and why? (3 marks)

(b) Find the dc gain of the system. Hence, or otherwise, determine the steady-state output of the system if the system input is $5u(t)$, where $u(t)$ is the unit step function.

(3 marks)

(c) Let $h(t)$ be the inverse Laplace transform of $H(s)$. Derive $h(t)$. In the present context, what is $h(t)$ called?

[**Hint:** *Decompose* $H(s)$ *into 2nd order partial fractions and then obtain their inverse with the help of the Laplace transform table*.]

(4 marks)

Q.2 Let $W(f)$ and B be the spectrum and bandwidth, respectively, of a signal $w(t)$. Using $w(t)$, we form three other signals:

$$x(t) = w(t) \times \sum_k e^{j2\pi kt}, \qquad y(t) = w(5t) \quad \text{and} \quad z(t) = w(t-0.2).$$

Let $X(f)$, $Y(f)$ and $Z(f)$ be the spectrum of $x(t)$, $y(t)$ and $z(t)$, respectively.

(a) Express $X(f)$ in terms of $W(f)$. (4 marks)

(b) Express $Y(f)$ in terms of $W(f)$. Hence, determine the bandwidths of $y(t)$ in terms of B.

(3 marks)

(c) Express $Z(f)$ in terms of $W(f)$. Hence, determine $\angle Z(f)$ in terms of $\angle W(f)$.

(3 marks)

Q.3 The signal $x(t)$ shown in Figure Q.3 is sampled at a sampling frequency of 0.5 Hz to give the sampled signal $x_s(t)$.

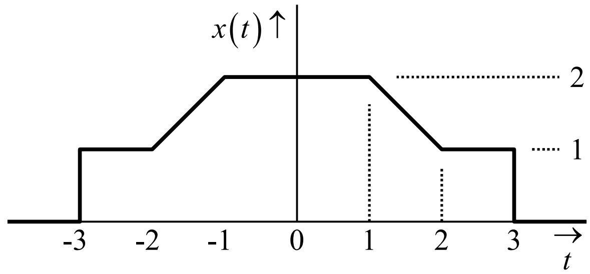

Figure Q.3

(a) Obtain the expression for $x(t)$. (3 marks)

(b) Derive the Fourier transform of $x(t)$. (3 marks)

(c) Derive the Fourier transform of the sampled signal $x_s(t)$. (4 marks)

Q.4 A second order system with input $x(t)$ and output $y(t)$ is described by the differential equation:

$$\frac{d^2}{dt^2}y(t)+6\frac{d}{dt}y(t)+25y(t)=50x(t).$$

(a) Determine the transfer function, $G(s)$, of this system. (4 marks)

(b) Determine the damping ratio, ζ, and undamped natural frequency, ω_n, of the system.

(2 marks)

(c) If the input signal is $x(t)=3\sin(6t)$, derive the steady state output response, $y_{ss}(t)$, of the system.

(4 marks)

SECTION B : Answer 3 out of the 4 questions in this section

Q.5 Loudness compensation is used in some hi-fi equipment to boost the level of bass (low frequencies) and treble (high frequencies) at low listening level. The intention is to equalize the frequency response of the ears, which are less sensitive to extreme low and high frequencies.

The Bode magnitude plot for a loudness compensation circuit with transfer function $G(s)$ is shown in Figure Q.5.

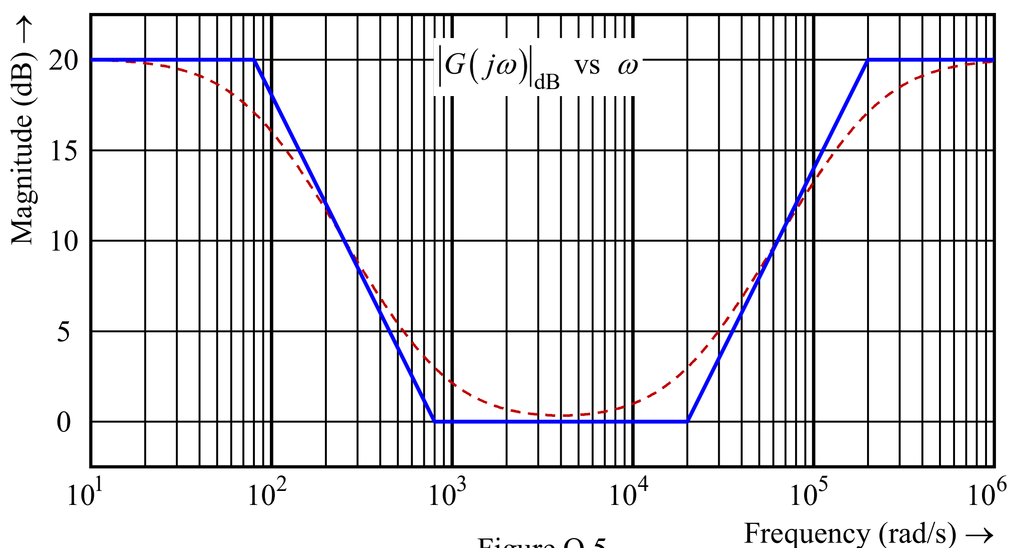

Figure Q.5

Assume all zeros have negative real parts.

(a) Determine the low-frequency and high-frequency asymptotic phase of $G(s)$? (4 marks)

(b) Identify $G(s)$. (6 marks)

(c) Sketch the pole-zero map for $G(s)$, assuming all zeros have negative real parts. (4 marks)

(d) Suppose we cascade $G(s)$ with an identical system to form a new system $H(s) = G(s) \cdot G(s)$. Sketch the straight-line Bode magnitude plot for $H(s)$. (6 marks)

Label all your sketches adequately.

Q.6 A periodic signal is modeled by $x(t) = 16\text{sinc}^2(16t) \ast \sum_n \delta(t - 0.25n)$, where '$\ast$' denotes convolution.

(a) What is the fundamental frequency of $x(t)$? (2 marks)

(b) Derive and draw a labeled sketch of the Fourier transform, $X(f)$, of $x(t)$. (8 marks)

(c) Based on the results in Part (b), or otherwise, express $x(t)$ without using the convolution $(\ast)$ and summation (Σ) operators. (4 marks)

(d) Suppose we pass $x(t)$ through a lowpass filter to produce the output $y(t)$. If the frequency response of the filter is $H(f) = \text{rect}\left(\frac{f}{12}\right)$, find the ratio:

$$P_{ratio} = \frac{\text{average power of } y(t)}{\text{average power of } x(t)}.$$

(6 marks)

Q.7 Consider the signal $x(t) = 5[1 + \cos(10\pi t)]\text{rect}\left(\frac{t}{2}\right)$.

(a) Obtain the Fourier transform of $x(t)$. (6 marks)

(b) Determine the energy spectral density, $E_x(f)$, of $x(t)$ and sketch it. (6 marks)
[*Hint*: Assume that overlapping sinc functions beyond the 4th null can be neglected.]

(c) Determine the energy, E, of $x(t)$. (4 marks)

(d) The signal $x(t)$ is applied to the input of an ideal low-pass filter with frequency response $H(f) = \text{rect}\left(\frac{f}{5}\right)$. Determine the spectrum of the output from the filter and comment on the shape of the output signal in the time domain.

(4 marks)

Q.8 Consider the electronic circuit shown in Figure Q.8

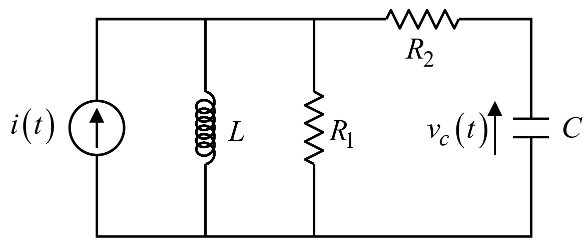

Figure Q.8

(a) Obtain the expression for the transfer function, $G(s) = \dfrac{V_c(s)}{I(s)}$, where $V_c(s)$ and $I(s)$ are the Laplace transforms of $v_c(t)$ and $i(t)$, respectively.

(10 marks)

(b) If $L = 1$ henry, $C = 1$ farad, and $R_2 = 1$ ohm, and for a critically damped system, determine the value of R_1.

(7 marks)

(c) What is the undamped natural frequency, ω_n, of the critically damped circuit of part (b).

(3 marks)

END OF QUESTIONS

Fourier Series: $\begin{cases} c_k = \dfrac{1}{T}\displaystyle\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k\,t/T)\,dt \\ x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(j2\pi k\,t/T) \end{cases}$

Fourier Transform: $\begin{cases} X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$

FOURIER TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}(fT)$
Triangle	$\text{tri}\left(\frac{t}{T}\right)$	$T\,\text{sinc}^2(fT)$
Sine Cardinal	$\text{sinc}\left(\frac{t}{T}\right)$	$T\,\text{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp\left(-\alpha^2\pi^2 f^2\right)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES		
	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi f t_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty}X_1(\zeta)X_2(f-\zeta)d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$ $\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n^{th} order Ramp	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES		
	Time-domain	***s*-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)u(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\frac{1}{s}X(s)$
Differentiation in the time-domain	$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\frac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\frac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\frac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\frac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

System Type	Transfer Function (Standard Form)	Unit Impulse and Unit Step Responses	Remarks
1st order system	$G(s)=\frac{K}{T}\cdot\frac{1}{s+1/T}$	$y_\delta(t)=\frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$ $y_{step}(t)=K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	T : Time-constant K : DC Gain Real Pole at $s=-\frac{1}{T}$
2nd order system $(\zeta>1)$ **Overdamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}$	$y_\delta(t)=\left[K_1\exp(-p_1t)+K_2\exp(-p_2t)\right]u(t)$ $y_{step}(t)=\left[K-\frac{K_1}{p_1}\exp(-p_1t)-\frac{K_2}{p_2}\exp(-p_2t)\right]u(t)$	K : DC Gain $\left.\begin{array}{l}p_1=\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\\ p_2=\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\end{array}\right\\}$ $K_1=-K_2=\frac{K\omega_n^2}{p_2-p_1}$ Real Distinct Poles at $s=-p_1$ and $s=-p_2$
2nd order system $(\zeta=1)$ **Critically damped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{(s+\omega_n)^2}$	$y_\delta(t)=K\omega_n^2t\exp(-\omega_nt)u(t)$ $y_{step}(t)=K\left[1-\exp(-\omega_nt)-\omega_nt\exp(-\omega_nt)\right]u(t)$	K : DC Gain Real Repeated Poles at $s=-\omega_n$
2nd order system $(0<\zeta<1)$ **Underdamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\left(\sigma^2+\omega_d^2\right)}{(s+\sigma)^2+\omega_d^2}$	$y_\delta(t)=K\frac{\omega_n^2}{\omega_d}\exp(-\sigma t)\sin(\omega_dt)u(t)$ $y_{step}(t)=K\left[1-\frac{\omega_n}{\omega_d}\exp(-\sigma t)\sin(\omega_dt+\phi)\right]u(t)$	K : DC Gain ω_n : Undamped Natural Frequency ζ : Damping Ratio ω_d : Damped Natural Frequency $\sigma=\zeta\omega_n$ $\omega_d^2=\omega_n^2\left(1-\zeta^2\right)$ $\omega_n^2=\sigma^2+\omega_d^2$ $\tan(\phi)=\frac{\omega_d}{\sigma}$ Complex Conjugate Poles at $s=-\sigma\pm j\omega_d$
2nd order system $(\zeta=0)$ **Undamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{s^2+\omega_n^2}$	$y_\delta(t)=K\omega_n\sin(\omega_nt)u(t)$ $y_{step}(t)=K(1-\cos\omega_nt)u(t)$	K : DC Gain ω_n : Undamped Natural Frequency Imaginary Conjugate Poles at $s=\pm j\omega_n$

2^{nd} order system RESONANCE $\left(0 \le \zeta < 1/\sqrt{2}\right)$ ➔ *RESONANCE FREQUENCY :* $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ *RESONANCE PEAK :* $M_r = \left|G\left(j\omega_r\right)\right| = \dfrac{K}{2\zeta\sqrt{1-\zeta^2}}$

Trigonometric Identities

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\theta) = 0.5\left(e^{j\theta} + e^{-j\theta}\right)$	$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
$\sin(\theta) = -0.5j\left(e^{j\theta} - e^{-j\theta}\right)$ $\sin^2(\theta) + \cos^2(\theta) = 1$	$\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta) = 0.5\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$	$\cos(\alpha)\cos(\beta) = 0.5\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
$\sin^2(\theta) = 0.5\left[1 - \cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta) = 0.5\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$
$\cos^2(\theta) = 0.5\left[1 + \cos(2\theta)\right]$	$C\cos(\theta) - S\sin(\theta) = \sqrt{C^2 + S^2}\cos\left[\theta + \tan^{-1}\left(\dfrac{S}{C}\right)\right]$

Complex Unit (j) ➔ $\left(j = \sqrt{-1} = e^{j\pi/2} = e^{j90^\circ}\right)$ $\left(-j = \dfrac{1}{j} = e^{-j\pi/2} = e^{-j90^\circ}\right)$ $\left(j^2 = -1\right)$

Definitions of Basic Functions

Rectangle:

$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1; & -T/2 \le t < T/2 \\ 0; & \text{elsewhere} \end{cases}$$

Triangle:

$$\text{tri}\left(\frac{t}{T}\right) = \begin{cases} 1 - |t|/T; & |t| \le T \\ 0; & |t| > T \end{cases}$$

Sine Cardinal:

$$\text{sinc}\left(\frac{t}{T}\right) = \begin{cases} \dfrac{\sin(\pi t/T)}{\pi t/T}; & t \ne 0 \\ 1; & t = 0 \end{cases}$$

Signum:

$$\text{sgn}(t) = \begin{cases} 1; & t \ge 0 \\ -1; & t < 0 \end{cases}$$

Unit Impulse:

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \ne 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

Unit Step:

$$u(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$