Outline of Lecture

1. Linear Time Invariant (LTI) Systems
   - System Models in the s-Domain - Transfer Functions
2. Standard Forms of Transfer Functions
   - First Order Systems
   - Second Order Systems
3. Four Types of Second Order Systems
   - Underdamped Systems $\zeta < 1$
   - Critically damped Systems $\zeta = 1$
   - Overdamped Systems $\zeta > 1$
   - Undamped Systems $\zeta = 0$
4. Relationship between Poles, ζ and ω_n

Linear Time Invariant (LTI) Systems

1. System Model in Time Domain

- Linear time invariant (LTI) systems are modeled in the time domain by linear constant-coefficient differential equations which have the general form :

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{1}$$

  where $x(t)$ is the input, $y(t)$ is the output, and a_k, b_k are real constants. We call this an N^{th}-order system in accordance with the order of the highest derivative of $y(t)$, assuming that $a_N \neq 0$.

- Many real world systems can be approximately modeled using (1). Examples are circuits involving R, L and C, spring-damper systems, DC motors, etc.

2. System Models in the s-Domain (also called Transfer Functions)

The LTI model in (1) can be converted to a s- domain model using Laplace transform.

Applying Laplace transform to (1) and assuming zero initial conditions :

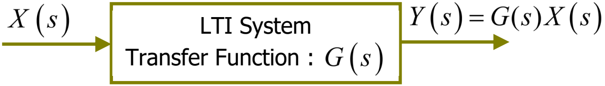
$$G(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{N} a_k s^k}{\sum_{k=0}^{M} b_k s^k} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{a_N s^N + b_{N-1} s^{N-1} + \cdots + a_0} \qquad (2)$$

$G(s)$ is called the transfer function of the LTI system where $X(s) = \mathcal{L}[x(t)]$ and $Y(s) = \mathcal{L}[y(t)]$, $x(t)$ and $y(t)$ being inputs and outputs. $G(s)$ is an N^{th}-order LTI system in s-domain.

The differential equation in (1) has become an algebraic equation in (2). It is no longer a function of time t but a function of s which is complex.

The input-output model of the LTI system in the s-domain is thus :

$$G(s) = \frac{Y(s)}{X(s)}$$



Often $G(s)$ in (2) is factorized into :

$$G(s) = K\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots\left(\frac{s}{p_N}+1\right)} \qquad K = \frac{b_0}{a_0} \quad (3)$$

$$\text{or} \quad G(s) = K'\frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)} \qquad K' = \frac{b_M}{a_N} \quad (4)$$

For $k = 1, 2, \ldots, N$,

- $-p_k$ are the roots of the denominator of G(s) in (2) or equivalently, (4) ie they are the values of s that satisfy the equation :

$$a_N s^N + a_{N-1}s^{N-1} + \ldots + a_0 = 0.$$

- $G(s)|_{s=-p_k} = \infty$
- $-p_k$ are called the poles of $G(s)$.
- For an N^{th}-order $G(s)$, there are N number of poles of $G(s)$. These poles may be real or complex and either distinct or repeated.

For $k = 1, 2, \ldots, M$,

- $-z_k$ are the roots of the numerator of $G(s)$ in (2) or equivalently, (4) ie they are the values of s that satisfy the equation :

$$b_M s^M + b_{M-1} s^{M-1} + \ldots + b_0 = 0.$$

- $G(s)|_{s=-z_k} = 0$
- $-z_k$ are called the zeros of $G(s)$.
- The LTI system of $G(s)$ in (2) is said to have N poles and M zeros.
- For real practical systems, $M < N$. The difference $(N - M)$ is called the pole-zero excess.
- The role of poles and zeros in the response of LTI systems will be clear in the next set of lectures.

Example 1

A car travels along the road with an engine thrust of $x(t)$ Newtons. The car has a mass of m kg and its displacement from a reference position can be modeled as $y(t)$. The frictional force acting on the tyres may be assumed proportional to the car's velocity.

- *Write down the expression for the acceleration of the car.*
- *Find the transfer function between the car's displacement and thrust.*

Apply Newtons's second law of motion : $F = ma$.
Since velocity is $\frac{dy}{dt}$ and acceleration is $\frac{d^2y}{dt^2}$, then

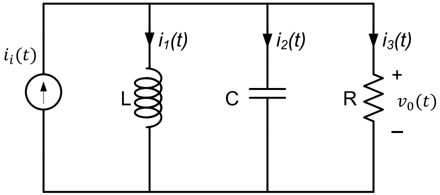
$$x(t) - k\frac{dy}{dt} = m\frac{d^2y}{dt^2}$$

where k is the proportionality constant. Taking Laplace transform and assuming zero initial conditions ie car at rest at reference point,

$$\begin{aligned} X(s) - ksY(s) &= ms^2Y(s) \\ \frac{Y(s)}{X(s)} &= \frac{1}{s(ms+k)} \quad \ldots \text{transfer function} \end{aligned}$$

Example 2 (Parallel RLC Circuit)

For the circuit on the right, find the transfer function between the output $v_0(t)$ *and input current* $i_i(t)$.



Applying KCL :

$$
\begin{aligned}
i_i(t) &= i_1(t) + i_2(t) + i_3(t) \\
&= \frac{1}{L}\int_0^t v_0(\tau)d\tau + C\frac{dv_0(t)}{dt} + \frac{v_0(t)}{R} \\
\frac{di_i(t)}{dt} &= \frac{1}{L}v_0(t) + C\frac{d^2v_0(t)}{dt^2} + \frac{1}{R}\frac{dv_0(t)}{dt}
\end{aligned}
$$

Taking Laplace transform and assuming zero initial conditions :

$$
\begin{aligned}
sI_i(s) &= \frac{1}{L}V_0(s) + Cs^2V_0(s) + \frac{1}{R}sV_0(s) \\
\frac{V_0(s)}{I_i(s)} &= \frac{RLs}{s^2RLC + sL + R} \quad \ldots\text{transfer function}
\end{aligned}
$$

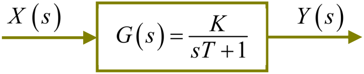
Exercise 1 (Write your answer in this space)

A motor shaft rotates at $\omega(t)$ *rad/s. Write down the expression for the angular position* $\theta(t)$ *radians of the shaft. Find the transfer function,* $\dfrac{\Theta(s)}{\Omega(s)}$ *where* $\Theta(s) = \mathcal{L}[\theta(t)]$ *and* $\Omega(s) = \mathcal{L}[\omega(t)]$.

Standard Forms of Transfer Functions

1. First Order Systems

- Time domain model : $T\dfrac{dy}{dt} + y(t) = Kx(t)$.
- s-domain model or transfer function : $G(s) = \dfrac{Y(s)}{X(s)} = \dfrac{K}{sT+1}$



- Pole : $s = -\dfrac{1}{T}$, no zero for this first order system.
- Parameters are K and T where K is the steady state / DC / static gain and T is the time constant of $G(s)$.
- Common examples of first order systems (try to derive them) :

Series RC Circuit

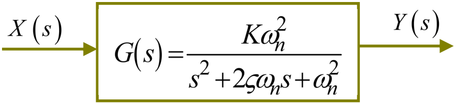
$$G(s) = \frac{V_c(s)}{V_{in}(s)} = \frac{1}{RCs+1}$$

Series RL Circuit

$$G(s) = \frac{I_L(s)}{I_{in}(s)} = \frac{R}{sL+R}$$

2. Second Order Systems

- Time domain model : $\dfrac{d^2y}{dt^2} + 2\zeta\omega_n \dfrac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t).$
- Transfer function : $G(s) = \dfrac{Y(s)}{X(s)} = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$



- Poles (2 in total) obtained as follows :

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\begin{aligned} s_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \end{aligned} \tag{5}$$

- Parameters are : K, the steady state gain, ζ the damping ratio, and ω_n the natural frequency.

- Common examples of second order systems :

RLC circuit in Ex. 2 Slide 7

$$\frac{V_0(s)}{I_i(s)} = \frac{LRs}{LCRs^2 + Ls + R}$$

mass-spring-damper system

$$\frac{P(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

- Damping ratio ζ is related to how energy dissipates in the system. Higher damping ratio implies faster energy dissipation while lower damping ratio implies slower dissipation.
- Natural frequency ω_n is the frequency at which a system vibrates when it is not excited by an external energy source.
- Resonant frequency is different from natural frequency but they are related. See later how the resonant frequency is derived.
- The steady state gain K is also given by

$$K = \lim_{s \to 0} G(s) = G(s)\big|_{s=0}.$$

This formula applies to all LTI systems.

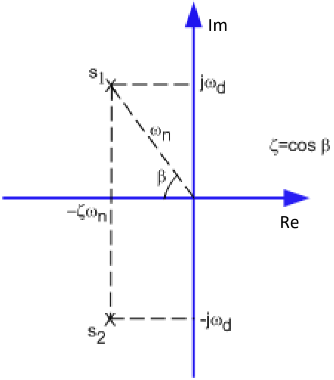
Four Types of Second Order Systems

There are 4 possible types of poles that 2nd order systems can have, depending on the value of the damping ratio, ζ.

1. Underdamped system where $\zeta < 1$

   From (5), poles are complex conjugate of one another :

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \\ &= -\sigma \pm j\omega_d \end{aligned}$$



Poles on complex s-plane

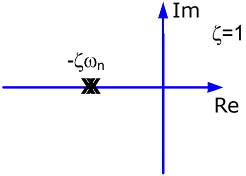
- Transfer function :

$$G(s) = \frac{K\omega_n^2}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)}$$

- ω_d is the damped natural frequency.
- $\zeta = \cos\beta$ - can be deduced from the poles.
- Magnitude of poles : $|s_{1,2}| = \omega_n$

2. Critically damped system where $\zeta = 1$

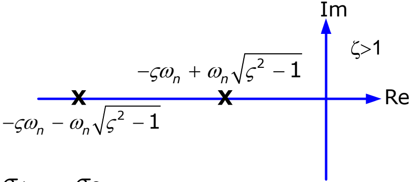
From (5), poles are real and repeated :



$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\zeta\omega_n, -\zeta\omega_n \\ |s_{1,2}| &= \omega_n \quad \because \zeta = 1 \end{aligned}$$

Transfer function : $G(s) = \dfrac{K\omega_n^2}{(s+\omega_n)^2}$

3. Overdamped system where $\zeta > 1$



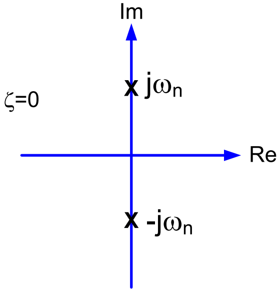
From (5), poles are real and distinct at

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\sigma_1,\ -\sigma_2 \\ |s_{1,2}| &= |-\sigma_1|, |-\sigma_2| \neq \omega_n \\ G(s) &= \frac{K\omega_n^2}{(s+\sigma_1)(s+\sigma_2)} \end{aligned}$$

4. Undamped system where $\zeta = 0$

From (5), poles are complex conjugate and purely imaginary at

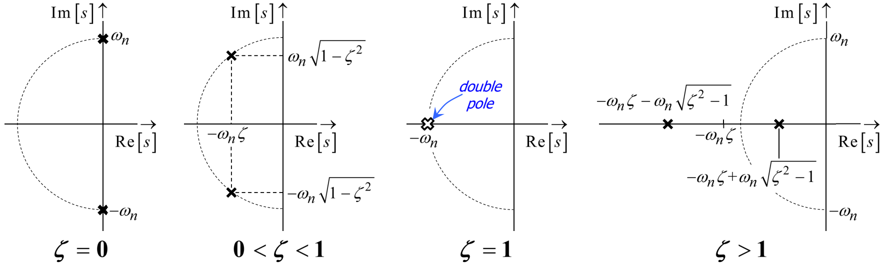
$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= +j\omega_n, -j\omega_n \\ |s_{1,2}| &= \omega_n \end{aligned}$$



Transfer function : $G(s) = \dfrac{K\omega_n^2}{(s^2 + \omega_n^2)}$

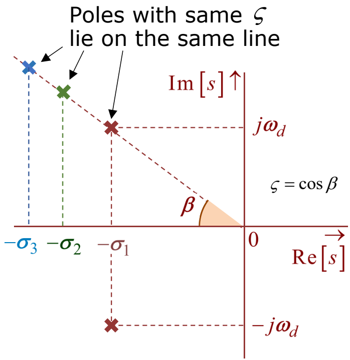
With zero damping, the response of this type of second order system is oscillatory, as you will see later.

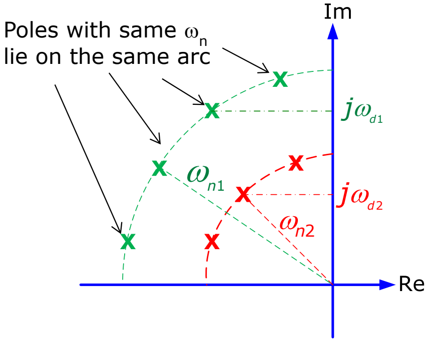
In summary, the pole diagrams for the 4 types of 2nd order systems are :



$\zeta = 0:$	Poles are imaginary conjugate pairs $\rightarrow$ $s_{1,2} = \pm j\omega_n$ System is said to be undamped.
$0 < \zeta < 1:$	Poles are complex conjugate pairs $\rightarrow$ $s_{1,2} = -\sigma \pm j\omega_d$ System is said to be underamped.
$\zeta = 1:$	Poles are real and repeated $\rightarrow$ $s_{1,2} = -\omega_n, -\omega_n$ System is said to be critically damped.
$\zeta > 1:$	Poles are real and distinct $\rightarrow$ $s_{1,2} = -\sigma_1, -\sigma_2$ System is said to be overdamped.

Relationship between Poles, ζ and ω_n





- Poles with same damping ratio, ζ, are located on the same line with $\beta = \cos^{-1}\zeta$.
- Poles with the same ω_n are located on the same arc with radius of ω_n.

Example 3

For the RLC circuit in Example 2 on Slide 7, assume that $R = 2\Omega$, $L = 2$ H, and $C = 1$ F. Determine the damping ratio of the RLC circuit. Find its poles and zeros.

From Slide 7, the transfer function of the circuit is :

$$\begin{aligned} G(s) &= \frac{RLs}{s^2RLC + sL + R} = \frac{4s}{4s^2 + 2s + 2} \\ &= \frac{s}{s^2 + 0.5s + 0.5} \quad \text{compare denominator with } s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}$$

Then $2\zeta\omega_n = 0.5$ and $\omega_n^2 = 0.5$ or $\omega_n = \sqrt{0.5}$.

$$\text{Hence} \quad \zeta = \frac{0.5}{2\sqrt{0.5}} = \frac{\sqrt{0.5}}{2} < 1$$

Hence the RLC circuit is underdamped. One zero at $s = 0$ and poles at

$$s_{1,2} = \frac{-0.5 \pm \sqrt{0.5^2 - 4(1)(0.5)}}{2} = -0.25 \pm 0.66j$$

Exercise 2 (Write your answer in this space)

The transfer function of a LTI system has a zero at $s = +2$ *and poles at* $s_{1,2} = \pm 3j$. *What is the transfer function,* $G(s)$, *for this system?*