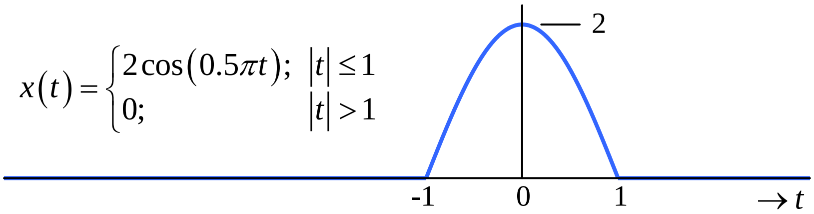
EE2023 TUTORIAL 3 (SOLUTIONS)

Solution to Q.1

$$x(t) = \begin{cases} 2\cos(0.5\pi t); & |t| \le 1 \\ 0; & |t| > 1 \end{cases}$$



(a)

Method 1: By applying direct Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt$$

$$= \int_{-\infty}^{-1} 0\exp(-j2\pi ft)\,dt + \int_{-1}^{1} 2\cos(0.5\pi t)\exp(-j2\pi ft)\,dt + \int_{1}^{\infty} 0\exp(-j2\pi ft)\,dt$$

$$= 2\int_{-1}^{1} \underbrace{\cos(0.5\pi t)\cos(2\pi ft)}_{even\ \ function\ of\ t}\,dt - \cancel{j2\int_{-1}^{1} \underbrace{\cos(0.5\pi t)\sin(2\pi ft)}_{odd\ \ function\ of\ t}\,dt}$$

$$= 4\int_{0}^{1} \cos(0.5\pi t)\cos(2\pi ft)\,dt$$

$$\cdots\ applying\ \cos(A)\cos(B) = \frac{1}{2}\cos\left(\frac{A-B}{2}\right) + \frac{1}{2}\cos\left(\frac{A+B}{2}\right)$$

$$= 2\int_{0}^{1} \cos\big((2\pi f - 0.5\pi)t\big) + \cos\big((2\pi f + 0.5\pi)t\big)\,dt$$

$$= 2\left[\frac{\sin\big((2\pi f - 0.5\pi)t\big)}{2\pi f - 0.5\pi} + \frac{\sin\big((2\pi f + 0.5\pi)t\big)}{2\pi f + 0.5\pi}\right]_0^1$$

$$= 2\left(\frac{\sin(2\pi f - 0.5\pi)}{2\pi f - 0.5\pi} + \frac{\sin(2\pi f + 0.5\pi)}{2\pi f + 0.5\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f - 0.5} + \frac{\cos(2\pi f)}{2f + 0.5}\right) = \frac{2\cos(2\pi f)}{\pi\left(0.25 - 4f^2\right)}$$

Method 2: By applying Fourier transform properties:

The half-cosine pulse can be modeled as $x(t) = 2\cos(0.5\pi t)\cdot \mathrm{rect}(0.5t)$

$$\Im\{2\cos(0.5\pi t)\} = \delta(f - 0.25) + \delta(f + 0.25)$$

$$\Im\{\mathrm{rect}(0.5t)\} = 2\mathrm{sinc}(2f)$$

Applying the 'Multiplication in time-domain' property of the Fourier transform

$$\left[x(t) = \underbrace{2\cos(0.5\pi t)\cdot \mathrm{rect}(0.5t)}_{\substack{\text{Multiplication in} \\ \text{time-domain}}}\right] \rightleftarrows \left[X(f) = \underbrace{\Im\{2\cos(0.5\pi t)\} * \Im\{\mathrm{rect}(0.5t)\}}_{\substack{\text{Convolution in} \\ \text{frequency-domain}}}\right]$$

we get

$$
\begin{aligned}
\therefore X(f) &= \left[\delta(f-0.25)+\delta(f+0.25)\right] * 2\operatorname{sinc}(2f) \\
&= 2\operatorname{sinc}\big(2(f-0.25)\big) + 2\operatorname{sinc}\big(2(f+0.25)\big) \\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{\pi(2f-0.5)} + \frac{\sin(2\pi f+0.5\pi)}{\pi(2f+0.5)}\right) \\
&= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5} + \frac{\cos(2\pi f)}{2f+0.5}\right) \\
&= \frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)} \quad \cdots\cdots \text{ Same result obtained by } \textbf{Method 1}
\end{aligned}
$$

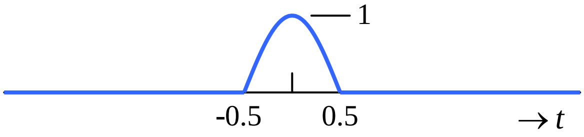
(b) From Part (a): $X(f) = \dfrac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}$

Define an intermediate function $\tilde{y}(t) = 0.5x(2t)$

Applying the ***scaling property***:

$$
\begin{aligned}
\tilde{Y}(f) &= \Im\{0.5x(2t)\} \\
&= 0.5\left[\frac{1}{2}X\left(\frac{f}{2}\right)\right] = \frac{1}{4}X\left(\frac{f}{2}\right) \quad \cdots\cdots \quad (*)
\end{aligned}
$$

$\tilde{y}(t) = 0.5x(2t)$

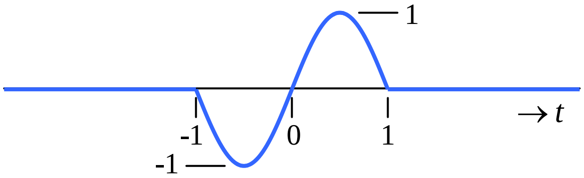


Now, $y(t) = \tilde{y}(t-0.5) - \tilde{y}(t+0.5)$

Applying the ***time-shifting property***:

$$
\begin{aligned}
Y(f) &= \tilde{Y}(f)\exp\left(-j2\pi f\left(\frac{1}{2}\right)\right) \\
&\quad -\tilde{Y}(f)\exp\left(j2\pi f\left(\frac{1}{2}\right)\right) \quad \cdots\cdots\cdots \quad (**)
\end{aligned}
$$

$y(t) = \tilde{y}(t-0.5) - \tilde{y}(t+0.5)$



Substituting $(*)$ into $(**)$, we get

$$
\begin{aligned}
Y(f) &= \frac{1}{4}X\left(\frac{f}{2}\right)\exp(-j\pi f) - \frac{1}{4}X\left(\frac{f}{2}\right)\exp(j\pi f) \\
&= \frac{1}{4}X\left(\frac{f}{2}\right)\left\{\underbrace{\left[\cos(\pi f) - j\sin(\pi f)\right]}_{\exp(-j\pi f)} - \underbrace{\left[\cos(\pi f) + j\sin(\pi f)\right]}_{\exp(j\pi f)}\right\} \\
&= -j\frac{1}{2}X\left(\frac{f}{2}\right)\sin(\pi f) \\
&= \frac{1}{j2}\underbrace{\left[\frac{2\cos(\pi f)}{\pi\left(0.25-f^2\right)}\right]}_{X(f/2)}\sin(\pi f) = \frac{1}{j2}\left[\frac{\sin(2\pi f)}{\pi\left(0.25-f^2\right)}\right]
\end{aligned}
$$

OBSERVATION: $y(t)$ is real & odd and $Y(f)$ is pure imaginary & odd.

Solution to Q.2

(a) Fig.Q.2(a)(I) is a plot of $u(t-\gamma)$ against t:

$$\left[u(t)=\begin{cases}1; & t\geq 0\\ 0; & t<0\end{cases}\right] \xrightarrow[t \text{ to } t-\gamma]{\textit{change}} \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\geq 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & t\geq \gamma\\ 0; & t<\gamma\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } t \text{ while}\\ \text{treating } \gamma \text{ as a parameter}\\ \text{Fig.Q2(a)(I)}}}$$

Fig.Q.2(a)(II) is a plot of $u(t-\gamma)$ against γ:

$$\left[u(t)=\begin{cases}1; & t\geq 0\\ 0; & t<0\end{cases}\right] \xrightarrow[t \text{ to } t-\gamma]{\textit{change}} \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\geq 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & \gamma\leq t\\ 0; & \gamma>t\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } \gamma \text{ while}\\ \text{treating } t \text{ as a parameter}\\ \text{Fig.Q2(a)(II)}}}$$

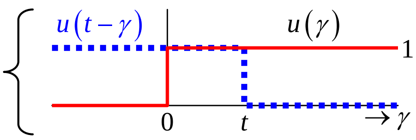
On the γ-axis, since $x(\gamma)=x(\gamma)u(t-\gamma)$ in the integration interval $(-\infty,t]$, we have

$$\int_{-\infty}^{t} x(\gamma)d\gamma = \underbrace{\int_{-\infty}^{t} x(\gamma)u(t-\gamma)d\gamma = \int_{-\infty}^{\infty} x(\gamma)u(t-\gamma)d\gamma}_{\because\, u(t-\gamma)=0 \text{ when } \gamma>t} = x(t)\ast u(t)$$

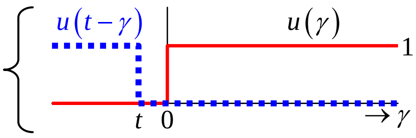
(b) $[\cos(t)u(t)]\ast u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)d\gamma$

Consider the term $u(\gamma)u(t-\gamma)$ in the integrand as a function of γ.

When $t\geq 0$, we have $u(\gamma)u(t-\gamma)=\begin{cases}1; & 0\leq\gamma\leq t\\ 0; & \textit{elsewhere}\end{cases}$



When $t<0$, we have $u(\gamma)u(t-\gamma)=0 \quad \forall\gamma$



$$\therefore \cos(t)u(t)\ast u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)d\gamma=\begin{cases}\int_0^t\cos(\gamma)d\gamma; & t\geq 0\\ 0; & t<0\end{cases}$$

$$=\begin{cases}\sin(t); & t\geq 0\\ 0; & t<0\end{cases}$$

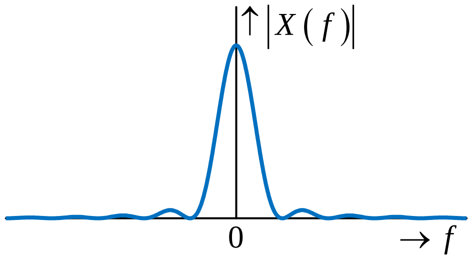
$$=\sin(t)u(t)$$

Solution to Q.3

Spectrum of $x(t)$:

The given triangular pules may be expressed as $x(t) = \alpha \cdot \mathrm{tri}\left(\frac{t}{\alpha}\right)$. Applying the Fourier transform pair $\mathrm{tri}\left(\frac{t}{T}\right) \rightleftarrows T\mathrm{sinc}^2(Tf)$, it is easy to see that

$$X(f) = \Im\left\{\alpha \cdot \mathrm{tri}\left(\frac{t}{\alpha}\right)\right\} = \alpha^2 \mathrm{sinc}^2(\alpha f).$$

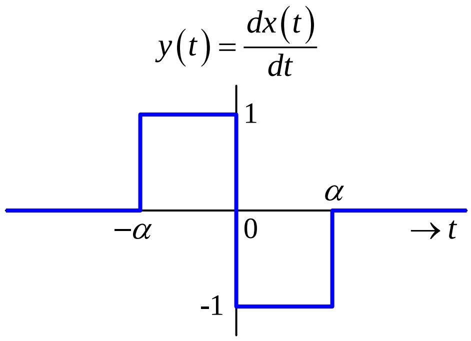


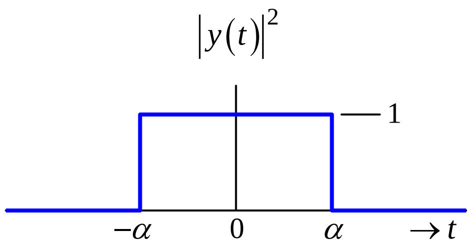
Hence,

Magnitude spectrum: $|X(f)| = \alpha^2 \mathrm{sinc}^2(\alpha f)$

Phase spectrum: $\angle X(f) = 0$.

ESD and Energy of $y(t) = \frac{dx(t)}{dt}$:





Applying the 'Differentiation in Time-Domain' property of the Fourier transform:

$$Y(f) = j2\pi f \cdot X(f) = j2\pi f \alpha^2 \mathrm{sinc}^2(\alpha f)$$

Hence,

ESD:
$$\begin{aligned} E_y(f) &= |Y(f)|^2 = Y(f)Y^*(f) \\ &= 4\pi^2 f^2 \alpha^4 \mathrm{sinc}^4(\alpha f) \end{aligned} .$$

Total Energy:
$$E = \overbrace{\int_{-\infty}^{\infty} E_y(f)\,df = \underbrace{\int_{-\infty}^{\infty} |y(t)|^2\,dt = 2\alpha}_{\text{By inspection of the plot of } |y(t)|^2}}^{\text{Rayleigh energy theorem}}$$

Solution to Q.4

Spectrum of $\boldsymbol{x}(\boldsymbol{t})$: $\quad X(f) = \exp(-\alpha|f|); \quad \alpha > 0$

ESD of $\boldsymbol{x}(\boldsymbol{t})$: $\quad E_x(f) = |X(f)|^2 = \exp(-2\alpha|f|)$

(a) ***Energy of $x(t)$ contained within a bandwidth of B:***

$$e(B) = \int_{-B}^{B} E_x(f)\,df = 2\int_0^B \exp(-2\alpha f)\,df = 2\left[\frac{\exp(-2\alpha f)}{-2\alpha}\right]_0^B = \frac{1}{\alpha}\left[1-\exp(-2\alpha B)\right] \quad \cdots\cdots \ (\clubsuit)$$

Total energy of $x(t)$ is equal to energy of $x(t)$ contained within a bandwidth of ∞.

Substituting $B=\infty$ into $(\clubsuit)$, we get

$$e(\infty) = \int_{-\infty}^{\infty} E_x(f)\,df = \frac{1}{\alpha} \quad \cdots\cdots \ \text{TOTAL ENERGY}$$

Let W denote the 99% energy containment bandwidth of $x(t)$. Then

Energy contained in bandwidth $W = 0.99 \times$ Total Energy

$$\to \ e(W) = 0.99 \times e(\infty)$$

$$\to \ \frac{1}{\alpha}\left[1-\exp(-2\alpha \cdot W)\right] = 0.99 \times \frac{1}{\alpha}$$

$$\to \ \exp(2\alpha W) = 100$$

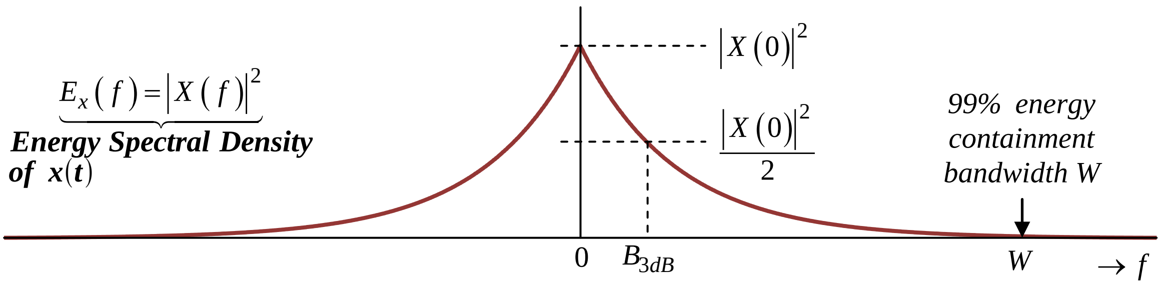
$$\to \ W = \frac{1}{\alpha}\ln(10) \ \text{Hz}$$

(b) ***Let B_{3dB} denote the 3dB bandwidth of $x(t)$. Then***

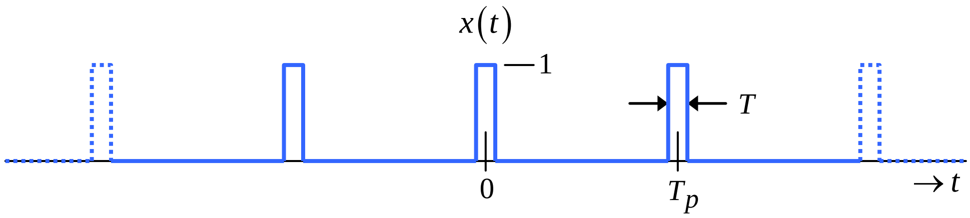
$$\frac{E_x(B_{3dB})}{E_x(0)} = \frac{1}{2} \quad \to \quad \frac{\exp(-2\alpha B_{3dB})}{\exp(0)} = \frac{1}{2} \quad \to \quad B_{3dB} = \frac{1}{2\alpha}\ln(2) \ \text{Hz}$$

Percent energy contained within the 3dB bandwidth:

$$\frac{e(B_{3dB})}{e(\infty)} \times 100 = \frac{\cancel{\frac{1}{\alpha}}\left[1-\exp\left(-\cancel{2\alpha}\frac{\ln(2)}{\cancel{2\alpha}}\right)\right]}{\cancel{\frac{1}{\alpha}}} \times 100 = 50\%$$



Solution to Q.5



(a) Expressing $x(t)$ as $x(t) = A\text{rect}\left(\frac{t}{T}\right) * \sum_n \delta\left(t - nT_p\right)$, the Fourier transform of $x(t)$ is obtained as

$$X(f) = AT\text{sinc}(fT) \times \frac{1}{T_p}\sum_k \delta\left(f - \frac{k}{T_p}\right) = \sum_k \underbrace{A\frac{T}{T_p}\text{sinc}\left(k\frac{T}{T_p}\right)}_{c_k}\delta\left(f - \frac{k}{T_p}\right)$$

Therefore,

$$\textbf{Power Spectral Density of } x(t):\ P_x(f) = \sum_{k=-\infty}^{\infty} A^2\frac{T^2}{T_p^2}\text{sinc}^2\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$

(b) The average power of $x(t)$ can be computed using

$$P = \int_{-\infty}^{\infty} P_x(f)df = \sum_{k=-\infty}^{\infty} \left|c_k\right|^2 = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} \left|x(t)\right|^2 dt\,.$$

Based on the $x(t)$ given, and the expressions for c_k and $P_x(f)$ found in Part (a), it would be easier to compute P in the time-domain, i.e.

$$\textbf{Average Power of } x(t):\ P = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} \left|x(t)\right|^2 dt = \frac{A^2}{T_p}\int_{-0.5T}^{0.5T} dt = A^2\frac{T}{T_p}$$

(c) Using the Fourier series coefficients, c_k, and average power, P, of $x(t)$ found in Parts-(a) and (b), respectively, the 99% power containment bandwidth of $x(t)$ is given by $W = \frac{K}{T_p}(\text{Hz})$ where K satisfies:

$$\sum_{k=-(K-1)}^{K-1} \left|c_k\right|^2 < 0.99P \le \sum_{k=-K}^{K} \left|c_k\right|^2$$

.

Solution to S.1

(a) $x(t)=\cos(2\pi f_c t)u(t)$

$$\begin{cases} X(f)=\Im\{\cos(2\pi f_c t)\}*\Im\{u(t)\}=\frac{1}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right]*\left[\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)\right] \\ \qquad =\frac{1}{4}\left[\frac{1}{j\pi(f-f_c)}+\delta(f-f_c)+\frac{1}{j\pi(f+f_c)}+\delta(f+f_c)\right] \\ \qquad =\frac{1}{4}\left[\delta(f-f_c)+\delta(f+f_c)\right]-\frac{jf}{2\pi(f^2-f_c^2)} \end{cases}$$

(b) $x(t)=\sin(2\pi f_c t)u(t)$

$$X(f)=\frac{j}{4}\left[\delta(f+f_c)-\delta(f-f_c)\right]-\frac{f_c}{2\pi(f^2-f_c^2)} \quad \cdots \quad (\text{Same approach as part } (a))$$

(c) $x(t)=\exp(-\alpha t)\cos(\omega_c t)u(t);\quad \alpha>0$

$$\begin{cases} X(\omega)=\Im\{\exp(-\alpha t)\cos(\omega_c t)u(t)\}=\Im\left\{\frac{1}{2}\left[\exp(-\alpha t+j\omega_c t)+\exp(-\alpha t-j\omega_c t)\right]u(t)\right\} \\ \qquad =\frac{1}{2}\int_0^\infty\left[\exp(-\alpha t+j\omega_c t)+\exp(-\alpha t-j\omega_c t)\right]\exp(-j\omega t)\,dt \\ \qquad =\frac{1}{2}\int_0^\infty \exp\left[(-\alpha-j\omega+j\omega_c)t\right]+\exp\left[(-\alpha-j\omega-j\omega_c)t\right]dt \\ \qquad =\frac{1}{2}\left[\frac{\exp\left[(-\alpha-j\omega+j\omega_c)t\right]}{-\alpha-j\omega+j\omega_c}+\frac{\exp\left[(-\alpha-j\omega-j\omega_c)t\right]}{-\alpha-j\omega-j\omega_c}\right]_0^\infty \\ \qquad =\frac{1}{2}\left[\frac{1}{(\alpha+j\omega)-j\omega_c}+\frac{1}{(\alpha+j\omega)+j\omega_c}\right]=\frac{\alpha+j\omega}{(\alpha+j\omega)^2+\omega_c^2} \end{cases}$$

(d) $x(t)=\exp(-\alpha t)\sin(\omega_c t)u(t);\quad \alpha>0$

$$\begin{cases} X(\omega)=\Im\{\exp(-\alpha t)\sin(\omega_c t)u(t)\}=\Im\left\{\frac{1}{j2}\left[\exp(-\alpha t+j\omega_c t)-\exp(-\alpha t-j\omega_c t)\right]u(t)\right\} \\ \qquad =\frac{1}{j2}\int_0^\infty\left[\exp(-\alpha t+j\omega_c t)-\exp(-\alpha t-j\omega_c t)\right]\exp(-j\omega t)\,dt \\ \qquad =\frac{1}{j2}\int_0^\infty \exp\left[(-\alpha-j\omega+j\omega_c)t\right]-\exp\left[(-\alpha-j\omega-j\omega_c)t\right]dt \\ \qquad =\frac{1}{j2}\left[\frac{\exp\left[(-\alpha-j\omega+j\omega_c)t\right]}{-\alpha-j\omega+j\omega_c}-\frac{\exp\left[(-\alpha-j\omega-j\omega_c)t\right]}{-\alpha-j\omega-j\omega_c}\right]_0^\infty \\ \qquad =\frac{1}{j2}\left[\frac{1}{(\alpha+j\omega)-j\omega_c}-\frac{1}{(\alpha+j\omega)+j\omega_c}\right]=\frac{\omega_c}{(\alpha+j\omega)^2+\omega_c^2} \end{cases}$$

Solution to S.2

Start with the Fourier transform pair: $\exp(-\alpha t)u(t) \rightleftarrows \dfrac{1}{\alpha + j2\pi f}$

Applying the **duality** property of FT: $\dfrac{1}{\alpha - j2\pi t} \rightleftarrows \exp(-\alpha f)u(f)$

Applying the **differentiation** property of FT:

$$\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{1}{\alpha - j2\pi t}\right) = (j2\pi)^{n-1}(n-1)!\frac{1}{(\alpha - j2\pi t)^n}\right] \rightleftarrows (j2\pi f)^{n-1}\exp(-\alpha f)u(f)$$

$$\frac{1}{(\alpha - j2\pi t)^n} \rightleftarrows \frac{f^{n-1}}{(n-1)!}\exp(-\alpha f)u(f)$$

Applying the **duality** property of FT: $\dfrac{f^{n-1}}{(n-1)!}\exp(-\alpha t)u(t) \rightleftarrows \dfrac{1}{(\alpha + j2\pi f)^n}$

$$\therefore\ \Im^{-1}\left\{\frac{1}{(\alpha + j2\pi f)^n}\right\} = \frac{t^{n-1}}{(n-1)!}\exp(-\alpha t)u(t)$$

Solution to S.3

$$\frac{1}{2-\omega^2 + j3\omega} = -\frac{1}{(\omega - j)(\omega - 2j)} = -\frac{j}{\omega - j} + \frac{j}{\omega - 2j} = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2}$$

Given: $\Im\{\exp(-\alpha t)u(t)\} = \dfrac{1}{j\omega + \alpha}$.

$$\therefore\ \Im^{-1}\left\{\frac{1}{2-\omega^2 + j3\omega}\right\} = \exp(-t)u(t) - \exp(-2t)u(t)$$

Solution to S.4

Given: $\Im\{x(t)\} = \mathrm{rect}(\pi f)$ and $y(t) = \dfrac{dx(t)}{dt}$

Applying the **Differentiation** property of the FT: $Y(f) = \Im\left\{\dfrac{d}{dt}x(t)\right\} = j2\pi f \cdot \mathrm{rect}(\pi f)$

Applying the **Rayleigh energy theorem**:

$$E = \int_{-\infty}^{\infty}|y(t)|^2\,dt = \int_{-\infty}^{\infty}|Y(f)|^2\,dt$$

$$= \int_{-\infty}^{\infty} 4\pi^2 f^2 \mathrm{rect}^2(\pi f)\,df$$

$$= \int_{-1/(2\pi)}^{1/(2\pi)} 4\pi^2 f^2\,df = \left[\frac{4\pi^2 f^3}{3}\right]_{-1/(2\pi)}^{1/(2\pi)} = \frac{1}{3\pi}$$

Solution to S.5

Given: $\Im\left\{\frac{\pi}{\alpha}\exp\left(-2\pi\alpha|t|\right)\right\}=\frac{1}{\alpha^2+f^2}$ and $x(t)=\frac{1}{\alpha^2+t^2}$.

Applying the **duality** property of FT: $\frac{1}{\alpha^2+t^2}\rightleftarrows\frac{\pi}{\alpha}\exp\left(-2\pi\alpha|f|\right)$

$\therefore\; X(f)=\frac{\pi}{\alpha}\exp\left(-2\pi\alpha|f|\right)$

Let B be the 99% energy containment bandwidth of $x(t)=\frac{1}{\alpha^2+t^2}$. It follows that

$$0.99=\frac{\int_0^B |X(f)|^2\,df}{\int_0^\infty |X(f)|^2\,df}=\frac{\int_0^B \exp(-4\pi\alpha f)\,df}{\int_0^\infty \exp(-4\pi\alpha f)\,df}=1-\exp(-4\pi\alpha B)$$

which yields

$$\exp(-4\pi\alpha B)=0.01 \quad \text{or} \quad B=\frac{\ln(100)}{4\pi\alpha}=\frac{0.366}{\alpha}.$$

Solution to S.6

Let $X(f)$ and $X(\omega)$ denote the Fourier transform of $x(t)$ in cyclic frequency and angular frequency, respectively. Then by definition, we have

$$x(t)=\underbrace{\int_{-\infty}^{\infty}X(f)\exp(j2\pi ft)\,df}_{\text{inverse FT of } X(f)}=\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\exp(j\omega t)\,d\omega}_{\text{inverse FT of } X(\omega)}.$$

$$\text{Now:}\left\{\begin{array}{l}\Im^{-1}\{\delta(f)\}=\underbrace{\int_{-\infty}^{\infty}\delta(f)\exp(j2\pi ft)\,df}_{\substack{1\\ [\text{sifting property of } \delta(f)]}}=1\\[2em] \Im^{-1}\{2\pi\delta(\omega)\}=\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty}2\pi\delta(\omega)\exp(j\omega t)\,d\omega}_{\substack{2\pi\\ [\text{sifting property of } \delta(\omega)]}}=1\end{array}\right\}\cdots(\clubsuit)$$

Since the Fourier transform is a one-to-one transformation, we conclude from $(\clubsuit)$ that $\boldsymbol{\delta(f)=2\pi\delta(\omega)}$ if $\omega=2\pi f$.
