Outline of Lecture

Sinusoidal Responses of LTI Systems

Consider an LTI system with transfer function $G(s)$ and input $x(t) = e^{st}$ where s is a complex variable. The output $y(t)$ at steady state can be computed using the convolution integral :

$$
\begin{aligned}
y_{ss}(t) = \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} x(t) * g(t) = \lim_{t\to\infty} \int_0^t g(\tau) e^{s(t-\tau)} d\tau \\
&= \lim_{t\to\infty} \int_0^t g(\tau) e^{st} e^{-s\tau} d\tau = G(s) e^{st} \qquad (1)
\end{aligned}
$$

Equation (1) suggests that :

- If $s = j\omega$, then for an input $x(t) = e^{j\omega t}$, $y_{ss}(t) = G(j\omega)e^{j\omega t}$.
- If $s = -j\omega$, then for an input $x(t) = e^{-j\omega t}$, $y_{ss}(t) = G(-j\omega)e^{-j\omega t}$.
- If input $x(t) = A\left[e^{j\omega t} - e^{-j\omega t}\right] = 2Aj\sin\omega t$, then

$$
y_{ss}(t) = \lim_{t\to\infty} y(t) = A\left[G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}\right] \qquad (2)
$$

Since $G(s)$ is a function of the complex variable, s, then $G(j\omega)$ is a complex number which can be written as

$$G(j\omega) = M_\omega e^{j\phi_\omega} \text{ and } G(-j\omega) = M_\omega e^{-j\phi_\omega}$$

where $M_\omega > 0$ and ϕ_ω are the magnitude (also referred to as gain) and phase, respectively, of $G(j\omega)$. Then from (2),

$$\begin{aligned} y_{ss}(t) &= A\left[G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}\right] \\ &= A\left[M_\omega e^{j\phi_\omega} e^{j\omega t} - M_\omega e^{-j\phi_\omega} e^{-j\omega t}\right] \\ &= AM_\omega \left[e^{j(\omega t + \phi_\omega)} - e^{-j(\omega t + \phi_\omega)}\right] \\ &= 2jAM_\omega \sin(\omega t + \phi_\omega) \end{aligned}$$

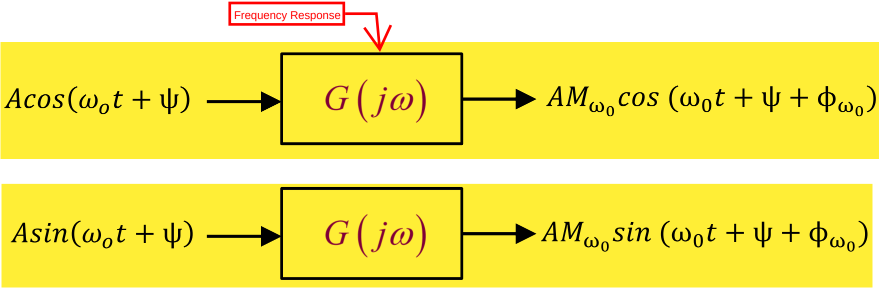
This gives the all important result that for an input $x(t) = 2Aj \sin \omega t$, the corresponding steady state output is

$$y_{ss}(t) = 2jAM_\omega \sin(\omega t + \phi_\omega)$$

Hence for an input $x(t) = A \sin \omega t$,

$$y_{ss}(t) = AM_\omega \sin(\omega t + \phi_\omega)$$

Summary of Steady State Sinusoidal Response



- Note that the inputs above have a non-zero phase ψ at $t = 0$.
- $M_{\omega_0} = |G(j\omega_0)|$ and $\phi_{\omega_0} = \angle G(j\omega_0)$ is the gain and phase of the system, respectively.
- $G(j\omega)$ is referred to as the frequency response of the LTI system.
- $G(s)$ is the transfer function of the LTI system.

Example 1

Find the steady state response of $G(s) = \frac{20}{s+1}$ *to the input* $x(t) = \sin 10t$.

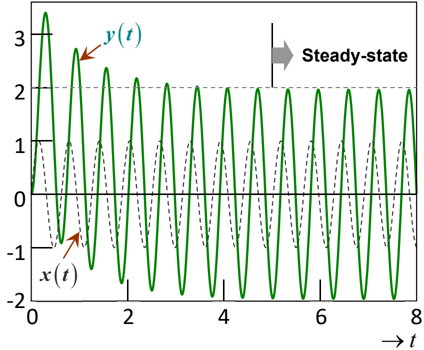
Method A : Using the formula $Y(s) = G(s)X(s)$

$$
\begin{aligned}
X(s) &= \frac{10}{s^2+100} \\
Y(s) &= G(s)X(s) = \frac{200}{(s+1)(s^2+100)} \\
&= \frac{200}{101}\frac{1}{s+1} + \frac{20}{101}\frac{10}{s^2+100} - \frac{200}{101}\frac{s}{s^2+100} \\
y(t) &= \frac{200}{101}e^{-t} + \frac{20}{101}\left[\sin 10t - 10\cos 10t\right] \\
&= \underbrace{\frac{200}{101}e^{-t}}_{\text{transient solution}} + \underbrace{\frac{20}{\sqrt{101}}\sin\left(10t - \tan^{-1} 10\right)}_{\text{steady state solution}} \quad \text{full solution} \\
\therefore\ y_{ss}(t) &= \frac{20}{\sqrt{101}}\sin\left(10t - \tan^{-1} 10\right)
\end{aligned}
$$

Method B : Using the formula $y_{ss}(t) = AM_\omega \sin(\omega t + \phi_\omega)$

Input $x(t) = \sin 10t$ has a frequency of $\omega = 10$ rad/s, amplitude $A = 1$.

$$\left.\begin{aligned} G(j\omega) &= \tfrac{20}{j\omega+1} = M_\omega e^{j\phi_\omega} \\ M_{10} &= \left|\tfrac{20}{j10+1}\right| = \tfrac{20}{\sqrt{10^2+1}} = \tfrac{20}{\sqrt{101}} \\ \phi_{10} &= \angle G(j10) = -\tan^{-1} 10 \end{aligned}\right\} y_{ss}(t) = \frac{20}{\sqrt{101}} \sin\left(10t - \tan^{-1} 10\right)$$



Full solution :

$$y(t) = \frac{200}{101}e^{-t} + \frac{20}{\sqrt{101}} \sin\left(10t - \tan^{-1} 10\right)$$

Steady state solution :

$$y_{ss}(t) = \frac{20}{\sqrt{101}} \sin\left(10t - \tan^{-1} 10\right)$$

Method B is easier than A to compute steady state sinusoidal response!

Bode Diagrams

- A Bode diagram is used to visualize the frequency response, $G(j\omega)$.
- A Bode diagram consists of two plots :
  - The magnitude plot is a plot of $|G(j\omega)|_{dB} = 20\log_{10}|G(j\omega)|$ dB versus $\log_{10}\omega$
  - The phase plot is a plot of $\angle G(j\omega)$ (in degrees) versus $\log_{10}\omega$
- The magnitude, when expressed in dB, is frequently referred to as Gain since dB is a relative measure. For instance, if v_0 and v_i are the input and output of a system, respectively, then we say that the system has a gain of $20\log_{10}|v_0/v_i|$ dB. Therefore, $\log_{10}|v_0|$ is the system gain measured with reference to a unit input, i.e. $|v_i| = 1$.
- A Bode diagram is usually plotted on semilog graph paper where the x-axis is scaled by $\log_{10}\omega$
- A Bode diagram is usually approximated by straight lines plots

The construction of straight line approximations of Bode diagrams starts with transfer functions. Examples of transfer functions :

Transfer Functions	Type
$G(s) = Ks$	Pure differentiator system
$G(s) = \frac{K}{s}$	Pure integrator system
$G(s) = \frac{K}{sT+1}$	First order system
$G(s) = \frac{K}{s}\frac{1}{Ts+1}$	First order sys with 1 integrator
$G(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	2^{nd} order underdamped system
$G(s) = \frac{K(s/z_1+1)\ldots(s/z_M+1)}{(s/p_1+1)\ldots(s/p_N+1)}$	n^{th} order system, $M < N$
$G(s) = \frac{K}{s}\frac{(s/z_1+1)\ldots(s/z_M+1)}{(s/p_1+1)\ldots(s/p_{N-1}+1)}$	n^{th} order sys with 1 integrator
$G(s) = \frac{Ks(s/z_1+1)\ldots(s/z_{M-1}+1)}{(s/p_1+1)\ldots(s/p_N+1)}$	n^{th} order sys with 1 differentiator

Bode diagrams of higher order transfer functions can be constructed from the knowledge of some basic building blocks. The basic building blocks which we will show are in the table below.

Basic building blocks	Type
$G(s) = K$	Constant DC gain, assume $K > 0$
$G(s) = K_d s$	Differentiator with gain K_d
$G(s) = \dfrac{K_i}{s}$	Integrator with gain K_i
$G(s) = (s/z_1 + 1)$	Zero factor with unity DC gain
$G(s) = \dfrac{1}{s/p_1 + 1}$	Pole factor with unity DC gain
$G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$	2^{nd} order factor with unity DC gain

Note that the DC gain is computed from $G(s)|_{s=0}$.

1. Bode diagram of $G(s) = K_{dc},\ K_{dc} > 0$

Transfer function : $G(s) = K_{dc}$

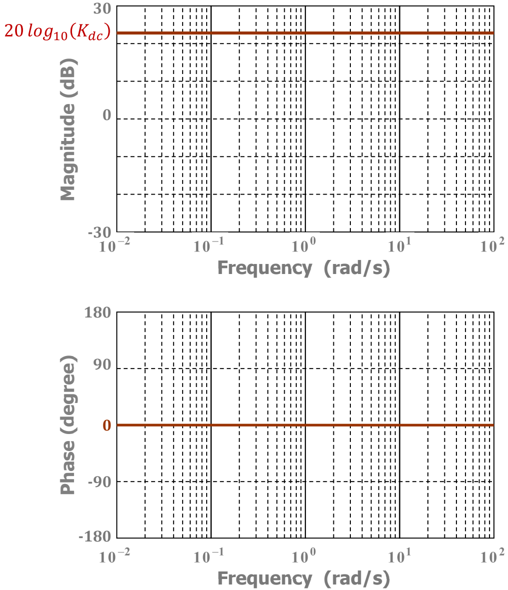
Freq. Response : $G(j\omega) = K_{dc}$

Magnitude Plot : $|G(j\omega)| = K_{dc}$

$$|G(j\omega)|_{dB} = 20\log_{10} K_{dc}$$

Phase Plot : $\angle G(j\omega) = 0^0$

The magnitude and phase plots are both straight lines with zero gradient.



2. Bode diagram of $G(s) = K_d s,\ K_d > 0$

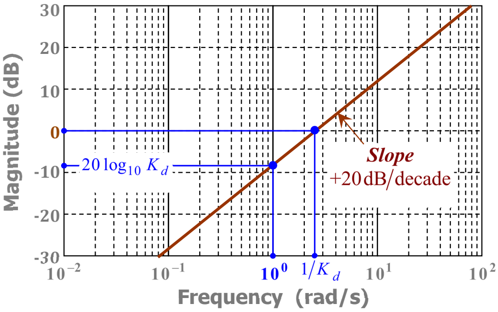
Freq. Response :

$$G(j\omega) = K_d j\omega = K_d\omega \underbrace{e^{j\frac{\pi}{2}}}_{j}$$

Magnitude Plot : $|G(j\omega)| = K_d\omega$

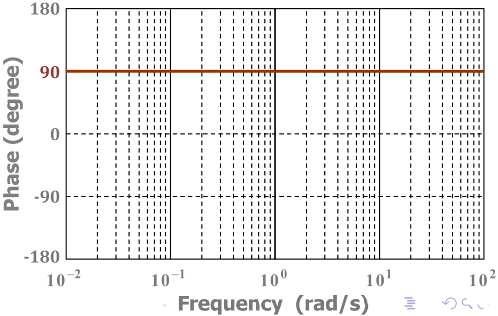
$$|G(j\omega)|_{dB} = 20\log_{10} K_d + \underbrace{20\log_{10}\omega}_{\text{20dB/decade}}$$

The magnitude plot is a straight line with slope 20 dB/decade passing through the points $(1/K_d, 0)$ and $(1, 20\log_{10} K_d)$.



Phase Plot : $\angle G(j\omega) = \frac{\pi}{2}$ or 90^0

The phase plot is a horizontal straight line located at $+90^0$.



3. Bode diagram of $G(s) = K_i/s,\ K_i > 0$

Freq. Response :

$$G(j\omega) = \frac{K_i}{j\omega} = \frac{K_i}{\omega}\underbrace{e^{-j\frac{\pi}{2}}}_{1/j}$$

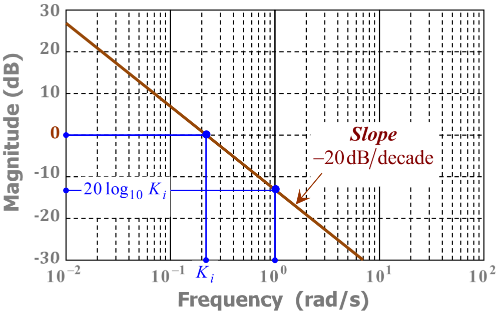
Magnitude Plot : $|G(j\omega)| = K_i/\omega$

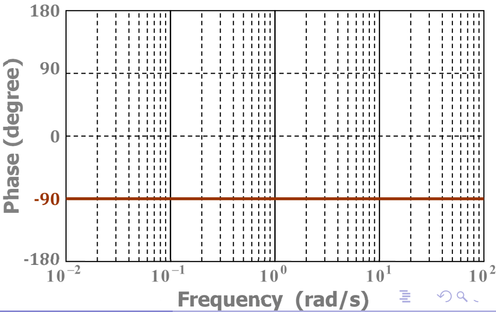
$$|G(j\omega)|_{dB} = 20\log_{10} K_i - \underbrace{20\log_{10}\omega}_{\text{-20dB/decade}}$$

The magnitude plot is a straight line with slope -20 dB/decade passing through the points $(K_i, 0)$ and $(1, 20\log_{10} K_i)$.

Phase Plot : $\angle G(j\omega) = -\frac{\pi}{2}$ or -90^0

The phase plot is a horizontal straight line located at -90^0.





4. Bode diagram of zero factor $G(s) = (s/z_m + 1),\ z_m > 0$

Freq. Response : $G(j\omega) = \frac{j\omega}{z_m} + 1$

Magnitude Plot :

$$|G(j\omega)| = \sqrt{\omega^2/z_m^2 + 1}$$

$$|G(j\omega)|_{dB} = 20\log_{10}\sqrt{\omega^2/z_m^2 + 1}$$

$\omega << z_m : |G(j\omega)|_{dB} \to 0$ dB

$\omega >> z_m$:

$$|G(j\omega)|_{dB} \to \underbrace{20\log_{10}\omega/z_m}_{\text{20dB/decade}} \text{ dB}$$

3 dB corner freq : $\omega = z_m$ rad/s

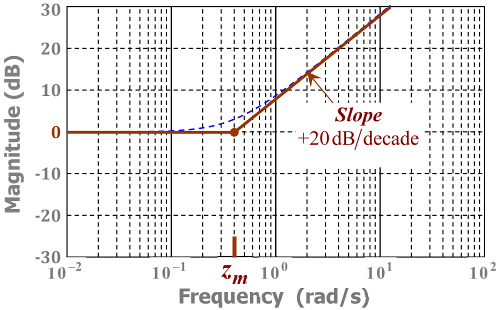
Phase Plot :

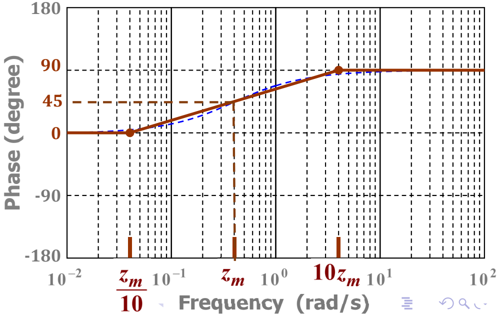
$$\angle G(j\omega) = \tan^{-1}(\omega/z_m)$$

$$\omega << z_m : \lim_{\omega\to 0}\angle G(j\omega) = 0$$

$$\omega >> z_m : \lim_{\omega\to\infty}\angle G(j\omega) = 90^0$$

At $\omega = z_m : \angle G(jz_m) = 45^0$





5. Bode diagram for pole factor $G(s) = \frac{1}{s/p_n+1}$, $p_n > 0$

Freq. Response : $G(j\omega) = \frac{1}{j\omega/p_n+1}$

Magnitude Plot :

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2/p_n^2+1}}$$

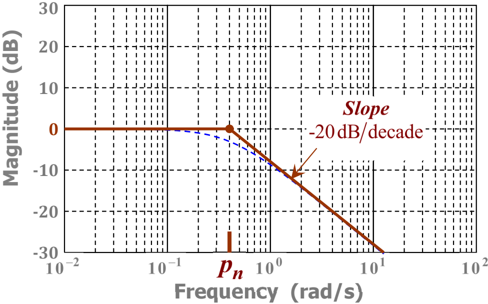
$$|G(j\omega)|_{dB} = -20\log_{10}\sqrt{\omega^2/p_n^2+1}$$

$\omega << p_n$: $|G(j\omega)|_{dB} \rightarrow 0$ dB

$\omega >> p_n$:

$$|G(j\omega)|_{dB} \rightarrow \underbrace{-20\log_{10}\omega/p_n}_{-20\text{dB/decade}} \text{ dB}$$

3 dB corner freq : $\omega = p_n$ rad/s



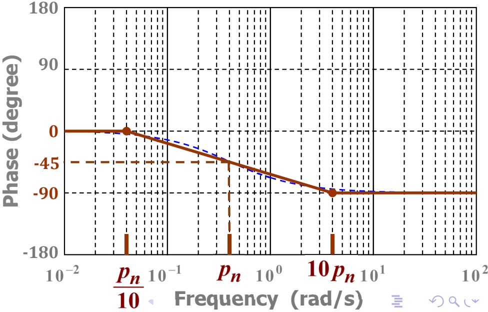
Phase Plot :

$$\angle G(j\omega) = -\tan^{-1}(\omega/p_n)$$

$$\omega << p_n : \lim_{\omega\to 0}\angle G(j\omega) = 0$$

$$\omega >> p_n : \lim_{\omega\to\infty}\angle G(j\omega) = -90^0$$

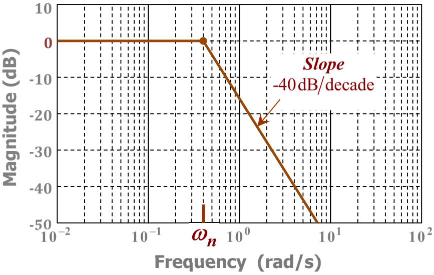
At $\omega = p_n$: $\angle G(jp_n) = -45^0$

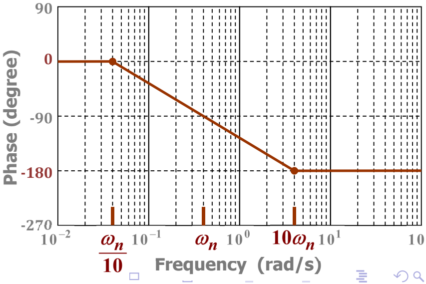


6. Bode diagram for 4 different 2^{nd} order factors : $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

- Overdamped system with $\zeta > 1$
  The 2 poles are real and distinct and $G(s) = \frac{K}{(s/\sigma_1+1)(s/\sigma_2+1)}$. $G(s)$ consists of two pole factors and its Bode diagram can be constructed from these two pole factors.

- Critically damped system with $\zeta = 1$
  In this case, $G(s) = \frac{K}{(s/\omega_n+1)^2}$ which contains two repeated pole factors. Applying the same approach as the single pole factor, the corner frequency is at ω_n but the straight line slope is now -40 dB/decade.



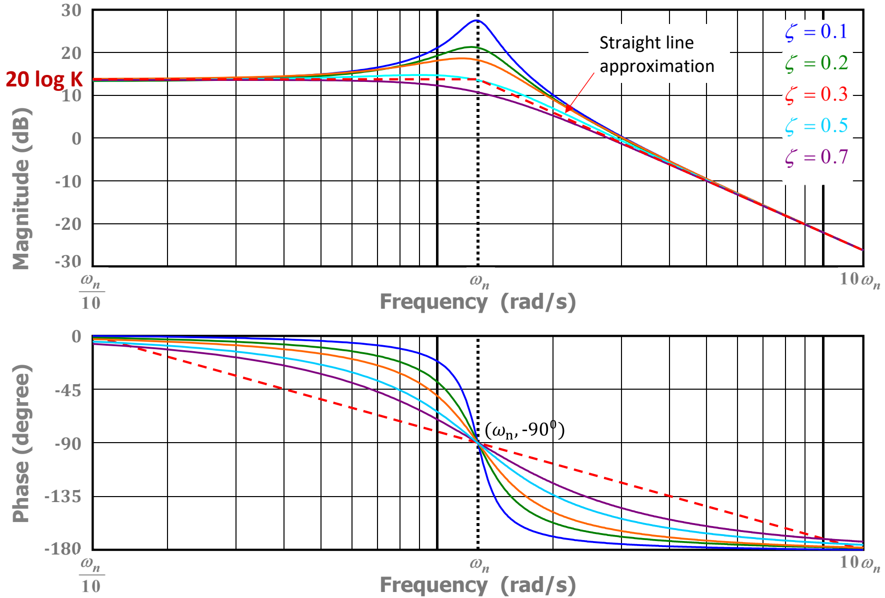


- Underdamped and zero damping systems with $0 \leq \zeta < 1$

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{(\omega_n^2 - \omega^2) + 2j\zeta\omega_n\omega} \text{ for } s = j\omega$$

Poles are complex and the single pole factor approach cannot be used.

  - The Bode diagrams vary significantly for different values of ζ.
  - The straight line approximation or asymptotes are the same as that for the critically damped system with $\zeta = 1$ in the previous slide.
  - The straight line approximated Bode diagram for $\zeta = 1$ forms the basis for the sketch of the Bode diagram for all other $\zeta < 1$.
  - Peaks are seen on the magnitude plots for $\zeta < 0.7$. These are resonance peak which can derived (see slides later). The lower the ζ, the higher the resonant peak.
  - Corner frequency of the straight line approximated system is at ω_n.
  - At $\omega = \omega_n$, the phase plots go thru $(\omega_n, -90^0)$ for all values of ζ.



30
20
20 log K
10
0
-10
-20
-30
Magnitude (dB)
$\frac{\omega_n}{10}$
ω_n
$10\omega_n$
Frequency (rad/s)
Straight line approximation
$\zeta = 0.1$
$\zeta = 0.2$
$\zeta = 0.3$
$\zeta = 0.5$
$\zeta = 0.7$
0
-45
-90
-135
-180
Phase (degree)
$(\omega_n, -90^0)$
$\frac{\omega_n}{10}$
ω_n
$10\omega_n$
Frequency (rad/s)

The magnitude plot reaches a maximum at some frequency very close to ω_n. This maximum peak increases as ζ decreases. The magnitude of the peak is called the resonance peak and the frequency at which it occurs is called the resonant frequency, denoted as ω_r.

Derivation of the resonant peak and frequency

$$\text{Magnitude : } |G(j\omega)| = \frac{K\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

Maximizing $|G(j\omega)|$ is the same as minimizing $(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2$.

$$\begin{aligned}
\text{Hence } \frac{d}{d\omega}\left[(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\right] &= -4\omega(\omega_n^2-\omega^2) + 8\zeta^2\omega_n^2\omega = 0 \\
\omega &= 0,\ \pm\omega_n\sqrt{1-2\zeta^2} = \omega_r \\
|G(j\omega_r)| &= \frac{K}{2\zeta\sqrt{1-\zeta^2}}
\end{aligned}$$

$$\left.\begin{array}{ll}
\text{Resonant frequency :} & \omega_r = \omega_n\sqrt{1-2\zeta^2} \\
\text{Resonant peak :} & M_r = \frac{K}{2\zeta\sqrt{1-\zeta^2}}
\end{array}\right\} \text{valid for } \zeta < 1/\sqrt{2}$$

Example 2

Sketch the Bode diagram of an overdamped second order system consisting of two real poles with

$$G(s) = \frac{10}{(10s+1)(s+1)}$$

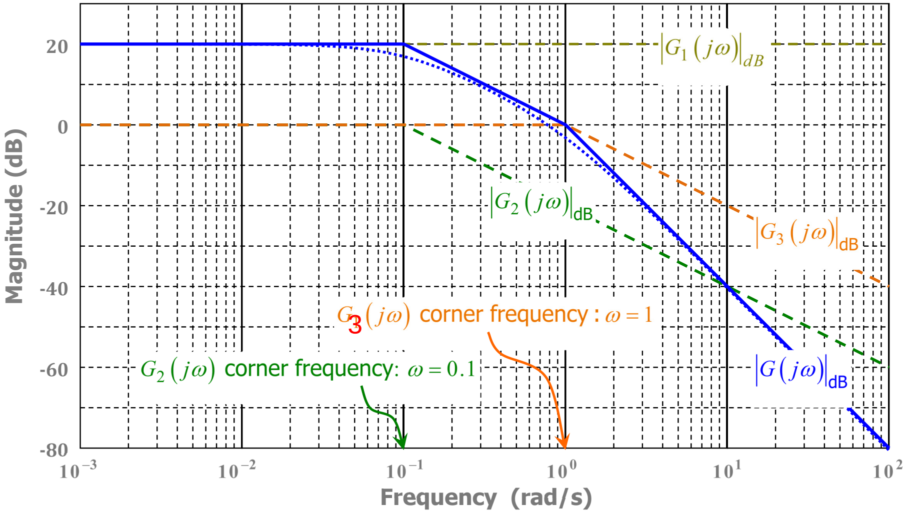
$$\begin{aligned} G(s) &= G_1(s)G_2(s)G_3(s) = 10\left(\frac{1}{\frac{s}{0.1}+1}\right)\left(\frac{1}{s+1}\right) \\ G_1(s) &= 10, \;\; G_2(s) = \frac{1}{\frac{s}{0.1}+1}, \;\; G_3(s) = \frac{1}{s+1} \end{aligned}$$

Steps :

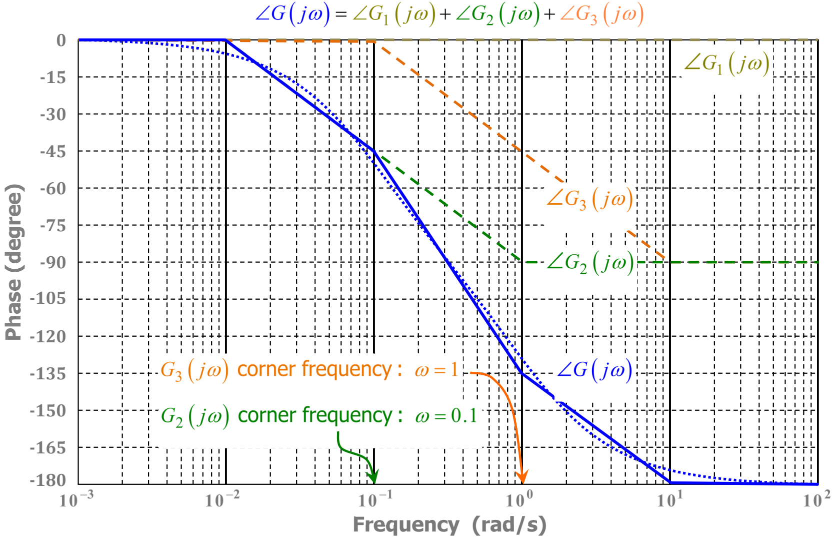
1. Sketch $|G_1(j\omega)|_{dB} = 20\log_{10} 10 = 20$ dB - should be a straight line
2. Sketch $|G_2(j\omega)|_{dB} = -20\log_{10}\sqrt{100\omega^2+1}$ dB - corner freq at $\omega = 0.1$
3. Sketch $|G(j\omega)|_{dB} =$ -20log $\sqrt{\omega^2+1}$ dB - corner freq $\omega = 1$
4. Then add all 3 sketches :

$$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} + |G_3(j\omega)|_{dB}$$

$$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} + |G_3(j\omega)|_{dB}$$



Phase plot of $G(s) = \frac{10}{(10s+1)(s+1)}$: use same approach as magnitude plot. However as $\omega \to \infty$, $\lim_{s\to\infty} G(s) \to \frac{1}{s^2}$ which is a double integrator and hence $\lim_{\omega\to\infty} \angle G(j\omega) \to -180^0$



Example 3

Sketch the Bode diagram of a first order system with 1 integrator

$$G(s) = \frac{1}{s(0.1s+1)}$$

$$\begin{aligned} G(s) &= G_1(s)G_2(s) = \left(\frac{1}{s}\right)\left(\frac{1}{\frac{s}{10}+1}\right) \\ G_1(s) &= \frac{1}{s}, \;\; G_2(s) = \left(\frac{1}{\frac{s}{10}+1}\right) \end{aligned}$$

Magnitude Plot

$G_1(s)$: no corner frequency, straight line slope of -20 dB/decade

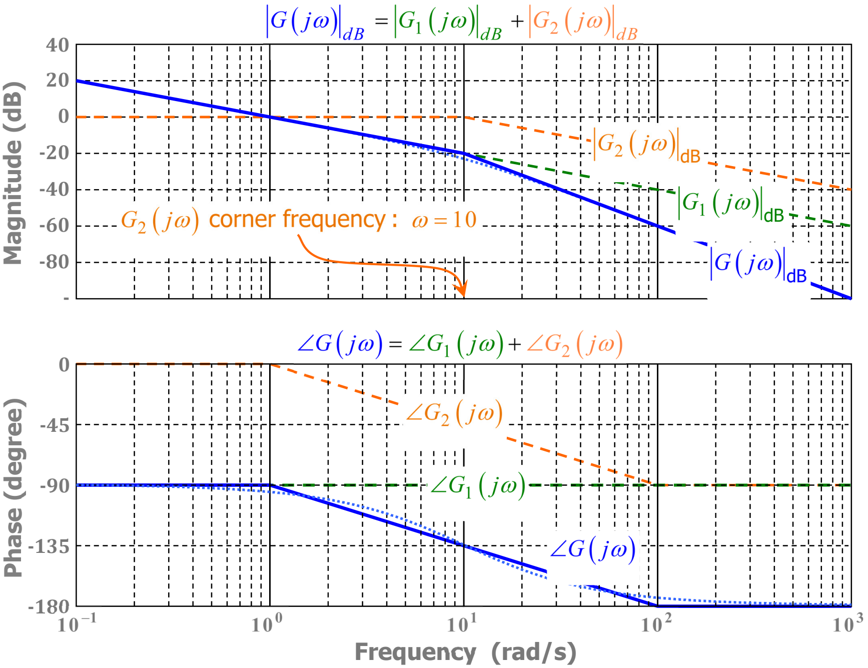
$G_2(s)$: corner freq at $\omega = 10$ rad/s

Phase Plot

$G_1(s)$: Phase is constant at -90^0

$G_2(s)$: corner frequency at $\omega = 10$ rad/s

$\lim_{s\to\infty} G(s) \to \frac{1}{s^2}$ - a double integrator and hence $\lim_{\omega\to\infty} \angle G(j\omega) \to -180^0$


$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB}$
Magnitude (dB)
40
20
0
-20
-40
-60
-80
$|G_2(j\omega)|_{\mathrm{dB}}$
$|G_1(j\omega)|_{\mathrm{dB}}$
$|G(j\omega)|_{\mathrm{dB}}$
$G_2(j\omega)$ corner frequency : $\omega = 10$
$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$
Phase (degree)
0
-45
-90
-135
-180
$\angle G_2(j\omega)$
$\angle G_1(j\omega)$
$\angle G(j\omega)$
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

Example 4

Sketch the Bode diagram of an underdamped 2^{nd} order system

$$G(s) = \frac{1000}{s^2 + 10s + 100}$$

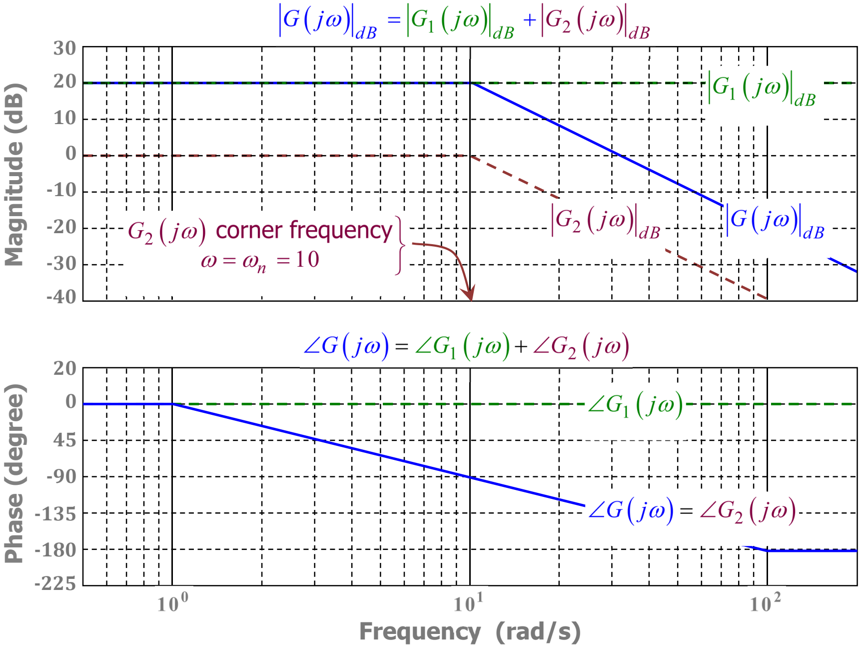
Comparing with the standard form : $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$$K = 10, \quad \omega_n = 10, \quad \zeta = 0.5$$

$$\begin{aligned} G(s) &= G_1(s)G_2(s) = \frac{1000}{s^2 + 10s + 100} \\ G_1(s) &= 10, \quad G_2(s) = \frac{100}{s^2 + 10s + 100} \end{aligned}$$

Sketch the Bode diagrams of $G_1(s)$ and $G_2(s)$ separately and combine them, noting the following points :

- Slope of magnitude plot of $G_2(s)$ is -40 dB/dec.
- As $\zeta = 0.5$, hence the resonance peak is not very pronounced.
- Phase of phase plot of $G_2(s)$ is -180^0 as $\omega \to \infty$.



$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB}$
Magnitude (dB)
30
20
10
0
-10
-20
-30
-40
$|G_1(j\omega)|_{dB}$
$|G_2(j\omega)|_{dB}$
$|G(j\omega)|_{dB}$
$G_2(j\omega)$ corner frequency
$\omega = \omega_n = 10$
$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$
Phase (degree)
20
0
45
-90
-135
-180
-225
$\angle G_1(j\omega)$
$\angle G(j\omega) = \angle G_2(j\omega)$
10^0
10^1
10^2
Frequency (rad/s)

Example 5

Underdamped 2^{nd} order system with resonance :

$$G(s) = \frac{250}{s^2 + 3s + 25}$$

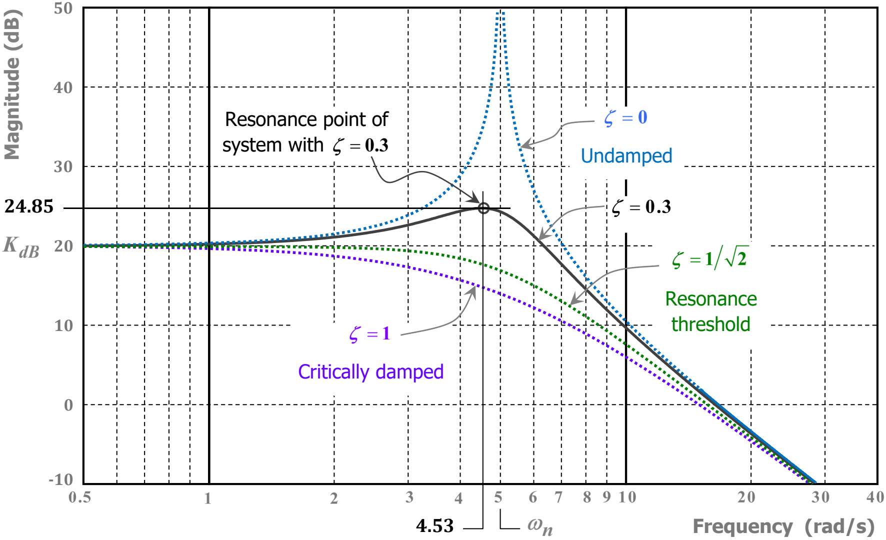
Comparing with the standard form : $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$$K = 10, \quad \omega_n = 5, \quad \zeta = 0.3$$

The straight line approximated Bode diagram will look very similar to that in Example 4, except that corner frequency is at $\omega_n = 5$ instead of 10.

Resonance freq : $\omega_r = \omega_n\sqrt{1 - 2\zeta^2} \approx 4.53$

Resonance peak : $M_r = \left.\frac{K}{2\zeta\sqrt{1-\zeta^2}}\right|_{dB} \approx 24.85$ dB



Magnitude (dB)
50
40
30
24.85
K_{dB}
20
10
0
-10
Resonance point of
system with $\zeta = 0.3$
$\zeta = 0$
Undamped
$\zeta = 0.3$
$\zeta = 1/\sqrt{2}$
Resonance
threshold
$\zeta = 1$
Critically damped
0.5
1
2
3
4
5
6
7
8
9
10
20
30
40
4.53
ω_n
Frequency (rad/s)

Asymptotic Characteristics of Bode Diagrams (Optional)

General form of $G(s)$:

$$G(s) = \frac{K(s/z_1+1)(s/z_2+1)\dots(s/z_M+1)}{s^n(s/p_1+1)(s/p_2+1)\dots(s/p_{N-n}+1)}$$

which is an N^{th} order system with n integrators (n poles at $s=0$), M zeros, and $(N-n)$ non-zero poles. We also say that it has $(N-M)$ pole-zero excess.

- Aysmptotic slope of magnitude plot $[|G(j\omega)|_{dB}]$

  Low frequency : $\omega \to 0$

  $$\lim_{\omega\to 0} G(s) \quad \to \quad \frac{K}{s^n} \quad n \text{ integrators}$$

  $$\text{Slope at low frequency} \quad \to \quad -20n \text{ dB/decade}$$

  High frequency : $\omega \to \infty$

  $$\lim_{\omega\to\infty} G(s) \to \frac{K}{s^{N-M}} \quad N-M \text{ (pole-zero excess) integrators}$$

  $$\text{Slope at high frequency} \to -20(N-M)\,\text{dB/decade}$$

- Asymptotic phase of phase plots $[\angle G(j\omega)]$

Low frequency : $\omega \to 0$

$$\lim_{\omega \to 0} G(s) \quad \to \quad \frac{K}{s^n} \quad n \text{ integrators}$$

$$\text{Phase at low frequency} \quad \to \quad -90n^0$$

High frequency : $\omega \to \infty$

$$\lim_{\omega \to \infty} G(s) \to \frac{K}{s^{N-M}} \quad N - M \text{ (pole-zero excess) integrators}$$

$$\text{Phase at high frequency} \to -90(N-M)^0$$

How do you use these asymptotic characteristics? You use them to sketch (approximate) the low and high frequency parts of the Bode diagrams.

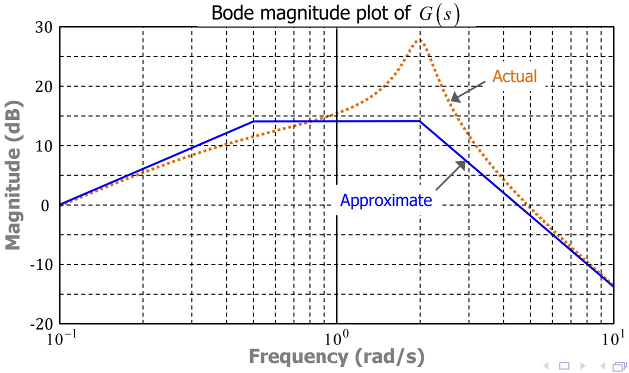
Transfer Function Identification from Bode Magnitude Plot

Example 6

The Bode magnitude plot of a LTI system with the transfer function

$$G(s) = \frac{K(s+a)}{(s/b+1)(s^2+2\zeta\omega_n s+\omega_n^2)}$$ *is shown below.*

(a) Identify $K,\ a,\ b$ and ω_n in the $G(s)$. (b) What values will the low and high frequency asymptotes of the phase plot converge to?



Rewrite $G(s)$ as $\quad G(s) = \frac{K}{\omega_n^2}(s+a).\frac{1}{\frac{s}{b}+1}.\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$

- The magnitude plot has a low frequency asymptote of $+20$ dB/dec, indicating that there is a single differentiator of the form $K_d s$. Hence $a = 0$ and we may write $G(s) = \frac{K}{\omega_n^2}s.\frac{1}{\frac{s}{b}+1}.\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$

- The low freq asymptote passes thru (0.1 rad/s, 0 dB), indicating that

$$\frac{K}{\omega_n^2}0.1 = 1 \text{ (corresponding to 0 dB) } \implies K = 10\omega_n^2$$

- At $\omega = 0.5$, the straight line approximation shows a corner. The slope changes from $+20$ dB/dec to 0 dB/dec, indicating that there is a pole factor which is contributed by the term $\frac{1}{\frac{s}{b}+1}$. Since the corner freq is $\omega = 0.5$, then $b = 0.5$.

- Another corner exists at $\omega = 2$ and the slope changes from 0 dB/dec to -40 dB/dec and this is contributed by the 2^{nd} order factor. The corner corresponds to ω_n and hence $\omega_n = 2$. Hence it follows that $K = 40$.

  We have identified $G(s) = \dfrac{40s}{(2s+1)(s^2+4\zeta s+4)}$.

- Phase plot

  Since $G(s)$ contains 1 differentiator, the low frequency asymptote of the phase plot starts at $+90^0$.

  The high frequency asymptote behaves according to :

$$\lim_{s\to\infty} G(s) \to \frac{1}{s^2} \quad \text{pole-zero excess of 2}$$

  Hence at high frequency, the phase plot tends to -180^0.