EE2023 Signals & Systems
Chapter 2 – Discrete-Frequency Spectrum (Fourier Series)

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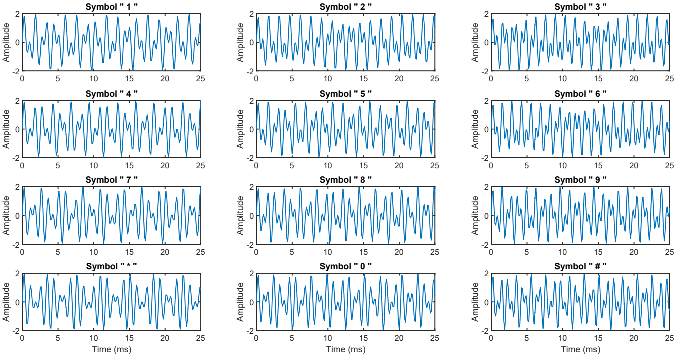
Time and Frequency Domain Representation of Signals

Time-domain representation

- A **time-domain signal** is a function which represents the time variation of a physical variable.
- A **continuous-time** signal is one that is defined for all time instants, t, in an interval of interest. It is usually represented as a waveform.

Example - Time-domain representation of telephone's keypad tones



- Each keypad tone consists of two "simultaneous" real sinusoidal signals, $x(t) = \sin(2\pi f_l t) + \sin(2\pi f_h t)$.
  - Row in which the key appears determines the low-frequency component, f_l.
  - Column in which the key appears determines the high-frequency component, f_h.

f_l \ f_h	1209 Hz	1336 Hz	1477 Hz
697 Hz	1	2	3
770 Hz	4	5	6
852 Hz	7	8	9
941 Hz	$*$	0	$\#$

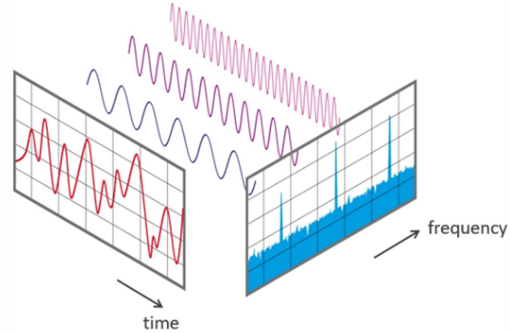
  - From the table, the sound generated when "1" is pressed may be represented by

$$x(t) = \sin(2\pi \cdot \underbrace{697}_{f_l}\, t) + \sin(2\pi \cdot \underbrace{1209}_{f_h}\, t)$$

Instead of using waveforms, is there another way to describe continuous-time analgoue signals that consists of multiple sinsuoids?

Frequency-domain representation

Continuous-time analgoue signals signal can also be defined using "characteristics" of its sinusoidal components!



- Parameters of real sinusoidal signals are the cyclic/angular frequency, amplitude and phase.

$$\underbrace{\mu}_{\text{amplitude}} \sin(2\pi \underbrace{f_o}_{\text{cyclic freq}} t + \underbrace{\phi}_{\text{phase}}) \quad \text{or} \quad \underbrace{\mu}_{\text{amplitude}} \cos(\underbrace{2\pi f_o}_{\text{angular freq}} t + \underbrace{\phi}_{\text{phase}})$$

- The more "general" complex exponential function, a complex signal, is also characterised by the cyclic/angular frequency, magnitude/amplitude and phase.

$$\underbrace{\mu}_{\text{magnitude}} \exp[j(\underbrace{2\pi f_o}_{\text{angular freq}} t + \underbrace{\phi}_{\text{phase}})] = \underbrace{\mu \exp(j\phi)}_{\text{spectrum}} \cdot \exp(j2\pi \underbrace{f_o}_{\text{cyclic freq}} t)$$

- A time-domain signal comprising N complex exponential functions has the following spectral representation:

$f_{o,1}$	$f_{o,2}$	$\ldots$	$f_{o,N}$
$\mu_1 e^{j\phi_1}$	$\mu_2 e^{j\phi_2}$	$\ldots$	$\mu_N e^{j\phi_N}$

- The spectral representation can be presented using two graphs:
  - Magnitude Spectrum: Plot of μ_i versus $f_{o,i}$
  - Phase Spectrum: Plot of ϕ_i versus $f_{o,i}$

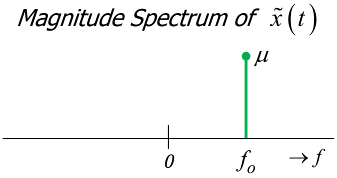
The spectral plots (Magnitude Spectrum and Phase Spectrum) are **frequency-domain** representation of the **time-domain** signal, $x(t)$.

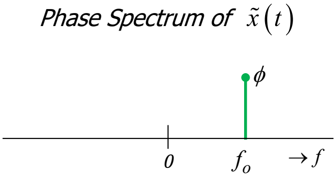
Example

Sketch the magnitude and phase spectral of $\tilde{x}(t) = \mu\, e^{j(2\pi f_o t + \phi)}$

$$\begin{aligned}\tilde{x}(t) &= \underbrace{\mu \exp[j(2\pi f_o t + \phi)]}_{\text{complex exponential}} \\ &= \underbrace{\mu}_{\text{magnitude}} \exp(j \underbrace{\phi}_{\text{phase}}) \exp(j2\pi \underbrace{f_o}_{\text{freq}} t)\end{aligned}$$

Spectral Plot:





Magnitude and phase spectra of $\tilde{x}(t)$ are **discrete-frequency** plots.

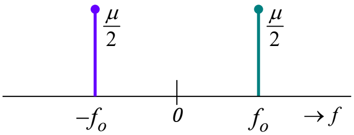
Example Sketch the magnitude and phase spectral of $x_c(t) = \mu\, cos(2\pi f_o t + \phi)$

Converting the cosine function into its complex exponential form, $\cos(\theta) = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$,

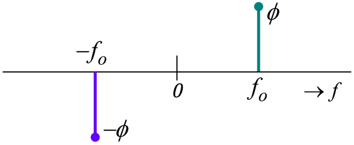
$$
\begin{aligned}
x_c(t) &= \mu\, cos(2\pi f_o t + \phi) \\
&= \frac{1}{2}\mu \exp[j(2\pi f_o t + \phi)] + \frac{1}{2}\mu \exp[-j(2\pi f_o t + \phi)] \\
&= \underbrace{\frac{\mu}{2}}_{\text{magnitude}} \exp(j \underbrace{\phi}_{\text{phase}}) \exp(j2\pi \underbrace{f_o}_{\text{freq}} t) + \underbrace{\frac{\mu}{2}}_{\text{magnitude}} \exp(j \underbrace{(-\phi)}_{\text{phase}}) \exp(j2\pi \underbrace{(-f_o)}_{\text{freq}} t)
\end{aligned}
$$

Spectral Plot:

Magnitude Spectrum of $x_c(t)$



Phase Spectrum of $x_c(t)$



Example Sketch the magnitude and phase spectral of $x_s(t) = \mu\, sin(2\pi f_o t + \phi)$

Converting the sine function into its complex exponential form, $\sin(\theta) = \frac{1}{j2}\left[e^{j\theta} - e^{-j\theta}\right]$,

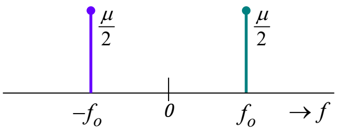
$$x_s(t) = \mu\,\sin(2\pi f_o t + \phi) = \frac{1}{j2}\mu\exp[j(2\pi f_o t + \phi)] - \frac{1}{j2}\mu\exp[-j(2\pi f_o t + \phi)]$$

The polar form of j is $e^{j\frac{\pi}{2}}$. Hence,

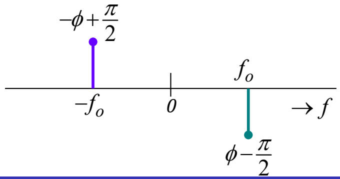
$$x_s(t) = \underbrace{\frac{\mu}{2}}_{\text{magnitude}}\exp(j\underbrace{\left(\phi - \frac{\pi}{2}\right)}_{\text{phase}})\exp(j2\pi\underbrace{f_o}_{\text{freq}}t) + \underbrace{\frac{\mu}{2}}_{\text{magnitude}}\exp(j\underbrace{\left(-\phi + \frac{\pi}{2}\right)}_{\text{phase}})\exp(j2\pi\underbrace{(-f_o)}_{\text{freq}}t)$$

Spectral Plot:

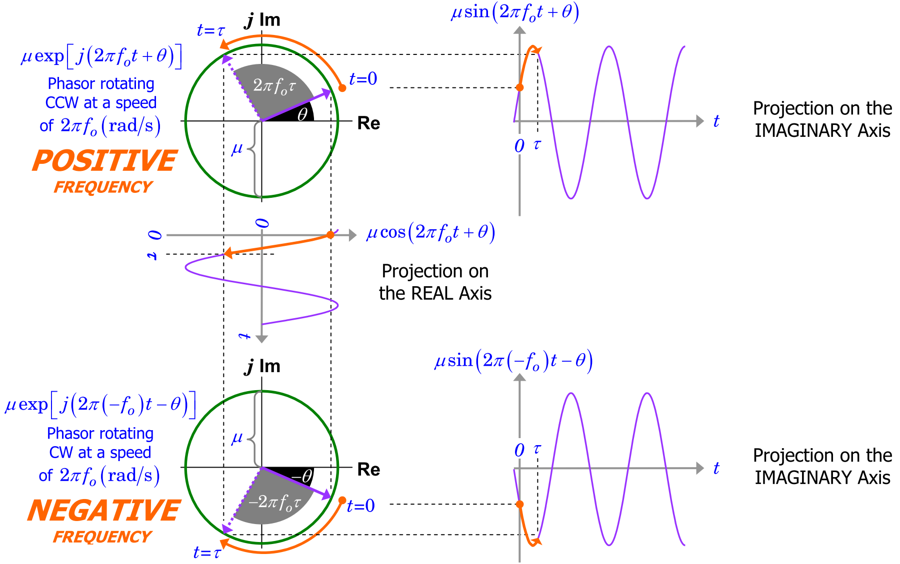
Magnitude Spectrum of $x_s(t)$



Phase Spectrum of $x_s(t)$



Concept of negative frequency

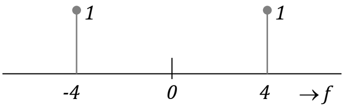


Example Sketch the magnitude and phase spectral of $x(t) = 2\sin\left(8\pi t + \frac{\pi}{6}\right)$

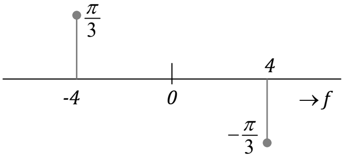
Expressing $x(t)$ in terms of complex exponentials using $\sin(\theta) = \frac{1}{j2}\left[e^{j\theta} - e^{-j\theta}\right]$,

$$\begin{aligned}
x(t) &= 2 \cdot \frac{1}{j2}\left\{\exp\left[j\left(2\pi(4)t + \frac{\pi}{6}\right)\right] - \exp\left[-j\left(2\pi(4)t + \frac{\pi}{6}\right)\right]\right\} \\
&= \exp\left(-j\frac{\pi}{2}\right)\exp\left(j\frac{\pi}{6}\right)\exp\left(j2\pi(4)t\right) + \exp\left(j\frac{\pi}{2}\right)\exp\left(-j\frac{\pi}{6}\right)\exp\left(j2\pi(-4)t\right) \\
&= \exp\left(-j\frac{\pi}{3}\right)\exp\left(j2\pi(4)t\right) + \exp\left(j\frac{\pi}{3}\right)\exp\left(j2\pi(-4)t\right)
\end{aligned}$$

Magnitude Spectrum



Phase Spectrum



Question : Can any time-domain signal be represented in frequency-domain?

Fourier Series : Complex Exponential Representation

- Any bounded **periodic signal**, $x_p(t)$, can be represented by the sum of an infinite number of **harmonically related** complex exponential functions (sinusoids).

$$\underbrace{x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_p} t}}_{\text{Fourier series synthesis/expansion}}$$

  - T_P is the fundamental period
  - $f_p = \frac{1}{T_p}$ is the fundamental cyclic frequency
  - k is an integer known as the frequency index
  - kf_p is the k^{th} **harmonic** of the fundamental cyclic frequency, f_p
  - $c_k = |c_k| e^{j\angle c_k}$ are the **Fourier series coefficients** of $x_p(t)$.
- Graphical representations of c_k are the **discrete-frequency** spectra plots of $x_p(t)$.
  - Magnitude Spectrum: Plot of $|c_k|$ versus frequency or the frequency index k.
  - Phase Spectrum: Plot of $\angle c_k$ versus frequency or the frequency index k.

Given a periodic signal $x_p(t)$, the characteristics of its complex exponential components may be determined (i.e. derive the Fourier series coefficients c_k) by multiplying $x_p(t)$ by $e^{-j2\pi\frac{k}{T_p}t}$, and then integrating the product over any one period:

$$\begin{aligned}
\int_{t_o}^{t_o+T_p} x_p(t)e^{-j2\pi\frac{k}{T_p}t}\,dt &= \int_{t_o}^{t_o+T_p} e^{-j2\pi\frac{k}{T_p}t}\sum_{m=-\infty}^{\infty} c_m e^{j2\pi\frac{m}{T_p}t}\,dt \\
&= \sum_{m=-\infty}^{\infty} c_m \int_{t_o}^{t_o+T_p} e^{-j2\pi\frac{k-m}{T_p}t}dt \\
&= \sum_{m=-\infty}^{\infty} c_m \left[\frac{e^{-j2\pi\frac{(k-m)}{T_p}t}}{-j2\pi(k-m)/T_p}\right]_{t_o}^{t_o+T_p} \\
&= \sum_{m=-\infty}^{\infty} c_m \left[\frac{e^{-j2\pi\frac{(k-m)t_o}{T_p}}\left(e^{-j2\pi(k-m)}-1\right)}{-j2\pi(k-m)/T_p}\right]
\end{aligned}$$

- When $m \neq k$, $e^{-j2\pi(k-m)} = 1$. Hence, $\dfrac{e^{-j2\pi\frac{(k-m)t_o}{T_p}}\left(e^{-j2\pi(k-m)} - 1\right)}{-j2\pi(k-m)/T_p} = 0$
- When $m = k$, $\displaystyle\int_{t_o}^{t_o+T_p} e^{-j2\pi\frac{k-m}{T_p}t} dt = \int_{t_o}^{t_o+T_p} 1\, dt = T_p$

Since $\dfrac{e^{-j2\pi\frac{(k-m)t_o}{T_p}}\left(e^{-j2\pi(k-m)} - 1\right)}{-j2\pi(k-m)/T_p} = \begin{cases} T_p; & m = k \\ 0; & m \neq k \end{cases}$,

$$\int_{t_o}^{t_o+T_p} x_p(t)e^{-j2\pi\frac{k}{T_p}t}\, dt = c_k T_p$$

Re-arranging, the **Fourier series analysis** expression is

$$c_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)e^{-j2\pi\frac{k}{T_p}t}\, dt = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)e^{-j2\pi k f_p t}\, dt \qquad k = 0, \pm 1, \pm 2, \ldots$$

Fourier Series : Trigonometric Representation

- The Fourier series expansion of $x_p(t)$ can also be expressed in terms of real sinusoidal signals (cosine and sine functions).
- Relationship between the two representations of the Fourier series expansion is the Euler's Identity, $e^{j\phi} = \cos\phi + j\sin\phi$.

$$
\begin{aligned}
x_p(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_p} t} = \sum_{k=-\infty}^{-1} c_k e^{j2\pi \frac{k}{T_p} t} + c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi \frac{k}{T_p} t} \\
&= c_0 + \sum_{k=1}^{\infty} \left[c_{-k} \cos\left(2\pi \frac{k}{T_p} t\right) - j c_{-k} \sin\left(2\pi \frac{k}{T_p} t\right) \right. \\
&\qquad \left. + c_k \cos\left(2\pi \frac{k}{T_p} t\right) + j c_k \sin\left(2\pi \frac{k}{T_p} t\right) \right] \\
&= c_0 + \sum_{k=1}^{\infty} \left[(c_k + c_{-k}) \cos\left(2\pi \frac{k}{T_p} t\right) + j (c_k - c_{-k}) \sin\left(2\pi \frac{k}{T_p} t\right) \right]
\end{aligned}
$$

- Let $c_k = a_k - jb_k$ and $c_{-k} = a_k + jb_k$, and substitute into

$$x_p(t) = c_0 + \sum_{k=1}^{\infty} \left[(c_k + c_{-k}) \cos\left(2\pi \frac{k}{T_p} t\right) + j\,(c_k - c_{-k}) \sin\left(2\pi \frac{k}{T_p} t\right) \right]$$

The resulting expression is the trigonometric Fourier series

$$x_p(t) = a_0 + 2\sum_{k=1}^{\infty} \left[a_k \cos\left(2\pi \frac{k}{T_p} t\right) + b_k \sin\left(2\pi \frac{k}{T_p} t\right) \right]$$

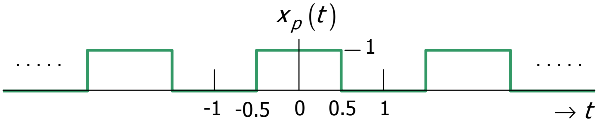
Fourier series coefficients for the trigonometric form are defined as

$$\begin{aligned} a_k &= \frac{c_{-k} + c_k}{2} = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x_p(t) \cos\left(2\pi \frac{k}{T_p} t\right) \, dt; \qquad k \geq 0 \\ b_k &= \frac{c_{-k} - c_k}{j2} = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x_p(t) \sin\left(2\pi \frac{k}{T_p} t\right) \, dt; \qquad k > 0 \end{aligned}$$

Complex Exponential **Fourier series** complex exp kernel	**Analysis** (Compute Fourier series coefficients from $x_p(t)$) $c_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)e^{-j2\pi\frac{k}{T_p}t}\,dt \quad k = 0, \pm1, \pm2, \ldots$ **Synthesis** (Construct $x_p(t)$ from Fourier series coefficients) $x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi\frac{k}{T_p}t}$
Trigonometric **Fourier series** cos & sin kernels	**Analysis** (Compute Fourier series coefficients from $x_p(t)$) $a_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\cos\left(2\pi\frac{k}{T_p}t\right)dt; \qquad k \geq 0$ $b_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\sin\left(2\pi\frac{k}{T_p}t\right)dt; \qquad k > 0$ **Synthesis** (Construct $x_p(t)$ from Fourier series coefficients) $x_p(t) = a_0 + 2\sum_{k=1}^{\infty}\left[a_k\cos\left(2\pi\frac{k}{T_p}t\right) + b_k\sin\left(2\pi\frac{k}{T_p}t\right)\right]$

Example

Derive the discrete-frequency spectra of the square wave, $x_p(t)$

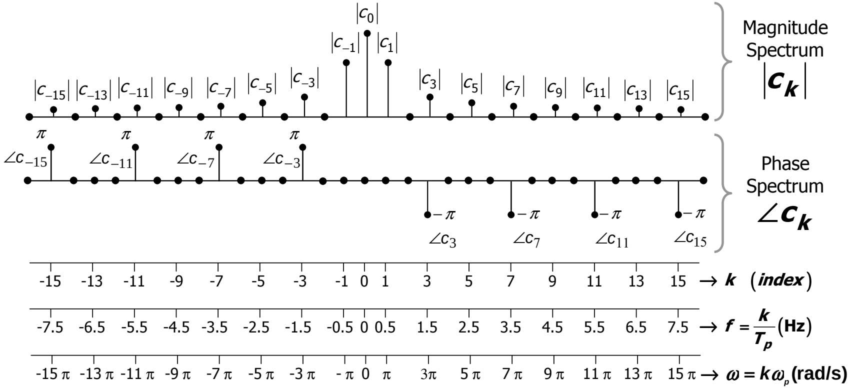


$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi\frac{k}{T_p}t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t} \qquad \because T_p = 2$$

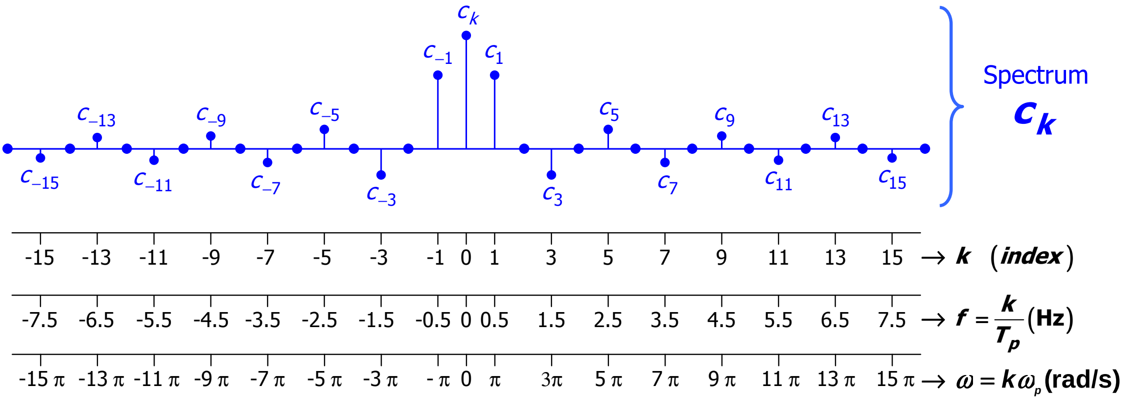
where

$$\begin{aligned}
c_k &= \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\, e^{-j2\pi\frac{k}{T_p}t}\,dt = \frac{1}{2}\int_{-0.5}^{1.5} x_p(t)\, e^{-jk\pi t}\,dt \\
&= \frac{1}{2}\left[\int_{-0.5}^{0.5} 1\cdot e^{-jk\pi t}\,dt + \int_{0.5}^{1.5} 0\cdot e^{-jk\pi t}\,dt\right] \\
&= \frac{1}{2}\left.\frac{e^{-jk\pi t}}{-jk\pi}\right|_{-0.5}^{0.5} = \frac{1}{j2k\pi}\left[e^{j\frac{k\pi}{2}} - e^{-j\frac{k\pi}{2}}\right] \\
&= \begin{cases} \frac{1}{2}\frac{\sin(0.5k\pi)}{0.5k\pi}; & k \neq 0 \\ \frac{1}{2} & k = 0 \end{cases} = \frac{1}{2}\,\mathrm{sinc}\,(0.5k)
\end{aligned}$$

k	$\ldots$	-3	-2	-1	0	1	2	3	$\ldots$
$c_k = \frac{1}{2}\,\text{sinc}\,(0.5k)$	$\ldots$	$-\frac{1}{3\pi}$	0	$\frac{1}{\pi}$	0.5	$\frac{1}{\pi}$	0	$-\frac{1}{3\pi}$	$\ldots$
$\lvert c_k \rvert$	$\ldots$	$\frac{1}{3\pi}$	0	$\frac{1}{\pi}$	0.5	$\frac{1}{\pi}$	0	$\frac{1}{3\pi}$	$\ldots$
$\angle c_k$	$\ldots$	π	0	0	0	0	0	$-\pi$	$\ldots$



Since c_k are real numbers, the amplitude and phase spectra can be combined into the single spectrum shown below :



- Phase of π and $-\pi$ denote a negative number.
- Phase of 0 denotes a positive number.

Example

Consider the signal

$$x(t) = (1+j)e^{-j6t} + j3e^{-j4t} + 4 - j3e^{j4t} + (1-j)e^{j6t}$$

Show whether or not $x(t)$ is real and periodic. If $x(t)$ is periodic, find its complex exponential Fourier series coefficients, c_k, and sketch its magnitude and phase spectra.

- $x(t)$ is a real signal because, except for a constant term, it composes purely of complex sinusoids that come in conjugate pairs. This characteristic means that $x(t)$ can be re-written as

$$\begin{aligned} x(t) &= (1+j)e^{-j6t} + j3e^{-j4t} + 4 - j3e^{j4t} + (1-j)e^{j6t} \\ &= \sqrt{2}e^{j0.25\pi}e^{-j6t} + 3e^{j0.5\pi}e^{-j4t} + 4 + 3e^{-j0.5\pi}e^{-j4t} + \sqrt{2}e^{-j0.25\pi}e^{-j6t} \\ &= 4 + 3\left[e^{-j(4t-0.5\pi)} + e^{j(4t-0.5\pi)}\right] + \sqrt{2}\left[e^{-j(6t-0.25\pi)} + e^{j(6t-0.5\pi)}\right] \\ &= 4 + 6\cos(4t-0.5\pi) + 2\sqrt{2}\cos(6t-0.25\pi) \end{aligned}$$

- **Periodicity**

  Sinusoidal components of a periodic signal must be **harmonically related**, i.e. their frequencies must must at least one common factor. The fundamental frequency of the periodic signal is given by the highest common factor of the component frequencies.

  In this case, $\mathrm{HCF}\{-6, -4, 4, 6\} = 2$ rad/s.
  $\Longrightarrow x(t)$ is periodic with a fundamental angular frequency, ω_p of 2 rad/s.

- Complex exponential Fourier series coefficients of $x(t)$ To derive c_k, compare sinusoids with the same frequency in

$$x(t) = (1+j)e^{-j6t} + j3e^{-j4t} + 4 - j3e^{j4t} + (1-j)e^{j6t}$$

with its Fourier series expansion

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_p} t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_p t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt} \\ &= \ldots + c_{-3} e^{j(-3).2t} + c_{-2} e^{j(-2).2t} + \ldots + c_o + \ldots + c_2 e^{j2.2t} + c_3 e^{j3.2t} + \ldots \end{aligned}$$

Result is

$$c_k = \begin{cases} 1+j = \sqrt{2}e^{0.25\pi} & ; k=-3 \\ j3 = 3e^{0.5\pi} & ; k=-2 \\ 4 & ; k=0 \\ -j3 = 3e^{-0.5\pi} & ; k=2 \\ 1-j = \sqrt{2}e^{-0.25\pi} & ; k=3 \\ 0 & ; \text{otherwise} \end{cases}$$