EE2023 Signals & Systems
Chapter 4 – ESP, PSP and Bandwidth

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Recap – Time-domain definitions of energy & power

- As introduced in Chapter 1, the notion of “strength” or “size” of a time-domain signal is captured by the following concepts:

  Energy signal

  - The **total energy**, E, of a complex signal $x(t)$ is defined as

$$E = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} |x(t)|^2 \, dt \text{ Joules}$$

  - $x(t)$ is said to be an energy signal if and only if $0 < E < \infty$.

  Power signal

  - The **average power**, P, of a complex signal $x(t)$ is defined as

$$P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt \text{ Watts}$$

  - $x(t)$ is said to be a power signal if and only if $0 < P < \infty$.

- Relationship between total energy and average power is $P = \dfrac{E}{2 \cdot \infty}$

Frequency-domain definition of energy and ESD

The total energy of a time-domain signal, $x(t)$, can be computed from its frequency-domain representation, $X(f) = \mathcal{F}\{x(t)\}$, via the **Rayleigh Energy Theorem**

$$E = \underbrace{\lim_{\mathcal{T}\to\infty} \int_{-\mathcal{T}}^{\mathcal{T}} |x(t)|^2 \, dt}_{\text{time-domain}} = \underbrace{\int_{-\infty}^{\infty} |X(f)|^2 \, df = \int_{-\infty}^{\infty} E_x(f) \, df}_{\text{frequency-domain}}$$

where $E_x(f) = |X(f)|^2$ Joules/Hz

$E_x(f)$ is known as the **energy spectral density** (ESD) of the time-domain signal $x(t)$.

- ESD describes how the energy of a time-domain signal is distributed across its frequency components.
- Total energy of a time-domain signal is the total area under its ESD.
- Complexity of calculating E in the time- or frequency-domain is dependent on the nature of the time-domain signal and its spectrum.

Energy Spectral Density (ESD) – Properties

By definition, the energy spectral density of a time-domain signal $x(t)$ is

$$E_x(f) = |X(f)|^2$$

As the magnitude spectrum, $|X(f)|$, is always real and non-negative,

- $E_x(f)$ is a real function of f.
- $E_x(f) \geq 0 \ \forall f$

The spectrum of a **real** time-domain signal, $x_r(t)$, has conjugate symmetric property i.e.

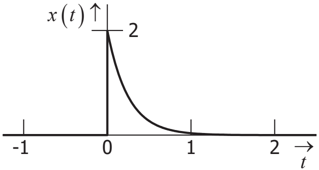
$$\text{If } \mathcal{F}\{x_r(t)\} = X_r(f), \text{ then } X_r^*(f) = X_r(-f).$$

Therefore, $|X_r(f)|$ is an even function and

- $E_x(f)$ is an even function of f if the time-domain signal is real.

Example

Consider the signal, $x(t) = 2e^{-4t}u(t)$.
Find the spectrum, $X(f)$, and energy spectral density, $E_x(f)$, of $x(t)$. Calculate the total energy, E, of $x(t)$ using the time-domain and frequency-domain approaches.

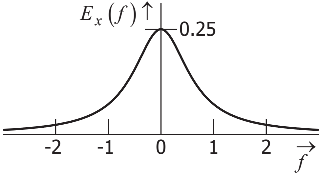


- Fourier transform of $x(t)$ is

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt = \int_0^{\infty} 2e^{(-4t)} e^{-j2\pi ftdt} = 2\left[\frac{e^{-(4+j2\pi f)t}}{-(4+j2\pi f)}\right]_0^{\infty} = \frac{2}{4+j2\pi f}$$

- Energy spectral density is

$$E_x(f) = |X(f)|^2 = \frac{4}{16+4\pi^2 f^2} = \frac{1}{4+\pi^2 f^2}$$



- Time-domain approach :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{0}^{\infty} 4e^{-8t} \, dt = \left. \frac{4e^{-8t}}{-8} \right|_{0}^{\infty} = 0.5$$

- Frequency-domain approach :

$$\begin{aligned} E &= \int_{-\infty}^{\infty} E_x(f) \, df \\ &= \int_{-\infty}^{\infty} \frac{1}{4 + \pi^2 f^2} \, df = \frac{1}{4} \int_{-\infty}^{\infty} \frac{1}{1 + (0.5\pi f)^2} \, df \end{aligned}$$

Let $\tan\theta = 0.5\pi f$. Therefore, $\sec^2\theta \, d\theta = 0.5\pi \, df$ and

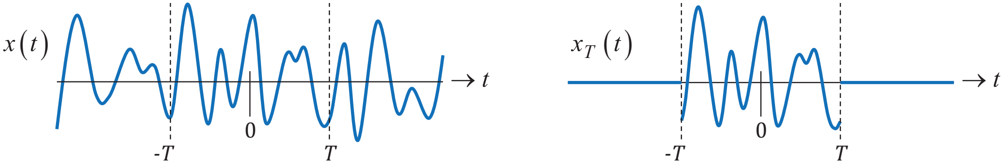
$$E = \frac{1}{4} \int_{-0.5\pi}^{0.5\pi} \frac{1}{1 + \tan^2\theta} \frac{\sec^2\theta}{0.5\pi} \, d\theta = \frac{1}{4} \int_{-0.5\pi}^{0.5\pi} \frac{1}{0.5\pi} d\theta = 0.5$$

- Results are consistent with Rayleigh Energy Theorem. In this example, it is much easier to compute E via the time-domain approach !

Frequency-domain definition of power and PSD

In order to express the average power of a time-domain signal, $x(t)$, in terms its spectrum, let's consider the following truncated version of $x(t)$

$$x_T(t) = x(t)\,\mathrm{rect}\left(\frac{t}{2T}\right)$$



Note that $\lim_{T\rightarrow\infty} x_T(t) = x(t)$.

Since $x_T(t)$ is an energy signal, the total energy of $x_T(t)$ may be determined using the Rayleigh Energy Theorem i.e.

$$\int_{-\infty}^{\infty} |x_T(t)|^2\,dt = \int_{-\infty}^{\infty} |X_T(f)|^2\,df$$

Since $x_T(t) = \begin{cases} x(t); & -T < t < T \\ 0; & \text{otherwise} \end{cases}$,

$$\int_{-\infty}^{\infty} |x_T(t)|^2 \, dt = \int_{-T}^{T} |x(t)|^2 \, dt .$$

The expression derived from the Rayleigh Energy Theorem can then be written as

$$\int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X_T(f)|^2 \, df$$

Dividing by $2T$ and taking the limit $T \to \infty$ leads to the **Parseval Power Theorem**.

$$\begin{aligned} \underbrace{P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}_{\text{time-domain def of average power}} &= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2 \, df \\ &= \int_{-\infty}^{\infty} P_x(f) \, df \end{aligned}$$

where $P_x(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2$ (Watts/Hz)

Power Spectral Density (PSD)

According to the Parseval Power Theorem, $P = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2\, df = \int_{-\infty}^{\infty} P_x(f)\, df$

- The integrand $P_x(f) = \lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2$ Watts/Hz may be interpreted as the power density of the signal at frequency f. Hence, $P_x(f)$ is known as the **Power Spectral Density** (PSD).
- *Remark:* Except for "special" signals that may be constructed by summing sinusoids (e.g. periodic signals), $P_x(f) = \lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2$ is challenging to evaluate.

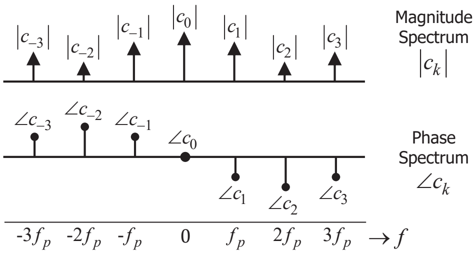
Properties of Power Spectral Density

Since $E_x(f)$ and $P_x(f)$ are defined using the magnitude spectrum, they have the same properties i.e.

- $P_x(f)$ is a real function of f.
- $P_x(f) \geq 0\ \forall f$
- $P_x(f)$ is an even function of f if the time-domain signal is real.

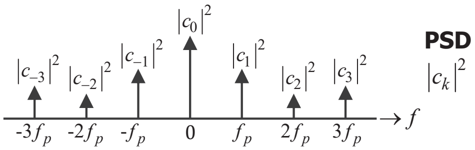
Average Power and PSD of Periodic Signals

Consider a periodic signal, $x_p(t)$. The fundamental frequency, period and Fourier series coefficients of $x_p(t)$ are denoted as f_p, T_p and c_k.



The continuous-frequency spectrum of $x_p(t)$ is

$$X_p(f) = \sum_{-\infty}^{\infty} c_k\, \delta\left(f - \frac{k}{T_p}\right) = \sum_{-\infty}^{\infty} c_k\, \delta(f - kf_p)$$



Power Spectral Density (PSD) of $x_p(t)$:

$$P_x(f) = \sum_{-\infty}^{\infty} |c_k|^2 \delta\left(f - kf_p\right)$$

Average power of $x_p(t)$:

$$P = \int_{-\infty}^{\infty} P_x(f)\, df = \sum_{-\infty}^{\infty} |c_k|^2$$

Average power and PSD of periodic signals – Proof (Optional)

$$\boldsymbol{P} = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} \left|x_p(t)\right|^2 dt \qquad \left(\begin{array}{l} Since\ x_p(t)\ is\ periodic, its\ power\ may\ be\ obtained\ by \\ averaging\ over\ 1\ period. \end{array}\right)$$

$$= \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} \mathfrak{I}^{-1}\left\{\sum_{k=-\infty}^{\infty} c_k \delta\left(f - k/T_p\right)\right\}\left[\mathfrak{I}^{-1}\left\{\sum_{l=-\infty}^{\infty} c_l \delta\left(f - l/T_p\right)\right\}\right]^* dt$$

$$= \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} \left[\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_k \delta\left(f - k/T_p\right)\cdot e^{j2\pi f t} df\right]\left[\int_{-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_l^* \delta\left(\tilde{f} - l/T_p\right)\cdot e^{-j2\pi \tilde{f} t} d\tilde{f}\right] dt$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^* \delta\left(f - k/T_p\right)\left[\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p}\left\{\int_{-\infty}^{\infty}\delta\left(\tilde{f} - l/T_p\right) e^{j2\pi\left(f-\tilde{f}\right)t} d\tilde{f}\right\} dt\right]\right\} df$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^* \delta\left(f - k/T_p\right)\left[\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} e^{j2\pi\left(f-l/T_p\right)t} dt\right]\right\} df$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^* \delta\left(f - k/T_p\right)\operatorname{sinc}\left(fT_p - l\right)\right\} df$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^* \delta\left(f - k/T_p\right)\operatorname{sinc}\left(k - l\right)\right\} df$$

$$= \int_{-\infty}^{\infty}\underbrace{\left\{\sum_{k=-\infty}^{\infty}\left|c_k\right|^2 \delta\left(f - k/T_p\right)\right\}}_{\mathbf{PSD}:\, \boldsymbol{P_x}(f)} df = \sum_{k=-\infty}^{\infty}\left|c_k\right|^2 \int_{-\infty}^{\infty}\delta\left(f - k/T_p\right) df = \underbrace{\sum_{k=-\infty}^{\infty}\left|c_k\right|^2}_{\mathbf{Power}:\, \boldsymbol{P}}$$

Example

Consider the signal, $x(t) = 2 + 4e^{j8\pi t} + 6\cos(16\pi t)$.
Find the spectrum, $X(f)$, and power spectral density, $P_x(f)$, of $x(t)$. Calculate the average power, P, of $x(t)$.

- Fundamental frequency of $x(t)$, $f_p = \text{HCF}\{4{,}8\} = 4$ Hz
- Replacing $6\cos(16\pi t)$ by its complex exponential representation,

$$x(t) = 2 + 4e^{j8\pi t} + 6\cos(16\pi t) = 2 + 4e^{j8\pi t} + 3e^{j16\pi t} + 3e^{-j16\pi t}$$

- Comparing with the Fourier series expansion $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}$ where $f_p = 4$,

$$c_k = \begin{cases} 2; & k = 0 \\ 4; & k = 1 \\ 3; & k = \pm 2 \\ 0; & \text{otherwise} \end{cases}$$

- Spectrum of $x(t)$ is

$$\begin{aligned} X(f) &= \sum_{k=-\infty}^{\infty} c_k \, \delta(f-4k) \\ &= 3\delta(f+8)+2\delta(f)+4\delta(f-4)+3\delta(f-8) \end{aligned}$$

- Power spectral density is

$$\begin{aligned} P_x(f) &= \sum_{k=-\infty}^{\infty} |c_k|^2 \delta\,(f-4k) \\ &= 9\delta(f+8)+4\delta(f)+16\delta(f-4)+9\delta(f-8) \end{aligned}$$

- Power is

$$\begin{aligned} P &= \int_{-\infty}^{\infty} P_x(f)\,df \\ &= \sum_{k=-\infty}^{\infty} |c_k|^2 = 4+16+9+9 = 38 \end{aligned}$$

Bandwidth – Definition & Bandwidth of Bandlimited Signals

Definition

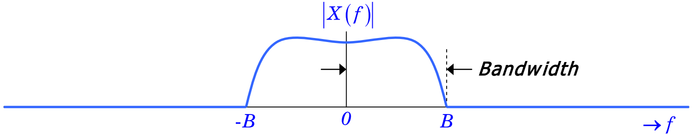
The **bandwidth** of a signal, $x(t)$, is a measure of the width of the range of frequencies occupied by its magnitude spectrum, $|X(f)|$.

- **Bandlimited Lowpass Signal**

  A real signal, $x(t)$, is said to be a **bandlimited lowpass signal** if its magnitude spectrum is concentrated around 0 Hz and satisfies

$$|X(f)| = 0; \quad |f| > B$$

  where B is defined as the bandwidth of the signal



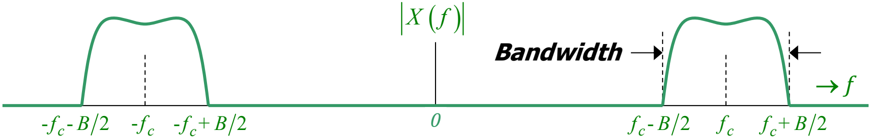
  Note that $|X(f)|$ has even symmetry if $x(t)$ is a real signal.

Bandlimited Bandpass Signal

A real signal, $x(t)$, is said to be a **bandlimited bandpass signal** if its magnitude spectrum is concentrated around a non-zero **center frequency**, f_c and satisfies

$$|X(f)| = 0; \quad ||f| - f_c| > \frac{B}{2}$$

where B is defined as the **bandwidth** of the signal.



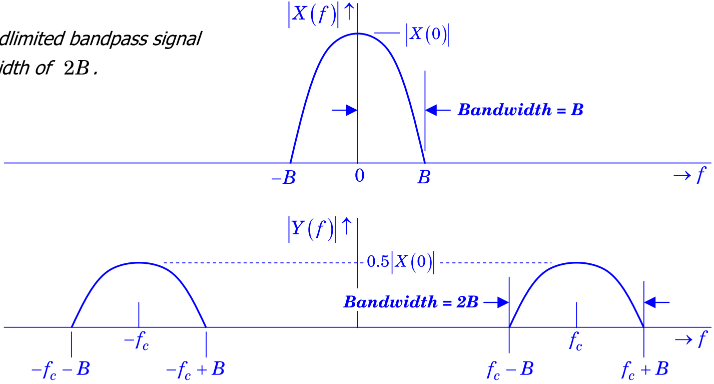
- $|X(f)|$ has even symmetry if $x(t)$ is a real signal.
- The definition assumes that $|X(f)|$, $f \geq 0$ is symmetric about $f = f_c$ and $|X(f)|$, $f \leq 0$ is symmetric about $f = -f_c$. This is usually the case in many practical situations.

Example

Suppose B is the bandwidth of a bandlimited lowpass signal, $x(t)$. Derive the bandwidth of $y(t) = x(t)\cos(2\pi f_c t)$, $f_c >> 2B$. Express your answer in terms of B.

$$Y(f) = X(f) * 0.5[\delta(f - f_c) + \delta(f + f_c)] = 0.5X(f - f_c) + 0.5X(f + f_c)$$

$y(t)$ is a bandlimited bandpass signal with a bandwidth of $2B$.



Bandwidth – Unrestricted Band

- In general, practical signals have infinite frequency extent. Such signals are said to have unrestricted band.
  - For signals that have infinite frequency extent, from the signal processing and system design standpoint, it is often useful to define a bandwidth measure that includes only the "important" frequency components of the signal.
  - Unfortunately, there is no universally applicable measure of "importance". In this course, three "measures of importance" and the corresponding bandwidth definitions are introduced.

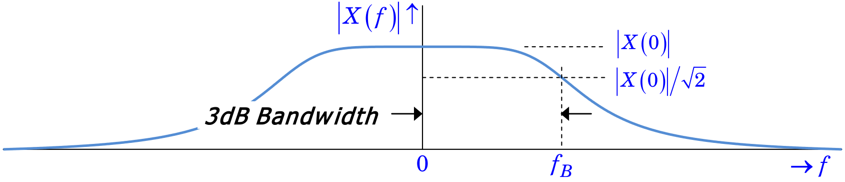
3-dB Bandwidth

The 3-dB bandwidth (B_{3dB}) is defined to be the frequency at which the spectral density drops to 50% or 0.5 of the spectral density component at zero frequency i.e.

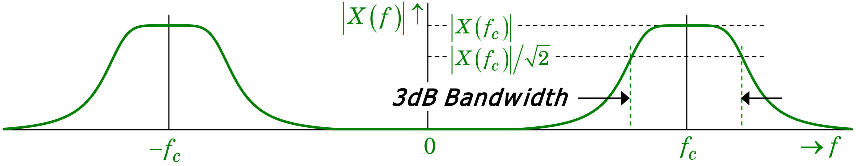
$$|X(B_{3dB})|^2 = 0.5|X(0)|^2 \qquad \text{or} \qquad \frac{|X(B_{3dB})|}{|X(0)|} = \frac{1}{\sqrt{2}}$$

- The term "3-dB" comes about because $20\log_{10}\dfrac{1}{\sqrt{2}} \approx -3$ dB

- **Lowpass signal** :
  The 3-dB bandwidth of a lowpass signal, $x(t)$, is defined as the frequency at which $|X(f)| = |X(0)|/\sqrt{2}$ first occurs when f is increased from 0.



- **Bandpass signal** :
  Similarly, the 3-dB bandwidth of a bandpass signal, $x(t)$, with center frequency f_c is illustrated below.



Example

Compute the 3-dB bandwidth, B_{3dB}, of the Gaussian pulse $x(t) = e^{-\frac{t^2}{2}}$, given that its energy spectral density is

$$E_x(f) = 2\pi e^{-4\pi^2 f^2}.$$

Since B_{3dB} be the 3-dB bandwidth of $x(t)$, then by definition

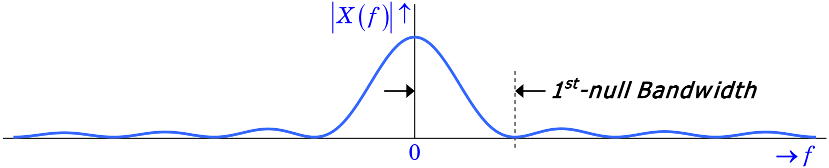
$$\frac{|X(B_{3dB})|}{|X(0)|} = \frac{1}{\sqrt{2}} \qquad \text{or} \qquad \frac{|X(B_{3dB})|^2}{|X(0)|^2} = \frac{1}{2}$$

As the energy spectral density $E_x(f) = |X(f)|^2$,

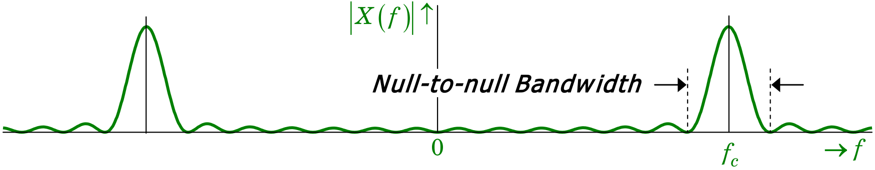
$$\begin{aligned} \frac{E_x(B_{3dB})}{E_x(0)} = \frac{2\pi e^{-4\pi^2 B_{3dB}^2}}{2\pi} &= e^{-4\pi^2 B_{3dB}^2} = \frac{1}{2} \\ 4\pi^2 B_{3dB}^2 &= \ln(2) \\ B_{3dB} &= \frac{\sqrt{\ln(2)}}{2\pi} \end{aligned}$$

1st-null Bandwidth, B_{null}

- Lowpass Signal :
  The 1st-null bandwidth of a lowpass signal, $x(t)$, is defined as the frequency at which $|X(f)| = 0$ first occurs when f is increased from 0.



- Likewise, the 1st-null (a.k.a null-to-null) bandwidth of a bandpass signal, $x(t)$ with center frequency f_c is illustrated below.



Example

What is the 1st-null bandwidth of $x(t) = 5 \cdot \text{tri}(4t - 8)$?

- First, express $x(t) = 5 \cdot \text{tri}(4t - 8)$ as $5 \cdot \text{tri}\left(\frac{t-2}{0.25}\right)$.
- From the FT table, $\text{tri}\left(\frac{t}{T}\right) \leftrightarrows T\text{sinc}^2(Tf)$.

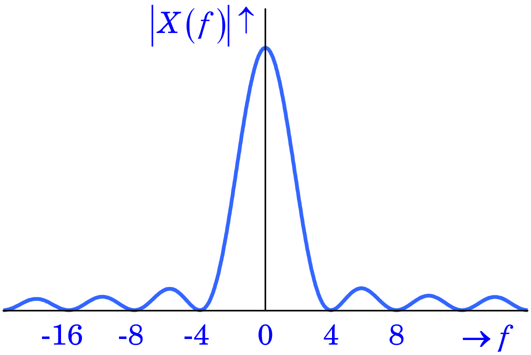
  The time-shifting property states that $x(t - t_0) \leftrightarrows X(f)e^{-j2\pi t_0 f}$.

  Hence, the spectrum of $x(t)$ is

$$X(f) = \frac{5}{4}\text{sinc}^2\left(\frac{f}{4}\right)\, e^{-j4\pi f}$$

- The magnitude spectrum of $x(t)$ is

$$|X(f)| = \frac{5}{4}\text{sinc}^2\left(\frac{f}{4}\right)$$

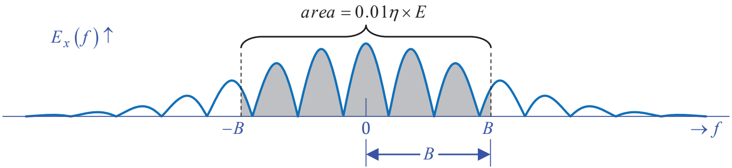


- The nulls of $|X(f)|$ occur at $f = \pm 4, \pm 8, \pm 12, \ldots$ Hz.
- Since the 1st-null of $|X(f)|$ occurs at 4 Hz, the 1st-null bandwidth of $x(t)$ is 4 Hz.

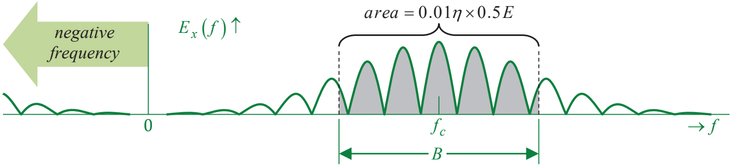
η% Energy Containment Bandwidth

The η% energy containment bandwidth, B, of a real energy signal is the smallest bandwidth that contains at least η% of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f)\, df$.

LOWPASS SIGNAL $x(t)$:



BANDPASS SIGNAL $x(t)$:



η% Power Containment Bandwidth

The η% power containment bandwidth, B, of a real power signal is the smallest bandwidth that contains at least η% of the average signal energy $P = \int_{-\infty}^{\infty} P_x(f)\, df$.