National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals and Systems Notes 4

1 Bode Diagrams of Frequency Responses

The frequency response of a system is the measure of how the system responds to a sinusoidal input of a certain frequency. The measure is in terms of the gain and phase of the transfer function at that frequency. The frequency response of a system is therefore defined by :

$$Gain = |G(j\omega)|,\ Phase = \angle G(j\omega)$$

where $|.|$ and $\angle.$ denote the magnitude and phase of the complex quantity, $G(j\omega)$, respectively, while ω is the frequency of interest.

The frequency response is an intrinsic property of linear time invariant systems as it characterises how sinusoidal signals are altered in going through the system. Via the Fourier transforms, input signals may be written as a bunch of sinuosoids and thus, the frequency response of a system, together with the superposition principle for linear systems, plays a direct role in processing the inputs to form the output.

There are two ways to visualize the frequency response of a system. One way is via the Bode diagrams and the other is via the polar plot of the complex quantity, $G(j\omega)$. Polar plots are not discussed here as it is not in the syllabus of EE2023.

1.1 Bode Diagrams

Bode diagrams consist of two plots : gain vs frequency and phase vs frequency. Furthermore, such plots are normally plotted on a semilogx graph paper which is a graph paper where the x-axis is not a linear frequency scale but instead, is a log frequency axis. Gain is generally plotted in dB units while the phase in degrees instead of radians.

The conversion between absolute gain and dB is as follows :

$$\text{Absolute gain} : |G(j\omega)|, \quad \text{Gain in dB} : 20\log_{10}|G(j\omega)|.$$

Bode magnitude diagrams are also approximated by straight line asymptotes. Extensive examples are given as follows :

1. $G(s) = Ks$ or $G(j\omega) = jK\omega$: $G(s)$ is a differentiator. In this case,

$$|G(j\omega)| = K\omega = (20\log_{10}K + 20\log_{10}\omega) \text{ dB} \tag{1}$$

$$\angle G(j\omega) = 90^0. \tag{2}$$

Equation (1) tells us that when the magnitude (dB) is plotted against the $\log_{10}\omega$ axis, a straight line is obtained. The slope of the straight line is +20dB/decade where a decade means a 10 times step in frequency. In other words, for every 10 times increase in frequency, the magnitude (or gain - same thing) increases by 20 dB.

Some examples of differentiators are given in Figure 1. K can be found from the magni-

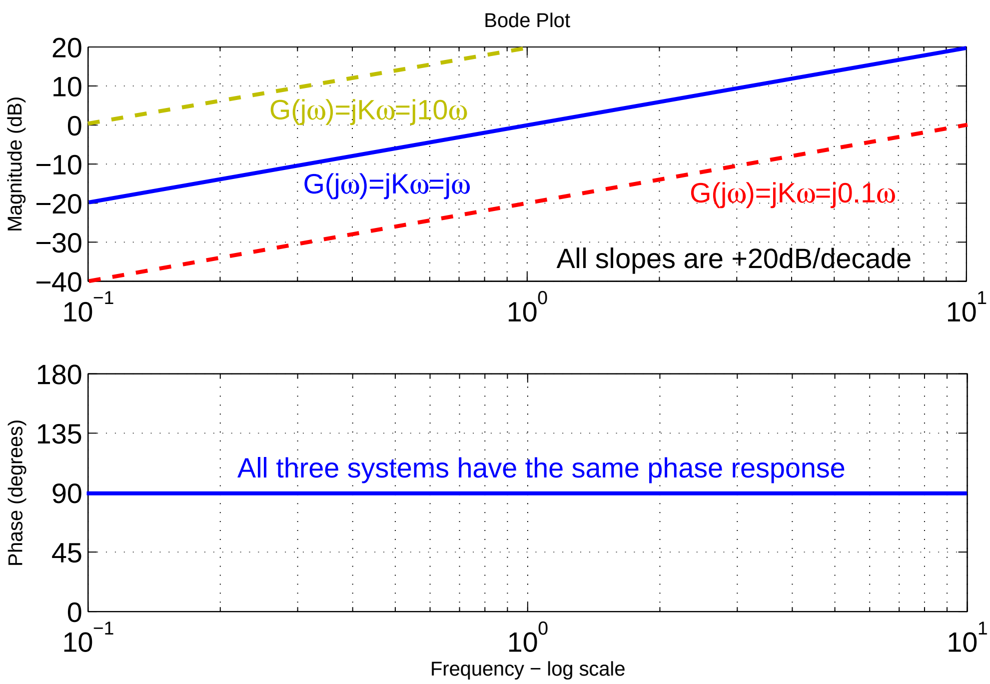

Fig. 1: Bode Diagrams of Differentiators of the form $G(s) = Ks$

tude response of $G(j\omega)$. This can be found as follows :

Locate the frequency, ω, where $|G(j\omega)| = 1$ or 0 dB. Suppose this frequency is ω_0, then $K\omega_0 = 1$ and therefore $K = 1/\omega_0$. The red dashed line in Figure 1 is an example where $K = 0.1$ and $\omega_0 = 10$.

When there is more than one differentiator, example N differentiators and hence $G(s) = Ks^N$, then the Bode diagram will have the following characteristics :

- Magnitude response is a straight line with slope of $+20N$dB/decade.
- Phase response is a constant at $+90N^0$ for all frequencies.

The value of K can be determined by locating the frequency where

$$K\omega^N = 1 \text{ and thus } K = \frac{1}{\omega^N}.$$

2. $G(s) = K/s$ or $G(j\omega) = -jK/\omega$: $G(s)$ is an integrator. In this case,

$$|G(j\omega)| = K/\omega = (20\log_{10} K - 20\log_{10}\omega) \text{ dB} \tag{3}$$

$$\angle G(j\omega) = -90^0. \tag{4}$$

In the same way as the differentiator, (3) tells us that the slope of the straight line is -20dB/decade or for every 10 times increase in frequency, the magnitude <u>decreases</u> by 20 dB. Examples of the Bode diagrams of integrators are given in Figure 2. K can be found from the magnitude response of $G(j\omega)$. This can be found as follows :

Locate the frequency, ω, where $|G(j\omega)| = 1$ or 0 dB. Suppose this frequency is ω_0, then $K/\omega_0 = 1$ and therefore $K = \omega_0$. The green line in Figure 2 is an example where $K = 10$ and $\omega_0 = 10$.

When there is more than one integrator, example N integrators and hence $G(s) = \frac{K}{s^N}$, then the Bode diagram will have the following characteristics :

- Magnitude response is a straight line with slope of $-20N$dB/decade.
- Phase response is a constant at $-90N^0$ for all frequencies.

The value of K can be determined by locating the frequency where

$$\frac{K}{\omega^N} = 1 \text{ and thus } K = \omega^N.$$

Example : Suppose $G(s) = K/s^2$. Find K using the Bode diagram in Figure 3.

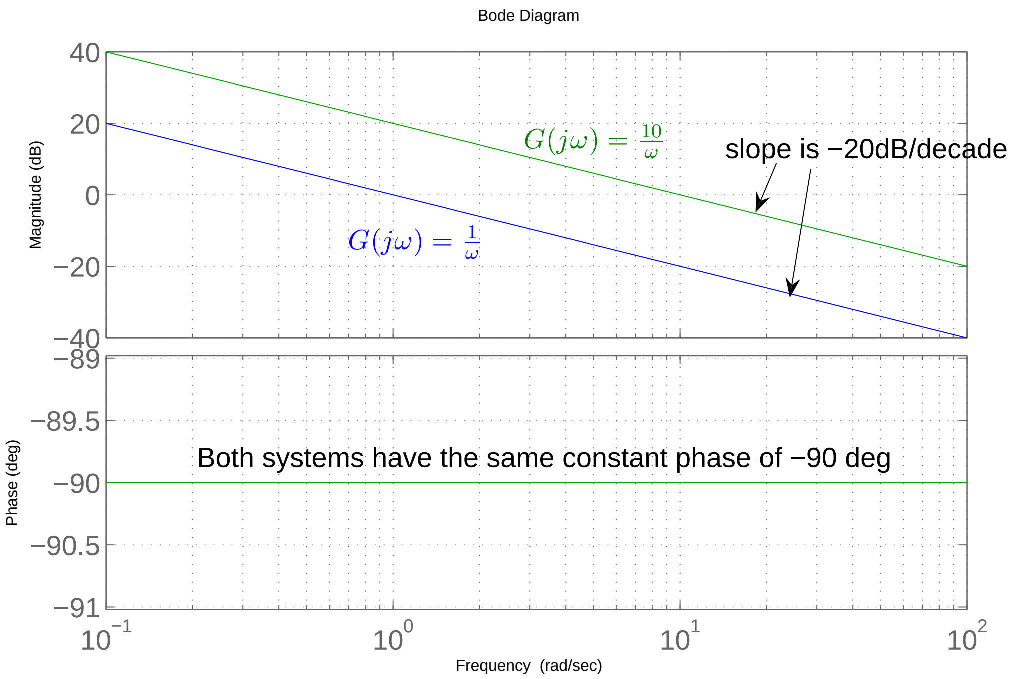

Fig. 2: Bode Diagrams of Integrators of the form $G(s) = K/s$

3. $G(s)$ is a first order transfer function of the form $G(s) = \frac{K}{sT+1}$. Pole of $G(s)$ is at $s = -1/T$. The frequency response of this system is given by :

$$G(j\omega) = \frac{K}{jT\omega + 1}.$$

The magnitude and phase responses of $G(j\omega)$ are given by :

$$|G(j\omega)| = \frac{K}{\sqrt{T^2\omega^2 + 1}} \text{ and } \angle G(j\omega) = -\tan^{-1}\omega T.$$

At frequencies, $\omega << 1/T$,

$$\begin{aligned} |G(j\omega)| &\approx K \text{ or } 20\log_{10} K \text{ dB} \\ \angle G(j\omega) &= -\tan^{-1}\omega T << -45^0. \end{aligned} \tag{5}$$

This means that the low frequency gain is the same as the DC gain of $G(s)$.

At exactly $\omega = 1/T$,

$$\begin{aligned} |G(j1/T)| &= \frac{K}{\sqrt{2}} = 20\log_{10} K - 10\log_{10} 2 = 20\log_{10} K - 3.01\ dB \\ \angle G(j1/T) &= -45^0 \end{aligned}$$

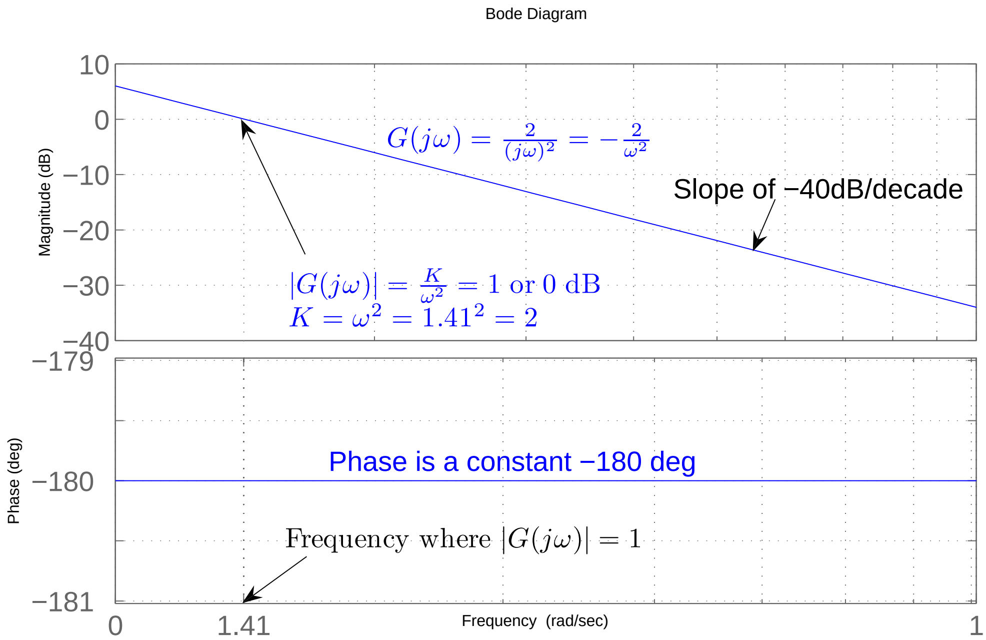

Fig. 3: Bode Diagram of $G(s) = K/s^2$

At higher frequencies, $\omega >> 1/T$,

$$|G(j\omega)| \approx \frac{K}{\omega T} \text{ or } (20\log_{10} K - 20\log_{10} \omega T) \text{ dB} \quad (6)$$

$$\angle G(j\omega) = -\tan^{-1} \omega T \to -90^0. \quad (7)$$

Notice that (6) is similar to (3). This implies that at high frequency, a first order system behaves like an integrator as seen from the magnitude response. Thus the high frequency slope is also -20 dB/decade.

In summary, for a first order system, the magnitude response has two parts :

- At low frequency, the magnitude can be constant and equal to the DC gain of $20\log_{10} K$ dB.
- At high frequency, specifically after $\omega = 1/T$, the magnitude response look like that of an integrator with a slope of -20 dB/decade.
- $\omega = 1/T$ is called the <u>corner</u> frequency and at this frequency, the phase is -45^0 while the magnitude is -3.01 dB below the DC gain. The corner frequency corresponds to the pole at $s = -1/T$.

The phase plots can be sketched as follows :

- Low frequency approximation of 0^0 up to $\omega = 0.1/T$
- Between $\omega = 0.1/T$ and $\omega = 10/T$, draw a straight line through the $(1/T, -45^0)$ point
- High frequency approximation of -90^0 for $\omega > 10/T$.

An example is given in Figure 4 for $G(s) = \frac{2}{s+1}$. The corner frequency is at $\omega = 1$ corresponding to the pole at $s = -1$. The DC gain is at $20 \log_{10} 2$.

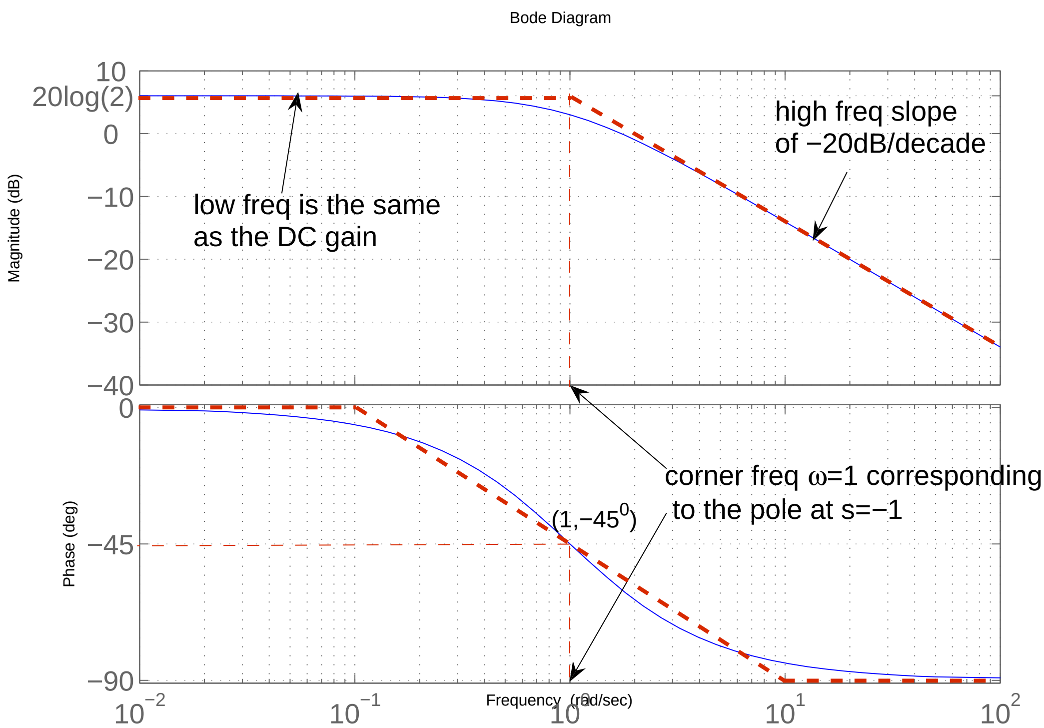

Fig. 4: Bode Diagrams of $G(s) = \frac{2}{s+1}$

4. A second order system with 2 real poles, $G(s) = \frac{K}{(sT_1+1)(sT_2+1)}$. The two poles are at $s = -1/T_1$ and $s = -1/T_2$. Suppose $T_1 > T_2$.

   The magnitude response of $G(j\omega)$ in dB is :

$$|G(j\omega)|_{dB} = K_{dB} + \left|\frac{1}{j\omega T_1 + 1}\right|_{dB} + \left|\frac{1}{j\omega T_2 + 1}\right|_{dB}$$

   where the subscript dB denotes the units in dB. The magnitude response can be summarized as follows :

   - low frequency magnitude is the same as DC gain ie $K_{dB} = 20\log_{10} K$ dB
   - Since this is a second order overdamped system, there are two corner frequencies. One corresponding to each pole. Since $T_1 > T_2$, the first corner frequency is at $1/T_1$ and the second at $1/T_2$ because $1/T_1 < 1/T_2$.
   - There are 3 straight line sections :
     - For $0 < \omega < 1/T_1$, straight horizontal line at K_{dB}.
     - For $1/T_1 < \omega < 1/T_2$, straight line with slope -20dB/decade.
     - For $\omega > 1/T_2$, straight line with slope -40dB/decade. This high frequency portion has a slope of -40dB/decade because at high frequencies, $G(s)$ behaves like a double integrator.

     In terms of the phase response, the straight line asymptotes (approximations) are as follows:

     - K factor does not introduce any phase
     - Each factor $1/(j\omega T_1 + 1)$ and $1/(j\omega T_2 + 1)$ contributes a phase response which is the same as the first order phase response in the previous example.
     - Sum the two phase responses accordingly.

   Figures 5 and 6 show how the magnitude and phase responses of $G(s) = \frac{10}{(10s+1)(s+1)}$ are constructed.

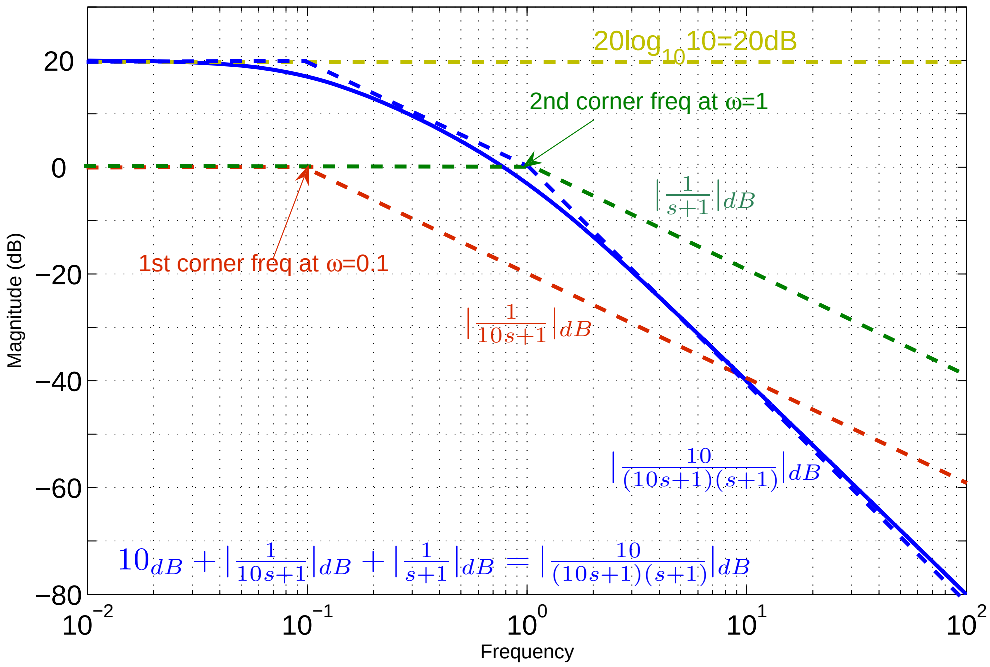

Fig. 5: Magnitude Response of $G(s) = \frac{10}{(10s+1)(s+1)}$

5. A second order system with an integrator and a first order term : $G(s) = \frac{1}{s(Ts+1)}$.

   The same method of constructing the magnitude and phase responses applies. The complete response is the sum of responses of $1/s$ and $\frac{1}{Ts+1}$. Thus

$$\begin{aligned}
\left|\frac{1}{j\omega(j\omega T+1)}\right|_{dB} &= \left|\frac{1}{\omega}\right|_{dB} + \left|\frac{1}{(j\omega T+1)}\right|_{dB} \\
&= -20\log_{10}\omega + \left|\frac{1}{(j\omega T+1)}\right|_{dB} \\
\angle\frac{1}{j\omega(j\omega T+1)} &= \angle\frac{1}{j\omega} + \angle\frac{1}{(j\omega T+1)} \\
&= -90^0 + \angle\frac{1}{(j\omega T+1)}
\end{aligned}$$

   An example for $G(s) = \frac{1}{s(0.1s+1)}$ is given in Figure 7.

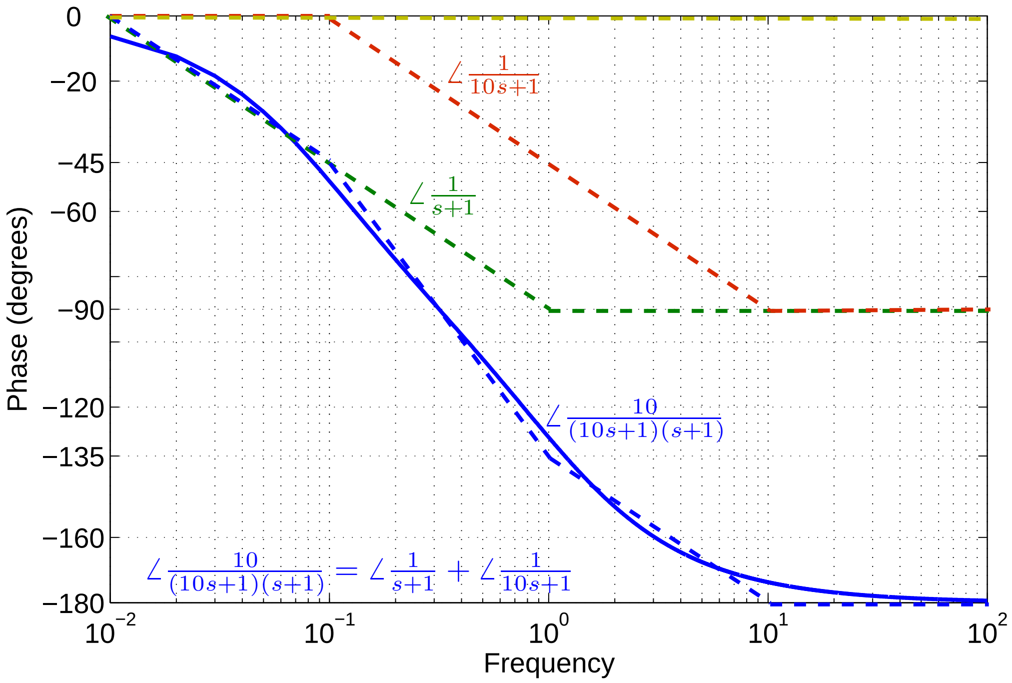

Fig. 6: Phase Response of $G(s) = \frac{10}{(10s+1)(s+1)}$

6. A second order underdamped system $G(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. Poles are complex and damping ratio $\zeta < 1$. $G(s)$ can be rewritten in terms of the <u>normalized</u> frequency, ω/ω_n. Thus $G(s)$ becomes

$$\begin{aligned} G(s) &= \frac{K}{\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1} \\ &= \frac{K}{s'^2 + 2\zeta s' + 1} \end{aligned}$$

where $s' = s/\omega_n$. The frequency response is thus

$$G(j\omega) = \frac{K}{-\frac{\omega}{\omega_n}^2 + 2j\zeta\frac{\omega}{\omega_n} + 1}$$

Bode diagrams can be drawn in terms of the normalized frequency ω/ω_n. In terms of the magnitude response, when $\omega/\omega_n << 1$,

$$\begin{aligned} |G(j\omega)| &= \frac{K}{\sqrt{\left(1 - \cancel{\left(\frac{\omega}{\omega_n}\right)^2}\right)^2 + 4\zeta^2\cancel{\left(\frac{\omega}{\omega_n}\right)^2}}} \\ &\approx K \end{aligned} \tag{8}$$

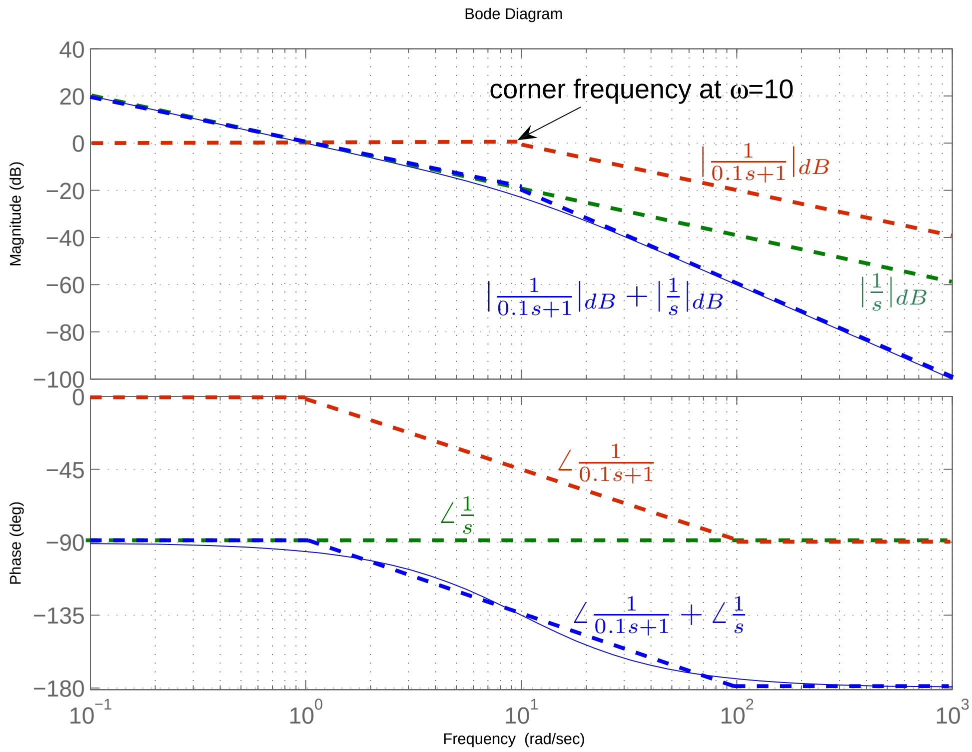

Fig. 7: Bode Diagram of $G(s) = \frac{1}{s(0.1s+1)}$

Hence at frequencies $\omega << \omega_n$, the low frequency approximation is the same as the DC gain of K or $20\log_{10} K$.

At high frequencies where $\omega/\omega_n >> 1$,

$$\begin{aligned} |G(j\omega)| &= \frac{K}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \\ &\approx \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2} \\ &= -40\log_{10}\left(\frac{\omega}{\omega_n}\right) \end{aligned}$$

Thus at high frequencies where $\omega >> \omega_n$, the straight line approximation (asymptote) has a slope of -40 db per decade. The corner frequency is at $\omega/\omega_n = 1$ or $\omega = \omega_n$.

An example is given Figure 8. Note that the magnitude response has a 'hump' or peak around $\omega = \omega_n$.

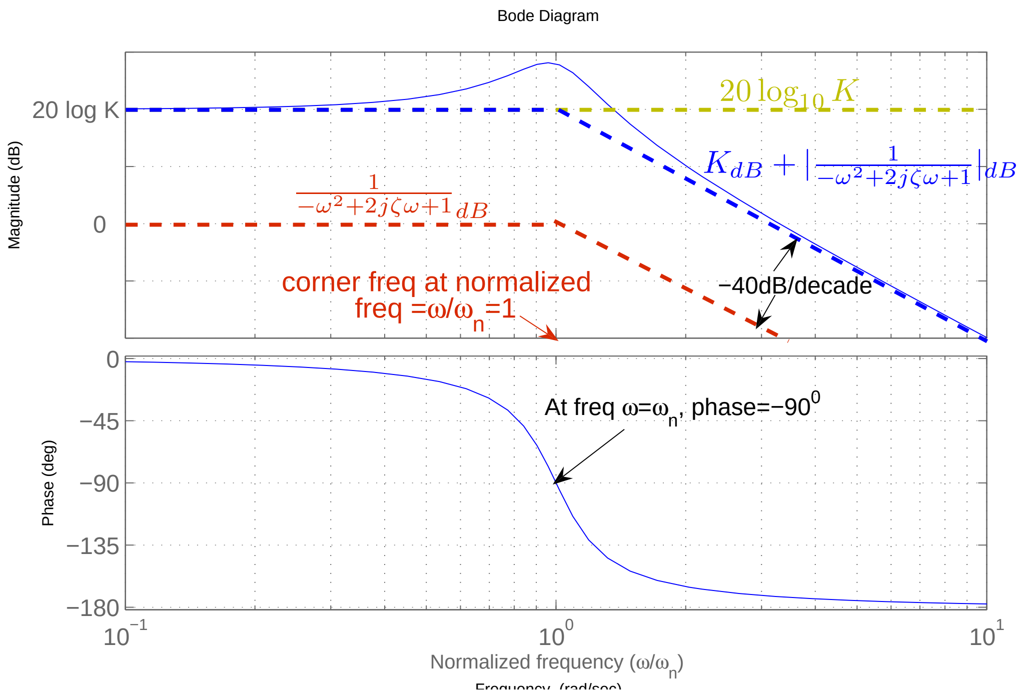

Fig. 8: Bode Diagram of $G(s) = \frac{K}{s'^2+2\zeta s'+1}$

In general, the smaller the damping ratio, ζ, the larger is this peak. The family of Bode diagrams for different values of damping ratio is shown in Figure 9.

There is another concept related to underdamped second order systems when ζ is small. In particular, when $\zeta < 1/\sqrt{2}$, the 'hump' or the peak in the magnitude response is much more pronounced and this is associated with the phenomenon of resonance. The resonance frequency, ω_r, is given in terms of the natural frequency, ω_n, and the damping ratio, ζ as follows :

$$\begin{aligned}
\omega_r &= \omega_n\sqrt{1-2\zeta^2} \text{ valid only for } \zeta < \frac{1}{\sqrt{2}} = 0.7071 \\
M_r &= |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}
\end{aligned}$$

Note from the equations above that $\omega_r \to \omega_n$ and $M_r \to \infty$ when $\zeta \to 0$.

The idea of resonance is one which allows receivers to be tuned to certain carrier frequencies.

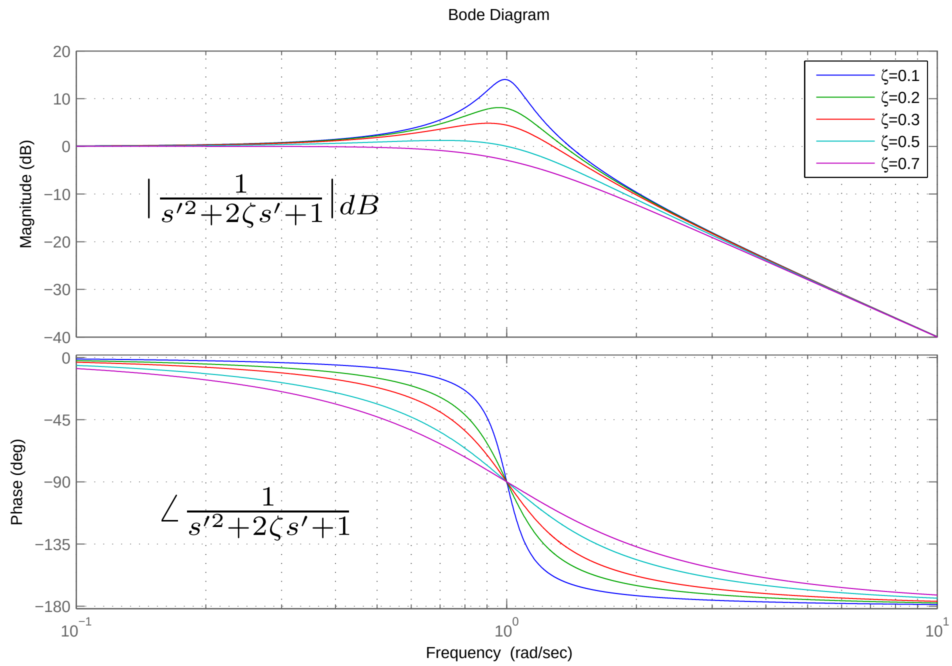

Fig. 9: Family of Bode Diagram for different ζ