EE2023 Signals & Systems
Chapter 9 – Unit Step Responses of LTI Systems

TAN Woei Wan

Dept. of Electrical & Computer Engineering
National University of Singapore

Overview : Useful input-output relationships of LTI Systems

Three useful types of input-output mappings are :

- **Unit Impulse Response**, $y_\delta(t)$ is the output signal when the input is an unit impulse function.
- **Unit Step Response**, $y_{step}(t)$ is the output response when the input is a step signal.
- **Sinusoidal Response**, $y_{ss}(t)$ is the output response when $t \to \infty$ and the input is a sinusoidal signal .

Each of these responses is important to LTI systems because they define certain behaviours that can be generalized for such systems.

- Unit impulse response characterizes a LTI system in time-domain.
- Unit step response is very commonly encountered in practice and they give info about some physical parameters in the LTI system.
- Sinusoidal response is related to the frequency response of the system.

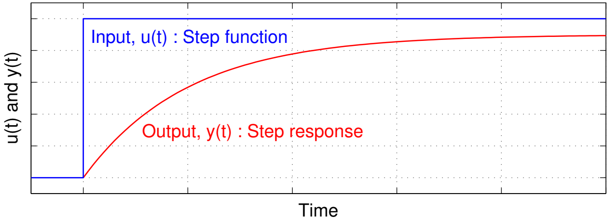
Unit Step Response : Definition

The **unit step response**, $y_{step}(t)$, of a continuous LTI system is defined as the response of the system when the input, $x(t)$, is the unit step function, $U(t)$.

$$x(t) = u(t) \longrightarrow \boxed{G(s)} \longrightarrow$$ Output, $y_{step}(t)$, is the unit step response

Step response is a mathematical expression that describes the output when the input is system changed from the current value to a new level. Many daily tasks/actions are examples of step responses e.g.

- Switching on a fan/kettle
- F1 driver pressing the pit lane speed limiter
- Student receiving a piece of CA homework



When the input $x(t)$ is the unit step function $u(t)$, then $X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}$.

If the system transfer function is $G(s)$, then $Y_{step}(s) = G(s) \times \frac{1}{s}$.

$$\text{Unit Step response, } y_{step}(t) = \mathcal{L}^{-1}\left\{G(s) \cdot \frac{1}{s}\right\}$$

Let $G(s) = K' \dfrac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$, then

$$\begin{aligned} Y_{step}(s) &= K' \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} \times \frac{1}{s} \\ y_{step}(t) &= \mathcal{L}^{-1}\left\{\frac{\alpha_1}{s+p_1} + \ldots + \frac{\alpha_n}{s+p_n} + \frac{\beta}{s}\right\} = \left[\sum_{i=1}^{n} \alpha_i e^{-p_i t} + \beta\right] u(t) \end{aligned}$$

- Note that system poles are $-p_1, \ldots, -p_n$ and $s = 0$ is an input pole.
- If system is BIBO stable, then $\sum_{i=1}^{n} A_i e^{-p_i t} = 0$ when $t \to \infty$ and $\lim_{t\to\infty} y_{step}(t) =$ constant.

Relationship between Unit Step and Unit Impulse Responses

The s-domain expression for the unit step response of a LTI system, $G(s)$, is

$$Y_{step}(s) = G(s)\frac{1}{s}$$

Applying the following concepts to $Y_{step}(s) = \dfrac{G(s)}{s}$,

- the impulse response, $\mathcal{L}\left\{y_\delta(t)\right\} = G(s)$
- Transform of an Integral Property states $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \dfrac{F(s)}{s}$ where $\mathcal{L}\{f(t)\} = F(s)$

the relationship between the unit step and unit impulse responses of a LTI system is

$$\boxed{\begin{aligned} y_{step}(t) &= \int_0^t y_\delta(\tau)\,\mathrm{d}\tau \\ y_\delta(t) &= \frac{\mathrm{d}y_{step}(t)}{\mathrm{d}t} \end{aligned}}$$

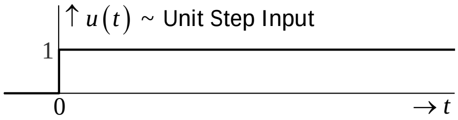
Unit Step Response – First order system

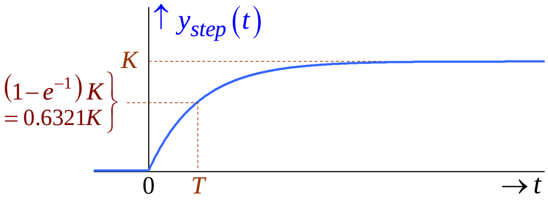
- Unit impulse response of a first order system is $y_\delta(t) = \frac{K}{T} e^{-\frac{t}{T}}\, u(t)$. Hence,

$$y_{step}(t) = \int_0^t y_\delta(\tau)\, d\tau = \int_0^t \frac{K}{T} e^{-\frac{\tau}{T}}\, u(\tau)\, d\tau = \left[-Ke^{-\frac{\tau}{T}}\right]_0^t = \left[K - Ke^{-\frac{t}{T}}\right] u(t)$$

- Unit step response of a first order sytem, $G(s) = \dfrac{K}{Ts+1}$ is:

$$\begin{aligned} Y(s) &= \frac{K}{s\,(Ts+1)} \\ &= \frac{K}{s} - \frac{KT}{Ts+1} \\ y_{step}(t) &= \left[K - Ke^{-\frac{t}{T}}\right] u(t) \end{aligned}$$





Behaviour of $y_{step}(t)$ when $t \to \infty$

- Applying Final Value Theorem, "final value" of the unit step response when $t \to \infty$ is

$$\lim_{t\to\infty} y_{step}(t) = \lim_{s\to 0} sY_{step}(s) = \lim_{s\to 0} G(s) = G(0) = K$$

- When the input is $A \cdot u(t)$, then final value of the step response when $t \to \infty$ is

$$\lim_{t\to\infty} y_{step}(t) = G(0) \times A = K \cdot A$$

Relationship between $y_{step}(t)$ and system pole

- System pole is $s = -\frac{1}{T}$. Hence, $y_{step}(t)$ contains a $\alpha e^{\frac{-t}{T}}$ term.
  - $\alpha e^{\frac{-t}{T}}$ term decays to zero faster if T is smaller.
- When $t = T$, $y_{step}(t) = K[1 - e^{-1}] = 0.632K$
  Time constant, T, can be determined by finding the time taken by the step response to reach 63.2% of the final value.

Unit Step Response – Second order system

Transfer function of standard second order system is $\dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$.

Hence, unit step response of a second order system may be expressed as

$$y_{step}(t) = \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \times \frac{1}{s}\right\}$$

Initial conditions : $y_{step}(0) = \dot{y}_{step}(0) = 0$

- **$y_{step}(t)$ of overdamped ($\zeta > 1$) 2^{nd} sys**: Distinct real poles

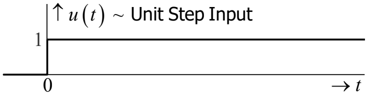
$$\begin{aligned} y_{step}(t) &= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \times \frac{1}{s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{(s+a)(s+b)} \times \frac{1}{s}\right\} \end{aligned}$$

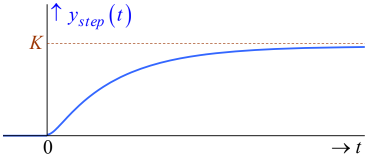
$a = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$ and $b = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

- **$y_{step}(t)$ of overdamped ($\zeta > 1$) 2^{nd} sys:** Distinct real poles (con't)

$$y_{step}(t) = \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{(s+a)(s+b)} \times \frac{1}{s}\right\}$$
$$= \left[K - \frac{K_1}{a}e^{-at} - \frac{K_2}{b}e^{-bt}\right]u(t)$$

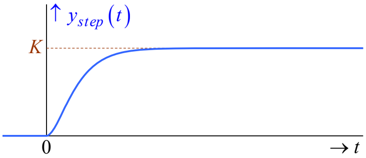
$$K_1 = \frac{K\omega_n^2}{b-a} \text{ and } K_2 = \frac{K\omega_n^2}{a-b}$$





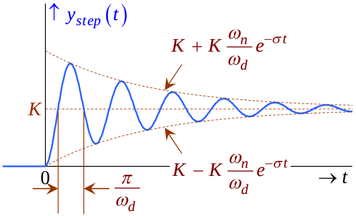
- **$y_{step}(t)$ of critically damped ($\zeta = 1$) 2^{nd} sys:** Repeated real poles

$$y_{step}(t) = \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{(s+\omega_n)^2} \times \frac{1}{s}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{K}{s} - \frac{K}{s+\omega_n} - \frac{K\omega_n}{(s+\omega_n)^2}\right\}$$
$$= K\left(1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right)u(t)$$



- **$y_{step}(t)$ of underdamped $(0 < \zeta < 1)$ 2^{nd} sys**: Complex conjugate poles

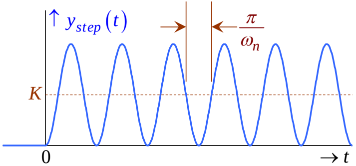
$$y_{step}(t) = \mathcal{L}^{-1}\left\{\frac{K\left(\sigma^2+\omega_d^2\right)}{\left[(s+\sigma)^2+\omega_d^2\right]} \times \frac{1}{s}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{K}{s} - \frac{K(s+\sigma)}{(s+\sigma)^2+\omega_d^2} - \frac{K\sigma}{(s+\sigma)^2+\omega_d^2}\right\}$$



$$y_{step}(t) = K - Ke^{-\sigma t}\left[\cos\left(\omega_d\right)t + \frac{\sigma}{\omega_d}\sin\left(\omega_d t\right)\right]u(t) = K\left[1 - \frac{\omega_n}{\omega_d}e^{-\sigma t}\sin\left(\omega_d t + \phi\right)\right]u(t)$$

- **$y_{step}(t)$ of undamped $(\zeta = 0)$ 2^{nd} sys**: Imaginery conjugate poles

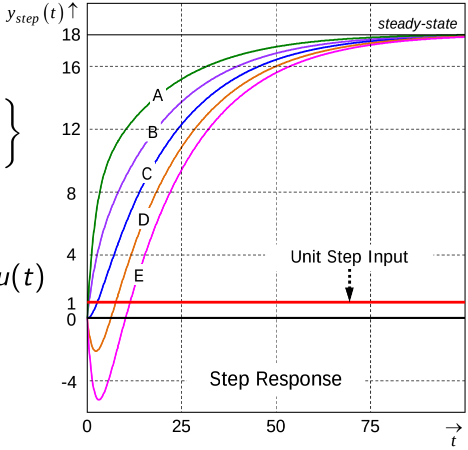
$$y_{step}(t) = \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2+\omega_n^2} \times \frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{K}{s} - \frac{Ks}{s^2+\omega_n^2}\right\}$$
$$= K(1-\cos\omega_n t)\,u(t)$$



Unit Step Response – Effect of an Additional Zero

Consider a system $G(s)$ which has a DC gain of 18, a zero at $s = -\frac{1}{\gamma}$, and poles $s = -0.5$ and $s = -0.05$. Derive the unit step response.

$$
\begin{aligned}
G(s) &= \frac{18(\gamma s + 1)}{40s^2 + 22s + 1} \\
y_{step}(t) &= \mathcal{L}^{-1}\{\frac{G(s)}{s}\} = \mathcal{L}^{-1}\left\{\frac{18(\gamma s + 1)}{s(40s^2 + 22s + 1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{18}{s} + \frac{2-\gamma}{s+0.5} - \frac{20-\gamma}{s+0.05}\right\} \\
&= \left[18 + (2-\gamma)e^{-0.5t} - (20-\gamma)e^{-0.05t}\right] u(t)
\end{aligned}
$$



- Location of system zero does not affect the "final" value.
- $\lim_{t\to\infty} y_{step}(t) = \lim_{s=0} sY_{step}(s) = G(0) = 18.$.

Transient and Steady-state Response

Recall that

$$
\begin{aligned}
Y_{step}(s) &= K' \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} \times \frac{1}{s} \\
y_{step}(t) &= \mathcal{L}^{-1}\left\{ \frac{\alpha_1}{s+p_1} + \dots + \frac{\alpha_n}{s+p_n} + \frac{\beta}{s} \right\} = \underbrace{\sum_{i=1}^{n} \alpha_i e^{-p_i t}\, u(t)}_{\text{transient}} + \underbrace{\beta\, u(t)}_{\text{steady-state}}
\end{aligned}
$$

The step response of a LTI system may be decomposed into 2 parts:-

- **Transient Response** comprising terms that map to the system poles.
  - If the system is stable, this component decays to zero when $t \to \infty$ so it exists only "temporarily".
  - If the system is unstable, this component will "blow up".
- **Steady-state Response** is a constant if the input is a step function. For an arbitrary input, terms in this component map to the input pole(s) so it is a scaled form of the input.

Definition of Steady-State/DC/Static Gain

- Often times, it is useful to know how the step response behaves at steady-state by inspecting $G(s)$.
  - If the system is unstable, then the step response will "blow up".
  - If the system is marginally stable, then the steady-state step response is a sinusoid.
  - If the system is stable, then the "final" value is a constant.
- Suppose the input $x(t) = Au(t)$, i.e. a step function of magnitude A, is applied to a **stable** LTI system. Then,

$$X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{A\,u(t)\} = \frac{A}{s}$$

Using the Final Value Theorem, the step response of a stable LTI system at steady-state ($t \to \infty$) can be expressed as

$$\lim_{t\to\infty} y_{step}(t) = \lim_{s\to 0} sY_{step}(s) = \lim_{s\to 0} sG(s)\frac{A}{s} = A \cdot G(0)$$

$G(0)$ is known as the **Steady-state/DC/Static Gain**. It is a transfer function parameter that can be used to ascertain the "final" value of $y_{step}(t)$ without the need to compute $y_{step}(t)$ or apply the Final Value Theorem.