NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION FOR

(Semester II : 2014/2015)

EE2023 – SIGNALS & SYSTEMS

Apr/May 2015 - Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. This paper contains **EIGHT (8)** questions and comprises **TEN (10)** printed pages.
2. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.
3. This is a **CLOSED BOOK** examination.
4. Programmable calculators are not allowed.
5. Tables of Fourier Transforms, Laplace Transforms and trigonometric identities are provided in Pages 8, 9 and 10, respectively.

SECTION A : Answer ALL questions in this section

Q1. Consider the following differential equation which describes the motion of a vehicle :

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = x(t),$$

where $y(t)$ is the vehicle's position from a reference point, and $x(t)$ represents thrust.

(a) Derive the transfer function of this vehicle, relating the thrust to the vehicle's position. Is the system stable? Justify your answer.

(4 marks)

(b) Find the vehicle's steady state (final) position if a thrust of $x(t) = 2u(t)$ is applied while the initial conditions of the vehicle are $y(0) = 6$ and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$. You may assume $u(t)$ to be the unit step function.

(3 marks)

(c) Give an alternative method to find the vehicle's final position for the same input in part (b) above.

(3 marks)

Q2. Consider the underdamped 2nd order system with a transfer function, $G(s)$, given by

$$G(s) = \frac{80}{s^2 + 4s + 16}.$$

(a) Determine the damping ratio of the system.

(2 marks)

(b) Suppose a step function of magnitude 7 is applied to the system, what is the steady state value of the output signal? Sketch the response of the system, assuming zero initial conditions. You <u>do not need to derive</u> the response due to the DC signal of 7.

(5 marks)

(c) What is the frequency at which the magnitude of the frequency response of the system is at a maximum? Find this maximum frequency response value.

(3 marks)

Q3. The spectrum, $X(f)$, of an energy signal, $x(t)$, is given by

$$X(f)=\cos(\pi f)\mathrm{rect}(f)\otimes\left[-\frac{1}{2}\delta(f+1)+\delta(f)-\frac{1}{2}\delta(f-1)\right].$$

(a) Sketch the magnitude spectrum, $|X(f)|$, and phase spectrum, $\angle X(f)$, of $x(t)$. Label your sketch clearly and adequately.

(6 marks)

(b) Compute the energy of $x(t)$ contained within its 1^{st}-null bandwidth.

(4 marks)

Q4. The signal $x(t) = \mathrm{sinc}^2(5t)$ is sampled at 15Hz to produce the signal $x_s(t)$.

(a) Derive the Fourier transform of the sampled signal $x_s(t)$.

(6 marks)

(b) Sketch the spectrum of the sampled signal $x_s(t)$.

(4 marks)

SECTION B : Answer 3 out of the 4 questions in this section

Q5. Answer **ALL** the short questions (a) to (d).

(a) Find the Laplace transform of $x(t) = te^{-2t}\cos 3t\, u(t)$ where $u(t)$ represents the unit step function.

(4 marks)

(b) The unit step response of an electrical circuit is given by

$$y(t) = 6t - 9e^{-t} + 5e^{-3t} - 12 \qquad t \geq 0.$$

Determine the unit impulse response of the circuit.

(4 marks)

(c) The poles and zeros of a system are given as follows :

Poles : $s = -1,\ -1 \pm j4$
Zeros : $s = \pm j1$

If the steady state response of the system to a unit step input is 1, determine the transfer function of the system.

(6 marks)

(d) If the input and total response of a linear time invariant system are given by $x(t) = 2\cos 10t$ and $y(t) = 5e^{-10t} + 5\sqrt{2}\cos\left(10t + \dfrac{\pi}{4}\right)$ respectively, show that the transfer function of the system is given by

$$G(s) = \frac{5s}{s+10}.$$

(6 marks)

Q6. An energy pulse is modeled by $q(t)=\text{sinc}^2(t)*\left[1.1\,\text{sinc}(1.1t)\right]$, where '$*$' denotes convolution. The spectrum, $Q(f)$, of $q(t)$ is sampled in the frequency domain to form $X(f)=Q(f)\sum_k \delta(f-0.1k)$. Let $x(t)$ denote the inverse Fourier transform of $X(f)$.

(a) Draw a labeled sketch of the spectrum, $Q(f)$, of $q(t)$.

(4 marks)

(b) Draw a labeled sketch of the spectrum, $X(f)$, of $x(t)$.

(4 marks)

(c) By inspection of the sketch in Part (b), or otherwise, determine whether or not $x(t)$ is periodic. If $x(t)$ is periodic, find its fundamental frequency, its Fourier series coefficients, c_k, and its average power.

(12 marks)

Q.7 Consider the periodic signal $x(t)$ shown in Figure Q.7.

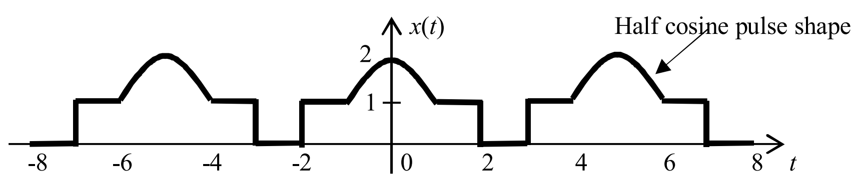

Figure Q.7

(a) Derive the Fourier transform of $x(t)$.

(6 marks)

(b) Derive the Fourier series coefficients of $x(t)$.

(3 marks)

(c) Derive an expression for the average power P of $x(t)$.

(4 marks)

(d) The 98% power containment bandwidth, W, is defined as the smallest bandwidth that contains at least 98% of the average power of the signal. Derive an expression for the 98% power containment bandwidth W.

(7 marks)

Q.8 Figure Q.8 shows the Bode magnitude plot of a system, whose transfer function is $G(s)$.

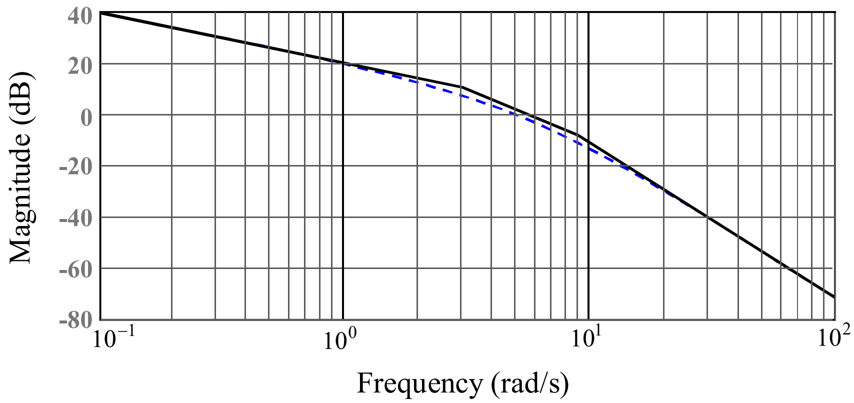

Figure Q.8: Bode magnitude plot of a system with transfer function $G(s)$

(a) Identify the transfer function, $G(s)$.

(6 marks)

(b) Describe the low frequency asymptote of the Bode phase plot for $G(s)$.

(2 marks)

(c) Determine the output signal of the system when the input signal $x(t) = 13\sin\left(0.8t + \frac{\pi}{4}\right)$ is applied.

(6 marks)

(d) Derive the impulse response of the system.

(6 marks)

END OF QUESTIONS

This page is intentionally left blank to facilitate detachment of the formula sheet for easy reference.

Fourier Series: $\begin{cases} X_k = \dfrac{1}{T}\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k\,t/T)\,dt \\ x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(j2\pi k\,t/T) \end{cases}$

Fourier Transform: $\begin{cases} X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$

FOURIER TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
Triangle	$\mathrm{tri}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}^2(fT)$
Sine Cardinal	$\mathrm{sinc}\left(\frac{t}{T}\right)$	$T\,\mathrm{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp\left(-\alpha^2\pi^2 f^2\right)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES		
	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi f t_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(\zeta)X_2(f-\zeta)\,d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
		$\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n[th] order Ramp	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES

	Time-domain	*s*-domain
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)u(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\frac{1}{s}X(s)$
Differentiation in the time-domain	$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\frac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\frac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\frac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\frac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

	$y_{step}(t)$	$Y_{step}(s)$	**SYSTEM PARAMETERS**
Step response of 1st order system	$K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	$\frac{1}{s}\cdot\frac{K}{(sT+1)}$	T: System Time-constant K: System Steady-state (or DC) Gain
Step response of 2nd order <u>underdamped</u> system: $(0<\zeta<1)$	$K\left[1-\frac{\exp(-\omega_n\zeta t)}{(1-\zeta^2)^{0.5}}\sin\left(\omega_n(1-\zeta^2)^{0.5}t+\phi\right)\right]u(t)$	$\frac{1}{s}\cdot\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	ω_n: System Undamped Natural Frequency ζ: System Damping Factor ω_d: System Damped Natural Frequency K: System Steady-state (or DC) Gain $\sigma=\omega_n\zeta$ $\omega_d^2=\omega_n^2(1-\zeta^2)$ $\omega_n^2=\sigma^2+\omega_d^2$ $\tan(\phi)=\omega_d/\sigma$
	$K\left[1-\left(\frac{\sigma^2+\omega_d^2}{\omega_d^2}\right)^{0.5}\exp(-\sigma t)\sin(\omega_d t+\phi)\right]u(t)$	$\frac{1}{s}\cdot\frac{K(\sigma^2+\omega_d^2)}{(s+\sigma)^2+\omega_d^2}$	

2nd order system - RESONANCE - $\left(0\le\zeta<1/\sqrt{2}\right)$	*RESONANCE FREQUENCY :* $\omega_r=\omega_n(1-2\zeta^2)^{0.5}$	*RESONANCE PEAK :* $M_r=\lvert H(j\omega_r)\rvert=\frac{K}{2\zeta(1-\zeta^2)^{0.5}}$

TRIGONOMETRIC IDENTITIES	
$\exp(\pm j\theta)=\cos(\theta)\pm j\sin(\theta)$	$\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$
$\cos(\theta)=\frac{1}{2}\left[\exp(j\theta)+\exp(-j\theta)\right]$	$\cos(\alpha\pm\beta)=\cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
$\sin(\theta)=\frac{1}{j2}\left[\exp(j\theta)-\exp(-j\theta)\right]$	$\tan(\alpha\pm\beta)=\frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$
$\sin^2(\theta)+\cos^2(\theta)=1$	
$\sin(2\theta)=2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$	$\cos(\alpha)\cos(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
$\sin^2(\theta)=\frac{1}{2}\left[1-\cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta)=\frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$
$\cos^2(\theta)=\frac{1}{2}\left[1+\cos(2\theta)\right]$	$\mathbf{C}\cos(\theta)-\mathbf{S}\sin(\theta)=\sqrt{\mathbf{C}^2+\mathbf{S}^2}\cos\left[\theta+\tan^{-1}\left(\frac{\mathbf{S}}{\mathbf{C}}\right)\right]$