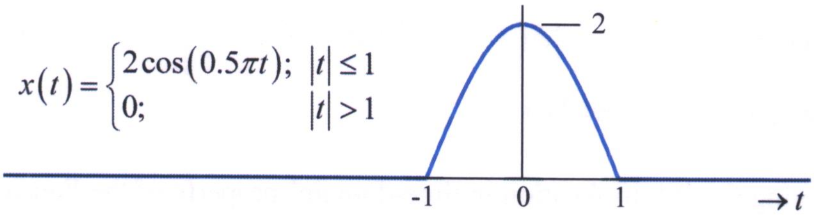
EE2023/TEE2023 TUTORIAL 3 (SOLUTIONS)

Solution to Q.1

$$x(t)=\begin{cases}2\cos(0.5\pi t); & |t|\le 1\\ 0; & |t|>1\end{cases}$$



(a)

Method 1: By applying direct Fourier transform:

$$X(f)=\int_{-\infty}^{\infty}x(t)\exp(-j2\pi ft)\,dt=\int_{-1}^{1}2\cos(0.5\pi t)\exp(-j2\pi ft)\,dt$$

$$=2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\cos(2\pi ft)}_{even\ function\ of\ t}dt-\cancel{j2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\sin(2\pi ft)}_{odd\ function\ of\ t}dt}$$

Using:
$\exp(\pm j\theta)=\cos(\theta)\pm j\sin(\theta)$

$$=4\int_{0}^{1}\cos(0.5\pi t)\cos(2\pi ft)\,dt$$

$$=2\int_{0}^{1}\cos\big((2\pi f-0.5\pi)t\big)+\cos\big((2\pi f+0.5\pi)t\big)\,dt$$

Using:
$\cos(a)\cos(b)=\frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$

$$=2\left[\frac{\sin\big((2\pi f-0.5\pi)t\big)}{2\pi f-0.5\pi}+\frac{\sin\big((2\pi f+0.5\pi)t\big)}{2\pi f+0.5\pi}\right]_0^1$$

$$=2\left(\frac{\sin(2\pi f-0.5\pi)}{2\pi f-0.5\pi}+\frac{\sin(2\pi f+0.5\pi)}{2\pi f+0.5\pi}\right)$$

$$=\frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5}+\frac{\cos(2\pi f)}{2f+0.5}\right)$$

Using:
$\sin(a-b)=\sin(a)\cos(b)-\cos(a)\sin(b)$

$$=\frac{2\cos(2\pi f)}{\pi}\left(\frac{-1}{2f-0.5}+\frac{1}{2f+0.5}\right)$$

$$=\frac{2\cos(2\pi f)}{\pi}\left(\frac{-2f-0.5+2f-0.5}{4f^2-0.25}\right)$$

$$=\frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}$$

Method 2: By applying Fourier transform properties:

$$x(t) = 2\cos(0.5\pi t)\cdot \mathrm{rect}(0.5t)$$

$$\Im\{2\cos(0.5\pi t)\} = 2\left[\frac{1}{2}\{\delta(f-0.25)+\delta(f+0.25)\}\right] = \delta(f-0.25)+\delta(f+0.25)$$

$$\Im\{\mathrm{rect}(0.5t)\} = 2\mathrm{sinc}(2f)$$

Applying the 'Multiplication in time-domain' property of the Fourier transform

$$\left[x(t) = \underbrace{2\cos(0.5\pi t)\cdot \mathrm{rect}(0.5t)}_{\substack{\text{Multiplication in}\\ \text{time-domain}}}\right] \rightleftarrows \left[X(f) = \underbrace{\Im\{2\cos(0.5\pi t)\} * \Im\{\mathrm{rect}(0.5t)\}}_{\substack{\text{Convolution in}\\ \text{frequency-domain}}}\right]$$

we get

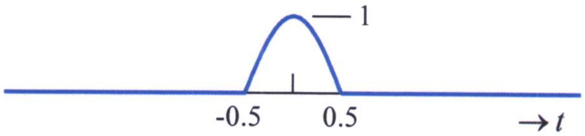
$$\begin{aligned} X(f) &= [\delta(f-0.25)+\delta(f+0.25)] * 2\mathrm{sinc}(2f) \\ &= 2\mathrm{sinc}(2(f-0.25)) + 2\mathrm{sinc}(2(f+0.25)) \\ &= 2\mathrm{sinc}(2f-0.5) + 2\mathrm{sinc}(2f+0.5) \\ &= 2\left(\frac{\sin(2\pi f - 0.5\pi)}{\pi(2f-0.5)} + \frac{\sin(2\pi f + 0.5\pi)}{\pi(2f+0.5)}\right) \end{aligned}$$

...... Same result obtained by **Method 1**

(b)

From Part (a): $X(f)=\dfrac{2\cos(2\pi f)}{\pi(0.25-4f^2)}$

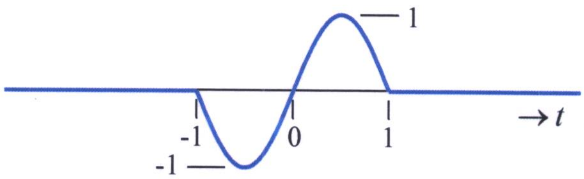
$\tilde{y}(t)=0.5x(2t)$



Applying the ***scaling property***:

$$\tilde{Y}(f)=0.5\left[\frac{1}{2}X\left(\frac{f}{2}\right)\right]=\frac{1}{4}X\left(\frac{f}{2}\right) \qquad \cdots\cdots\cdots (*)$$

$y(t)=\tilde{y}(t-0.5)-\tilde{y}(t+0.5)$



Applying the ***time-shifting property***:

$$Y(f)=\tilde{Y}(f)\exp\left(-j2\pi f\left(\frac{1}{2}\right)\right)-\tilde{Y}(f)\exp\left(j2\pi f\left(\frac{1}{2}\right)\right) \qquad \cdots\cdots\cdots (**)$$

Substituting $(*)$ into $(**)$:

$$\begin{cases} Y(f)=\frac{1}{4}X\left(\frac{f}{2}\right)\exp(-j\pi f)-\frac{1}{4}X\left(\frac{f}{2}\right)\exp(j\pi f) \\ \quad =-j\frac{1}{2}X\left(\frac{f}{2}\right)\sin(\pi f) \\ \quad =\frac{1}{j2}\left[\frac{2\cos(\pi f)}{\pi(0.25-f^2)}\right]\sin(\pi f) \\ \quad =\frac{1}{j2}\left[\frac{\sin(2\pi f)}{\pi(0.25-f^2)}\right] \end{cases}$$

Solution to Q.2

(a)

$$\frac{E_x(f_{3dB})}{E_x(0)} = \frac{1}{2}$$

$$\frac{16e^{-2f_{3dB}}}{16} = \frac{1}{2}$$

$$e^{-2f_{3dB}} = \frac{1}{2}$$

$$e^{2f_{3dB}} = 2$$

$$\therefore f_{3dB} = \frac{1}{2}\ln(2)$$

(b)

$$E_x(f) = |X(f)|^2 = 16e^{-2|f|}$$

$$|X(f)| = 4e^{-|f|}$$

$$\angle X(f) = -0.5f$$

$$X(f) = |X(f)|e^{j\angle X(f)} = 4e^{-|f|}e^{j(-0.5f)} = 4e^{-|f|}e^{-j0.5f}$$

Solution to Q.3

Spectrum of $x(t)$:

$$x(t) = \alpha \operatorname{tri}\left(\frac{t}{\alpha}\right)$$

$$X(f) = \alpha\left[\alpha \operatorname{sinc}^2(\alpha f)\right] = \alpha^2 \operatorname{sinc}^2(\alpha f)$$

$$|X(f)| = \alpha^2 \operatorname{sinc}^2(\alpha f)$$

$$\angle X(f) = 0$$

Spectrum of $x'(t) = \dfrac{dx(t)}{dt}$:

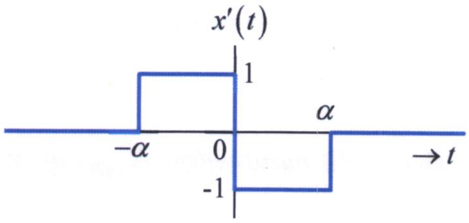
$$x'(t) = \frac{dx(t)}{dt}$$

$$X'(f) = j2\pi f X(f) = j2\pi f\left[\alpha^2 \operatorname{sinc}^2(\alpha f)\right]$$

$$\text{Energy spectral density} = |X'(f)|^2 = \left|j2\pi f \alpha^2 \operatorname{sinc}^2(\alpha f)\right| = 4\pi^2 f^2 \alpha^4 \operatorname{sinc}^4(\alpha f)$$

Total energy of $x'(t)$:

The signal $x'(t)$ is shown to be :



Hence the total energy is :

$$E = \int_{-\infty}^{\infty} |x'(t)|^2 dt = \int_{-\alpha}^{\alpha} 1.dt = [t]_{-\alpha}^{\alpha} = 2\alpha$$

Solution to Q.4

Given: $X(f) = \exp(-\alpha|f|);\ \ \alpha > 0$

(a) ***Energy Spectral Density of $x(t)$:***

$$E_x(f) = |X(f)|^2 = \exp(-2\alpha|f|)$$

Energy of $x(t)$ contained within a bandwidth of B:

$$E_B = \int_{-B}^{B} E_x(f)\,df = 2\int_0^B \exp(-2\alpha f)\,df = 2\left[\frac{\exp(-2\alpha f)}{-2\alpha}\right]_0^B = \frac{1}{\alpha}\left[1-\exp(-2\alpha B)\right]$$

Total energy of $x(t)$:

$$E = \underbrace{\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df}_{\text{Rayleigh Energy Theorem}} = \int_{-\infty}^{\infty} E_x(f)\,df = E_B\big|_{B=\infty} = \frac{1}{\alpha}$$

99% energy containment bandwidth, W, of $x(t)$:

$$\left[\underbrace{\frac{1}{\alpha}\left[1-\exp(-2\alpha W)\right]}_{E_B|_{B=W}} = 0.99E = \frac{0.99}{\alpha}\right] \rightarrow \exp(2\alpha W) = 100$$

$$\rightarrow W = \frac{1}{\alpha}\ln(10)\ \text{Hz}$$

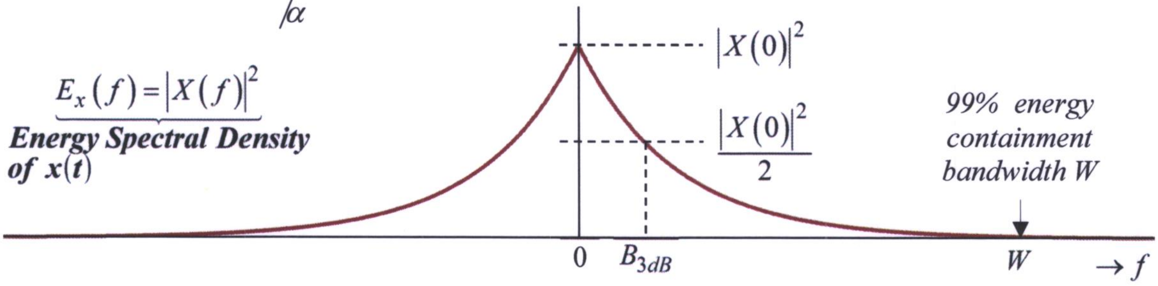
(b) ***3dB bandwidth, B_{3dB}, of $x(t)$:***

By definition, $|X(B_{3dB})| = \dfrac{|X(0)|}{\sqrt{2}}$.

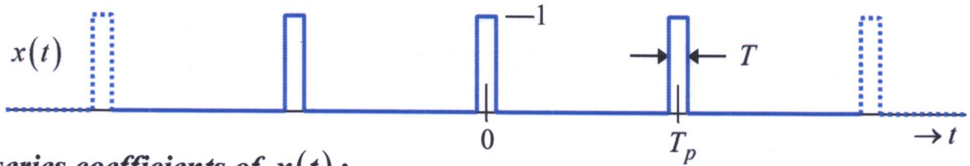
$$\text{Solving: } \left\{\begin{array}{l} |X(f)| = \exp(-\alpha|f|) \\ \left.\begin{array}{l} |X(B_{3dB})| = \exp(-\alpha B_{3dB}) \\ |X(0)| = 1 \end{array}\right\} \rightarrow \exp(-\alpha B_{3dB}) = \dfrac{1}{\sqrt{2}} \\ \qquad\qquad\qquad\qquad\qquad \rightarrow B_{3dB} = \dfrac{1}{2\alpha}\ln(2)\ \text{Hz} \end{array}\right.$$

Percent energy contained within the 3dB bandwidth:

$$\frac{E_B}{E}\times 100\Bigg|_{B=B_{3dB}} = \frac{\cancel{\frac{1}{\alpha}}\left[1-\exp\left(-\cancel{2\alpha}\frac{\ln(2)}{\cancel{2\alpha}}\right)\right]}{\cancel{\frac{1}{\alpha}}}\times 100 = \left[1-\exp(-\ln(2))\right]\times 100 = \left[1-\exp(\ln(1/2))\right]\times 100 = 5$$



Solution to Q.5



(a) ***Fourier series coefficients of $x(t)$:***

$$X_k = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x(t)\exp\left(-j2\pi kt/T_p\right)dt = \frac{1}{T_p}\int_{-0.5T}^{0.5T} \exp\left(-j2\pi kt/T_p\right)dt$$

$$= \frac{1}{T_p}\left[\frac{\exp\left(-j2\pi kt/T_p\right)}{-j2\pi k/T_p}\right]_{-0.5T}^{0.5T}$$

$$= \frac{1}{T_p}\left[\frac{\exp(-j\pi kT/T_p)}{-j2\pi k/T_p} - \frac{\exp(j\pi kT/T_p)}{-j2\pi k/T_p}\right]$$

$$= \frac{1}{T_p}\left[\frac{\exp(j\pi kT/T_p)}{j2\pi k/T_p} - \frac{\exp(-j\pi kT/T_p)}{j2\pi k/T_p}\right]$$

$$= \frac{1}{T_p}\frac{1}{\pi k/T_p}\frac{1}{2j}\left[\exp(j\pi kT/T_p) - \exp(-j\pi kT/T_p)\right]$$

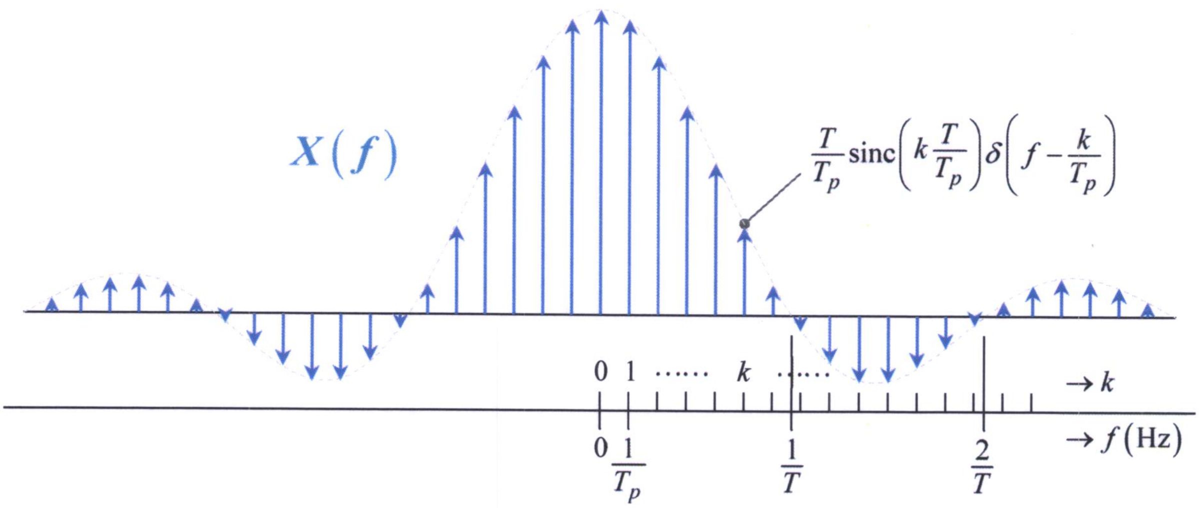
$$= \frac{1}{T_p}\frac{1}{\pi k/T_p}\sin\left(\pi kT/T_p\right)$$

$$= \frac{T}{T_p}\left[\frac{\sin\left(\pi kT/T_p\right)}{\pi kT/T_p}\right]$$

$$= \frac{T}{T_p}\operatorname{sinc}\left(k\frac{T}{T_p}\right)$$

Frequency spectrum (or Fourier transform) of $x(t)$:

$$X(f) = \sum_{k=-\infty}^{\infty} X_k\delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T}{T_p}\operatorname{sinc}\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$



(b) ***Power Spectral Density of*** $\boldsymbol{x(t)}$**:**

$$P_x(f) = \sum_{k=-\infty}^{\infty} |X_k|^2 \delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T^2}{T_p^2} \text{sinc}^2\left(k\frac{T}{T_p}\right) \delta\left(f - \frac{k}{T_p}\right)$$

Average power of $\boldsymbol{x(t)}$**:**

$$P = \underbrace{\int_{-\infty}^{\infty} P_x(f)\,df = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} |x(t)|^2\,dt}_{\text{Parseval Power Theorem}} = \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = \frac{1}{T_p}[t]_{-0.5T}^{0.5T} = \frac{T}{T_p}$$

99% power containment bandwidth, W, of $\boldsymbol{x(t)}$**:**

$$W = \frac{K}{T_p}(\text{Hz}) \quad \cdots \quad \left(\begin{array}{l} \text{where } K \text{ satisfies } \displaystyle\sum_{k=-K}^{K} |X_k|^2 \geq 0.99P > \sum_{k=-(K-1)}^{(K-1)} |X_k|^2 \\ \text{in which } |X_k|^2 = \dfrac{T^2}{T_p^2}\text{sinc}^2\left(k\dfrac{T}{T_p}\right) \text{ and } P = \dfrac{T}{T_p}. \end{array} \right).$$

Supplementary Questions (Solutions)

S1(a) $x(t) = \cos\left(2\pi f_c t\right)u(t)$

$$X(f) = \frac{1}{2}\left[\delta\left(f - f_c\right) + \delta\left(f + f_c\right)\right] \otimes \left[\frac{1}{2}\left\{\delta(f) + \frac{1}{j\pi f}\right\}\right]$$

$$= \frac{1}{4}\left[\delta(f - f_c) + \frac{1}{j\pi(f - f_c)}\right] + \frac{1}{4}\left[\delta(f + f_c) + \frac{1}{j\pi(f + f_c)}\right]$$

$$= \frac{1}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] + \frac{1}{4}\left[\frac{1}{j\pi(f - f_c)} + \frac{1}{j\pi(f + f_c)}\right]$$

$$= \frac{1}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] + \frac{1}{4}\left[\frac{j\pi(f + f_c) + j\pi(f - f_c)}{-\pi^2\left(f^2 - f_c^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] + \frac{1}{4}\left[\frac{2j\pi f}{\pi^2\left(f_c^2 - f^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] + \left[\frac{jf}{2\pi\left(f_c^2 - f^2\right)}\right]$$

S1(b) $x(t) = \sin\left(2\pi f_c t\right)u(t)$

$$X(f) = \frac{1}{2j}\left[\delta\left(f - f_c\right) - \delta\left(f + f_c\right)\right] \otimes \left[\frac{1}{2}\left\{\delta(f) + \frac{1}{j\pi f}\right\}\right]$$

$$= \frac{1}{4j}\left[\delta(f - f_c) + \frac{1}{j\pi(f - f_c)}\right] - \frac{1}{4}\left[\delta(f + f_c) + \frac{1}{j\pi(f + f_c)}\right]$$

$$= \frac{1}{4j}\left[\delta(f - f_c) - \delta(f + f_c)\right] + \frac{1}{4j}\left[\frac{1}{j\pi(f - f_c)} - \frac{1}{j\pi(f + f_c)}\right]$$

$$= \frac{1}{4j}\left[\delta(f - f_c) - \delta(f + f_c)\right] + \frac{1}{4j}\left[\frac{j\pi(f + f_c) - j\pi(f - f_c)}{-\pi^2\left(f^2 - f_c^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f - f_c) - \delta(f + f_c)\right] + \frac{1}{4}\left[\frac{2j\pi f_c}{\pi^2\left(f_c^2 - f^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f - f_c) - \delta(f + f_c)\right] + \left[\frac{jf}{2\pi\left(f_c^2 - f^2\right)}\right]$$

S1(c) $s(t) = e^{-\alpha t}\cos\left(\omega_c t\right)u(t)$

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \\
&= \int_{-\infty}^{\infty} e^{-\alpha t}\cos\left(\omega_c t\right)u(t)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\cos\left(\omega_c t\right)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\left\{\frac{1}{2}\left[e^{f\omega_c t} + e^{-f\omega_c t}\right]\right\}e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_{0}^{\infty}\left[e^{(-\alpha + j\omega_c - j\omega)t} + e^{(-\alpha - j\omega_c - j\omega)t}\right]dt \\
&= \frac{1}{2}\int_{0}^{\infty}\left[e^{-(\alpha - j\omega_c + j\omega)t} + e^{-(\alpha + j\omega_c + j\omega)t}\right]dt \\
&= \frac{1}{2}\left[\frac{e^{-(\alpha - j\omega_c + j\omega)t}}{-(\alpha - j\omega_c + j\omega)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-(\alpha + j\omega_c + j\omega)t}}{-(\alpha + j\omega_c + j\omega)}\right]_0^{\infty} \\
&= \frac{1}{2}\frac{1}{\alpha - j\omega_c + j\omega} + \frac{1}{2}\frac{1}{\alpha + j\omega_c + j\omega} \\
&= \frac{1}{2}\frac{\alpha + j\omega_c + j\omega + \alpha - j\omega_c + j\omega}{\left(\alpha + j\omega\right)^2 + \omega_c^2} \\
&= \frac{\alpha + j\omega}{\left(\alpha + j\omega\right)^2 + \omega_c^2}
\end{aligned}
$$

S1(d) $s(t) = e^{-\alpha t}\sin\left(\omega_c t\right)u(t)$

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \\
&= \int_{-\infty}^{\infty} e^{-\alpha t}\sin\left(\omega_c t\right)u(t)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\sin\left(\omega_c t\right)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\left\{\frac{1}{2j}\left[e^{f\omega_c t} - e^{-f\omega_c t}\right]\right\}e^{-j\omega t}\,dt \\
&= \frac{1}{2j}\int_{0}^{\infty}\left[e^{(-\alpha + j\omega_c - j\omega)t} - e^{(-\alpha - j\omega_c - j\omega)t}\right]dt \\
&= \frac{1}{2j}\int_{0}^{\infty}\left[e^{-(\alpha - j\omega_c + j\omega)t} - e^{-(\alpha + j\omega_c + j\omega)t}\right]dt \\
&= \frac{1}{2j}\left[\frac{e^{-(\alpha - j\omega_c + j\omega)t}}{-(\alpha - j\omega_c + j\omega)}\right]_0^{\infty} - \frac{1}{2j}\left[\frac{e^{-(\alpha + j\omega_c + j\omega)t}}{-(\alpha + j\omega_c + j\omega)}\right]_0^{\infty} \\
&= \frac{1}{2j}\frac{1}{\alpha - j\omega_c + j\omega} - \frac{1}{2j}\frac{1}{\alpha + j\omega_c + j\omega} \\
&= \frac{1}{2j}\frac{\alpha + j\omega_c + j\omega - \alpha + j\omega_c - j\omega}{\left(\alpha + j\omega\right)^2 + \omega_c^2} \\
&= \frac{\omega_c}{\left(\alpha + j\omega\right)^2 + \omega_c^2}
\end{aligned}
$$

S2 $e^{-\alpha t}u(t) \Leftrightarrow \dfrac{1}{\alpha + j2\pi f} = \dfrac{1}{\alpha + j\omega}$

Given: $tx(t) \Leftrightarrow j\dfrac{d}{d\omega}X(j\omega)$

Let: $x(t) \Leftrightarrow \dfrac{1}{\alpha + j\omega}$

$$tx(t) \Leftrightarrow j\frac{d}{d\omega}\left[\frac{1}{\alpha + j\omega}\right] = j.j.(-1).\frac{1}{(\alpha + j\omega)^2} = \frac{1}{(\alpha + j\omega)^2}$$

$$t^2x(t) \Leftrightarrow j\frac{d}{d\omega}\left[\frac{(1)}{(\alpha + j\omega)^2}\right] = j.j.\frac{(1)(-2)}{(\alpha + j\omega)^3} = \frac{(1)(2)}{(\alpha + j\omega)^3}$$

$$t^3x(t) \Leftrightarrow j\frac{d}{d\omega}\left[\frac{(1)(2)}{(\alpha + j\omega)^3}\right] = j.j\frac{(1)(2)(-3)}{(\alpha + j\omega)^4} = \frac{(1)(2)(3)}{(\alpha + j\omega)^4}$$

In general, we have: $t^{n-1}x(t) \Leftrightarrow \dfrac{(n-1)!}{(\alpha + j\omega)^n}$, hence : $\dfrac{t^{n-1}}{(n-1)!} \Leftrightarrow \dfrac{1}{(\alpha + j\omega)^n}$

S3 $e^{-\alpha t}u(t) \Leftrightarrow \dfrac{1}{\alpha + j\omega}$

$$\frac{1}{2 - \omega^2 + j2\omega} = \frac{1}{(2 + j\omega)(1 + j\omega)}$$

Let: $\dfrac{1}{(1 + j\omega)(2 + j\omega)} = \dfrac{A}{2 + j\omega} + \dfrac{B}{1 + j\omega} = \dfrac{A(2 + j\omega) + B(1 + j\omega)}{(1 + j\omega)(2 + j\omega)}$

Hence for the numerators, we have: $A(1 + j\omega) + B(2 + j\omega) = 1$

Comparing constants: $2A + B = 1$

Comparing j terms: $A + B = 0 \rightarrow A = -B$

Substituting $A = -B$ into the constants equations, we have: $2A + (-A) = 1 \rightarrow A = 1$ & $B = -1$

Hence:

$$\frac{1}{(1 + j\omega)(2 + j\omega)} = \frac{1}{1 + j\omega} - \frac{1}{2 + j\omega}$$

Taking the inverse Fourier transform:

$$\frac{1}{1 + j\omega} - \frac{1}{2 + j\omega} \Leftrightarrow e^{-t}u(t) - e^{-2t}u(t)$$

S4 $x(t) \Leftrightarrow rect(\pi f)$; $y(t) = \dfrac{d}{dt}x(t)$

Fourier transform of $y(t)$ is: $Y(f) = j2\pi f.X(f) = j2\pi f \,\mathrm{rect}(\pi f)$

Energy of $y(t)$: $E_y = \int_{-\infty}^{\infty}|Y(f)|^2 df = \int_{-\infty}^{\infty}|j2\pi f \,\mathrm{rect}(\pi f)|^2 df = \int_{-1/2\pi}^{1/2\pi} 4\pi^2 f^2 df = \dfrac{4\pi^2}{3}\left[f^3\right]_{-1/2\pi}^{1/2\pi} = \dfrac{1}{3\pi}$

S5 Given: $\frac{\pi}{\alpha}e^{-2\pi\alpha|t|} \Leftrightarrow \frac{1}{\alpha^2+f^2}$

Using duality: $\frac{1}{\alpha^2+t^2} \Leftrightarrow \frac{\pi}{\alpha}e^{-2\pi\alpha|f|}$

The total energy is:

$$E = \int_{-\infty}^{\infty}\left|\frac{\pi}{\alpha}e^{-2\pi\alpha|f|}\right|^2 df = \frac{2\pi}{\alpha}\int_0^{\infty} e^{-4\pi\alpha f}\,df = \frac{2\pi}{\alpha}\left[\frac{e^{-4\pi\alpha f}}{-4\pi\alpha}\right]_0^{\infty} = \frac{2\pi}{\alpha}\left[\frac{1}{4\pi\alpha}\right] = \frac{1}{2\alpha^2}$$

The energy up to a bandwidth of B Hz is:

$$E_B = \int_{-B}^{B}\left|\frac{\pi}{\alpha}e^{-2\pi\alpha|f|}\right|^2 df = \frac{2\pi}{\alpha}\int_0^{B} e^{-4\pi\alpha f}\,df = \frac{2\pi}{\alpha}\left[\frac{e^{-4\pi\alpha f}}{-4\pi\alpha}\right]_0^{B} = \frac{2\pi}{\alpha}\left[\frac{1}{4\pi\alpha} - \frac{e^{-4\pi\alpha B}}{4\pi\alpha}\right] = \frac{1}{2\alpha^2}\left[1-e^{-4\pi\alpha B}\right]$$

For the 99% energy containment bandwidth, we have $E_B = 0.99E$:

$$\frac{1}{2\alpha^2}\left[1-e^{-4\pi\alpha B}\right] = 0.99\left[\frac{1}{2\alpha^2}\right]$$

$$1-e^{-4\pi\alpha B} = 0.99$$

$$e^{-4\pi\alpha B} = 0.01$$

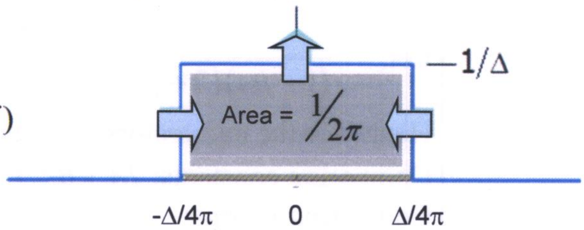
$$e^{4\pi\alpha B} = 100$$

$$B = \frac{1}{4\pi\alpha}\ln(100) = \frac{0.366}{\alpha}$$

S6 Suppose the Dirac-δ function being defined as: $d(f) = \lim_{f\to 0}\frac{1}{\Delta}rect\left(\frac{f}{\Delta}\right)$

Then for $\delta(\omega f) = \delta(2\pi f)$, we can define:

$$\delta(2\pi f) = \lim_{f\to 0}\frac{1}{\Delta}rect\left(\frac{f}{\Delta/2\pi}\right) = \frac{1}{2\pi}\lim_{f\to 0}\frac{1}{\Delta}rect\left(\frac{f}{\Delta}\right) = \frac{1}{2\pi}\delta(f)$$



Hence: $\delta(f) = 2\pi\delta(2\pi f) = 2\pi\delta(\omega)$
