NATIONAL UNIVERSITY OF SINGAPORE

EE2023/EE2023E/TEE2023 – SIGNALS AND SYSTEMS

(Semester II : 2018/2019)

Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write only your student number. Do not write your name.

2. Students should write the answers for each question on a new page.

3. This paper contains **EIGHT (8)** questions and comprises **TWELVE (12)** printed pages.

4. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.

5. This is a **CLOSED BOOK** examination. However you are allowed to bring one self-prepared A4-size help sheet to the examination hall.

6. Programmable and/or graphic calculators are not allowed.

7. Tables of formulas are provided on Pages 9 to 12.

SECTION A : Answer ALL questions in this section

Q1. The signal $x(t) = 4\operatorname{sinc}^2(2t)\cos(8\pi t)$ is sampled at 4Hz to obtain the sampled signal $x_s(t)$.

(a) Sketch the spectrum $X(f)$ of the signal $x(t)$.

(4 marks)

(b) Derive the Fourier Transform, $X_s(f)$, of the sampled signal $x_s(t)$ and sketch its spectrum.

(4 marks)

(c) Can the signal $x(t)$ be recovered from the sampled signal $x_s(t)$, and if so how?

(2 marks)

Q2. Consider the RLC series circuit shown in Figure Q2 below.

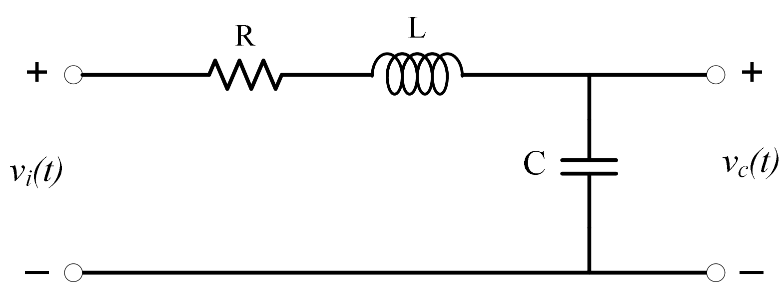


Figure Q2 : RLC Series Circuit

Let R = 2Ω, L = 1H and C = 1F.

(a) Derive the transfer function between the output $V_c(s) = L\{v_c(t)\}$ and input $V_i(s) = L\{v_i(t)\}$.

(3 marks)

(b) Find the poles and zeros of the transfer function.

(2 marks)

(c) Write down the differential equation which describes the circuit. Suppose the initial conditions of the circuit are $v_c(0) = 1$ V and $\left.\dfrac{dv_c(t)}{dt}\right|_{t=0} = 0$ V/s. Sketch the output voltage, $v_c(t)$, for an input $v_i(t) = u(t)$ where $u(t)$ is the unit step function.

(5 marks)

Q3. The signal $x(t) = 3\text{tri}\left(\frac{t}{2}\right) * [\delta(t-2) + \delta(t+2)]$ where '*' denotes the convolution operation.

(a) Sketch $x(t)$ with sufficient labelling and find the Fourier transform $X(f)$ of $x(t)$.

(3 marks)

(b) Find the energy of $x(t)$.

(3 marks)

(c) Find the 1st -null bandwidth of $x(t)$.

(4 marks)

Q4. Consider a critically damped second order system with an undamped natural frequency of 3 rad/s and a DC gain of 7.

(a) Find the poles of the critically damped second order system.

(2 marks)

(b) Derive the output signal of the critically damped second order system when the input signal is $2\delta(t)$. Assume all initial conditions are zero.

(3 marks)

(c) Suppose the output signal of the critically damped second order system is

$$y(t) = 3.5[1 - e^{-3t} - 3te^{-3t}]u(t) \; .$$

i. Identify the transient response.

(2 marks)

ii. What is the input signal?

(3 marks)

SECTION B : Answer 3 out of the 4 questions in this section

Q5. (a) Determine the Fourier Transform of the signal $x(t)$ shown in Figure Q5(a).

(6 marks)

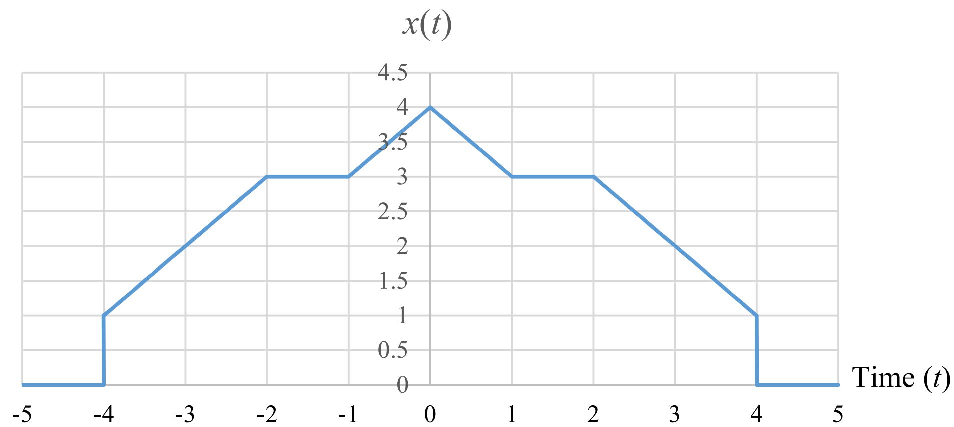


Figure Q5(a)

(b) Determine the Fourier Transform of the periodic signal $x_p(t)$ shown in Figure Q5(b), where $x_p(t)$ comprises repetitions of the generating signal $x(t)$ at a period of 10 seconds.

(6 marks)

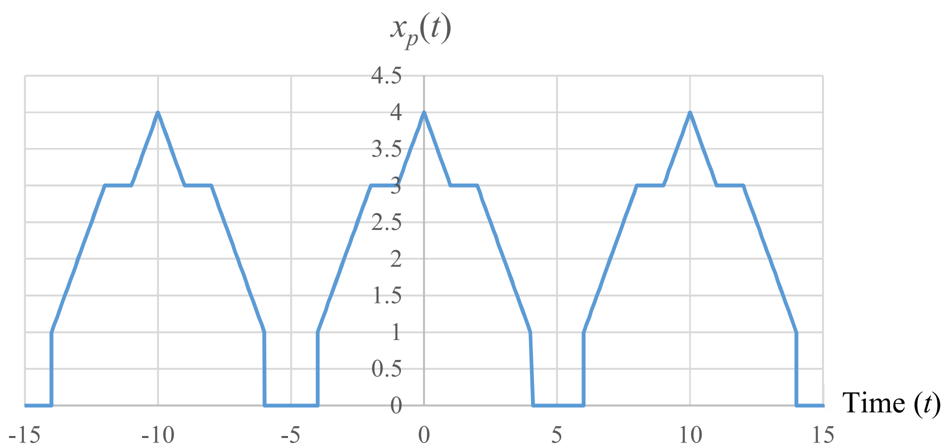


Figure Q5(b)

(c) Determine the Fourier series coefficients $X_{p,k}$ of the periodic signal $x_p(t)$.

(2 marks)

(d) Obtain the average power of the periodic signal $x_p(t)$.

(3 marks)

(e) Determine the first null bandwidth.

(3 marks)

Q6. (a) Find the Laplace transform of $x(t) = t\cos 3t$.

(4 marks)

(b) A second order system has a transfer function given by:

$$G(s) = \frac{9}{2s^2 + 6s + 18}.$$

i. Determine if the system is underdamped, overdamped, critically damped or undamped.

(2 marks)

ii. Find the poles of the system. What are the values of its DC gain and natural frequency?

(4 marks)

iii Sketch the output when the input to the system is a step function of magnitude 2. Label your sketch clearly. Include the final steady state value of the output and if there are any oscillations, indicate the frequency of the oscillations on your sketch. You may assume zero initial conditions.

(4 marks)

(c) A system is observed to have an output of $y(t) = t + \frac{1}{2}\left(e^{-2t} - 1\right)$ when an input of $x(t) = 2u(t)$ was injected into the system with zero initial conditions.

i. Find the transfer function of the system.

(4 marks)

ii. Is the system stable, marginally stable or unstable? Explain your answer.

(2 marks)

Q7. (a) In the transmitter of a communication system, the signal $m(t)$, shown in Figure Q7(a), is modulated by $c(t)$ to obtain $x_c(t)$ as shown in Figure Q7(b). Suppose $A_c = 0.5$, $f_c = 10{,}000$ Hz, and ⊗ represents multiplication.

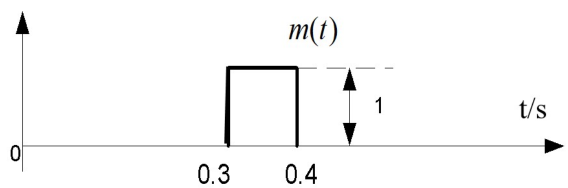


Figure Q7(a)

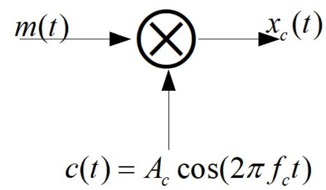


Figure Q7(b)

i. Find the expression for $m(t)$.

(3 marks)

ii. Determine the Fourier transform, $M(f)$, of signal $m(t)$.

(3 marks)

(b) The receiver carries out the operations shown in Figure Q7(c), where $y_d(t) = x_c^2(t)$ is subjected to an ideal low pass filter to recover the signal $\hat{m}(t)$.

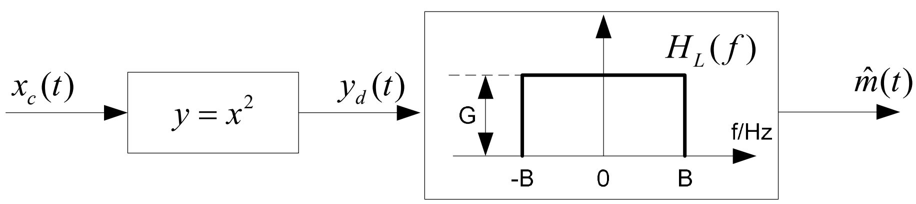


Figure Q7(c)

i. Find the expression for $y_d(t)$.

(4 marks)

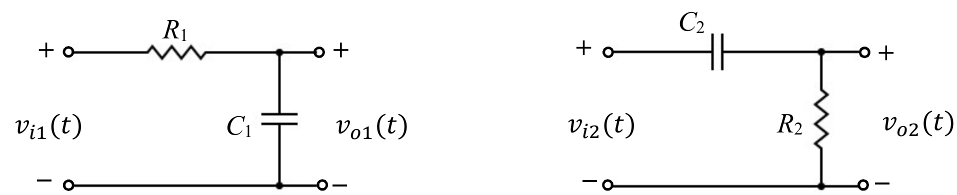
ii. Determine the Fourier transform, $Y_d(f)$, of signal $y_d(t)$.

(4 marks)

iii. If the spectra of the recovered signal $\hat{m}(t)$ and that of the original signal $m(t)$ are required to be the same within their 1st null bandwidth, determine the gain G and the lowest cut-off frequency B of the low pass filter in Figure Q7(c).

(6 marks)

Q8. (a) Figure Q8(a) shows two filter circuits. One of the circuits is a low pass filter, while the other is a high pass filter.



Filter circuit 1 Filter circuit 2

Figure Q8(a) : Filter circuits

i. The differential equation model of filter circuit 1 is

$$R_1 C_1 \frac{dv_{o1}(t)}{dt} + v_{o1}(t) = v_{i1}(t).$$

Derive $G_1(s)$, the transfer function of filter circuit 1. Clearly state the assumption(s) used to derive the transfer function.

(3 marks)

ii. Derive $G_2(s) = \frac{V_{o2}(s)}{V_{i2}(s)}$, the transfer function of filter circuit 2, where $L\{v_{i2}(t)\} = V_{i2}(s)$ and $L\{v_{o2}(t)\} = V_{o2}(s)$

(3 marks)

iii. Which circuit is the high pass filter?

(1 mark)

Question 8 continues on Page 8

(b) The Bode Magnitude plot of a bandpass filter formed by cascading filter circuit 1 to filter circuit 2, i.e. $v_{o1}(t) = v_{i2}(t)$, is shown in Figure Q8(b).

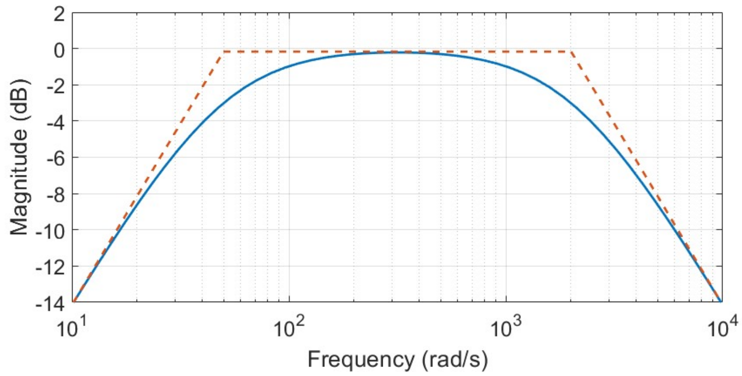


Figure Q8(b): Bode Magnitude plot of the bandpass filter, $G_1(s)\ G_2(s)$

i. Using Figure Q8(b), derive the transfer function of the bandpass filter, $G_1(s)\ G_2(s)$.
(5 marks)

ii. What are the time constants of filter circuit 1 (R_1C_1) and filter circuit 2 (R_2C_2)?
(2 marks)

iii. Suppose the input signal is $25 + 10\cos(800t + 0.1)$. What is the steady-state output of the bandpass filter?
(6 marks)

END OF QUESTIONS

Fourier Series:
$$\begin{cases} X_k = \dfrac{1}{T}\displaystyle\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k\,t/T)\,dt \\ x(t) = \displaystyle\sum_{k=-\infty}^{\infty} X_k \exp(j2\pi k\,t/T) \end{cases}$$

Fourier Transform:
$$\begin{cases} X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$$

FOURIER TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}(fT)$
Triangle	$\text{tri}\left(\frac{t}{T}\right)$	$T\,\text{sinc}^2(fT)$
Sine Cardinal	$\text{sinc}\left(\frac{t}{T}\right)$	$T\,\text{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp(-\alpha^2\pi^2 f^2)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES

	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi ft_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty}X_1(\zeta)X_2(f-\zeta)d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$ $\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n[th] order Ramp	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES		
	Time-domain	***s*-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)u(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\frac{1}{s}X(s)$
Differentiation in the time-domain	$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\frac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\frac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\frac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\frac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

System Type	Transfer Function (Standard Form)	Unit Impulse and Unit Step Responses	Remarks
1st order system	$G(s)=\frac{K}{T}\cdot\frac{1}{s+1/T}$	$y_\delta(t)=\frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$ $y_{step}(t)=K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	T : Time-constant K : DC Gain Real Pole at $s=-\frac{1}{T}$
2nd order system **$(\zeta>1)$** **Overdamped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}$	$y_\delta(t)=\left[K_1\exp(-p_1t)+K_2\exp(-p_2t)\right]u(t)$ $y_{step}(t)=\left[K-\frac{K_1}{p_1}\exp(-p_1t)-\frac{K_2}{p_2}\exp(-p_2t)\right]u(t)$	K : DC Gain $\left.\begin{array}{l}p_1=\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\\ p_2=\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\end{array}\right\}$ $K_1=-K_2=\frac{K\omega_n^2}{p_2-p_1}$ Real Distinct Poles at $s=-p_1$ and $s=-p_2$
2nd order system **$(\zeta=1)$** **Critically damped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K\omega_n{}^2}{(s+\omega_n)^2}$	$y_\delta(t)=K\omega_n^2t\exp(-\omega_nt)u(t)$ $y_{step}(t)=K\left[1-\exp(-\omega_nt)-\omega_nt\exp(-\omega_nt)\right]u(t)$	K : DC Gain Real Repeated Poles at $s=-\omega_n$
2nd order system **$(0<\zeta<1)$** **Underdamped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K\left(\sigma^2+\omega_d^2\right)}{(s+\sigma)^2+\omega_d^2}$	$y_\delta(t)=K\frac{\omega_n^2}{\omega_d}\exp(-\sigma t)\sin(\omega_dt)u(t)$ $y_{step}(t)=K\left[1-\frac{\omega_n}{\omega_d}\exp(-\sigma t)\sin(\omega_dt+\phi)\right]u(t)$	K : DC Gain ω_n : Undamped Natural Frequency ζ : Damping Ratio ω_d : Damped Natural Frequency $\sigma=\zeta\omega_n$ $\omega_d^2=\omega_n^2\left(1-\zeta^2\right)$ $\omega_n^2=\sigma^2+\omega_d^2$ $\tan(\phi)=\frac{\omega_d}{\sigma}$ Complex Conjugate Poles at $s=-\sigma\pm j\omega_d$
2nd order system **$(\zeta=0)$** **Undamped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K\omega_n{}^2}{s^2+\omega_n^2}$	$y_\delta(t)=K\omega_n\sin(\omega_nt)u(t)$ $y_{step}(t)=K(1-\cos\omega_nt)u(t)$	K : DC Gain ω_n : Undamped Natural Frequency Imaginary Conjugate Poles at $s=\pm j\omega_n$

2nd order system RESONANCE $\left(0 \le \zeta < 1/\sqrt{2}\right)$ ➔ *RESONANCE FREQUENCY :* $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ *RESONANCE PEAK :* $M_r = \left|G(j\omega_r)\right| = \dfrac{K}{2\zeta\sqrt{1-\zeta^2}}$

Trigonometric Identities

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\theta) = 0.5\left(e^{j\theta} + e^{-j\theta}\right)$	$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
$\sin(\theta) = -0.5j\left(e^{j\theta} - e^{-j\theta}\right)$	$\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$
$\sin^2(\theta) + \cos^2(\theta) = 1$	
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta) = 0.5\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$	$\cos(\alpha)\cos(\beta) = 0.5\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
$\sin^2(\theta) = 0.5\left[1 - \cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta) = 0.5\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$
$\cos^2(\theta) = 0.5\left[1 + \cos(2\theta)\right]$	$C\cos(\theta) - S\sin(\theta) = \sqrt{C^2+S^2}\cos\left[\theta + \tan^{-1}\left(\frac{S}{C}\right)\right]$

Complex Unit (j) ➔ $\left(j = \sqrt{-1} = e^{j\pi/2} = e^{j90^\circ}\right)$ $\left(-j = \frac{1}{j} = e^{-j\pi/2} = e^{-j90^\circ}\right)$ $\left(j^2 = -1\right)$

Definitions of Basic Functions

Rectangle:

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1; & -T/2 \le t < T/2 \\ 0; & \text{elsewhere} \end{cases}$$

Triangle:

$$\mathrm{tri}\left(\frac{t}{T}\right) = \begin{cases} 1-|t|/T; & |t| \le T \\ 0; & |t| > T \end{cases}$$

Sine Cardinal:

$$\mathrm{sinc}\left(\frac{t}{T}\right) = \begin{cases} \dfrac{\sin(\pi t/T)}{\pi t/T}; & t \ne 0 \\ 1; & t = 0 \end{cases}$$

Signum:

$$\mathrm{sgn}(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$

Unit Impulse:

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \ne 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

Unit Step:

$$u(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$