NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION FOR

(Semester I : 2017/2018)

Names of examiners: AP Loh, C S Ng, WW Tan, Lawrence Wong, and J Zhang

EE2023 – SIGNALS & SYSTEMS

Nov/Dec 2017 - Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. This paper contains **EIGHT (8)** questions and comprises **TEN (10)** printed pages.
2. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.
3. This is a **CLOSED BOOK** examination. However you are allowed to bring one self-prepared A4-size crib sheet to the examination hall.
4. Programmable and/or graphic calculators are not allowed.
5. Tables of formulas are provided on Pages 7 to 10.

SECTION A : Answer ALL questions in this section

Q.1 The signal $x(t) = \text{sinc}(2t)$ is sampled at 4 Hz to obtain the sampled signal, $x_s(t)$.

(a) Derive the Fourier transform, $X_s(f)$, of the sampled signal $x_s(t)$ and sketch its spectrum.

(6 marks)

(b) What is the Nyquist sampling frequency?

(2 marks)

(c) If $x(t)$ is sampled at a frequency of 2 Hz, sketch and label the sampled signal.

(2 marks)

Q.2 A system is modelled by a steady state or DC gain of one and has zeros and poles given by:

$$z = -2, \;\; p_1 = +j2, \;\; p_2 = -j2$$

where z denotes the zero while p_1 and p_2 are the poles.

(a) What is the order of the transfer function model of the system?

(2 marks)

(b) Derive the transfer function of the system.

(5 marks)

(c) What is the damping ratio and natural frequency of the system?

(3 marks)

Q.3 The response of a second order system to the input signal, $2\delta(t)$, is

$$8\, e^{-2t} \sin\left(2\sqrt{3}t\right).$$

(a) Derive the system transfer function.

(5 marks)

(b) Find the system poles.

(2 marks)

(c) What is the damping ratio, undamped natural frequency and DC gain of the system?

(3 marks)

Q.4 The continuous-frequency spectrum of a signal $x(t)$ is given by

$$X(f) = 2\delta(f+40) + 2\delta(f+30) + 3\delta(f+20) + 4 + 3\delta(f-20) + 2\delta(f-30) + 2\delta(f-40).$$

(a) Draw an adequately labeled sketch of the power spectral density, $P_x(f)$, of $x(t)$.

(4 marks)

(b) What is the average power of $x(t)$?

(2 marks)

(c) The 80% power containment bandwidth of a power signal is defined as the smallest bandwidth that contains at least 80% of the average signal power. What is the 80% power containment bandwidth of $x(t)$?

(4 marks)

SECTION B : Answer 3 out of the 4 questions in this section

Q.5 Let signals $x(t) = 40\ \mathrm{sinc}(20t-1)$ and $y(t) = x(t)\cos\left(2\pi \times 10^3 t\right)$.

(a) Find the Fourier transform of $x(t)$.

(7 marks)

(b) Find the Fourier transform of $y(t)$ and plot its spectrum with proper labeling.

(13 marks)

Q.6 A thermal cycling chamber is often used to test satellite components to ensure the reliability of the components when they are operating in space. The function of the thermal cycling chamber is to provide heating and cooling cycles to the satellite components. Hence this chamber must be able to respond well to input signals to achieve the desired heating and cooling cycles. The input and output of the chamber are illustrated in Figure Q6 below.

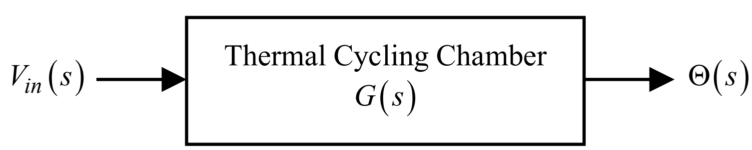

Figure Q6

Q.6 is continued on Page 4

$V_{in}(s) = \mathcal{L}\{v_{in}(t)\}$ is the input signal to the chamber, in Volts and $\Theta(s) = \mathcal{L}\{\theta(t)\}$ is the measured chamber temperature in degree Celsius.

Suppose the chamber can be modeled by a first order transfer function given by

$$G(s) = \frac{\Theta(s)}{V_{in}(s)} = \frac{4}{3s+2}.$$

(a) What is the steady state gain and time constant of the chamber?

(2 marks)

(b) If the input signal to the chamber is $v_{in}(t) = 100u(t)$ Volts, what will be the steady state temperature in the chamber? Sketch the response to this input. You may assume $u(t)$ to be the unit step function and all initial conditions are zero.

(3 marks)

(c) If the input signal is $v_{in}(t) = 50\sin(0.1t)\,u(t)$ Volts, what will be the steady state temperature in the chamber? Assume zero initial conditions. Sketch the response.

(4 marks)

(d) The satellite engineer wishes to set the desired chamber temperature between 50 degree Celsius to 150 degree Celsius with a sinusoidal waveform of frequency 2 rad/s.

(i) If the input signal is of the form $v_{in}(t) = [x_0 + x_1 \sin(2t)]\,u(t)$ Volts, what values of x_0 and x_1 should the engineer choose in order to achieve the desired chamber temperature profile?

(3 marks)

(ii) With the values of x_0 and x_1 chosen above, find the steady state temperature of the chamber. What are the maximum and minimum temperatures that the engineer is able to achieve in the chamber? Assume all initial conditions to be zero.
(*Hint : If you are unable to give the right values of x_0 and x_1, you may just assume any value and show how you can solve this part.*)

(5 marks)

(iii) Is the engineer able to achieve his desired temperature profile in the chamber? Why?

(3 marks)

Q.7 (a) Consider a second order system with DC gain of K, unity damping ratio, undamped natural frequency of 3 rad/s and transportation delay of 0.1 seconds i.e.

$$G(s) = \frac{9Ke^{-0.1s}}{s^2 + 6s + 9}.$$

(i) Derive the steady state output signal of the second order system if the input signal is $7\cos(5t)$ and the DC gain, K, is 17.

(6 marks)

(ii) Suppose the desired amplitude of the steady state sinusoidal output signal is 36 when the input signal is $16\cos(5t + 15°)$. What should be the system DC gain?

(3 marks)

(b) Figure Q7 shows the Bode magnitude plot of a system with transfer function $H(s)$. Suppose the system is stable and does not have transportation delay.

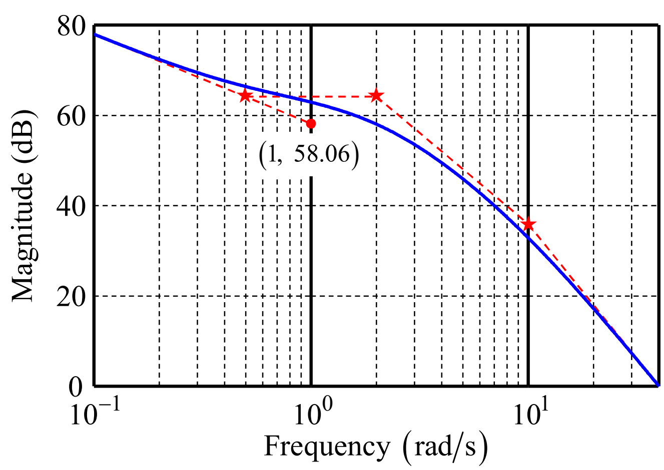

Figure Q7: Bode magnitude plot of $H(s)$

(i) Identify the transfer function, $H(s)$.

(7 marks)

(ii) Identify the error(s) in the following statement:

The low frequency asymptote of the Bode phase plot for $H(s)$ is a horizontal line at 90 degrees.

(2 marks)

(iii) Describe the high frequency asymptote of the Bode phase plot for $H(s)$.

(2 marks)

Q.8 Figure Q8(a) shows a vowel synthesizer which consists of a linear time-invariant (LTI) filter driven by an impulse train

$$e(t) = 2\sum_{n=-\infty}^{\infty} \delta(t-10n).$$

The impulse response, $h(t)$, of the LTI filter is plotted in Figure Q8(b) where $h(t)=0$ for $t<0$, and

$$H(f) = \Im\{h(t)\} = \frac{40}{\left(13-20f^2\right)+j4f}$$

is the Fourier transform of $h(t)$.

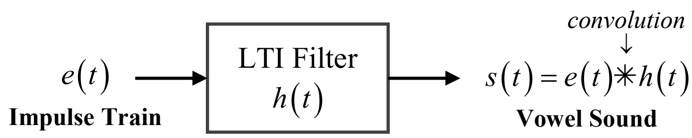

Figure Q8(a)

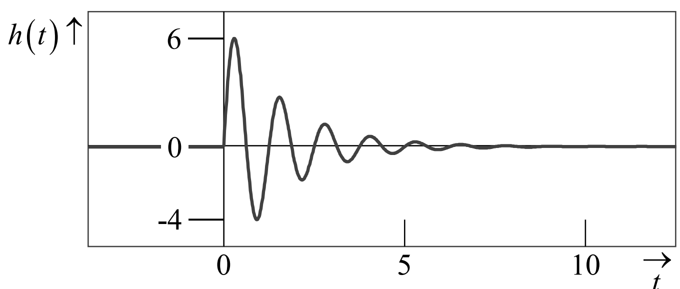

Figure Q8(b)

(a) Draw an adequately labeled sketch of $s(t)$.

(5 marks)

(b) Derive the spectrum, $S(f)$, of $s(t)$.

(6 marks)

(c) i. Find the complex exponential Fourier series coefficients, c_k, of $s(t)$.

(5 marks)

ii Based on the value of c_0 alone, can we claim with absolute certainty that the average power of $s(t)$ is greater than the average value of $s(t)$? Explain your answer.

(4 marks)

END OF QUESTIONS

Fourier Series: $$\begin{cases} X_k = \dfrac{1}{T}\displaystyle\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k t/T)\,dt \\ x(t) = \displaystyle\sum_{k=-\infty}^{\infty} X_k \exp(j2\pi k t/T) \end{cases}$$

Fourier Transform: $$\begin{cases} X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$$

FOURIER TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
Triangle	$\mathrm{tri}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}^2(fT)$
Sine Cardinal	$\mathrm{sinc}\left(\frac{t}{T}\right)$	$T\,\mathrm{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp(-\alpha^2\pi^2 f^2)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES		
	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi f t_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty}X_1(\zeta)X_2(f-\zeta)\,d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)\,d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
		$\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
nth order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES		
	Time-domain	***s*-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)u(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\dfrac{1}{\lvert\alpha\rvert}X\left(\dfrac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\dfrac{1}{s}X(s)$
Differentiation in the time-domain	$\dfrac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\dfrac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\dfrac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\dfrac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\dfrac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

System Type	Transfer Function (Standard Form)	Unit Impulse and Unit Step Responses	Remarks
1^{st} order system	$G(s)=\frac{K}{T}\cdot\frac{1}{s+1/T}$	$y_\delta(t)=\frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$ $y_{step}(t)=K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	T : Time-constant K : DC Gain Real Pole at $s=-\frac{1}{T}$
2^{nd} order system $(\zeta>1)$ Overdamped	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}$	$y_\delta(t)=\left[K_1\exp(-p_1t)+K_2\exp(-p_2t)\right]u(t)$ $y_{step}(t)=\left[K-\frac{K_1}{p_1}\exp(-p_1t)-\frac{K_2}{p_2}\exp(-p_2t)\right]u(t)$	K : DC Gain $\left.\begin{array}{l}p_1=\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\\ p_2=\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\end{array}\right\\} K_1=-K_2=\frac{K\omega_n^2}{p_2-p_1}$ Real Distinct Poles at $s=-p_1$ and $s=-p_2$
2^{nd} order system $(\zeta=1)$ Critically damped	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{(s+\omega_n)^2}$	$y_\delta(t)=K\omega_n^2 t\exp(-\omega_n t)u(t)$ $y_{step}(t)=K\left[1-\exp(-\omega_n t)-\omega_n t\exp(-\omega_n t)\right]u(t)$	K : DC Gain Real Repeated Poles at $s=-\omega_n$
2^{nd} order system $(0<\zeta<1)$ Underdamped	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\left(\sigma^2+\omega_d^2\right)}{(s+\sigma)^2+\omega_d^2}$	$y_\delta(t)=K\frac{\omega_n^2}{\omega_d}\exp(-\sigma t)\sin(\omega_d t)u(t)$ $y_{step}(t)=K\left[1-\frac{\omega_n}{\omega_d}\exp(-\sigma t)\sin(\omega_d t+\phi)\right]u(t)$	K : DC Gain ω_n : Undamped Natural Frequency ζ : Damping Ratio ω_d : Damped Natural Frequency $\sigma=\zeta\omega_n$ $\quad\omega_d^2=\omega_n^2\left(1-\zeta^2\right)$ $\quad\omega_n^2=\sigma^2+\omega_d^2$ $\quad\tan(\phi)=\frac{\omega_d}{\sigma}$ Complex Conjugate Poles at $s=-\sigma\pm j\omega_d$
2^{nd} order system $(\zeta=0)$ Undamped	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{s^2+\omega_n^2}$	$y_\delta(t)=K\omega_n\sin(\omega_n t)u(t)$ $y_{step}(t)=K(1-\cos\omega_n t)u(t)$	K : DC Gain ω_n : Undamped Natural Frequency Imaginary Conjugate Poles at $s=\pm j\omega_n$

2ndorder system RESONANCE $\left(0 \le \zeta < 1/\sqrt{2}\right)$ ➔ *RESONANCE FREQUENCY :* $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ *RESONANCE PEAK :* $M_r = \left|G(j\omega_r)\right| = \dfrac{K}{2\zeta\sqrt{1-\zeta^2}}$

Trigonometric Identities

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\theta) = 0.5\left(e^{j\theta} + e^{-j\theta}\right)$	$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
$\sin(\theta) = -0.5j\left(e^{j\theta} - e^{-j\theta}\right)$	$\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$
$\sin^2(\theta) + \cos^2(\theta) = 1$	
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta) = 0.5\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$	$\cos(\alpha)\cos(\beta) = 0.5\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
$\sin^2(\theta) = 0.5\left[1 - \cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta) = 0.5\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$
$\cos^2(\theta) = 0.5\left[1 + \cos(2\theta)\right]$	$C\cos(\theta) - S\sin(\theta) = \sqrt{C^2+S^2}\cos\left[\theta + \tan^{-1}\left(\dfrac{S}{C}\right)\right]$

Complex Unit (j) ➔ $\left(j = \sqrt{-1} = e^{j\pi/2} = e^{j90^\circ}\right)$ $\left(-j = \dfrac{1}{j} = e^{-j\pi/2} = e^{-j90^\circ}\right)$ $\left(j^2 = -1\right)$

Definitions of Basic Functions

Rectangle:

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1; & -T/2 \le t < T/2 \\ 0; & \text{elsewhere} \end{cases}$$

Triangle:

$$\mathrm{tri}\left(\frac{t}{T}\right) = \begin{cases} 1 - |t|/T; & |t| \le T \\ 0; & |t| > T \end{cases}$$

Sine Cardinal:

$$\mathrm{sinc}\left(\frac{t}{T}\right) = \begin{cases} \dfrac{\sin(\pi t/T)}{\pi t/T}; & t \ne 0 \\ 1; & t = 0 \end{cases}$$

Signum:

$$\mathrm{sgn}(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$

Unit Impulse:

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \ne 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

Unit Step:

$$u(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$