EE2023 TUTORIAL 2 (SOLUTIONS)

Solution to Q.1

(a) $x(t)$ has an average (or DC) value of 2 $\because$ Zero-frequency component has value 2

(b) $x(t)$ is a POWER SIGNAL $\because \begin{cases}\text{Spectrum is defined only at discrete} \\ \text{frequency points (sum of sinusoids)}\end{cases}$

(c) $x(t)$ is APERIODIC $\because \{\pi,\ \pi^2, \pi^3\} \cdots\cdots$ has no common factor

Therefore, $x(t)$ does not have a Fourier series expansion.

Solution to Q.2

(a) $x(t)=\cos(3.2t)+\sin(1.6t)+\exp(j2.8t) \quad \cdots \quad \begin{cases}\cos(3.2t) & \text{has a frequency of } 3.2\ rad/s \\ \sin(1.6t) & \text{has a frequency of } 1.6\ rad/s \\ \exp(j2.8t) & \text{has a frequency of } 2.8\ rad/s\end{cases}$

Highest common factor (HCF) of $\{3.2,\ 1.6,\ 2.8\}$ exists and is equal to 0.4. Thus, $x(t)$ is periodic with a fundamental frequency of ω_p=0.4 rad/s. The period of $x(t)$ is $T_p=\dfrac{2\pi}{0.4}=5\pi\ s$.

REMARKS: *Although $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$, it does not contain the fundamental frequency component itself.*

(b) $x(t)=\cos(4t)+\sin(\pi t) \quad \cdots \quad \begin{cases}\cos(4t) & \text{has a frequency of } 4\ rad/s \\ \sin(\pi t) & \text{has a frequency of } \pi\ rad/s\end{cases}$

The set of frequencies $\{4,\pi\}$ has no common factor. Thus, $x(t)$ is not periodic.

REMARKS: *Summing sinusoids does not necessarily lead to a periodic signal unless the frequencies of the sinusoids are harmonics of a common fundamental frequency.*

Solution to Q.3

(a) The fundamental frequency of $x(t)=6\sin(12\pi t)+4\exp\left(j\left(8\pi t+\frac{\pi}{4}\right)\right)+2$ is $\begin{cases} \boldsymbol{f_p = HCF\{6,4\} = 2} \\ \boldsymbol{T_p = 0.5} \end{cases}$.

Re-write $x(t)$ as a sum of weighted zero-phase complex exponentials and arrange the terms in ascending frequency order:

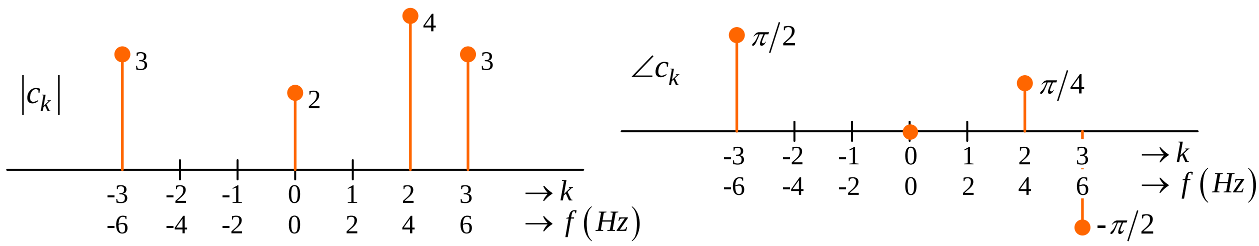
$$\begin{aligned} x(t) &= \frac{6}{j2}\left[\exp(j12\pi t)-\exp(-j12\pi t)\right]+4\exp(j\pi/4)\exp(j8\pi t)+2 \\ &= j3\exp(-j12\pi t)+2+4\exp(j\pi/4)\exp(j8\pi t)-j3\exp(j12\pi t) \end{aligned} \qquad (1)$$

Express $x(t)$ as a complex exponential Fourier series:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k \exp\left(j2\pi\frac{k}{T_p}t\right) = \sum_{k=-\infty}^{\infty} c_k \exp(j4\pi kt) \\ &= \left(\begin{array}{c} \cdots + c_{-3}\exp(-j12\pi t)+c_{-2}\exp(-j8\pi t)+c_{-1}\exp(-j4\pi t) \\ +c_0 \\ +c_1\exp(j4\pi t)+c_2\exp(j8\pi t)+c_3\exp(j12\pi t)+\cdots\cdots \end{array}\right) \end{aligned} \qquad (2)$$

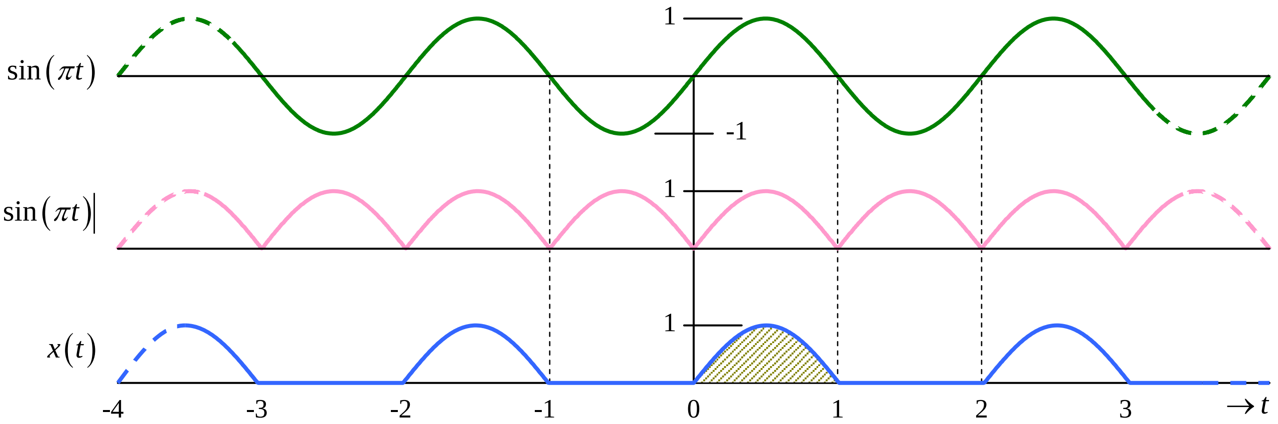
Comparing coefficients of complex exponential terms in (1) and (2), we conclude that:

$$\boldsymbol{c_{-3} = j3,\ c_0 = 2,\ c_2 = 4\exp\left(j\frac{\pi}{4}\right),\ c_3 = -j3} \text{ and } \boldsymbol{[c_k = 0\ ;\ k \neq 0,\ 2,\ \pm 3]}.$$



Remarks: *If a periodic signal is given as a sum of sinusoids, then its Fourier series coefficients can be evaluated using the above method without the need to perform any integration.*

(b) $x(t)=\frac{1}{2}\left(|\sin(\pi t)|+\sin(\pi t)\right)$: Half-wave rectification of $\sin(\pi t)$.



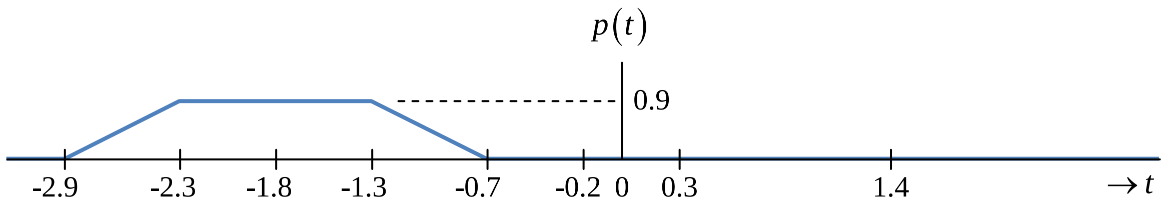
Period: $T = 2$ s

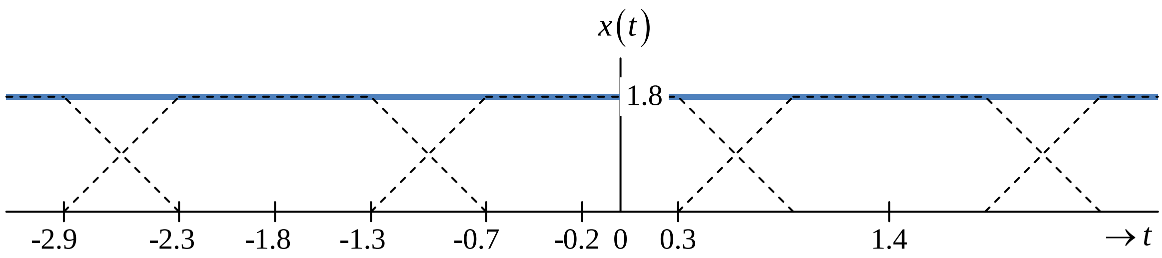
Fundamental frequency : $f_1 = \frac{1}{T} = 0.5$ Hz

4^{th} harmonic frequency : $f_4 = 4f_1 = 2$ Hz

Solution to Q.4

Graphically, we observe that $x(t) = \sum_{n=-\infty}^{\infty} 2p(t-1.6n) = 1.8$.





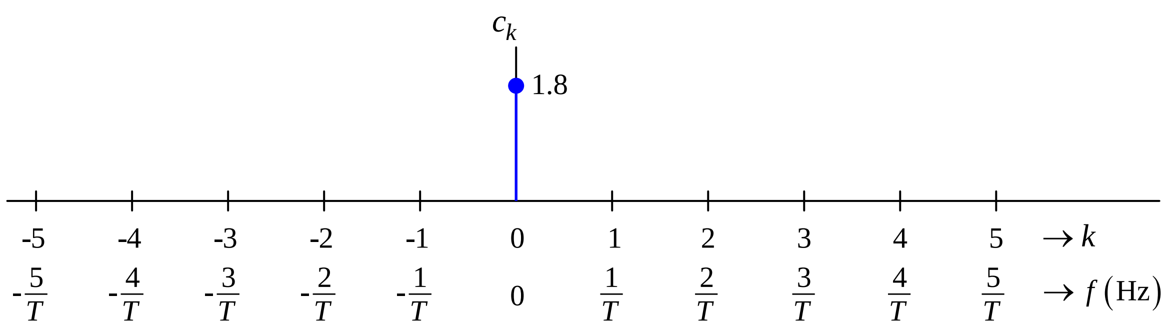
By Deduction:

- $x(t)$ has a *zero-frequency* component of value 1.8, which implies that $c_0 = 1.8$.
- $x(t)$ has no *non-zero frequency* components, which implies that $c_k = 0;\ \ k \neq 0$.

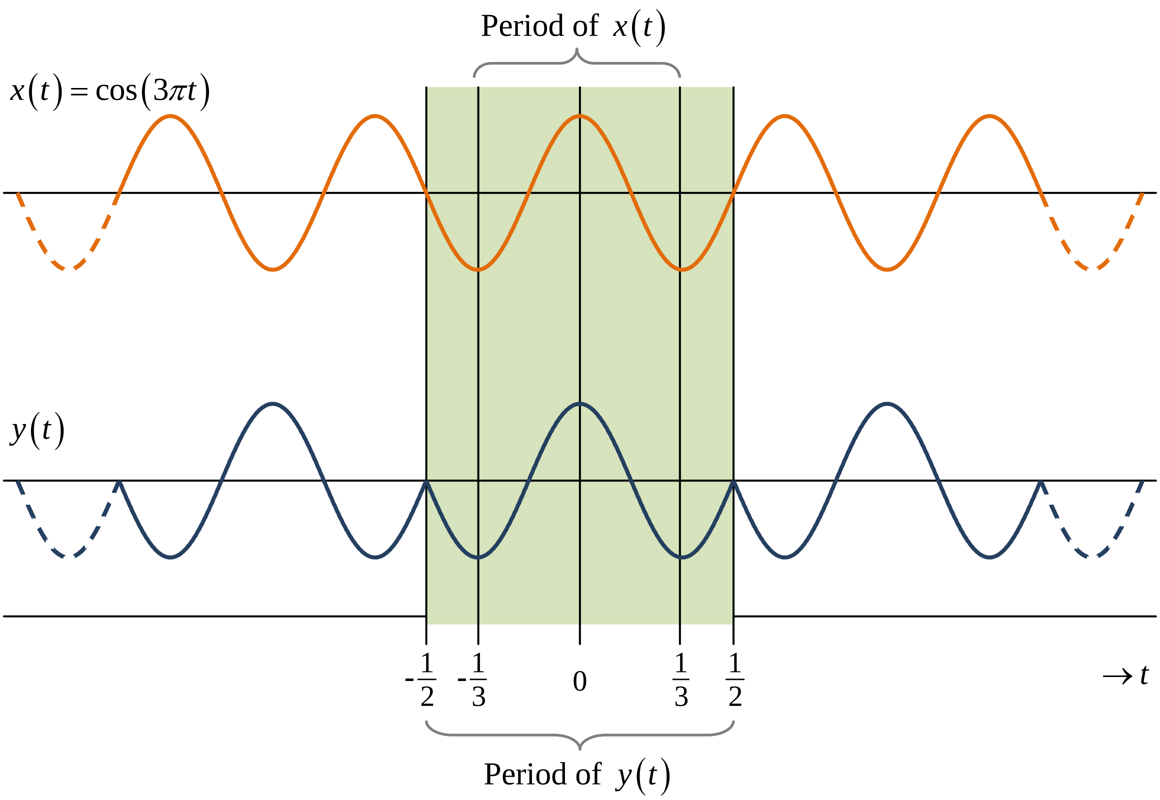
By Derivation:

Since $x(t)$ is a constant (or a DC signal), it may be treated as a periodic signal of arbitrary period *T*, where $0 < T < \infty$. Its Fourier series coefficients can thus be computed as

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{-T/2}^{T/2} 1.8\exp\left(-j2\pi\frac{k}{T}t\right)dt = \frac{1.8}{T}\left[\frac{\exp(-j2\pi kt/T)}{-j2\pi k/T}\right]_{-T/2}^{T/2} \\
&= \frac{1.8}{T}\left[\frac{\exp(-j\pi k)}{-j2\pi k/T} - \frac{\exp(j\pi k)}{-j2\pi k/T}\right] \\
&= 1.8\frac{\sin(\pi k)}{\pi k} = 1.8\ \mathrm{sinc}(k) = \begin{cases} 1.8; & k = 0 \\ 0; & k \neq 0 \end{cases}
\end{aligned}
$$

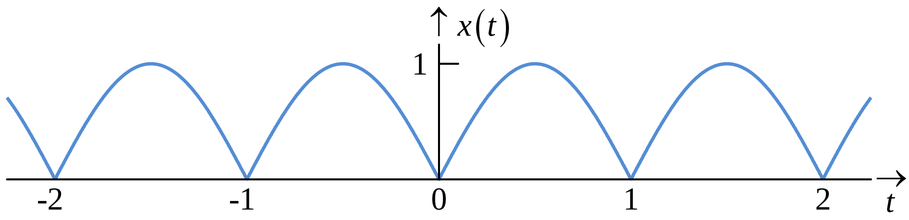


Solution to Q.5



Solution to S.1

(a)



$Period = 1$

(b)

$$
\begin{aligned}
c_k &= \frac{1}{1}\int_0^1 \sin(\pi t)\exp(-j2\pi kt)\,dt \\
&= \frac{1}{j2}\int_0^1 \left[\exp(j\pi t)-\exp(-j\pi t)\right]\exp(-j2\pi kt)\,dt \\
&= \frac{1}{j2}\int_0^1 \exp\left[j\pi(1-2k)t\right]-\exp\left[-j\pi(1+2k)t\right]dt \\
&= \frac{1}{j2}\left[\frac{\exp\left[j\pi(1-2k)t\right]}{j\pi(1-2k)}-\frac{\exp\left[-j\pi(1+2k)t\right]}{-j\pi(1+2k)}\right]_0^1 \\
&= \frac{1}{j2}\left[\frac{\exp\left[j\pi(1-2k)\right]-1}{j\pi(1-2k)}-\frac{\exp\left[-j\pi(1+2k)\right]-1}{-j\pi(1+2k)}\right] \\
&= \frac{1}{\pi(2k+1)}-\frac{1}{\pi(2k-1)} = -\frac{2}{\pi}\cdot\frac{1}{4k^2-1}
\end{aligned}
$$

$$
x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(j2\pi kt) = -\frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{4k^2-1}\exp(j2\pi kt)
$$

(c)

$$
a_k = \frac{c_{-k}+c_k}{2} = -\frac{2}{\pi}\cdot\frac{1}{4k^2-1}
$$

$$
b_k = \frac{c_{-k}-c_k}{j2} = 0
$$

$$
x(t) = a_0 + 2\sum_{k=1}^{\infty} a_k\cos(2\pi kt)+b_k\sin(2\pi kt) = \frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos(2\pi kt)
$$

Solution to S.2

$$
\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 \exp(-jkt)\,dt = \frac{1}{2\pi}\left(\left[t^2\frac{\exp(-jkt)}{-jk}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2t\frac{\exp(-jkt)}{-jk}\,dt\right) \\
&= \frac{1}{2\pi}\left(\left[t^2\frac{\exp(-jkt)}{-jk}\right]_{-\pi}^{\pi} - \left[2t\frac{\exp(-jkt)}{(-jk)^2}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2\frac{\exp(-jkt)}{(-jk)^2}\,dt\right) \\
&= \frac{1}{2\pi}\left(\left[t^2\frac{\exp(-jkt)}{-jk}\right]_{-\pi}^{\pi} - \left[2t\frac{\exp(-jkt)}{(-jk)^2}\right]_{-\pi}^{\pi} + \left[2\frac{\exp(-jkt)}{(-jk)^3}\right]_{-\pi}^{\pi}\right) \\
&= \pi\left[\frac{\sin(\pi k)}{k}\right] + \left[\frac{2\cos(\pi k)}{k^2}\right] - \frac{2}{\pi}\left[\frac{\sin(\pi k)}{k^3}\right] \\
&= \pi^2\left[\frac{\sin(\pi k)}{\pi k}\right] + \left[\frac{2\pi k\cos(\pi k) - 2\sin(\pi k)}{\pi k^3}\right] = \begin{cases} 2(-1)^k/k^2; & k \neq 0 \\ \pi^2/3; & k = 0 \end{cases}
\end{aligned}
$$

$$
x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(jkt) = \frac{\pi^2}{3} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2(-1)^k}{k^2}\exp(jkt) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kt)
$$

Solution to S.3

$$
\begin{aligned}
x(t) &= h_0 + \sum_{k=1}^{\infty} h_k \cos\left(2\pi\frac{k}{T_0}t - \theta_k\right) \\
&= \underbrace{h_0}_{a_0} + 2\sum_{k=1}^{\infty} \underbrace{0.5h_k\cos(\theta_k)}_{a_k}\cos\left(2\pi\frac{k}{T_0}t\right) + \underbrace{0.5h_k\sin(\theta_k)}_{b_k}\sin\left(2\pi\frac{k}{T_0}t\right)
\end{aligned}
$$

$$
\left.\begin{aligned}
c_{-k} &= c_k^* \quad (\because x(t) \text{ is real}) \\
a_k &= \frac{c_{-k}+c_k}{2} = \mathrm{Re}[c_k] \\
b_k &= \frac{c_{-k}-c_k}{j2} = -\mathrm{Im}[c_k]
\end{aligned}\right\} \rightarrow
\left\{\begin{aligned}
h_0 &= a_0 = c_0 \\
h_k &= 2\sqrt{a_k^2+b_k^2} = \sqrt{\mathrm{Re}^2[c_k]+\mathrm{Im}^2[c_k]} = 2|c_k| \\
\theta_k &= \tan^{-1}\left(\frac{b_k}{a_k}\right) = -\tan^{-1}\left(\frac{\mathrm{Im}[c_k]}{\mathrm{Re}[c_k]}\right)
\end{aligned}\right.
$$
