NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION FOR

(Semester I : 2013/2014)

EE2023 – SIGNALS & SYSTEMS

Nov/Dec 2013 - Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. This paper contains **EIGHT (8)** questions and comprises **TWELVE (12)** printed pages.
2. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.
3. This is a **CLOSED BOOK** examination.
4. Programmable calculators are not allowed.
5. Tables of Fourier Transforms, Laplace Transforms and trigonometric identities are provided in Pages 9, 10 and 11, respectively.

Examiners :
A/Prof Loh Ai Poh
A/Prof Ng Chun Sum
A/Prof Tan Woei Wan
Prof Lawrence Wong

SECTION A : Answer ALL questions in this section

Q1. The dynamic model of a system is given by

$$\frac{d^2x}{dt^2} + K\frac{dx}{dt} + \alpha x(t) = T(t).$$

where $T(t)$ and $x(t)$ are the input and output respectively, and $K, \alpha > 0$.

(a) Find the transfer function $\left(G(s) = \frac{X(s)}{T(s)}\right)$ of the system in terms of K and α.

(2 marks)

(b) Let $K = 2$ and $\alpha = 5$. What happens to the steady state gain of the system if K is doubled?

(2 marks)

(c) For the same values of K and α in part (b), show what happens to the poles of $G(s)$ if K is doubled. Show the effect in relation to the position of the poles on the complex plane. Hence explain the effect on the response to a unit step input. You do not need to calculate the exact unit step response.

(3 marks)

(d) For the same values of K and α in part (b), show what happens to the damping ratio of $G(s)$ if α is doubled while keeping K constant. Explain the effect on the response to a unit step input.

(3 marks)

Q2. Given the periodic signal $x(t) = 5\cos(4t) + 2\cos\left(6t + \frac{\pi}{6}\right) + 10$.

(a) Determine the complex Fourier series coefficients of $x(t)$. (4 marks)

(b) Determine the Fourier transform of $x(t)$. (3 marks)

(c) Determine the average power of $x(t)$. (3 marks)

Q3. Consider the signal $x(t) = t\exp(-t)u(t)$ where $u(t)$ is the unit step function. The spectrum of $x(t)$ is given by $X(f) = \dfrac{1}{(1 + j2\pi f)^2}$.

(a) Let $y(t) = 2t\exp(-0.5t)u(t)$.

(i) Express $y(t)$ in terms of $x(t)$.

$$\left[Hint:\ u(\alpha t) = \begin{cases} u(t); & \alpha > 0 \\ u(-t); & \alpha < 0 \\ 1; & \alpha = 0 \end{cases} \right]$$

(3 marks)

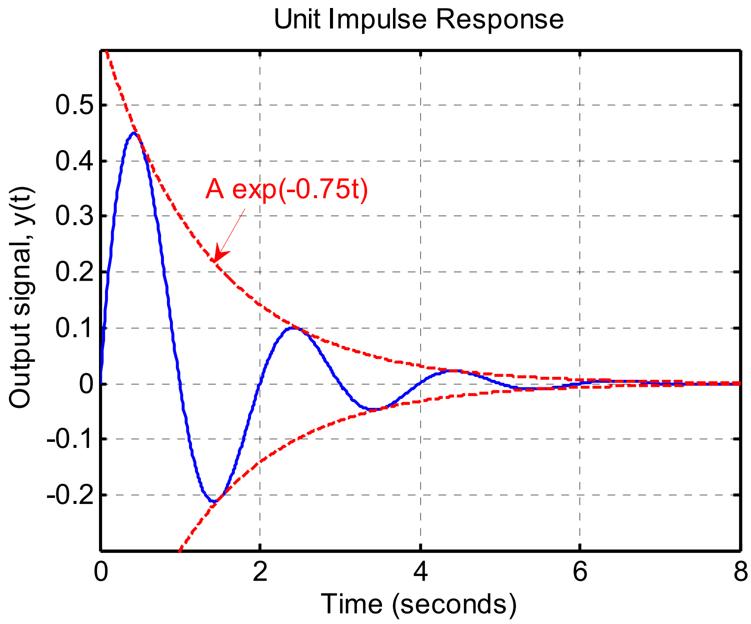
(ii) Based on the result obtained in Part (i), or otherwise, which of the two signals, $x(t)$ or $y(t)$, would you expect to have a larger bandwidth, and why?

(3 marks)

(b) Determine the spectrum of $z(t) = \begin{cases} x(t); & t \ge 0 \\ x(-t); & t < 0 \end{cases}$.

(4 marks)

Q4. The unit impulse response of a standard second order system, $G(s)$, is shown in Figure Q4.



(a) Determine the poles of $G(s)$.

(4 mark)

(b) Is the system stable, marginally stable or unstable? Justify your answer.

(2 mark)

(c) Identify all the functions from the following list that may be terms in the output signal when the quadratic signal, t^2, is applied to the system?

- Decaying exponential function, $Ae^{-at}U(t)$
- Growing exponential function, $Be^{at}U(t)$
- Decaying complex exponential function, $e^{-at}(C_1 \sin \omega t + C_2 \cos \omega t)U(t)$
- Growing complex exponential function, $e^{at}(C_1 \sin \omega t + C_2 \cos \omega t)U(t)$
- Step function, $DU(t)$
- Ramp function, $EtU(t)$
- Quadratic function, $Ft^2U(t)$

$U(t)$ is the unit step function, A, B, C_1, C_2, D, E and F are constants.

(4 mark)

SECTION B : Answer 3 out of the 4 questions in this section

Q5. Figure Q5 shows a Butterworth filter which can be designed using a resistor, an inductor and a capacitor.

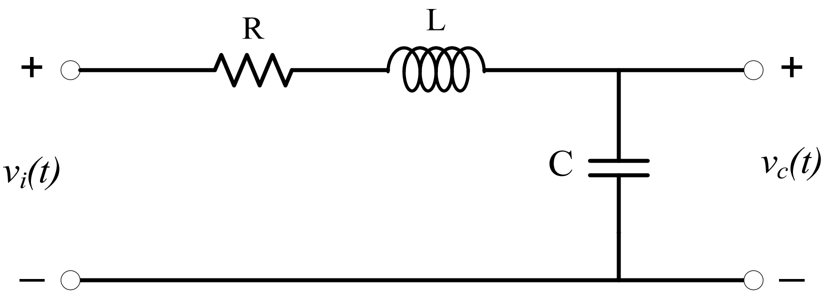


Figure Q5: A Butterworth Filter

Let the input and output of the filter be $v_i(t)$ and $v_c(t)$, respectively.

(a) Show that the transfer function of the filter is given by

$$G(s) = \frac{V_c(s)}{V_i(s)} = \frac{1}{s^2 LC + sRC + 1}$$

where $V_i(s)$ and $V_c(s)$ are the Laplace transforms of $v_i(t)$ and $v_c(t)$, respectively.

(3 marks)

(b) Derive the relationship between R, L and C if the squared magnitude response of the filter has the following characteristics :

$$|G(j\omega)|^2 = \frac{1}{1+\omega^4 L^2 C^2}.$$

(4 marks)

(c) Assume $R = \sqrt{\frac{2L}{C}}$.

i. Find the damping ratio of the filter.

(4 marks)

ii. Suppose the 3 dB bandwidth of the filter is ω_c rad/s. Find ω_c in terms of L and C.

(3 marks)

iii. Design a Butterworth filter with a 3 dB bandwidth of 10^6 rad/s and $R = 1$ kΩ.

(6 marks)

Q.6 (a) Figure Q6(a) shows an emergency signaling system. The transmitter transmits a signal, $m(t) = x(t)c(t)$, where

$$x(t) = \cos(10\pi t)$$

is the emergency tone and

$$c(t) = \cos(1600\pi t)$$

is the carrier wave. The transmitter and receiver are connected by an ideal communication channel.

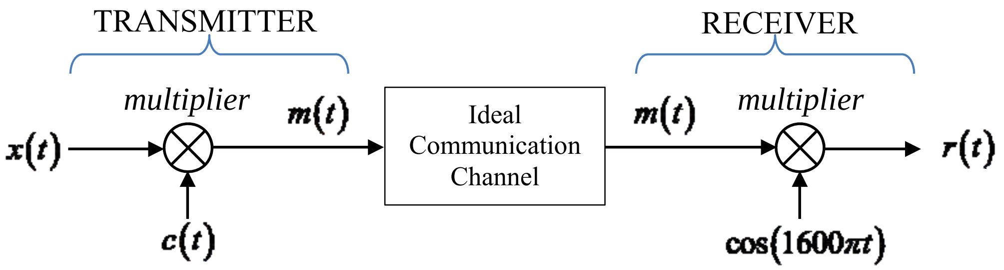


Figure Q6(a)

i. Sketch the spectrum of $m(t)$.

(4 marks)

ii. Derive the expression for the receiver output, $r(t)$.

(4 marks)

iii. Can $x(t)$ be recovered from $r(t)$ by passing $r(t)$ through a lowpass filter? If 'YES', specify the filter. If 'NO', explain why?

(2 marks)

(b) Derive the Fourier transform of the periodic signal shown in Figure Q6(b) of which the double-hump generating function is a full-wave rectified sine pulse.

(10 marks)

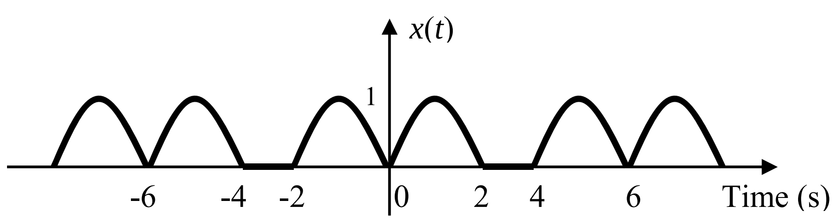


Figure Q6(b)

Q7. Figure Q7(a) shows the block diagram of a signal generator where $x(t)$ is the source signal, $h(t)$ is the impulse response of the filter, and $y(t)$ is the desired signal.

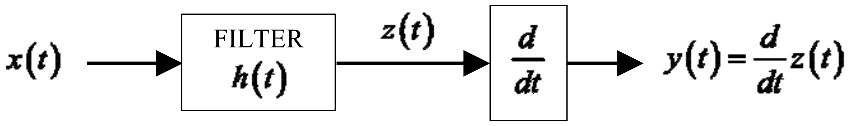


Figure Q7(a): Signal Generator

The impulse response of the filter is given by

$$h(t) = \frac{3}{4}\text{sinc}(150t)\cos(2\pi f_o t),$$

where $100 \text{ Hz} \le f_o \le 1000 \text{ Hz}$, and the spectrum of $x(t)$ is shown in Figure Q7(b).

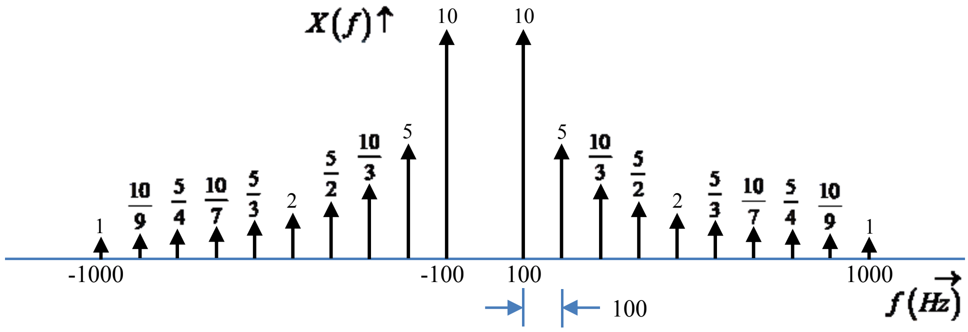


Figure Q7(b): Spectrum of $x(t)$

(a) (i) Is $x(t)$ an energy or power signal?

(ii) Is $x(t)$ a real, imaginary or complex signal?

(iii) Is $x(t)$ an odd or even function of t?

(iv) Is $x(t)$ periodic? If 'YES', what is the period?

(v) What is the DC value of $x(t)$?

(5 marks)

(b) Determine the frequency response, $H(f) = \Im\{h(t)\}$, of the filter.

(5 marks)

(c) Suppose $f_o = 450 \text{ Hz}$. Using the result of Part (b), or otherwise, find $y(t)$. $\left[\textit{Hint : Find } z(t) \textit{ first}\right]$

(6 marks)

(d) How many different signals can the signal generator produce if the value of f_o can be continuously adjusted between 100 Hz and 1000 Hz?

(4 marks)

Q8. Consider a system, $G(s)=\dfrac{Y(s)}{X(s)}$, whose transient behavior is dominated by a first order plus dead-time factor $G_a(s)=\dfrac{Ke^{-sL}}{(\tau s+1)}$ i.e. $G(s)=G_a(s)\cdot G_b(s)$.

(a) The following observations were obtained from experiments conducted to identify the system model :

- The steady-state output value of the system is 26 if the input signal is $2u(t)$ where $u(t)$ is the unit step function.
- When the input signal is $x(t)=7\cos(5t+30°)$, the steady-state output signal is $\lim_{t\to\infty} y(t)=\dfrac{91}{\sqrt{2}}\cos(5t-40°)$.

Using this information, determine K, τ and L, the parameters of the dominant factor $G_a(s)$.

(10 marks)

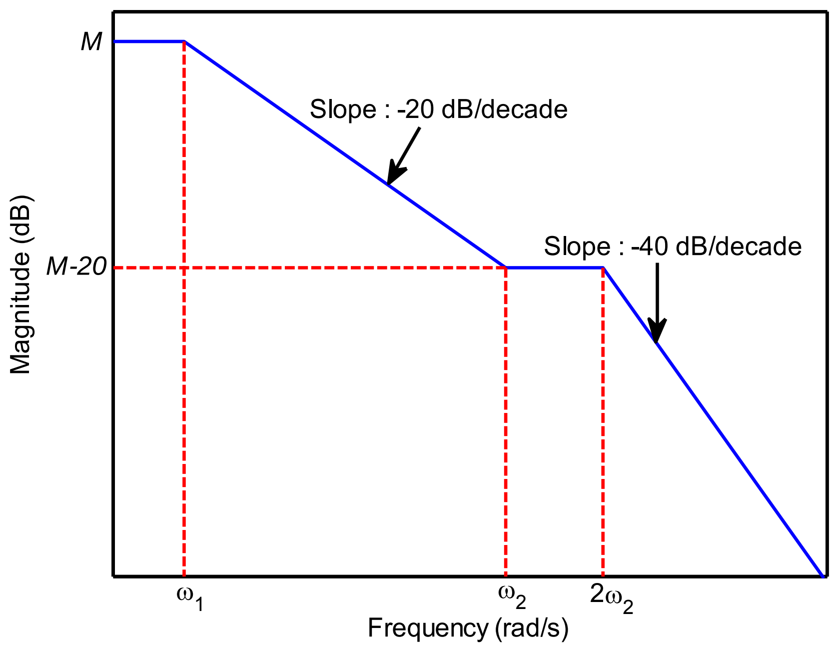
(b) The Bode magnitude plot of $G(s)$ is shown in Figure Q8.

- Using results from part (a), show that the value of M and the first corner frequency, ω_1, in the Bode magnitude plot are 22.3 dB and 5 rad/s respectively.

  [*Hint: The contribution of $G_b(s)$ to M and ω_1 is negligible compared to that of $G_a(s)$*]

- What is the transfer function $G(s)$?

(10 marks)



END OF QUESTIONS

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Fourier Series:
$$\begin{cases} X_k = \dfrac{1}{T}\displaystyle\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k\,t/T)\,dt \\ x(t) = \displaystyle\sum_{k=-\infty}^{\infty} X_k \exp(j2\pi k\,t/T) \end{cases}$$

Fourier Transform:
$$\begin{cases} X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$$

FOURIER TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	$\mathbf{1}$
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
Triangle	$\mathrm{tri}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}^2(fT)$
Sine Cardinal	$\mathrm{sinc}\left(\frac{t}{T}\right)$	$T\,\mathrm{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp\left(-\alpha^2\pi^2 f^2\right)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES

	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi f t_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty}X_1(\zeta)X_2(f-\zeta)\,d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)\,d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
		$\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s)=\int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS		
	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n^th^ order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES		
	Time-domain	***s*-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\dfrac{1}{\lvert\alpha\rvert}X\left(\dfrac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\dfrac{1}{s}X(s)$
Differentiation in the time-domain	$\dfrac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\dfrac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\dfrac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\dfrac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\dfrac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

	$y_{step}(t)$	$Y_{step}(s)$	**SYSTEM PARAMETERS**
Step response of 1^{st} order system	$K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	$\frac{1}{s}\cdot\frac{K}{(sT+1)}$	$\begin{pmatrix} T: & \text{System Time-constant} \\ K: & \text{System Steady-state (or DC) Gain}\end{pmatrix}$
Step response of 2^{nd} order <u>underdamped</u> system: $(0<\zeta<1)$	$K\left[1-\frac{\exp(-\omega_n\zeta t)}{(1-\zeta^2)^{0.5}}\sin\left(\omega_n(1-\zeta^2)^{0.5}t+\phi\right)\right]u(t)$	$\frac{1}{s}\cdot\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\begin{pmatrix} \omega_n: & \text{System Undamped Natural Frequency} \\ \zeta: & \text{System Damping Factor} \\ \omega_d: & \text{System Damped Natural Frequency} \\ K: & \text{System Steady-state (or DC) Gain}\end{pmatrix}$ $\begin{pmatrix} \sigma=\omega_n\zeta \\ \omega_d^2=\omega_n^2(1-\zeta^2) \\ \omega_n^2=\sigma^2+\omega_d^2 \\ \tan(\phi)=\omega_d/\sigma \end{pmatrix}$
	$K\left[1-\left(\frac{\sigma^2+\omega_d^2}{\omega_d^2}\right)^{0.5}\exp(-\sigma t)\sin(\omega_d t+\phi)\right]u(t)$	$\frac{1}{s}\cdot\frac{K(\sigma^2+\omega_d^2)}{(s+\sigma)^2+\omega_d^2}$	
2^{nd} order system - RESONANCE - $(0\le\zeta<1/\sqrt{2})$	*RESONANCE FREQUENCY :* $\omega_r=\omega_n(1-2\zeta^2)^{0.5}$		*RESONANCE PEAK :* $M_r=\left\|H(j\omega_r)\right\|=\frac{K}{2\zeta(1-\zeta^2)^{0.5}}$

TRIGONOMETRIC IDENTITIES	
$\exp(\pm j\theta)=\cos(\theta)\pm j\sin(\theta)$	$\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$
$\cos(\theta)=\frac{1}{2}\left[\exp(j\theta)+\exp(-j\theta)\right]$	$\cos(\alpha\pm\beta)=\cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
$\sin(\theta)=\frac{1}{j2}\left[\exp(j\theta)-\exp(-j\theta)\right]$	$\tan(\alpha\pm\beta)=\frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$
$\sin^2(\theta)+\cos^2(\theta)=1$	
$\sin(2\theta)=2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$	$\cos(\alpha)\cos(\beta)=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
$\sin^2(\theta)=\frac{1}{2}\left[1-\cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta)=\frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$
$\cos^2(\theta)=\frac{1}{2}\left[1+\cos(2\theta)\right]$	$\mathbf{C}\cos(\theta)-\mathbf{S}\sin(\theta)=\sqrt{\mathbf{C}^2+\mathbf{S}^2}\cos\left[\theta+\tan^{-1}\left(\frac{\mathbf{S}}{\mathbf{C}}\right)\right]$