Outline of Lecture

Continuous-Frequency Spectrum (Fourier Transform)

1. Fourier Transform

- In the last lecture, we have shown that the discrete-frequency spectrum of a periodic signal, $x_p(t)$ is given by its complex exponential Fourier series coefficient, c_k.
- The Fourier series expansion of an aperiodic signal does not exist. Instead we make use of another mathematical tool called Fourier transform to derive the spectrum of an aperiodic signal in the continuous-frequency domain, f.
- The continuous-frequency spectrum of an aperiodic signal $x(t)$ is given by its Fourier transform, $X(f)$:

$$\left.\begin{array}{c}\text{forward}\\ \text{Fourier transform}\\ \text{t-domain to f-domain}\end{array}\right\} X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{1}$$

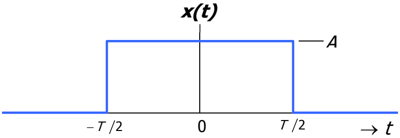
$$\left.\begin{array}{c}\text{inverse}\\ \text{Fourier transform}\\ \text{f-domain to t-domain}\end{array}\right\} x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \tag{2}$$

- $x(t)$ and $X(f)$ are Fourier transform pairs, denoted by $x(t) \leftrightarrow X(f)$.

Example 1

Spectrum of $x(t) = A\mathit{rect}\left(\frac{t}{T}\right)$

$$
\begin{aligned}
x(t) &= A\mathit{rect}\left(\frac{t}{T}\right) \\
&= \begin{cases} A & -\frac{T}{2} \le t < \frac{T}{2} \\ 0 & \mathit{elsewhere} \end{cases}
\end{aligned}
$$

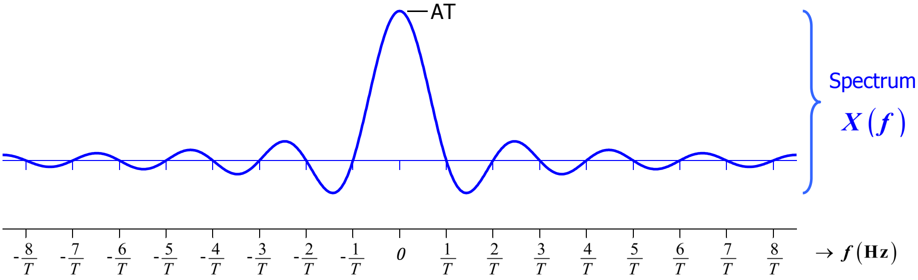


The Fourier transform of $x(t)$ *is :*

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\
&= \int_{-0.5T}^{0.5T} A e^{-j2\pi f t} dt \\
&= \begin{cases} AT\dfrac{\sin(\pi f T)}{\pi f T} & f \neq 0 \\ AT & f = 0 \end{cases} \\
&= AT\mathit{sinc}(fT)
\end{aligned}
$$

$$
\underbrace{\left[A\mathrm{rect}\left(\frac{t}{T}\right) \leftrightarrow AT\mathrm{sinc}(fT) \right]}_{Fourier\ transform\ pair}
$$

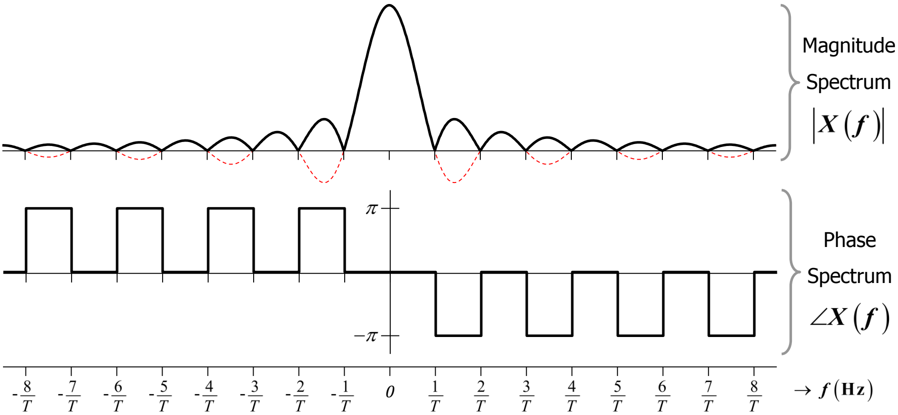
Since $X(f)$ is real, the spectrum can be plotted in one single plot :



$X(f)$ can also be written in terms of its magnitude and phase :

$$X(f) = AT\text{sinc}(fT) = |X(f)|e^{j\angle X(f)} \text{ where } \begin{cases} |X(f)| = AT|\text{sinc}(fT)| \\ \angle X(f) = \begin{cases} 0 & X(f) > 0 \\ \pm\pi & X(f) < 0 \end{cases} \end{cases}$$

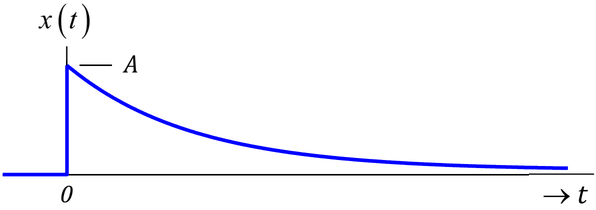
where the corresponding plots are :



Example 2

Spectrum of an exponentially decaying pulse : $x(t) = Ae^{-\alpha t}u(t)$

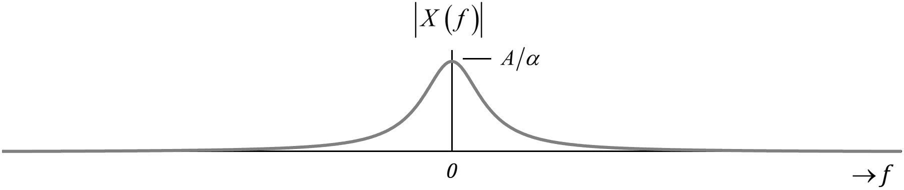
$$x(t) = Ae^{-\alpha t}u(t) = \begin{cases} Ae^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases} \ldots \textit{assume that } \alpha > 0$$



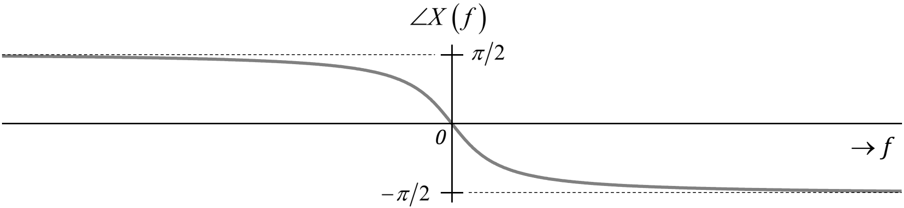
The Fourier transform of $x(t)$ is as follows :

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} Ae^{-\alpha t}e^{-j2\pi ft}dt \\ &= \int_{0}^{\infty} Ae^{-(\alpha+j2\pi f)t}dt = A\left[\frac{e^{-(\alpha+j2\pi f)t}}{-(\alpha+j2\pi f)}\right]_{0}^{\infty} = \frac{A}{\alpha+j2\pi f} \end{aligned}$$

Magnitude spectrum : $|X(f)| = \sqrt{X(f)X^*(f)} = \dfrac{A}{\sqrt{\alpha^2 + 4\pi^2 f^2}}$



Phase spectrum : $\angle X(f) = \tan^{-1}\left(\dfrac{Im[X(f)]}{Re[X(f)]}\right) = -\tan^{-1}\left(\dfrac{2\pi f}{\alpha}\right)$



Spectrum of $X(f)$ cannot be combined into one plot as $X(f)$ is complex.

2. Fourier Transform Properties

Let $\begin{cases} X(f) = \mathcal{F}\{x(t)\} & \text{denote Fourier transform of } x(t) \\ x(t) \leftrightarrow X(f) & \text{denote Fourier transform pair} \end{cases}$

Property A (Linearity)

$$\alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(f) + \beta X_2(f)$$

Example 3

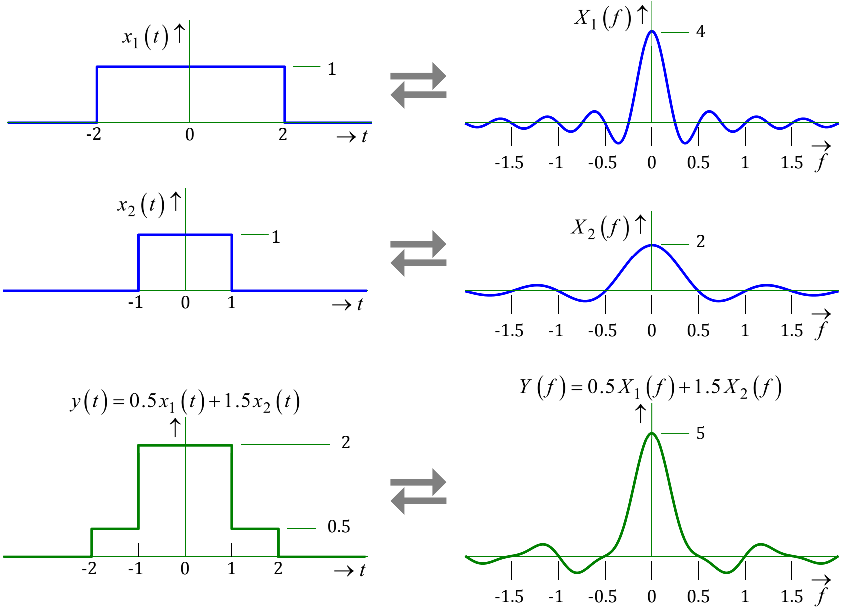
Given $\begin{array}{ccc} [x_1(t) = \mathit{rect}\,(t/4)] & \leftrightarrow & [X_1(f) = 4\,\mathit{sinc}\,(4f)] \\ [x_2(t) = \mathit{rect}\,(t/2)] & \leftrightarrow & [X_2(f) = 2\,\mathit{sinc}\,(2f)] \end{array}$

Find the Fourier transform of $Y(f) = 0.5x_1(t) + 1.5x_2(t)$.

Answer : Applying linearity property :

$$\begin{aligned} Y(f) &= 0.5X_1(f) + 1.5X_2(f) \\ &= 0.5\,[4\,\mathrm{sinc}(4f)] + 1.5\,[2\,\mathrm{sinc}(2f)] \\ &= 2\,\mathrm{sinc}(4f) + 3\,\mathrm{sinc}(2f) \end{aligned}$$

Time domain Frequency domain



Property B (Time Scaling)

$$x(\beta t) \leftrightarrow \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right)$$

Example 4

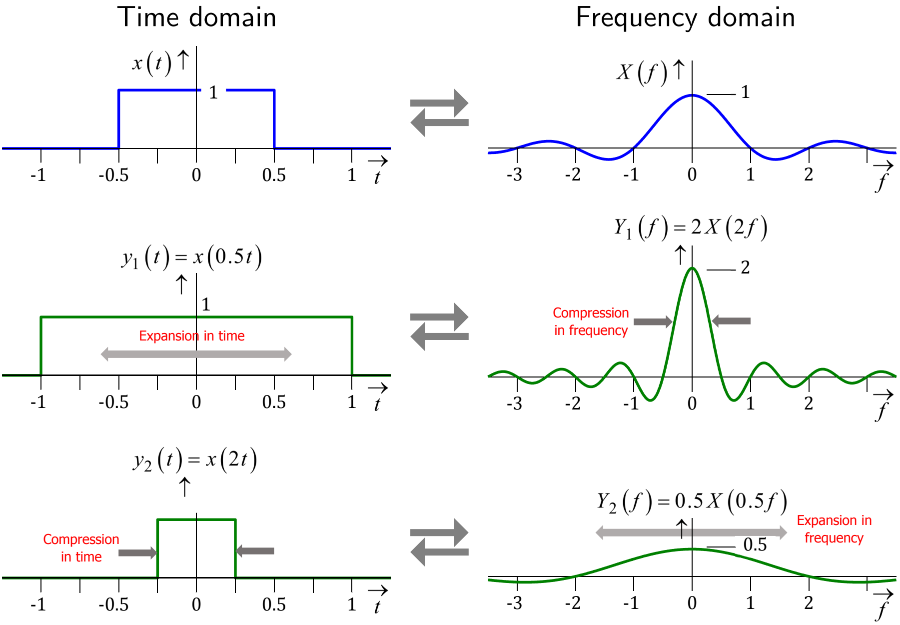
Given $[x(t) = \textit{rect}\,(t)] \leftrightarrow [X(f) = \textit{sinc}\,(f)]$

Find the Fourier transform of $\begin{cases} y_1(t) = x(0.5t) & \cdots \textit{expansion in time domain} \\ y_2(t) = x(2t) & \cdots \textit{compression in time domain} \end{cases}$

Answer : Applying time scaling property :

$$\begin{aligned} Y_1(f) &= 2X(2f) = 2\,\mathrm{sinc}\,(2f) && \cdots \text{compression in frequency domain} \\ Y_2(f) &= 0.5X(0.5f) = 0.5\,\mathrm{sinc}\,(0.5f) && \cdots \text{expansion in frequency domain} \end{aligned}$$

- Expansion in time domain leads to compression in frequency domain.
- Compression in time domain leads to expansion in frequency domain.
- Hence time-spread is inversely proportional to frequency-spread.



Time domain
Frequency domain
$x(t)\uparrow$
1
-1
-0.5
0
0.5
1
$\vec{t}$
$X(f)\uparrow$
1
-3
-2
-1
0
1
2
$\vec{f}$
$y_1(t) = x(0.5t)$
Expansion in time
$Y_1(f) = 2X(2f)$
2
Compression in frequency
$y_2(t) = x(2t)$
Compression in time
$Y_2(f) = 0.5X(0.5f)$
0.5
Expansion in frequency

Property C (Duality)

$$X(t) \leftrightarrow x(-f) \text{ or } X(-t) \leftrightarrow x(f)$$

Example 5

Given $\left[x(t) = \frac{1}{2}\text{tri}\left(\frac{t}{2}\right)\right] \leftrightarrow \left[X(f) = \text{sinc}^2(2f)\right]$

Find the Fourier transform of $y(t) = 2\,\text{sinc}^2(2t)$.

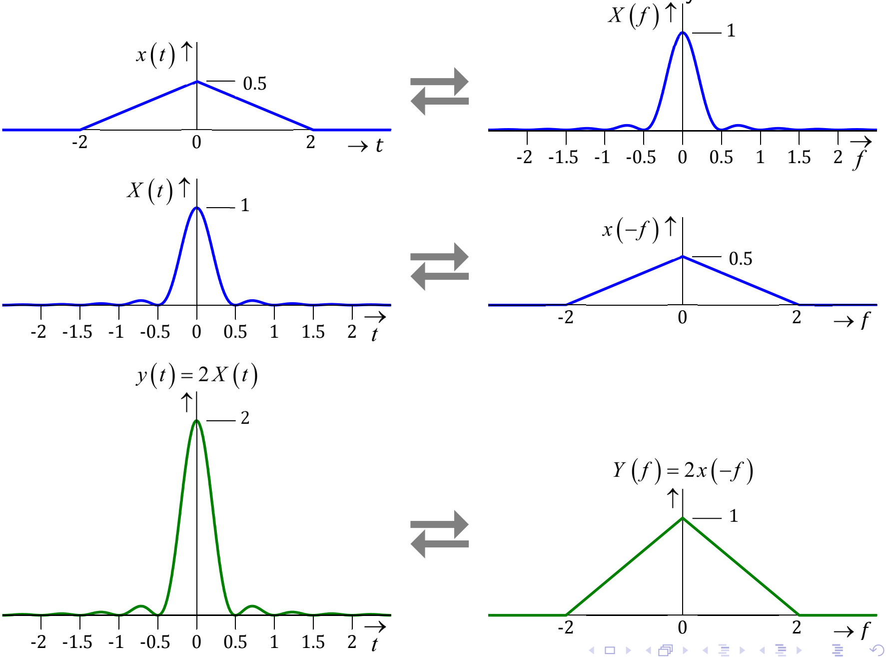
Answer : Note that $y(t) = 2X(t)$. Applying duality property :

$$\underbrace{\left[\overbrace{\frac{1}{2}\text{tri}\left(\frac{t}{2}\right)}^{x(t)} \leftrightarrow \overbrace{\text{sinc}^2(2f)}^{X(f)}\right]}_{\text{Given}} \xrightarrow[\textit{property}]{\textit{by duality}} \left[\overbrace{\text{sinc}^2(2t)}^{X(t)} \leftrightarrow \overbrace{\frac{1}{2}\text{tri}\left(\frac{-f}{2}\right)}^{x(-f)}\right]$$

Hence $Y(f) = \mathcal{F}\{2X(t)\} = \mathcal{F}\left\{2\,\text{sinc}^2(2t)\right\} = \text{tri}\left(\frac{f}{2}\right)$

Note that the negative sign in tri(.) is dropped because it is an even function i.e. $\text{tri}(-f) = \text{tri}(f)$.

Time domain　　　　　　　　　　Frequency domain



Property D (Time Shifting)

$$x(t-t_0) \leftrightarrow X(f)e^{-j2\pi f t_0}$$

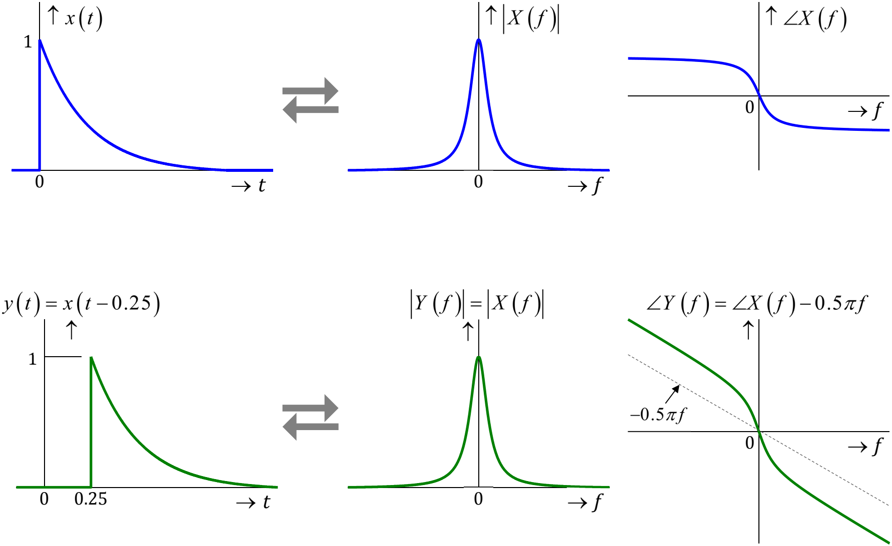
Example 6

Given $\left[x(t)=e^{-t}u(t)\right] \leftrightarrow \left[X(f)=\dfrac{1}{1+j2\pi f}\right]$

Find the Fourier transform of $y(t)=x(t-0.25)$.

Answer : Applying the time-shifting property :

$$\begin{aligned}
Y(f) &= \mathcal{F}\{x(t-0.25)\} = X(f)e^{-j2\pi f(0.25)} \\
|Y(f)| &= |X(f)|.\underbrace{\left|e^{-j2\pi f(0.25)}\right|}_{1} = |X(f)| = \frac{1}{\sqrt{1+4\pi^2 f^2}} \\
\angle Y(f) &= \angle X(f) + \angle e^{-j2\pi f(0.25)} \\
&= \angle X(f) - 0.5\pi f \\
&= -\tan^{-1}(2\pi f) - 0.5\pi f
\end{aligned}$$



Notice that the time shifted function $y(t)$ has the same magnitude spectrum as $x(t)$ but the phase spectrum is shifted by $-2\pi f t_0 = -0.5\pi f$.

Property E (Frequency Shifting (Modulation))

$$x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$$

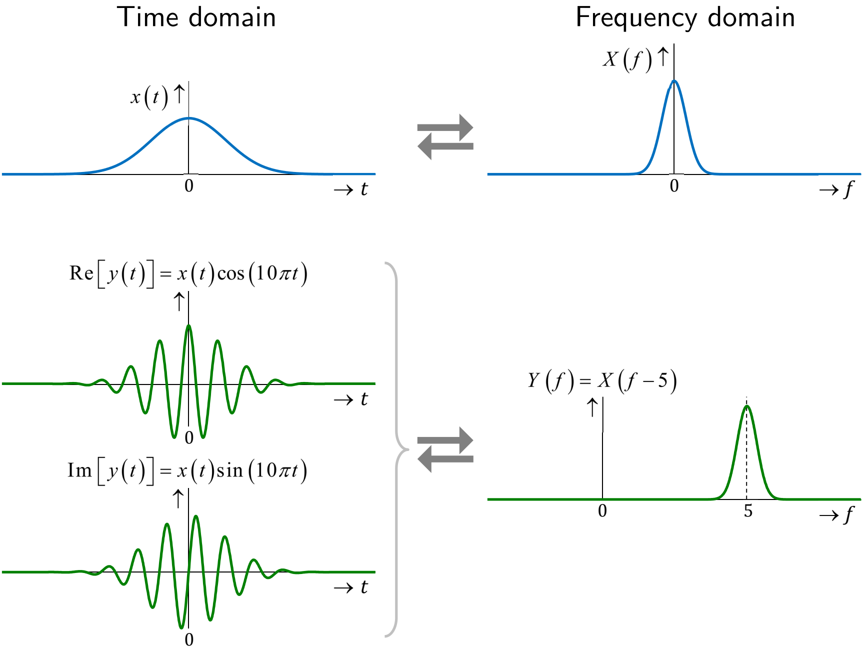
Example 7

Given $\left[x(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right] \leftrightarrow \left[X(f) = e^{-2\pi^2 f^2}\right]$

$$\begin{aligned} \textit{Find the Fourier transform of } y(t) &= x(t)e^{j2\pi 5t} \\ &= \underbrace{x(t)\cos(2\pi 5t)}_{\textit{Re}[y(t)]} + \underbrace{jx(t)\sin(2\pi 5t)}_{\textit{Im}[y(t)]} \end{aligned}$$

<u>Answer</u> : Applying the frequency-shifting property :

$$\begin{aligned} Y(f) &= \mathcal{F}\left\{x(t)e^{j2\pi 5t}\right\} \\ &= X(f-5) \\ &= e^{-2\pi^2 (f-5)^2} \end{aligned}$$


Time domain
Frequency domain
$x(t)\uparrow$
0
$\rightarrow t$
$X(f)\uparrow$
0
$\rightarrow f$
$\mathrm{Re}[y(t)] = x(t)\cos(10\pi t)$
0
$\rightarrow t$
$\mathrm{Im}[y(t)] = x(t)\sin(10\pi t)$
0
$\rightarrow t$
$Y(f) = X(f-5)$
0
5
$\rightarrow f$

Property F (Differentiation in the Time Domain)

$$\frac{d}{dt}x(t) \leftrightarrow j2\pi f X(f)$$

Example 8

Given $[x(t) = \textit{tri}\,(t)] \leftrightarrow \left[X(f) = \textit{sinc}^2(f)\right]$

Find the Fourier transform of $y(t) = \dfrac{dx(t)}{dt}$.

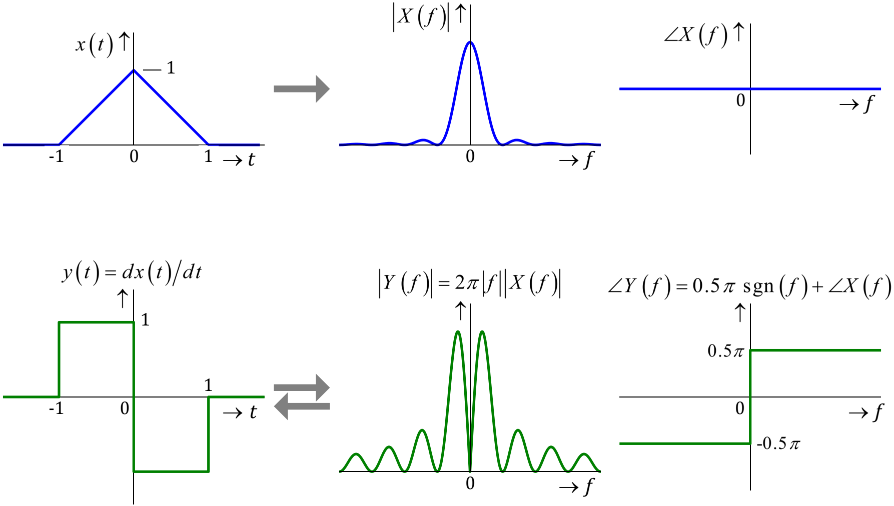
<u>Answer</u> : Applying the differentiation in time domain property :

$$\begin{aligned}
Y(f) &= \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f X(f) \\
|Y(f)| &= 2\pi|f||X(f)| = 2\pi|f|\text{sinc}^2(f) \\
\angle Y(f) &= \angle j2\pi f + \angle X(f) \\
&= 0.5\pi\text{sgn}\,(f) + \underbrace{\angle X(f)}_{0} \\
&= 0.5\pi\text{sgn}\,(f)
\end{aligned}$$

$$\text{sgn}(f) = \begin{cases} +1 & f > 0 \\ -1 & f < 0 \end{cases}$$

$$\begin{aligned}
\angle j2\pi f &= \angle jf \\
&= \begin{cases} 0.5\pi & f > 0 \\ -0.5\pi & f < 0 \end{cases}
\end{aligned}$$

Time Domain Magnitude Spectrum Phase Spectrum



Property G (Integration in the Time Domain)

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$$

where $\delta(f)$ *is the unit impulse and* $X(0) = \int_{-\infty}^{\infty} x(t)dt$.

Example 9

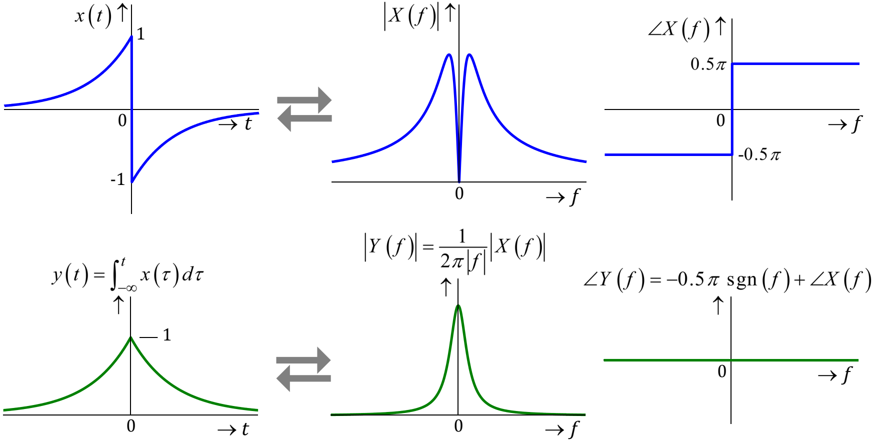
$$\left[x(t) = e^{t}u(-t) - e^{-t}u(t)\right] \leftrightarrow X(f) = \frac{j4\pi f}{1+4\pi^2 f^2} \rightarrow \begin{array}{l} |X(f)| = \frac{4\pi|f|}{1+4\pi^2 f^2} \\ \angle X(f) = \frac{1}{2}\pi \mathit{sgn}(f) \end{array}$$

Find the Fourier transform of $y(t) = \displaystyle\int_{-\infty}^{t} x(\tau)d\tau$.

<u>Answer</u> : Applying the integration in the time domain property :

$$\begin{aligned} Y(f) &= \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{1}{j2\pi f}X(f) + \frac{1}{2}\underbrace{X(0)}_{0}\delta(f) = \frac{2}{1+4\pi^2 f^2} \\ |Y(f)| &= \frac{2}{1+4\pi^2 f^2}, \quad \angle Y(f) = 0 \text{ because } Y(f) > 0\ \forall\ f \end{aligned}$$

Time Domain | Magnitude Spectrum | Phase Spectrum

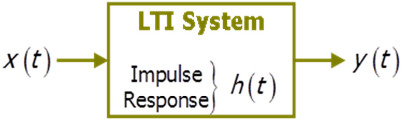


Property H (Convolution in Time equiv. to Multiplication in Freq.)

$$\underbrace{\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau}_{x_1(t)*x_2(t)} \leftrightarrow \underbrace{X_1(f).X_2(f)}_{\textit{multiplication in f}}$$

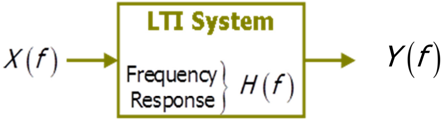
Note that

- $*$ denotes the convolution operator
- $w(t) * v(t) = \int_{-\infty}^{\infty} w(\tau)v(t-\tau)d\tau = \int_{-\infty}^{\infty} v(\tau)w(t-\tau)d\tau$
- This property is used extensively in the study of linear time-invariant systems.



TIME-DOMAIN
convolution

$y(t) = x(t) * h(t)$



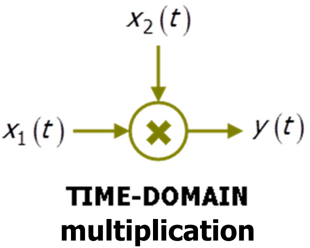
FREQUENCY-DOMAIN
multiplication

$Y(f) = H(f)X(f)$

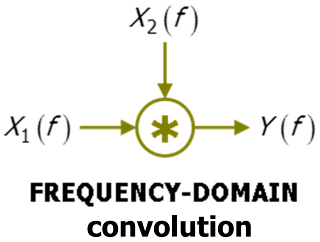
Property I (Multiplication in Time equiv. to Convolution in Freq.)

$$\underbrace{x_1(t).x_2(t)}_{\textit{multiplication in time}} \leftrightarrow \underbrace{\int_{-\infty}^{\infty} X_1(\tau)X_2(f-\tau)d\tau}_{X_1(f)*X_2(f)}$$

This property is used extensively in radio communication systems.



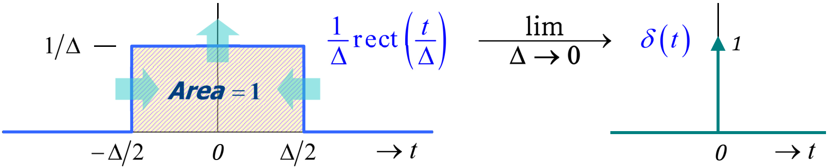
$$y(t) = x_1(t).x_2(t)$$



$$Y(f) = X_1(f) * X_2(f)$$

Fourier Transform of Common Signals

1. The unit impulse or Dirac-δ function, $\delta(t)$ defined in Lecture 2.

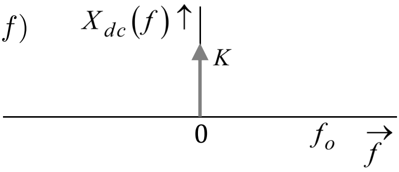


$$
\begin{aligned}
\delta(t) &= \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad \int_{-\epsilon}^{\epsilon} \delta(t)dt = 1 \ \forall\ \epsilon > 0 \\
\mathcal{F}[\delta(t)] &= \tilde{\Delta}(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt \\
&= \lim_{\Delta \to 0} \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta} e^{-j2\pi ft} dt = \lim_{\Delta \to 0} \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta} \cancelto{1}{e^{-j2\pi ft}} dt = 1
\end{aligned}
$$

Therefore $\mathcal{F}[\delta(t)] = 1$.

2. The DC signal $x_{dc}(t) = K$.

$$\mathcal{F}[K\delta(t)] = K \xrightarrow{\text{duality}} \mathcal{F}[K] = K\delta(f)$$

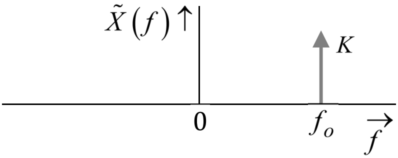


Therefore

$$X_{dc}(f) = \mathcal{F}[x_{dc}(t) = K] = K\delta(f)$$

3. Complex exponential signal

$$\tilde{x}(t) = Ke^{j2\pi f_0 t}$$

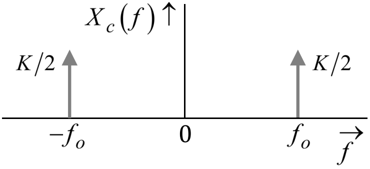


Therefore

$$\underbrace{\tilde{X}(f) = \mathcal{F}[\tilde{x}(t)] = X_{dc}(f - f_0)}_{\text{frequency-shifting property}} = K\delta(f - f_0)$$

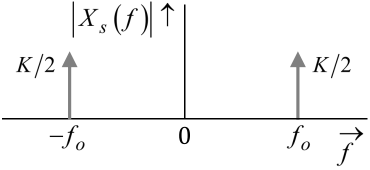
4. Cosine : $x_c(t) = K\cos(2\pi f_0 t)$.

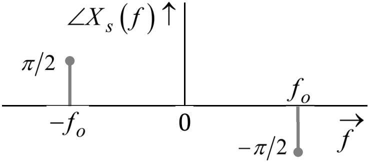
$$\begin{aligned} x_c(t) &= K\cos(2\pi f_0 t) \\ &= \frac{K}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right] \\ X_c(f) &= \frac{K}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \end{aligned}$$



5. Sine : $x_s(t) = K\sin(2\pi f_0 t)$.

$$\begin{aligned} x_s(t) &= K\sin(2\pi f_0 t) \\ &= \frac{K}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] \\ X_s(f) &= \frac{K}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right] \\ &= \frac{K}{2}e^{-j\frac{\pi}{2}}\delta(f - f_0) \\ &\quad + \frac{K}{2}e^{j\frac{\pi}{2}}\delta(f + f_0) \end{aligned}$$





6. Arbitrary Periodic Signal, $x_p(t)$, period T_p

$$\underbrace{x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}}_{\text{Fourier series expansion}} \qquad \underbrace{c_k = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x_p(t) e^{-j2\pi k f_p t} dt}_{\text{Fourier series coefficients}}$$

applying Fourier transform to $x_p(t)$,

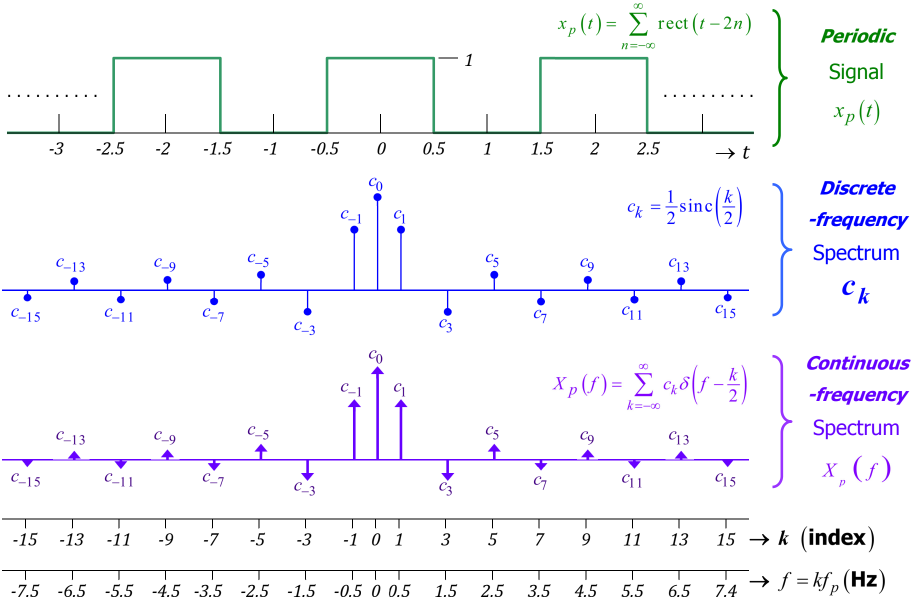
$$X_p(f) = \mathcal{F}[x_p(t)] = \mathcal{F}\left[\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}\right] = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left[e^{j2\pi k f_p t}\right]$$

Substituting $\mathcal{F}\left[e^{j2\pi k f_p t}\right] = \delta(f - k f_p)$, we get :

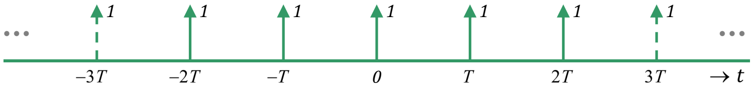
$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_p) \tag{3}$$

Important Note The Fourier transform $X_p(f)$ of any periodic signal $x_p(t)$ can be obtained by first computing the Fourier series coefficient c_k and substituting into (3).

Illustrating using the square wave you have seen before : $T_p = 2, f_p = 0.5$ Hz

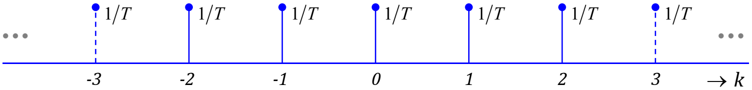


7. An interesting periodic signal which we will encounter often is the Dirac Comb, defined as $s_T(t) = \sum_n \delta(t - nT)$ where T is the fundamental period.



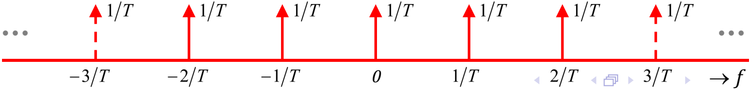
The discrete-frequency spectrum c_k [Fourier series coeff of $s_T(t)$]

$$c_k = \frac{1}{T}\int_{-0.5T}^{0.5T} s_T(t) e^{-j2\pi k \frac{t}{T}} dt = \frac{1}{T}\int_{-0.5T}^{0.5T} \delta(t) e^{-j2\pi k \frac{t}{T}} dt = \frac{1}{T}$$



The continuous-freq. spectrum $S_T(f)$ [Fourier transform of $s_T(t)$]

$$S_T(f) = \sum_k c_k \delta(f - \frac{k}{T}) = \frac{1}{T}\sum_k \delta(f - \frac{k}{T})$$



Fourier Transforms of Common Signals

Signal	**x(t)**	**X(f)**
Constant (DC)	K	$K\delta(f)$
Impulse	$\delta(t)$	1
Step function	$u(t)$	$\frac{1}{2}\left[\delta(f) + \frac{1}{j\pi f}\right]$
Sign or Signum	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangular	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\mathrm{sinc}\left(Tf\right)$
Triangular	$\mathrm{tri}\left(\frac{t}{T}\right)$	$T\mathrm{sinc}^2\left(Tf\right)$
Cardinal sine	$\mathrm{sinc}\left(\frac{t}{T}\right)$	$T\mathrm{rect}\left(Tf\right)$
Complex exponential	$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
Cosine	$\cos(2\pi f_0 t)$	$0.5[\delta(f - f_0) + \delta(f + f_0)]$
Sine	$\sin(2\pi f_0 t)$	$-0.5j[\delta(f - f_0) - \delta(f + f_0)]$
Gaussian	$e^{-\left(\frac{t^2}{\alpha^2}\right)}$	$\alpha\sqrt{0.5}e^{-\alpha^2\pi^2 f^2}$
Comb	$\sum_k \delta(t - kT)$	$\frac{1}{T}\sum_k \delta\left(f - \frac{k}{T}\right)$

Exercise 1

The waveforms of 4 pulses, $x_1(t)$, $x_2(t)$, $x_3(t)$ *and* $x_4(t)$ *are shown below. Derive the spectrum* $X_1(f)$, $X_2(f)$, $X_3(f)$ *and* $X_4(f)$ *and express the spectra of* $X_2(f)$, $X_3(f)$ *and* $X_4(f)$ *in terms of* $X_1(f)$.