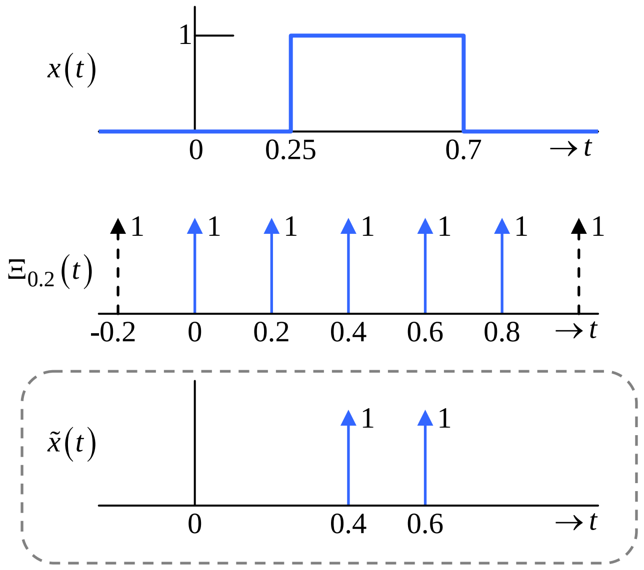
EE2023 TUTORIAL 4 (SOLUTIONS)

Solution to Q.1



$$\tilde{x}(t) = \underbrace{\mathrm{rect}\left(\frac{t-0.475}{0.45}\right)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-0.2n)}_{\Xi_{0.2}(t)}$$

$$= \delta(t-0.4) + \delta(t-0.6)$$

$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$

$$\tilde{X}(f) = \exp(-j2\pi f 0.4) + \exp(-j2\pi f 0.6)$$

$$= \begin{cases} \cos(0.8\pi f) - j\sin(0.8\pi f) + \\ \quad \cos(1.2\pi f) - j\sin(1.2\pi f) \end{cases}$$

$$= \begin{cases} [\cos(0.8\pi f) + \cos(1.2\pi f)] - \\ \quad j[\sin(0.8\pi f) + \sin(1.2\pi f)] \end{cases}$$

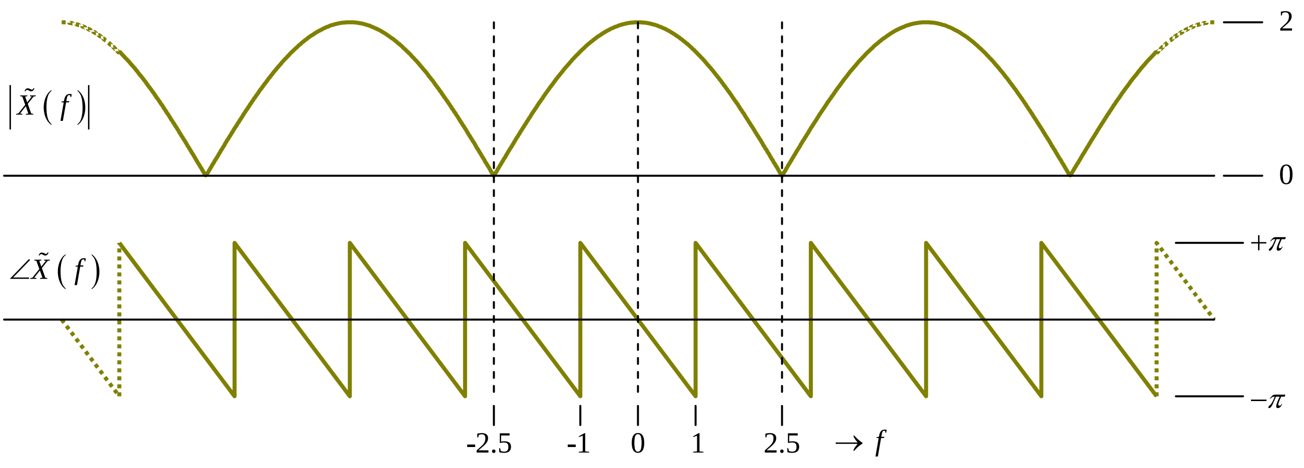
$$= 2\cos(\pi f)\cos(0.2\pi f) - j2\sin(\pi f)\cos(0.2\pi f)$$

$$\left|\tilde{X}(f)\right|^2 = 4\cos^2(\pi f)\cos^2(0.2\pi f) + 4\sin^2(\pi f)\cos^2(0.2\pi f)$$

$$= 4\cos^2(0.2\pi f)$$

- $\left|\tilde{X}(f)\right| = 2\left|\cos(0.2\pi f)\right|$

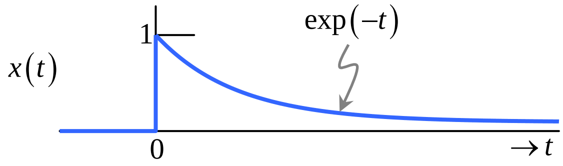
- $\angle\tilde{X}(f) = -\tan^{-1}\left(\dfrac{2\sin(\pi f)\cos(0.2\pi f)}{2\cos(\pi f)\cos(0.2\pi f)}\right) = -\tan^{-1}(\tan(\pi f)) = -\pi f$



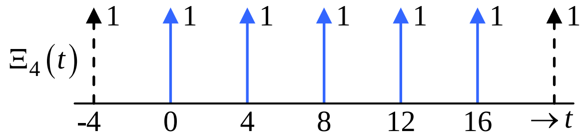
Solution to Q.2

$$\tilde{x}(t) = \underbrace{\exp(-t)u(t)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-4n)}_{\Xi_4(t)}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\delta(t-4n)$$



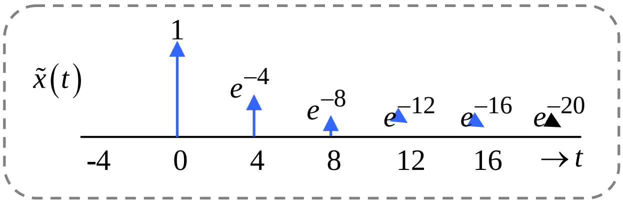
$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$



$$\tilde{X}(f) = \sum_{n=0}^{\infty} \exp(-4n)\Im\{\delta(t-4n)\}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j2\pi f 4n)$$

$$= \underbrace{\sum_{n=0}^{\infty} \exp(-4n(j2\pi f+1))}_{\text{Geometric Series}}$$



$$= \frac{1}{1-\exp(-4(j2\pi f+1))} = \frac{1}{1-\exp(-4)\exp(-j8\pi f)}$$

When a signal is sampled in the time domain, the spectrum of the sampled signal is periodic with a period equal to the sampling frequency, which in this case is 0.25 Hz.

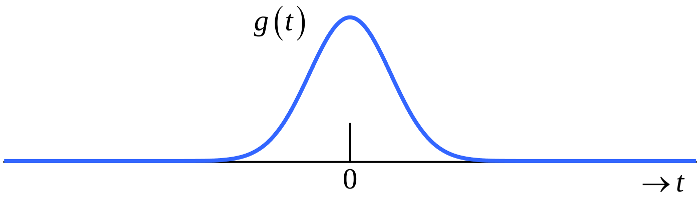
Thus, to show that $\tilde{X}(f)$ is periodic, all we need to show is $\tilde{X}(f+0.25) = \tilde{X}(f)$ as follows:

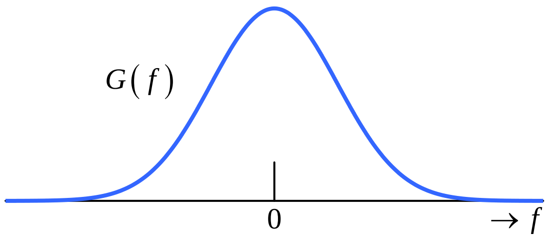
$$\tilde{X}(f+0.25) = \frac{1}{1-\exp(-4)\exp(-j8\pi(f+0.25))}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)\underbrace{\exp(-j2\pi)}_{1}}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)} = \tilde{X}(f)$$

Solution to Q.3





Rewrite $x(t)=\sum_{n=-\infty}^{\infty} g\left(t-nT_p\right)$ in convolution form:

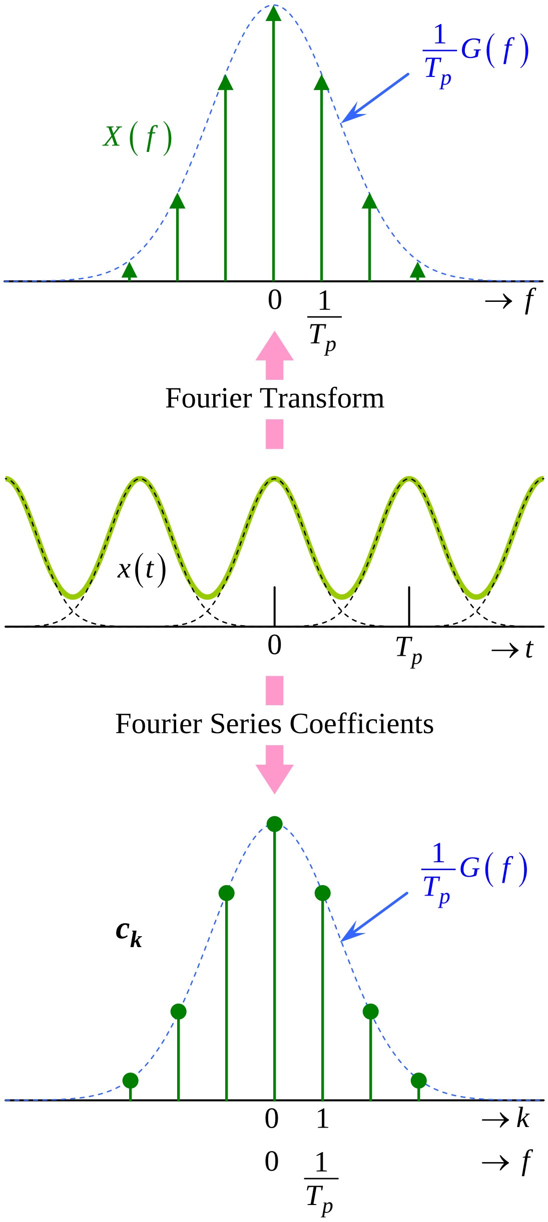
$$x(t)=g(t)*\sum_{n=-\infty}^{\infty}\delta\left(t-nT_p\right) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (*)$$

Applying the 'Convolution' property of the Fourier transform to $(*)$:

$$X(f)=G(f)\cdot\frac{1}{T_p}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T_p}\right) \quad \cdots\cdots\cdots\cdots \quad (**)$$

Conclusion:

$X(f)$ can be obtained by sampling $G(f)/T_p$ in the frequency-domain at regular spacings of $1/T_p$ Hz.



Relationship between c_k and $G(f)$

Rewrite $(**)$ as: $X(f)=\sum_{k=-\infty}^{\infty}\frac{1}{T_p}G\left(\frac{k}{T_p}\right)\delta\left(f-\frac{k}{T_p}\right)$

In terms of c_k: $\quad X(f)=\sum_{k=-\infty}^{\infty}c_k\delta\left(f-\frac{k}{T_p}\right)$

Hence, $c_k=\frac{1}{T_p}G(f)\Big|_{f=k/T_p}=\frac{1}{T_p}G\left(\frac{k}{T_p}\right)$

Uniqueness of a generating function

The generating function of a periodic signal is NOT unique. For instance, one period of a periodic signal can be used as its generating function.

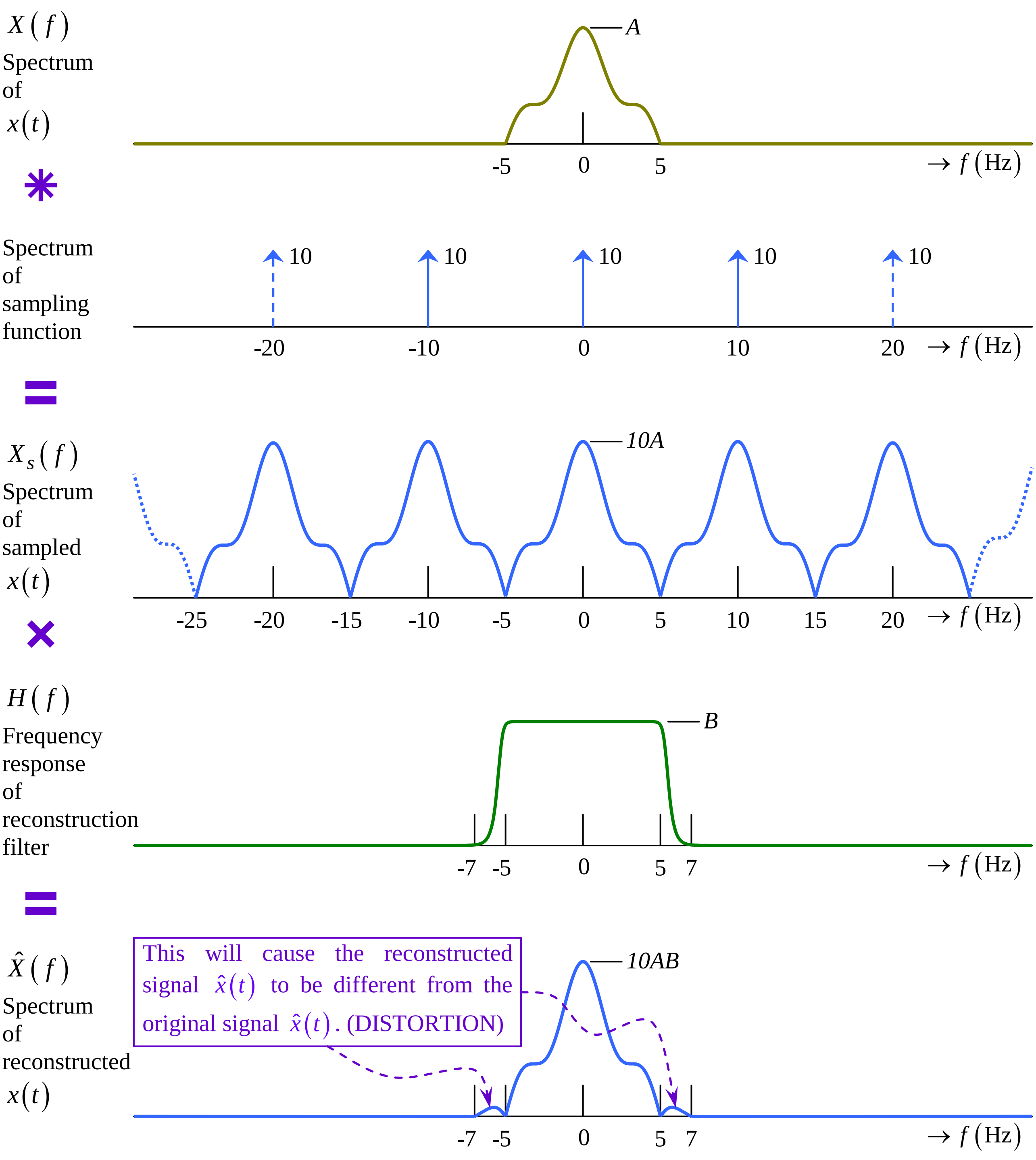
Remarks: From the sampling theorem, we observed that when a signal is sampled at regular time-intervals of T_s (sec)*, the spectrum corresponding to the sampled signal is periodic with a period of* $f_s=1/T_s$ (Hz)*. In this exercise, we showed that when a spectrum is sampled at regular frequency-intervals of* f_p (Hz)*, the signal corresponding to the sampled spectrum is periodic with a period of* $T_p=1/f_p$ (sec)*. Hence, sampling a signal leads to a periodic spectrum and sampling a spectrum leads to periodic signal. This phenomenon is due to the convolution property of the Fourier Transform and the fact that* $\left[\Xi_T(t)\rightleftarrows\frac{1}{T}\Xi_{1/T}(f)\right]$.

Solution to Q.4

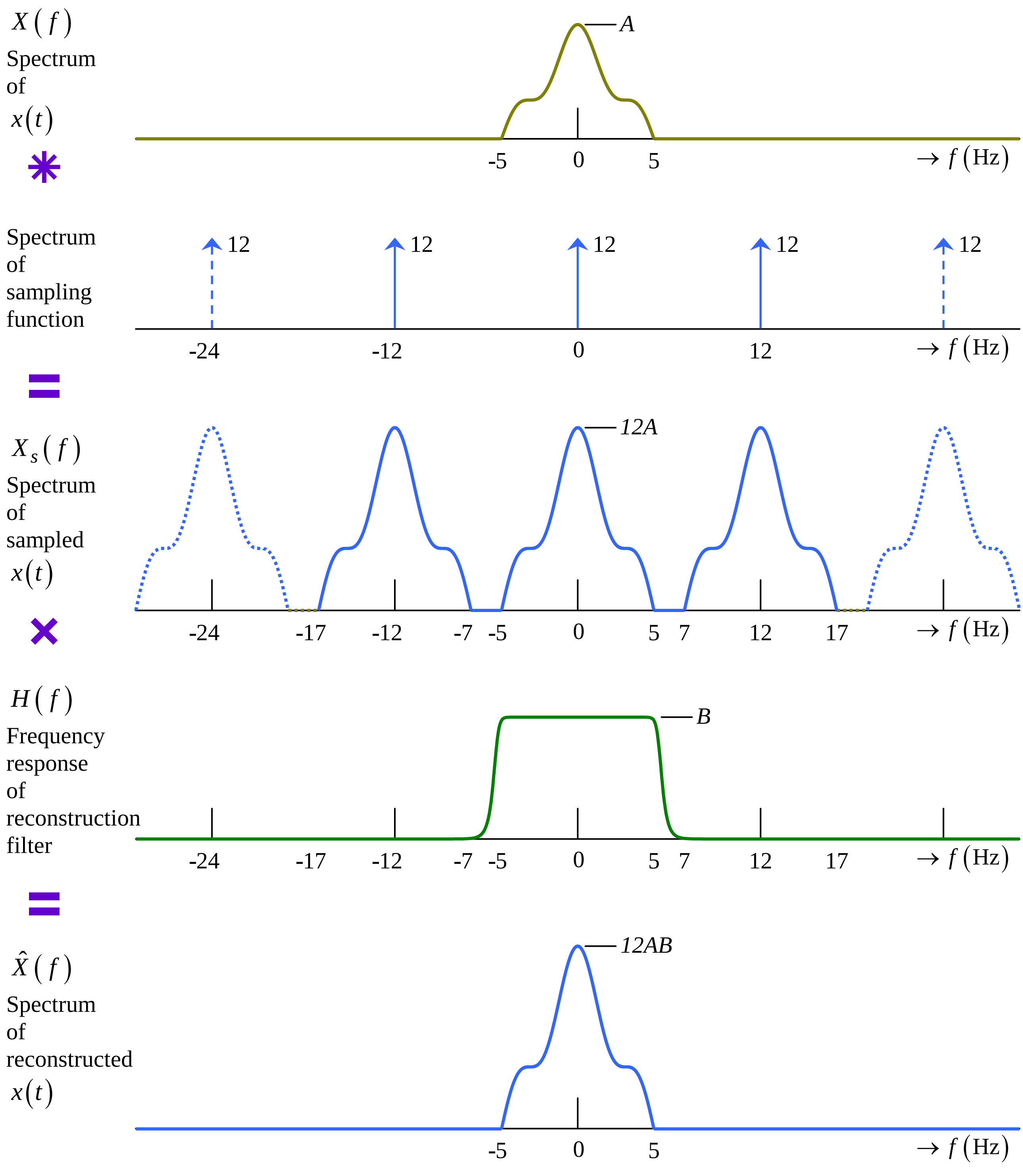
- Nyquist sampling frequency: $2 \times 5 = 10$ Hz
- Recommended sampling frequency: 12 Hz

  The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.

***Sampling frequency:* 10 Hz (Nyquist Rate)**



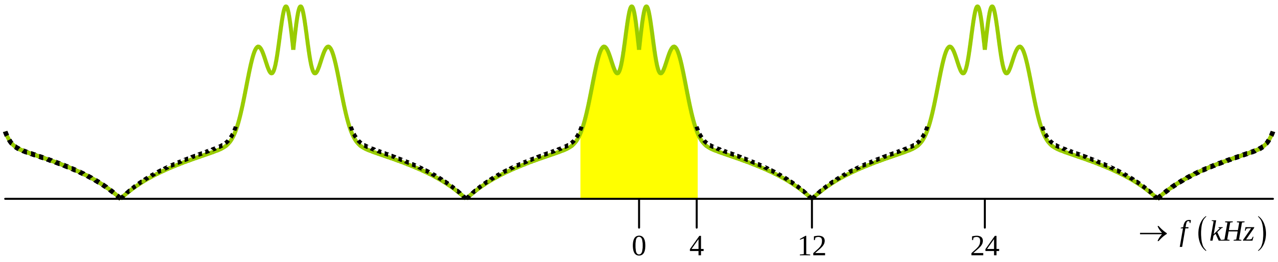
***Sampling frequency:* 12 Hz**



Remarks: *Sampling frequencies higher that 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.*

Solution to Q.5

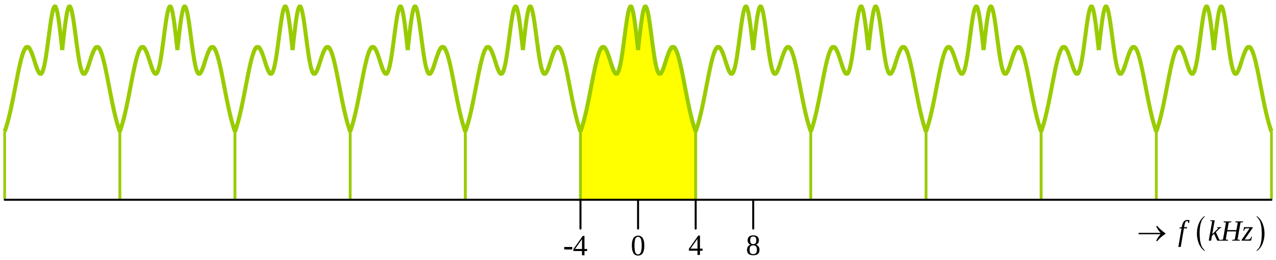
Situation A: $\left(\begin{array}{c}\textbf{\textit{No anti-aliasing lowpass filter}}\\ \textbf{\textit{and no frequency aliasing}}\end{array}\right)$: $\left\{\textbf{Sampling frequency} = \mathbf{2 \times 12 = 24}\ \boldsymbol{kHz}\right.$



Advantage: No anti-aliasing LPF needed.

Disadvantage: Sampling frequency is higher than necessary to preserve the *TQ-Band*.

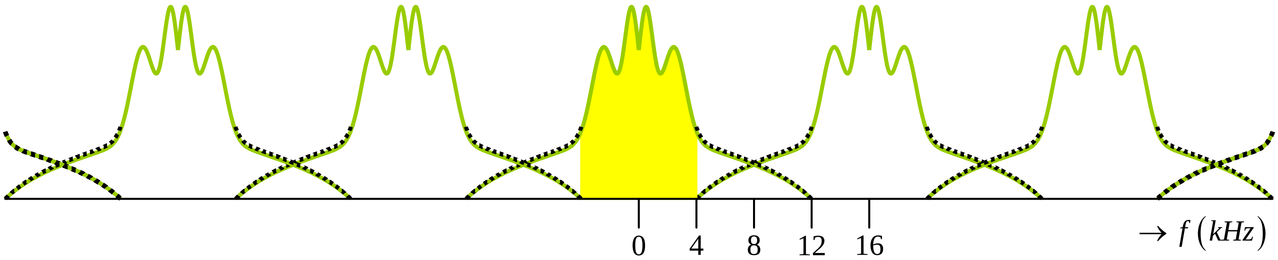
Situation B: ***Lowest sampling frequency***: $\left\{\begin{array}{l}\textbf{Ideal anti-aliasing filter of bandwidth } \mathbf{4}\ \boldsymbol{kHz}\\ \textbf{Sampling frequency} = \mathbf{2 \times 4 = 8}\ \boldsymbol{kHz}\end{array}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band.*

Disadvantage: Require anti-aliasing LPF with sharp cutoff.

Situation C: ***No anti-aliasing lowpass filter***: $\left\{\textbf{Sampling frequency} = \mathbf{2} \times \left(\dfrac{\mathbf{12+4}}{\mathbf{2}}\right) = \mathbf{16}\ \boldsymbol{kHz}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

Disadvantage: Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.

Solution to S.1

$f_s = 2 \times 20 = 40\ kHz$

$$\left[x_s(t) = x(t) \cdot \sum_n \delta\left(t - \frac{n}{40000}\right) \right] \rightleftarrows \left[\begin{aligned} X_s(f) &= X(f) * 40000 \sum_k \delta(f - 40000k) \\ &= 40000 \sum_k X(f - 40000k) \end{aligned} \right]$$

Specifying $H(f) = \frac{1}{40000} \cdot \mathrm{rect}\left(\frac{f}{40000}\right)$ will lead to $X_s(f) H(f) = X(f)$.

Solution to S.2

(a)

$$\begin{cases} x_s(t) = \sum_{n=-5}^{5} x(5n)\delta(t - 5n) \quad \text{(5: sampling period)} \\ \therefore \text{ Sampling frequency: } f_s = \frac{1}{5} = 0.2\ Hz \end{cases}$$

(b) Perfect reconstruction is of $x(t)$ from $x_s(t)$ only possible if $x(t)$ is bandlimited to $f_s/2 = 0.1$, i.e. $X(f) = 0;\ f > 0.1$.

Since $X(f) = \Im^{-1}\{\mathrm{tri}(t)\} = \mathrm{sinc}^2(f)$ has infinite frequency extent, perfect reconstruction is not possible due to frequency aliasing.

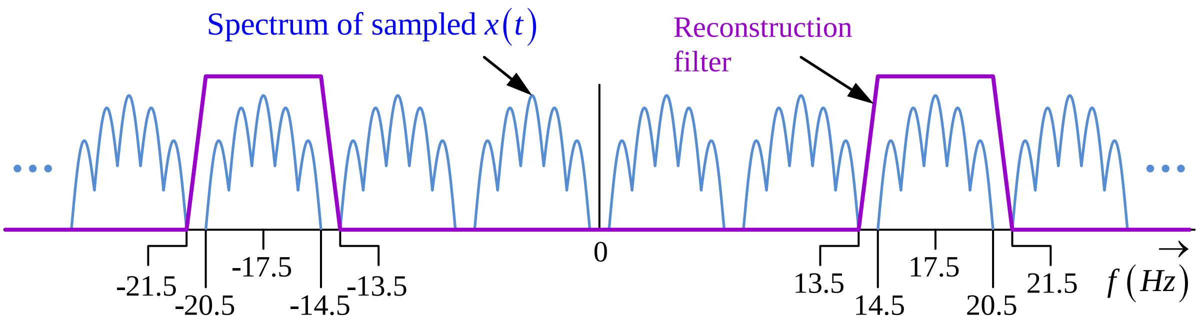
Solution to S.3

(a) Nyquist Sampling Frequency: $f_s = 2 \times 20.5 = 41\ Hz$

(b) Center frequency: $f_c = 17.5\ Hz$ Bandwidth: $B = 6\ Hz$

Possible sampling frequencies : $f_s = 2f_c/k;\quad k = 1,\ 2,\ \cdots,\ \lfloor 2f_c/B \rfloor$

Lowest sampling frequency : $f_s = \frac{2f_c}{\lfloor 2f_c/B \rfloor} = \frac{35}{5} = 7\ Hz$



$$\text{Reconstruction Filter Specs}: \begin{cases} \text{Ideal passband: } 14.5 < |f| < 20.5 \\ \text{Ideal stopband: } |f| < 13.5 \ \ or \ \ |f| > 21.5 \end{cases}$$