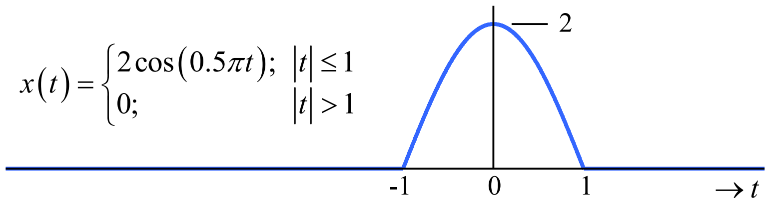
EE2023 TUTORIAL 3 (SOLUTIONS)

Solution to Q.1

$$x(t)=\begin{cases}2\cos(0.5\pi t); & |t|\le 1\\ 0; & |t|>1\end{cases}$$



(a)

Method 1: By applying direct Fourier transform:

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt = \int_{-1}^{1} 2\cos(0.5\pi t)\exp(-j2\pi ft)\,dt\\
&= 2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\cos(2\pi ft)}_{even\ function\ of\ t}dt - \cancel{j2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\sin(2\pi ft)}_{odd\ function\ of\ t}dt}\\
&= 4\int_{0}^{1}\cos(0.5\pi t)\cos(2\pi ft)\,dt\\
&= 2\int_{0}^{1}\cos((2\pi f-0.5\pi)t)+\cos((2\pi f+0.5\pi)t)\,dt\\
&= 2\left[\frac{\sin((2\pi f-0.5\pi)t)}{2\pi f-0.5\pi}+\frac{\sin((2\pi f+0.5\pi)t)}{2\pi f+0.5\pi}\right]_0^1\\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{2\pi f-0.5\pi}+\frac{\sin(2\pi f+0.5\pi)}{2\pi f+0.5\pi}\right)\\
&= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5}+\frac{\cos(2\pi f)}{2f+0.5}\right)\\
&= \frac{2\cos(2\pi f)}{\pi}\left(\frac{-1}{2f-0.5}+\frac{1}{2f+0.5}\right)\\
&= \frac{2\cos(2\pi f)}{\pi}\left(\frac{-2f-0.5+2f-0.5}{4f^2-0.25}\right)\\
&= \frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}
\end{aligned}$$

Using:
$\exp(\pm j\theta)=\cos(\theta)\pm j\sin(\theta)$

Using:
$\cos(a)\cos(b)=\frac{1}{2}[\cos(a-b)+\cos(a+b)]$

Using:
$\sin(a-b)=\sin(a)\cos(b)-\cos(a)\sin(b)$

Method 2: By applying Fourier transform properties:

$$x(t) = 2\cos(0.5\pi t)\cdot \mathrm{rect}(0.5t)$$

$$\Im\{2\cos(0.5\pi t)\} = 2\left[\frac{1}{2}\{\delta(f-0.25)+\delta(f+0.25)\}\right] = \delta(f-0.25)+\delta(f+0.25)$$

$$\Im\{\mathrm{rect}(0.5t)\} = 2\mathrm{sinc}(2f)$$

Applying the 'Multiplication in time-domain' property of the Fourier transform

$$\left[x(t) = \underbrace{2\cos(0.5\pi t)\cdot \mathrm{rect}(0.5t)}_{\substack{\text{Multiplication in}\\ \text{time-domain}}}\right] \rightleftarrows \left[X(f) = \underbrace{\Im\{2\cos(0.5\pi t)\} \ast \Im\{\mathrm{rect}(0.5t)\}}_{\substack{\text{Convolution in}\\ \text{frequency-domain}}}\right]$$

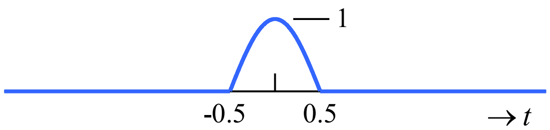
we get

$$\begin{aligned} X(f) &= [\delta(f-0.25)+\delta(f+0.25)] \ast 2\mathrm{sinc}(2f) \\ &= 2\mathrm{sinc}(2(f-0.25)) + 2\mathrm{sinc}(2(f+0.25)) \\ &= 2\mathrm{sinc}(2f-0.5) + 2\mathrm{sinc}(2f+0.5) \\ &= 2\left(\frac{\sin(2\pi f - 0.5\pi)}{\pi(2f-0.5)} + \frac{\sin(2\pi f + 0.5\pi)}{\pi(2f+0.5)}\right) \cdots\cdots \text{ Same result obtained by } \textbf{Method 1} \end{aligned}$$

(b)

From Part (a): $X(f) = \dfrac{2\cos(2\pi f)}{\pi\left(0.25 - 4f^2\right)}$

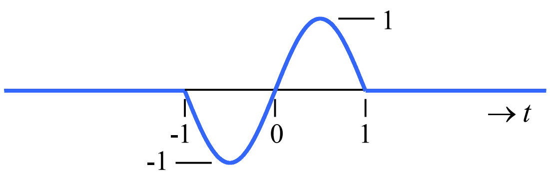
$$\tilde{y}(t) = 0.5x(2t)$$



Applying the ***scaling property***:

$$\tilde{Y}(f) = 0.5\left[\frac{1}{2}X\left(\frac{f}{2}\right)\right] = \frac{1}{4}X\left(\frac{f}{2}\right) \qquad \cdots\cdots\cdots\cdots (*)$$

$$y(t) = \tilde{y}(t-0.5) - \tilde{y}(t+0.5)$$



Applying the ***time-shifting property***:

$$Y(f) = \tilde{Y}(f)\exp\left(-j2\pi f\left(\frac{1}{2}\right)\right) - \tilde{Y}(f)\exp\left(j2\pi f\left(\frac{1}{2}\right)\right) \qquad \cdots\cdots\cdots (**)$$

Substituting $(*)$ into $(**)$:
$$\begin{cases} Y(f) = \dfrac{1}{4}X\left(\dfrac{f}{2}\right)\exp(-j\pi f) - \dfrac{1}{4}X\left(\dfrac{f}{2}\right)\exp(j\pi f) \\ \quad = -j\dfrac{1}{2}X\left(\dfrac{f}{2}\right)\sin(\pi f) \\ \quad = \dfrac{1}{j2}\left[\dfrac{2\cos(\pi f)}{\pi\left(0.25 - f^2\right)}\right]\sin(\pi f) \\ \quad = \dfrac{1}{j2}\left[\dfrac{\sin(2\pi f)}{\pi\left(0.25 - f^2\right)}\right] \end{cases}$$

Solution to Q.2

(a)

Fig.Q.2(a)(I) is a plot of $u(t-\gamma)$ against t:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \rightarrow \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & t\ge \gamma\\ 0; & t<\gamma\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } t \text{ while}\\ \text{treating } \gamma \text{ as a parameter}}}$$

Fig.Q.2(a)(II) is a plot of $u(t-\gamma)$ against γ:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \rightarrow \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & \gamma\le t\\ 0; & \gamma>t\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } \gamma \text{ while}\\ \text{treating } t \text{ as a parameter}}}$$

We can express the following:

$$\int_{-\infty}^{t} x(\gamma)\,d\gamma = \underbrace{\int_{-\infty}^{t} x(\gamma)u(t-\gamma)\,d\gamma = \int_{-\infty}^{\infty} x(\gamma)u(t-\gamma)\,d\gamma}_{\because\ u(t-\gamma)=0 \textit{ when } \gamma>t} = x(t)\ast u(t)$$

(b)

$$\cos(t)u(t)\ast u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)\,d\gamma=\begin{cases}\int_0^t\cos(\gamma)\,d\gamma; & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\begin{cases}\sin(t); & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\sin(t)u(t)$$

(c)

Using the forward Fourier transform equation, it is straightforward to derive the Fourier transform pair:

$$\operatorname{rect}\left(\frac{t}{\alpha}\right) \rightleftarrows \alpha\cdot\operatorname{sinc}(\alpha f) \quad \cdots\cdots\cdots \quad (\ast)$$

Applying the 'Duality' property of the Fourier transform to $(\ast)$:

$$\alpha\cdot\operatorname{sinc}(\alpha t) \rightleftarrows \operatorname{rect}\left(\frac{f}{\alpha}\right) \quad \cdots\cdots \quad (\ast\ast)$$

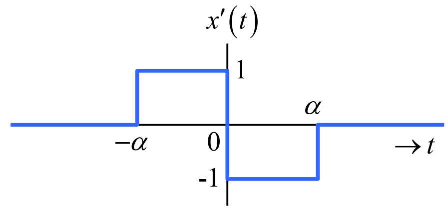
Taking the limit $\alpha\to\infty$ on both sides of $(\ast\ast)$:

$$\lim_{\alpha\to\infty}\alpha\cdot\operatorname{sinc}(\alpha t) \rightleftarrows \lim_{\alpha\to\infty}\operatorname{rect}\left(\frac{f}{\alpha}\right)=1$$

Hence, $\boldsymbol{\lim_{\alpha\to\infty}\alpha\cdot\operatorname{sinc}(\alpha t)=\Im^{-1}\{1\}=\delta(t)}$

Solution to Q.3

Spectrum of $x'(t) = \dfrac{dx(t)}{dt}$:



$$x'(t) = \frac{dx(t)}{dt} = \text{rect}\left(\frac{t+0.5\alpha}{\alpha}\right) - \text{rect}\left(\frac{t-0.5\alpha}{\alpha}\right)$$

Applying the 'Linearity' property of the Fourier transform:

$$\Im\{x'(t)\} = \Im\left\{\text{rect}\left(\frac{t+0.5\alpha}{\alpha}\right)\right\} - \Im\left\{\text{rect}\left(\frac{t-0.5\alpha}{\alpha}\right)\right\}$$

$$\Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} = \alpha \cdot \text{sinc}(\alpha f)$$

Applying the 'Time-shifting' property of the Fourier transform:

$$\begin{aligned}\Im\{x'(t)\} &= \alpha \cdot \text{sinc}(\alpha f)\left[\exp(j\pi\alpha f) - \exp(-j\pi\alpha f)\right] \\ &= \alpha \cdot \text{sinc}(\alpha f)\left(j2\sin(\pi\alpha f)\right) \\ &= j2\pi f \alpha^2 \cdot \frac{\sin(\pi\alpha f)}{\pi\alpha f} \cdot \frac{\sin(\pi\alpha f)}{\pi\alpha f} \\ &= j2\pi f \alpha^2 \text{sinc}^2(\alpha f)\end{aligned}$$

Spectrum of $x(t)$:

$$\Im\{x(t)\} = \Im\left\{\int_{-\infty}^{t} x'(\tau)\,d\tau\right\} \quad \cdots \quad \text{Noting: } \int_{-\infty}^{\infty} x'dt = 0$$

Applying the 'Integration' property of the Fourier transform:

$$\begin{aligned}\Im\{x(t)\} &= \frac{1}{j2\pi f}\Im\{x'(t)\} \\ &= \frac{1}{j2\pi f} \cdot j2\pi f \alpha^2 \text{sinc}^2(\alpha f) \\ &= \alpha^2 \text{sinc}^2(\alpha f)\end{aligned}$$

Expressing $x(t)$ as a function of $\text{rect}(\cdot)$:

$$\begin{aligned}\Im\{x(t)\} &= \alpha^2 \text{sinc}^2(\alpha f) = \alpha\text{sinc}(\alpha f) \cdot \alpha\text{sinc}(\alpha f) \\ &= \Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} \cdot \Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (*)\end{aligned}$$

Applying the 'Convolution' property of the Fourier transform:

$$\Im\left\{\text{rect}\left(\frac{t}{\alpha}\right) \ast \text{rect}\left(\frac{t}{\alpha}\right)\right\} = \Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} \cdot \Im\left\{\text{rect}\left(\frac{t}{\alpha}\right)\right\} \quad \cdots\cdots \quad (**)$$

Comparing $(*)$ and $(**)$, we have $x(t) = \text{rect}\left(\dfrac{t}{\alpha}\right) \ast \text{rect}\left(\dfrac{t}{\alpha}\right)$

Solution to Q.4

Given: $X(f) = \exp(-\alpha|f|);\quad \alpha > 0$

(a) ***Energy Spectral Density of*** $\boldsymbol{x(t)}$***:***

$$E_x(f) = |X(f)|^2 = \exp(-2\alpha|f|)$$

Energy of $\boldsymbol{x(t)}$ ***contained within a bandwidth of*** B***:***

$$E_B = \int_{-B}^{B} E_x(f)\,df = 2\int_0^B \exp(-2\alpha f)\,df = 2\left[\frac{\exp(-2\alpha f)}{-2\alpha}\right]_0^B = \frac{1}{\alpha}\left[1-\exp(-2\alpha B)\right]$$

Total energy of $\boldsymbol{x(t)}$***:***

$$E = \underbrace{\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df}_{\text{Rayleigh Energy Theorem}} = \int_{-\infty}^{\infty} E_x(f)\,df = E_B\big|_{B=\infty} = \frac{1}{\alpha}$$

99% energy containment bandwidth, W, of $\boldsymbol{x(t)}$***:***

$$\left[\underbrace{\frac{1}{\alpha}\left[1-\exp(-2\alpha W)\right]}_{E_B|_{B=W}} = 0.99E = \frac{0.99}{\alpha}\right] \rightarrow \exp(2\alpha W) = 100$$

$$\rightarrow W = \frac{1}{\alpha}\ln(10) \text{ Hz}$$

(b) ***3dB bandwidth,*** $\boldsymbol{B_{3dB}}$***, of*** $\boldsymbol{x(t)}$***:***

By definition, $|X(B_{3dB})| = \dfrac{|X(0)|}{\sqrt{2}}$.

$$\text{Solving: } \left\{\begin{aligned} &|X(f)| = \exp(-\alpha|f|) \\ &\left.\begin{aligned} &|X(B_{3dB})| = \exp(-\alpha B_{3dB}) \\ &|X(0)| = 1 \end{aligned}\right\} \rightarrow \exp(-\alpha B_{3dB}) = \frac{1}{\sqrt{2}} \\ &\qquad\qquad\qquad\qquad\quad \rightarrow B_{3dB} = \frac{1}{2\alpha}\ln(2) \text{ Hz} \end{aligned}\right.$$

Percent energy contained within the 3dB bandwidth:

$$\left.\frac{E_B}{E}\times 100\right|_{B=B_{3dB}} = \frac{\cancel{\tfrac{1}{\alpha}}\left[1-\exp\left(-\cancel{2\alpha}\dfrac{\ln(2)}{\cancel{2\alpha}}\right)\right]}{\cancel{\tfrac{1}{\alpha}}}\times 100 = \left[1-\exp(-\ln(2))\right]\times 100 = \left[1-\exp(\ln(1/2))\right]\times 100 = 50\%$$

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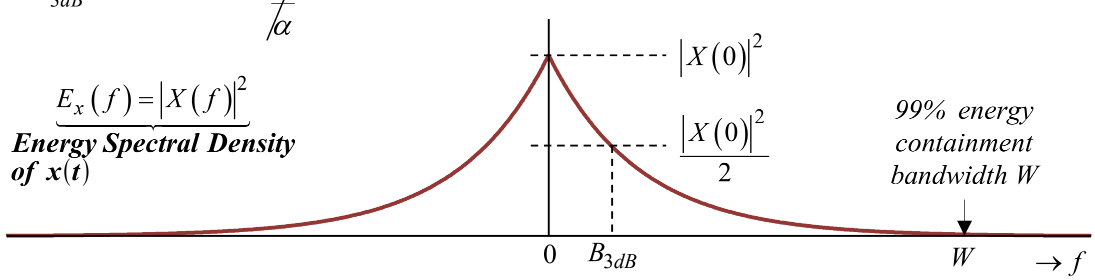
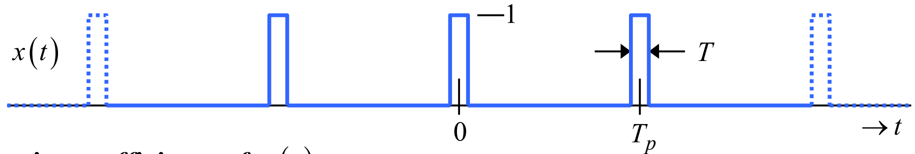


Solution to Q.5



(a) ***Fourier series coefficients of $x(t)$:***

$$X_k = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x(t)\exp\left(-j2\pi kt/T_p\right)dt = \frac{1}{T_p}\int_{-0.5T}^{0.5T}\exp\left(-j2\pi kt/T_p\right)dt$$

$$= \frac{1}{T_p}\left[\frac{\exp\left(-j2\pi kt/T_p\right)}{-j2\pi k/T_p}\right]_{-0.5T}^{0.5T}$$

$$= \frac{1}{T_p}\left[\frac{\exp(-j\pi kT/T_p)}{-j2\pi k/T_p} - \frac{\exp(j\pi kT/T_p)}{-j2\pi k/T_p}\right]$$

$$= \frac{1}{T_p}\left[\frac{\exp(j\pi kT/T_p)}{j2\pi k/T_p} - \frac{\exp(-j\pi kT/T_p)}{j2\pi k/T_p}\right]$$

$$= \frac{1}{T_p}\frac{1}{\pi k/T_p}\frac{1}{2j}\left[\exp(j\pi kT/T_p) - \exp(-j\pi kT/T_p)\right]$$

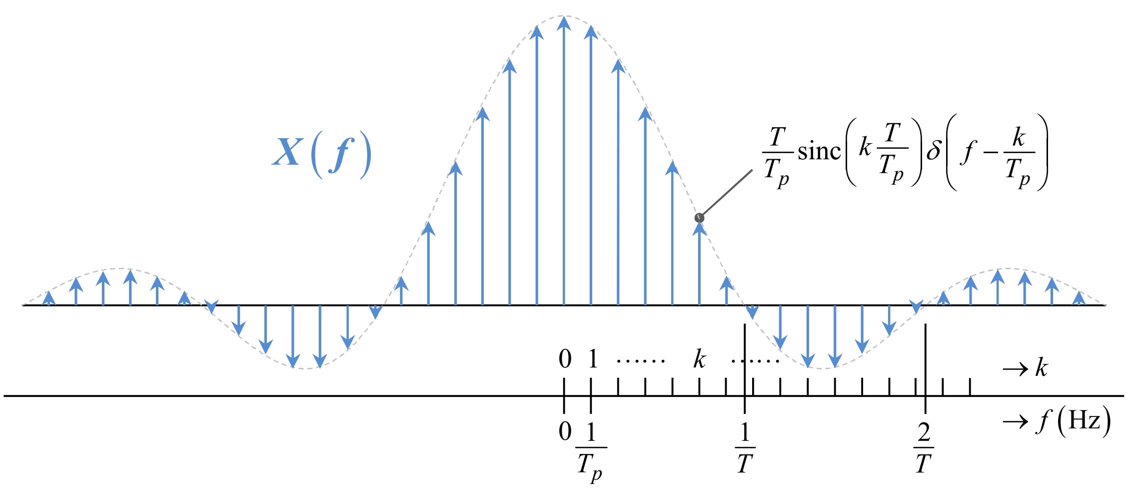
$$= \frac{1}{T_p}\frac{1}{\pi k/T_p}\sin\left(\pi kT/T_p\right)$$

$$= \frac{T}{T_p}\left[\frac{\sin\left(\pi kT/T_p\right)}{\pi kT/T_p}\right]$$

$$= \frac{T}{T_p}\operatorname{sinc}\left(k\frac{T}{T_p}\right)$$

Frequency spectrum (or Fourier transform) of $x(t)$:

$$X(f) = \sum_{k=-\infty}^{\infty} X_k\delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty}\frac{T}{T_p}\operatorname{sinc}\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$



(b) ***Power Spectral Density of $x(t)$:***

$$P_x(f) = \sum_{k=-\infty}^{\infty} |X_k|^2 \delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T^2}{T_p^2} \mathrm{sinc}^2\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$

Average power of $x(t)$:

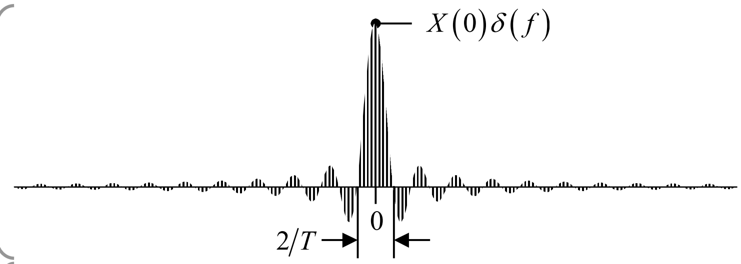
$$P = \underbrace{\int_{-\infty}^{\infty} P_x(f)\,df = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} |x(t)|^2\,dt}_{\text{Parseval Power Theorem}} = \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = \frac{1}{T_p}[t]_{-0.5T}^{0.5T} = \frac{T}{T_p}$$

99% power containment bandwidth, W, of $x(t)$:

$$W = \frac{K}{T_p}(\mathrm{Hz}) \quad \cdots \quad \left(\begin{array}{l} \text{where } K \text{ satisfies } \displaystyle\sum_{k=-K}^{K} |X_k|^2 \geq 0.99P > \sum_{k=-(K-1)}^{(K-1)} |X_k|^2 \\ \text{in which } \displaystyle |X_k|^2 = \frac{T^2}{T_p^2}\mathrm{sinc}^2\left(k\frac{T}{T_p}\right) \text{ and } P = \frac{T}{T_p}. \end{array}\right).$$

(c)

$$X(f)=\sum_{k=-\infty}^{\infty} X_k \delta\left(f-k/T_p\right)$$

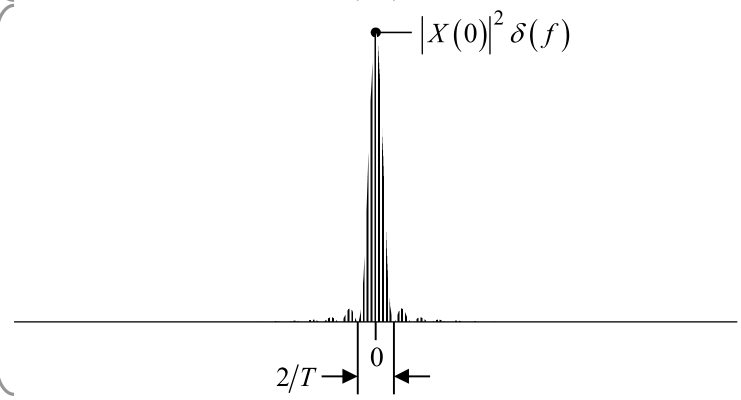


Power Spectral Density of $x(t)$

$$P_x(f)=\sum_{k=-\infty}^{\infty} \left|X_k\right|^2 \delta\left(f-k/T_p\right)$$

$$\text{Average Power } =\int_{-\infty}^{\infty} P_x(f)\,df$$

$$\left(\textit{from Part } (b)\right)=\frac{T}{T_p}$$

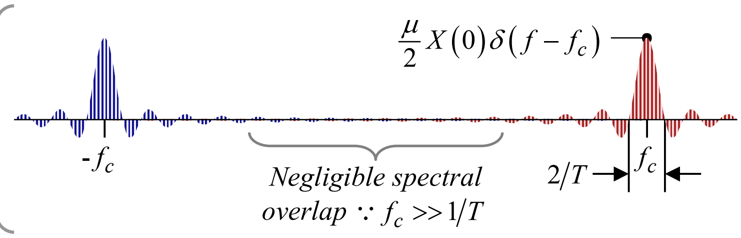


$$\Im\left\{\mu\cos\left(2\pi f_c t\right)\right\}=\frac{\mu}{2}\begin{bmatrix}\delta\left(t+f_c\right)+\\ \delta\left(t-f_c\right)\end{bmatrix}$$

$$Y(f)=X(f)\ast\Im\left\{\mu\cos\left(2\pi f_c t\right)\right\}$$

$$=\left[\frac{\mu}{2}X\left(f+f_c\right)+\frac{\mu}{2}X\left(f-f_c\right)\right]$$

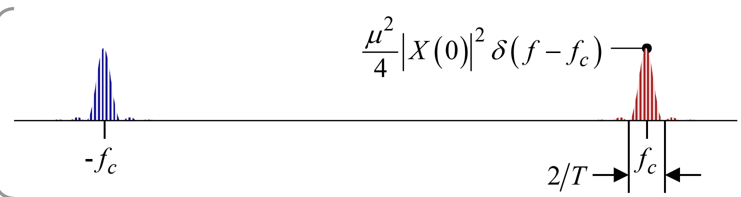
$$=\sum_{k=-\infty}^{\infty}\frac{\mu}{2}X_k\begin{bmatrix}\delta\left(f+f_c-\dfrac{k}{T_p}\right)+\\ \delta\left(f-f_c-\dfrac{k}{T_p}\right)\end{bmatrix}$$



Power Spectral Density of $y(t)$

Owing to negligible overlap between the +ve and −ve frequency spectral lobes, the PSD of $y(t)$ is given by :

$$P_y(f)=\sum_{k=-\infty}^{\infty}\frac{\mu^2}{4}\left|X_k\right|^2\begin{bmatrix}\delta\left(f+f_c-\dfrac{k}{T_p}\right)+\\ \delta\left(f-f_c-\dfrac{k}{T_p}\right)\end{bmatrix}$$



$$\text{Average power of } y(t)=\int_{-\infty}^{\infty}P_y(f)\,df=2\int_{-\infty}^{\infty}\frac{\mu^2}{4}P_x(f)\,df=\frac{\mu^2}{2}\int_{-\infty}^{\infty}P_x(f)\,df$$

$$=0.5\mu^2\times\text{Average power of } x(t)=0.5\mu^2\,T/T_p$$

$$\left(\begin{matrix}\textit{Assuming that } \mu \textit{ cannot be changed, the laser pointer output power can}\\ \textit{only be controlled by changing the duty cycle } T/T_p \textit{ of the control signal}\end{matrix}\right)\left(\begin{matrix}\textit{Allows estimation}\\ \textit{of the battery life}\end{matrix}\right)$$

$$\text{99\% power containment bandwidth of } y(t)=\underbrace{2\times\left(\text{99\% power containment bandwidth of } x(t)\right)}_{\textit{Due to bandwidth expansion by a factor of 2}}$$

Supplementary Questions (Solutions)

S1(a) $x(t) = \cos\left(2\pi f_c t\right)u(t)$

$$X(f) = \frac{1}{2}\left[\delta\left(f-f_c\right)+\delta\left(f+f_c\right)\right]\otimes\left[\frac{1}{2}\left\{\delta(f)+\frac{1}{j\pi f}\right\}\right]$$

$$= \frac{1}{4}\left[\delta(f-f_c)+\frac{1}{j\pi(f-f_c)}\right]+\frac{1}{4}\left[\delta(f+f_c)+\frac{1}{j\pi(f+f_c)}\right]$$

$$= \frac{1}{4}\left[\delta(f-f_c)+\delta(f+f_c)\right]+\frac{1}{4}\left[\frac{1}{j\pi(f-f_c)}+\frac{1}{j\pi(f+f_c)}\right]$$

$$= \frac{1}{4}\left[\delta(f-f_c)+\delta(f+f_c)\right]+\frac{1}{4}\left[\frac{j\pi(f+f_c)+j\pi(f-f_c)}{-\pi^2\left(f^2-f_c^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f-f_c)+\delta(f+f_c)\right]+\frac{1}{4}\left[\frac{2j\pi f}{\pi^2\left(f_c^2-f^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f-f_c)+\delta(f+f_c)\right]+\left[\frac{jf}{2\pi\left(f_c^2-f^2\right)}\right]$$

S1(b) $x(t) = \sin\left(2\pi f_c t\right)u(t)$

$$X(f) = \frac{1}{2j}\left[\delta\left(f-f_c\right)-\delta\left(f+f_c\right)\right]\otimes\left[\frac{1}{2}\left\{\delta(f)+\frac{1}{j\pi f}\right\}\right]$$

$$= \frac{1}{4j}\left[\delta(f-f_c)+\frac{1}{j\pi(f-f_c)}\right]-\frac{1}{4}\left[\delta(f+f_c)+\frac{1}{j\pi(f+f_c)}\right]$$

$$= \frac{1}{4j}\left[\delta(f-f_c)-\delta(f+f_c)\right]+\frac{1}{4j}\left[\frac{1}{j\pi(f-f_c)}-\frac{1}{j\pi(f+f_c)}\right]$$

$$= \frac{1}{4j}\left[\delta(f-f_c)-\delta(f+f_c)\right]+\frac{1}{4j}\left[\frac{j\pi(f+f_c)-j\pi(f-f_c)}{-\pi^2\left(f^2-f_c^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f-f_c)-\delta(f+f_c)\right]+\frac{1}{4}\left[\frac{2j\pi f_c}{\pi^2\left(f_c^2-f^2\right)}\right]$$

$$= \frac{1}{4}\left[\delta(f-f_c)-\delta(f+f_c)\right]+\left[\frac{jf}{2\pi\left(f_c^2-f^2\right)}\right]$$

S1(c) $s(t) = e^{-\alpha t}\cos\left(\omega_c t\right)u(t)$

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \\
&= \int_{-\infty}^{\infty} e^{-\alpha t}\cos\left(\omega_c t\right)u(t)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\cos\left(\omega_c t\right)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\left\{\frac{1}{2}\left[e^{f\omega_c t} + e^{-f\omega_c t}\right]\right\}e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_{0}^{\infty}\left[e^{(-\alpha + j\omega_c - j\omega)t} + e^{(-\alpha - j\omega_c - j\omega)t}\right]dt \\
&= \frac{1}{2}\int_{0}^{\infty}\left[e^{-(\alpha - j\omega_c + j\omega)t} + e^{-(\alpha + j\omega_c + j\omega)t}\right]dt \\
&= \frac{1}{2}\left[\frac{e^{-(\alpha - j\omega_c + j\omega)t}}{-\left(\alpha - j\omega_c + j\omega\right)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-(\alpha + j\omega_c + j\omega)t}}{-\left(\alpha + j\omega_c + j\omega\right)}\right]_0^{\infty} \\
&= \frac{1}{2}\frac{1}{\alpha - j\omega_c + j\omega} + \frac{1}{2}\frac{1}{\alpha + j\omega_c + j\omega} \\
&= \frac{1}{2}\frac{\alpha + j\omega_c + j\omega + \alpha - j\omega_c + j\omega}{\left(\alpha + j\omega\right)^2 + \omega_c^2} \\
&= \frac{\alpha + j\omega}{\left(\alpha + j\omega\right)^2 + \omega_c^2}
\end{aligned}
$$

S1(d) $s(t) = e^{-\alpha t}\sin\left(\omega_c t\right)u(t)$

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \\
&= \int_{-\infty}^{\infty} e^{-\alpha t}\sin\left(\omega_c t\right)u(t)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\sin\left(\omega_c t\right)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-\alpha t}\left\{\frac{1}{2j}\left[e^{f\omega_c t} - e^{-f\omega_c t}\right]\right\}e^{-j\omega t}\,dt \\
&= \frac{1}{2j}\int_{0}^{\infty}\left[e^{(-\alpha + j\omega_c - j\omega)t} - e^{(-\alpha - j\omega_c - j\omega)t}\right]dt \\
&= \frac{1}{2j}\int_{0}^{\infty}\left[e^{-(\alpha - j\omega_c + j\omega)t} - e^{-(\alpha + j\omega_c + j\omega)t}\right]dt \\
&= \frac{1}{2j}\left[\frac{e^{-(\alpha - j\omega_c + j\omega)t}}{-\left(\alpha - j\omega_c + j\omega\right)}\right]_0^{\infty} - \frac{1}{2j}\left[\frac{e^{-(\alpha + j\omega_c + j\omega)t}}{-\left(\alpha + j\omega_c + j\omega\right)}\right]_0^{\infty} \\
&= \frac{1}{2j}\frac{1}{\alpha - j\omega_c + j\omega} - \frac{1}{2j}\frac{1}{\alpha + j\omega_c + j\omega} \\
&= \frac{1}{2j}\frac{\alpha + j\omega_c + j\omega - \alpha + j\omega_c - j\omega}{\left(\alpha + j\omega\right)^2 + \omega_c^2} \\
&= \frac{\omega_c}{\left(\alpha + j\omega\right)^2 + \omega_c^2}
\end{aligned}
$$

S2 $e^{-\alpha t}u(t) \Leftrightarrow \dfrac{1}{\alpha + j2\pi f} = \dfrac{1}{\alpha + j\omega}$

Given: $tx(t) \Leftrightarrow j\dfrac{d}{d\omega}X\left(j\omega\right)$

Let: $x(t) \Leftrightarrow \dfrac{1}{\alpha + j\omega}$

$$tx(t) \Leftrightarrow j\frac{d}{d\omega}\left[\frac{1}{\alpha + j\omega}\right] = j.j.(-1).\frac{1}{\left(\alpha + j\omega\right)^2} = \frac{1}{\left(\alpha + j\omega\right)^2}$$

$$t^2x(t) \Leftrightarrow j\frac{d}{d\omega}\left[\frac{(1)}{\left(\alpha + j\omega\right)^2}\right] = j.j.\frac{(1)(-2)}{\left(\alpha + j\omega\right)^3} = \frac{(1)(2)}{\left(\alpha + j\omega\right)^3}$$

$$t^3x(t) \Leftrightarrow j\frac{d}{d\omega}\left[\frac{(1)(2)}{\left(\alpha + j\omega\right)^3}\right] = j.j\frac{(1)(2)(-3)}{\left(\alpha + j\omega\right)^4} = \frac{(1)(2)(3)}{\left(\alpha + j\omega\right)^4}$$

In general, we have: $t^{n-1}x(t) \Leftrightarrow \dfrac{\left(n-1\right)!}{\left(\alpha + j\omega\right)^n}$, hence : $\dfrac{t^{n-1}}{\left(n-1\right)!} \Leftrightarrow \dfrac{1}{\left(\alpha + j\omega\right)^n}$

S3 $e^{-\alpha t}u(t) \Leftrightarrow \dfrac{1}{\alpha + j\omega}$

$$\frac{1}{2-\omega^2 + j2\omega} = \frac{1}{\left(2 + j\omega\right)\left(1 + j\omega\right)}$$

Let: $\dfrac{1}{\left(1 + j\omega\right)\left(2 + j\omega\right)} = \dfrac{A}{2 + j\omega} + \dfrac{B}{1 + j\omega} = \dfrac{A(2 + j\omega) + B(1 + j\omega)}{\left(1 + j\omega\right)\left(2 + j\omega\right)}$

Hence for the numerators, we have: $A(1 + j\omega) + B(2 + j\omega) = 1$

Comparing constants: $2A + B = 1$

Comparing j terms: $A + B = 0 \;\rightarrow\; A = -B$

Substituting $A = -B$ into the constants equations, we have: $2A + (-A) = 1 \;\rightarrow\; A = 1$ & $B = -1$

Hence:

$$\frac{1}{\left(1 + j\omega\right)\left(2 + j\omega\right)} = \frac{1}{1 + j\omega} - \frac{1}{2 + j\omega}$$

Taking the inverse Fourier transform:

$$\frac{1}{1 + j\omega} - \frac{1}{2 + j\omega} \Leftrightarrow e^{-t}u(t) - e^{-2t}u(t)$$

S4 $x(t) \Leftrightarrow rect(\pi f)$; $y(t) = \dfrac{d}{dt}x(t)$

Fourier transform of $y(t)$ is: $Y(f) = j2\pi f.X(f) = j2\pi f\,\mathrm{rect}(\pi f)$

Energy of $y(t)$: $E_y = \int_{-\infty}^{\infty}\left|Y(f)\right|^2 df = \int_{-\infty}^{\infty}\left|j2\pi f\,\mathrm{rect}(\pi f)\right|^2 df = \int_{-1/2\pi}^{1/2\pi} 4\pi^2 f^2 df = \dfrac{4\pi^2}{3}\left[f^3\right]_{-1/2\pi}^{1/2\pi} = \dfrac{1}{3\pi}$

S5 Given: $\frac{\pi}{\alpha} e^{-2\pi\alpha|t|} \Leftrightarrow \frac{1}{\alpha^2 + f^2}$

Using duality: $\frac{1}{\alpha^2 + t^2} \Leftrightarrow \frac{\pi}{\alpha} e^{-2\pi\alpha|f|}$

The total energy is:

$$E = \int_{-\infty}^{\infty} \left| \frac{\pi}{\alpha} e^{-2\pi\alpha|f|} \right|^2 df = \frac{2\pi}{\alpha} \int_0^{\infty} e^{-4\pi\alpha f} df = \frac{2\pi}{\alpha} \left[\frac{e^{-4\pi\alpha f}}{-4\pi\alpha} \right]_0^{\infty} = \frac{2\pi}{\alpha} \left[\frac{1}{4\pi\alpha} \right] = \frac{1}{2\alpha^2}$$

The energy up to a bandwidth of B Hz is:

$$E_B = \int_{-B}^{B} \left| \frac{\pi}{\alpha} e^{-2\pi\alpha|f|} \right|^2 df = \frac{2\pi}{\alpha} \int_0^{B} e^{-4\pi\alpha f} df = \frac{2\pi}{\alpha} \left[\frac{e^{-4\pi\alpha f}}{-4\pi\alpha} \right]_0^{B} = \frac{2\pi}{\alpha} \left[\frac{1}{4\pi\alpha} - \frac{e^{-4\pi\alpha B}}{4\pi\alpha} \right] = \frac{1}{2\alpha^2}\left[1 - e^{-4\pi\alpha B}\right]$$

For the 99% energy containment bandwidth, we have $E_B = 0.99E$:

$$\frac{1}{2\alpha^2}\left[1 - e^{-4\pi\alpha B}\right] = 0.99\left[\frac{1}{2\alpha^2}\right]$$

$$1 - e^{-4\pi\alpha B} = 0.99$$

$$e^{-4\pi\alpha B} = 0.01$$

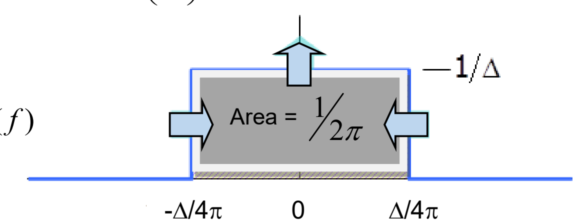
$$e^{4\pi\alpha B} = 100$$

$$B = \frac{1}{4\pi\alpha}\ln(100) = \frac{0.366}{\alpha}$$

S6 Suppose the Dirac-δ function being defined as: $d(f) = \lim_{f\to 0} \frac{1}{\Delta} rect\left(\frac{f}{\Delta}\right)$

Then for $\delta(\omega f) = \delta(2\pi f)$, we can define:

$$\delta(2\pi f) = \lim_{f\to 0} \frac{1}{\Delta} rect\left(\frac{f}{\Delta/2\pi}\right) = \frac{1}{2\pi} \lim_{f\to 0} \frac{1}{\Delta} rect\left(\frac{f}{\Delta}\right) = \frac{1}{2\pi}\delta(f)$$



Hence: $\delta(f) = 2\pi\delta(2\pi f) = 2\pi\delta(\omega)$
