Outline of Lecture

1. Discrete-Frequency Spectrum (Fourier Series)
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3. Spectrum of a Sinusoid
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   - Spectrum of a Cosine Signal
   - Spectrum of a Sine Signal
4. Fourier Series
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   - Trigonometric Fourier Series
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Discrete-Frequency Spectrum (Fourier Series)

1. What is the spectrum of a signal?

    - The frequency domain representation of a signal is called the spectrum of the signal.
    - The frequency domain refers to the analysis of signals with respect to frequency, rather than time. Put simply, a time-domain graph shows how a signal changes over time, whereas a frequency-domain graph or spectrum shows the frequency components in the signal.
    - Frequency spectrum of a signal is the range of frequencies contained in the signal.
    - The graphical representation of a spectrum consists of two plots : *magnitude spectrum* and the *phase spectrum*.

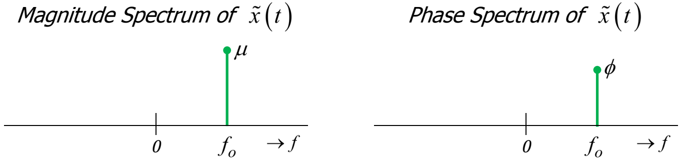
2. Spectrum of a Sinusoid

2.1 Spectrum of a complex exponential signal

$$\tilde{x}(t) = \underbrace{\mu e^{j(2\pi f_0 t + \phi)}}_{\text{complex exponential}} = \underbrace{\mu e^{j\phi}}_{\text{spectrum}} \; e^{j2\pi f_0 t}$$

Magnitude spectrum : μ
Phase spectrum : ϕ
Frequency : f_0

Spectral Plot



Exercise 1

Try plotting the magnitude and phase spectrum of $\tilde{x}^*(t)$ *here.*

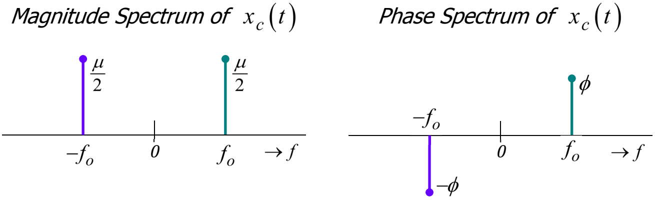
2.2 Spectrum of a Cosine signal

$$x_c(t) = \overbrace{\underbrace{\mu\cos(2\pi f_0 t + \phi)}_{cosine} = \frac{1}{2}\underbrace{\mu e^{j(2\pi f_0 t+\phi)}}_{\tilde{x}(t)} + \frac{1}{2}\underbrace{\mu e^{-j(2\pi f_0 t+\phi)}}_{\tilde{x}^*(t)}}^{applying\ Euler's\ formula}$$

$$= \frac{\mu}{2}e^{j\phi}e^{j2\pi f_0 t} + \frac{\mu}{2}e^{-j\phi}e^{-j2\pi f_0 t}$$

We can see from the above that $x_c(t)$ has 2 frequency components, f_0 and $-f_0$.

Spectral Plot



Next let's see what spectrum we get for a sine signal.

2.3 Spectrum of a Sine signal

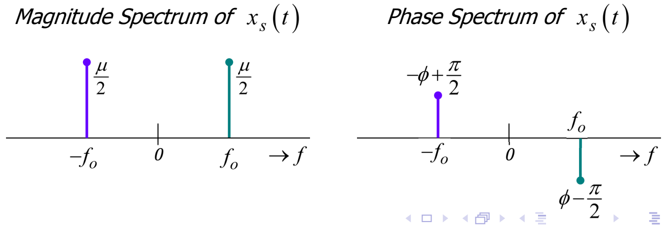
$$x_s(t) = \overbrace{\underbrace{\mu\sin(2\pi f_0 t + \phi)}_{sine} = \frac{1}{2j}\underbrace{\mu e^{j(2\pi f_0 t + \phi)}}_{\tilde{x}(t)} - \frac{1}{2j}\underbrace{\mu e^{-j(2\pi f_0 t + \phi)}}_{\tilde{x}^*(t)}}^{applying\ Euler's\ formula}$$

$$\ldots \text{with } j = e^{j\frac{\pi}{2}} \text{ and } -j = e^{-j\frac{\pi}{2}}$$

$$= \frac{\mu}{2}e^{j(\phi - \frac{\pi}{2})}e^{j2\pi f_0 t} + \frac{\mu}{2}e^{j(-\phi + \frac{\pi}{2})}e^{-j2\pi f_0 t}$$

This shows that $x_s(t)$ also has 2 frequency components with same magnitude spectrum as the cosine signal $x_c(t)$ but the phase spectrum is different.

Spectral Plot



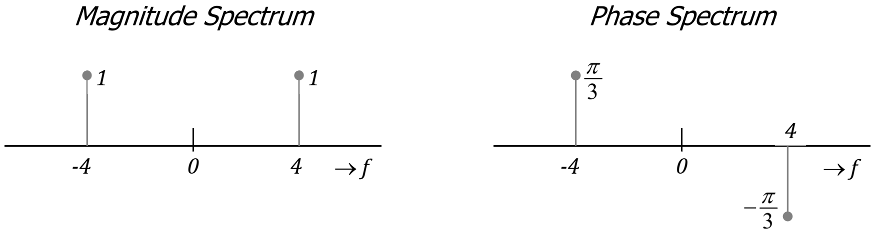
Example 1

Sketch the magnitude and phase spectra of $x(t) = 2\sin\left(8\pi t + \frac{\pi}{6}\right)$.

$$\left.\begin{array}{rcl} e^{j\theta} & = & \cos\theta + j\sin\theta \\ e^{-j\theta} & = & \cos\theta - j\sin\theta \end{array}\right\} \rightarrow \left\{\begin{array}{rcl} \cos\theta & = & \frac{1}{2}[e^{j\theta} + e^{-j\theta}] \\ \sin\theta & = & \frac{1}{2j}[e^{j\theta} - e^{-j\theta}] \end{array}\right.$$

Expressing $x(t)$ *in terms of complex exponentials :*

$$\begin{aligned} x(t) &= 2\sin\left(8\pi t + \frac{\pi}{6}\right) = 2.\frac{1}{2j}\left[e^{j(2\pi(4)t+\frac{\pi}{6})} - e^{-j(2\pi(4)t+\frac{\pi}{6})}\right] \\ &= e^{-j\frac{\pi}{2}} e^{j\frac{\pi}{6}} e^{j2\pi(4)t} + e^{j\frac{\pi}{2}} e^{-j\frac{\pi}{6}} e^{j2\pi(-4)t} \\ &= e^{-j\frac{\pi}{3}} e^{j2\pi(4)t} + e^{j\frac{\pi}{3}} e^{j2\pi(-4)t} \end{aligned}$$



Fourier Series

3. Unlike sinusoids, the spectra of non-sinusoidal periodic signals such as square wave, sawtooth wave, etc., cannot be determined simply by inspection. The spectra of such signals are derived using a mathematical tool called Fourier series, which is an expansion of a periodic function into a sum of complex exponentials.

3.1 Complex Exponential Fourier Series
Any bounded periodic signal $x_p(t)$ of period T_p can be represented by a sum of complex sinusoids.

$$x_p(t) = \underbrace{\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k\left(\frac{1}{T_p}\right)t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}}_{\text{Fourier series expansion}}$$

where f_p is the fundamental frequency and kf_p is the k^{th} harmonic of f_p.

c_k's are called the Fourier series coefficients of $x_p(t)$, and they constitute the discrete-frequency spectrum of $x_p(t)$.

3.2 Given $x_p(t)$, how do we determine the k^{th} Fourier series coeff., c_k?
To determine, c_k, multiply $x_p(t)$ by $e^{-j2\pi k f_p t}$ and integrate over one period :

$$\begin{aligned}\int_{-0.5T_p}^{0.5T_p} x_p(t)e^{-j2\pi k f_p t}dt &= \int_{-0.5T_p}^{0.5T_p} e^{-j2\pi k f_p t} \sum_{m=-\infty}^{\infty} c_m e^{j2\pi m f_p t}dt \\ &= \sum_{m=-\infty}^{\infty} c_m \int_{-0.5T_p}^{0.5T_p} e^{-j2\pi(k-m)f_p t}dt \\ &= c_k T_p \quad \text{...try to show this yourself}\end{aligned}$$

This gives the formula for c_k:

$$c_k = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x_p(t)e^{-j2\pi k f_p t}dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

special case for $k = 0$:

$$c_0 = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x_p(t)dt = \text{average value of } x_p(t)$$

3.3 Trigonometric Fourier Series - alternative form of Fourier Series

The exponential form of the Fourier series can also be expressed in terms of cosine and sine functions as follows :

$$
\begin{aligned}
x_p(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} = \sum_{k=-\infty}^{-1} c_k e^{j2\pi k f_p t} + c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_p t} \\
&= c_0 + \sum_{k=1}^{\infty} c_{-k} e^{-j2\pi k f_p t} + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_p t} \\
&= c_0 + \sum_{k=1}^{\infty} [c_{-k}\cos(2\pi k f_p t) - jc_{-k}\sin(2\pi k f_p t) + \\
&\qquad c_k \cos(2\pi k f_p t) + jc_k \sin(2\pi k f_p t)] \\
&= c_0 + \sum_{k=1}^{\infty} [(c_k + c_{-k})\cos(2\pi k f_p t) + j(c_k - c_{-k})\sin(2\pi k f_p t)]
\end{aligned}
$$

If $c_k = a_k - jb_k$ and $c_{-k} = a_k + jb_k$, then we get the following form :

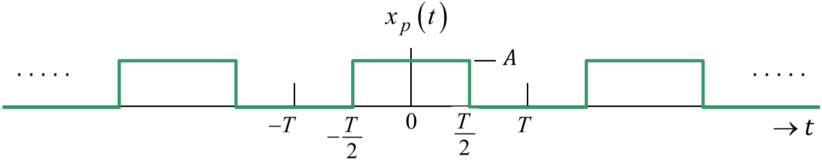
$$
x_p(t) = a_0 + 2\sum_{k=1}^{\infty} [a_k \cos(2\pi k f_p t) + b_k \sin(2\pi k f_p t)], \quad a_0 = c_0
$$

3.4 Summary and Examples

Complex Exponential Fourier Series	Analysis (Fourier Series Coefficients) $c_k = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x_p(t)e^{-j2\pi k f_p t}dt, \forall k$ Synthesis (Fourier Series Expansion) $x_p(t) = \sum_{-\infty}^{\infty} c_k e^{j2\pi k f_p t}$
Trigonometric Fourier Series	Analysis (Fourier Series Coefficients) $a_k = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x_p(t)\cos(2\pi k f_p t)dt,\ k \geq 0$ $b_k = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x_p(t)\sin(2\pi k f_p t)dt,\ k > 0$ Synthesis (Fourier Series Expansion) $x_p(t) = a_0 + 2\sum_{k=1}^{\infty} [a_k\cos(2\pi k f_p t) + b_k\sin(2\pi k f_p t)]$

Example 2

Find and sketch the discrete-freq. spectrum (c_k) of a square wave, $x_p(t)$.



Period = $T_p = 2T$, Pulse Width = T

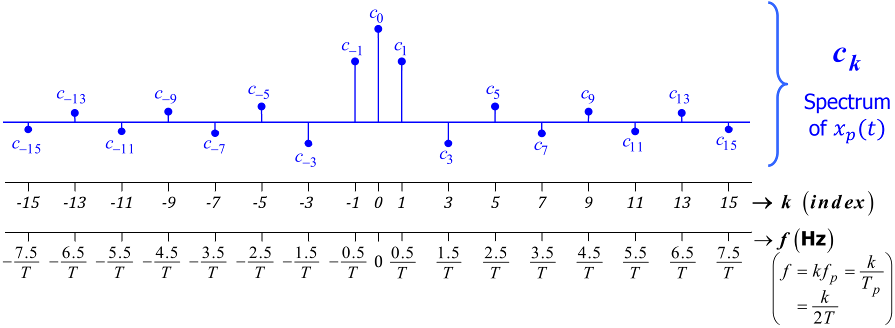
$$
\begin{aligned}
c_k &= \frac{1}{2T}\int_{-T}^{T} x_p(t)e^{-j2\pi k\frac{1}{2T}t}dt = \frac{A}{2T}\int_{-0.5T}^{0.5T} e^{-j\pi k\frac{1}{T}t}dt \\
&= \frac{A}{2T}\left[\frac{e^{-jk\pi\frac{1}{T}t}}{-j\pi k\frac{1}{T}}\right]_{-0.5T}^{0.5T} = \frac{A}{2}\left[\frac{\sin 0.5\pi k}{0.5\pi k}\right] = \frac{A}{2}\text{sinc}\left(\frac{k}{2}\right)
\end{aligned}
$$

Evaluating c_k, we get :

$c_{-\infty}, \ldots, c_{-3} = c_3 = -\frac{A}{3\pi}, c_{-2} = c_2 = 0, c_{-1} = c_1 = \frac{A}{\pi}, c_0 = \frac{A}{2}, \ldots, c_\infty$

In fact, for all k, $c_{-k} = c_k$ and for even $k = \pm 2, \pm 4, \pm 6, \ldots, c_k = 0$.

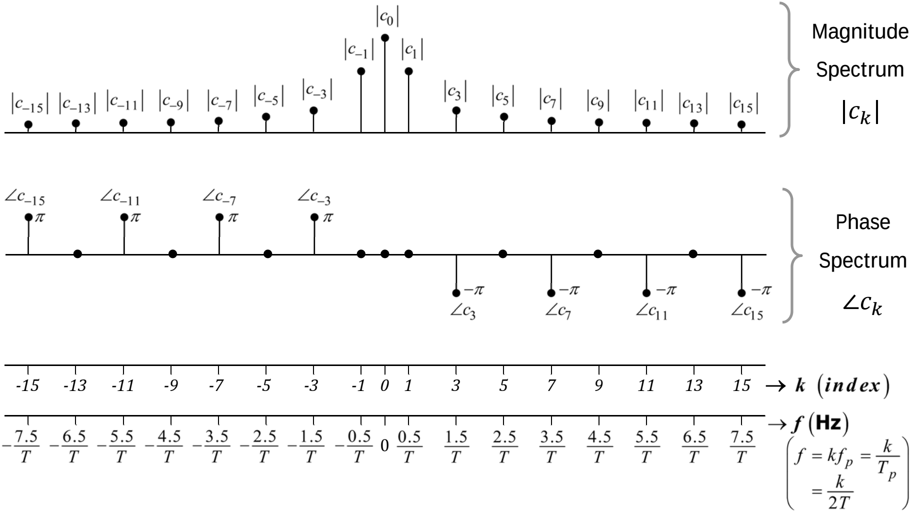
Since c_k is real, we can plot the spectrum in one single plot :



We may also write c_k in terms of its magnitude and phase :

$$c_k = \frac{A}{2}\text{sinc}\left(\frac{k}{2}\right) \text{ where } \begin{cases} |c_k| = \frac{A}{2}\left|\text{sinc}\left(\frac{k}{2}\right)\right| \\ \\ \angle c_k = \begin{cases} 0 & \text{if } c_k \geq 0 \\ \pm\pi & \text{if } c_k < 0 \end{cases} \end{cases}$$

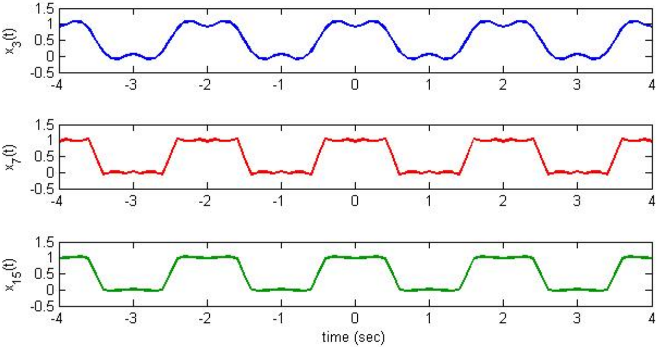
c_k can be visualized using the magnitude and phase spectra as follows :



We have seen that the square wave from Example 2 can be written in terms of the Fourier Series as follows :

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\pi \frac{k}{T} t}, \text{ where } c_k = \frac{A}{2} \text{sinc}\left(\frac{k}{2}\right)$$

How can we reconstruct $x_p(t)$? Notice that the summation in $x_p(t)$ is from $k = -\infty$ to ∞. What happens if we sum up to only a finite k? The following graphs demonstrate this.

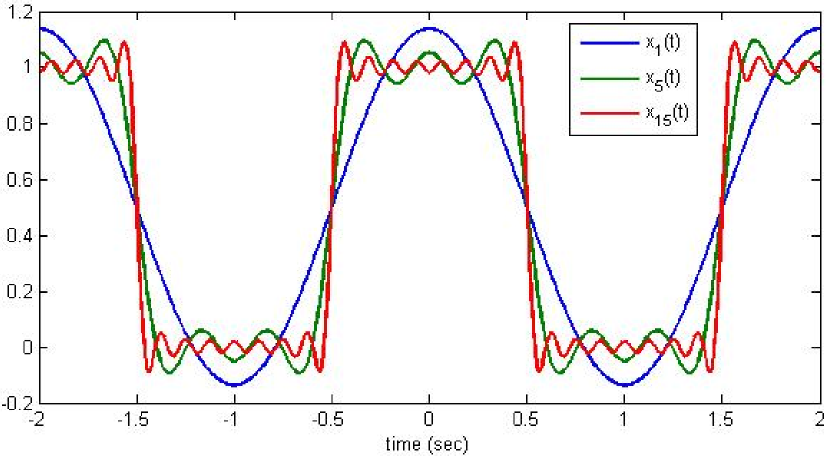


$$x_m(t) = \sum_{k=-m}^{m} c_k e^{j\pi \frac{k}{T} t}$$

summing up to the c_m term

Do you see a perfectly square wave like $x_p(t)$? Why?

A zoomed in view :



Exercise 2 (You may write your answer below)

Why do the reconstructed signals look like the above?

Example 3

$$x(t) = (1+j)e^{-j6t} + j3e^{-j4t} + 4 - j3e^{j4t} + (1-j)e^{j6t}$$

Show whether $x(t)$ is real and periodic. If $x(t)$ is periodic, find its complex exponential Fourier series coefficients, c_k, and sketch its magnitude and phase spectra.

Answer $x(t)$ is real. Reason : $x(t)$ is composed purely of complex sinusoids that come in conjugate pairs. This allows $x(t)$ to be re-written as :

$$\begin{aligned}
x(t) &= (1+j)e^{-j6t} + j3e^{-j4t} + 4 - j3e^{j4t} + (1-j)e^{j6t} \\
&= \sqrt{2}e^{j\frac{1}{4}\pi}e^{-j6t} + 3e^{j\frac{1}{2}\pi}e^{-j4t} + 4 + 3e^{-j\frac{1}{2}\pi}e^{j4t} + \sqrt{2}e^{-j\frac{1}{4}\pi}e^{j6t} \\
&= 4 + 3\left[e^{-j(4t-\frac{1}{2}\pi)} + e^{j(4t-\frac{1}{2}\pi)}\right] + \sqrt{2}\left[e^{-j(6t-\frac{1}{4}\pi)} + e^{j(6t-\frac{1}{4}\pi)}\right] \\
&= 4 + 6\cos\left(4t - \frac{1}{2}\pi\right) + 2\sqrt{2}\cos\left(6t - \frac{1}{4}\pi\right) \rightarrow \text{real values for all } t
\end{aligned}$$

This shows that $x(t)$ is real.

Answer $x(t)$ is periodic.

Note : In order for a signal which is made up of sinusoids to be periodic, the frequency components must be harmonically related.

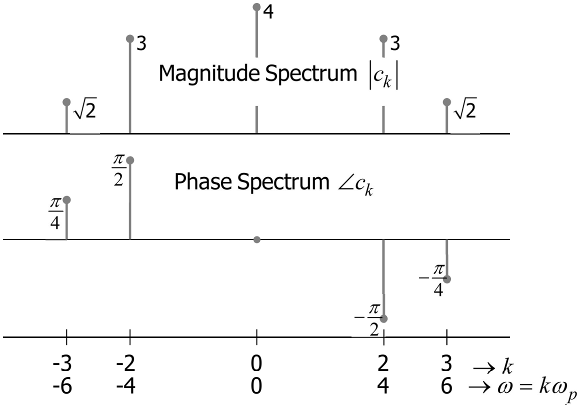
Harmonically related frequencies means that they must be multiples of the fundamental frequency. So the question is whether there exists a fundamental frequency which is a common factor to the component frequencies. The question is : Does $HCF\{f_1, f_2, f_3, \ldots\}$ exists? HCF stands for highest common factor and $f_1, f_2, \ldots$ are the component frequencies in the signal.

In Example 3, the component frequencies are 4 and 6 rad/s. The $HCF\{4, 6\} = 2$ and thus the fundamental frequency exists and is $\omega_p = 2$ rad/s. Since the fundamental frequency exists, $x(t)$ is periodic.

Let's see how we can now find the Fourier Series coefficient, c_k for $x(t)$.

$$x(t) = \underbrace{(1+j)}_{c_{-3}} e^{-j6t} + \underbrace{j3}_{c_{-2}} e^{-j4t} + \underbrace{4}_{c_0} + \underbrace{-j3}_{c_2} e^{j4t} + \underbrace{(1-j)}_{c_3} e^{j6t}$$

$$c_k = \begin{cases} 1+j & k=-3 \\ 3j & k=-2 \\ 4 & k=0 \\ -3j & k=2 \\ 1-j & k=3 \end{cases} = \begin{cases} \sqrt{2}e^{\pm j0.25\pi} & k=\mp 3 \\ 3e^{\pm j0.5\pi} & k=\mp 2 \\ 4 & k=0 \\ 0 & k \text{ otherwise} \end{cases}$$



Exercise 3 (you may write your answer below)

Consider the signal $x(t) = 2\sin(5t + 0.1\pi) + e^{j3t} - e^{-j3t}$

1. *Is* $x(t)$ *real or complex?*
2. *What are the frequency components in* $x(t)$*?*
3. *Sketch the spectrum of* $x(t)$*.*

Exercise 4 (you may write your answer below)

The discrete-frequency spectrum of $x(t)$ is given in the figure below.

1. *Write a mathematical expression for $x(t)$.*
2. *Let $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}$ denote the Fourier series of $x(t)$ where f_p is the fundamental frequency. Evaluate c_k.*

Spectrum of $x(t)$