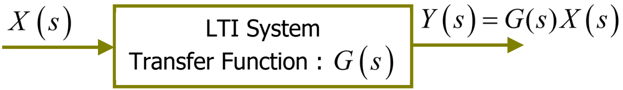
Outline of Lecture

Output Responses of LTI Systems



In general, if an only if initial conditions are zero, $Y(s) = G(s)X(s)$.

Thus, by the property of the Laplace transform, for any general input $x(t)$, the output $y(t)$ can be computed using convolution :

$$y(t) = \mathcal{L}^{-1}[G(s)X(s)] = g(t) * x(t) \tag{1}$$

where $g(t) = \mathcal{L}^{-1}[G(s)]$ and * represents convolution.

In this lecture, we consider 3 types of responses :

1. Impulse Response, $y_\delta(t)$, when input is an impulse, $x(t) = \delta(t)$
2. Step Response, $y_{step}(t)$ when input is a step function, $x(t) = u(t)$
3. Sinusoidal Response, $y_s(t)$ when input is a sinusoid, $x(t) = A \sin \omega t$

Impulse Response

For a unit impulse input $x(t) = \delta(t)$,

$$\begin{array}{rrcl} & X(s) & = & 1 \\ \text{Hence} & Y_\delta(s) & = & G(s).1 = G(s) \\ \text{Therefore} & y_\delta(t) & = & \mathcal{L}^{-1}[G(s)] = g(t) \end{array}$$

- It follows that $g(t) \leftrightarrow G(s)$ is a Laplace transform pair.
- $g(t)$ is called the impulse response of the LTI system with transfer function $G(s)$.
- You may think of $g(t)$ as the time domain equivalent of the transfer function and thus it is the time domain characteristic of the LTI system.
- What can the impulse response tell us about the LTI system?

Notion of BIBO Stability - analysis from impulse response

The transfer function $G(s)$ may be partial factorized as

$$G(s) = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \ldots + \frac{\alpha_N}{s+p_N} \tag{2}$$

where α_i in (2) are constants, while the poles of $G(s)$ are at $s = -p_1, -p_2, \ldots, -p_N$. Hence the impulse response is given by

$$y_\delta(t) = \alpha_1 e^{-p_1 t} + \alpha_2 e^{-p_2 t} + \ldots + \alpha_N e^{-p_N t} \quad \text{for } t \geq 0$$

1. If all poles have negative real parts i.e. Re$(-p_i) < 0$, then

   $$\lim_{t\to\infty} y_\delta(t) = 0 \tag{3}$$

   Systems with impulse response which satisfies (3) are said to be bounded-input-bounded-output (BIBO) stable.

   In fact, it can be shown that for any general bounded input $x(t)$, the output of $G(s)$ will be bounded. Hence $G(s)$ is considered to be BIBO stable.

2. If any of the poles have positive real parts i.e. $\text{Re}(-p_i) > 0$, then

$$\lim_{t\to\infty} y_\delta(t) = \infty \tag{4}$$

Systems with impulse response satisfying (4) are said to be unstable.

3. If any of the poles have zero real parts i.e. $\text{Re}(-p_i) = 0$ while all others have negative real parts, then

$$\lim_{t\to\infty} y_\delta(t) = \text{constant} \tag{5}$$

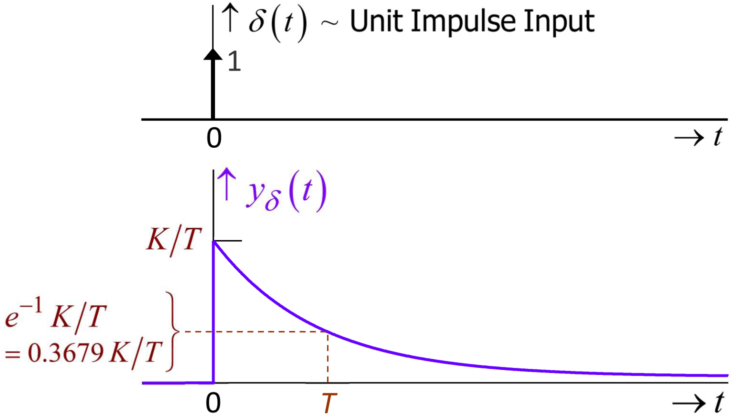
Systems with impulse response satisfying (5) are marginally stable.

- We conclude that poles determine the stability of the LTI system. They provide a convenient way to determine if outputs will be bounded or unbounded without calculating $y(t)$ at all.
- It should be noted that if the input $x(t)$ is unbounded, then outputs $y(t)$ will also be unbounded, regardless of the poles of $G(s)$.
- Note that systems with repeated poles with $\text{Re}(-p_i) = 0$ are unstable.

1. Impulse Response of First Order Systems

First order transfer function : $G(s) = \dfrac{K}{sT+1} = \dfrac{K}{T}\dfrac{1}{s+\frac{1}{T}}$

Impulse response : $y_\delta(t) = \mathcal{L}^{-1}[G(s)] = \dfrac{K}{T}e^{-\frac{t}{T}}$



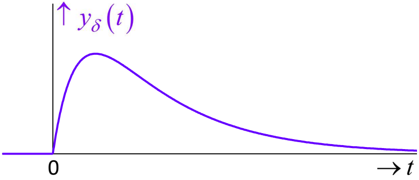
2. Impulse Response of Second Order Systems

Second order transfer function : $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Overdamped system : $\zeta > 1$

$$G(s) = \frac{K\omega_n^2}{(s+\sigma_1)(s+\sigma_2)}$$

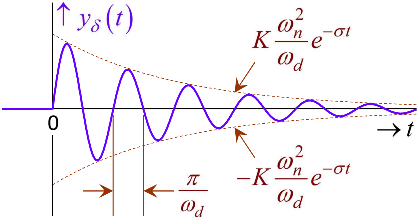
$$g(t) = K'e^{-\sigma_1 t} - K'e^{-\sigma_2 t}$$



Underdamped system : $\zeta < 1$

$$G(s) = \frac{K\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

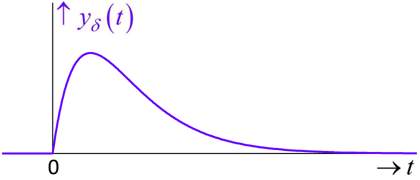
$$g(t) = \frac{K\omega_n^2}{\omega_d} e^{-\sigma t} \sin \omega_d t$$



Frequency of oscillation $= \omega_d$

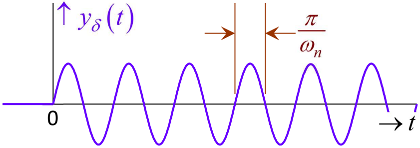
Critically damped system : $\zeta = 1$

$$G(s) = \frac{K\omega_n^2}{(s+\omega_n)^2}$$
$$g(t) = K\omega_n^2 t e^{-\omega_n t}$$



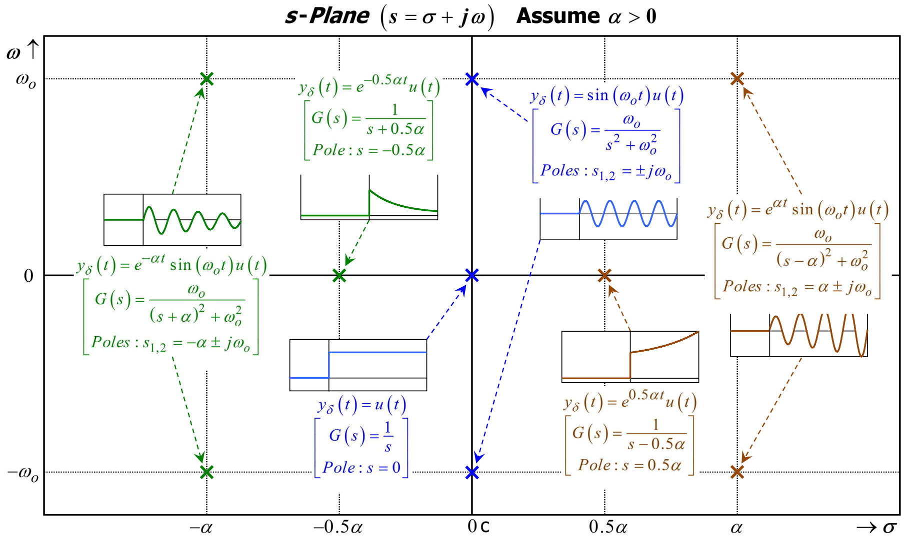
Zero damping : $\zeta = 0$

$$G(s) = \frac{K\omega_n^2}{(s^2+\omega_n^2)}$$
$$g(t) = K\omega_n \sin\omega_n t$$



Frequency of oscillation $= \omega_n$

Relationship between Impulse Response and Pole Locations



Step Response

For a step input $x(t) = u(t)$ where $u(t)$ is the unit step function :

$$
\begin{aligned}
X(s) &= \frac{1}{s} \\
\text{Hence} \qquad Y_{step}(s) &= G(s).\frac{1}{s} \\
\text{Therefore} \qquad y_{step}(t) &= \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \int_0^t g(\tau)d\tau \\
&= \int_0^t y_\delta(\tau) \text{ (or unit impulse response) } d\tau \\
\text{Conversely} \qquad y_\delta(t) &= \frac{dy_{step}(t)}{dt}
\end{aligned}
$$

$$
\begin{aligned}
\text{Using FVT,} \qquad \lim_{t\to\infty} y_{step}(t) &= \lim_{s\to 0} sY_{step}(s) \\
&= \lim_{s\to 0} s.G(s)\frac{1}{s} = \lim_{s\to 0} G(s) \\
&= \text{DC gain of } G(s)
\end{aligned}
$$

1. Unit Step Response of First Order Systems

First order transfer function : $G(s) = \dfrac{K}{sT+1}$

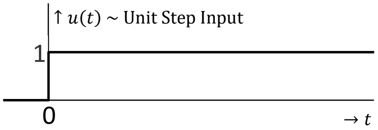
$$\text{Step response} \quad y_{step}(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] = \mathcal{L}^{-1}\left[\frac{K}{s(sT+1)}\right]$$

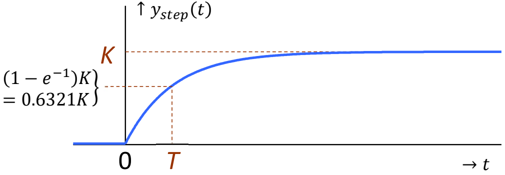
$$= \mathcal{L}^{-1}\left[\frac{K}{s} - \frac{K}{s+\frac{1}{T}}\right] = K\left[1 - e^{-\frac{1}{T}t}\right]$$

$$\lim_{t\to\infty} y_{step}(t) = K = \lim_{s\to 0} G(s) = \lim_{s\to 0} sY_{step}(s)$$

At $t = T$, $y_{step}(T) = K\left[1 - e^{-1}\right] = 0.6321K$

T is defined to be the time constant of the first order system.



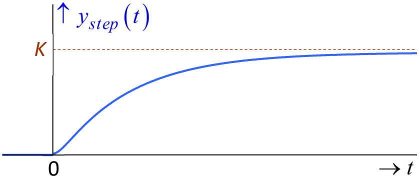


2. Step Response of Second Order Systems

Overdamped system : $\zeta > 1$

$$G(s) = \frac{K\omega_n^2}{(s+\sigma_1)(s+\sigma_2)}$$

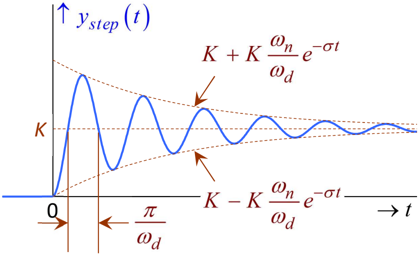
$$y_{step}(t) = K + K'e^{-\sigma_1 t} - K''e^{-\sigma_2 t}$$



Underdamped system : $\zeta < 1$

$$G(s) = \frac{K\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$y_{step}(t) = K - K\frac{\omega_n}{\omega_d}e^{-\sigma t}\sin(\omega_d t + \phi)$$

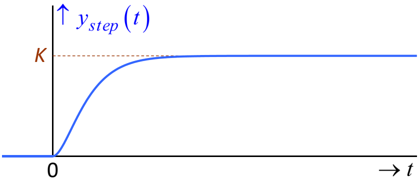


Frequency of oscillation $= \omega_d$

Critically damped system : $\zeta = 1$

$$G(s) = \frac{K\omega_n^2}{(s+\omega_n)^2}$$

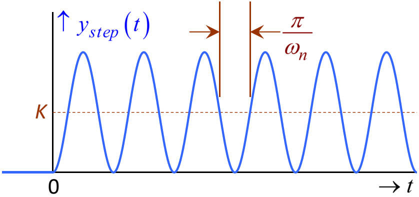
$$y_{step}(t) = K\left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right]$$



Zero damping : $\zeta = 0$

$$G(s) = \frac{K\omega_n^2}{(s^2+\omega_n^2)}$$

$$y_{step}(t) = K(1 - \cos\omega_n t)$$



Frequency of oscillation $= \omega_n$

Example 1 (DC Motor)

A DC motor can be modeled as $\frac{\Omega(s)}{V_{in}(s)} = \frac{K}{Js+B}$ *where* $\Omega(s) = \mathcal{L}[\omega(t)]$ *and* $V_{in}(s) = \mathcal{L}[v_{in}(t)]$ *and* $\omega(t)$ *and* $v_{in}(t)$ *denote the angular speed of the motor and power supply voltage, respectively.* K, J *and* B *are positive constants. Describe the response of the motor to a constant input,* $v_{in}(t) = 5V$*. You do not need to perform any calculations.*

- Since K, J and B are positive constants, the pole of the system is at $s = -B/J < 0$ and hence the motor is BIBO stable.
- When $v_{in}(t) = 5$V, the angular velocity, $\omega(t)$ will also reach a steady state value (bounded) because the system is BIBO stable.
- Furthermore, rewrite $G(s)$ in the standard 1^{st} order form $\frac{K}{sT+1}$, we have $\frac{\Omega(s)}{V_{in}(s)} = \frac{K}{Js+B} = \frac{K/B}{sJ/B+1}$. This tells us that the time constant is $T = J/B$ and hence we know how "quickly" or "slowly" the motor will respond to a step input.

This shows how LTI systems theory is useful. We can get a good idea of how the output will respond to a particular input without any calculations.

Example 2

Continuing with Example 1, suppose we now want to model the angular position $\theta(t)$, instead of the angular velocity $\omega(t)$. Find the transfer function, $G_1(s)$ between $\Theta(s)$ and $V_{in}(s)$. What is the nature of stability of $G_1(s)$?

The relationship between $\omega(t)$ and $\theta(t)$ is :

$$\begin{aligned}
\omega(t) &= \frac{d\theta(t)}{dt} \quad \text{(angular velocity = derivative of angular position)} \\
\Omega(s) &= s\Theta(s) \\
\frac{\Omega(s)}{\Theta(s)} &= s \quad \text{or} \quad \frac{\Theta(s)}{\Omega(s)} = \frac{1}{s} \qquad \left(\theta(t) = \int_0^t \omega(\tau)d\tau\right)
\end{aligned}$$

From Example 1, $\frac{\Omega(s)}{V_{in}(s)} = \frac{K}{Js+B}$. Hence

$$G_1(s) = \frac{\Theta(s)}{V_{in}(s)} = \frac{\Theta(s)}{\Omega(s)} \times \frac{\Omega(s)}{V_{in}(s)} = \frac{1}{s} \times \frac{K}{Js+B} = \frac{K}{s(JS+B)}$$

Since there is a pole at $s = 0$ ie Re$[-p_1] = 0$, we conclude from Slide 5 that $G_1(s)$ is marginally stable.

Example 3

Referring to the motor example in Slide 14, suppose that $J = 2$ and $B = 10$ and $K = 1000$. What is the motor speed at steady state? How long will this motor take to reach 63.2% of its steady state rotational speed if the input voltage is 5 V applied at $t \geq 0$? Sketch the motor response.

The transfer function is given by $\dfrac{\Omega(s)}{V_{in}(s)} = \dfrac{K}{Js + B} = \dfrac{1000}{2s + 10}$.

With $v_{in}(t) = 5u(t)$, $V_{in}(s) = 5\frac{1}{s}$. Hence,

$$\begin{aligned}
\Omega(s) &= \frac{1000}{2s+10} \times 5\frac{1}{s} \\
&= \frac{2500}{s(s+5)} = \frac{500}{s} - \frac{500}{s+5} \\
\omega(t) &= \underbrace{500}_{\text{steady state}} - \underbrace{500e^{-5t}}_{\text{transient response}} \qquad t \geq 0
\end{aligned}$$

Steady state : $\lim_{t \to \infty} \omega(t) = 500$ rad/s or approximately 955 rpm.

To compute the time that the motor takes to reach 63.2% of its steady state, solve the following equation :

$$\begin{aligned} 361 = 0.632 \times 500 &= 500 - 500e^{-5t} \\ 0.632 &= 1 - e^{-5t} \implies t = 0.2 \end{aligned} \tag{6}$$

Recall that the motor has a transfer function

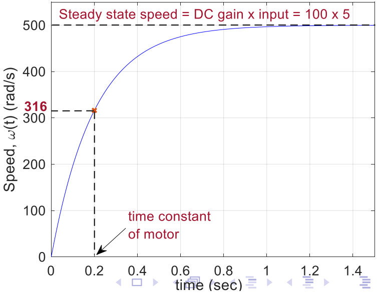
$$\frac{\Omega(s)}{V_{in}(s)} = \frac{1000}{2s+10} = \frac{100}{0.2s+1} \leftrightarrow \frac{K}{sT+1} \tag{7}$$

Hence (referring to Slide 11), and comparing coefficients, the time constant of the motor is $T = 0.2$ which is also verified by equation (6).

According to (7),
DC gain of motor $= K = 100$

Steady state speed = DC gain x input voltage

$\therefore$ steady state speed $=$ $100 \times 5 = 500$ since input is 5 V.



Role of Poles of $G(s)$

- In the s-domain, the relationship between the input, $x(t)$, and output, $y(t)$, of a LTI system is given by

$$Y(s) = G(s)X(s) \tag{8}$$

where $$G(s) = K' \frac{(s+z_1)(s+z_2)\ldots(s+z_M)}{(s+p_1)(s+p_2)\ldots(s+p_N)}$$

is the transfer function with poles $p_1, \ldots, p_N$ and zeros $z_1, \ldots, z_M$.

- The Laplace transform of the input signal, $X(s)$ may also contain poles and zeros but mostly we are concerned with the poles.
- The poles associated with $G(s)$ are called the system poles.
- The poles associated with $X(s)$ are called the input poles.
- Let's see how the poles play a role in the responses.

Role of Poles in Transient Responses

Besides determining stability of $G(s)$, poles of $G(s)$ also give a good idea of the transient response of $G(s)$. Let's illustrate this via an example.

Example 4

Consider $G(s) = \frac{1}{(s+1)(s+2)}$. *Suppose* $G(s)$ *is has an input* $x(t) = 2u(t)$. *Find the output response of the* $G(s)$, *assuming zero initial conditions.*

Note that the system poles are at $s_{1,2} = -1, -2$ and input pole is at $s = 0$. Using equation (8), $Y(s) = G(s)X(s)$.

$$
\begin{aligned}
Y(s) &= \frac{1}{(s+1)(s+2)}\frac{2}{s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \\
&= \underbrace{\frac{1}{s}}_{\text{input pole}} - \underbrace{\frac{2}{s+1} + \frac{1}{s+2}}_{\text{system poles}} \\
y(t) &= (\underbrace{1}_{\substack{\text{steady state} \\ \text{comes from input pole}}} - \underbrace{2e^{-t} + e^{-2t}}_{\substack{\text{transient response} \\ \text{comes from system poles}}})u(t) \quad \rightarrow \text{full response of } G(s)
\end{aligned}
$$

Example 5

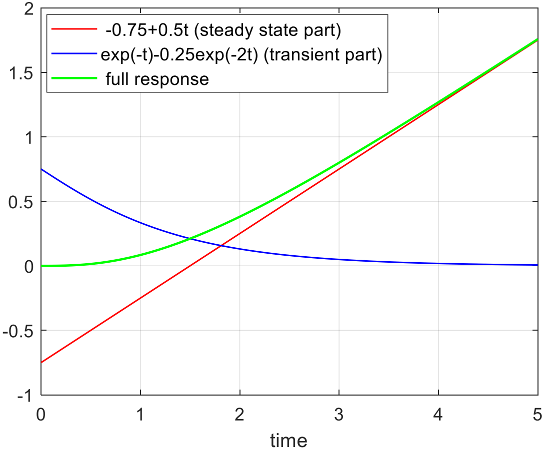
Using the $G(s)$ from Example 4, find the output response of $G(s)$ when the input is a ramp function ie $x(t) = t$.

$$
\begin{aligned}
Y(s) &= \frac{1}{(s+1)(s+2)}\frac{1}{s^2} = \frac{As+B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \\
&= \underbrace{\frac{-0.75s+0.5}{s^2}}_{\text{input pole}} + \underbrace{\frac{1}{s+1} - \frac{0.25}{s+2}}_{\text{system poles}} \qquad \text{(input pole at } s = 0,\ 0\text{)} \\
y(t) &= (\underbrace{-0.75+0.5t}_{\text{steady state}} + \underbrace{e^{-t} - 0.25e^{-2t}}_{\text{transient response}})u(t) \qquad \rightarrow \text{full response of } G(s)
\end{aligned}
$$

The transient response (due to system poles) decays to zero as $t \to \infty$. The steady state response will be

$$\lim_{t\to\infty} y(t) = -0.75 + 0.5t \quad t \geq 0$$

Plot of the response is given in the next slide.



- The transient part is characterised by the system poles ie if the system pole is at $s = -p$, then the transient part will contain a e^{-pt} term.
- The steady state part follows the input functions (since it is developed from the input pole). In this case, the input was a ramp function ($x(t) = t$) and hence the output response contains a t-term as well as a t^0 or constant term.

Role of Zeros of $G(s)$

- The zeros do not play a role in the stability of $G(s)$.
- The zeros only affect the α_i in (2) during partial factorization. They affect the transient response of $y(t)$ which decays to zero (for stable systems) as $t \to \infty$.

Example 6

Consider a second order system $G(s)$ with DC gain of 18 and zero at $s = -\frac{1}{\gamma}$ with

$$G(s) = 18.\frac{\gamma s + 1}{40s^2 + 22s + 1}$$

Analyse its impulse and step responses with respect to variations in γ.

Transfer function :

$$
\begin{aligned}
G(s) &= 18.\frac{\gamma s+1}{40s^2+22s+1} \\
&= 18.\frac{\gamma s+1}{(2s+1)(20s+1)} = \frac{0.5\gamma-1}{s+0.5} - \frac{0.05\gamma-1}{s+0.05}
\end{aligned}
$$

Impulse Response :

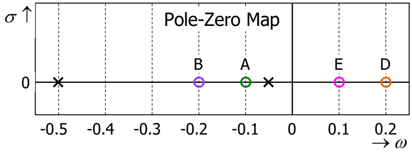
$$
y_\delta(t) = (0.5\gamma-1)e^{-0.5t} - (0.05\gamma-1)e^{-0.05t} \qquad t \geq 0
$$

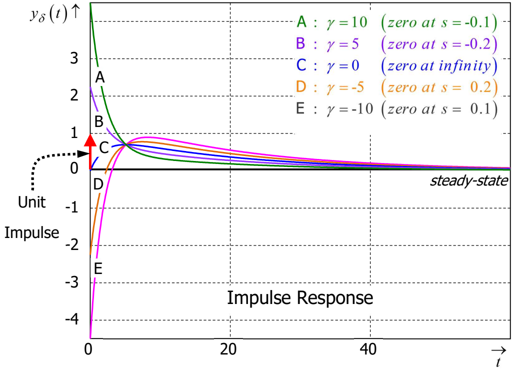
Step Response (according to Slide 10) :

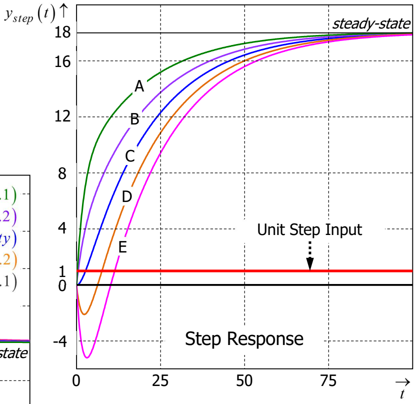
$$
\begin{aligned}
Y_{step}(s) &= G(s).\frac{1}{s} = \frac{18(\gamma s+1)}{s(2s+1)(20s+1)} \\
&= \frac{(2-\gamma)}{s+0.5} - \frac{(20-\gamma)}{s+0.05} + \frac{18}{s}
\end{aligned}
$$

$$
y_{step}(t) = \underbrace{(2-\gamma)e^{-0.5t} - (20-\gamma)e^{-0.05t}}_{\text{transient response}} + \underbrace{18}_{\text{steady state}} \qquad t \geq 0
$$

Effects of Zeros on Impulse and Step Responses







Notice how the location of the zero affects the transient part of the system responses without affecting the steady-state part.

Exercise 1

Find the transfer function, impulse response and the nature of stability of the systems represented by the following differential equations :

1. $\frac{dy(t)}{dt} = x(t - t_0)$
2. $3\frac{d^2y(t)}{dt^2} + y(t) = 0.5x(t)$

In both systems, $y(t)$ *and* $x(t)$ *denote the output and input, respectively.*

Exercise 2

Find the unit step response of the system :

$$\frac{d^2y(t)}{dt^2} + 1.5\frac{dy(t)}{dt} + 0.36y(t) = 0.1\frac{dx}{dt} + 0.7x(t)$$

when the initial conditions are $y(0^-) = 0.3$ *and* $\left.\frac{dy}{dt}\right|_{0^-} = -0.2$.

Find the transfer function of this system? Determine its poles and zeros.