National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems
Tutorial 5 Solutions

Section I

1. (a) Using : $\cos^2(\omega t) = \frac{1}{2}\left[\cos(2\omega t)+1\right]$

$$\mathcal{L}\left\{\cos^2(\omega t)\right\} = \frac{1}{2}\mathcal{L}\left\{\cos(2\omega t)+1\right\} = \frac{1}{2}\left[\frac{s}{s^2+4\omega^2}+\frac{1}{s}\right]$$

(b) $$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{A_1}{(s-1)}+\frac{A_2}{(s+2)}+\frac{A_3}{(s+4)}\right\}$$

The numerator gives:

$1 = A_1(s+2)(s+4) + A_2(s-1)(s+4) + A_3(s-1)(s+2)$

Set s = 1, we have: $1 = A_1(1+2)(1+4) \quad \Rightarrow \quad A_1 = 1/15$

Set s = -2, we have: $1 = A_2(-2-1)(-2+4) \quad \Rightarrow \quad A_2 = -1/6$

Set s = -4, we have: $1 = A_3(-4-1)(-4+2) \quad \Rightarrow \quad A_3 = 1/10$

Hence:

$$\mathcal{L}^{-1}\left\{\frac{1}{15}\frac{1}{(s-1)}-\frac{1}{6}\frac{1}{(s+2)}+\frac{1}{10}\frac{1}{(s+4)}\right\} = \frac{1}{15}e^{t}-\frac{1}{6}e^{-2t}+\frac{1}{10}e^{-4t}$$

(c) $$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s_1^2}\right\} \text{ where } s_1 = s+1$$

$$= te^{-t}$$

(d) $$\mathcal{L}^{-1}\left\{\frac{s+9}{(s^2+6s+13)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+9}{(s+3)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+3)+6}{(s+3)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+3)}{(s+3)^2+4}+3\frac{2}{(s+3)^2+4}\right\}$$

$$= e^{-3t}\left[\cos(2t)+3\sin(2t)\right]$$

(e) $$\mathcal{L}\left\{\frac{3}{5}-\frac{\sqrt{45}}{5}e^{-2t}\sin\left(t+\tan^{-1}0.5\right)\right\} = \mathcal{L}\left\{\frac{3}{5}-\frac{\sqrt{45}}{5}e^{-2t}\left[\sin(t)\cos(\tan^{-1}0.5)+\cos(t)\sin(\tan^{-1}0.5)\right]\right\}$$
$$= \mathcal{L}\left\{\frac{3}{5}-\frac{\sqrt{45}}{5}e^{-2t}\left[\sin(t)\frac{2}{\sqrt{5}}+\cos(t)\frac{1}{\sqrt{5}}\right]\right\}$$
$$= \mathcal{L}\left\{\frac{3}{5}-\frac{6}{5}e^{-2t}\sin(t)-\frac{3}{5}e^{-2t}\cos(t)\right\}$$
$$= \frac{3}{5s}-\frac{6}{5}\frac{1}{(s+2)^2+1}-\frac{3}{5}\frac{s+2}{(s+2)^2+1}$$
$$= \frac{3}{5s}-\frac{6}{5}\frac{1}{s^2+4s+5}-\frac{3}{5}\frac{s+2}{s^2+4s+5}$$
$$= \frac{3}{5}\frac{(s^2+4s+5)-2s-s(s+2)}{s(s^2+4s+5)}$$
$$= \frac{3}{5}\frac{s^2+4s+5-2s-s^2-2s}{s(s^2+4s+5)}$$
$$= \frac{3}{s(s^2+4s+5)}$$

(f) $$\mathcal{L}\left\{(t-1)^2u(t-1)\right\} = \frac{2}{s^3}e^{-s}$$

(g) $$\mathcal{L}\left\{t^2u(t-1)\right\} = \mathcal{L}\left\{(t-1)^2u(t-1)+2t.u(t-1)-u(t-1)\right\}$$
$$= \mathcal{L}\left\{(t-1)^2u(t-1)+2(t-1).u(t-1)+u(t-1)\right\}$$
$$= \frac{2}{s^3}e^{-s}+\frac{2}{s^2}e^{-s}+\frac{1}{s}e^{-s}$$

(h) $$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+\pi^2}\right\} = \cos\left[\pi\left(t-2\right)\right]u(t-2)$$
$$= \left[\cos(\pi t)\cos(2\pi)+\sin(\pi t)\sin(2\pi)\right]u(t-2)$$
$$= \cos(\pi t)u(t-2)$$

(i) $$\mathcal{L}\left\{te^{-t}\sin(t)\right\} = -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right] = -\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right] = \frac{2(s+1)}{(s^2+2s+2)^2}$$

(j) As $\mathcal{L}\left\{\sin(3t)\right\} = \dfrac{3}{s^2+9}$, then $\dfrac{d}{ds}\dfrac{3}{s^2+9} = \dfrac{6s}{\left(s^2+9\right)^2}$

$$\therefore\quad \mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+9\right)^2}\right\} = \frac{1}{6}t\sin(3t)$$

2. Taking the Laplace transform of : $\ddot{y}(t)+4\dot{y}(t)+3y(t)=2r(t)$, we obtain :

$$\left[s^2Y(s)-sy(0)-\dot{y}(0)\right]+4\left[sY(s)-y(0)\right]+3Y(s)=\frac{2}{s}$$

Given $y(0)=1$ and $\dot{y}(0)=0$, we have:

$$s^2Y(s)-s+4sY(s)-4+3Y(s)=\frac{2}{s}$$

$$Y(s)\left[s^2+4s+3\right]-s-4=\frac{2}{s}$$

$$Y(s)\left[s^2+4s+3\right]=\frac{2}{s}+s+4=\frac{s^2+4s+2}{s}$$

$$Y(s)=\frac{s^2+4s+2}{s\left(s^2+4s+3\right)}=\frac{s^2+4s+2}{s\left(s+1\right)\left(s+3\right)}$$

Let: $Y(s)=\dfrac{A_1}{s}+\dfrac{A_2}{s+1}+\dfrac{A_3}{s+3}=\dfrac{A_1(s+1)(s+3)+A_2s(s+3)+A_3s(s+1)}{s\left(s+1\right)\left(s+3\right)}$

Hence: $A_1(s+1)(s+3)+A_2s(s+3)+A_3s(s+1)=s^2+4s+2$

Setting $s=0$, we obtain: $3A_1=2 \;\Rightarrow\; A_1=2/3$

Setting s=-1, we obtain: $-2A_2=-1 \;\Rightarrow\; A_2=1/2$

Setting $s=-3$, we obtain: $6A_3=-1 \;\Rightarrow\; A_3=-1/6$

Hence, we have:

$$Y(s)=\frac{2}{3s}+\frac{1}{2(s+1)}-\frac{1}{6(s+3)}$$

Taking the inverse Laplace Transform, we have:

$$y(t)=\frac{2}{3}+\frac{1}{2}\exp(-t)-\frac{1}{6}\exp(-3t)$$

Section II

1. In the given circuit, the input signal is the voltage source and the output signal is the current flowing in the circuit, $i(t)$. To solve the problem,

- Derive the differential equation relating the input and output signals using Kirchoff Voltage Law:

$$100i(t) + 2\frac{di(t)}{dt} = 100, \quad t < 0 \tag{1a}$$

$$25i(t) + 2\frac{di(t)}{dt} = 100, \quad t \geq 0 \tag{1b}$$

- The problem states that the circuit is operating in steady-state with the switch open prior to $t = 0$. This statement indicates that only the steady-state solution is needed for $t < 0$. Since the input signal is a constant, the steady-state output should also be a constant. Hence, $\frac{di(t)}{dt}$ in Equation 1a should be zero. Equation 1a reduces to $100i(t) = 100$ or $i(t) = 1$ when $t < 0$.
  Another way to find the steady-state current is to use the property that an inductor reduces to a short circuit at steady-state if the input signal is a constant. The steady-state current can then be found by solving a resistive circuit.
- Lastly, solve Equation 1b using Laplace Transform and the initial condition $i(0) = 1$.

$$25i(t) + 2\frac{di(t)}{dt} = 100.u(t)$$

$$25I(s) + 2\left[sI(s) - i(0)\right] = \frac{100}{s}$$

$$\left[2s + 25\right]I(s) = 2i(0) + \frac{100}{s}$$

$$\begin{aligned} I(s) &= \frac{2i(0)}{2s+25} + \frac{100}{s(2s+25)} \\ &= \frac{2}{2s+25} + \frac{100}{s(2s+25)} \\ &= \frac{1}{s+12.5} + \frac{50}{s(s+12.5)} \\ &= \frac{s+50}{s(s+12.5)} \end{aligned}$$

Let : $I(s) = \frac{A_1}{s} + \frac{A_2}{s+12.5}$, then for the numerators on the left and right side, we have:

$s + 50 = A_1(s+12.5) + A_2 s$

Let s = 0, then $50 = 12.5A_1 \quad \Rightarrow \quad A_1 = 4$

Let s = -12.5, then $37.5 = A_2(-12.5) \quad \Rightarrow \quad A_2 = -3$

Hence:

$$I(s) = \frac{4}{s} - \frac{3}{s+12.5}$$

Taking the inverse Laplace Transform:

$$i(t) = 4 - 3e^{-12.5t}$$

Note that only one initial condition is needed as the system model is a first order differential equation.

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2. (a) Using KVL, the differential equation relating $i(i)$ to $E(t)$ is

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

$$L\frac{d^2i(t)}{dt} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dE(t)}{dt}$$

(b) Transforming the differential equation into the s-domain using Laplace Transform,

$$s^2I(s) - si(0) - i'(0) + 6\left[sI(s) - i(0)\right] + 25I(s) = \frac{120s}{s^2+25}$$

Since initial conditions $i(0)$ and $i'(0)$ are zero,

$$s^2I(s) + 6sI(s) + 25I(s) = \frac{120s}{s^2+25}$$

$$I(s)\left[s^2 + 6s + 25\right] = \frac{120s}{s^2+25}$$

$$I(s) = \frac{120s}{(s^2+6s+25)(s^2+25)}$$

$$= \frac{120s}{\left[(s+3)^2+16\right](s^2+25)}$$

Let $I(s)=\dfrac{120s}{\left[(s+3)^2+16\right](s^2+25)}=\dfrac{A_1}{\left[(s+3)^2+16\right]}+\dfrac{A_2\left(s+3\right)}{\left[(s+3)^2+16\right]}+\dfrac{A_3}{s^2+25}+\dfrac{A_4 s}{s^2+25}$

Then the numerator on the left and right sight are:

$$120s=A_1(s^2+25)+A_2(s+3)\left(s^2+25\right)+A_3\left[(s+3)^2+16\right]+A_4 s\left[(s+3)^2+16\right]$$

$$=\left(A_2+A_4\right)s^3+\left(A_1+3A_2+A_3+6A_4\right)s^2+\left(25A_2+6A_3+25A_4\right)s+25(A_1+75A_2+25A_3)$$

Matching s^3, we have: $0=A_2+A_4$ (1)

Matching s^2, we have: $0=A_1+3A_2+A_3+6A_4$ (2)

Matching s, we have: $120=25A_2+6A_3+25A_4$ (3)

Matching constants, we have: $0=25A_1+75A_2+25A_3 \quad\Rightarrow\quad 0=A_1+3A_2+A_3$ (4)

Subst. (1) in (3), we obtain: $120=6A_3 \quad\Rightarrow\quad A_3=20$

Subst. $A_3=20$ in (4), we obtain: $A_1+3A_2=-20$ (5)

Subst. $A_3=20$ and (5) in (2), we obtain: $-20+20+6A_4=0 \quad\Rightarrow\quad A_4=0$

Subst. $A_4=0$ in (1), we obtain: $0=A_2+0 \quad\Rightarrow\quad A_2=0$

Subst. $A_2=0$ in (5), we obtain: $A_1+0=-20 \quad\Rightarrow\quad A_1=-20$

Hence:

$$I(s)=-\frac{20}{(s+3)^2+16}+\frac{20}{s^2+25}$$

$$=-5\frac{4}{(s+3)^2+16}+4\frac{5}{s^2+25}$$

Taking the inverse Laplace Transform:

$$i(t)=-5e^{-3t}\sin(4t)+4\sin(5t)$$

3. (a) Drugs taken in tablet form can be modelled by impulse function whose strength is equal to the quantity of drug ingested. To obtain the necessary mathematical function, first consider the definition of *impulse function* and *strength* of an impulse functions. By definition, $\delta(t - a)$ is non-zero when $t = a$. Since system is linear time invariant, the instant when first tablet is ingested can be assumed to be $t = 0$ and the second tablet is taken at $t = 1$ (1 day later or 25 hours later). Hence, $f(t) = 100\delta(t = 0) + 50\delta(t - 1) = 100\delta(t) + 50\delta(t - 1)$.

(b) System is modelled by a second order differential equation so two initial conditions are needed. Given that there is no stress relief drug in Ah Kow's bloodstream, $y(0^-) = \dot{y}(0) = 0$.

(c) To solve the problem, start by performing Laplace Transform on both sides of the differential equation. Then we factorize and obtain the inverse Laplace Transform.

$$\frac{d^2y(t)}{dt} + 3\frac{dy(t)}{dt} + 2y(t) = f(t) = 100\delta(t) + 50\delta(t-1)$$

Taking the Laplace Transform and with initial conditions being zero:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 100 + 50e^{-s}$$

$$Y(s)\left[s^2 + 3s + 2\right] = 100 + 50e^{-s}$$

$$Y(s) = \frac{100 + 50e^{-s}}{s^2 + 3s + 2} = \frac{100}{s^2 + 3s + 2} + \frac{50e^{-s}}{s^2 + 3s + 2} = \frac{100}{(s+1)(s+2)} + \frac{50e^{-s}}{(s+1)(s+2)}$$

Consider: $\dfrac{100}{(s+1)(s+2)} = \dfrac{A_1}{s+1} + \dfrac{A_2}{s+2}$

Then numerators are: $100 = A_1(s+2) + A_2(s+1)$

Substitute $s = -1$, we have: $100 = A_1(-1+2) = A_1 \quad \Rightarrow \quad A_1 = 100$

Substitute $s = -2$, we have: $100 = A_2(-2+1) = -A_2 \quad \Rightarrow \quad A_2 = -100$

Hence we have:

$$Y(s) = \left[\frac{100}{(s+1)} - \frac{100}{(s+2)}\right] + \left[\frac{50}{(s+1)} - \frac{50}{(s+2)}\right]e^{-s}$$

Taking the inverse Laplace Transform:

$$y(t) = 100\left[e^{-t} - e^{-2t}\right]u(t) + 50\left[e^{-(t-1)} - e^{-2(t-1)}\right]u(t-1)$$

At $t = 4$, we have:

$$y(4) = 100\left[e^{-4} - e^{-8}\right] + 50\left[e^{-3} - e^{-6}\right] = 4.163$$

Section III

1. - Input signal is the current source, i®, while the desired output signal is the voltage across the resistor, $v(t)$. Hence, objective is to derive a differential equation involving $v(t)$ and $i(t)$.
   - Other concepts:
     - Kirchoff Current Law : Sum of current flowing through resistor and capacitor is equal to the current provided by the source.
     - Current flowing through the resistor is $\frac{v(t)}{R}$.
     - Current flowing through the capacitor is $C\frac{dv(t)}{dt}$.

   Hence,

   $$C\frac{dv(t)}{dt}+\frac{v(t)}{R}=i(t)$$

   Solve the differential equation to derive $v(t)$.