EE2023 TUTORIAL 1 (SOLUTIONS)

Solution to Q.1

(a) $$z=\frac{1-j1}{1+j2}=\frac{\sqrt{2}e^{j\tan^{-1}(-1/1)}}{\sqrt{5}e^{j\tan^{-1}(2/1)}}=\sqrt{\frac{2}{5}}\frac{e^{-j\pi/4}}{e^{j1.1071}}=\sqrt{0.4}\frac{e^{-j0.7854}}{e^{j1.1071}}=\sqrt{0.4}e^{j(-0.7854-1.1071)}=0.6325e^{-j1.8925}$$

Magnitude = $\sqrt{0.4}$ or 0.6325

Phase = -1.8925 rads or -108.4°

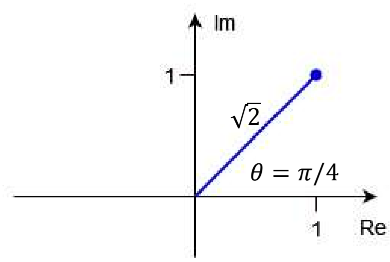
(b) $$\begin{aligned}z=(-1+j1)\times(1+j2)&=\left(\sqrt{2}e^{j(\mathrm{an}^{-1}(-1/1))}\right)\times\left(\sqrt{5}e^{j\tan^{-1}(2/1)}\right)\\&=\sqrt{10}e^{j(\pi-\pi/4)}e^{j1.1071}\\&=\sqrt{10}e^{j(2.3562+1.1071)}\\&=\sqrt{10}e^{j3.4633}\\&=\sqrt{10}e^{-j(2\pi-3.4633)}\\&=\sqrt{10}e^{-j2.8199}\end{aligned}$$

Magnitude = $\sqrt{10}$ or 3.1623

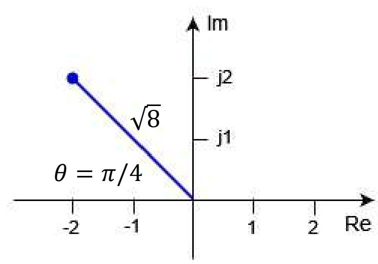
Phase = 3.4633 rads or 198.4° or -2.8199 rads or 161.6°

Solution to Q.2

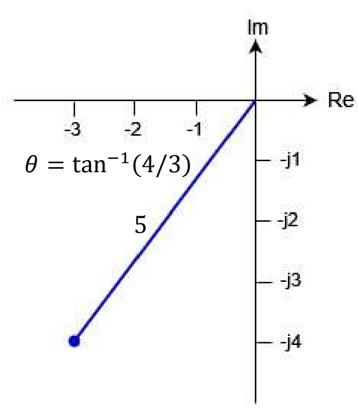
(a) $1+j1=\sqrt{2}e^{j\tan^{-1}(1/1)}=\sqrt{2}e^{j\pi/4}=\sqrt{2}e^{j0.7854}$



(b) $-2+j2=\sqrt{8}e^{j\tan^{-1}(2/-2)}=\sqrt{8}e^{j(\pi-\pi/4)}=\sqrt{8}e^{j3\pi/4}$



(c) $-3-j4=\sqrt{25}e^{j\tan^{-1}(-4/-3)}=5e^{j(\pi+0.9273)}=5e^{j4.0689}=5e^{-j(2\pi-4.0689)}=5e^{-j2.2143}$



Solution to Q.3

Write z in polar form:

$$z = x + jy = |z|\exp(j\angle z).$$

Since adding integer multiples of 2π to $\angle z$ does not affect the value of z, we may also express z as

$$z = |z|\exp\big(j(\angle z + 2k\pi)\big)$$

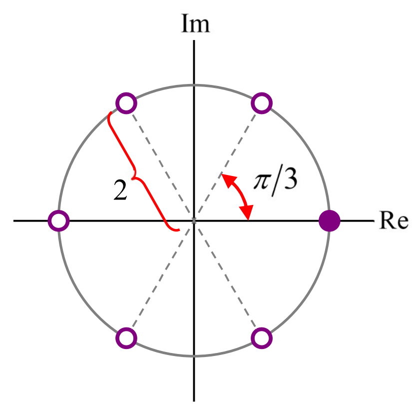
where k is an integer. This leads to

$$\sqrt[N]{z} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right); \quad k = 0,1,\cdots,N-1,$$

which yields the N distinct values of $\sqrt[N]{z}$.

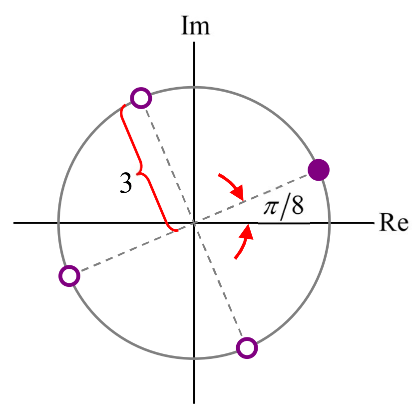
$\sqrt[6]{\mathbf{64}}$ **:**

$$\begin{cases} z = 64 \;\rightarrow\; \begin{cases} |z| = 64 \\ \angle z = 0 \end{cases} \\ \sqrt[6]{64} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=64,\,N=6} \\ \quad = 2\exp\left(j\left(\dfrac{k\pi}{3}\right)\right); \quad k = 0,1,\cdots,5 \\ \quad = \begin{cases} (\;\;2),\; 2\exp\left(j\left(\dfrac{\pi}{3}\right)\right);\; 2\exp\left(j\left(\dfrac{2\pi}{3}\right)\right); \\ (-2),\; 2\exp\left(j\left(\dfrac{4\pi}{3}\right)\right);\; 2\exp\left(j\left(\dfrac{5\pi}{3}\right)\right) \end{cases} \end{cases}$$



$\sqrt[4]{\boldsymbol{j81}}$ **:**

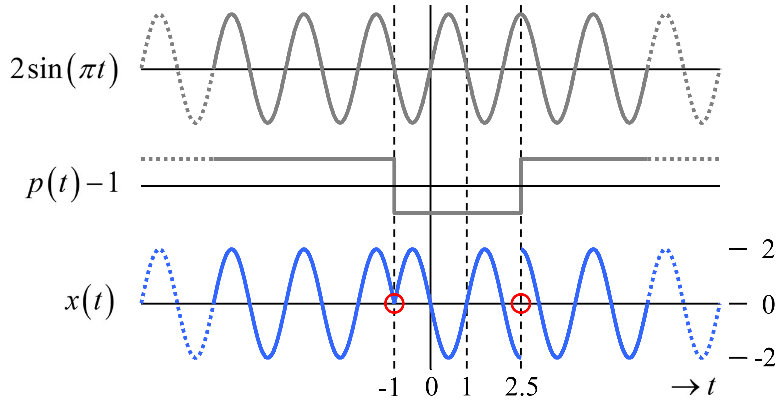
$$\begin{cases} z = j81 \;\rightarrow\; \begin{cases} |z| = 81 \\ \angle z = \dfrac{\pi}{2} \end{cases} \\ \sqrt[4]{j81} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=81,\,N=4} \\ \quad = 3\exp\left(j\left(\dfrac{\pi}{8} + \dfrac{k\pi}{2}\right)\right); \quad k = 0,1,\cdots,3 \\ \quad = \begin{cases} 3\exp\left(j\left(\dfrac{\pi}{8}\right)\right),\; 3\exp\left(j\left(\dfrac{5\pi}{8}\right)\right), \\ 3\exp\left(j\left(\dfrac{9\pi}{8}\right)\right),\; 3\exp\left(j\left(\dfrac{13\pi}{8}\right)\right) \end{cases} \end{cases}$$



Solution to Q.4

(a) $p(t) = 2 - 2\text{rect}\left(\frac{t-0.75}{3.5}\right)$

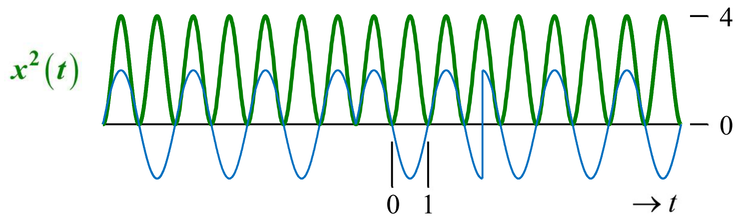
(b) By inspection, $x(t)$ is not periodic.



Notice the π rad (or $180°$) phase jumps in $x(t)$ occurring at the zero crossings of $p(t)-1$.

(c)

$$x^2(t) = 4\sin^2(\pi t)\underbrace{\left(p(t)-1\right)^2}_{1}$$
$$= 4\sin^2(\pi t)$$
$$= 2\left(1-\cos(2\pi t)\right)$$



Note that $x^2(t)$ is periodic with a period of $T = 1$.

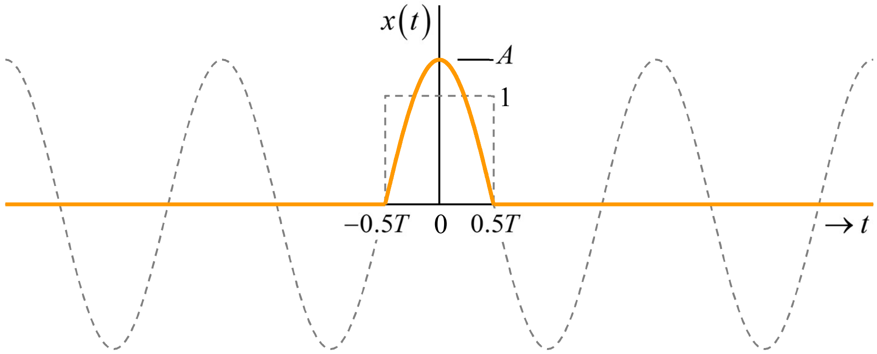
Total energy:
$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \sum_{n=-\infty}^{\infty} \underbrace{\int_{nT}^{(n+1)T} x^2(t)\,dt}_{\substack{\int \text{over one period} \\ \text{thus independent} \\ \text{of } n}} = \left(\underbrace{\int_0^T x^2(t)\,dt}_{\text{finite}}\right) \underbrace{\sum_{n=-\infty}^{\infty} 1}_{\infty} = \infty$$

Average Power:
$$P = \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt}_{\substack{x^2(t) \text{ is periodic. } \therefore \\ P \text{ can be obtained} \\ \text{by averaging over} \\ \text{one period.}}} = \int_{-0.5}^{0.5} 2\left(1-\cos(2\pi t)\right)dt = 2$$

Conclusion: $x(t)$ is an aperiodic power signal.

Solution to Q.5

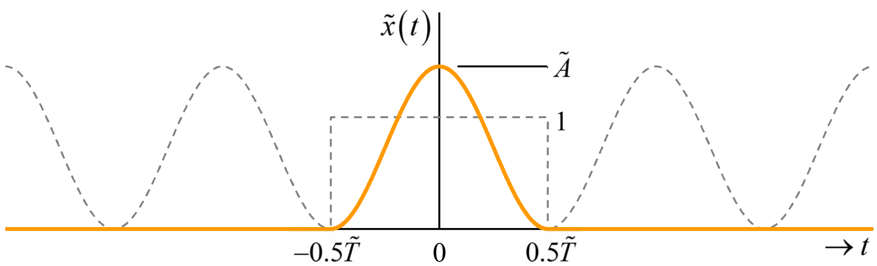
Half-cosine pulse: $x(t) = A\cos\left(\frac{\pi t}{T}\right)\text{rect}\left(\frac{t}{T}\right)$



$$x^2(t) = \frac{A^2}{2}\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]\text{rect}\left(\frac{t}{T}\right)$$

Energy: $E = \frac{A^2}{2}\int_{-0.5T}^{0.5T} 1+\underbrace{\cos\left(\frac{2\pi t}{T}\right)}_{\int \text{over one period} = 0} dt = \frac{1}{2}A^2T$

Raised-cosine pulse: $\tilde{x}(t) = \frac{\tilde{A}}{2}\left(1+\cos\left(\frac{2\pi t}{\tilde{T}}\right)\right)\text{rect}\left(\frac{t}{\tilde{T}}\right)$



$$\tilde{x}^2(t) = \frac{\tilde{A}^2}{4}\left[\frac{3}{2}+2\cos\left(\frac{2\pi t}{\tilde{T}}\right)+\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)\right]\text{rect}\left(\frac{t}{\tilde{T}}\right)$$

Energy: $\tilde{E} = \frac{\tilde{A}^2}{4}\int_{-0.5\tilde{T}}^{0.5\tilde{T}} \frac{3}{2}+\underbrace{2\cos\left(\frac{2\pi t}{\tilde{T}}\right)}_{\int \text{over one period} = 0}+\underbrace{\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)}_{\int \text{over two periods} = 0} dt = \frac{3}{8}\tilde{A}^2\tilde{T}$

Both $x(t)$ and $\tilde{x}(t)$ will have the same energy if $A^2T = \frac{3}{4}\tilde{A}^2\tilde{T}$.

Solution to Q.6

(a) $x(t)=\cos(3.2t)+\sin(1.6t)+\exp(j2.8t) \quad \cdots \quad \begin{cases} \cos(3.2t) & \text{has a frequency of } 3.2 \ rad/s \\ \sin(1.6t) & \text{has a frequency of } 1.6 \ rad/s \\ \exp(j2.8t) & \text{has a frequency of } 2.8 \ rad/s \end{cases}$

Highest common factor (HCF) of $\{3.2,\ 1.6,\ 2.8\}$ exists and is equal to 0.4. Thus, $x(t)$ is periodic and has a fundamental frequency of $0.4\ rad/s$ $(\text{or } 0.2/\pi\, Hz)$ and a fundamental period of $5\pi\ s$.

REMARKS: *Although $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$, it does not contain the fundamental frequency component itself.*

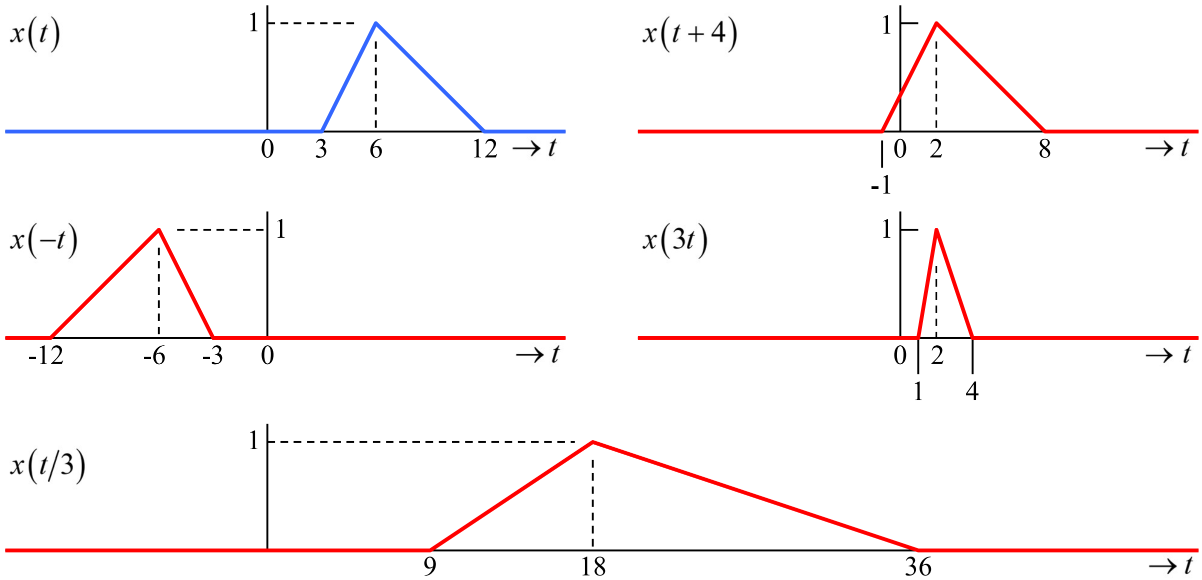
(b) $x(t)=\cos(4t)+\sin(\pi t) \quad \cdots \quad \begin{cases} \cos(4t) & \text{has a frequency of } 4 \ rad/s \\ \sin(\pi t) & \text{has a frequency of } \pi \ rad/s \end{cases}$

Highest common factor (HCF) of $\{4,\ \pi\}$ does not exist. Thus, $x(t)$ is not periodic.

REMARKS: *Summing sinusoids does not necessarily lead to a periodic signal unless the frequencies of the sinusoids are harmonics of a common fundamental frequency.*

Solution to Q.7

(a)



(b) We observe that $y(t)$ is a time-scaled, -reversed and –shifted version of $x(t)$.

For problems of this nature, we should start with time-scaling first since it involves linear warping of the time axis. If we were to start with time-shifting and/or time-reversal, we may have to redo them after time-scaling. However, this sequence of operation need not be followed if we are sketching the signal from the mathematical expression.

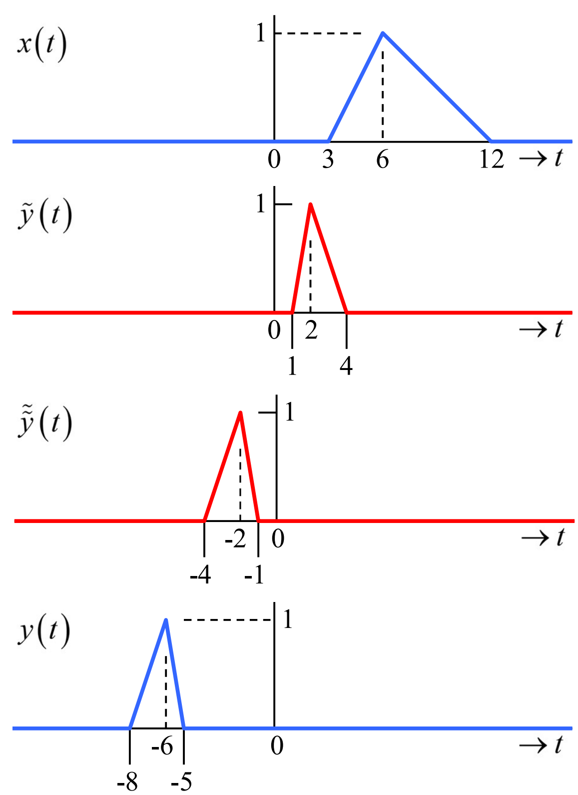
Comparing $x(t)$ and $y(t)$, we note that $y(t)$ involves time-scaling (or contraction) of $x(t)$ by a factor of 3.

Time-scaling of $x(t)$: $\tilde{y}(t) = x(3t)$

Time-reversal of $\tilde{y}(t)$: $\tilde{\tilde{y}}(t) = \tilde{y}(-t) = x(-3t)$

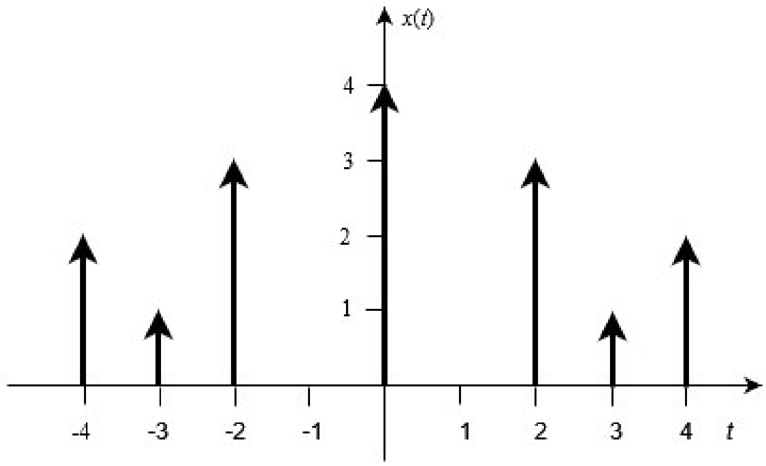
Time shifting of $\tilde{\tilde{y}}(t)$: $\begin{cases} y(t) = \tilde{\tilde{y}}(t+4) \\ \quad\;\; = x(-3(t+4)) \end{cases}$

$\therefore\ \boldsymbol{y(t) = x(-3(t+4))}$



Solution to Q.8

$$x(t) = 2\delta(t+4) + \delta(t+3) + 3\delta(t+2) + 4\delta(t) + 3\delta(t-2) + \delta(t-3) + 2\delta(t-4)$$



Solution to Q.9

$$X(f) = 2\,\mathrm{rect}\left(\frac{f}{4}\right) - \cos\left(2\pi\left(\frac{1}{4}\right)f\right)\mathrm{rect}\left(\frac{f}{2}\right) = 2\,\mathrm{rect}\left(\frac{f}{4}\right) - \cos\left(\frac{\pi f}{2}\right)\mathrm{rect}\left(\frac{f}{2}\right)$$

Supplementary Questions (Solutions)

S1(a) Given that integration of unit step function, $u(t)$, is a ramp, i.e. $t.u(t)$, the $x(t)$ is made up of:

$$\begin{aligned} x(t) &= \frac{1}{2}\left[t.u(t)\right].u(2-t) \\ &= \left[\int_{-\infty}^{\infty} \frac{1}{2} u(\tau)d\tau\right] u(2-t) \end{aligned}$$

S1(b) The signal $u(t)$ is observed to be made up of various $u(t)$ functions that are shifted in time and/or reversed in time. Hence:

$$x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$$

S2(a) Given: $x(t) = \cos(2t + 0.25\pi)$

$x(t)$ is periodic with an angular frequency of 2 rads/s.

Hence, its frequency is $\dfrac{2}{2\pi} = \dfrac{1}{\pi}$ and period of π.

$$\begin{aligned} P &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \left|x(t)\right|^2 dt \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\left(2t + 0.25\pi\right) dt \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[1 + \cos\left(4t + 0.5\pi\right)\right] dt \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2} dt + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\left(4t + 0.5\pi\right) dt \\ &= \frac{1}{2\pi}\left[t\right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] \\ &= \frac{1}{2} \end{aligned}$$

$$\boxed{\text{Note that } \int_{=\pi/2}^{\pi/2} \frac{1}{2}\cos\left(4t + 0.5\pi\right) dt = 0}$$

S2(b)

$$
\begin{aligned}
P &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |x(t)|^2\, dt \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \left|\frac{1}{2}\left[1+\cos(2t)\right]\right|^2 dt \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{4}(1+\cos^2(2t)dt \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} dt + \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \cos^2(2t)dt \\
&= \frac{1}{4\pi}\left[t\right]_{-\pi/2}^{\pi/2} + \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[1+\cos(4t)\right]dt \\
&= \frac{1}{4} + \frac{1}{8\pi}\int_{-\pi/2}^{\pi/2} dt + \frac{1}{8\pi}\int_{-\pi/2}^{\pi/2} \cos(4t)dt \\
&= \frac{1}{4} + \frac{1}{8\pi}\left[t\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{4} + \frac{1}{8} \\
&= \frac{3}{8}
\end{aligned}
$$

S2(c) $x(t)\cos(2\pi t)u(t)$ is not a periodic signal since $x(t)=0$ for $t<0$.

S2(d) $x(t)=e^{j\pi t}$; $f=0.5$; $T=2$; and $x(t)$ is periodic.

S3(a) $\int_{-\infty}^{t} \cos(\tau)u(\tau)d\tau = \int_{-\infty}^{t} \cos(\tau)d\tau = \sin(t)u(t)$

S3(b) $\int_{-\infty}^{t} \cos(\tau)\delta(\tau)d\tau = \int_{-\infty}^{t} \cos(0)\delta(\tau)d\tau = \int_{-\infty}^{t} 1.\delta(\tau)d\tau = u(t)$

S3(c)

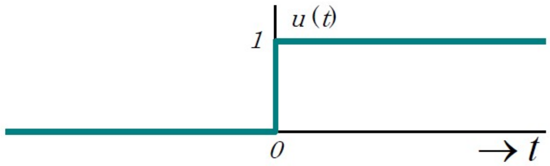
$$
\begin{aligned}
\int_0^{2\pi} t.\sin\left(\frac{t}{2}\right)\delta(\pi-t)dt &= \int_0^{2\pi} t.\sin\left(\frac{t}{2}\right)\delta\left(-(t-\pi)\right)dt \\
&= \int_0^{2\pi} \pi.\sin\left(\frac{\pi}{2}\right)\delta\left(-(t-\pi)\right)dt \\
&= \pi\int_0^{2\pi} \delta\left(-(t-\pi)\right)dt \\
&= \pi
\end{aligned}
$$

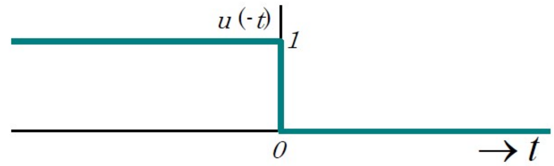
Note that $\int \delta\left(-(t-\pi)\right)dt = 1$, as it is the area within $\delta(-(t-\pi))$.

S4 $x(t) = x_x(t) + x_o(t)$; $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$; $x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$

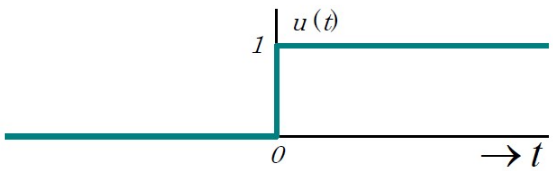
S4(a)

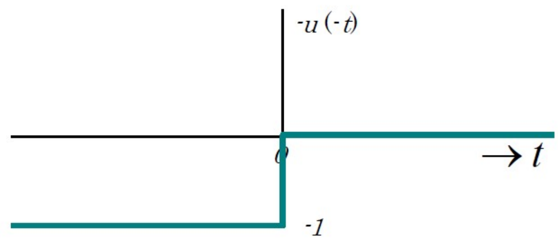
$$x_e(t) = \frac{1}{2}\left[u(t) + u(-t) = \begin{cases} 1, & t = 0 \\ 0.5, & t \neq 0 \end{cases}\right]$$





$$x_o(t) = \frac{1}{2}\left[u(t) - u(-t)\right] = \begin{cases} -0.5, & t < 0 \\ 0.5, & t \geq 0 \end{cases} = \frac{1}{2}\operatorname{sgn}(t)$$





S4(b)

$$x_e(t) = \frac{1}{2}\left[\sin(\omega_c t + \pi/4) + \sin(-\omega_c t + \pi/4)\right]$$
$$= \frac{1}{2}\left[2\sin\left\{\frac{1}{2}\left(\frac{\pi}{2}\right)\right\}\cos\left\{\frac{1}{2}\left(2\omega_c t\right)\right\}\right]$$
$$= \frac{1}{\sqrt{2}}\cos\left(\omega_c t\right)$$

Using:

$$\sin(A) + \sin(B) = 2\sin\left\{\frac{1}{2}\left(A+B\right)\right\}\cos\left\{\frac{1}{2}\left(A-B\right)\right\}$$

$$x_o(t) = \frac{1}{2}\left[\sin(\omega_c t + \pi/4) - \sin(-\omega_c t + \pi/4)\right]$$
$$= \frac{1}{2}\left[2\cos\left\{\frac{1}{2}\left(\frac{\pi}{2}\right)\right\}\sin\left\{\frac{1}{2}\left(2\omega_c t\right)\right\}\right]$$
$$= \frac{1}{\sqrt{2}}\sin\left(\omega_c t\right)$$

Using:

$$\sin(A) - \sin(B) = 2\cos\left\{\frac{1}{2}\left(A+B\right)\right\}\sin\left\{\frac{1}{2}\left(A-B\right)\right\}$$