Outline of Lecture

1. ESD, PSD and Bandwidth
2. Energy Spectral Density, ESD
3. Power Spectral Density, PSD
4. Bandwidth
    - 3-dB bandwidth
    - 1^{st} null bandwidth
    - $\eta\%$ energy and power containment bandwidths

ESD, PSD and Bandwidth

1. Energy Spectral Density (ESD) - essentially Energy Spectrum ESD describes how the energy of a signal is distributed across its frequency components. In time domain, the energy of a signal, $x(t)$, is given by :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x^*(t)x(t)dt \quad \text{(in Joules)}$$

The Rayleigh energy theorem provides an alternative method for computing the energy of a signal in the frequency domain as follows :

$$E = \overbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}^{\text{time domain}} = \overbrace{\int_{-\infty}^{\infty} |X(f)|^2 df}^{\text{frequency domain}} \quad \text{Rayleigh Energy Theorem}$$

where $X(f) = \mathcal{F}[x(t)]$. The integrand $|X(f)|^2$ can be interpreted as the energy density of $x(t)$ at frequency, f. Hence the ESD of a signal $x(t)$ is defined as :

$E_x(f) = |X(f)|^2$ ...Energy Spectral Density, ESD, in Joules/Hz

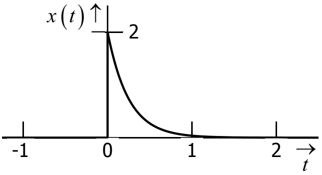
2. Properties of ESD, $E_x(f) = |X(f)|^2$
   - $E_x(f) = |X(f)|^2 \geq 0 \; \forall f$ is a real function of f because $|magnitude|$ is always positive and real.
   - $E_x(f)$ is an even function of f if $x(t)$ is real.

Example 1 (ESD and Energy of a Signal)

Consider the energy signal

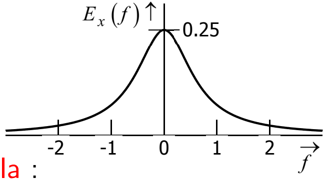
$$x(t) = 2e^{-4t}u(t).$$

Find the spectrum $X(f)$ *and ESD* $E_x(f)$ *of* $x(t)$. *Calculate the total energy* E *of* $x(t)$ *using the time- and frequency-domain formulae of* E.



$$\text{Spectrum}: \left\{ \begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} 2e^{-4t}e^{-j2\pi ft}dt \\ &= 2\frac{e^{-(2j\pi f+4)t}}{-(j2\pi f+4)}\Bigg|_0^{\infty} = \frac{1}{2+j\pi f} \end{aligned} \right.$$

ESD :
$$\begin{cases} E_x(f) &= |X(f)|^2 \\ &= \left|\dfrac{1}{2+j\pi f}\right|^2 = \dfrac{1}{4+\pi^2 f^2} \end{cases}$$



Energy calculated using the time domain formula :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} 4e^{-8t} dt = \left.\frac{4e^{-8t}}{-8}\right|_0^{\infty} = \frac{1}{2} \qquad (1)$$

Energy calculated using the frequency domain formula :

$$\begin{aligned} E &= \int_{-\infty}^{\infty} E_x(f) df = \int_{-\infty}^{\infty} \frac{1}{4+\pi^2 f^2} df \\ \ldots & \text{ let } \tan\theta = 0.5\pi f, \therefore \sec^2\theta d\theta = 0.5\pi df \\ &= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1}{1+\tan^2\theta}\frac{\sec^2\theta}{0.5\pi} d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1}{0.5\pi} d\theta = \frac{1}{2} \qquad (2) \end{aligned}$$

(1) and (2) verifies the Rayleigh theorem. The choice of calculating E in time or frequency domain depends on the complexity of the signal. In this example, computing E in the time domain is a little easier than computing E in the frequency domain.

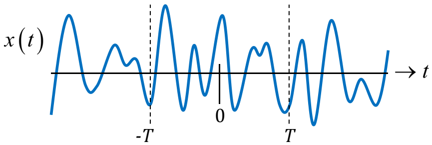
3. Power Spectral Density (PSD) - While the ESD applies to energy signal where $E < \infty$, for power signals where $E = \infty$, we define the power spectral density which describes how the power is distributed across its frequency components.

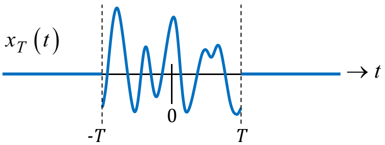
   In the time domain, power is defined as the average of the squared magnitude of the signal :

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

   To analyze the PSD, we begin with a truncated version of $x(t)$. We define :

$$x_T(t) = x(t)\text{rect}\left(\frac{t}{2T}\right) \text{ and } \lim_{T\to\infty} x_T(t) = x(t)$$





$x_T(t)$ is now an energy signal (since it is a pulse) and has a Fourier transform :

$$X_T(f) = \int_{-\infty}^{\infty} x_T(t) e^{-j2\pi f t} dt.$$

Applying Rayleigh theorem, we get :

$$\int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

Dividing by $2T$, and taking the limit as $T \to \infty$ leads to the Parseval power theorem :

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2 df$$

...Parseval power theorem

The integrand $\lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2$ can be interpreted as the spectral density (PSD) of the signal at frequency f. Hence

$$\text{PSD} \ = P_x(f) = \lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2 \quad \ldots \text{Watts / Hz}$$

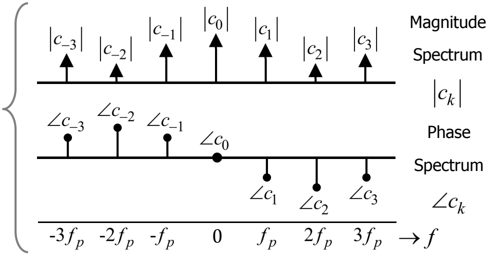
4. Properties of PSD
   - $P_x(f) = \lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2 \geq 0 \; \forall f$ is a real function of f because $|magnitude|$ is always positive and real.
   - $P_x(f)$ is an even function of f if $x(t)$ is real.
   - The formula for PSD appears complex but if the signal is periodic (and therefore is a power signal), the PSD can be calculated easily as seen in the next section.

5. PSD of Periodic Signals

Let f_p, T_p and c_k denote the fundamental frequency, period and Fourier series coefficient of a periodic signal $x_p(t)$.

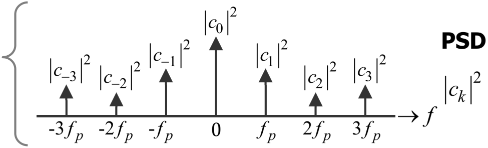
The continuous-frequency spectrum of $x_p(t)$ is :

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_p)$$



PSD of $x_p(t)$:

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - k f_p)$$



Average power of $x_p(t)$:

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

6. Proof of the average power formula, $P = \sum_{k=-\infty}^{\infty} |c_k|^2$

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x_p(t)|^2 dt \\
&= \frac{1}{2T_p} \int_{-T_p}^{T_p} x_p(t) x_p(t)^* dt \qquad \left(\begin{array}{c} x_p(t) \text{ is periodic, power can be} \\ \text{computed over 1 period} \end{array} \right) \\
&= \frac{1}{2T_p} \int_{-T_p}^{T_p} x_p(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi k f_p t} dt \\
&= \sum_{k=-\infty}^{\infty} c_k^* \underbrace{\frac{1}{2T_p} \int_{-T_p}^{T_p} x_p(t) e^{-j2\pi k f_p t} dt}_{\text{formula for } c_k} \\
&= \sum_{k=-\infty}^{\infty} c_k^* c_k \\
&= \sum_{k=-\infty}^{\infty} |c_k|^2
\end{aligned}
$$

Example 2 (PSD and Power of Periodic Signals)

Consider the periodic signal $x_p(t) = 2 + 4e^{j8\pi t} + 6\cos(16\pi t)$. Find its spectrum $X_p(f)$, spectral density $P_x(f)$ and average power, P.

Answer

$$x_p(t) = 2 + 4e^{j8\pi t} + 6\cos(16\pi t) = 2 + 4e^{j2\pi(4)t} + 3e^{j2\pi(8)t} + 3e^{-j2\pi(8)t}$$

Fundamental frequency : HCF $\{4, 8\} = 4$ Hz. $\therefore$ $f_p = 4$ Hz.
Comparing with the Fourier series expansion $\sum_k c_k e^{j2\pi k f_p t}$, we get :

$$c_k = \begin{cases} 2 & k = 0 \\ 4 & k = 1 \\ 3 & k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

Spectrum : $X_p(f) = \sum_k c_k \delta(f - 4k)$

PSD : $P_x(f) = \sum_k |c_k|^2 \delta(f - 4k)$

Ave. Power : $P = \sum_k |c_k|^2$

$$\begin{aligned} X_p(f) &= 3\delta(f+8) + 2\delta(f) + 4\delta(f-4) + 3\delta(f-8) \\ P_x(f) &= 9\delta(f+8) + 4\delta(f) + 16\delta(f-4) + 9\delta(f-8) \\ P &= 9 + 4 + 16 + 9 = 38 \text{ Watts} \end{aligned}$$

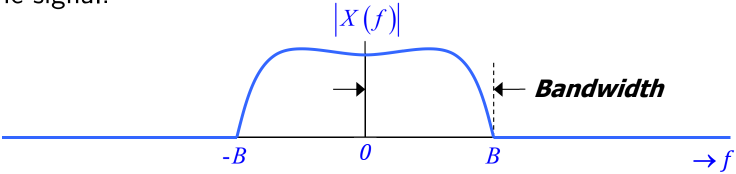
Bandwidth of Signals

The bandwidth of a signal $x(t)$ is a measure of the width of the range of frequencies occupied by its magnitude spectrum $X(f)$. It is a notion applied only to the frequency domain and not the time domain.

There are several definitions for bandwidth. The definitions are also dependent on the type of signals that you have : lowpass or bandpass

1. Bandlimited signals

- Lowpass Signal : A signal $x(t)$ is said to be a bandlimited lowpass signal if its magnitude spectrum is concentrated around 0 Hz, and satisfies $|X(f)| = 0, \ |f| > B$ where B is defined as the bandwidth of the signal.

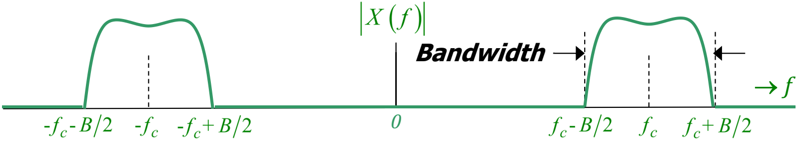


Note that when $x(t)$ is real, $|X(f)|$ is symmetric.

- Bandpass Signal : A signal $x(t)$ is said to be a bandlimited bandpass signal if its magnitude spectrum is concentrated around a non-zero center frequency, f_c, and at the same time satisfies

$$|X(f)| = 0, \; ||f| - f_c| > B/2$$

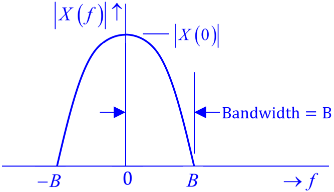
where B is defined as the bandwidth of the signal.

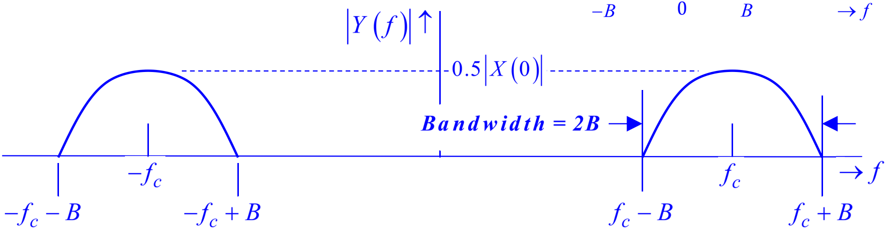


Example 3

Let B be the bandwidth of a bandlimited lowpass signal $x(t)$. Express the bandwidth of $y(t) = x(t)\cos(2\pi f_c t)$ in terms of B. Assume $f_c >> B$.

$$
\begin{aligned}
Y(f) &= X(f) * 0.5[\delta(f+f_c) + \delta(f-f_c)] \\
&= 0.5X(f+f_c) + 0.5X(f-f_c)
\end{aligned}
$$





Notice that $y(t)$ has been frequency shifted to around $\pm f_c$ and now has a bandwidth of $2B$.

2. Signals with Unrestricted Band

In practice, signals are seldom strictly bandlimited but have infinite frequency extent. Such signals are said to have unrestricted band.

The concept of infinite bandwidth presents difficulties in signal processing. For instant, consider the propagation of a signal with unrestricted band through a system with bandwidth B_s :

- If B_s is finite, the signal spectrum after going thru the system will be truncated and this may lead to an unacceptable level of signal distortion. To avoid signal distortion, $B_s \geq B$ where B is the input signal bandwidth.

- If $B_s \to \infty$ to accommodate the signal spectrum, the system noise will completely mask out the signal. This is because noise typically has a very wide bandwidth (sometimes refer to as white spectrum like white light which occupies the entire optical spectrum) because it tends to be random and hence contain high frequency components.

In signal processing, it is often useful to define a bandwidth measure to include only the "important part" of the signal spectrum. This bandwidth measure can then be used to estimate the bandwidth of the signal.

The choice of bandwidth measure is dependent on what we consider as the "important part" of the signal spectrum.

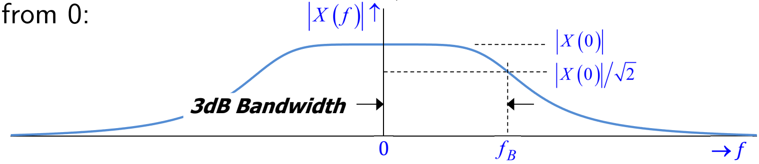
We will now introduce the notions of

- 3-dB bandwidth
- 1^{st} null bandwidth, and
- $\eta\%$ energy and power containment bandwidths

a. 3-dB bandwidth

- Lowpass signal, $x(t)$

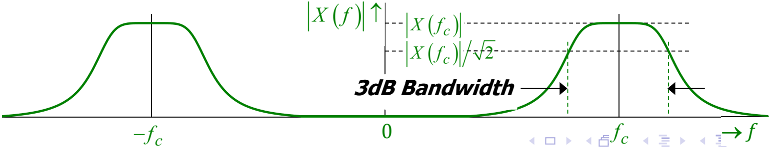
The 3-dB bandwidth of a lowpass signal $x(t)$ is defined as the frequency at which $|X(f)| = \dfrac{|X(0)|}{\sqrt{2}}$ first occurs when f is increased from 0:



f_B is the 3-dB b/w : $20\log_{10}|\frac{X(f_B)}{X(0)}| = 20\log_{10}(\frac{1}{\sqrt{2}}) = -3.01dB.$

- Bandpass signal, $x(t)$

Likewise, the 3dB bandwidth of a bandpass signal $x(t)$ with center frequency f_c is defined as illustrated below:



Example 4

Consider the lowpass Gaussian pulse $x(t) = e^{-t^2/2}$ *which has an energy spectral density given by*

$$E_x(f) = 2\pi e^{-4\pi^2 f^2}.$$

Find the 3-dB bandwidth of $x(t)$*.*

Answer

Let f_B be the 3-dB bandwidth of $x(t)$. By definition,

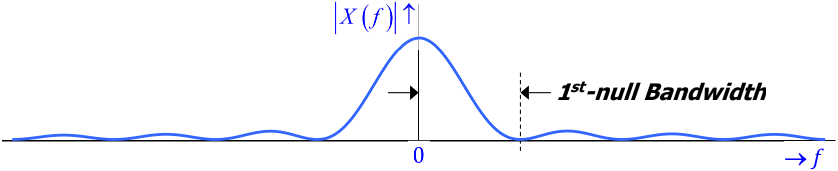
$$\frac{|X(f_B)|}{|X(0)|} = \frac{1}{\sqrt{2}} \quad \text{or} \quad \frac{|X(f_B)|^2}{|X(0)|^2} = \frac{1}{2}$$

Since $E_x(f) = |X(f)|^2$, it follows that

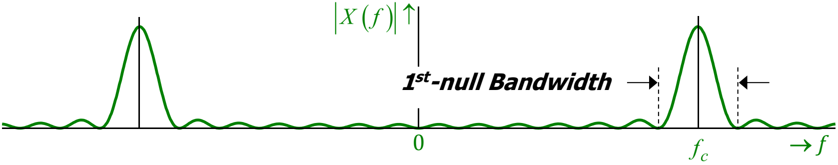
$$\frac{E_x(f_B)}{E_x(0)} = \frac{2\pi e^{-4\pi^2 f^2}}{2\pi} = \frac{1}{2} \implies f_B = \frac{\sqrt{\ln(2)}}{2\pi}$$

b. 1^{st}-null bandwidth

- Lowpass signal, $x(t)$ The 1^{st}-null bandwidth of a lowpass signal $x(t)$ is defined as the frequency at which $|X(f)| = 0$ first occurs when f is increased from 0:



- Bandpass signal, $x(t)$ Likewise, the 1^{st}-null bandwidth of a bandpass signal $x(t)$ with center frequency f_c is defined as illustrated below:



Example 5

What is the 1^{st}-null bandwidth of $x(t) = 5\textit{tri}(4t-8)$?

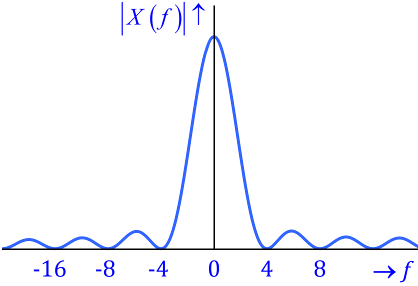
Answer

$x(t) = 5\text{tri}(4t-8) = 5\text{tri}\left(\frac{t-2}{0.25}\right)$

Recall $\mathcal{F}\left\{\text{tri}\left(\frac{t}{T}\right)\right\} = T\text{sinc}^2(fT)$.

Applying the time-shifting property of the Fourier transform, we obtain :

$$\begin{aligned} X(f) &= \frac{5}{4}\text{sinc}^2\left(\frac{f}{4}\right)e^{-j4\pi f} \\ \therefore |X(f)| &= \frac{5}{4}\text{sinc}^2\left(\frac{f}{4}\right) \end{aligned}$$



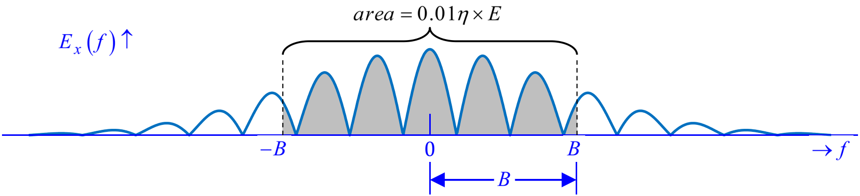
The nulls of $X(f)$ occur at $f = \pm 4, \pm 8, \pm 12, \ldots$ Hz.

Since the 1^{st}-null occurs at $f = 4$ Hz, the 1^{st}-null bandwidth of $x(t)$ is 4 Hz.

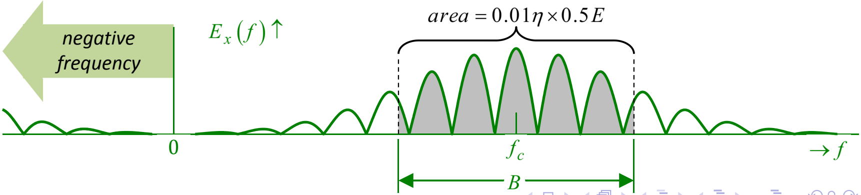
c. $\eta\%$ energy containment bandwidth

The $\eta\%$ energy containment bandwidth, B, of a real energy signal is the smallest bandwidth that contains at least $\eta\%$ of the total signal energy, $E = \int_{-\infty}^{\infty} E_x(f)df$.

- Lowpass signal, $x(t)$



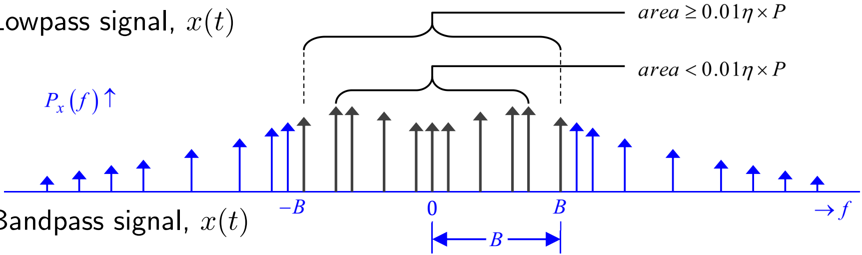
- Bandpass signal, $x(t)$



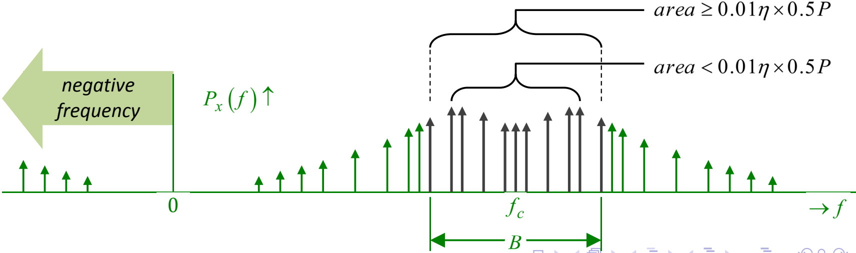
d. $\eta\%$ power containment bandwidth

The $\eta\%$ power containment bandwidth, B, of a real power signal is defined as the smallest bandwidth that contains at least $\eta\%$ of the average signal power $P = \int_{-\infty}^{\infty} P_x(f)df$.

- Lowpass signal, $x(t)$



- Bandpass signal, $x(t)$



Exercise 1

Determine whether each of the following signals is an energy signal, power signal or neither.

1. $x_1(t) = e^{-\alpha t}u(t),\ \alpha > 0$
2. $x_2(t) = \alpha t u(t),\ \alpha > 0$
3. $x_3(t) = \alpha\cos(2\pi t + \beta),\ \alpha,\ \beta > 0$

Exercise 2

The spectrum of a Gaussian pulse

$$x(t) = e^{-\frac{t^2}{2\sigma^2}} \quad \sigma > 0$$

has the form

$$X(f) = \sigma\sqrt{2\pi}e^{-2\sigma^2\pi^2 f^2}.$$

where σ *is a measure of the time spread of* $x(t)$*. Let* B *denote the 3-dB bandwidth of* $x(t)$*. Compute the time-bandwidth product* σB *and comment on the result.*