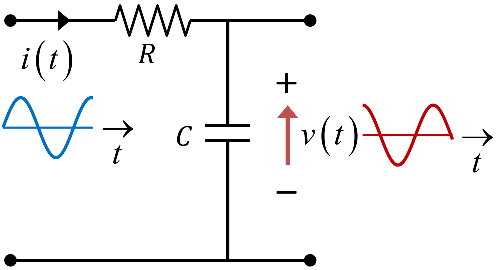
Outline of Lecture

Signals

- Signals can manifest in many forms such as electrical voltage or current, radio wave, infrared and ultraviolet rays, lightwave, sound wave, mechanical pressure, etc.
- When a signal is expressed as a function of time, we have what is called the time-domain representation of the signal

For example, in a RC circuit, the voltage $v(t)$ may represent the output signal and the current $i(t)$ may represent the input signal.

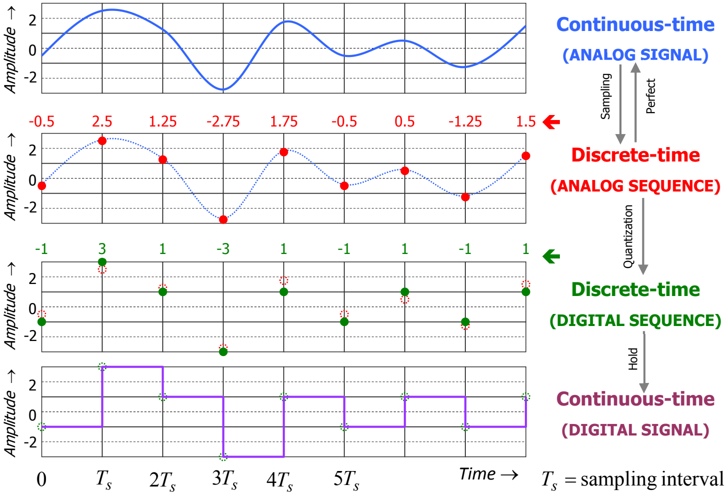


Classification of Signals

1. Continuous time signal : A continuous time signal is one that is defined for all time instants, t. It is usually represented as a waveform, $x(t)$.
    - Analog signal - If $x(t)$ can take any value in the interval $(-\infty, +\infty)$, then it is considered an analog signal
    - Digital signal - If $x(t)$ can take only values from a finite set $(\alpha_1, \alpha_2, ..., \alpha_m)$, where $\alpha_i \epsilon (-\infty, +\infty)$ at every time instant, then it is considered a digital signal

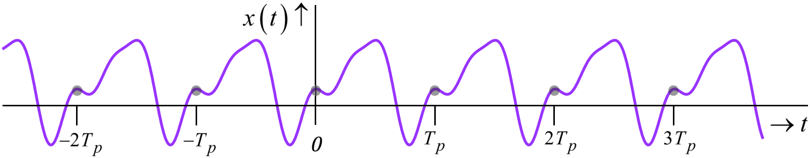
2. Discrete time signal : A discrete time signal is one that is defined only at specific time instants, t_i. It is usually represented as a sequence of numbers, $(n), n = 0, 1, 2, \ldots$.
    - If $x(n)$ can take on any values in the interval $(-\infty, +\infty)$, then it is considered an analog sequence
    - If $x(n)$ can take only values from a finite set $(\alpha_1, \alpha_2, ..., \alpha_m)$, where $\alpha_i \epsilon (-\infty, +\infty)$ at any time instant, then it is considered a digital sequence

Illustration of Continuous and Digital Signals



3. Periodic and Aperiodic Signals

A periodic signal $x_p(t)$ is one which satisfies the condition where $x_p(t) = x_p(t + T_p)$ where $T_p > 0$. The smallest value of T_p is also called the fundamental period or simply the period of $x_p(t)$.



This signal satisfies $x(t) = x(t + T_p)$.
Shortest time interval of repetition $= T_p$
Therefore fundamental period $= T_p$ seconds
Fundamental frequency $= 1/T_p$ Hz or $2\pi/T_p$ rad/sec

Any signal that is not periodic is aperiodic.

4. Real and Complex Signals
A complex signal $x(t)$ is a signal that can be expressed as $x(t) = a(t) + jb(t)$ where $a(t)$ and $b(t)$ are real signals and $j = \sqrt{-1}$.

- Cartesian or rectangular form of $x(t)$ where

$$\begin{aligned} x(t) &= a(t) + jb(t) \quad (1) \\ \text{where } a(t) &= \text{Re}[x(t)], \text{ Real part of } x(t) \\ b(t) &= \text{Im}[x(t)], \text{ Imaginary part of } x(t) \end{aligned}$$

- Exponential form of $x(t)$

$$\begin{aligned} x(t) &= r(t)e^{j\theta(t)} \quad (2) \\ \text{where } r(t) &= |x(t)|, \text{ magnitude or modulus of } x(t) \\ \theta(t) &= \angle x(t), \text{ Phase or argument of } x(t) \end{aligned}$$

- Relationship between Cartesian and exponential forms of $x(t)$

Applying Euler's formula,

$e^{j\theta(t)} = \cos\theta(t) + j\sin\theta(t)$

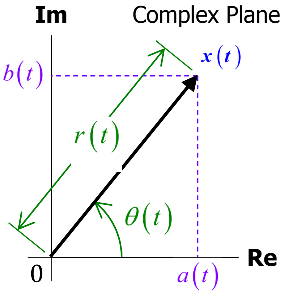
to (2), we get

$$\begin{aligned} x(t) &= r(t)e^{j\theta(t)} \\ &= r(t)[\cos\theta(t) + j\sin\theta(t)] \end{aligned}$$

Comparing this to (1), leads to

$$\begin{aligned} a(t) &= r(t)\cos\theta(t) \\ b(t) &= r(t)\sin\theta(t) \end{aligned}$$

$$\begin{aligned} r(t) &= \sqrt{a^2(t) + b^2(t)} \\ \theta(t) &= \tan^{-1}\left(\frac{b(t)}{a(t)}\right) \end{aligned}$$



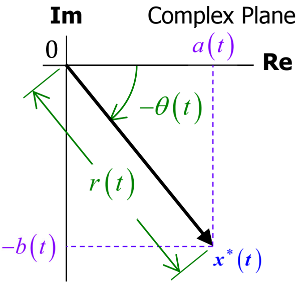
Phasor Diagram of $x(t)$

- The conjugate of a complex signal $x(t)$ is denoted by $x^*(t)$ and is obtained by replacing all j in $x(t)$ by $-j$ ie $x^*(t) = a(t) - jb(t)$. The consequence of replacing j by $-j$ is that the phase of $x^*(t)$ is negative of the phase of $x(t)$.

$$\begin{aligned} \angle x(t) &= \theta(t) \\ \angle x^*(t) &= -\theta(t) \end{aligned}$$

  But the magnitude of $x(t)$ is the same as that of $x^*(t)$ ie $|x(t)| = |x^*(t)|$

- A real signal is a special case of the complex signal where $[b(t) = 0]$ or $\theta(t) = \pm n\pi,\ n = 0, 1, 2, 3, \ldots$
- A real signal can be made up of 2 complex signals $x(t) = z_1(t) + z_2(t)$ if and only if the components are complex conjugate of one another ie $z_1(t) = z_2^*(t)$.



Phasor Diagram of $x^*(t)$

5. Deterministic and Random Signals
   - A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table. Because of this, the future values of the signal can be predicted with complete confidence.

Example 1

$x(t) = \cos 2\pi t$. *At any time* $t = t_0$, $x(t_0)$ *is exactly determined as* $\cos 2\pi t_0$

   - A random signal has a lot of uncertainty about its behavior. The future values of the signal cannot be accurately predicted and can only be guessed (or estimated) based on the signal statistics and observation of past outcomes.

Example 2

$x(t) = \cos(2\pi t + \phi)$ *where* ϕ *can take any values in the set* $\{0, 0.5\pi, \pi\}$ *with equal probabilities. At any time* $t = t_0$, $x(t_0)$ *can be any one of the values* $\cos(2\pi t_0)$, $\cos(2\pi t_0 + 0.5\pi)$ *or* $\cos(2\pi t_0 + \pi)$ *with equal probability.*

6. Energy and Power Signals
   - The total energy, E, of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

   $x(t)$ is said to be an energy signal if and only if $0 < E < \infty$.

   - The average power, P, of a signal $x(t)$ is defined as

$$P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt = \frac{1}{2\infty} E$$

   $x(t)$ is said to be a power signal if and only if $0 < P < \infty$.

   - Signals can be neither energy nor power signals. It can be either energy or power but not both.

Implications of $P = \frac{1}{2\infty} E$

Energy signals have zero average power because E=finite and $P = 0$
Power signals have infinite total energy because P=finite and $E = \infty$

Example 3

$$x(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Total energy, $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2\alpha t} dt = \left[\frac{e^{-2\alpha t}}{-2\alpha t}\right]_0^{\infty} = \frac{1}{2\alpha}$

Total finite energy implies zero average power $\Rightarrow$ $x(t)$ *is an energy signal.*

Example 4

$$x(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Total energy, $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} t^2 dt = \left[\frac{t^3}{3}\right]_0^{\infty} = \infty$

Infinite total energy and average power $\Rightarrow$ $x(t)$ *is neither an energy nor power signal.*

Example 5

$x(t) = \alpha \cos \frac{2\pi t}{T}$ - $x(t)$ is periodic with period T (prove it).
Is $x(t)$ an energy or power signal? Since $x(t)$ is periodic, its average power can be computed by averaging over 1 period.

$$
\begin{aligned}
P = \overbrace{\lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt}^{\textit{original defn of av. power}} &= \overbrace{\frac{1}{T} \int_0^T |x(t)|^2 dt}^{\textit{averaged over 1 period}} \\
&= \frac{1}{T} \int_0^T \alpha^2 \cos^2 \left(\frac{2\pi t}{T} \right) dt \\
&= \frac{\alpha^2}{2T} \int_0^T 1 + \cos \left(\frac{4\pi t}{T} \right) dt = \frac{\alpha^2}{2}
\end{aligned}
$$

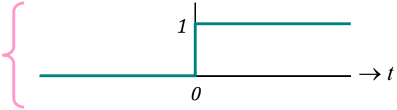
Conclude that $x(t)$ is a power signal.

In fact, all bounded periodic signals are power signals.

Basic Signals

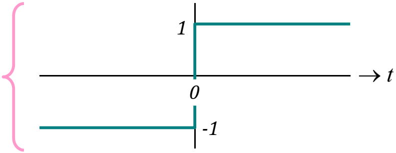
A. The Unit Step Function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



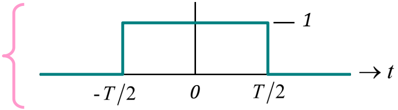
B. The Sign or Signum Function

$$\text{sign}(t) = \begin{cases} +1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$



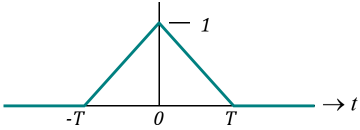
C. The Rectangle Function

$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{elsewhere} \end{cases}$$



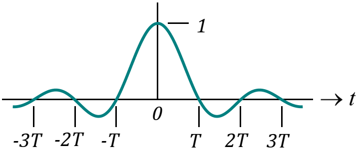
D. The Triangle Function

$$\text{tri}\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{|t|}{T} & |t| \leq T \\ 0 & |t| > T \end{cases}$$



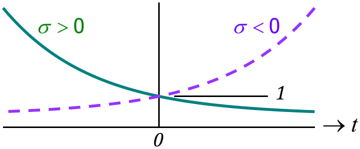
E. The Sinc or Sign Cardinal Function

$$\text{sinc}\left(\frac{t}{T}\right) = \begin{cases} \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} & t \neq 0 \\ 1 & t = 0 \end{cases}$$



E. The Real Exponential Function

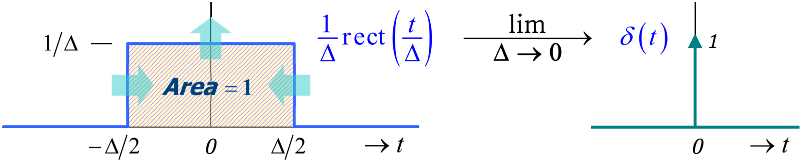
$$x(t) = e^{-\sigma t}$$



G. The Unit Impulse (or Dirac-δ) Function

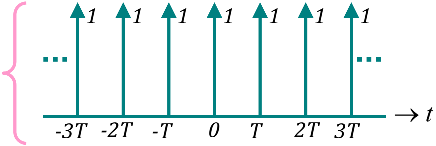
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\epsilon}^{\epsilon} \delta(t)dt = 1 \ \forall \epsilon$$

We may view the unit impulse as a limiting case of a rectangle pulse which has a unit area that is independent of its pulse width:



H. The Dirac Comb Function

$\Xi(t) = \Sigma_{n=-\infty}^{\infty} \delta(t - nT)$

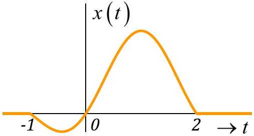


Time-Scaling, -Reversal and -Shifting of Signals

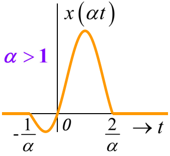
A. Time-scaling of signals

Time-scaling of $x(t)$ is effected by replacing the time t by αt, where α is a positive real number.

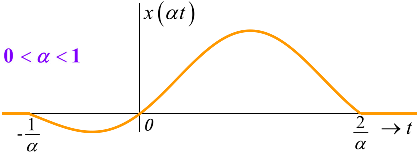
$$\begin{aligned} 0 < \alpha < 1 \quad &: \quad \text{uniform expansion of } x(t) \text{ along the time axis} \\ \alpha > 1 \quad &: \quad \text{uniform contraction of } x(t) \text{ along the time axis} \end{aligned}$$



original $x(t)$



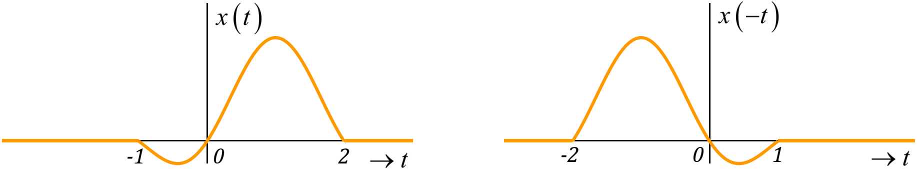
contracted $x(t)$



expanded $x(t)$

B. Time-reversal of signals

Time-reversal of $x(t)$ is effected by replacing the time t by $-t$

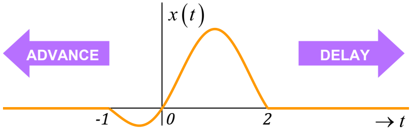


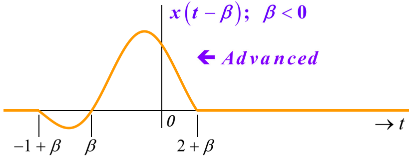
Notice how the signal has been flipped over the y-axis.

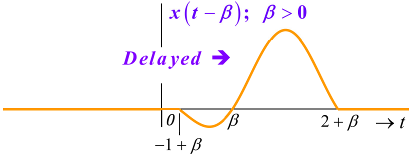
C. Time-shifting of signals

Time-shifting of $x(t)$ is effected by replacing the time t by $(t - \beta)$, where β is a real number.

$$\beta > 0 \quad : \quad x(t) \text{ delayed by } \beta \text{ unit of time}$$
$$\beta < 0 \quad : \quad x(t) \text{ advanced by } \beta \text{ unit of time}$$







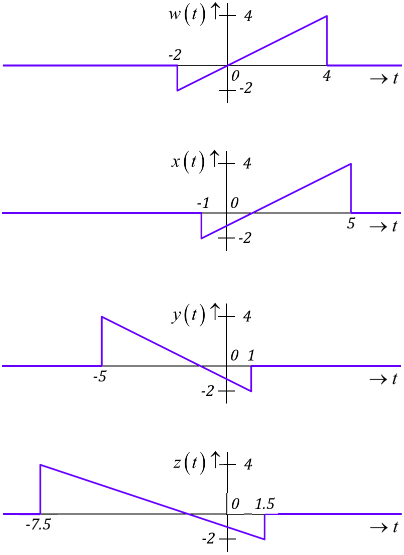
Example 6

$$\textit{Given } w(t) = \begin{cases} t & -2 \le t < 4 \\ 0 & \textit{elsewhere} \end{cases}$$

$$\textit{Sketch } \begin{cases} x(t) = w(t-1) \\ y(t) = x(-t) \\ z(t) = y\left(\frac{2}{3}t\right) \end{cases}$$

$$z(t) = y\left(\frac{2}{3}t\right) = \overbrace{x\left(-\frac{2}{3}t\right)}^{\textit{sub } t=\frac{2}{3}t \textit{ in } x(-t)}$$

$$= \underbrace{w\left(-\frac{2}{3}t - 1\right)}_{\textit{sub } t=-\frac{2}{3}t \textit{ in } w(t-1)}$$



Exercise 1 (Use the space below)

Sketch $x_1(t) = 3rect\left(\frac{t}{5}\right)$*. Then sketch* $x_1(0.2t)$*,* $x_1(5t)$ *and* $x_1(t-1)$*.*

Exercise 2 (Use the space below)

Sketch $x_2(t) = 2\,\textit{sinc}(4t)$. *Then sketch* $x_3(f) = 2\,\textit{sinc}(4f)$.

- *Identify the zero crossings of* $x_2(t)$ *and* $x_3(f)$.
- *What is the difference between* $x_2(t)$ *and* $x_3(f)$*?*
- *What will* $x_4(t) = |2\,\textit{sinc}(4t)|$ *look like?*