National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems Tutorial 6

Section I : Exercises that are straightforward applications of the concepts covered in class. Please attempt these problems on your own.

1. Consider the following transfer function :

$$G(s) = \frac{Y(s)}{X(s)} = \frac{s+9}{s^2+6s+13}$$

   (a) Write down the differential equation relating $y(t)$ and $x(t)$. What values should the output signal (initial conditions) assume when $t=0$ for the transfer function to hold ?

   ANSWER : $\ddot{y}(t) + 6\dot{y}(t) + 13y(t) = \dot{x}(t) + 9x(t)$; $\dot{y}(0) = y(0) = 0$

   (b) Suppose $x(t)$ is a step function of magnitude 2. Determine the steady-state value of $y(t)$

   - by performing inverse Laplace Transform.
   - using the Final Value Theorem.

   ANSWER : Steady-state value of $y(t) = \frac{18}{13}$

2. According to the convolution theorem, the unit step response of a system is

$$y(t) = \int_0^t 150e^{-0.5\tau}\sin(0.5\tau)U(t-\tau)d\tau$$

   where $U(t)$ is the unit step function. What is the system transfer function ?

   ANSWER : $\dfrac{75}{s^2+s+0.5}$

Section II : Problems that will be discussed in class.

1. Consider the electrical circuit shown in Figure 1. Derive the transfer function $\dfrac{I_1(s)}{I(s)}$, where $\mathcal{L}\{i(t)\} = I(s)$ and $\mathcal{L}\{i_1(t)\} = I_1(s)$. The assumptions made in the derivation of the transfer function should be clearly stated.

   ANSWER : $\dfrac{I_1(s)}{I(s)} = \dfrac{1}{LCs^2 + R_1Cs + 1}$

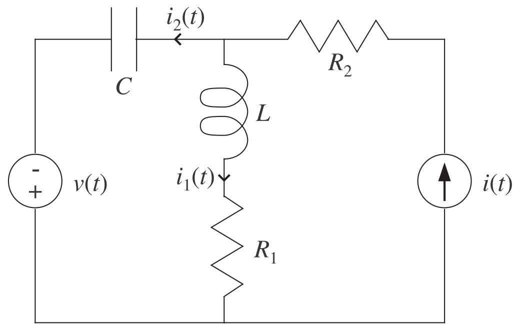

Figure 1: Electrical circuit

2. The input-output relationship of a thermometer can be modelled by the following transfer function :

$$5\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = 0.99x(t)$$

where $x(t)$ is the temperature of the environment in which the thermometer is placed,
$y(t)$ is the measured temperature.

The thermometer is inserted into a heat bath maintained at a constant temperature and the thermometer reading is allowed to stabilised before the temperature of the water in the heat bath is increased at a steady rate of 1^oC/second. Assume that $t = 0$ at the instant when the hot bath temperature starts to increase.

(a) Suppose the measured temperature is 24.75^oC when $t = 0$ i.e. $y(0) = 24.75^o$C. What is the temperature of the heat bath ? ANSWER : $x(0) = 25^o$C

(b) Write a mathematical expression to represent the temperature in the heat bath, $x(t)$. Then, solve the differential equation to obtain the time-domain expression for the measured temperature, $y(t)$. ANSWER : $y(t) = 19.8 + 0.99t + 4.95e^{-\frac{t}{5}}$

(c) What is the transfer function representation of the thermometer ? ANSWER : $G(s) = \frac{0.99}{5s+1}$

(d) Let $y(t) = y_1(t) + y(0)$ and $x(t) = x_1(t) + x(0)$. Derive an expression for the time-domain expression for the measured temperature, $y(t)$, using the transfer function of the thermometer obtained in part (c).

3. For the following linear time-invariant continuous time systems, determine if the system is BIBO stable, marginally stable or unstable.

(a) Transient response is $e^{-t} + e^{2t}$ for $t \geq 0$.

(b) Transient response is $\sin 2t$ for $t \geq 0$.

(c) Transient response is $e^{-t}\sin 2t$ for $t \geq 0$.

(d) Differential equation representation is $\ddot{y}(t) - \dot{y}(t) - 6y(t) = 4x(t)$.

(e) Transfer function is $\dfrac{s+3}{s^2+3}$.

(f) Transfer function is $\dfrac{4}{(s^2+4)^2}$.

(g) Transfer function is $\dfrac{2s-1}{s^2+2s+4}$.

(h) System response is $2t - \frac{2}{5} + \frac{2}{5}e^{-5t}$ when the input signal is the ramp function, t.

ANSWER : (a) Unstable; (b) Marginally stable; (c) Stable; (d) Unstable; (e) Marginally stable; (f) Unstable; (d) Stable; (h) Stable

4. The behaviour of an air heating system may be described by the following differential equation :

$$RC\frac{d\theta_o(t)}{dt} + \theta_o(t) = Rh(t)$$

where $h(t)$ is the heat input (system input), R is the thermal resistance, and C is the thermal capacitance.

Figure 2 shows the outlet air temperature, $\theta_o(t)$, when the system input is an unit impulse function, i.e. $h(t) = \delta(t)$, under zero initial conditions.

(a) Show that the unit impulse response of the air heating system is $\theta_o(t) = \frac{1}{C}e^{-\frac{t}{RC}}$.

(b) From Figure 2, estimate the thermal resistance, R, and the thermal capacitance, C, of the air heating system.

ANSWER : $R = 30$ and $C = 0.1$

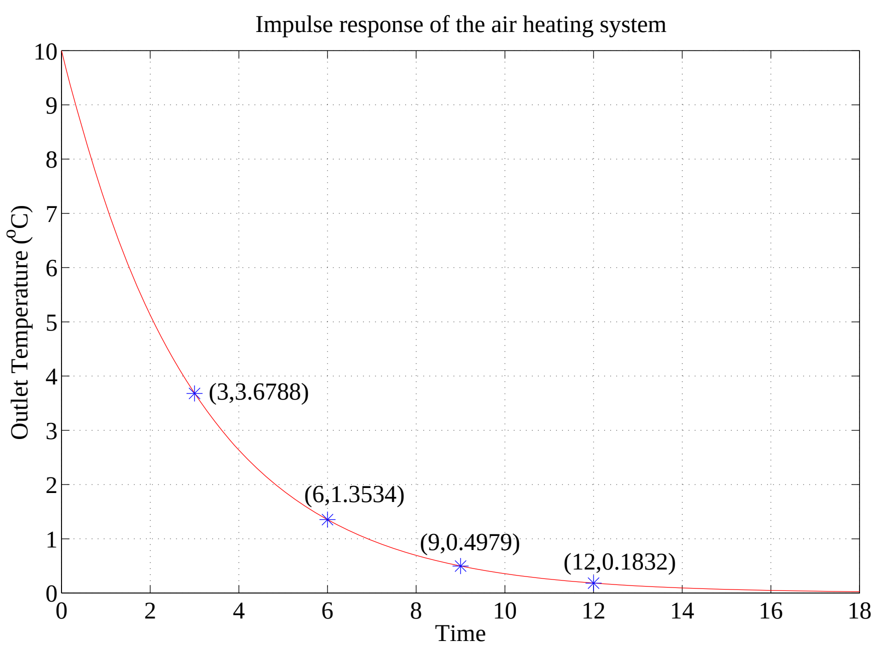

Figure 2: Impulse response of the air heating system

Section III : Practice Problems. These problems will not be discussed in class.

1. Consider the electrical circuit shown in Figure 1. Derive the transfer function $\dfrac{I_2(s)}{I(s)}$ and $\dfrac{I_1(s)}{V(s)}$, where $\mathcal{L}\{i(t)\} = I(s)$, $\mathcal{L}\{i_1(t)\} = I_1(s)$, $\mathcal{L}\{i_2(t)\} = I_2(s)$. and $\mathcal{L}\{V(t)\} = V(s)$.

$$\text{ANSWER}: \frac{I_2(s)}{I(s)} = \frac{s^2LC + sR_1C}{LCs^2 + R_1Cs + 1}$$
$$\frac{I_1(s)}{V(s)} = -\frac{sC}{LCs^2 + R_1Cs + 1}$$

2. Let the input signal, output signal and transfer function of a system be $u(t)$, $y(t)$ and $G(s)$ respectively. When the input signal is a step function of magnitude 4,

   - the steady-state output signal, $\lim_{t\to\infty} y(t)$, is 8, and
   - the poles of $Y(s) = \mathcal{L}\{y(t)\}$ are $s = 0, -3, -7 \pm 5j$.

   What is the system transfer function $G(s)$? Is the system stable, marginally stable or unstable ?

$$\text{ANSWER}: G(s) = \frac{444}{(s+3)(s^2+14s+74)}; \text{Stable}$$