# EE2023 Signals & Systems
# Chapter 7 – Laplace transform Review

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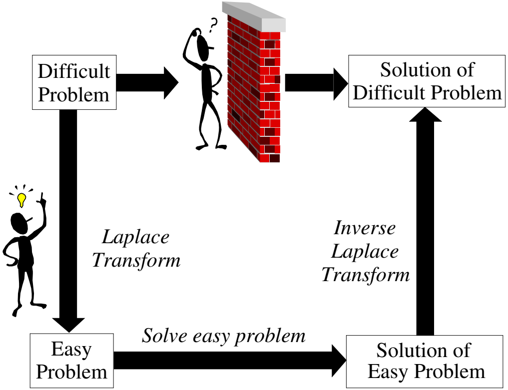
# What is Laplace transform ?

Laplace transform is an integral transform, that is particularly useful for solving linear ordinary differential equation.




Problem is simplified by converting a differential equation (time-domain) into an algebraic equation in the $s$-domain (complex frequency domain).

Differentiation ➠ Multiplication
Integration ➠ Division

- Since the input-output mapping of LTI systems is defined by linear differential equations, Laplace transform is the tool for analysising and characterisating LTI systems.

# Laplace transform – Definition

- The **unilateral** Laplace transform of a time-domain function, $f(t)$, is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\, dt$$

- If $f(t)$ are signals associated with a causal system, i.e. $f(t) = 0$ for $t < 0$, then the definition of the unilateral Laplace transform is equivalent to

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}\, dt$$

  The above equation is known as the **bilateral** Laplace transform.
- In this course, unless otherwise specified, we will assume that all time-domain functions are causal and adopt the unilateral Lapalce transform as "the Laplace transform".

# Laplace transform Table

| Time-domain functions, $x(t)$ | | $s$-domain functions, $X(s)$ |
|---|---|---|
| **Unit Impulse** | $\delta(t)$ | $1$ |
| **Unit Step** | $u(t)$ | $1/s$ |
| **Ramp** | $t\,u(t)$ | $1/s^2$ |
| **n$^{\text{th}}$ order Ramp** | $t^n u(t)$ | $\dfrac{n!}{s^{n+1}}$ |
| **Damped Ramp** | $t\exp(-\alpha t)u(t)$ | $1/(s+\alpha)^2$ |
| **Exponential** | $\exp(-\alpha t)u(t)$ | $1/(s+\alpha)$ |
| **Cosine** | $\cos(\omega_o t)u(t)$ | $s/(s^2+\omega_o^2)$ |
| **Sine** | $\sin(\omega_o t)u(t)$ | $\omega_o/(s^2+\omega_o^2)$ |
| **Damped Cosine** | $\exp(-\alpha t)\cos(\omega_o t)u(t)$ | $\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$ |
| **Damped Sine** | $\exp(-\alpha t)\sin(\omega_o t)u(t)$ | $\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$ |

# Laplace transform Properties

| | Time-domain | *s*-domain |
|---|---|---|
| **Linearity** | $\alpha x_1(t)+\beta x_2(t)$ | $\alpha X_1(s)+\beta X_2(s)$ |
| **Time shifting** | $x(t-t_o)u(t-t_o)$ | $\exp(-st_o)X(s)$ |
| **Shifting in the s-domain** | $\exp(s_o t)x(t)$ | $X(s-s_o)$ |
| **Time scaling** | $x(\alpha t)$ | $\frac{1}{\lvert\alpha\rvert}X\left(\frac{s}{\alpha}\right)$ |
| **Integration in the time-domain** | $\int_{0^-}^{t} x(\zeta)\,d\zeta$ | $\frac{1}{s}X(s)$ |
| **Differentiation in the time-domain** | $\frac{dx(t)}{dt}$ | $sX(s)-x(0^-)$ |
| | $\frac{d^n x(t)}{dt^n}$ | $s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\frac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$ |
| **Differentiation in the s-domain** | $-tx(t)$ | $\frac{dX(s)}{ds}$ |
| | $(-t)^n x(t)$ | $\frac{d^n X(s)}{ds^n}$ |
| **Convolution in the time-domain** | $\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$ | $X_1(s)X_2(s)$ |
| **Initial value theorem** | $x(0^+)=\lim_{s\to\infty} sX(s)$ | |
| **Final value theorem** | $\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$ | |

# Laplace transform Properties – Differentiation in time-domain

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = sF(s) - f(0^-) \qquad \mathcal{L}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^k(0^-)$$

**Example**

Differential equation relating the input, $v(t)$, to the output, $v_c(t)$, in a series $RC$ circuit is

$$RC\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} + v_c(t) = v(t)$$

Let $\mathcal{L}\{v_c(t)\} = V_c(s)$ and $\mathcal{L}\{v(t)\} = V(s)$. Applying the differentiation in time-domain property, the $s$-domain expression is

$$\underbrace{RC[sV_c(s) - v_c(0^-)] + V_c(s) = V(s)}_{\text{Algebraic equation}}$$

$$\begin{aligned}(sRC+1)V_c(s) &= RCv_c(0^-) + V(s)\\ V_c(s) &= \frac{RCv_c(0^-)}{sRC+1} + \frac{V(s)}{(sRC+1)}\end{aligned}$$

# Laplace transform Properties – Shift in $s$-domain

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$$

**Example**

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+3}\right\} = \frac{1}{\sqrt{3}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}\right\} \\
&= \frac{1}{\sqrt{3}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s_1^2+(\sqrt{3})^2}\right\};\ s_1 = s+1
\end{aligned}$$

As $\mathcal{L}\{\sin \omega t\} = \dfrac{\omega}{s^2+\omega^2}$ and $\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+4}\right\} = \frac{1}{\sqrt{3}}\, e^{-t} \sin \sqrt{3}t$$

# Laplace transform Properties – Final Value Theorem (FVT)

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- Let $f(t) = 1 + e^{-2t}$. Then, $F(s) = \mathcal{L}\{1 + e^{-2t}\} = \dfrac{1}{s} + \dfrac{1}{s+2}$.

$$\lim_{s=0} sF(s) = \lim_{s=0} s\left[\frac{1}{s} + \frac{1}{s+2}\right] = 1 = \lim_{t\to\infty} f(t)$$

  FVT correctly predicts the “final value” of the signal $f(t)$.

- Let $f(t) = \sin\omega t$. Then, $F(s) = \mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$.

$$\lim_{s=0} sF(s) = \lim_{s=0} s\frac{\omega}{s^2+\omega^2} = 0 \neq \lim_{t\to\infty} f(t)$$

  The answer derived via FVT “appears reasonable”, but $\lim\limits_{t\to\infty} \sin\omega t$ is not defined. Hence, it is essentially to check that the function tends to a final constant as $t \to \infty$ before applying FVT.

# Inverse Laplace transform

- The inverse Laplace transform, $\mathcal{L}^{-1}$, of $F(s)$ is defined as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\, ds$$

  where $c = Re[s]$ is a vertical line that lies inside the Region of Convergence (ROC) of $F(s)$. Evaluation of the integral requires knowledge of complex variable theory.

- **Special Case**: If $F(s)$ is a rational function of the form $\frac{C(s)}{D(s)}$, where $C(s)$ and $D(s)$ are polynomials in $s$, the alternative method is
  - Expand $F(s)$ into a sum of partial fractions, namely $F(s) = \dfrac{C(s)}{D(s)} = \displaystyle\sum_{n=1}^{N} \frac{C_k(s)}{D_k(s)}$
  - Due to the linearity propery of Laplace transform, the inverse tranform of $F(s)$ may be expressed as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\sum_{n=1}^{N} \frac{C_n(s)}{D_n(s)}\right\} = \sum_{n=1}^{N} \mathcal{L}^{-1}\left\{\frac{C_n(s)}{D_n(s)}\right\}$$

  where $\mathcal{L}^{-1}\left\{\dfrac{C_n(s)}{D_n(s)}\right\}$ is obtained from Laplace transform tables.

# Inverse Laplace transform – Partial Fraction Expansion

Let $F(s) = \dfrac{C(s)}{D(s)} = \dfrac{K'(s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_0)}{s^n + b_{n-1}s^{n-1} + \ldots b_1 s + b_0}$ where $K'$ = constant and $m < n$.

- $D(s) = (s+p_1)(s+p_2)\ldots(s+p_n)$ i.e. $D(s)$ can be factorised into distinct first order polynomials. Then, the partial fraction expansion is

$$F(s) = \frac{C(s)}{(s+p_1)(s+p_2)\ldots(s+p_N)} = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \ldots + \frac{\alpha_n}{s+p_n}$$

- $D(s) = (s+p_1)^r \ldots (s+p_{n-r})$ i.e. $D(s)$ has $r$ **repeated** first order factors and $(n-r)$ distinct first order factors. The partial fraction expansion is

$$F(s) = \frac{C(s)}{(s+p_1)^r \ldots (s+p_{n-r})} = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{(s+p_1)^2} + \ldots + \frac{\alpha_r}{(s+p_2)^r} + \ldots + \frac{\alpha_{n-r}}{s+p_{n-r}}$$

- $D(s) = (s^2+bs+c)\ldots(s+p_{n-2})$ i.e. $D(s)$ has a quadratic factor and $(n-2)$ distinct first order factors.

$$F(s) = \frac{C(s)}{(s^2+bs+c)\ldots(s+p_{n-2})} = \frac{\beta s + \alpha_1}{(s^2+bs+c)} + \ldots + \frac{\alpha_{n-2}}{s+p_{n-2}}$$

# Inverse Laplace transform via Partial Fraction Expansion – Example

Find the inverse LT of $F(s) = \dfrac{2}{(s+1)(s+2)}$

$$\begin{aligned} F(s) = \frac{2}{(s+1)(s+2)} &= \frac{A_1}{s+1} + \frac{A_2}{s+2} \\ &= \frac{A_1(s+2) + A_2(s+1)}{(s+1)(s+2)} \end{aligned}$$

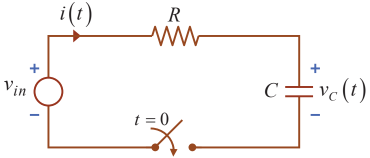
Comparing coefficients, $2 = A_1(s+2) + A_2(s+1)$

$$\begin{array}{lll} \text{Let } s = -2: & 2 = A_2(-2+1) & \Rightarrow A_2 = -2 \\ \text{Let } s = -1: & 2 = A_1(-1+2) & \Rightarrow A_1 = 2 \end{array}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{2}{s+2}\right\} \\ &= 2e^{-t} - 2e^{-2t} \end{aligned}$$

# Laplace transform method for solving DEs – Series RC circuit

Use Laplace Transform to derive the voltage across the capacitor, $v_c(t)$, if $v(t) = v_{in}u(t)$ and the initial condition is $v_c(0^-)$



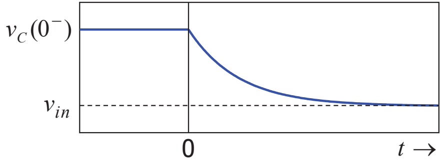

Differential equation model : $RC\dfrac{\mathrm{d}v_c(t)}{\mathrm{d}t} + v_c(t) = v_{in} \cdot u(t)$

Let $\mathcal{L}\{v_c(t)\} = V_c(s)$. Applying the differentiation in time-domain property, the $s$-domain expression is

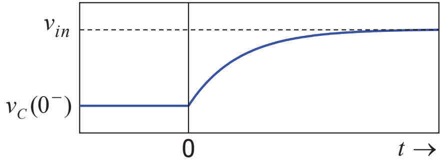
$$\begin{aligned} sRCV_c(s) - RCv_c(0^-) + V_c(s) &= \frac{v_{in}}{s} \\ (sRC+1)V_c(s) &= RCv_c(0^-) + \frac{v_{in}}{s} \\ V_c(s) &= \frac{RCv_c(0^-)}{sRC+1} + \frac{v_{in}}{s(sRC+1)} \end{aligned}$$

$$
\begin{aligned}
V_c(s) &= \frac{RCv_c(0^-)}{sRC+1} + \frac{v_{in}}{s(sRC+1)} \\
&= \frac{v_c(0^-)}{s+\frac{1}{RC}} + \frac{v_{in}}{s} - \frac{VRC}{sRC+1} \\
&= \frac{v_c(0^-)}{s+\frac{1}{RC}} + \frac{v_{in}}{s} - \frac{v_{in}}{s+\frac{1}{RC}} \\
v_c(t) &= \mathcal{L}^{-1}\{V_c(s)\} \\
&= [v_c(0^-) - v_{in}]e^{-\frac{t}{RC}} + v_{in}
\end{aligned}
$$

Plot of $v_C(t)$ for $v_C(0^-) > v_{in}$




Plot of $v_C(t)$ for $v_C(0^-) < v_{in}$

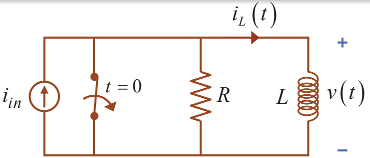



- Applying Initial Value Theorem (IVT): $\lim_{t\to 0} v_c(t) = \lim_{s\to\infty} sV_c(s) = v_c(0^-)$
- Applying Final Value Theorem (FVT): $\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} sV_c(s) = v_{in}$

IVT and FVT provide the means to determine $\lim_{t\to 0} v_c(t)$ and $\lim_{t\to\infty} v_c(t)$ without performing inverse Laplace transform.

# Laplace transform method for solving DEs – Parallel RL circuit

Derive the differential equation and use Laplace Transform to derive $i_L(t)$, the current flowing through the inductor, given that the initial condition is $i_L(0^-)$.




- Since $R$ and $L$ are connected in parallel, the voltage across R is equal to the voltage across $L$. Using the $I - V$ relationship for an inductor,

$$v(t) = L\frac{di_L(t)}{dt}$$

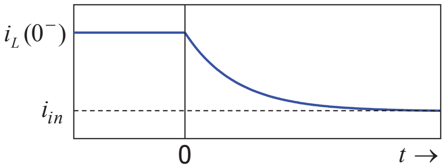
- Applying KCL,

$$\begin{aligned} \frac{v(t)}{R} + i_L(t) &= i_{in}u(t) \\ \frac{1}{R}\underbrace{L\frac{di_L(t)}{dt}}_{v(t)} + i_L(t) &= i_{in}u(t) \end{aligned}$$

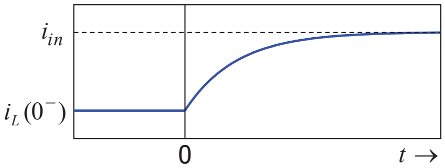
$$\frac{L}{R}\left[sI_L(s) - i_L(0^-)\right] + I_L(s) = \frac{i_{in}}{s}$$

$$\begin{aligned}
I_L(s) &= \frac{i_L(0^-)}{s+\frac{R}{L}} + \frac{i_{in}\frac{R}{L}}{s(s+\frac{R}{L})} \\
&= \frac{i_L(0^-)}{s+\frac{R}{L}} + \frac{i_{in}}{s} - \frac{i_{in}}{s+\frac{R}{L}} \\
i_L(t) &= \mathcal{L}^{-1}\{I_L(s)\} \\
&= [i_L(0^-) - i_{in}]e^{-\frac{R}{L}t} + i_{in}
\end{aligned}$$

Plot of $i_L(t)$ for $i_L(0^-) > i_{in}$



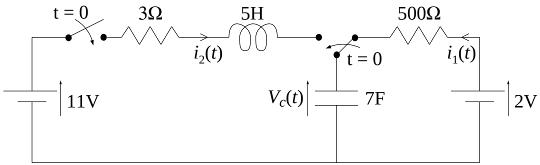

Plot of $i_L(t)$ for $i_L(0^-) > i_{in}$

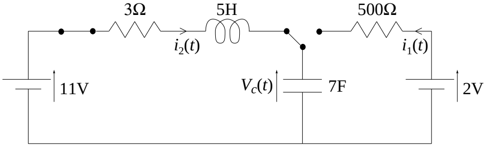



- Applying Initial Value Theorem (IVT): $\lim_{t\to 0} v_c(t) = \lim_{s\to\infty} sI_L(s) = i_L(0^-)$
- Applying Final Value Theorem (FVT): $\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} sI_L(s) = i_{in}$

# Laplace transform method for solving DEs – Non-zero initial conditions

Determine the voltage across the capacitor, $v_c(t)$, when $t \geq 0$ given that $v_c(0^-) = 2$ and $\dot{v}_c(0^-) = 0$.




When $t \geq 0$, circuit is a series RLC circuit. Hence, the same current $i_2(t)$ flows through all electric elements in the circuit.

- From $I - V$ relationship of the capacitor, current $i_2(t) = C\dfrac{\mathrm{d}v_c(t)}{\mathrm{d}t} = 7\dfrac{\mathrm{d}v_c(t)}{\mathrm{d}t}$.
- Voltage across the resistor is $i_2(t)R = RC\dfrac{\mathrm{d}v_c(t)}{\mathrm{d}t} = 21\dfrac{\mathrm{d}v_c(t)}{\mathrm{d}t}$.
- Voltage across the inductor is $L\dfrac{di_2(t)}{dt} = LC\dfrac{\mathrm{d}^2v_c(t)}{\mathrm{d}t^2} = 35\dfrac{\mathrm{d}^2v_c(t)}{\mathrm{d}t^2}$.

Applying KVL, the differential equation for determining $v_c(t)$ is

$$35\frac{\mathrm{d}^2 v_c(t)}{\mathrm{d}t^2} + 21\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} + v_c(t) = 11u(t)$$

Performing Laplace Transform,

$$35\left[s^2 V_c(s) - sv_c(0^-) - \dot{v}_c(0^-)\right] + 21\left[sV_c(s) - v_c(0^-)\right] + v_c(s) = \frac{11}{s}$$

$$\left[35s^2 + 21s + 1\right] V_c(s) = 70s + 42 + \frac{11}{s}$$

$$\begin{aligned}
V_c(s) &= \frac{70s+42}{35s^2+21s+1} + \frac{11}{s(35s^2+21s+1)} \\
v_c(t) &= \mathcal{L}^{-1}\left\{\frac{2s+\frac{6}{5}}{s^2+\frac{3}{5}s+\frac{1}{35}} + \frac{\frac{11}{35}}{s(s^2+\frac{3}{5}s+\frac{1}{35})}\right\} \\
&= \frac{2.2104}{s+0.052} - \frac{0.2104}{s+0.548} + \frac{11}{s} - \frac{12.1573}{s+0.052} + \frac{1.1573}{s+0.548} \\
&= 11 - 9.94e^{-0.052t} + 0.94e^{-0.548t}, t \geq 0
\end{aligned}$$