EE2023 Signals & Systems
Chapter 8 – Linear Time Invariant Systems

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DE model of LTI systems – Recap

- Relationship between the input $x(t)$ and output $y(t)$ of a LTI system is described by a linear differential equations with real constant coefficients.

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_o y(t) = b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \ldots + b_o x(t)$$

  where a_k and b_k are constants and $m < n$ if the LTI system is causal.
- The time-domain output signal, $y(t)$, of the LTI system is the solution of the linear differential equation.
- In Chapter 7, the Laplace transform method to solve differential equations is introduced.
- Laplace transform is also used to derive an alternative model for LTI systems. These are called transfer functions, and is the foundation for generalizing the behaviours of LTI systems.

Transfer Function of LTI Systems – Definition

Applying Laplace transform to both sides of the DE representing a LTI system,

$$\mathcal{L}\left\{a_n\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\ldots+a_o y(t)\right\}$$
$$=\quad\mathcal{L}\left\{b_m\frac{d^m x(t)}{dt^m}+b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}}+\ldots+b_o x(t)\right\}$$

Let $\mathcal{L}\{y(t)\}=Y(s)$ and $\mathcal{L}\{x(t)\}=X(s)$. The result of the Laplace transform operation is

$$a_n s^n Y(s)+a_{n-1}s^{n-1}Y(s)+\ldots+a_0 Y(s)=b_m s^m X(s)+b_{m-1}s^{m-1}X(s)+\ldots+b_0 X(s)$$

if **all initial conditions are zero** such that

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\}=s^n F(s)-\cancel{\sum_{k=0}^{n-1}s^{n-1-k}f^k(0^-)}$$

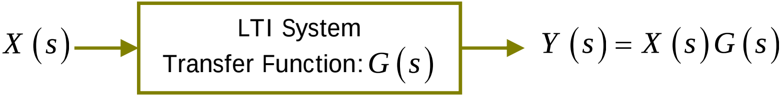
Re-arranging $a_n s^n Y(s) + \ldots + a_0 Y(s) = b_m s^m X(s) + \ldots + b_0 X(s)$ yields

$$\begin{aligned}
[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0] Y(s) &= [b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0] X(s) \\
\therefore\ G(s) = \frac{Y(s)}{X(s)} &= \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}
\end{aligned}$$

$G(s) = \dfrac{Y(s)}{X(s)}$ is called the **transfer function** of the LTI system, and is the s-domain representation of the LTI system.

The input-output relationship of a system can then be expressed as

$$\begin{aligned}
s\text{-domain} &: \quad Y(s) = G(s) \cdot X(s) \\
t\text{-domain} &: \quad y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s) \cdot X(s)\}
\end{aligned}$$



Transfer Function of LTI Systems - Exmaple

- Resistor : $V(t) = Ri(t)$

$$\Rightarrow G(s) = \frac{V(s)}{I(s)} = R$$

- Capacitor : $V_c(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$

  By applying the Transform of Integral Rule,

$$\Rightarrow G(s) = \frac{V_c(s)}{I(s)} = \frac{1}{sC}, \qquad V_c(0) = 0$$

- Inductor : $V_L(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$

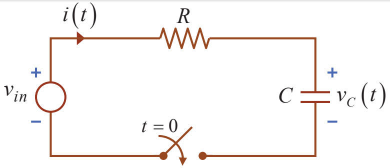
  By applying the Transform of Derivative Rule,

$$\Rightarrow G(s) = \frac{V_L(s)}{I(s)} = sL, \qquad i_L(0) = 0$$

Note that transfer function of the capacitor and inductor are also known as **complex impedances** in circuit theory.

Transfer Function of LTI Systems - Exmaple

Derive the transfer function, $G(s) = \dfrac{V_c(s)}{V_{in}(s)}$ of the series RC-circuit, and derive $v_c(t)$ using $G(s)$.



- Method 1 :

  The differential equation relating $v_c(t)$ to $v_{in}(t)$ is $RC\dfrac{\mathrm{d}v_c(t)}{\mathrm{d}t} + v_c(t) = v_{in}(t)$

  Performing Laplace Transform,

$$\begin{aligned} \mathcal{L}\left\{RC\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} + v_c(t)\right\} &= \mathcal{L}\{v_{in}(t)\} \\ RCsV_c(s) - RCv_c(0) + V_c(s) &= V_{in}(s) \end{aligned}$$

  Assuming zero initial conditions i.e. $v_c(0) = 0$ and re-arranging the equation,

$$\text{Transfer function, } G(s) = \frac{V_c(s)}{V_{in}(s)} = \frac{1}{sRC + 1}$$

- Method 2 (Complex impedances approach) :

  Transfer function of capacitor is $\frac{1}{sC}$; $v_c(0) = 0$

  Performing voltage division,

$$
\begin{aligned}
V_c(s) &= \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_{in}(s) \\
G(s) &= \frac{V_c(s)}{V_{in}(s)} = \frac{1}{sRC + 1}
\end{aligned}
$$

- Input signal is $v_{in}(t) = v_{in} \cdot u(t)$ or $V_{in}(s) = \mathcal{L}\{v_{in} \cdot u(t)\} = \frac{v_{in}}{s}$.

$$
\begin{aligned}
V_c(s) &= G(s) \times V_{in}(s) = \frac{1}{sRC + 1} \times \frac{v_{in}}{s} \\
v_c(t) &= \mathcal{L}^{-1}\{V_c(s)\} = v_{in}\left(1 - e^{-\frac{t}{RC}}\right) \qquad t \geq 0
\end{aligned}
$$

  DE has disappeared and calculus has been replaced by algebra!

Unit Impulse Response Model of LTI systems

- The impulse response, $y_\delta(t)$, of a continuous-time LTI system is defined as the response (or output) of the LTI system when the input is a unit impulse, $\delta(t)$.

$$\delta(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ \mathbf{T}\end{array}} \longrightarrow y_\delta(t) = \mathbf{T}\left[\delta(t)\right]$$

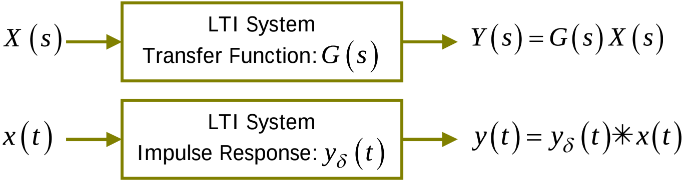
$$L\{\delta(t)\} = 1 \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ \text{Transfer Function: } G(s)\end{array}} \longrightarrow L\{y_\delta(t)\} = Y_\delta(s) = G(s)\cdot 1 = G(s)$$

- Since a continuous-time LTI system is characterized by its transfer function in the s-domain, the system is similiarly characterized in the time-domain by its impulse response, $y_\delta(t)$ because

$$\mathcal{L}\{y_\delta(t)\} = G(s) \qquad \text{or} \qquad y_\delta(t) = \mathcal{L}^{-1}\{G(s)\}$$

- Impulse reponse, being the time-domain characterization of the LTI system, may be used to derive the output signal.
- Suppose the input signal is an arbitrary signal, $x(t)$, and the output is represented as $y(t)$. The s-domain representation of the input $X(s) = \mathcal{L}^{-1}\{x(t)\}$, the output $Y(s) = \mathcal{L}\{y(t)\}$ and the LTI system $G(s) = \mathcal{L}\{y_\delta(t)\}$ have the following relationship:

$$
\begin{aligned}
Y(s) &= G(s) \cdot X(s) \\
y(t) &= \mathcal{L}^{-1}\{G(s)\} * \mathcal{L}^{-1}\{X(s)\} && \text{Convolution in time-domain property} \\
&= y_\delta(t) * x(t) \\
&= \int_{-\infty}^{\infty} y_\delta(t-\tau)x(\tau)\, d\tau && \text{Definition of convolution}
\end{aligned}
$$



Definition of System Poles and Zeros

In the study of a LTI system, it is often informative to express its transfer function in the following factorised forms:

$$
\begin{aligned}
G(s) &= \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0} \\
&= K\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\ldots\left(\frac{s}{z_m}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\ldots\left(\frac{s}{p_n}+1\right)}; \qquad K = \frac{b_0}{a_0} \\
&= K'\frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}; \qquad K' = \frac{b_m}{a_n}
\end{aligned}
$$

A transfer function $G(s)$ can be uniquely defined by the parameters $-p_i\,(i = 1, \ldots, n)$, $-z_j\,(j = 1, \ldots, m)$ and K (known as the **Steady-state/DC/Static Gain**).

Poles of G(s)

- $-p_i, i = 1, \ldots, n$ are the **system poles** of $G(s)$.
- $-p_i$ are the roots of the denominator polynomial of $G(s)$ i.e. poles are the values of s that satisfy $a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 = (s + p_1)(s + p_2) \ldots (s + p_n) = 0$.
- $G(-p_i) = \infty$.

Zeros of G(s)

- $-z_j, j = 1, \ldots, m$ are the **zeros** of $G(s)$.
- $-z_i$ are the roots of the numerator polynomial of $G(s)$ i.e. poles are the values of s that satisfy $b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0 = (s + z_1)(s + z_2) \ldots (s + z_m) = 0$.
- $G(-z_j) = 0$.

Pole-Zero Excess of G(s)

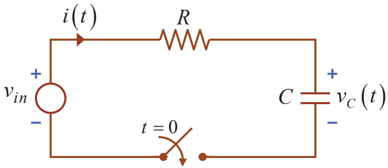
- The transfer function is said to have n poles and m zeros.
- For practical systems, $n > m$. The difference $n - m$ is called **pole-zero excess**.

In the s-domain, the relationship between the input, $X(s)$, and output, $Y(s)$, of a LTI system, $G(s)$, is given by $Y(s) = G(s)X(s)$.

- The system transfer function $G(s) = K' \dfrac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$ contains system poles $(-p_i)$ and system zeros $(-z_m)$.
- Likewise, $X(s)$ may contain poles and zeros too.
- To differentiate between poles of $G(s)$ and $X(s)$, poles of $X(s)$ are known as **input poles**.

Example

Derive the system pole(s) and input pole(s) of the series RC circuit.



- Transfer function, $G(s) = \dfrac{1}{sRC+1}$. Hence, system pole is $s = -\dfrac{1}{RC}$.
- Input signal is $v_{in} \cdot u(t)$. Since $\mathcal{L}\{v_{in} \cdot u(t)\} = \dfrac{v_{in}}{s}$, input pole is $s = 0$.

Significance of System Poles – System stabiity

Performing partial fraction decomposition,

$$G(s) = K' \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} + \dots + \frac{\alpha_n}{s+p_n}$$

The system impulse response is then given by

$$y_\delta(t) = \mathcal{L}^{-1}\{G(s)\} = \left[\alpha_1 e^{-p_1 t} + \alpha_2 e^{-p_2 t} + \dots + \alpha_n e^{-p_n t}\right] \, u(t)$$

- Terms in the impulse response can be mapped to the system poles.
  - Suppose all the system poles have negative real parts, i.e. $Re[-p_i] < 0 \, (i = 1, \dots, n)$. Each term in $y_\delta(t)$ is a right-sided exponentially decaying functions. Hence, $\lim_{t\to\infty} y_\delta(t) = 0$.
- It can be shown that convolving $y_\delta(t)$ with any bounded input will always result in a bounded output function. Consequently, system stability (whether the system output is bounded or not) can be defined using the behaviour of $y_\delta(t)$.
- Given the unique mapping between system poles and terms in $y_\delta(t)$, system poles provides a convenient way to check system stability by inspection. There is no need to compute the system output.

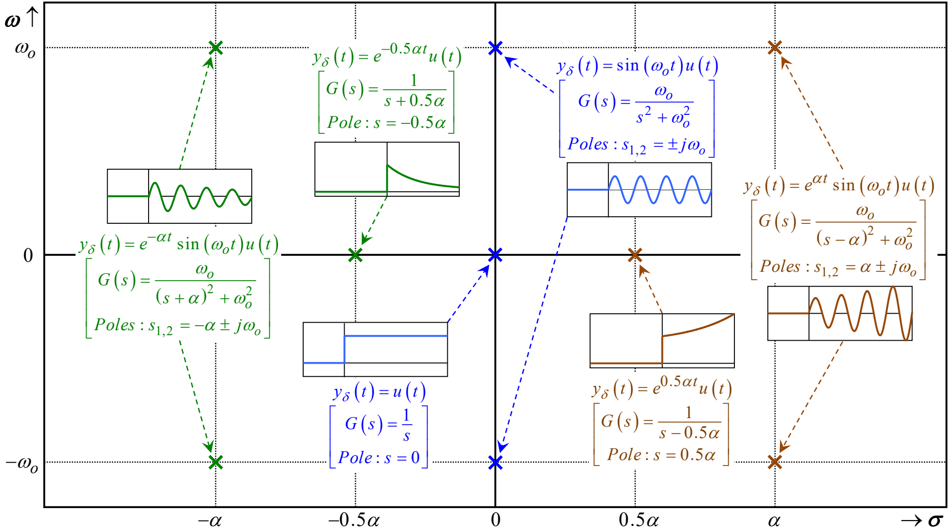
The different notions of system stability conceived based on the behavior of the system impulse response ($y_\delta(t)$) as $t \to \infty$, and the corresponding locations of the system poles, are:

- $G(s)$ is **BIBO stable**: $y_\delta(t)$ will converge to zero as t tends to infinity.
  - The condition needed for $\lim_{t\to\infty} y_\delta(t) = 0$ is $Re[-p_i] < 0$ for all $i = 1, \ldots, n$.
  - All system poles will lie on the left-half s-plane.
- $G(s)$ is **unstable**: $y_\delta(t)$ will "blow up" and become unbounded as t tends to infinity.
  - For $\lim_{t\to\infty} |y_\delta(t)| = \infty$ to occur, at least one term in $y_\delta(t)$ must grow exponentially as $t \to \infty$. The real part of the corresponding system pole must be positive i.e. $Re[-p_i] > 0$.
  - At least one system pole lie on the right-half s-plane.
- $G(s)$ is **marginally unstable**: $y_\delta(t)$ will not "blow up" and become unbounded, but neither will it converge to zero as t tends to infinity i.e. $\lim_{t\to\infty} y_\delta(t) \neq \infty$ and $\lim_{t\to\infty} y_\delta(t) \neq 0$.
  - $\lim_{t\to\infty} y_\delta(t) \neq \infty$ means $Re[-p_i] \leq 0$ for all $i = 1, \ldots, n$. For $\lim_{t\to\infty} y_\delta(t) \neq 0$, at least one term in $y_\delta(t)$ does not decay to zero as $t \to \infty$. The real part of the corresponding system pole must be 0 i.e. $Re[-p_i] = 0$.
  - At least one system pole will lie on the imaginary axis of the s-plane, with no system pole(s) lying on the right half s-plane.

Note: Systems with **repeated** poles on the imaginary axis are UNSTABLE.

Relationship between impulse response and pole location

s-Plane $(s = \sigma + j\omega)$ **Assume** $\alpha > 0$

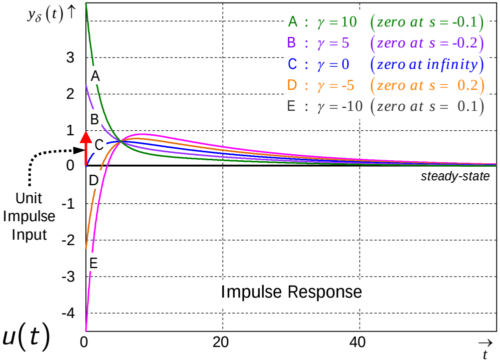


Role of system zeros

- System zeros ($-z_j$, $j = 1, \ldots, m$) do not play any role in determining whether a system is stable or not.
- Zeros only affect the values of the constants α_n during partial factorization, thus affecting only the manner in which impulse reponse $y_\delta(t)$ decays to zero as $t \to \infty$.

Consider a system $G(s)$ which has a DC gain of 18, a zero at $s = -\frac{1}{\gamma}$, and poles $s = -0.5$ and $s = -0.05$.

$$
\begin{aligned}
G(s) &= \frac{18(\gamma s + 1)}{40s^2 + 22s + 1} \\
&= \frac{0.5\gamma - 1}{s + 0.5} - \frac{0.05\gamma - 1}{s + 0.05} \\
y_\delta(t) &= \mathcal{L}^{-1}\{G(s)\} \\
&= \left[(0.5\gamma - 1)e^{-0.5t} - (0.05\gamma - 1)e^{-0.05t}\right] u(t)
\end{aligned}
$$



Types of Systems – First order system

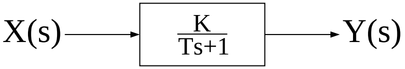
- D.E of a first-order system is generally written as

$$T\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = Kx(t)$$

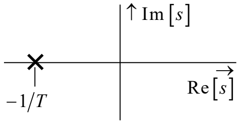
where K is the **steady-state/DC/static gain**
T is called the **time-constant**
$x(t)$ is the input signal
$y(t)$ is the output

- Performing Laplace Transform to both sides of the D.E. and assuming that initial condition is zero (i.e. $y(0) = 0$),

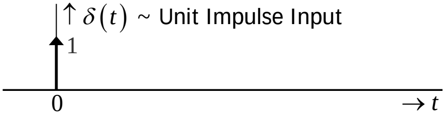
$$\begin{aligned} TsY(s) + Y(s) &= KX(s) \\ G(s) = \frac{Y(s)}{X(s)} &= \frac{K}{Ts+1} \end{aligned}$$

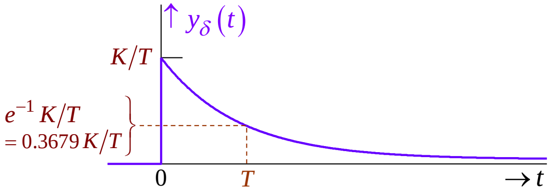


- Pole is located at $s = -\frac{1}{T}$.
- Unit impuluse response of a first order sytem is:



$$
\begin{aligned}
y_\delta(t) &= \mathcal{L}^{-1}\{G(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{K}{Ts+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{K}{T}\frac{1}{s+\frac{1}{T}}\right\} \\
&= \frac{K}{T}e^{-\frac{t}{T}}\,u(t)
\end{aligned}
$$





- Rate of exponential decay depends on the time constant, T. As T increases, the rate of decay decreases.
- When $t = T$, $y_\delta(t) = \frac{K}{T}e^{-1} = 0.3679\frac{K}{T}$.

Types of Systems – Second order system

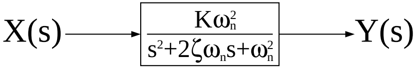
- Differential equation of a standard second-order system may be expressed as

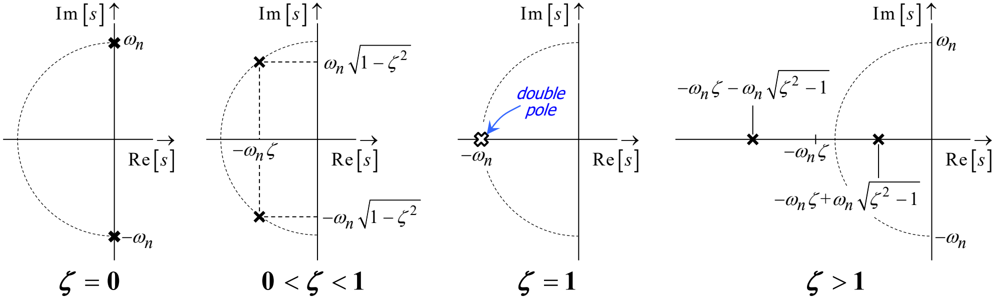
$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 2\zeta\omega_n \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \omega_n^2 y(t) = K\omega_n^2 x(t)$$

where K is the **steady-state/DC/static gain**
ζ is the **damping ratio**
ω_n is the **undamped natural frequency**
$x(t)$ is the input signal
$y(t)$ is the output signal.

- Performing Laplace Transform to both sides of the D.E. and assuming that initial conditions are zero (i.e. $y(0) = \dot{y}(0) = 0$),

$$\begin{aligned} s^2 Y(s) + 2\zeta\omega_n sY(s) + \omega_n^2 Y(s) &= K\omega_n^2 X(s) \\ G(s) = \frac{Y(s)}{X(s)} &= \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

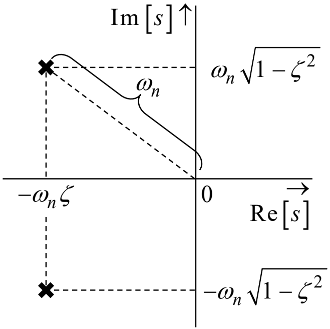


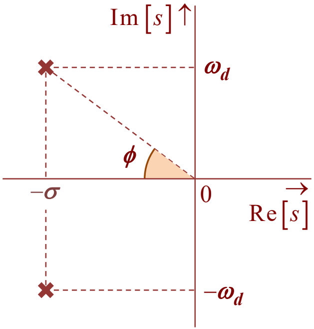


- $\zeta = 0$: Poles are an imaginary conjugate pair, $s_{1,2} = \pm j\omega_n$.
  System is said to be **undamped**.
- $0 < \zeta < 1$: Poles are a complex conjugate pair, $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$
  System is said to be **underdamped**.
- $\zeta = 1$: Poles are real and repeated, $s_{1,2} = -\omega_n, -\omega_n$.
  System is said to be **critically damped**.
- $\zeta > 1$: Poles are real and distinct, $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$
  System is said to be **overdamped**.

In addition to the standard parameters (K, ζ, ω_n), an **underdamped** second-order system is occassionally expressed using the decay rate parameter, $\sigma = \zeta\omega_n$, and the **damped natural frequency**, $\omega_d = \omega_n\sqrt{1-\zeta^2}$:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K(\sigma^2 + \omega_d^2)}{(s+\sigma)^2 + \omega_d^2}$$

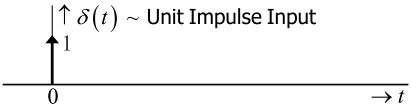


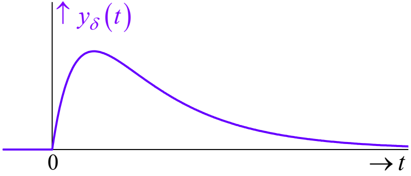


- $\sigma = \zeta\omega_n = \omega_n\cos(\phi)$
- $\omega_d = \omega_n\sqrt{1-\zeta^2}$
  $= \omega_n\sin(\phi)$
- $\phi = \tan^{-1}\dfrac{\omega_d}{\sigma}$
- $\omega_n^2 = \sigma^2 + \omega_d^2$

- **$y_\delta(t)$ of overdamped $(\zeta > 1)$ 2^{nd} order sys**: Distinct real poles

$$
\begin{aligned}
y_\delta(t) &= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{(s+a)(s+b)}\right\} \\
&= \left[\frac{K\omega_n^2}{b-a}e^{-at}+\frac{K\omega_n^2}{a-b}e^{-bt}\right]u(t)
\end{aligned}
$$

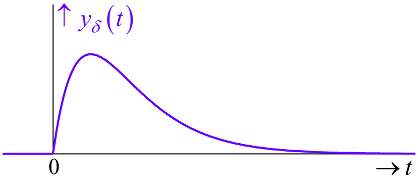




$a = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$ and $b = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

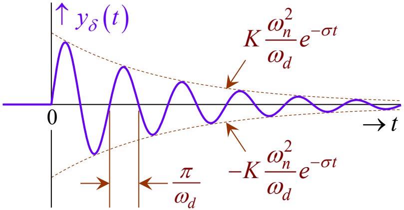
- **$y_\delta(t)$ of critically damped $(\zeta = 1)$ 2^{nd} order sys**: Repeated real poles

$$
\begin{aligned}
y_\delta(t) &= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{(s+\omega_n)^2}\right\} \\
&= K_1\omega_n^2 t e^{-\omega_n t}u(t)
\end{aligned}
$$



- **$y_\delta(t)$ of underdamped $(0 < \zeta < 1)$ 2^{nd} order sys**: Complex conjugate poles

$$
\begin{aligned}
y_\delta(t) &= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{(s+\sigma)^2+\omega_d^2}\right\} \\
&= K\frac{\omega_n^2}{\omega_d}e^{-\sigma t}\sin(\omega_d t)\,u(t) \\
&= \frac{K\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2})t\,u(t)
\end{aligned}
$$



- **$y_\delta(t)$ of undamped $(\zeta = 0)$ 2^{nd} order sys**: Imaginary conjugate poles

$$
\begin{aligned}
y_\delta(t) &= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{s^2+\omega_n^2}\right\} \\
&= K\omega_n\sin(\omega_n t)\,u(t)
\end{aligned}
$$