EE2023 Signals & Systems
Chapter 5 – Sampling & Reconstruction of Signals

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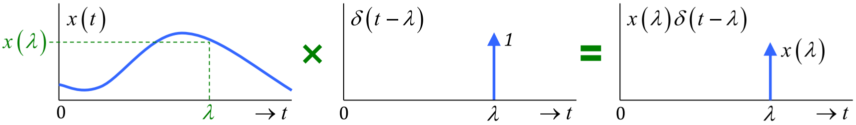
Sampling : Motivation

- Many naturally occuring signals are analogue in nature. However, it is increasingly common for appliances to use digital signal processing techniques e.g. CD/DVD/Blue Ray recordings, Digital TV and voice transmission over radio on mobile phones.
- Main reason for using digital signals is because they are more reliable to transmit.
- Digital signal processing (DSP) is concerned with the representation of discrete time signals by a sequence of numbers or symbols and the processing/manipulation of these signals.
- Objective of chapter:
  - Introduce the concept of **sampling** using a the Dirac Comb (impulse train) function to convert analogue signals into digital domain.
  - Derive the **Nyquist sampling theorem** which stipulates the minimum sampling frequency required in order to reconstruct the continuous time signal from its samples.

Review: Properties of Dirac delta function

- **Sampling**

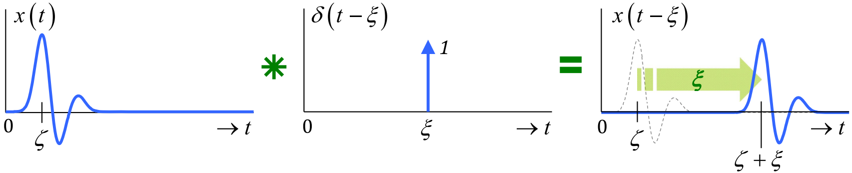
$$\boxed{x(t) \cdot \delta(t-\lambda) = x(\lambda)\delta(t-\lambda)}$$



Note that $X(f) \cdot \delta(f-\lambda) = X(\lambda)\delta(f-\lambda)$.

- **Replication**

$$\boxed{x(t) * \delta(t-\xi) = x(t-\xi)}$$



Note that $x(t) * \delta(t) = x(t)$ and $X(f) * \delta(f-\xi) = X(f-\xi)$.

Introduction to Ideal filters

- A "filter" is a linear time-invariant (LTI) system that has frequency-selective behaviour i.e. the filter "retains" some frequency components and "removes/rejects" other frequency components.
- Characteristics of a filter is defined by its **frequency response**, $H(f)$, in the frequency-domain or its **impulse response**, $h(t)$, in the time-domain.

$$h(t) = \mathcal{F}^{-1}\{H(f)\}$$

System ***Input/Output*** *relationship* →

Frequency-domain : $X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = H(f) \times X(f)$

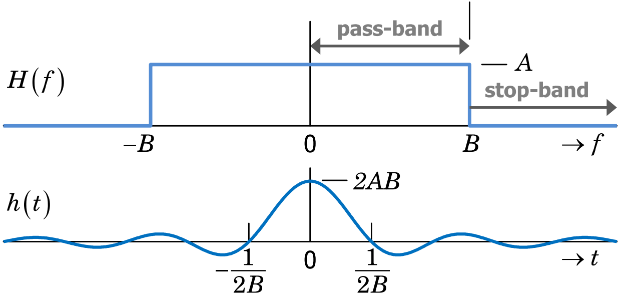
Time-domain : $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$

- The range of frequencies passed by a filter is referred to as the **pass-band** and the range of frequencies rejected by a filter is called the **stop-band**.
- An **ideal filter** is one that has full transmission in the pass-band, and complete signal rejection in the stop-band. In addition, the transition from pass-band to stop-band is abrupt.

- The spectrum of an ideal filter may be represented mathematically by rectangular functions because
  - Spectrum of the filter should be unity within the pass-band to provide full transmission.
  - Complete signal rejection can be achieved when spectrum of the filter is zero within the stop-band.
- **Ideal Low-Pass Filter (LPF)** :

  Frequency response of a low-pass filter : $H(f) = A\,\mathrm{rect}\left(\frac{f}{2B}\right)$

  Impulse response : $h(t) = 2AB\,\mathrm{sinc}(2Bt)$

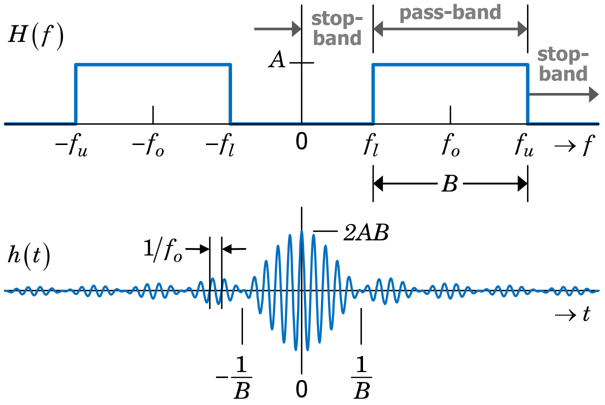


Bandwidth of the LPF is B.

Ideal Band-Pass Filter (BPF) :

Frequency response of a band-pass filter : $H(f) = A\left[\mathrm{rect}\left(\frac{f+f_o}{B}\right) + \mathrm{rect}\left(\frac{f-f_o}{B}\right)\right]$

Impulse response : $h(t) = 2AB\,\mathrm{sinc}(Bt)\cos(2\pi f_o t)$



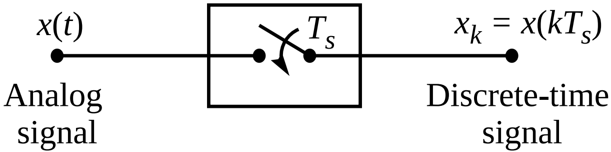
Upper cutoff freq $= f_u$
Lower cutoff freq $= f_l$
Center freq $= \frac{1}{2}(f_u + f_l)$
Bandwidth, $B = f_u - f_l$

Sampling : Definition

Let $x(t)$ be a continuous time signal and let $T_s > 0$ be a fixed number. From $x(t)$, the discrete-time sequence can be derived as :

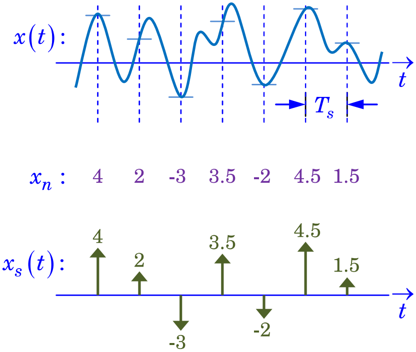
$$x_k = x(kT_s) \qquad \text{for } k = 0, 1, 2, \ldots$$

T_s is called the **sampling period**. The **sampling frequency**, f_s in Hertz is defined as $f_s = \dfrac{1}{T_s}$ Hz or $\omega_s = \dfrac{2\pi}{T_s} = 2\pi f_s$ radians/sec. The time $t = kT_s$ is called the sample instant. The process of extracting $x(kT_s)$ is called **sampling**.



- The sampler takes a snapshot of the continuous time signal $x(t)$ every T_s time units to produce a sequence, $x_k = x(kT_s)$.
- Information provided by the sequence x_k can also be modeled as

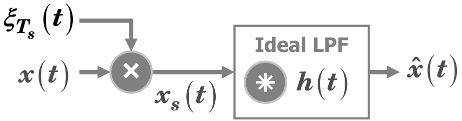
$$
\begin{aligned}
x_s(t) &= \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s) \\
&= x(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT_s)
\end{aligned}
$$

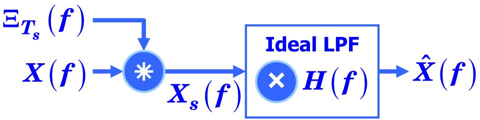


- $x_s(t)$ is a continuous-time function, so Fourier Transform may be used to aid our analysis.
- It can be also be shown that the spectrum (or discrete-time Fourier transform) of x_k is equal to the spectrum (or Fourier transform) of $x_s(t)$.

Continuous Time Sampling and Reconstruction Processes

- Sampling and reconstruction is necessary for digital implementation.
- Consider the digital communications system,
  - At the source, the audio signal was sampled, coded and then transmitted.
  - At the receiver, the digital signal was decoded and the audio signal recovered through low pass filtering (LPF).
- The block diagrams show the sampling and reconstruction process in time and frequency domains.





Sampling Re-construction
Time domain

Sampling Re-construction
Freq domain

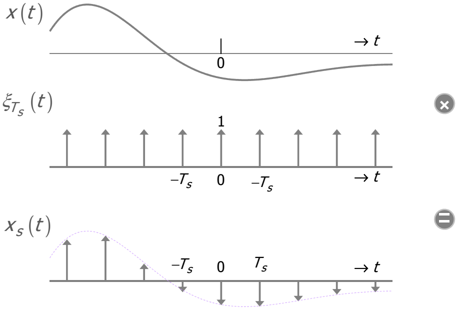
- Under the most ideal situation where there is no noise in the system, what are the conditions needed for $\hat{x}(t) = x(t)$?

Continuous Time Sampling Process

Continuous-time sampling process is modeled mathematically as by multiplying the signal, $x(t)$ with the Dirac comb function, $\xi_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$.

Sampled signal, $x_s(t)$, is

$$\begin{aligned} x_s(t) &= x(t) \cdot \xi_{T_s}(t) \\ &= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \end{aligned}$$



Question: What is the minimum $f_s = \dfrac{1}{T_s}$ for which the signal, $\hat{x}(t)$, reconstructed or recovered signal from $x_s(t)$ is perfect i.e. $\hat{x}(t) = x(t)$?

- Using the multiplication in time-domain FT property, spectrum of the sampled signal is:

$$[x_s(t) = x(t) \cdot \xi_{T_s}(t)] \leftrightarrows [X_s(f) = X(f) * \Xi_{T_s}(f)]$$

where $\mathcal{F}\{x(t)\} = X(f)$, $\mathcal{F}\{x_s(t)\} = X_s(f)$, $\mathcal{F}\{\xi_{T_s}(t)\} = \Xi_{T_s}(f)$.

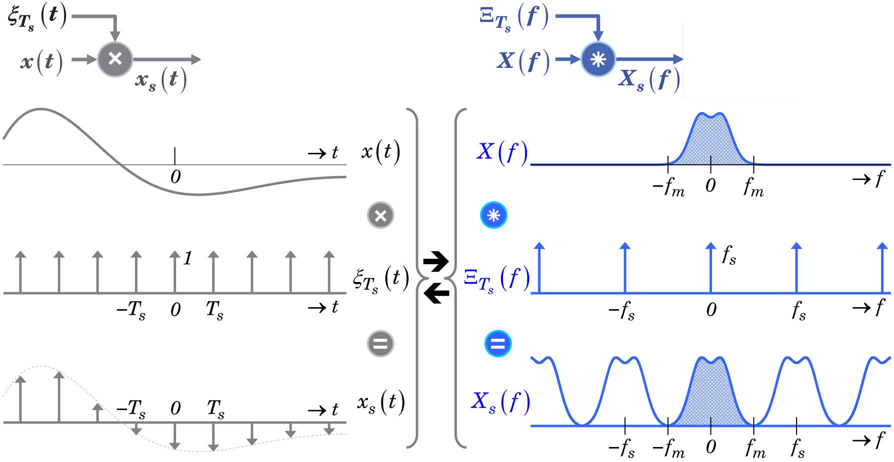
- Since $\Xi_{T_s}(f) = \dfrac{1}{T_s} \displaystyle\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right) = f_s \sum_{k=-\infty}^{\infty} \delta\left(f - kf_s\right)$,

$$\begin{aligned} X_s(f) &= X(f) * \left[f_s \sum_{k=-\infty}^{\infty} \delta\left(f - kf_s\right) \right] \\ &= \underbrace{f_s \sum_{k=-\infty}^{\infty} X(f) * \delta\left(f - kf_s\right) = f_s \sum_{k=-\infty}^{\infty} X\left(f - kf_s\right)}_{\text{Replication Property of } \delta(t)} \end{aligned}$$

$X_s(f)$ is **periodic with fundamental period $\mathbf{f_s}$** and comprises infinite replicas of $X(f)$. Each copy of $X(f)$ is centered around kf_s, $k = \ldots, -1, 0, 1, \ldots$.

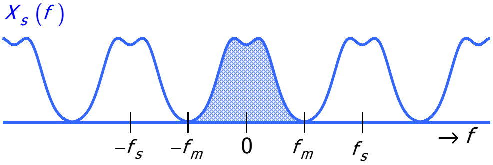
Process of sampling a lowpass signal

Sampling frequency, $f_s = \dfrac{1}{T_s} = 2f_m$

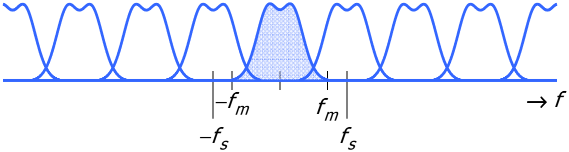


Spectrums of the sampled signal, $X_x(f)$, when a lowpass signal, $x(t)$, is sampled with different frequencies, f_s :

$f_s = 2f_m$

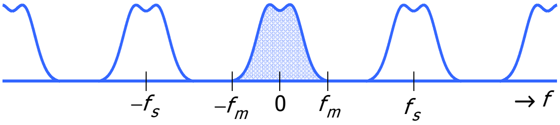


$f_s < 2f_m$



Spectral images overlap and consequently, perfect recovery of original analog signal is not possible. This phenomenon is called **signal aliaising**.

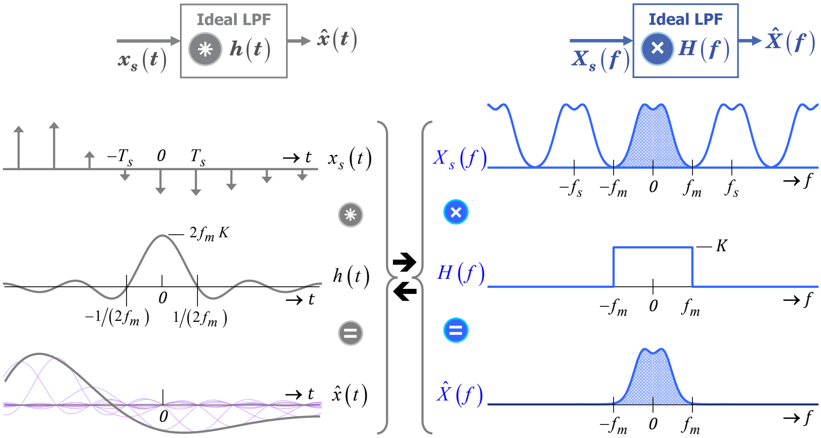
$f_s > 2f_m$



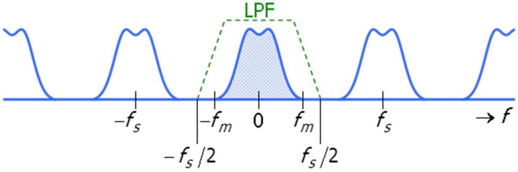
Spectral images do not overlap and gaps appear between spectral images due to **oversampling** i.e. sampling faster than the minimum rate.

Signal Reconstruction Process

Signal is reconstructed by filtering the sampled signal, $x_s(t)$. A lowpass filter is used to recover lowpass signals.

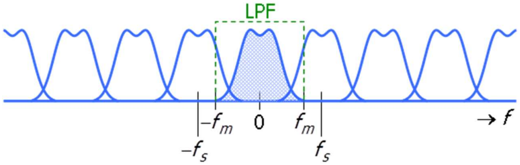


- Spectrum of the sampled signal, $X_s(f)$ for the case where $f_s > 2f_m$ is



- Perfect reconstruction is possible.
- Presence of spectral gaps makes the LPF design easier.
- **Oversampling** is more costly.

- Spectrum of the sampled signal, $X_s(f)$ for the case where $f_s < 2f_m$.



- Perfect recovery of original analog signal, $x(t)$, is not possible as the spectral images overlap (**signal aliaising**).

- *Conclusion*: The minimum sampling frequency is $f_s = 2f_m$ so that aliasing does not occur and consequently, signal reconstruction can be achieved.

Nyquist Sampling Theorem

Nyquist frequency

- **Nyquist frequency** f_{Nyquist}, is defined as twice the highest frequency component in the signal, f_m i.e. $f_{\text{Nyquist}} = 2f_m$.
- Aliasing will occur if $f_s < f_{\text{Nyquist}} = 2f_m$.

Nyquist Sampling Theorem

Conditions:

- $x(t)$ must be lowpass bandlimited i.e. $X(f) = 0$ if $|f| > f_m$.
- Sampling frequency, f_s, must be larger than the Nyquist frequency, f_{Nyquist} i.e.

$$\boxed{f_s > f_{\text{Nyquist}} = 2f_m}$$

- Characteristics of the reconstruction LPF :
  - Passband corner frequency is f_m.
  - Stopband corner frequency is $f_s/2$.

Example

A signal $x(t) = \text{sinc}^2(2t)$ is sampled at 8 Hz to produce the sampled signal $x_s(t)$. Sketch the spectra of $x(t)$ and $x_s(t)$. Can $x(t)$ be perfectly reconstructed from $x_s(t)$ using an ideal low-pass filter ? If yes, specify the filter. What is the Nyquist sampling frequency for $x(t)$?

- Since the sampling frequency, f_s, is 8 Hz, the continuous-time sampled signal is

$$x_s(t) = x(t) \cdot \sum_{k=-\infty}^{\infty} \delta\left(t - \frac{k}{8}\right) = \text{sinc}^2(2t) \sum_{k=-\infty}^{\infty} \delta\left(t - \frac{k}{8}\right)$$

- Spectrum of the sampled signal is

$$\begin{aligned} X_s(f) &= \mathcal{F}\{\text{sinc}^2(2t)\} * \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta\left(t - \frac{k}{8}\right)\right\} \\ &= X(f) * 8 \sum_{k=-\infty}^{\infty} \delta\left(f - 8k\right) \end{aligned}$$

- From the Fourier Transform table, $A\,\text{tri}\left(\frac{t}{T}\right) \leftrightarrows AT\text{sinc}^2(Tf)$. When $T=2$ and $A=0.5$,

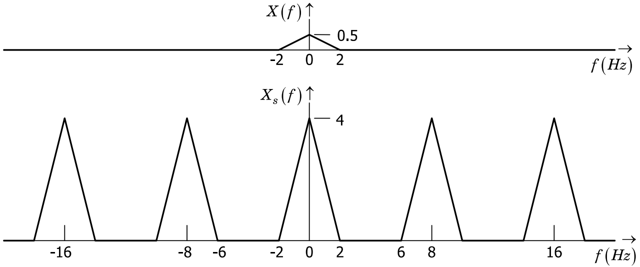
$$0.5\,\text{tri}\left(\frac{t}{2}\right) \leftrightarrows \text{sinc}^2(2f)$$

- From the Duality property of Fourier Transform and applying the even function property of a triangle function i.e. $\text{tri}\left(\frac{t}{T}\right) = \text{tri}\left(-\frac{t}{T}\right)$,

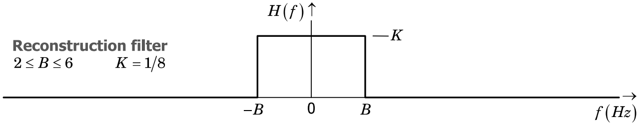
$$\text{sinc}^2(2t) \leftrightarrows X(f) = 0.5\,\text{tri}\left(\frac{f}{2}\right)$$

- Hence, spectrum of the sampled signal is

$$\begin{aligned} X_s(f) &= X(f) * 8 \sum_{k=-\infty}^{\infty} \delta(f-8k) \\ &= 4\,\text{tri}\left(\frac{f}{2}\right) * \sum_{k=-\infty}^{\infty} \delta(f-8k) \end{aligned}$$



- As the spectral images do not overlap, $x(t)$ can be perfectly reconstructed from $x_s(t)$ using an ideal low-pass filter with frequency response $H(f) = \dfrac{1}{8}\ \mathrm{rect}\left(\dfrac{f}{2B}\right)$, where $2 \leq B \leq 6$.



- The signal, $x(t) = \mathrm{sinc}^2(2t)$, does not have any frequency components higher than $f_m = 2Hz$. Hence, the Nyquist frequency is $f_{\text{Nyquist}} = 2 \times f_m = 4$ Hz.