NATIONAL UNIVERSITY OF SINGAPORE

EE2023 – SIGNALS AND SYSTEMS

(Semester I : 2019/2020)

Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write only your student number. Do not write your name.
2. Students should write the answers for each question on a new page.
3. This paper contains **EIGHT (8)** questions and comprises **TWELVE (12)** printed pages.
4. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.
5. This is a **CLOSED BOOK** examination. However you are allowed to bring one self-prepared A4-size help sheet to the examination hall.
6. Programmable and/or graphic calculators are not allowed.
7. Tables of formulas are provided on Pages 9 to 12.

SECTION A : Answer ALL questions in this section

Q.1 Let $X(f)$ be the Fourier transform of a signal $x(t)$.

(a) If $y(t) = 4x(-2(t-1))$, express its Fourier transform, $Y(f)$, in terms of $X(f)$.

(4 marks)

(b) If $X(f) = 2\text{rect}(f)\cos(\pi f)$, find the 3dB bandwidth and total energy of $x(t)$.

(6 marks)

Q.2 The discrete-frequency magnitude and phase spectra for the periodic signal $x(t)$ are shown in Figure Q.2.

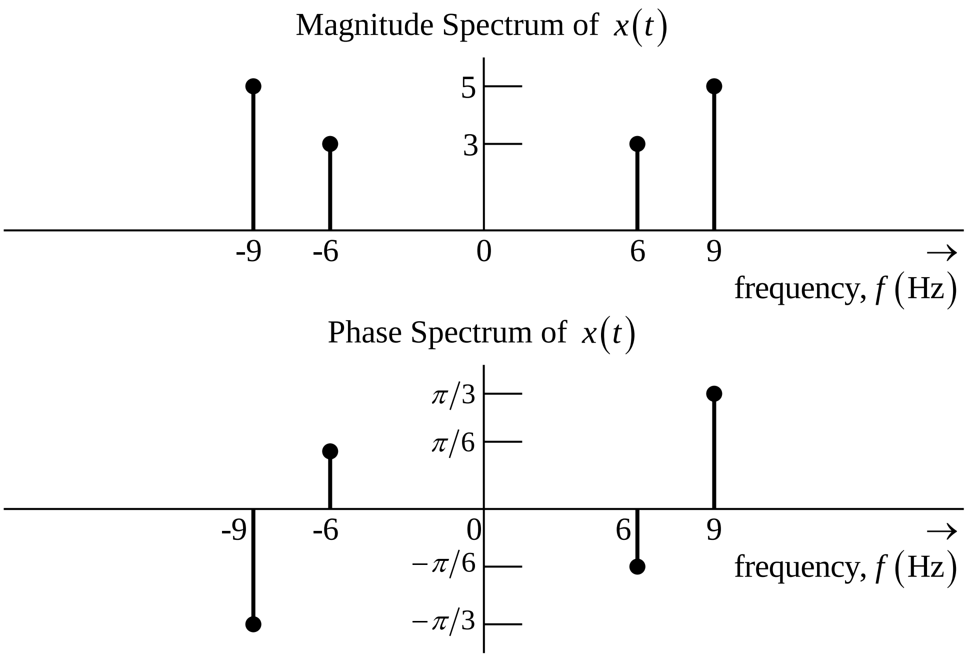

Figure Q.2

(a) What is the fundamental frequency and period of $x(t)$?

(2 marks)

(b) Derive an expression for the complex exponential Fourier Series representation for $x(t)$.

(6 marks)

(c) Determine the average power of $x(t)$.

(2 marks)

Q.3 The transfer function of a system is given by

$$G(s) = \frac{10s}{(s+10)^2}.$$

(a) Sketch the straight-line Bode magnitude and phase plots for $G(s)$. The sketches should be adequately labeled. You need not draw the semilog_x grid lines.

(6 marks)

(b) As frequency decreases, the frequency response of $G(s)$ tends to that of a certain system. Name this system.

(2 marks)

(c) As frequency increases, the frequency response of $G(s)$ tends to that of a certain system. Name this system.

(2 marks)

Q.4 The input-out relationship of a dynamic system with input $x(t)$ and output $y(t)$ is governed by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 85y(t) = 200x(t)$$

(a) Derive the system transfer function $G(s) = \frac{Y(s)}{X(s)}$, where $X(s)$ and $Y(s)$ are the Laplace transforms of the input $x(t)$ and output $y(t)$, respectively.

(3 marks)

(b) Determine the undamped natural frequency, ω_n, the damping ratio, ζ, and the DC gain of the system.

(3 marks)

(c) The input signal, $x(t) = 2\cos(4t)$, is applied to the dynamic system. Determine the steady-state output signal, $y_{ss}(t)$.

(4 marks)

SECTION B : Answer 3 out of the 4 questions in this section

Q.5 The block diagram of a communication system is shown in Figure Q.5 where f_s is the sampling frequency used at the receiver front-end and $H(f)$ is the frequency response of the lowpass filter.

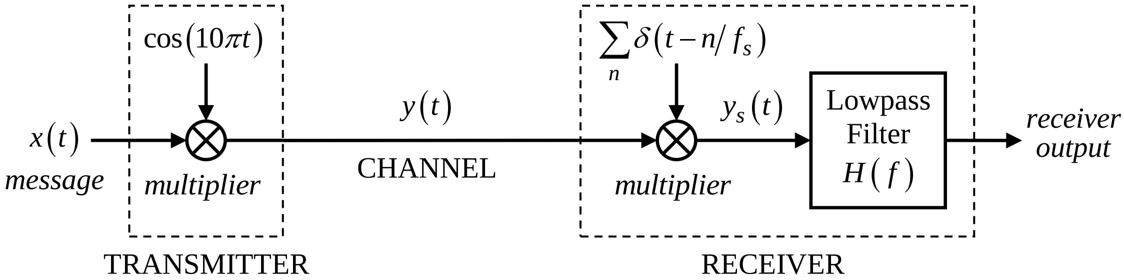

Figure Q.5

Let $X(f)$, $Y(f)$ and $Y_s(f)$ be the spectra of $x(t)$, $y(t)$ and $y_s(t)$, respectively, where

$$X(f) = \left[\cos(\pi f) + 2\right] \times \text{rect}\left(\frac{f}{2}\right).$$

The frequency response of the lowpass filter is

$$H(f) = A\,\text{rect}\left(\frac{f}{2B}\right).$$

(a) Sketch $X(f)$ and $Y(f)$ with adequate labeling. (8 marks)

(b) If $f_s = 6$ Hz, is it possible for the receiver to reproduce $x(t)$ at its output? Explain your answer with a sketch of $Y_s(f)$.

(5 marks)

(c) What is the smallest value of f_s that will enable the receiver to reproduce $x(t)$ at its output? In this case, find the values of the filter parameters A and B so that the receiver output is exactly equal to $x(t)$.

(7 marks)

Q.6 The signal $x(t) = A\,\text{tri}\left(\frac{t}{\alpha}\right) + B\,\text{tri}\left(\frac{t}{\beta}\right)$ is shown in Figure Q.6.

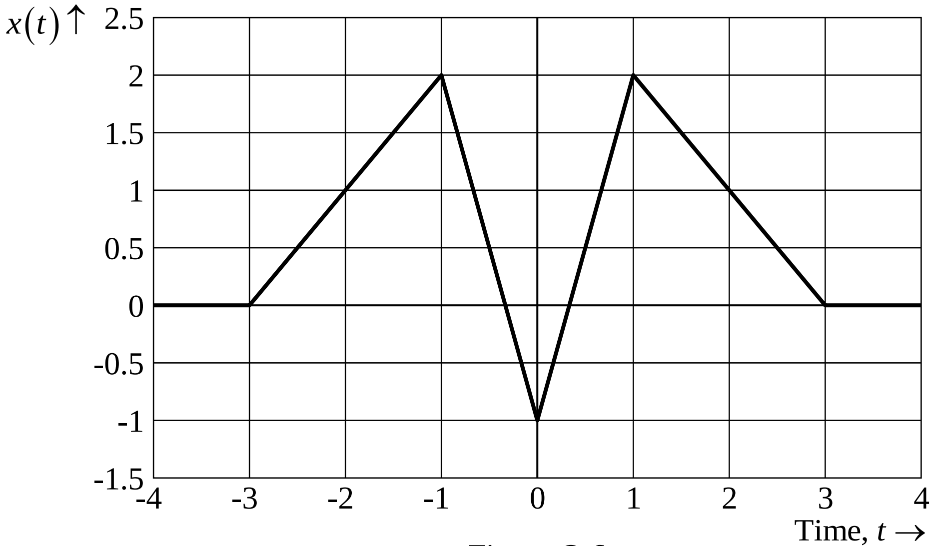

Figure Q.6

(a) Find the values of A, B, α and β.

(4 marks)

(b) Determine the Fourier transform, $X(f)$, of $x(t)$.

(3 marks)

(c) The signal $y(t) = x(t-5)$ is a delayed version of $x(t)$. Sketch the phase spectrum of $y(t)$.

(3 marks)

(d) The periodic signal $x_p(t)$ can be obtained by replicating $x(t)$ at a period of 6 seconds. Obtain an expression for $x_p(t)$ in terms of $x(t)$ and the Dirac δ-function.

(2 marks)

(e) Determine the Fourier transform, $X_p(f)$, of the periodic signal $x_p(t)$.

(4 marks)

(f) Determine the Fourier series coefficient, $X_{p,k}$, of the periodic signal $x_p(t)$.

(2 marks)

(g) Find an expression for the average power of $x_p(t)$.

(2 marks)

Q.7 The simplified model of a car suspension system shown in Figure Q.7(a) has a transfer function

$$G(s)=\frac{Y(s)}{X(s)}=K\frac{s+\alpha}{s^2+1.6\beta s+\beta^2}$$

where $X(s)$ and $Y(s)$ are, respectively, the Laplace transforms of the verticle displacements $x(t)$ and $y(t)$. The parameters K, α and β are positive constants.

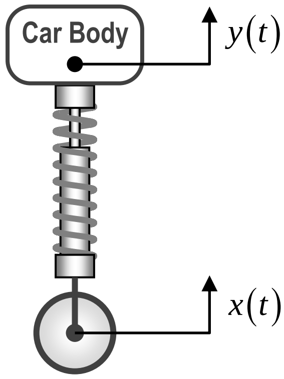

Figure Q.7(a)

Figure Q.7(b) shows the straightline Bode magnitude plot of $G(s)$.

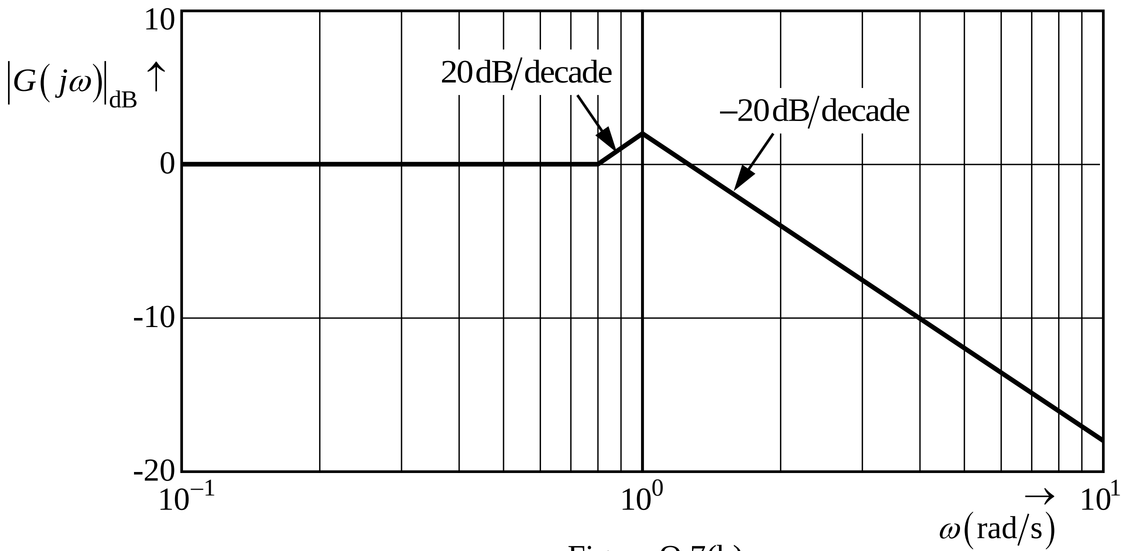

Figure Q.7(b)

(a) Find the poles of $G(s)$ in terms of β. Hence, determine whether the 2nd-order factor in $G(s)$ is undamped, underdamped, critically damped or overdamped.

(5 marks)

(b) What are the values of K, α and β? (5 marks)

(c) Derive the impulse response of the suspension system. (5 marks)

(d) Suppose the car is travelling on a path that has a kerb and the input to the suspension system may be modeled by $x(t)=0.4u(t)$. What will the steady-state value of $y(t)$ be?

(5 marks)

Q.8 Consider the circuit shown in Figure-Q.8, where L = 1/6 H, C = 1 F, $R1$ = 1/5 ohms and $R2$ = 5 ohms. Assume that switch S_1 was at position ① and S_2 was open for a long time before $t = 0$. At $t = 0$, S_1 is thrown to position ② while S_2 is closed.

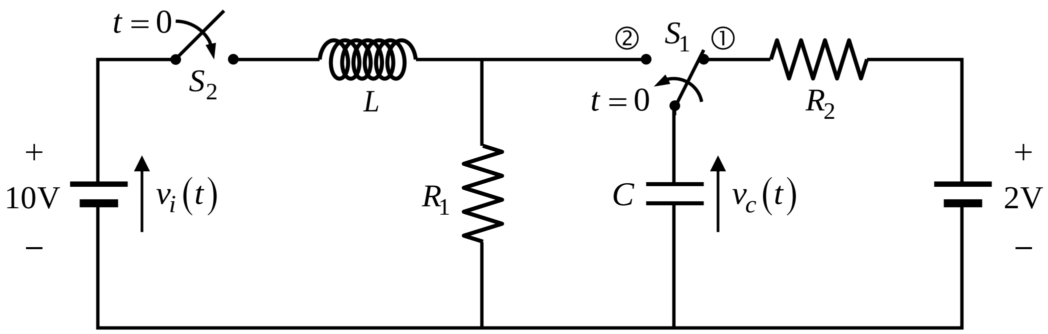

Figure Q.8

(a) Derive the transfer function $G(s) = \dfrac{V_c(s)}{V_i(s)}$ for the system for $t \geq 0$.

(8 marks)

(b) Derive the signal $v_c(t)$ for $t \geq 0$.

(12 marks)

END OF QUESTIONS

This page is intentionally left blank to facilitate detachment of the formula sheets for easy reference.

Fourier Series: $$\begin{cases} X_k = \dfrac{1}{T}\displaystyle\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k\,t/T)\,dt \\ x(t) = \displaystyle\sum_{k=-\infty}^{\infty} X_k \exp(j2\pi k\,t/T) \end{cases}$$

Fourier Transform: $$\begin{cases} X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$$

FOURIER TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
Triangle	$\mathrm{tri}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}^2(fT)$
Sine Cardinal	$\mathrm{sinc}\left(\frac{t}{T}\right)$	$T\,\mathrm{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp\left(-\alpha^2\pi^2 f^2\right)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES

	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi f t_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty}X_1(\zeta)X_2(f-\zeta)\,d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)\,d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$ $\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s)=\int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n[th] order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES

	Time-domain	*s*-domain
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)u(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\dfrac{1}{\lvert\alpha\rvert}X\left(\dfrac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\dfrac{1}{s}X(s)$
Differentiation in the time-domain	$\dfrac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\dfrac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\dfrac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\dfrac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\dfrac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

System Type	Transfer Function (Standard Form)	Unit Impulse and Unit Step Responses	Remarks
1st order system	$G(s)=\frac{K}{T}\cdot\frac{1}{s+1/T}$	$y_\delta(t)=\frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$ $y_{step}(t)=K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	T : Time-constant K : DC Gain Real Pole at $s=-\frac{1}{T}$
2nd order system $(\zeta>1)$ **Overdamped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}$	$y_\delta(t)=\left[K_1\exp(-p_1t)+K_2\exp(-p_2t)\right]u(t)$ $y_{step}(t)=\left[K-\frac{K_1}{p_1}\exp(-p_1t)-\frac{K_2}{p_2}\exp(-p_2t)\right]u(t)$	K : DC Gain $\left.\begin{array}{l}p_1=\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\\ p_2=\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\end{array}\right\\}$ $K_1=-K_2=\frac{K\omega_n^2}{p_2-p_1}$ Real Distinct Poles at $s=-p_1$ and $s=-p_2$
2nd order system $(\zeta=1)$ **Critically damped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K\omega_n{}^2}{(s+\omega_n)^2}$	$y_\delta(t)=K\omega_n^2 t\exp(-\omega_n t)u(t)$ $y_{step}(t)=K\left[1\text{-}\exp(\text{-}\omega_n t)\text{-}\omega_n t\exp(\text{-}\omega_n t)\right]u(t)$	K : DC Gain Real Repeated Poles at $s=-\omega_n$
2nd order system $(0<\zeta<1)$ **Underdamped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K\left(\sigma^2+\omega_d^2\right)}{(s+\sigma)^2+\omega_d^2}$	$y_\delta(t)=K\frac{\omega_n^2}{\omega_d}\exp(-\sigma t)\sin(\omega_d t)u(t)$ $y_{step}(t)=K\left[1-\frac{\omega_n}{\omega_d}\exp(-\sigma t)\sin(\omega_d t+\phi)\right]u(t)$	K : DC Gain ω_n : Undamped Natural Frequency ζ : Damping Ratio ω_d : Damped Natural Frequency $\sigma=\zeta\omega_n$ $\quad\omega_d^2=\omega_n^2\left(1-\zeta^2\right)$ $\quad\omega_n^2=\sigma^2+\omega_d^2$ $\quad\tan(\phi)=\frac{\omega_d}{\sigma}$ Complex Conjugate Poles at $s=-\sigma\pm j\omega_d$
2nd order system $(\zeta=0)$ **Undamped**	$G(s)=\frac{K\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$ $=\frac{K\omega_n{}^2}{s^2+\omega_n^2}$	$y_\delta(t)=K\omega_n\sin(\omega_n t)u(t)$ $y_{step}(t)=K\left(1-\cos\omega_n t\right)u(t)$	K : DC Gain ω_n : Undamped Natural Frequency Imaginary Conjugate Poles at $s=\pm j\omega_n$

2nd order system RESONANCE $\left(\mathbf{0 \le \zeta < 1/\sqrt{2}}\right)$ ➔ *RESONANCE FREQUENCY* : $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ *RESONANCE PEAK* : $M_r = \left|G\left(j\omega_r\right)\right| = \dfrac{K}{2\zeta\sqrt{1-\zeta^2}}$

Trigonometric Identities

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\theta) = 0.5\left(e^{j\theta} + e^{-j\theta}\right)$	$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
$\sin(\theta) = -0.5j\left(e^{j\theta} - e^{-j\theta}\right)$	$\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$
$\sin^2(\theta) + \cos^2(\theta) = 1$	
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta) = 0.5\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$	$\cos(\alpha)\cos(\beta) = 0.5\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
$\sin^2(\theta) = 0.5\left[1 - \cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta) = 0.5\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$
$\cos^2(\theta) = 0.5\left[1 + \cos(2\theta)\right]$	$C\cos(\theta) - S\sin(\theta) = \sqrt{C^2+S^2}\cos\left[\theta + \tan^{-1}\left(\dfrac{S}{C}\right)\right]$

Complex Unit $(\boldsymbol{j})$ ➔ $\left(j = \sqrt{-1} = e^{j\pi/2} = e^{j90^\circ}\right)$ $\left(-j = \dfrac{1}{j} = e^{-j\pi/2} = e^{-j90^\circ}\right)$ $\left(j^2 = -1\right)$

Definitions of Basic Functions

Rectangle:

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1; & -T/2 \le t < T/2 \\ 0; & \text{elsewhere} \end{cases}$$

Triangle:

$$\mathrm{tri}\left(\frac{t}{T}\right) = \begin{cases} 1 - |t|/T; & |t| \le T \\ 0; & |t| > T \end{cases}$$

Sine Cardinal:

$$\mathrm{sinc}\left(\frac{t}{T}\right) = \begin{cases} \dfrac{\sin(\pi t/T)}{\pi t/T}; & t \ne 0 \\ 1; & t = 0 \end{cases}$$

Signum:

$$\mathrm{sgn}(t) = \begin{cases} 1; & t \ge 0 \\ -1; & t < 0 \end{cases}$$

Unit Impulse:

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \ne 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

Unit Step:

$$u(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$