EE2023 Signals & Systems
Chapter 1 – Signals & Classification of Signals

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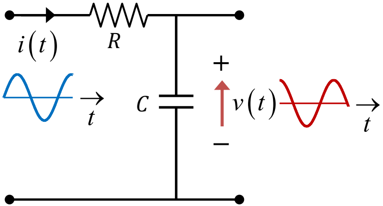
Definitions

- Signals
  - A **signal** is the mathematical representation of a physical quantity that conveys information about the behavior or nature of a phenomenon.
  - Signals can manifest in many forms such as electrical voltage or current, radio wave, infrared and ultraviolet rays, lightwave, sound wave, mechanical pressure,etc.
  - Mathematically, a signal is represented as a function of an **independent variable**.

A signal expressed as a function of time, t, is called the **time-domain** representation of the signal.



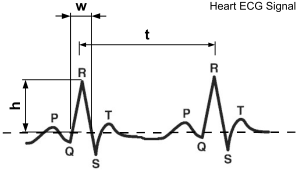
- Systems
  - A system generates a response, or output signal, for a given input signal. A system is thus a relationship between two signals.

Signal Classifications

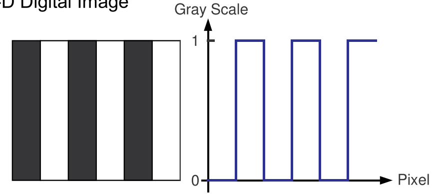
Different kinds of signals :

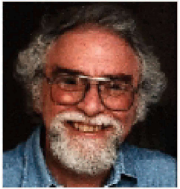
- Continuous-time and discrete-time signals
- Analog and digital signals
- Periodic and aperiodic signals
- Real and complex signals
- Deterministic and random signals
- Energy and Power signals

Sound Signal

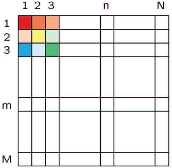


1-D Digital Image





2-D Digital Image

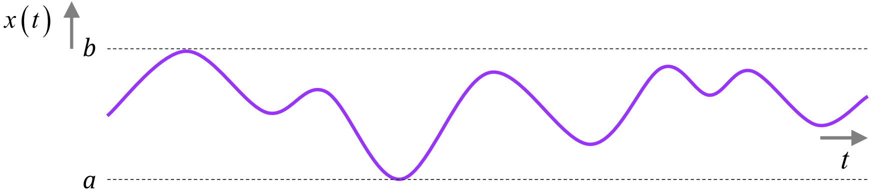


Continuous-time Signal: Analogue vs Digital Signal

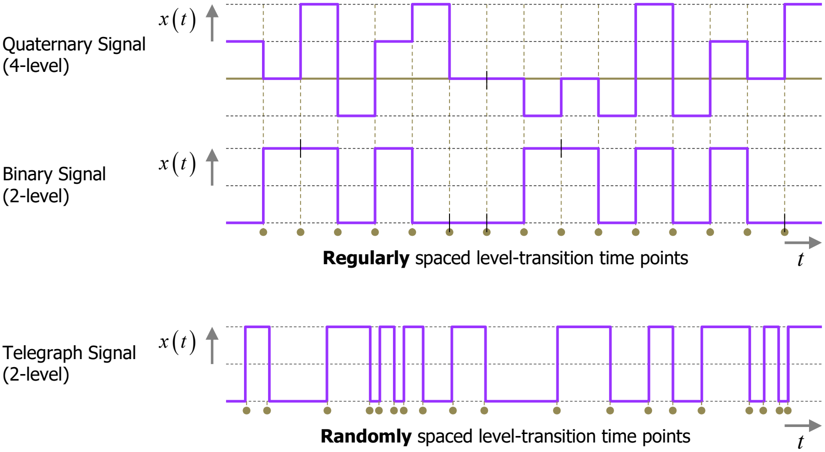
Continuous-time signal

A signal that is defined for *all time instants* in an interval of interest i.e. the independent variable is a continuous variable.

- **Analogue Signal** :
  A *continuous-time signal* that can take on any value in the continuous range (a, b) where a may be $-\infty$ and b may be $+\infty$.



- **Digital Signal** :
  A *continuous-time signal* that can take on only a finite number of values in the set $\{\alpha_1, \alpha_2, \cdots, \alpha_m, \cdots, \alpha_M\}$ at any time instant, where $\alpha_m \in (-\infty, \infty)$; $m = 1, 2, \cdots, M$.



Discrete-time Signal: Analogue vs Digital Sequence

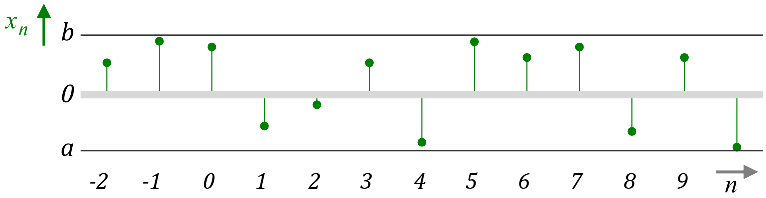
Discrete-time signal

Discrete-time signal is defined only at discrete time points. Usually denoted as a **sequence of numbers**, x_n, where n is an indexing integer.

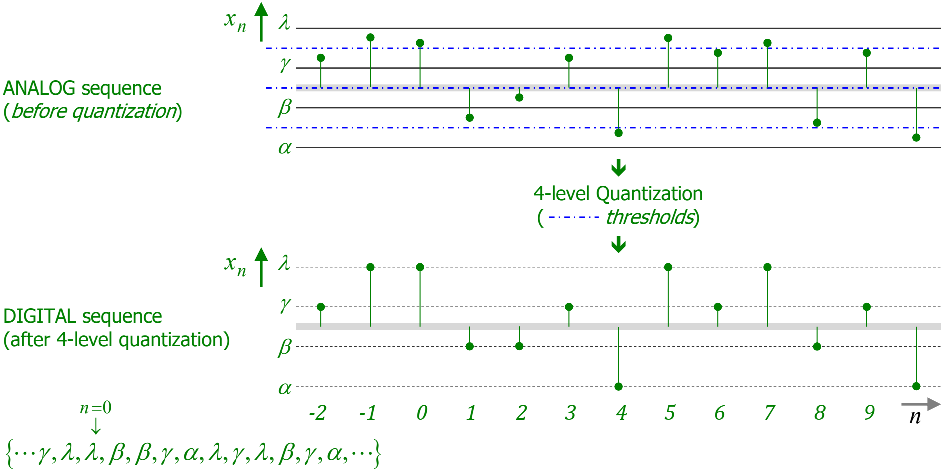
$$\{\ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots\}$$

Discrete-time signals may evolve naturally, or obtained by **sampling** a continuous-time signal, $x(t)$, such that $x_n = x(t_n)$ where t_n are discrete time points.

- **Analogue Sequence** :
  A *discrete-time signal* that can take on any value in the continuous range (a, b) where a may be $-\infty$ and b may be $+\infty$.



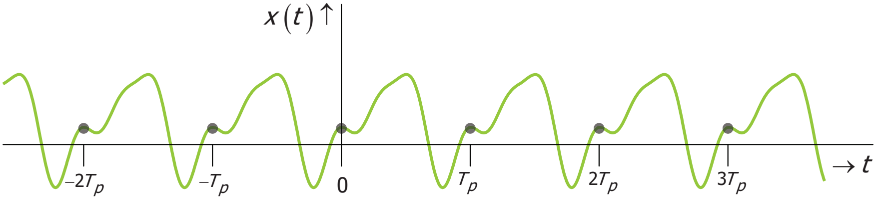
- **Digital Sequence** :
  A *discrete-time signal* that can only take on values in the set $\{\alpha_1, \alpha_2, \cdots, \alpha_m, \cdots, \alpha_M\}$ at specific time instances, where $\alpha_m \in (-\infty, \infty)$; $m = 1, 2, \cdots, M$.



Periodic vs Aperiodic Signals

- A signal is **periodic** if there is a non-zero positive value, T_p, satisfying

$$x(t) = x(t + T_p) \text{ for all values of } t .$$



- The smallest value of T_p is called the **fundamental period**, or simply **period** of $x(t)$.
- The reciprocal of the **fundamental period** (in seconds) is called the **fundamental cyclic frequency** (in Hertz) of $x(t)$.
  - Fundamental cyclic frequency : $f_p = \frac{1}{T_p}$ Hz
  - Fundamental angular frequency : $\omega_p = \frac{2\pi}{T_p}$ rad/sec
- Any signal that is not periodic is called **aperiodic**.

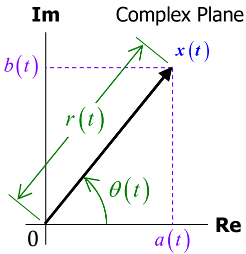
Real vs Complex Signals

A **complex signal**, $x(t)$, is a signal that may be expressed as

$$\underbrace{x(t) = a(t) + jb(t)}_{\text{Cartesian form}} \quad \text{or} \quad \underbrace{x(t) = r(t)e^{j\theta(t)}}_{\text{Polar form}} \quad \text{where } j = \sqrt{-1}.$$

$a(t)$, $b(t)$, $r(t)$ and $\theta(t)$ are real signals known as

$a(t)$	:	Real part of $x(t)$
$b(t)$	:	Imaginary part of $x(t)$
$r(t) = \lvert x(t)\rvert$	:	Magnitude (or Modulus) of $x(t)$
$\theta(t) = \angle x(t)$	:	Phase (or Argument) of $x(t)$



Phasor Diagram of $x(t)$

Relationship between Cartesian and Polar form of $x(t)$ may be derived using the **Euler's Identity**, $\boxed{e^{j\phi} = \cos\phi + j\sin\phi}$.

Replacing the $e^{j\theta(t)}$ term in the polar form of $x(t)$ by the Euler's Identity,

$$\begin{aligned} x(t) &= r(t)e^{j\theta(t)} \\ &= r(t)\cos[\theta(t)] + j\,r(t)\sin[\theta(t)] \end{aligned}$$

Comparing $x(t) = a(t) + jb(t)$ with $x(t) = r(t)e^{j\theta(t)} = r(t)\cos[\theta(t)] + j\,r(t)\sin[\theta(t)]$,

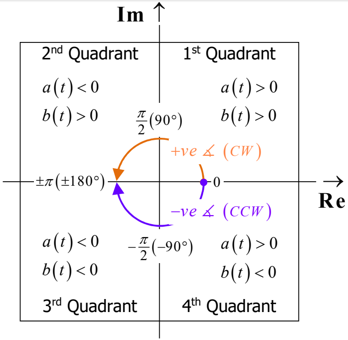
$$a(t) = r(t)\cos[\theta(t)] \qquad b(t) = r(t)\sin[\theta(t)]$$

or

$$r(t) = \sqrt{a(t)^2 + b(t)^2} \qquad \theta(t) = \tan^{-1}\left[\frac{b(t)}{a(t)}\right]$$

- $\theta(t)$, the angle formed by the phasor and positive real axis, may take on a value in the $[-180^o, 180^o]$ range. However, that the $\tan^{-1}(\cdot)$ function in a calculator returns a value in the $[-90^o, 90^o]$ range.

- To obtain the correct phase angle,
  - Compute $\tilde{\theta}(t) = \tan^{-1}\left[\dfrac{|b(t)|}{|a(t)|}\right] \in [0, 90^o]$
  - $$\theta(t) = \begin{cases} \tilde{\theta}(t) & \text{if } a(t) > 0, b(t) > 0 \\ -\tilde{\theta}(t) & \text{if } a(t) > 0, b(t) < 0 \\ 180^o - \tilde{\theta}(t) & \text{if } a(t) < 0, b(t) > 0 \\ \tilde{\theta}(t) - 180^o & \text{if } a(t) < 0, b(t) < 0 \end{cases}$$



- Special cases where phase angle, $\theta(t)$, are known include:

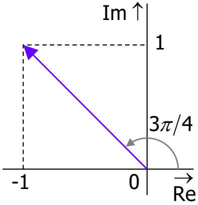
$$\begin{aligned} \theta(t) &= 0^o && \text{if } a(t) > 0 \text{ and } b(t) = 0 \\ \theta(t) &= 90^o && \text{if } a(t) = 0 \text{ and } b(t) > 0 \\ \theta(t) &= \pm 180^o && \text{if } a(t) < 0 \text{ and } b(t) = 0 \\ \theta(t) &= -90^o && \text{if } a(t) = 0 \text{ and } b(t) < 0 \end{aligned}$$

  - If $b(t) = 0$ or $\theta(t) = 0$ or $\pm n\pi$, then $x(t)$ is a **real signal**.

Example Write $z(t) = (1-j)e^{j2\pi t}$ in Cartesian and Polar forms. Calculate $\angle z(0.5)$ using both forms.

Cartesian form:

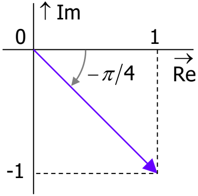
$$\begin{aligned} z(t) &= (1-j)e^{j2\pi t} = (1-j)[\cos(2\pi t) + j\sin(2\pi t)] \\ &= \cos(2\pi t) + j\sin(2\pi t) - j\cos(2\pi t) + \sin(2\pi t) \\ &= \underbrace{\sin(2\pi t) + \cos(2\pi t)}_{a(t)=Re[z(t)]} + j\underbrace{[\sin(2\pi t) - \cos(2\pi t)]}_{b(t)=Im[z(t)]} \\ z(0.5) &= -1 + j \\ \angle z(0.5) &= \tan^{-1}\frac{b(0.5)}{a(0.5)} = \tan^{-1}\frac{1}{-1} = \frac{3\pi}{4} \end{aligned}$$



Plot of $-1+j$

Polar form:

$$\begin{aligned} z(t) &= \underbrace{\sqrt{2}e^{-j\frac{\pi}{4}}}_{1-j} e^{j2\pi t} = \sqrt{2}e^{j\left(2\pi t - \frac{\pi}{4}\right)} \\ z(0.5) &= \sqrt{2}e^{j\frac{3\pi}{4}} \qquad \angle z(0.5) = \frac{3\pi}{4} \end{aligned}$$



Plot of $1-j$

Deterministic or Random Signals

Deterministic signal

- A deterministic signal is a signal in which each value is fixed by a mathematical expression, rule, or table. All future values of the signal can be predicted with complete confidence.
  - Example : $x(t) = \cos(2\pi t)$
    At any time instant t, $x(t)$ is exactly determined as $\cos(2\pi t)$.

Random signal

- A random signal has a lot of uncertainty about its behavior. The future values of the signal cannot beaccurately predicted and can usually only be guessed based on the signal statistics.
  - Example : $x(t) = \cos(2\pi t + \varphi)$ where φ can take on values in the set $\{0, 0.5\pi, \pi, 1.5\pi\}$ with equal probability.
    At any time instant t, $x(t)$ cannot be exactly determined since it can assume the values of $\cos(2\pi t)$, $-\sin(2\pi t)$, $-\cos(2\pi t)$ and $\sin(2\pi t)$ with equal probabilities. In this case, $x(t)$ can only be described by its statistics, such as: $\left[\text{Mean of } x(t) = 0,\ \text{Var of } x(t) = \frac{1}{2}\right]$.

Energy and Power Signals

- Idea of "strength" or "size" of a signal is crucial to many applications. For instance, it is useful to know
  - how much electricity can be used in a defibrillator without ill effects.
  - if the signal driving a set of headphones is enough to create a sound.

Energy signal

- The **total energy**, E, of a complex signal $x(t)$ is defined as

$$E = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} |x(t)|^2 \, dt$$

- $x(t)$ is said to be an energy signal if and only if $0 < E < \infty$.

Power signal

- The **average power**, P, of a complex signal $x(t)$ is defined as

$$P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt$$

- $x(t)$ is said to be a power signal if and only if $0 < P < \infty$.

- Relationship between the total energy and average power of $x(t)$ may be obtained by "combining" their definitions:

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \\
&= \frac{1}{2\cdot\infty} \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\
&= \frac{E}{2\cdot\infty} \qquad \because \; E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt
\end{aligned}
$$

Implication of $P = \dfrac{E}{2\cdot\infty}$ relationship are
  - Energy signals have zero average power, since a finite E implies that $P = 0$.
  - Power signals have infinite total energy, since a finite P implies $E = \infty$.
- Signals that satisfy neither $0 < E < \infty$ nor $0 < P < \infty$ are neither energy nor power signals.

Example Consider the signal $x(t) = \begin{cases} e^{-\alpha t} & ; t \geq 0 \\ 0 & ; t < 0 \end{cases} \quad ; \alpha > 0$

$$\text{Total Energy, } E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_0^{\infty} e^{-2\alpha t} dt = \left[\frac{e^{-2\alpha t}}{-2\alpha} \right]_0^{\infty} = \frac{1}{2\alpha}$$

Since E is finite, $x(t)$ is an **energy signal**. **Average Power** of $x(t)$ is zero.

Example Consider the signal $x(t) = \begin{cases} \alpha t & ; t \geq 0 \\ 0 & ; t < 0 \end{cases} \quad ; \alpha \neq 0$

$$\begin{aligned}
\text{Total Energy, } E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_0^{\infty} \alpha^2 t^2 dt = \left[\frac{\alpha^2 t^3}{3} \right]_0^{\infty} = \infty \\
\text{Average Power, } P &= \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt \\
&= \lim_{\tau \to \infty} \frac{1}{2\tau} \int_0^{\tau} \alpha^2 t^2 dt = \lim_{\tau \to \infty} \frac{1}{2\tau} \left[\frac{\alpha^2 t^3}{3} \right]_0^{\tau} = \lim_{\tau \to \infty} \frac{\alpha^2 \tau^2}{6} = \infty
\end{aligned}$$

$x(t)$ is neither an energy nor a power signal.

Example Consider the periodic signal $x_p(t) = \alpha \cos\left(\frac{2\pi t}{T}\right)$

- Instead of dividing total energy by "all time", the average power of a periodic signal, $x_p(t)$, may be computed over one-period T_p i.e.

$$P = \underbrace{\lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2\, dt}_{\text{general signal}} \xrightarrow{\text{equivalent to}} \underbrace{\frac{1}{T_p}\int_0^{T_p} |x_p(t)|^2\, dt}_{\text{periodic signal}}$$

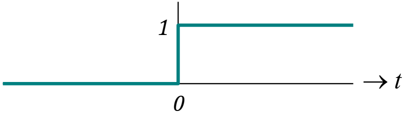
- Since $x_p(t) = \alpha \cos\left(\frac{2\pi t}{T}\right)$ is a periodic signal with period T,

$$\begin{aligned} P &= \frac{1}{T_p}\int_0^{T_p} |x_p(t)|^2\, dt = \frac{1}{T}\int_0^{T} \alpha^2 \cos^2\left(\frac{2\pi t}{T}\right) dt = \frac{\alpha^2}{2T}\int_0^{T} 1 + \cos\left(\frac{4\pi t}{T}\right) dt \\ &= \frac{\alpha^2}{2T}\left[t + \frac{\sin(\frac{4\pi t}{T})}{\frac{4\pi}{T}}\right]_0^T = \frac{\alpha^2}{2T}\left[T + \frac{\sin(4\pi)}{\frac{4\pi}{T}} - 0 - \frac{\sin(0)}{\frac{4\pi}{T}}\right] = \frac{\alpha^2}{2} \end{aligned}$$

Basic Signals – Unit Step, Signum, Rectangular

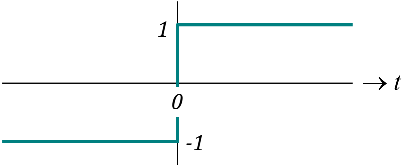
- **Unit Step function**

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



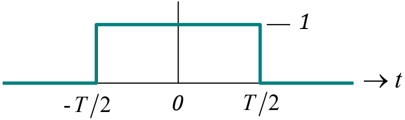
- **Signum (Sign) function**

$$\text{sgn}(t) = \begin{cases} +1, & t \geq 0 \\ -1, & t < 0 \end{cases}$$



- **Rectangle function**

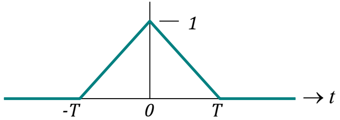
$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & \frac{-T}{2} \leq t < \frac{T}{2} \\ 0 & \text{elsewhere} \end{cases}$$



Basic Signals – Triangular, Cardinal Sine, Real Exponential

- **Triangle function**

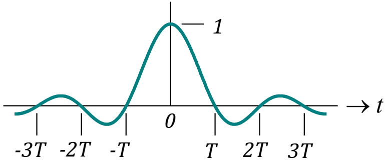
$$\text{tri}\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \leq T \\ 0 & |t| > T \end{cases}$$



- **Sine Cardinal (Sinc) function**

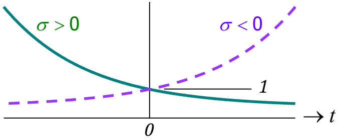
$$\text{sinc}\left(\frac{t}{T}\right) = \begin{cases} \frac{\sin(\pi t/T)}{\pi t/T}, & t \neq 0 \\ 1 & t = 0 \end{cases}$$

Note that $\text{sinc}\left(\frac{t}{T}\right) = 0$ at $t = \pm nT$ where n is a non-zero integer.



- **Real Exponential function**

$e^{-\sigma t}$

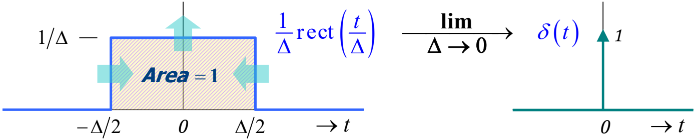


Basic Signals - Unit Impluse, Dirac Comb

Unit Impulse (Dirac Delta) function

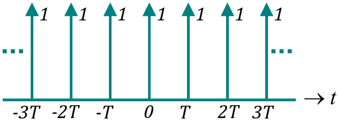
$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\epsilon}^{\epsilon} \delta(t)\, dt = 1;\ \forall \epsilon > 0$$

The unit impulse function may be defined as the limiting case of a regtangular function which has unit area and infinitesimally small base.



Dirac Comb function

$$\xi_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Basic Signals – Sinusoids

Sinusoidal function is a collective term for a general class of periodic signals that have the following forms:

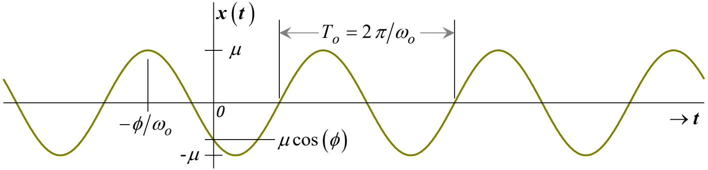
- **Cosine**: $x(t) = \mu\cos(\omega_o t + \phi) = \frac{\mu}{2}\left[e^{j(\omega_o t+\phi)} + e^{-j(\omega_o t+\phi)}\right]$
- **Sine**: $x(t) = \mu\sin(\omega_o t + \phi) = \frac{\mu}{2j}\left[e^{j(\omega_o t+\phi)} - e^{-j(\omega_o t+\phi)}\right]$
- **Complex Exponential**: $x(t) = \mu\, e^{j(\omega_o t+\phi)} = \mu\left[\cos(\omega_o t + \phi) + j\sin(\omega_o t + \phi)\right]$

where

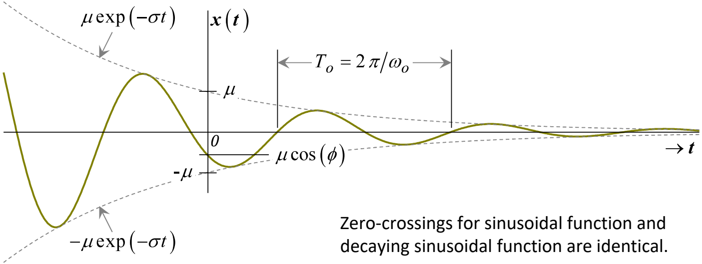
$\mu\ (> 0)$	:	amplitude (or magnitude)
ω_o	:	angular frequency (rad/s)
$f_o = \frac{\omega_o}{2\pi}$	:	cyclic frequency (Hz)
ϕ	:	phase (radians)
$\omega_o t + \phi$	:	instaneous phase (radians)

Cosine and Sine functions represent real signals, while the complex exponential function represents a complex signal.

Example Sketch $x(t) = \mu\cos(\omega_o t + \phi) = \mu\cos\left[\omega_o\left(t + \dfrac{\phi}{\omega_o}\right)\right]$, a real sinusoid.



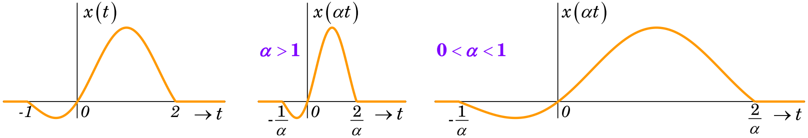
Example Sketch $x(t) = \mu e^{-\sigma t}\cos(\omega_o t + \phi)$, an exponentially decaying real sinusoid.



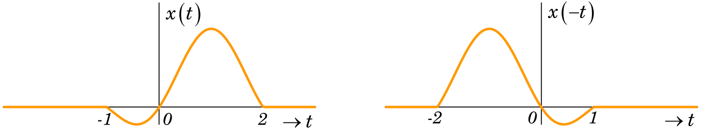
Zero-crossings for sinusoidal function and decaying sinusoidal function are identical.

Time-Scaling, Time-Reversal, and Time-Shifting of Signals

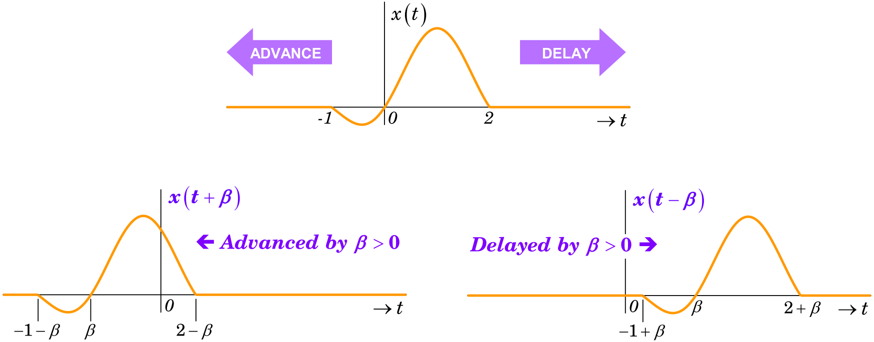
- **Time-Scaling**
  Time-scaling of a signal $x(t)$ is effected by replacing the time variable t by αt, where α is a positive real number.
  - $0 < \alpha < 1$: uniform **expansion** of $x(t)$ along the time axis.
  - $\alpha > 1$: uniform **contraction** of $x(t)$ along the time axis.



- **Time-Reversal**
  Time-reversal of a signal $x(t)$ is effected by replacing the time variable t by $-t$.



- **Time-shifting**
  Time-shifting of a signal $x(t)$ is effected by replacing the time variable t by $t - \beta$, where β is a real number.
  - $\beta > 0$: **Delaying** $x(t)$ by β unit of time.
  - $\beta < 0$: **Advancing** $x(t)$ by β unit of time.



Example

Give $w(t) = \begin{cases} t; & -2 \le t < 4 \\ 0; & \text{elsewhere} \end{cases}$

Sketch $w(t) = \begin{cases} x(t) = w(t-1) \\ y(t) = x(-t) \\ z(t) = y\left(\frac{2}{3}t\right) \end{cases}$

To sketch $z(t)$

- Replace t in $y(t) = x(-t)$ by $\frac{2}{3}t$

$$z(t) = y\left(\tfrac{2}{3}t\right) = x\left(-\tfrac{2}{3}t\right)$$

- Replace t in $x(t) = w(t-1)$ by $-\frac{2}{3}t$

$$z(t) = x\left(-\tfrac{2}{3}t\right) = w\left(-\tfrac{2}{3}t - 1\right)$$