National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems Notes 2

This chapter focuses on

- Classification of systems
- Models of linear time invariant systems

1 Classification of Systems

Physical systems, in the broadest sense are interconnections of components, devices or subsystems. Examples are communication systems, mechanical systems, electronic systems, chemical processing systems, etc. A system can be viewed as a process in which input signals are "transformed" or "processed" by the system, resulting in other signals which can be considered as outputs.

In general, systems can be classified as continuous time or discrete time systems. In this course, the focus is only on continuous time signals. In such systems, continuous-time input signals are applied which results in continuous-time output signals. They are further classified into the following categories according to their basic properties :

1. Systems with and without Memory

   A system is said to be *static* or *memoryless* if its output at a given time is dependent only on the input at that same time. For example, a resistor which obeys Ohm's law is a *memoryless* device because $V_r(t) = I(t)R$ where R is the resistance value. The voltage across it depends only on the current at time t. On the other hand, a capacitor which behaves according to

$$i(t) = C\frac{dV_c(t)}{dt} \text{ or equivalently } V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

is a *dynamic* system or system with *memory* because the voltage across the capacitor, $V_c(t)$ depends on all values of its currents from $-\infty$ to t.

In many physical systems, memory is directly associated with storage of energy. Examples are capacitors and inductors. As a result of them having memory, they are also dynamic because they change their response according to their past history of their inputs. Again, as a result of the memory, their response do not change abruptly and thus their outputs or response are also continuous in time.

2. Causal and Non-Causal Systems

   A system is *causal* if the output at any time depends only on values of the input at the present time and in the past (ie does not depend on future values). Such systems are also referred to as non-anticipative. All memoryless or static systems are causal.

3. Stable and Unstable Systems

   A system is said to be *stable* if a bounded input produces a bounded output. Examples of a stable system includes an inverted pendulum in which a small force applied to the pendulum results in oscillations with amplitudes which become smaller and smaller over time. Eventually, the inverted pendulum will stop swinging. In terms of physics, we understand that this happens because energy from the pendulum is lost through its mechanical bearings and air resistance as the pendulum swings. After a small period of time, all the energy will be lost and the pendulum stops swinging. This is an example of a stable system.

   Examples of an unstable system includes the "screeching" sound that one often hears when there is feedback between the loudspeaker and the microphone. The "screeching" sound gets louder and louder even though the input remains the same. In terms of signals, the signal amplitudes becomes larger and larger, resulting in the increasing loudness of the "screeching" sound.

   There are some very specific notions in systems theory that describes different forms of stability. In this course, we are only interested in bounded-input bounded-output notion of stability.

4. Time Varying and Time Invariant Systems

A system is *time invariant* if the behaviour and characteristics of the system are fixed over time. For example, an RC-circuit is time invariant because we expect to get the same results from an experiment with this circuit if we ran an identical experiment today and tomorrow. On the other hand, if we run a car with a full load of fuel, the behaviour of the car changes over time because as the fuel is used up, the mass of the car is different and this changes the characteristics of the car. The car may be more responsive as its mass reduces. Hence the characteristic of the car changes as it moves over time. As a result, it is much more complex to try to model a car accurately.

In mathematical terms, a time invariant system can be described by differential equations with constant coefficients. Examples :

- Time invariant system with first order differential equation as its model :

$$k\frac{dy}{dt} + y(t) = u(t)$$

  where k is a constant coefficient and $u(t)$ and $y(t)$ are the input and output, respectively, of the system.

- Time varying system with 2nd order differential equation as its model :

$$m(t)\frac{d^2y}{dt^2} + k(t)\frac{dy}{dt} + y(t) = u(t)$$

  where $m(t)$ and $k(t)$ are coefficients which are time dependent and $u(t)$ and $y(t)$ are the input and output, respectively, of the system. When $m(t)$ and $k(t)$ takes different values in time, the characteristics of the solution to the differential equation also changes.

5. Linear and Non-Linear Systems

A *linear* system is one that has the properties of :

- Additivity : $y(t) = f(x_1(t) + x_2(t)) = f(x_1(t)) + f(x_2(t))$
- Scaling : If $y(t) = f(x(t))$, then $\alpha y(t) = f(\alpha x(t)) = \alpha f(x(t))$

The property of additivity is also known as the *superposition* property which holds for all linear systems. Examples of linear system are in electrical circuits which contain linear components. Examples of linear components are resistors, inductors and capacitors. Example of a *non-linear* component is the diode.

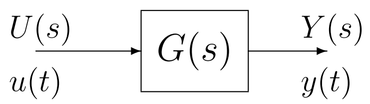
A *non-linear* system is one where the above properties do not hold.

We often use mathematical tools to help us describe or model the behaviour of many different types of systems. However, not many systems can be described precisely by mathematics. But it is always possible to approximate their behaviour based on physical or natural laws of Physics. Even when it is possible to write mathematical equations to describe the systems, it is not always the case that they can be solved easily. In particular, non-linear systems cannot be generalized easily. On the other hand, linear time-invariant systems can be described elegantly by mathematics. Their behaviours can be generalized easily and these lead to some nice properties that can be deduced for such systems, in many instances without having to solve their mathematical equations explicitly. The focus of this module is therefore on this class of systems.

2 Models of Linear Time Invariant Systems

Linear time invariant (LTI) systems are generally modeled using differential equations in the time domain or transfer functions in the frequency domain. Transfer functions are defined by the ratio of the output response to the input response in the s-domain where s is the Laplace variable. When $s = j\omega$, the model gives the frequency response of the system.

This is illustrated as follows :



$$\text{Transfer function : } G(s) = \frac{Y(s)}{U(s)}$$

Fig. 1: Input-Output Model of a Plant or System

In time domain, the general differential equation of the plant which is given by :

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dt^{n-2}} + \ldots + a_0 y(t) = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + b_{m-2}\frac{d^{m-2}u}{dt^{m-2}} + \ldots + b_0 u(t) \quad (1)$$

where $m < n$, $u(t)$ and $y(t)$ are the input and output, respectively, while $a_i, i = 1, \ldots, n$, $b_i, i = 1, \ldots, m$ are constant coefficients of the differential equation. The transfer function, $G(s)$ of the plant is obtained by taking Laplace transform on both sides of (1), assuming zero initial conditions as follows:

$$(a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0)Y(s) = (b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_0)U(s)$$

$$\text{Transfer function : } G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0}$$

$$= K\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\ldots\left(\frac{s}{z_m}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\ldots\left(\frac{s}{p_n}+1\right)} \quad (2)$$

$$= \frac{Kp_1p_2\ldots p_n}{z_1z_2\ldots z_m}\frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} \quad (3)$$

$$= K'\frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} \quad (4)$$

where $K = \frac{b_0}{a_0}$, $K' = \frac{Kp_1p_2\ldots p_n}{z_1z_2\ldots z_m}$, $-p_i$ and $-z_i$ are the system poles and zeros, respectively, of $G(s)$. Note, from (2) to (4), the different ways that a transfer function can be factorized. The general transfer function, $G(s)$, in (2) to (4) is said to

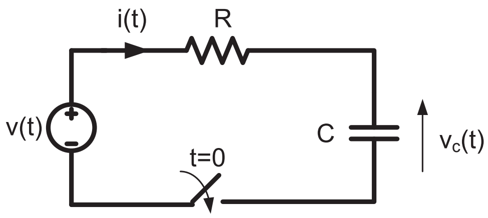
- be an n^{th} order system with n number of poles. Poles are the roots obtained by solving $a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0 = 0$ or $\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\ldots\left(\frac{s}{p_n}+1\right) = 0$ or $(s+p_1)(s+p_2)\ldots(s+p_n) = 0$.

- have m number of zeros. Zeros are the roots obtained by solving $b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_0 = 0$ or $\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\ldots\left(\frac{s}{z_m}+1\right) = 0$ or $(s+z_1)(s+z_2)\ldots(s+z_m) = 0$.

- have a pole-zero excess of $(n-m)$. For practical systems, $n > m$.

Examples of transfer functions of common circuits :

RC circuit



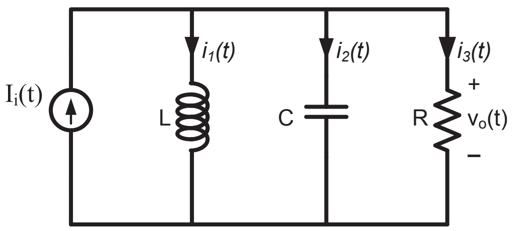
Derivation of transfer function

Using voltage division rule :

$$
\begin{aligned}
V_c(s) &= \frac{\frac{1}{sC}}{R+\frac{1}{sC}}V(s) \\
\frac{V_c(s)}{V(s)} &= G(s) = \frac{1}{sRC+1}
\end{aligned}
$$

Thus, an RC circuit has a first order transfer function.

RLC circuit



Derivation of transfer function

Using current division rule :

$$
\begin{aligned}
I_3(s) &= \frac{Z_{LC}}{Z_{LC}+R}I_i(s) \\
Z_{LC} &= \frac{1}{\frac{1}{sL}+sC} \\
&= \frac{sL}{s^2LC+1} \\
I_3(s) &= \frac{\frac{sL}{s^2LC+1}}{\frac{sL}{s^2LC+1}+R}I_i(s) \\
&= \frac{sL}{s^2RLC+sL+R}I_i(s) \\
V_0(s) &= \frac{sL}{s^2RLC+sL+R}R \\
\frac{V_0(s)}{I_i(s)} &= G(s) = \frac{sLR}{s^2RLC+sL+R}
\end{aligned}
$$

Thus, an RC circuit has a second order transfer function.

To understand the role of poles and zeros, consider the output :

$$
Y(s) = G(s)U(s) = \frac{Kp_1p_2\dots p_n}{z_1z_2\dots z_m}\frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}U(s)
$$

Note that alternative form of $G(s)$ such as (2) or (4) may also be used.

$U(s)$ is the LT of the input signal. If $u(t)$ is a unit step signal, then $U(s) = 1/s$. If $u(t) = e^{-at}$, then $U(s) = \frac{1}{s+a}$. Thus $U(s)$ may also contain poles and zeros. We distinguish the poles of $U(s)$ from the poles of $G(s)$ by referring to the former as input poles while the latter as system poles. Suppose we take the example of $U(s) = 1/s$, then

$$\begin{aligned} Y(s) &= \frac{Kp_1p_2\dots p_n}{z_1z_2\dots z_m}\frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}\frac{1}{s} \\ &= \frac{A_1}{(s+p_1)} + \frac{A_2}{(s+p_2)} + \dots + \frac{A_n}{(s+p_n)} + \frac{K}{s} \end{aligned}$$

where A_i are the constants obtained after partial factorization. Taking the inverse LT of $Y(s)$,

$$\begin{aligned} y(t) &= \underbrace{A_1e^{-p_1t} + A_2e^{-p_2t} + \dots + A_ne^{-p_nt}}_{y_{tr}(t)} + \underbrace{K}_{y_{ss}(t)} \\ &= y_{tr}(t) + y_{ss}(t) \end{aligned} \qquad (5)$$

where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response. Notice that $y_{tr}(t)$ is a function of the system poles $(-p_i, i = 1, \dots, n)$ while $y_{ss}(t)$ is similar to the input signal, which in this case, a constant unit step signal produces a constant term in the output $y(t)$. Furthermore,

$$\lim_{t\to\infty} y_{tr}(t) = 0, \quad \lim_{t\to\infty} y_{ss}(t) = K$$

From this example, we deduce that if the input signal is bounded, $y_{ss}(t)$ will remain bounded. However, for $y_{tr}(t)$, its final value depends on the system poles $-p_i$. If all the system poles have negative real parts ie $-p_i < 0$, then $y_{tr}(\infty) = 0$. If at least one of the system poles is positive $(-p_i > 0)$, then $y_{tr}(\infty) = \infty$.

The example above immediately tells us that the output of a system will be bounded if all poles $(-p_i)$ of its transfer function $G(s)$ are negative and the input signal is bounded.

In summary, we conclude that, in general, the output signal will be bounded if

- Input $u(t)$ is bounded
- All system poles have negative real parts.

This leads us to the definition of system stability. Firstly, stability is defined in the context of bounded-input bounded-output (BIBO). The different notions of stability are as follows :

- BIBO stable if $\lim_{t\to\infty} y_{tr}(t) = 0$. The condition for this is that all system poles must have negative real parts or simply negative.
- Unstable if $\lim_{t\to\infty} y_{tr}(t) = \infty$. This occurs when at least one system pole has positive real parts or simply positive.
- Marginally stable if $\lim_{t\to\infty} y_{tr}(t)$ has no fixed value or is non-zero. For example, $y_{tr}(t)$ could be sinusoidal and hence has no fixed value. This happens when the system poles are on the imaginary axis, including the origin.

Thus the test for BIBO stability of a system is simply to check the position of the system poles.

- If system poles are ALL negative, then system is BIBO stable.
- If at least one system pole is positive, then system is unstable.
- If at least one system pole is on the imaginary axis, then the system is marginally stable.

This becomes an easy and convenient way to check if a system will give a bounded output given a bounded input without having to calculate the output response $y(t)$ at all. System transfer functions are therefore very useful from this point of view.

An overall view of the transient response in relation to the pole locations on the complex plane is shown in Figure 3.

Notice that the zeros, $-z_i$, do not play any role in the stability of the system. This means that even if any z_i is positive, the output does not become unbounded. The zeros only affect the values of the constants A_i during partial factorization, thus affecting only the transient response.

Consider the following two examples which differ only by the presence of a zero at $s = -0.2$ in Example 2. In these 2 examples, you will see that the transient responses differ because A_i have changed when the two solutions of $y(t)$ are compared. Their step responses are given in Figure 4 for different zeros in the $G(s)$. Notice that when the zero is at $s = +0.2$,

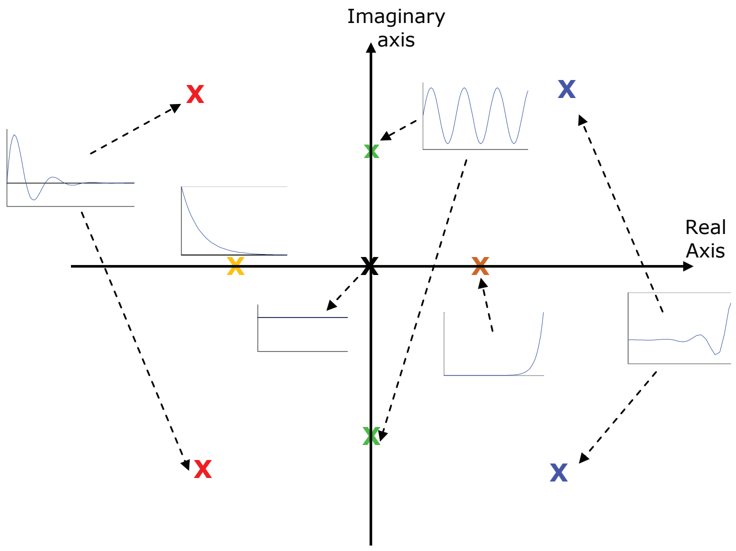


Fig. 2: Transient Response in relation to Pole Locations

the initial output response goes in the opposite direction (negative) as the final steady state output. What other conclusions can you make about the effect of zeros of $G(s)$?

Example 1

$$\begin{aligned} G(s) &= \frac{1}{35s^2+21s+1} \\ &= \frac{0.0285}{(s+0.548)(s+0.052)} \end{aligned}$$

If $u(t) = 9$, $U(s) = 9/s$, then

$$\begin{aligned} Y(s) &= \frac{0.0285}{(s+0.548)(s+0.052)}\frac{9}{s} \\ &= \frac{0.94}{s+0.548} - \frac{9.94}{s+0.052} + \frac{9}{s} \end{aligned}$$

Time domain output,

$$y(t) = \underbrace{0.94e^{-0.548t} - 9.94e^{-0.052t}}_{y_{tr}(t)} + \underbrace{9}_{y_{ss}(t)}$$

Example 2

$$\begin{aligned} G(s) &= \frac{5s+1}{35s^2+21s+1} \\ &= \frac{0.0285(5s+1)}{(s+0.548)(s+0.052)} \end{aligned}$$

If $u(t) = 9$, $U(s) = 9/s$, then

$$\begin{aligned} Y(s) &= \frac{0.0285(5s+1)}{(s+0.548)(s+0.052)}\frac{9}{s} \\ &= \frac{-1.642}{s+0.548} - \frac{7.359}{s+0.052} + \frac{9}{s} \end{aligned}$$

Time domain output,

$$y(t) = \underbrace{-1.642e^{-0.548t} - 7.359e^{-0.052t}}_{y_{tr}(t)} + \underbrace{9}_{y_{ss}(t)}$$

As a check, you may use the following checklist to determine if you have learned some important concepts so far :

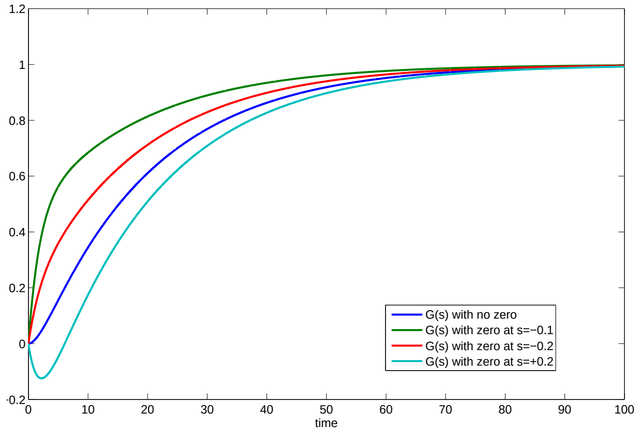


Fig. 3: Step responses with different zeros in the $G(s)$.

☐ I have learnt about the different classifications of systems.

☐ I have learnt that Laplace Transform are used to solve linear time invariant differential equations and convert such DE into a transfer function model of a system.

☐ I have learnt that transfer functions are s-domain models and when $s = j\omega$, the model is equivalent to a frequency response model of the system

☐ I have learnt that during the conversion of the DE to transfer functions, I have to set all initial conditions to zero.

☐ I have learnt what poles and zeros are. I can convert between different forms of the transfer functions, $G(s)$, like those shown in (2) and (3).

☐ I have learnt that poles determine the stability of a system. I have also learnt that zeros do not determine whether a system is stable or otherwise. It affects the coefficients of the final output expression, $y(t)$.

☐ I have learnt that poles which have negative real parts makes a system stable. Poles with positive real parts makes a system unstable while those which are purely imaginary makes a system marginally stable. A special case of the purely imaginary poles include poles which are at the origin ie $s = 0$ and such systems are also marginally stable.

In the next chapter, the focus is on some important physical parameters of first and second order transfer functions of LTI systems as well as their output responses for different types of inputs.

