EE2023 Signals & Systems
Chapter 10 – Frequency Response of LTI Systems

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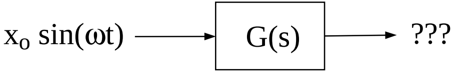
System response due to sinusoidal input

What is the steady-state output signal of a stable system when the input signal is a sinusoidal function?



- Intutively, the steady-state output will also be a sinusoidal signal as the solution to the system differential equations can be interpreted using two properties of sinusoidal signal :
  - Sinusoids can be differentiated indefinitely. Each differentiation results in another sinusoidal signal.

$$\sin\omega t \xrightarrow{\frac{d}{dt}} \omega\cos\omega t \xrightarrow{\frac{d}{dt}} -\omega^2\sin\omega t \xrightarrow{\frac{d}{dt}} \ldots$$

  - The result of adding two sinusoids of the same frequency together is another sinusoid with different amplitude and phase. However, the frequency of the signals is the same.

Conclusion is consistent with the observation that the steady-state response is related to the input pole.

Derivation of steady-state sinusoidal response using Laplace Transform

Consider a system with the transfer function $G(s) = \dfrac{Y(s)}{X(s)}$. Let the input $x(t)$ be

$$x(t) = Ae^{j(\omega_o t+\phi)}u(t) = Ae^{j\phi}e^{j\omega_o t}u(t) \qquad X(s) = \mathcal{L}\{x(t)\} = \frac{Ae^{j\phi}}{s - j\omega_o}$$

- Laplace Transform of the system output, $y(t)$, is

$$Y(s) = G(s) \cdot X(s) = G(s) \cdot \frac{Ae^{j\phi}}{s - j\omega_o}$$

- Assume that $G(s)$ is stable, and its system poles are located at $s = -p_1, -p_2, \ldots, -p_N$, then it can be shown via partial fractorisation that

$$Y(s) = Ae^{j\phi} \cdot \frac{G(j\omega_o)}{s - j\omega_o} + \sum_{n=1}^{N} \frac{A_n}{s + p_n} \tag{1}$$

$G(j\omega_o) = |G(j\omega_o)|e^{j\angle G(j\omega_o)} = G(s)|_{s=j\omega_o}$ is the system **frequency response** at ω_o rad/s.

- The time-domain output response is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = Ae^{j\phi}G(j\omega_o)e^{j\omega_o t}u(t) + \sum_{n=1}^{N} A_n e^{-p_n t}u(t)$$

- When $t \to \infty$, the steady-state output signal is

$$\begin{aligned} y_{ss}(t) &= \lim_{t\to\infty} y(t) = \underbrace{\lim_{t\to\infty} Ae^{j\phi}G(j\omega_o)e^{j\omega_o t}u(t)}_{Ae^{j\phi}G(j\omega_o)e^{j\omega_o t}} + \underbrace{\lim_{t\to\infty}\sum_{n=1}^{N} A_n e^{-p_n t}u(t)}_{=0} \\ &= A|G(j\omega_o)|e^{j[\omega_o t+\phi+\angle G(j\omega_o)]} = \hat{y}e^{j[\omega_o t+\Phi]} \end{aligned}$$

  - The steady-state response of $G(s)$ to an input sinusoid of frequency ω_o is obtained by multiplying the amplitude of the input sinusoid by $|G(j\omega_o)|$ and adding $\angle G(j\omega_o)$ to its phase.
  - Amplitude of the output sinusoid, $\hat{y} = A|G(j\omega_o)|$ or $|G(j\omega_o)| = \dfrac{\hat{y}}{A}$.
  - Phase of the output sinusoid, $\Phi = \phi + \angle G(j\omega_o)$ or $\angle G(j\omega_o) = \Phi - \phi$

Frequency Response Theorem

$$\underbrace{Ae^{j(\omega_o t+\varphi)}}_{\text{INPUT at steady-state}} \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ \text{Frequency Response: } G(j\omega)\end{array}} \longrightarrow \underbrace{A\left|G(j\omega_o)\right| e^{j[\omega_o t+\varphi+\angle G(j\omega_o)]}}_{\text{OUTPUT at steady-state}}$$

The result of expanding $x(t) = Ae^{j(\omega_o t+\phi)}u(t)$ and $y_{ss}(t) = A|G(j\omega_o)|e^{j[\omega_o t+\phi+\angle G(j\omega_o)]}$ using Euler's Identity is summarised below:

$$Ae^{j(\omega_o t+\psi)} \longrightarrow \boxed{G(j\omega)} \longrightarrow A\left|G(j\omega_o)\right| e^{j(\omega_o t+\psi+\angle G(j\omega_o))}$$

$$A\cos(\omega_o t+\psi) \longrightarrow \boxed{G(j\omega)} \longrightarrow A\left|G(j\omega_o)\right| \cos(\omega_o t+\psi+\angle G(j\omega_o))$$

$$A\sin(\omega_o t+\psi) \longrightarrow \boxed{G(j\omega)} \longrightarrow A\left|G(j\omega_o)\right| \sin(\omega_o t+\psi+\angle G(j\omega_o))$$

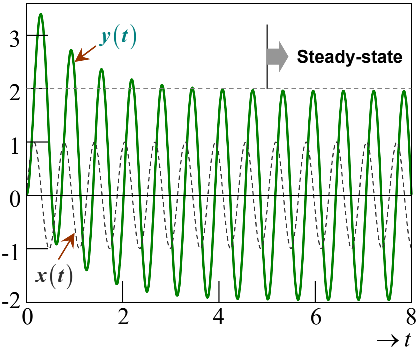
Example Calculate the output signal given that $G(s) = \dfrac{20}{s+1}$ and $x(t) = \sin(10t)u(t)$

- Method A: Using Laplace Transform

$$
\begin{aligned}
X(s) &= \mathcal{L}\{\sin(10t)u(t)\} = \frac{10}{s^2+100} \\
Y(s) &= G(s)X(s) = \frac{200}{(s+1)(s^2+100)} \\
&= \frac{200}{101}\left[\frac{1}{s+1}\right] + \frac{20}{101}\left[\frac{10}{s^2+100}\right] - \frac{200}{101}\left[\frac{s}{s^2+100}\right] \\
y(t) &= \frac{200}{101}e^{-t}u(t) + \frac{20}{101}\left[\sin(10t) - 10\cos(10t)\right]u(t) \\
&= \underbrace{\frac{200}{101}e^{-t}u(t)}_{\text{Transient}} + \underbrace{\frac{20}{\sqrt{101}}\sin\left[10t - \tan^{-1}(10)\right]u(t)}_{\text{Steady-State}} \\
y_{ss}(t) &= \lim_{t\to\infty} y(t) = \frac{20}{\sqrt{101}}\sin\left[10t - \tan^{-1}(10)\right]u(t)
\end{aligned}
$$

- Method B: Using Frequency Response Theorem

$$
\begin{aligned}
|G(j\omega_o)|_{\omega_o=10} &= \left|\frac{20}{10j+1}\right| \\
&= \frac{20}{\sqrt{101}} \\
\angle G(j\omega_o)|_{\omega_o=10} &= -\tan^{-1}(10) = -1.47 \text{ rad} \\
y_{ss}(t) &= A|G(j\omega_o)|\sin\left(\omega_o t + \angle G(j\omega_o)\right) \\
&= 1.\frac{20}{\sqrt{101}}\sin(10t - \tan^{-1}(10)) \\
&= \frac{20}{\sqrt{101}}\sin(10t - 1.47)
\end{aligned}
$$



Method B is an easier way to calculate the steady-state response of a system to a sinusoidal input, if the transient response is not required.

Can the Final Value Theorem be used to predict the steady-state output if the input is a sinusoidal function ?

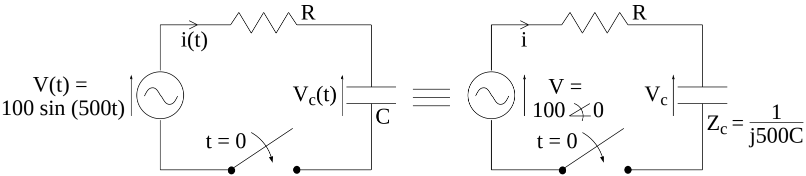
Electric Circuits : Steady-state Analysis

The transfer functions for capacitors and inductors may be extended for steady-state analysis of AC circuits by replacing s with $j\omega_o$, where ω_o is the angular frequency of the sinusoidal input signal, to obtain the impedance of the dynamic electric element.

	Transfer function	Impedance
Capacitor	$\frac{V(s)}{I(s)} = \frac{1}{sC}$	$Z_c(j\omega_o) = \frac{1}{j\omega_o C}$
Inductor	$\frac{V(s)}{I(s)} = sL$	$Z_L(j\omega_o) = j\omega_o L$

- Impedance, $Z(j\omega_o)$, is the effective resistance of the dynamic electrical components for a sinusoidal input of frequency ω_o.
- Impedance extends the concept of resistance and Ohm's Law to the steady-state analysis of AC circuits.

Example Given that $R = 75\Omega$ and $C = 40\mu F$, what is $V_c(t)$ at steady-state ?



Using voltage division,

$$V_c(j\omega_o)\,|_{\omega_o=500} = \left.\frac{Z_c(j\omega_o)}{R+Z_c(j\omega_o)}V\right|_{\omega_o=500} = \frac{\frac{1}{j500C}}{R+\frac{1}{j500C}} \times 100\angle 0$$

$$= \frac{1}{1+1.5j} \times 100\angle 0$$

$$= 0.5547\angle -56.3^o \times 100\angle 0$$

$$= 55.47\angle -56.3^o$$

At steady-state $(t \to \infty)$, $V_c(t) = 55.47\sin(500t - 56.3^o)$

Frequency Response : Definition and Significance

Frequency response of a LTI system, $G(j\omega_o) = |G(j\omega_o)|e^{j\angle G(j\omega_o)}$, is a quantitative measure relating the steady-state output of the system to the sinusoidal stimulus of angular frequency ω_o, and is used to characterize the dynamics of the system.

Significance of frequency response :

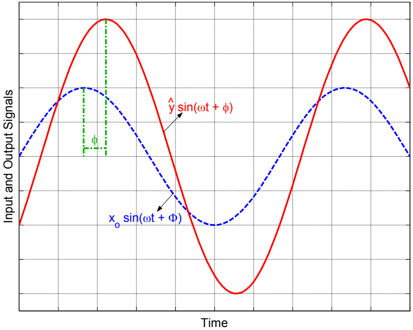
The steady-state output signal of a LTI system that is excited by an arbitrary input signal may be derived using the following steps:-

- Decompose the arbitrary (periodic) input signal, $x(t)$, into its sinusoidal components, $Ae^{j(\omega_o t+\psi)}$ using Fourier Transform (Series).
- For each sinusoidal component of $x(t)$, determine $G(j\omega_o)$. The corresponding steady-state output for the component is $A|G(j\omega_o)|e^{j(\omega_o t+\psi+\angle G(j\omega_o))}$.
- As the system is linear, the Principle of Superposition applies. Hence, the steady-state output signal of the LTI system is the sum of the individual sinusoidal responses.

Can the frequency response of a LTI system be obtained experimentally?

Procedure for collecting frequency response data

1. Use a sinusoidal waveform of frequency, ω_o, as the input signal.
2. Measure the gain, $|G(j\omega_o)|$, and phase, $\angle G(j\omega_o)$.
3. Repeat steps 1 and 2 for frequencies within range of interest



Freq (ω)	Gain $\frac{\hat{y}}{x_o}$ $\lvert G(j\omega)\rvert$	Phase Φ $\angle G(j\omega)$
ω_1	?	?
ω_2	?	?
ω_3	?	?
ω_4	?	?

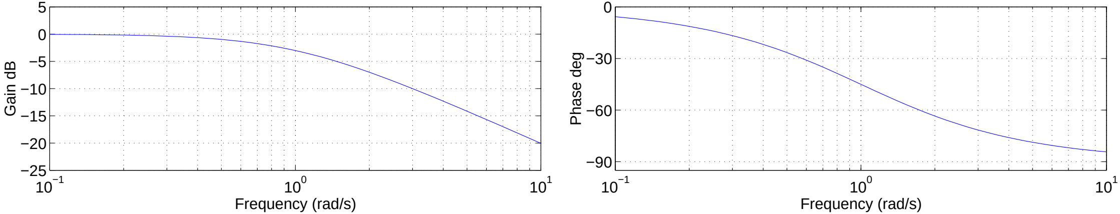
Since the frequency response of a LTI system, $G(j\omega_o) = |G(j\omega_o)|e^{\angle G(j\omega)}$, can be obtained experimentally, the system characteristics and steady-state output can be obtained without the need for mathematical modelling.

Bode Diagram

Frequency response data of a LTI system $G(j\omega) = |G(j\omega)|e^{\angle G(j\omega)}$ $(0 < \omega < \infty)$ may displayed graphically via a **Bode diagram**, which consists of two plots:

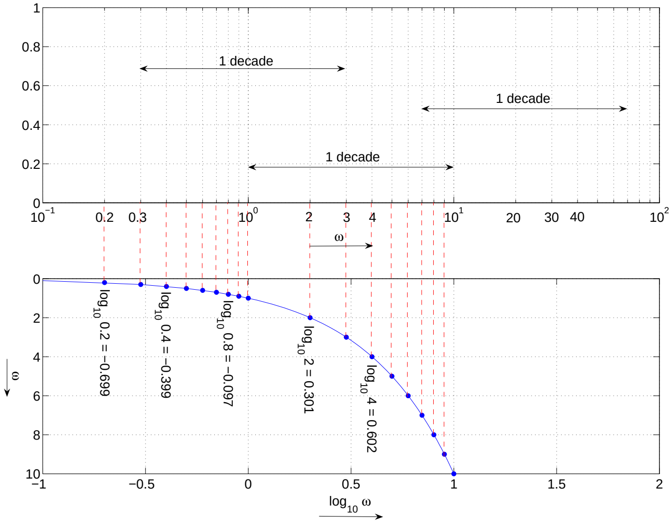
- **Magnitude Plot** – Plot of the gain in decibel (dB), $20\log_{10}|G(j\omega)|$, versus $\log_{10}\omega$.
- **Phase Plot** – Plot of the phase, $\angle G(j\omega)$, in degrees versus $\log_{10}\omega$.

$20\log_{10}|G(j\omega)|$ dB vs $\log_{10}\omega$ | $\angle G(j\omega)$ vs $\log_{10}\omega$



The magnitude and phase plots are normally plotted on a semilog-x graph paper, where the x-axis is logarithmically scaled.

Semilog-x graph paper: As the x-axis divisions is in logarithmic scale, the labels on the x-axis are written in frequency for convenience.



Construction of straight-line Bode Plots

Bode diagrams of common factors in the transfer function of a system have simple shapes that are commonly approximated by straight lines. The straight-line Bode plots are easy to construct and provide an easy means to ascertain system characteristics.

To systematize the process of constructing straight-line Bode plots, transfer functions are first re-written into one of the following three cases:

- N^{th}-order system with a differentiator

$$G(s) = K_d s \cdot \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\dots\left(\frac{s}{z_{M-1}}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\dots\left(\frac{s}{p_N}+1\right)}$$

  - K_d is the gain of the differentiator.
  - $z_m \neq 0 \;\; \forall \;\; m \in [1, M-1]$ and $p_n \neq 0 \;\; \forall \;\; n \in [1, N]$.
  - **Steady-state gain** of the system is $G(0) = 0$.

- N^{th}-order system with an integrator

$$G(s) = \frac{K_i}{s} \cdot \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\dots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\dots\left(\frac{s}{p_{N-1}}+1\right)}$$

  - K_i is the gain of the integrator.
  - $z_m \neq 0 \;\; \forall \;\; m \in [1, M]$ and $p_n \neq 0 \;\; \forall \;\; n \in [1, N-1]$.
  - **Steady-state gain** of the system is $G(0) = \infty$.
- N^{th}-order system without integrator or differentiator

$$G(s) = K_{dc} \cdot \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\dots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\dots\left(\frac{s}{p_N}+1\right)}$$

  - $K_{dc} = G(0)$ is the **steady-state gain** of the system.
  - $z_m \neq 0 \;\; \forall \;\; m \in [1, M]$ and $p_n \neq 0 \;\; \forall \;\; n \in [1, N]$.

The standardised transfer functions comprises the following basic systems:

- Steady-state gain (constant), $G(s) = K_{dc}$
- Differentiator with gain K_d, $G(s) = K_d s$
- Integrator with gain K_i, $G(s) = \dfrac{K_i}{s}$
- Zero factor with unity steady-state gain, $G(s) = \dfrac{s}{z_m} + 1$
- Pole factor with unity steady-state gain, $G(s) = \dfrac{1}{\frac{s}{p_n} + 1}$
- Second order factor with unity steady-state gain, $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

The Bode straight-line plots for the systems listed above will first be discussed. Next, how the straight-line Bode plots for higher order systems can be obtained by summing the plots obtained for their constituent basic systems will be introduced.

Bode plots for steady-state gain: $G(s) = K_{dc}$

- Frequency Response:

$$G(j\omega) = K_{dc}$$

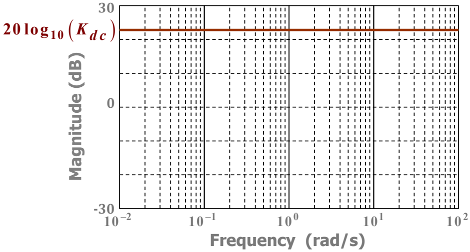
- Magnitude Response:

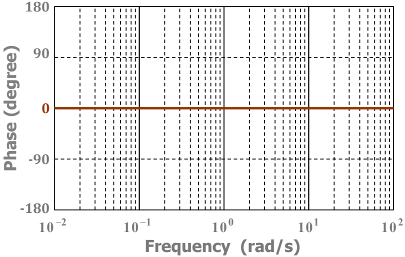
$$\begin{aligned} |G(j\omega)| &= K_{dc} \\ |G(j\omega)|_{dB} &= 20\log_{10}(K_{dc})\,\mathrm{dB} \end{aligned}$$

- Phase Response:

$$\angle G(j\omega) = 0^o$$

The magnitude and phase responses are both straight lines with zero gradient.





Bode plots for Differentiator: $G(s) = K_d s$

- Frequency Response:

$$G(j\omega) = K_d \cdot j\omega = K_d\, \omega\, e^{j90}$$

- Magnitude Response:

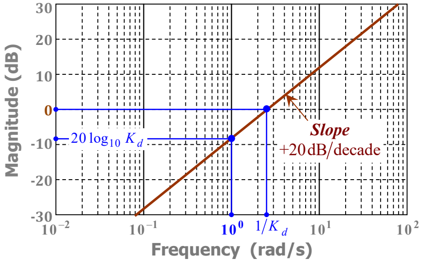
$$\begin{aligned} |G(j\omega)| &= K_d\, \omega \\ |G(j\omega)|_{dB} &= 20\log_{10}(K_d) + 20\log_{10}\omega \,\mathrm{dB} \end{aligned}$$

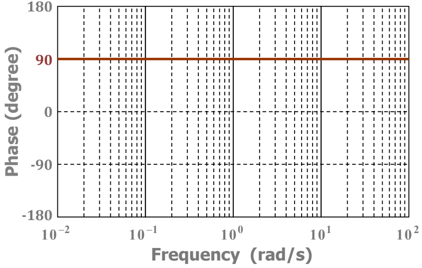
Magnitude response is a straight line with slope 20 dB/decade passing through the points $\left(\frac{1}{K_d}, 0\text{ dB}\right)$ and $(1, 20\log_{10} K_d)$.

- Phase Response:

$$\angle G(j\omega) = 90^o$$

Phase response is a horizontal straight line located at 90^o.





Bode plots for Integrator: $G(s) = \frac{K_i}{s}$

- Frequency Response:

$$G(j\omega) = \frac{K_i}{j\omega} = \frac{K_i}{\omega} e^{-j90}$$

- Magnitude Response:

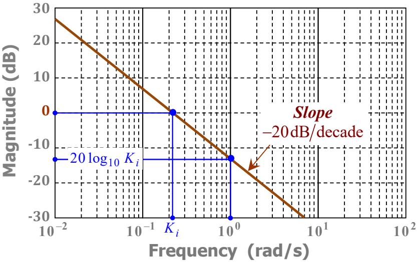
$$|G(j\omega)| = \frac{K_i}{\omega}$$
$$|G(j\omega)|_{dB} = 20\log_{10}(K_i) - 20\log_{10}\omega \text{ dB}$$

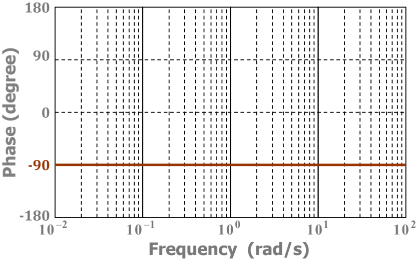
Magnitude response is a straight line with slope -20 dB/decade passing through the points $(K_i, 0 \text{ dB})$ and $(1, 20\log_{10} K_i)$.

- Phase Response:

$$\angle G(j\omega) = -90^o$$

Phase response is a horizontal straight line located at -90^o.



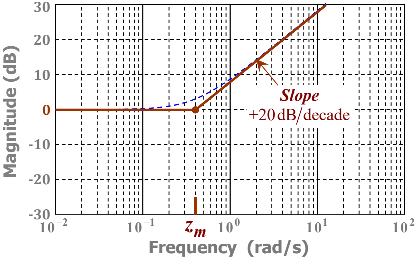


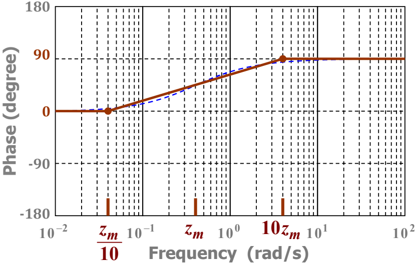
Bode plots for Zero Factor: $G(s) = \frac{s}{z_m} + 1; z_m > 0$

- Frequency Response: $G(j\omega) = \frac{j\omega}{z_m} + 1$
- Magnitude Response: $|G(j\omega)| = \sqrt{\frac{\omega^2}{z_m^2} + 1}$

$$|G(j\omega)|_{dB} = 20\log_{10}\left(\sqrt{\frac{\omega^2}{z_m^2} + 1}\right) \text{ dB}$$

  - When $\omega << z_m$, $|G(j\omega)|_{dB} \to 0$
  - When $\omega >> z_m$, $|G(j\omega)|_{dB} \to 20\log_{10}\left(\frac{\omega}{z_m}\right)$ which is a straight line with slope of 20 dB/decade.
  - 3 dB corner frequency, $\omega_c = z_m$ rad/s
- Phase Response: $\angle G(j\omega) = \tan^{-1}\frac{\omega}{z_m}$
  - Low Frequency asymptote: $\lim_{\omega\to 0}\angle G(j\omega) = 0^o$
  - High Frequency asymptote: $\lim_{\omega\to\infty}\angle G(j\omega) = 90^o$



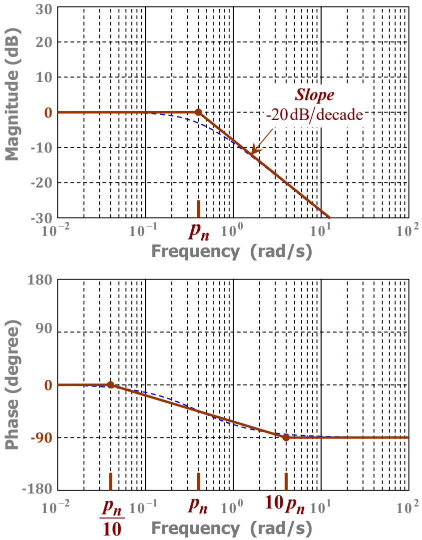


Bode plots for Pole Factor: $G(s) = \frac{1}{s/p_n+1}$; $p_n > 0$

- Frequency Response: $G(j\omega) = \frac{1}{j\omega/p_n+1}$
- Magnitude Response: $|G(j\omega)| = \dfrac{1}{\sqrt{\omega^2/p_n^2+1}}$

$$|G(j\omega)|_{dB} = -20\log_{10}\left(\sqrt{\frac{\omega^2}{p_n^2}+1}\right) \text{ dB}$$

  - When $\omega << z_m$, $|G(j\omega)|_{dB} \to 0$
  - When $\omega >> z_m$, $|G(j\omega)|_{dB} \to -20\log_{10}\left(\frac{\omega}{p_n}\right)$ which is a straight line with slope of -20 dB/decade.
  - 3 dB corner frequency, $\omega_c = p_m$ rad/s
- Phase Response: $\angle G(j\omega) = -\tan^{-1}\dfrac{\omega}{p_n}$
  - Low Frequency asymptote: $\lim_{\omega\to 0} \angle G(j\omega) = 0^o$
  - High Frequency asymptote: $\lim_{\omega\to\infty} \angle G(j\omega) = -90^o$



Bode plots for 2^{nd}-order factor: $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$; $\omega_n > 0$ and $\zeta \geq 0$

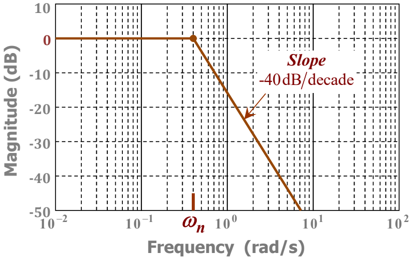
Overdamped ($\zeta > 1$) or Critically damped ($\zeta = 1$) systems

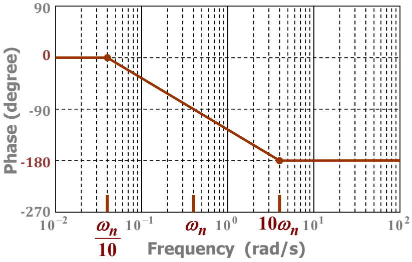
- Both system poles are real. By expressing $G(s)$ as a cascade of two pole factors, the straight-line Bode plots can be derived by graphically summing those constructed for each of the pole factors.

Underdamped ($0 < \zeta < 1$) systems

- System poles are complex. Straight-line Bode plots are constructed by assuming $\zeta = 1$ such that

$$G(s) = \frac{1}{\frac{s}{\omega_n}+1} \cdot \frac{1}{\frac{s}{\omega_n}+1}$$



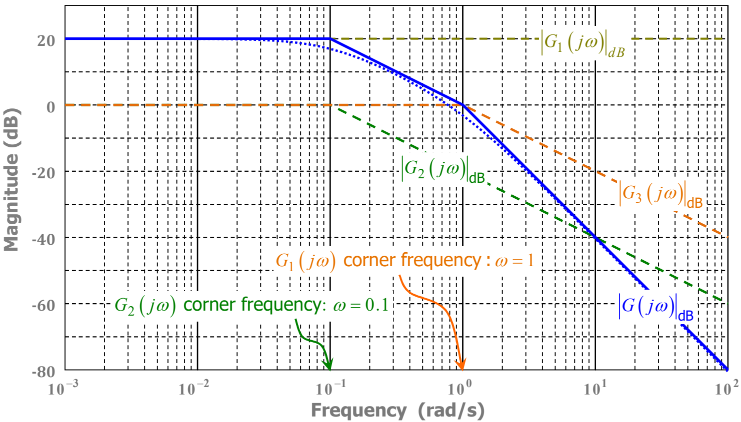


Example – Cascade of two pole factors (Overdamped 2^{nd} system)

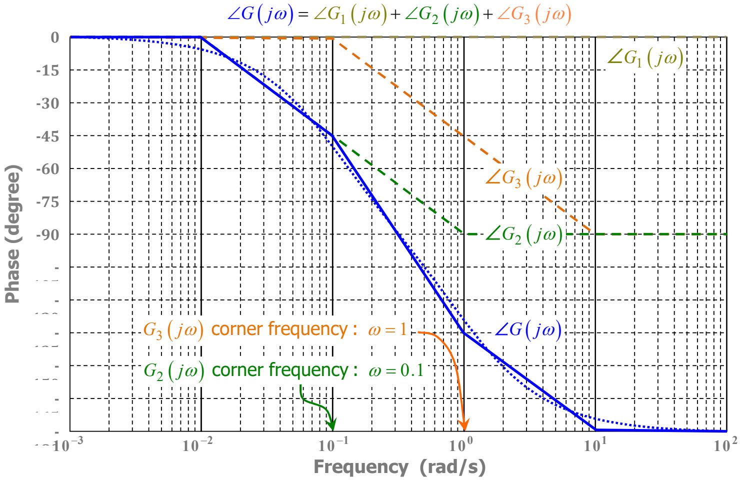
$$G(s) = \frac{10}{(10s+1)(s+1)} = 10 \cdot \frac{1}{\frac{s}{0.1}+1} \cdot \frac{1}{s+1} = G_1(s) \cdot G_2(s) \cdot G_3(s)$$

$$\begin{aligned} G_1(s) &= 10 \\ G_2(s) &= \frac{1}{\frac{s}{0.1}+1} \\ G_3(s) &= \frac{1}{s+1} \end{aligned}$$

$$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} + |G_3(j\omega)|_{dB}$$



$$G_1(s) = 10$$
$$G_2(s) = \frac{1}{\frac{s}{0.1} + 1}$$
$$G_3(s) = \frac{1}{s+1}$$



Since $\lim_{s\to\infty} G(s) = \frac{1}{s^2}$ and $s = j\omega$, the system behaves like a double integrator with unity combined gain as $\omega \to \infty$.

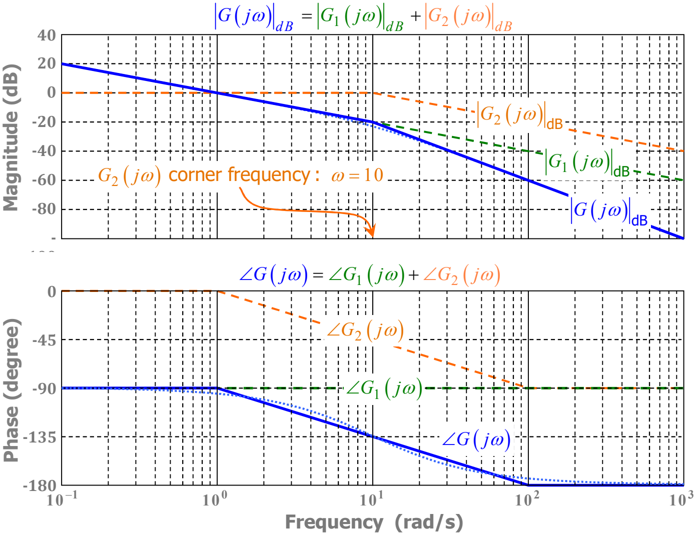
Example – Cascade of one Integrator with a pole factor

$$G(s) = \frac{1}{s(0.1s+1)} = G_1(s)\cdot G_2(s)$$

$$\text{where } G_1(s) = \frac{1}{s}$$

$$G_2(s) = \frac{1}{\frac{s}{10}+1}$$

Since $\lim_{s\to\infty} G(s) = \frac{10}{s^2}$ and $s = j\omega$, the system behaves like a double integrator with a combined gain of 10 as $\omega \to \infty$.

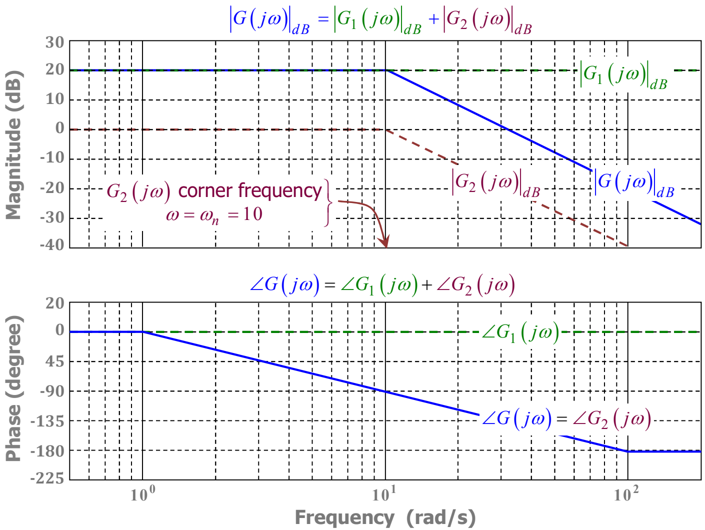


Example – Underdamped 2^{nd}-order system

Let $G(s) = \dfrac{1000}{s^2 + 10s + 100}$.

Comparing with the standard form $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, $\zeta = 0.5$, $\omega_n = 10$ and $K = 10$.

$$
\begin{aligned}
G(s) &= 10 \cdot \frac{100}{s^2 + 10s + 100} \\
&= G_1(s) \cdot G_2(s) \\
G_1(s) &= 10 \\
G_2(s) &= \frac{100}{s^2 + 10s + 100}
\end{aligned}
$$

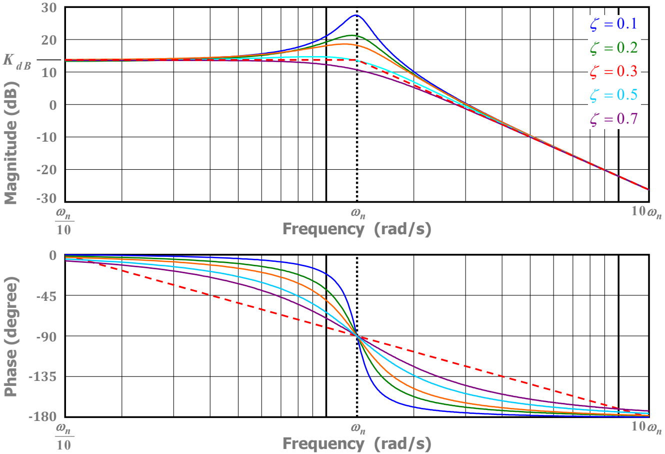


Resonance in underdamped 2^{nd}-order system

Bode diagrams of underdamped 2^{nd}-order systems $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ with different values of damping ratio, ζ.

Note that the magnitude response has a "hump" around $\omega = \omega_n$. The "hump" is more pronounce when ζ is small and disappears when ζ rises above a certain value.

The "hump" is associated with the phenomenon of **resonance**. The magnitude of the "hump" is called **resonant peak** and the frequency at which it occurs is called **resonant frequency**.



Magnitude response of an underdamped 2^{nd}-order system is

$$|G(j\omega)| = \frac{K\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}$$

Maximising $|G(j\omega)|$ is equivalent to minimizing $\left(\omega_n^2-\omega^2\right)^2+(2\zeta\omega_n\omega)^2$ or solving

$$\frac{d}{d\omega}\left(\omega_n^2-\omega^2\right)^2+(2\zeta\omega_n\omega)^2=0 \qquad \text{or} \qquad -4\omega\left(\omega_n^2-\omega^2\right)+8\zeta^2\omega_n^2\omega=0$$

Hence, values of ω where $|G(j\omega)|$ is a maximum point are

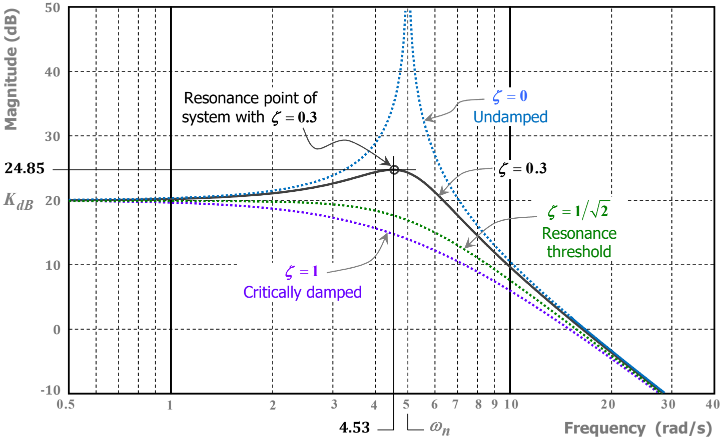
$$\omega = 0,\ \ -\omega_n\sqrt{1-2\zeta^2},\ \ \omega_n\sqrt{1-2\zeta^2}$$

- Resonant frequency, $\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}, \qquad \zeta < \frac{1}{\sqrt{2}}}$
- Resonant peak, $\boxed{M_r = |G(j\omega)|_{max} = |G(j\omega_r)| = \frac{K}{2\zeta\sqrt{1-\zeta^2}}, \qquad \zeta < \frac{1}{\sqrt{2}}}$

Example Find the resonant frequency and resonant magnitude of $\dfrac{250}{s^2+3s+25}$.

Comparing with the standard form $G(s) = \dfrac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $\zeta = 0.3$, $\omega_n = 5$ and $K = 10$.

$$\begin{aligned}
\omega_r &= \omega_n\sqrt{1-2\zeta^2} \\
&= 5\sqrt{1-2(0.3)^2} \\
&= 4.53 \text{ rad/s} \\
M_r &= \frac{K}{2\zeta\sqrt{1-\zeta^2}} \\
&= \frac{10}{2(0.3)\sqrt{1-0.3^2}} \\
&= 17.47 \\
&= 24.85 \text{ dB}
\end{aligned}$$



Asymptotic Behaviour of straight-line Bode Plots (Optional)

Asymptotic SLOPE of Magnitude Plot, $|G(j\omega)|_{dB}$

- High frequency:

$$\lim_{\omega\to\infty}[\text{slope of } |G(j\omega)|_{dB}] = \underbrace{[\text{No of Poles - No of Zeros}]}_{\text{Pole-Zero Excess}} \times (-20 \text{ dB/decade})$$

- Low frequency:

$$\lim_{\omega\to 0}[\text{slope of } |G(j\omega)|_{dB}] = [\text{No of Integrators - No of Differentiators}] \times (-20 \text{ dB/decade})$$

Asymptotic VALUE of Phase Plot, $\angle G(j\omega)$

- High frequency: Assuming no zero on the right-half of s-plane

$$\lim_{\omega\to\infty} \angle G(j\omega) = \underbrace{[\text{No of Poles - No of Zeros}]}_{\text{Pole-Zero Excess}} \times (-90^o)$$

- Low frequency:

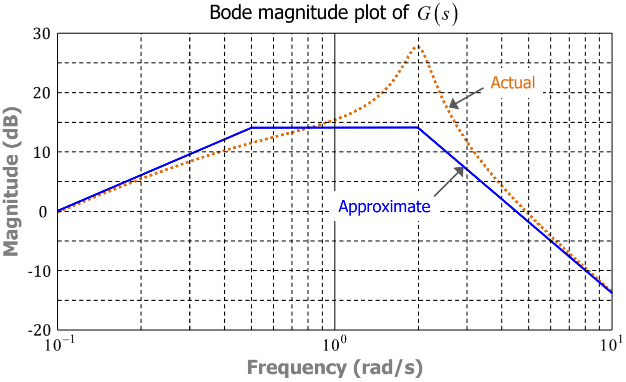
$$\lim_{\omega\to 0} \angle G(j\omega) = [\text{No of Integrators - No of Differentiators}] \times (-90^o)$$

System Identification from Bode Magnitude Plot

The Bode magnitude plot of the following LTI system is shown below.

$$G(s) = \frac{K(s+a)}{\left(\frac{s}{b}+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

- Identify the values of a, b, ω_n and K.
- What values will the high and low frequency asymptotes of the Bode phase plot for converge to?



Rewrite the transfer funcation as : $G(s) = \frac{K}{\omega_n^2} \cdot (s+a) \cdot \frac{1}{\frac{s}{b}+1} \cdot \frac{\omega_n^w}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

- The existence of a low frequency asymptote of 20 dB/decade indicates the presence of a differentiator $K_d s$. Hence, $a = 0$ and $G(s)$ may be rewritten as

$$G(s) = K_d s \cdot \frac{1}{\frac{s}{b}+1} \cdot \frac{\omega_n^w}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \text{where } K_d = \frac{K}{\omega_n^2} = \text{ differentiator gain}$$

  The low frequency asymptote passes through the $(0.1 \text{ rad/s}, 0 \text{ dB})$ point indicates that the derivative gain $\frac{1}{K_d} = 0.1$. Hence,

$$\frac{\omega_n^2}{K} = 0.1 \qquad\qquad K = 10\omega_n^2$$

- At $\omega = 0.5$ rad/s, a slope change of -20 dB/decade indicates the presence of a pole factor $\left(\frac{s}{0.5}+1\right)^{-1}$. Therefore, $b = 0.5$.

- At $\omega = 2$ rad/s , a slope change of -40 dB/decade together with the existence of a resonant peak in the actual Bode plot indicates which indicates the presence of an underdamped 2nd-order factor where $0 < \zeta < 1$ and $\omega_n = 2$ rad/s.
- Since $K = 10\omega_n^2$ and $\omega_n = 2$ rad/s, $K = 40$.
- Transfer function of the LTI system is

$$G(s) = \frac{20s}{(s+0.5)\left(s^2 + 4\zeta s + 4\right)}$$

- $G(s)$ has a pole-zero excess of 2. Hence, the high frequency asymptote of the Bode phase plot will converge to $2 \times (-90^o) = -180^o$.
- The presence of a differentiator indicates that the low frequency phase plot will converge to 90^o.