EE2023 TUTORIAL 2 (SOLUTIONS)

Solution to Q.1

Description of $x(t)$:

- $x(t)$ is a REAL & EVEN function of t $\because$ Spectrum is REAL and SYMMETRIC
- $x(t)$ has an average (or DC) value of 2 $\because$ Zero-frequency component has value 2
- $x(t)$ is APERIODIC $\because$ $\left\{\pi,\ \pi^2, \pi^3\right\}\ \cdots$ has no common factor
- $x(t)$ is a POWER SIGNAL $\because$ $\begin{cases}\text{Spectrum is defined only at discrete} \\ \text{frequency points } (\text{sum of sinusoids})\end{cases}$

Since $x(t)$ is non-periodic, it does not have a Fourier series expansion.

Solution to Q.2

(a) The frequencies in $x(t) = \cos(3.2t) + \sin(1.6t) + \exp(j2.8t)$ are 3.2 rad/s, 1.6 rad/s and 2.8 rad/s. Hence the HCF is 0.4 rad/s, and the fundamental frequency is $0.4/(2\pi) = 0.2/\pi$, and the period is 5π.

(b) The frequencies in $x(t) = \cos(4t) + \sin(\pi t)$ are 4 rad/s, 1.6 rad/s and π rad/s. There is no HCF, and hence there is no fundamental frequency nor period.

Solution to Q.3

(a) The fundamental frequency of $x(t)=6\sin(12\pi t)+4\exp\left(j\left(8\pi t+\frac{\pi}{4}\right)\right)+2$ is $\begin{cases} \boldsymbol{f_p = HCF\{6,4\}=2} \\ \boldsymbol{T_p = 0.5} \end{cases}$.

Re-write $x(t)$ as a sum of complex exponentials:

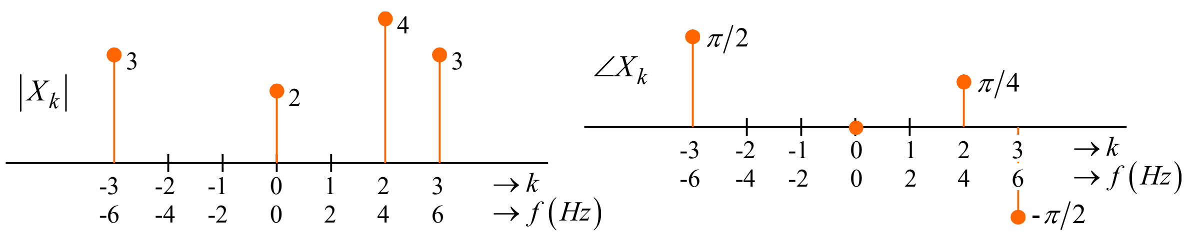
$$\begin{aligned} x(t) &= \frac{6}{j2}\left[\exp(j12\pi t)-\exp(-j12\pi t)\right]+4\exp(j\pi/4)\exp(j8\pi t)+2 \\ &= j3\exp(-j12\pi t)+2+4\exp(j\pi/4)\exp(j8\pi t)-j3\exp(j12\pi t) \end{aligned} \tag{1}$$

Express $x(t)$ as a complex exponential Fourier series:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T_p}t\right) = \sum_{k=-\infty}^{\infty} X_k \exp(j4\pi k t) \\ &= \begin{pmatrix} \cdots + X_{-3}\exp(-j12\pi t)+X_{-2}\exp(-j8\pi t)+X_{-1}\exp(-j4\pi t) \\ +X_0 \\ +X_1\exp(j4\pi t)+X_2\exp(j8\pi t)+X_3\exp(j12\pi t)+\cdots\cdots \end{pmatrix} \end{aligned} \tag{2}$$

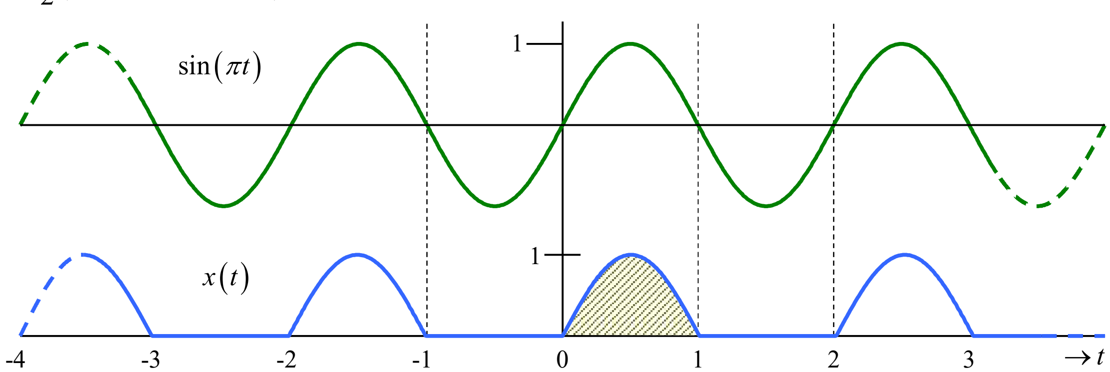
Comparing coefficients of complex exponential terms in (1) and (2), we conclude that:

$\boldsymbol{X_{-3}=j3}$, $\boldsymbol{X_0=2}$, $\boldsymbol{X_2=4\exp\left(j\frac{\pi}{4}\right)}$, $\boldsymbol{X_3=-j3}$ and $\boldsymbol{\left[X_k=0;\ k\neq 0,\ 2,\ \pm 3\right]}$.



Remarks: *If a periodic signal is given as a sum of sinusoids, then its Fourier series coefficients can be evaluated using the above method without the need to perform any integration.*

(b) $x(t)=\frac{1}{2}\left(|\sin(\pi t)|+\sin(\pi t)\right)$: Half-wave rectification of $\sin(\pi t)$.



Period of $x(t)$: $T=2$; Fundamental frequency = 0.5

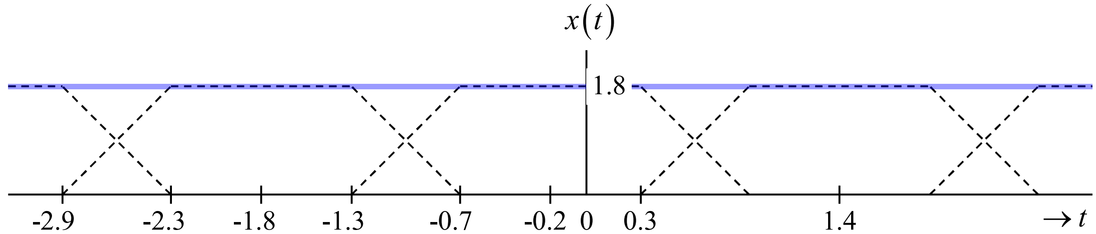
Frequency of 4th harmonic = 4 x 0.5 = 2 Hz

Coefficients of complex exponential Fourier series expansion of $x(t)$:

$$
\begin{aligned}
X_k &= \frac{1}{T}\int_0^T x(t)\exp(-j2\pi\, kt/T)\,dt = \frac{1}{2}\int_0^2 x(t)\exp(-j\pi kt)\,dt \\
&= \frac{1}{2}\int_0^1 \sin(\pi t)\exp(-j\pi kt)\,dt = \frac{1}{2}\int_0^1 \frac{1}{j2}\left[\exp(j\pi t) - \exp(-j\pi t)\right]\exp(-j\pi kt)\,dt \\
&= \frac{1}{j4}\int_0^1 \exp\left(-j\pi(k-1)t\right) - \exp\left(-j\pi(k+1)t\right)dt \\
&= \frac{1}{j4}\left[\frac{\exp\left(-j\pi(k-1)t\right)}{-j\pi(k-1)} - \frac{\exp\left(-j\pi(k+1)t\right)}{-j\pi(k+1)}\right]_0^1 \\
&= \frac{1}{j4}\left[\exp(-j\pi k)\left(\frac{\exp(j\pi)}{-j\pi(k-1)} - \frac{\exp(-j\pi)}{-j\pi(k+1)}\right) - \left(\frac{1}{-j\pi(k-1)} - \frac{1}{-j\pi(k+1)}\right)\right] \\
&= \frac{1}{j4}\left[(-1)^k\left(\frac{1}{j\pi(k-1)} - \frac{1}{j\pi(k+1)}\right) + \left(\frac{1}{j\pi(k-1)} - \frac{1}{j\pi(k+1)}\right)\right] \\
&= \frac{(-1)^k}{4\pi}\left[-\frac{1}{(k-1)} + \frac{1}{(k+1)}\right] + \frac{1}{4\pi}\left[-\frac{1}{(k-1)} + \frac{1}{(k+1)}\right] \\
&= \frac{(-1)^k}{4\pi}\left[-\frac{(k+1)}{\left(k^2-1\right)} + \frac{(k-1)}{\left(k^2+1\right)}\right] + \frac{1}{4\pi}\left[-\frac{(k+1)}{\left(k^2-1\right)} + \frac{(k-1)}{\left(k^2+1\right)}\right] \\
&= \frac{(-1)^k}{4\pi}\left[\frac{2}{\left(1-k^2\right)}\right] + \frac{1}{4\pi}\left[\frac{2}{\left(1-k^2\right)}\right] \\
&= \begin{cases} \dfrac{1}{\pi\left(1-k^2\right)}; & k = even \\ 0; & k = \ odd \end{cases} \\
&= \frac{1+(-1)^k}{2\pi\left(1-k^2\right)}
\end{aligned}
$$

Solution to Q.4

Graphically, we observe that $x(t) = \sum_{n=-\infty}^{\infty} 2p(t-1.6n) = 1.8$.



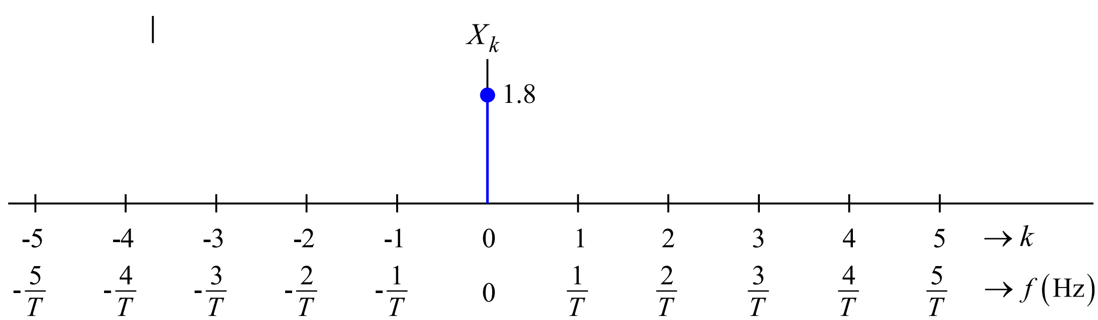
By Deduction:

- $x(t)$ has a *zero-frequency* component of value 1.8, which implies that $X_0 = 1.8$.
- $x(t)$ has no *non-zero frequency* components, which implies that $X_k = 0; \;\; k \neq 0$.

By Derivation:

Since $x(t)$ is a constant (or a DC signal), it may be treated as a periodic signal of arbitrary period *T*, where $0 < T < \infty$. Its Fourier series coefficients can thus be computed as

$$\begin{aligned}
X_k &= \frac{1}{T}\int_{-T/2}^{T/2} 1.8\exp\left(-j2\pi\frac{k}{T}t\right)dt \\
&= \frac{1.8}{T}\left[\frac{\exp(-j2\pi\, kt/T)}{-j2\pi\, k/T}\right]_{-T/2}^{T/2} \\
&= \frac{1.8}{T}\left[\frac{\exp(-j\pi k)}{-j2\pi\, k/T} - \frac{\exp(j\pi k)}{-j2\pi\, k/T}\right] \\
&= 1.8\frac{\sin(\pi k)}{\pi k} \\
&= 1.8\ \text{sinc}(k) \\
&= \begin{cases} 1.8; & k = 0 \\ 0; & k \neq 0 \end{cases}
\end{aligned}$$



Solution to Q.5

(a) The analysis subsystem assumes that the input $x(t)$ has a period of 1 and computes its Fourier series coefficients μ_k over the interval $[-0.5, 0.5]$. The synthesis subsystem uses μ_k as Fourier series coefficients to synthesize a periodic signal of period equal to 1.

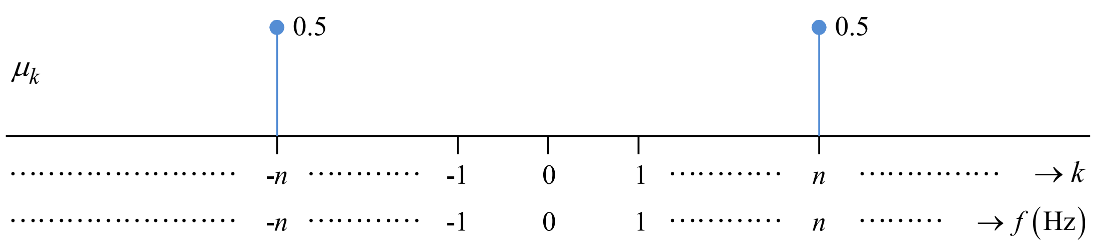
(b) i.

$$
\begin{aligned}
\mu_k &= \int_{-0.5}^{0.5} x(t)\exp(-j2\pi kt)\,dt = \int_{-0.5}^{0.5} \underbrace{\cos(2\pi nt)}_{x(t)}\exp(-j2\pi kt)\,dt \\
&= \int_{-0.5}^{0.5} \frac{1}{2}\left[\exp(j2\pi nt)+\exp(-j2\pi nt)\right]\exp(-j2\pi kt)\,dt \\
&= \frac{1}{2}\int_{-0.5}^{0.5} \exp(-j2\pi(k-n)t)+\exp(-j2\pi(k+n)t)\,dt \\
&= \frac{1}{2}\left[\frac{\exp(-j2\pi(k-n)t)}{-j2\pi(k-n)}+\frac{\exp(-j2\pi(k+n)t)}{-j2\pi(k+n)}\right]_{-0.5}^{0.5} \\
&= \frac{1}{2}\left[\frac{\exp(-j\pi(k-n))}{-j2\pi(k-n)}+\frac{\exp(-j\pi(k+n))}{-j2\pi(k+n)}-\frac{\exp(j\pi(k-n))}{-j2\pi(k-n)}-\frac{\exp(j\pi(k+n))}{-j2\pi(k+n)}\right] \\
&= \frac{1}{2}\left[\frac{\sin(\pi(k-n))}{\pi(k-n)}+\frac{\sin(\pi(k+n))}{\pi(k+n)}\right] \\
&= \frac{1}{2}\mathrm{sinc}(k-n)+\frac{1}{2}\mathrm{sinc}(k+n)
\end{aligned}
$$

Note that sinc(x) = 1 when x=0 and zero everywhere else for the case of x being an integer

(b) ii. ***n* (INTEGER)**

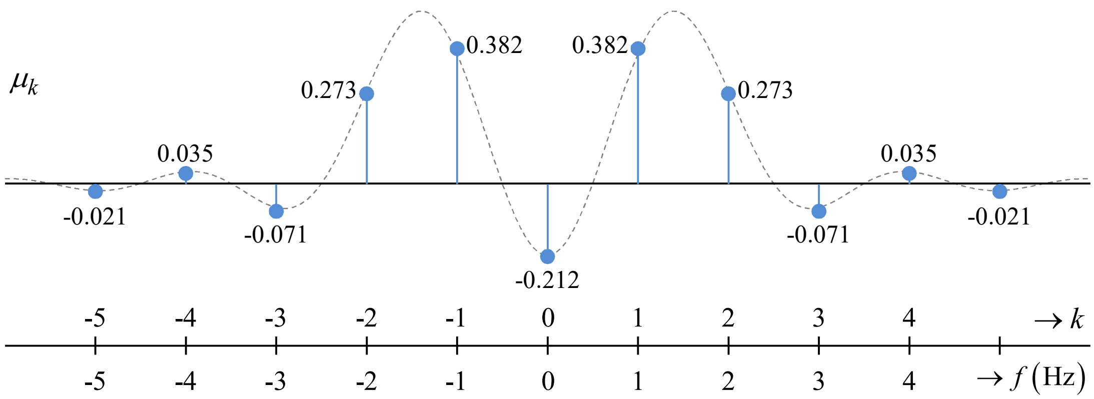
$$
\mu_k = \frac{1}{2}\mathrm{sinc}(k-n)+\frac{1}{2}\mathrm{sinc}(k+n) = \begin{cases} 0.5; & k=\pm n \\ 0; & k \neq \pm n \end{cases}
$$



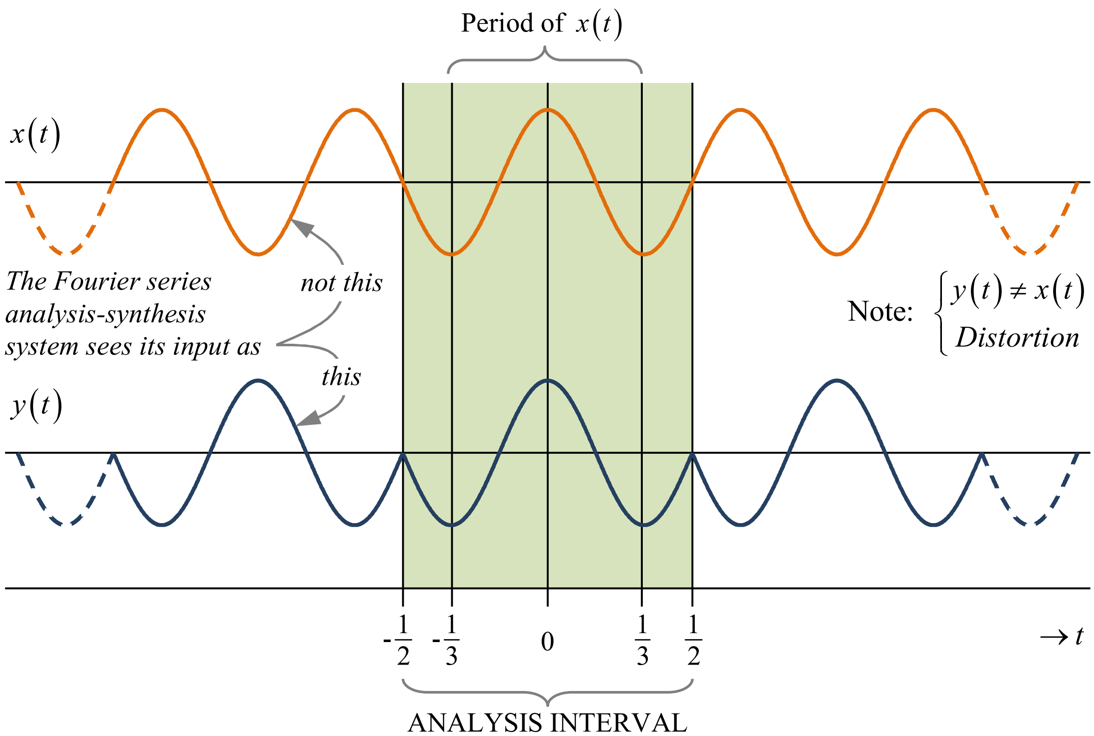
$$
\left.
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} \mu_k \exp(j2\pi kt) \\
&= \mu_{-n}\exp(-j2\pi nt)+\mu_n\exp(j2\pi nt) \\
&= 0.5\exp(j2\pi nt)+0.5\exp(-j2\pi nt) \\
&= \cos(2\pi nt)
\end{aligned}
\right\} \rightarrow \underbrace{y(t)=x(t);\ \forall \text{integer } n}_{\textit{No Distortion}}
$$

(b) iii. $n = 1.5$ **(NON-INTEGER)**

$$\mu_k = \frac{1}{2}\text{sinc}(k-1.5) + \frac{1}{2}\text{sinc}(k+1.5) = \frac{1}{2}\left[\frac{\sin(\pi k - 1.5\pi)}{\pi(k-1.5)} + \frac{\sin(\pi k + 1.5\pi)}{\pi(k+1.5)}\right]$$
$$= \frac{1}{2}\left[\frac{\sin(\pi k)\cos(1.5\pi) - \cos(\pi k)\sin(1.5\pi)}{\pi(k-1.5)} + \frac{\sin(\pi k)\cos(1.5\pi) + \cos(\pi k)\sin(1.5\pi)}{\pi(k+1.5)}\right]$$
$$= \frac{1}{2}\left[\frac{\cos(\pi k)}{\pi(k-1.5)} - \frac{\cos(\pi k)}{\pi(k+1.5)}\right] = \frac{3\cos(\pi k)}{2\pi\left(k^2 - 1.5^2\right)}$$



The analysis subsystem uses an analysis interval of 1 (from -0.5 to 0.5). Thus, the segment $[x(t);\ |t| \le 0.5]$ is implicitly treated by the system as one period of the input signal although the actual period of $x(t)$ is 2/3. Since μ_k have not been modified, the output signal is simply obtained by replicating the segment $[x(t);\ |t| \le 0.5]$ at regular intervals of duration 1. With this notion we may sketch $y(t)$ without the need to compute $y(t) = \sum_{k=-\infty}^{\infty} \mu_k \exp(j2\pi kt)$.



(c) The result of (b)(ii) shows that the system can extract the spectrum of any sinusoid of integer frequency and reconstruct the sinusoid without distortion.

The result of (b)(iii) shows that the system introduces distortion when applied to a sinusoid of non-integer frequency.

The system is thus ***applicable to only periodic signals that have integer fundamental frequencies*** because

- A periodic signal that has an integer fundamental frequency is made up of sinusoids that have integer frequencies.
- A periodic signal that has a non-integer fundamental frequency is made up of sinusoids, of which at least one has a non-integer frequency.

Solution to Q.6

Let the fundamental frequencies of $x(t)$ and $y(t)$ be f_x and f_y, respectively. Since $z(t)=x(t)+y(t)$ is periodic, its fundamental frequency f_z is given by the highest common factor (HCF) of $\{f_x, f_y\}$.

The corresponding Fourier series expansions are:

$$\left.\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi k f_x t\right) \\ y(t) &= \sum_{k=-\infty}^{\infty} Y_k \exp\left(j2\pi k f_y t\right) \\ z(t) &= \sum_{k=-\infty}^{\infty} \left[X_k \exp\left(j2\pi k f_x t\right) + Y_k \exp\left(j2\pi k f_y t\right)\right] = \sum_{k=-\infty}^{\infty} Z_k \exp\left(j2\pi k f_z t\right) \end{aligned}\right\} \rightarrow \begin{array}{c} Z_k \neq X_k + Y_k \\ (\textit{In general}) \end{array}$$

If $f_x = f_y = f_o$, then $f_z = \text{HCF}(f_o, f_o) = f_o$. It follows that

$$\left.z(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left[X_k \exp\left(j2\pi k f_o t\right) + Y_k \exp\left(j2\pi k f_o t\right)\right]}_{(X_k+Y_k)\exp(j2\pi k f_o t)} = \sum_{k=-\infty}^{\infty} Z_k \exp\left(j2\pi k f_o t\right)\right\} \rightarrow Z_k = X_k + Y_k$$

Conclusion:

If $z(t)=x(t)+y(t)$, then $Z_k = X_k + Y_k$ if and only if $x(t)$ and $y(t)$ have the same fundamental frequency, i.e. $f_x = f_y$.

Explanation:

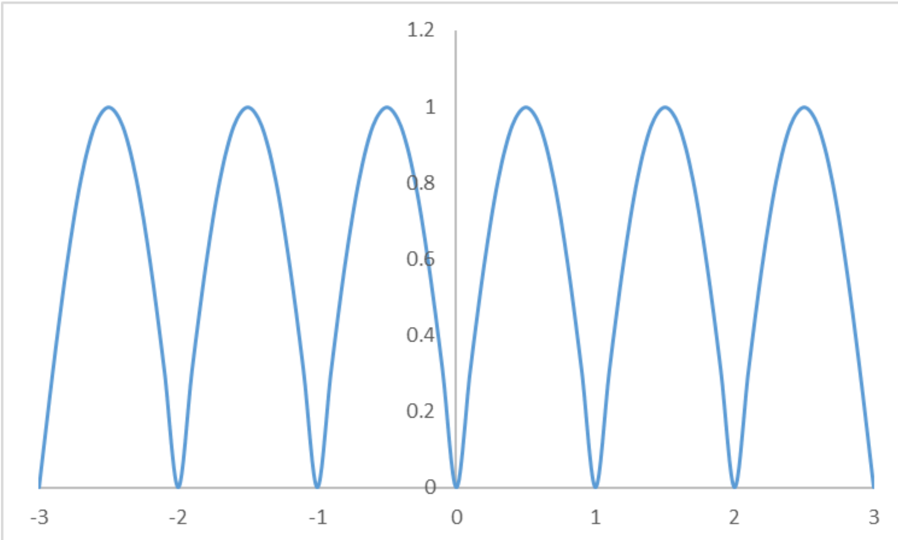
X_k is associated with $kf_x(\text{Hz})$ and Y_k is associated with $kf_y(\text{Hz})$. If $f_x \neq f_y$, then $X_k + Y_k$ is equivalent to adding Fourier series coefficients associated with different frequencies, which is erroneous. The Fourier series pair is linear in the frequency $f(\text{Hz})$ sense, and **not** in the index (k) sense. In other words, if $f_z = \text{HCF}\{f_x, f_y\}$ where $f_x = mf_z$ and $f_y = nf_z$, then

$$\underbrace{\boldsymbol{Z_k = X_{k/m} + Y_{k/n}}}_{\textit{matching frequencies}} \quad \text{and} \quad \underbrace{\boldsymbol{Z_k \neq X_k + Y_k}}_{\textit{unmatched frequencies}}$$

with the understanding that if k/m $(\text{or } k/n)$ is not an integer, then $X_{k/m} = 0$ $\left(\text{or } Y_{k/n} = 0\right)$.

Supplementary Questions (Solutions)

S1 The signal $x(t) = |\sin(\pi t)|$ is as follows:



Hence it has a period of 1 second.

$$c_0 = \int_0^1 \sin(\pi t)dt = \frac{1}{\pi}\left[\cos(\pi t)\right]_0^1 = -\frac{2}{\pi}$$

$$\begin{aligned}
c_k &= \int_0^1 \sin(\pi t) e^{-j2\pi kt} dt \\
&= \int_0^1 \frac{1}{2j}\left[e^{j\pi t} - e^{-j\pi t}\right] e^{-j2\pi kt} dt \\
&= \frac{1}{2j}\int_0^1 \left[e^{j\pi t(1-2k)} - e^{-j\pi t(1+2k)}\right] dt \\
&= \frac{1}{2j}\left[\frac{e^{j\pi t(1-2k)}}{j\pi(1-2k)} + \frac{e^{-j\pi t(1+2k)}}{-j\pi(1+2k)}\right]_0^1 \\
&= -\frac{1}{2\pi}\left[\frac{e^{j\pi(1-2k)}-1}{(1-2k)} + \frac{e^{-j\pi(1+2k)}-1}{(1+2k)}\right] \\
&= -\frac{1}{2\pi}\left[\frac{-1-1}{(1-2k)} + \frac{-1-1}{(1+2k)}\right] \\
&= -\frac{1}{2\pi}\left[\frac{-2}{(1-2k)} + \frac{-2}{(1+2k)}\right] \\
&= -\frac{1}{2\pi}\left[\frac{-2(1+2k)-2(1-2k)}{1-4k^2}\right] \\
&= -\frac{1}{2\pi}.\frac{4}{4k^2-1}
\end{aligned}$$

Hence, we have: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt} = -\frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{4k^2-1}e^{j2\pi kt}$

We have:

$$c_0 = a_0 = \frac{2}{\pi}$$
$$A_k = |c_k| = \frac{2}{\pi}\frac{1}{4k^2-1}$$
$$\angle c_k = \theta_k = -\pi$$

$$x(t) = a_0 + 2\sum A_k \cos\left(2\pi kt + \theta_k\right)$$
$$= \frac{2}{\pi} + 2\sum_{k=1}^{\infty}\frac{2}{\pi}\frac{1}{4k^2-1}\cos\left(2\pi kt - \pi\right)$$
$$= \frac{2}{\pi} - \sum_{k=1}^{\infty}\frac{4}{\pi}\frac{1}{4k^2-1}\cos\left(2\pi kt - \pi\right)$$

S2 Given $x(t) = t^2 \quad ; \quad -\pi < t < \pi$ and $x(t+2\pi) = x(t)$, we have a periodic signal with period of 2π.

$$c_0 = \frac{1}{2\pi}\int_{-\vartheta}^{\pi} t^2 dt = \frac{1}{2\pi}\left[\frac{1}{3}t^3\right]_{-\pi}^{\pi} = \frac{1}{6\pi}\left[\pi^3 + \pi^3\right] = \frac{\pi^2}{3}$$
$$c_k = \frac{1}{2\pi}\int_{=\pi}^{\pi} t^2 e^{-j2\pi\left(\frac{k}{2\pi}\right)t} dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-jkt} dt$$

Consider $\int t^2 e^{-jkt} dt$, we have:

$$\int t^2 e^{-jkt} dt = t^2\left(\frac{e^{-jkt}}{-jk}\right) - \int \frac{e^{-jkt}}{jk}.2t.dt$$
$$= \frac{jt^2}{k}e^{-jkt} - \frac{2j}{k}\int te^{-jkt}$$
$$= \frac{jt^2}{k}e^{-jkt} - \frac{2j}{k}\left[t\frac{e^{-jkt}}{-jk} - \int\frac{e^{-jkt}}{-jk}dt\right]$$
$$= \frac{jt^2}{k}e^{-jkt} - \frac{2j}{k}\left[-\frac{te^{-jkt}}{jk} + \frac{1}{jk}\frac{e^{-jkt}}{-jk}\right]$$
$$= \frac{jt^2}{k}e^{-jkt} + \frac{2}{k^2}te^{-jkt} - \frac{2j}{k^3}e^{-jkt}$$

Hence:

$$c_k = \frac{1}{2\pi}\left[\frac{jt^2}{k}e^{-jkt} + \frac{2}{k^2}te^{-jkt} - \frac{2j}{k^3}e^{-jkt}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi}\left[\frac{j\pi^2}{k}e^{-j\pi k} + \frac{2\pi}{k^2}e^{-j\pi k} - \frac{2j}{k^3}e^{-j\pi k} - \frac{j\pi^2}{k}e^{j\pi k} + \frac{2\pi}{k^2}e^{j\pi k} - \frac{2j}{k^3}e^{j\pi k}\right]$$
$$= \frac{1}{2\pi}\left[\frac{2\pi}{k^2}e^{-j\pi k} + \frac{2\pi}{k^2}e^{j\pi k}\right] = \frac{1}{2\pi}\left[\frac{4\pi}{k^2}(-1)^k\right] = \frac{2}{k^2}(-1)^k$$

Hence we have: $x(t) = \dfrac{\pi^2}{3} + \sum_k \dfrac{2}{k^2}(-1)^k e^{jkt}$

S3 $$x(t) = c_0 + \sum c_k \cos\left(2\pi \frac{k}{T_0} t - \theta_k\right)$$

Hence: $X_0 = c_0$ and $\left| X_k = \frac{1}{2} c_k \right|$ and $\angle X_k = \tan(-\theta_k) = -\tan(\theta_k)$