Outline of Lecture

1. Laplace Transform Review
   - Definition of Laplace Transform
   - Laplace Transform Table
   - Properties of Laplace Transform
   - Inverse Laplace Transform and Partial Factorization
2. Circuit Theory Review
   - Resistors, Capacitors and Inductors
   - Series RC Circuit
   - Parallel RL Circuit
   - Parallel RLC Circuit

Laplace Transform Review

1. Laplace Transforms

   The Laplace transform is an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

- Definition of Laplace Transforms The Laplace transform $F(s)$ of a time-domain function $f(t)$ is defined by

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{1}$$

where $f(t)$ is defined for all t and $s = \sigma + j\omega$ is a complex variable. If $f(t)$ is causal ie $f(t) = 0$ for $t < 0$, then

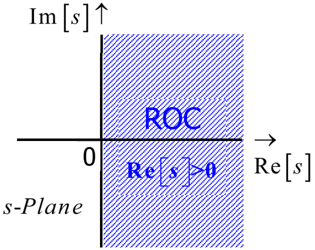
$$F(s) = \mathcal{L}[f(t)] = \int_{0}^{\infty} f(t)e^{-st}dt \tag{2}$$

(1) and (2) are, respectively known as the bilateral and unilateral Laplace transform. For the rest of this module, the unilateral LT in (2) will be used as we will only be dealing with causal systems.

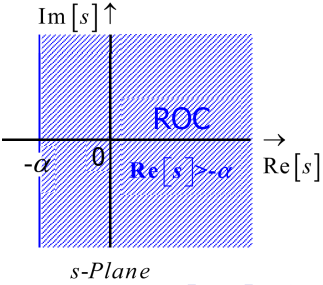
Example 1

Find the Laplace transform of (i) $u(t)$ and (ii) $e^{-\alpha t}u(t)$.

$$\begin{aligned}
\mathcal{L}[u(t)] &= \int_0^\infty 1e^{-st}dt \\
&= \left[\frac{e^{-st}}{-s}\right]_0^\infty \\
&= \begin{cases} \frac{1}{s} & \text{if } \mathrm{Re}[s] > 0 \\ \text{undefined} & \text{if } \mathrm{Re}[s] \leq 0 \end{cases}
\end{aligned}$$



$$\begin{aligned}
\mathcal{L}[e^{-\alpha t}u(t)] &= \int_0^\infty e^{-\alpha t}e^{-st}dt \\
&= \left[-\frac{e^{-(s+\alpha)t}}{s+\alpha}\right]_0^\infty \\
&= \begin{cases} \dfrac{1}{s+\alpha} & \text{if } \mathrm{Re}[s] > -\alpha \\ \text{undefined} & \text{if } \mathrm{Re}[s] \leq -\alpha \end{cases}
\end{aligned}$$



ROC = region of convergence

2. Laplace Transform Table

Table 1: Laplace Transform Pairs

$\mathbf{f}(\mathbf{t})$	$\mathbf{F}(\mathbf{s})$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin(at)u(t)$	$\frac{a}{s^2+a^2}$
$\cos(at)u(t)$	$\frac{s}{s^2+a^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{(s+a)}{(s+a)^2+b^2}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$

3. Properties of the Laplace Transform ($F(s) = \mathcal{L}[f(t)]$ or $f(t) \leftrightarrow F(s)$)

Properties	t-domain, $f(t)$	s-domain, $F(s)$
Linearity	$\alpha f_1(t) + \beta f_2(t)$	$\alpha F_1(s) + \beta F_2(s)$
Time Shifting	$f(t - t_0)$	$F(s)e^{-st_0}$
Shifting in s	$e^{s_0 t} f(t)$	$F(s - s_0)$
Time Scaling	$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert} F(\frac{s}{\alpha})$
Integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
Differentiation in t	$\frac{df(t)}{dt}$	$sF(s) - f(t)\rvert_{t=0}$
n^{th} order diff.	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} \left.\frac{d^k f}{dt^k}\right\rvert_{t=0}$
Differentiation in s	$(-1)^n f(t)$	$\frac{d^n F(s)}{ds^n}$
Convolution in t	$\int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) d\tau$	$F_1(s)F_2(s)$

- Initial and final value theorems

$$\text{Initial Value Theorem (IVT)}: \quad \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$\text{Final Value Theorem (FVT)}: \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The Final Value Theorem (FVT) is particularly useful for computing the final values or steady state of output responses of a LTI system.

4. The Inverse Laplace Transform
   The inverse Laplace transform is defined as :

$$f(t) = \mathcal{L}[F(s)] = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

However this inverse formula is never used. Instead, most of the time, you will use Table 1 in Slide 4 to help you.

Example 2

Find the inverse Laplace Transform of $\dfrac{1}{s^2+2s+4}$.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+3}\right\} \\
&= \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}\right\} \\
&= \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s_1^2+(\sqrt{3})^2}\right\} \text{ where } s_1 = s+1
\end{aligned}$$

Since $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$ and $\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)$,

it follows that $\mathcal{L}\left\{\dfrac{1}{s^2+2s+4}\right\} = \dfrac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t.$

5. Inverse Laplace Transform and Partial Factorization

- Laplace transforms are usually of the form : $F(s) = \dfrac{N(s)}{D(s)}$ where $N(s)$ and $D(s)$ are polynomials in the s-variable.
  Examples are $F(s) = \frac{1}{s}$, $F(s) = \frac{s+1}{(s+1)^2+3}$, $F(s) = \frac{1}{s+2}$, etc.
- The approach to obtaining the inverse LT of $F(s)$ is to first partial factorize $F(s)$ so that the partial factors consists of only polynomials of orders at most 2. The inverse LT of $F(s)$ is then the inverse LT of each partial factor which can be easily deduced from table in Slide 4. This approach works because of the linearity property of the LT.
- There are 3 different forms of partial factors :
  - (a) Functions involving only distinct linear factors in the $D(s)$:

$$F(s) = \frac{N(s)}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_n)} = \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2} + \ldots + \frac{A_n}{s+\alpha_n}$$

where $\alpha_i \neq \alpha_j,\ i \neq j$.
For these distinct linear factors, the inverse LT is given by :

$$f(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} + \ldots + A_n e^{-\alpha_n t}.$$

- Functions involving repeated linear factors in the $D(s)$:

$$F(s) = \frac{N(s)}{(s+\alpha)^n} = \frac{A_1}{s+\alpha} + \frac{A_2}{(s+\alpha)^2} + \ldots + \frac{A_n}{(s+\alpha)^n}$$

  For these repeated linear factors, the inverse LT is given by :

$$f(t) = \left(A_1 + A_2 t + \frac{1}{2} A_3 t^2 + \ldots + \frac{1}{n!} A_n t^{n-1} \right) e^{-\alpha t}.$$

- Functions involving quadratic factors with complex roots :

$$\begin{aligned} F(s) &= \frac{N(s)}{(s^2 + 2\beta_1 s + \gamma_1^2)(s^2 + 2\beta_2 s + \gamma_2^2)} \\ &= \frac{A_1 s + B_1}{(s^2 + 2\beta_1 s + \gamma_1^2)} + \frac{A_2 s + B_2}{(s^2 + 2\beta_2 s + \gamma_2^2)} \end{aligned}$$

For such factors, the general form of the inverse LT is given by :

$$f(t) = R_1 e^{-\beta_1 t} \sin\left(\sqrt{\gamma_1^2 - \beta_1^2}\, t + \phi_1 \right) + R_2 e^{-\beta_2 t} \sin\left(\sqrt{\gamma_2^2 - \beta_2^2}\, t + \phi_2 \right).$$

The R_i are functions of β_i and γ_i. For details on how to calculate A_i, R_i, β_i and γ_i, please refer to Revision Notes 0.

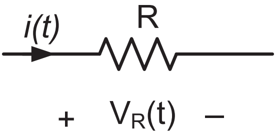
6. Review of Circuit Theory

- Resistors, Capacitors and Inductors

Resistors

$$\text{Time Domain}: i(t) = \frac{V_R(t)}{R}$$

$$\text{Frequency Domain}: I(s) = \frac{V_R(s)}{R}$$



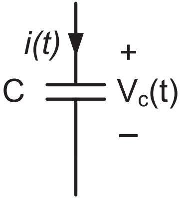
Capacitors

$$\text{Time Domain}: i(t) = C\frac{dV_c(t)}{dt}$$

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

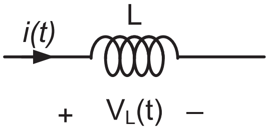
$$\text{Frequency Domain}: V_c(s) = \frac{I(s)}{sC}$$

$$Z_c(s) = \frac{V_c(s)}{I(s)} = \frac{1}{sC}$$



Inductors

$$
\begin{aligned}
\text{Time Domain}: V_L(t) &= L\frac{di(t)}{dt} \\
i(t) &= \frac{1}{L}\int_{-\infty}^{t} V_L(\tau)d\tau \\
\text{Frequency Domain}: V_L(s) &= sLI(s) \\
Z_L(s) &= \frac{V_L(s)}{I(s)} = sL
\end{aligned}
$$

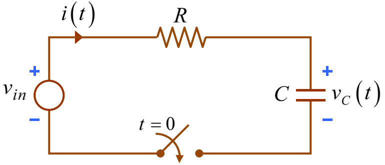


- In all the circuit elements, above, we assume zero initial conditions. If initial conditions are non-zero, then they have to be accounted for in the equations. An example of how this is done is shown in Slide 12.

- The Laplace transform is very useful in solving systems equations involving ordinary differential equation. See example in Slide 12.

Example 3 (Series RC Circuit)

For the RC circuit shown on the right, find the voltage $v_c(t)$ across the capacitor, C. The input v_{in} is a constant DC voltage source.



$$\underbrace{R\,C\frac{dv_c(t)}{dt}}_{i(t)} + v_c(t) = v_{in} \quad (v_{in}\ \text{constant}) \tag{3}$$

Taking Laplace transform on both sides of (3),

$$RC \quad \underbrace{[sV_c(s) - v_c(0^-)]}_{\mathcal{L}[\frac{dv_c(t)}{dt}]} + V_c(s) = \frac{v_{in}}{s}, \quad [v_c(0^-) = \text{initial condition}]$$

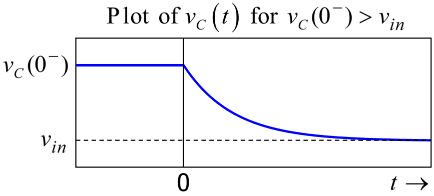
$$\begin{aligned} V_c(s) &= \frac{RCv_c(0^-)}{sRC+1} + \frac{v_{in}}{s(sRC+1)} = \frac{v_c(0^-) - v_{in}}{s + 1/(RC)} + \frac{v_{in}}{s} \\ v_c(t) &= \mathcal{L}^{-1}[V_c(s)] = [v_c(0^-) - v_{in}]e^{-\frac{t}{RC}} + v_{in} \ldots \text{for } t \geq 0 \end{aligned} \tag{4}$$

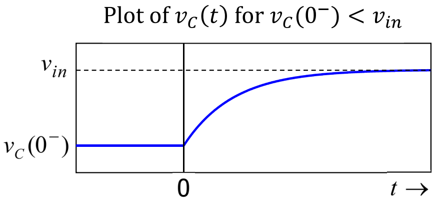
Re-arranging $v_c(t)$ in (4),

$$v_c(t) = v_{in}\left[1 - e^{-\frac{t}{RC}}\right] + v_c(0^-)e^{-\frac{t}{RC}} \ldots \text{for } t \geq 0 \qquad (5)$$

(5) is a typical charging equation for a capacitor, assuming non-zero initial condition.

Sketching the response curves for 2 possible initial conditions of $v_c(0^-)$:



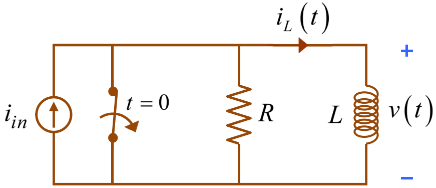


Applying IVT : $\lim_{t\to 0} v_c(t) = \lim_{s\to\infty} sV_c(s) = v_c(0^-)$

Applying FVT : $\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} sV_c(s) = v_{in}$

Example 4 (Parallel RL Circuit)

For the RL circuit shown on the right, find the current $i(t)$ through the inductor. Assume that i_{in} is a constant current source.



$$\frac{1}{R}\underbrace{L\frac{di_L(t)}{dt}}_{v(t)}+i_L(t)=i_{in} \quad \ldots\text{and taking Laplace transform :}$$

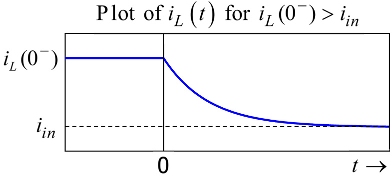
$$\frac{L}{R}\underbrace{[sI_L(s)-i_L(0^-)]}_{\mathcal{L}[\frac{di_L(t)}{dt}]}+I_L(s)=\frac{i_{in}}{s} \quad [i_L(0^-)\text{ is the initial condition}]$$

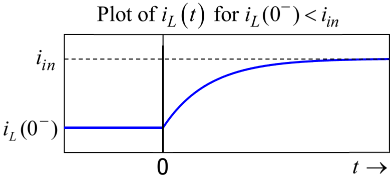
$$\begin{aligned} I_L(s) &= \frac{i_{in}R/L}{s(s+R/L)}+\frac{i_L(0^-)}{s+R/L}=\frac{i_L(0^-)-i_{in}}{s+R/L}+\frac{i_{in}}{s} \\ i_L(t) &= \mathcal{L}^{-1}[I_L(s)]=[i_L(0^-)-i_{in}]e^{-\frac{R}{L}t}+i_{in} \\ &= i_{in}[1-e^{-\frac{R}{L}t}]+i_L(0^-)e^{-\frac{R}{L}t} \quad \ldots\text{for } t\geq 0 \end{aligned} \tag{6}$$

$$i_L(t) = i_{in}[1 - e^{-\frac{R}{L}t}] + i_L(0^-)e^{-\frac{R}{L}t} \quad \ldots \text{for } t \geq 0 \tag{7}$$

Notice that (7) is very similar to (5) - both solutions come from first order differential equations with different coefficients.

Sketching the response curves for 2 possible initial conditions of $i_L(0^-)$:

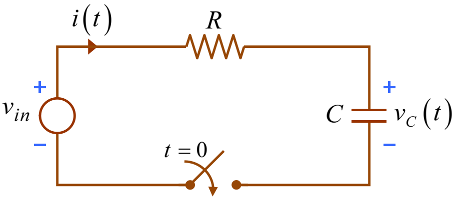


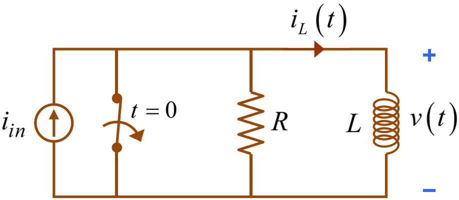


Applying IVT : $\lim_{t \to 0} i_L(t) = \lim_{s \to \infty} sI_L(s) = i_L(0^-)$

Applying FVT : $\lim_{t \to \infty} i_L(t) = \lim_{s \to 0} sI_L(s) = i_{in}$

Comparing series RC and parallel RL circuits :





$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_{in}$$

$$v_c(t) = v_{in}\left[1 - e^{-\frac{t}{RC}}\right] + v_c(0^-)e^{-\frac{t}{RC}}$$

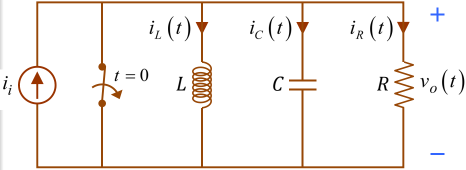
$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_{in}$$

$$i_L(t) = i_{in}[1 - e^{-\frac{R}{L}t}] + i_L(0^-)e^{-\frac{R}{L}t}$$

Based on this comparison, it can be concluded that the coefficients of the differential equation (RC and $\frac{L}{R}$) clearly plays a role in the expressions for $v_c(t)$ and $i_L(t)$ respectively.

Example 5 (Parallel RLC Circuit)

For the RLC circuit shown, write down the differential equation for solving $v_0(t)$ and the expression for $V_0(s)$. Determine $v_0(t)$ for the case when all initial conditions are zero. Assume $i_i = 1$ and $C = L = 1$ and $R = 5/6$.



Writing the differential equation for general input current $i_i(t)$,

$$i_i(t) = \underbrace{\frac{1}{L}\int_0^t v_0(\tau)d\tau}_{i_L(t)} + \underbrace{C\frac{dv_0(t)}{dt}}_{i_c(t)} + \underbrace{\frac{v_0(t)}{R}}_{i_R(t)}$$

$$\text{Differentiating : } \frac{di_i(t)}{dt} = \frac{1}{L}v_0(t) + C\frac{d^2v_0(t)}{dt^2} + \frac{1}{R}\frac{dv_0(t)}{dt} \qquad (8)$$

(8) is a second order differential equation which can be solved using Laplace transform, to obtain $v_0(t)$, given a particular $i_i(t)$.

Taking Laplace transforms of (8), and assuming zero initial conditions :

$$sI_i(s) - \cancel{i_i(0^-)}^{\,0} = \frac{V_0(s)}{L} + C\left[s^2V_0(s) - \cancel{\frac{dv(0^-)}{dt}}^{\,0} - s\cancel{v(0^-)}^{\,0}\right] + \frac{1}{R}\left[sV_0(s) - \cancel{v(0^-)}^{\,0}\right]$$

Substituting $I_i(s) = \frac{1}{s}$ (because $i_i = 1$ and hence $I_i(s) = \frac{1}{s}$), $L = C = 1$ and $R = 5/6$, we get,

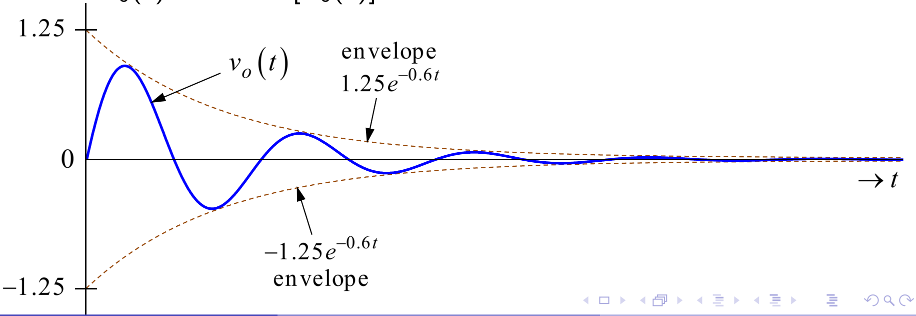
$$\begin{aligned} 1 &= \frac{V_0(s)}{L} + Cs^2V_0(s) + \frac{1}{R}sV_0(s) \\ &= V_0(s)\frac{R + s^2RLC + sL}{LR} \\ V_0(s) &= \frac{LR}{s^2RLC + sL + R} = \frac{1}{s^2 + \frac{6}{5}s + 1} \end{aligned}$$

$$V_0(s) = \frac{LR}{s^2RLC + sL + R} = \frac{1}{s^2 + \frac{6}{5}s + 1}$$

$$V_0(s) = \underbrace{\frac{1}{s^2 + \frac{6}{5}s + 1} = \frac{1}{(s+\frac{3}{5})^2 + \left[1 - \left(\frac{3}{5}\right)^2\right]}}_{\text{completing the square}}$$

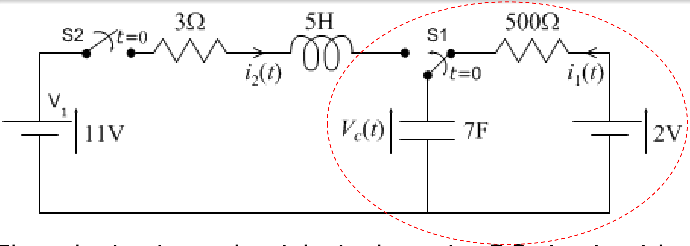
$$= \frac{1}{(s+\frac{3}{5})^2 + \left(\frac{4}{5}\right)^2} = \underbrace{\frac{1}{0.8}}_{=1.25} \times \frac{0.8}{(s+0.6)^2 + 0.8^2}$$

$$v_0(t) = \mathcal{L}^{-1}[V_0(s)] = 1.25e^{-0.6t}\sin 0.8t$$



Example 6 (Series RLC Circuit with non-Zero I.C.)

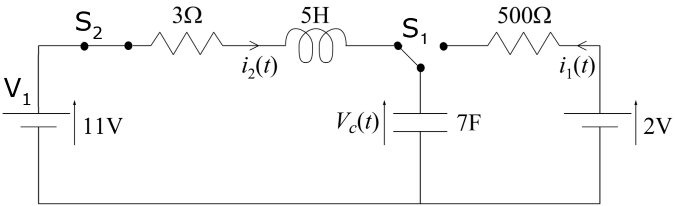
Consider the series RLC circuit below. Assume that the switch S1 was closed for a long time before $t = 0$. At $t = 0$, S1 is thrown the other way while S2 is closed. Derive the initial condition of the circuit. Hence find the resulting $v_c(t)$ when $t \geq 0$.



The sub-circuit on the right is the series RC circuit with a battery source of 2V. If S1 has been closed for a long time, the capacitor will be charged to 2V. There is no current flowing in the left circuit because S2 is open $t < 0$. Hence the initial conditions observed are :

$$v_c(0^-) = 2, \ \frac{dv_c(t)}{dt} = 0, \ i_1(0^-) = 0, \ \frac{di_1(t)}{dt} = \frac{di_2(t)}{dt} = 0$$

When $t \geq 0$, the circuit of interest is a RLC circuit :



KVL around the loop : $v_1(t) = 3i_2(t) + 5\dfrac{di_2(t)}{dt} + v_c(t)$

$$\left.\begin{array}{l} i_2(t) = C\frac{dv_c(t)}{dt} = 7\frac{dv_c(t)}{dt} \\ \\ v_L(t) = L\frac{di_2(t)}{dt} = 5\frac{di_2(t)}{dt} \end{array}\right\} v_1(t) = 3i_2(t) + 5\frac{di_2(t)}{dt} + v_c(t)$$

$$\begin{aligned} &= 3\times 7\frac{dv_c(t)}{dt} + 5\frac{d}{dt}\left(7\frac{dv_c(t)}{dt}\right) + v_c(t) \\ &= 21\frac{dv_c(t)}{dt} + 35\frac{d^2v_c(t)}{dt^2} + v_c(t) \qquad (9) \end{aligned}$$

Taking Laplace transform of (9), with a general battery source of $v_1(t)$,

$$\begin{aligned}
V_1(s) &= 21\underbrace{\left[sV_c(s) - v_c(0^-)\right]}_{\mathcal{L}(\frac{dv_c(t)}{dt})} + 35\underbrace{\left[s^2V_c(s) - sv_c(0^-) - \frac{dv_c(0^-)}{dt}\right]}_{\mathcal{L}(\frac{d^2v_c(t)}{dt^2})} + V_c(s) \\
&= (35s^2 + 21s + 1)V_c(s) - 21v_c(0^-) - 35sv_c(0^-) - 35\frac{dv_c(0^-)}{dt} \\
&= (35s^2 + 21s + 1)V_c(s) - 42 - 70s \left(v_c(0^-) = 2, \frac{dv_c(0^-)}{dt} = 0\right) \\
V_c(s) &= \frac{V_1(s) + 42 + 70s}{35s^2 + 21s + 1}
\end{aligned}$$

Since the battery source is a DC source of 11 V, then $V_1(s) = 11/s$. Hence

$$V_c(s) = \frac{11 + 42s + 70s^2}{s(35s^2 + 21s + 1)}$$

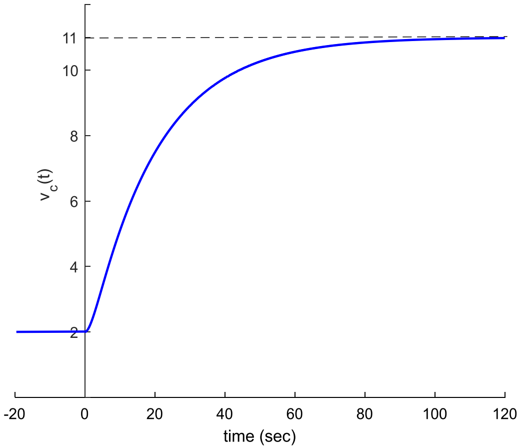
$v_c(t)$ can be obtained by taking the inverse Laplace transform of $V_c(s)$.

$$
\begin{aligned}
V_c(s) &= \frac{11+42s+70s^2}{s(35s^2+21s+1)} \\
&= \frac{A}{s}+\frac{Bs+C}{35s^2+21s+1} = \frac{(35A+B)s^2+(21A+C)s+A}{s(35s^2+21s+1)} \\
&= \frac{11}{s}-\frac{315s+189}{35s^2+21s+1} \qquad A=11, B=-315, C=-189 \\
&= \frac{11}{s}-\frac{9s+5.4}{s^2+0.6s+1/35} \\
&= \frac{11}{s}-\frac{9.94}{s+0.052}+\frac{0.94}{s+0.548} \\
v_c(t) &= \mathcal{L}[V_c(s)] = 11\underbrace{-9.94e^{-0.052t}+0.94e^{-0.548t}}_{\text{transient response}} \quad \text{for } t \geq 0 \\
\lim_{t\to\infty} v_c(t) &= 11 \quad \leftarrow \text{steady state value of } v_c(t) - \text{substitute } t=\infty
\end{aligned}
$$

Applying IVT : $\lim_{t\to 0} v_c(t) = \lim_{s\to\infty} sV_c(s) = 2$

Applying FVT : $\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} sV_c(s) = 11$

Plot of $v_c(t)$ is given below.



Note the initial condition $v_c(0^-) = 2$.

Final value or steady state value $v_c(t)|_{t\to\infty} = 11$.

Exercise 1 (Write your answer in this space)

For the RC circuit shown on the right, assume that the switch closed at $t = 0$ and open again at $t = 10$. Find the voltage $v_c(t)$ across the capacitor, C. Assume that $R = 2\,M\Omega$, $C = 1\,\mu F$ and v_{in} is a constant source of 5V.