NATIONAL UNIVERSITY OF SINGAPORE

EE2023/EE2023E/TEE2023 – SIGNALS AND SYSTEMS

Online Examination

Semester II : 2019/2020 Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. Please ensure that you have downloaded the correct e-examination paper which is **EE2023 Signals & Systems.**
2. This paper contains **EIGHT (8)** questions and comprises **TWELVE (12)** printed pages.
3. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.
4. Students should write the answers for each question on a new page.
5. This is an **OPEN BOOK** examination.
6. Tables of formulas are provided on Pages 9 to 12.
7. For the purpose of this exam, please take note of the digits *a*, *b*, *c* and *d* in your student number A0xx***abcd***X. These digits will be used in the questions in this examination.

At the end of this e-examination, please do the following :

1. On a blank sheet of paper (referred to as the Cover Page) write down your student number and total number of pages clearly. Then copy and sign the honor code statement below.

   Student Number : ______________________ Total Pages : ____________________
   (including cover page)

   I have read and abided by the "Declaration of Academic Integrity in the Online Examination" when writing this exam.

   Signature : ____________________________________

2. Please scan the Cover Page and all answer scripts. Ensure that you have page numbers on every page and scan all pages into one pdf file. **Name your file : student number-EE2023.pdf**

3. Please upload a copy of your answer scripts file to **LumiNUS → EE2023 → Files → Exam Submission,** and also email a copy to **ELEBOX38@nus.edu.sg**.

SECTION A : Answer ALL questions in this section

Before you begin, please take note of your *a, b, c* and *d* digits in your student number.

Q1. The signal $x(t)$ in Figure Q1 can be represented as $x(t) = \alpha.rect\left(\frac{t}{\beta}\right)[1 - \gamma cos(2\pi f_o t)]$. The axes labels in Figure Q1 are defined as T = (*a* + 1) and L = (*c* + 2), where *a* and *c* are the values from your student number.

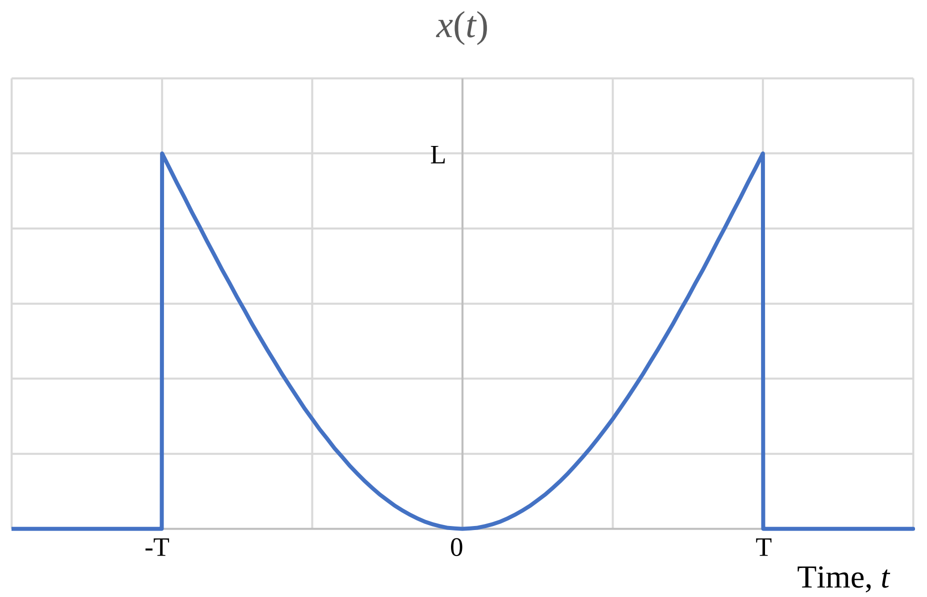

Figure Q1.

(a) What are the values of α, β, γ and f_o?

(2 marks)

(b) Determine the Fourier transform, $X(f)$, of the signal $x(t)$.

(4 marks)

(c) The periodic signal $x_p(t)$ is obtained by replications of the generating signal $x(t)$ with a period of 5 seconds.

i. Obtain an expression for $x_p(t)$ in terms of $x(t)$ and the unit impulse function $\delta(t)$.

(1 marks)

ii. Determine the Fourier transform, $X_p(f)$, of the periodic signal $x_p(t)$.

(3 marks)

Q2. Consider the periodic signal $x(t) = cos\left(8\pi t + \frac{\pi c}{3}\right) - 2e^{j12\pi t} + 3d$, where c and d are the values from your student number.

(a) What are the fundamental frequency, f_o, and the period of $x(t)$?

(2 marks)

(b) Determine the Fourier series coefficients, X_k, of the periodic signal $x(t)$.

(4 marks)

(c) Determine the Fourier transform, $X(f)$, of the periodic signal $x(t)$.

(4 marks)

Q3. A first order linear time invariant system is given in Figure Q3a where K and T are positive constants. The step response of the system to an input $x(t) = (a + 0.5)u(t)$ is given in Figure Q3b, where a is the value from your student number and $u(t)$ denotes the unit step function.

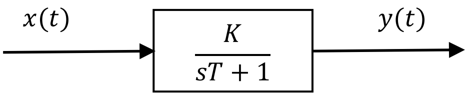

Fig. Q3a

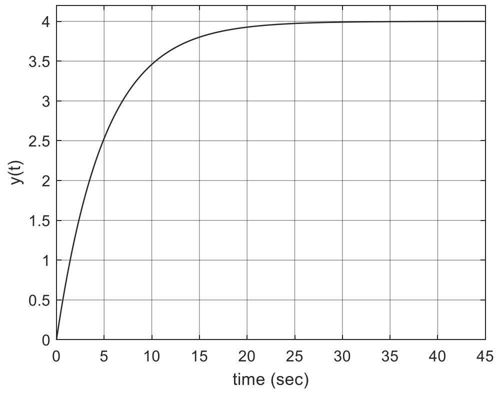

Fig. Q3b : Step Response

(a) Determine the parameters K and T.

(3 marks)

(b) Assume T remains constant. Sketch the output $y(t)$ if the value of K is <u>half the value obtained in part (a) above</u>. Label your sketch with appropriate values.

(2 marks)

(c) Assume K is the same value as that in part (a). Sketch the output $y(t)$ if the value of T is <u>twice the value obtained in part (a)</u>. Label your sketch with appropriate values.

(2 marks)

(d) With the parameters of K and T from part (a) above, find the steady state output response if the input is $x(t) = (a + 2)\sin(4t)$. In case you are not able to find the values of K and T in part (a), you may demonstrate your answer to this part by assuming that $K = T = 1$.

(3 marks)

Q4. The transfer function of a second order system is

$$G(s) = \frac{3(d+1)^2}{s^2 + 2(d+1)s + (d+1)^2} .$$

where d is the value from your student number.

(a) What is the steady-state gain of the system?

(2 marks)

(b) Find the system poles. Hence, or otherwise, determine if the system is underdamped, critically damped or overdamped.

(3 marks)

(c) Identify all the functions from the following list that may be terms in the output signal when the signal, $4u(t)$, is applied to the system?

- Decaying exponential function, $Ae^{-\alpha t}u(t)$
- Growing exponential function, $Be^{\alpha t}u(t)$
- Damped ramp function, $Cte^{-\alpha t}u(t)$
- Decaying complex exponential function, $e^{-\alpha t}\left(D_1 \sin \omega_o t + D_2 \cos \omega_o t\right)u(t)$
- Growing complex exponential function, $e^{\alpha t}\left(D_1 \sin \omega_o t + D_2 \cos \omega_o t\right)u(t)$
- Step function, $3u(t)$
- Step function, $4u(t)$
- Step function, $12u(t)$
- Step function, $75u(t)$
- Step function, $225u(t)$

$u(t)$ is the unit step function. α ($\alpha > 0$), ω_o, A, B, C, D_1 and D_2 are constants. For the given transfer function, what value would α assume?

(5 marks)

SECTION B : Answer 3 out of the 4 questions in this section

Q5. Consider the system shown in Figure Q5 where the input signal is $x(t) = 2(b + c) + 6\text{sinc}(6t)$, and $y(t) = \cos(20\pi t)$, where b and c are values from your student number. The symbol $\otimes$ represents multiplication.

(a) Determine the Fourier transform, $Z(f)$, of the signal $z(t)$, and sketch its magnitude spectrum.

(6 marks)

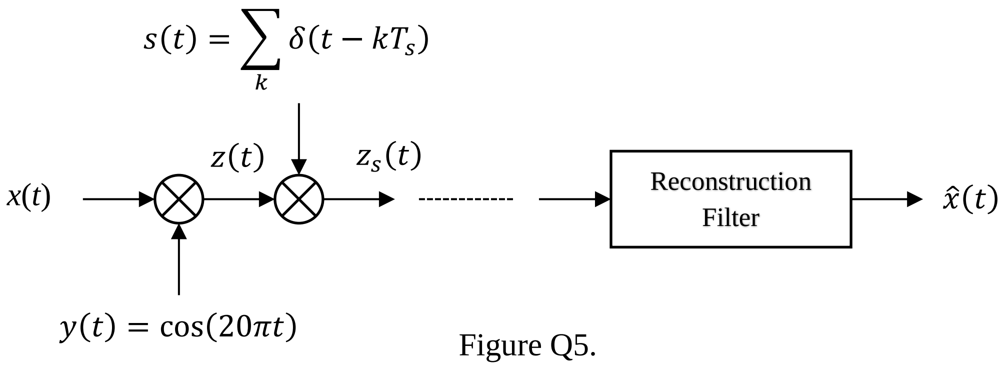

Figure Q5.

(b) The signal $s(t)$ is a sampling Comb function to obtain the sampled signal $z_s(t)$, in which the sampling frequency is 10 Hz.

i. What is the Nyquist frequency for signal $z(t) = x(t).y(t)$?

(2 marks)

ii. Determine the expression for $z_s(t)$ in terms of $z(t)$.

(2 marks)

iii. Determine the Fourier transform, $Z_s(f)$, of the signal $z_s(t)$, and sketch its magnitude spectrum.

(7 marks)

iii. Determine the characteristics of the reconstruction filter that will recover the signal $x(t)$ from the sampled signal $z_s(t)$.

(3 marks)

Q6. A virus is known to multiply according to the differential equation given by :

$$\frac{dv(t)}{dt} - (b + 0.5)v(t) = x(t)$$

where b is the value from your student number, $v(t)$ represents the amount of virus in the body over time, t (measured in days), while $x(t)$ represents the amount of virus introduced into the body from external sources.

Suppose a small sample of the virus was introduced into the body of John at $t = 0$. You may model this external input using $x(t) = \delta(t)$ where $\delta(t)$ represents a unit impulse.

When the viral particles reach $0.5\text{x}10^6$, the first response of the body's immune system is a fever. With no medical intervention, the person dies when the amount of viral particles reaches 10^7.

When the immune system responses, the rise of the body temperature reduces the rate of replication of the virus. At the same time the viral medication which John takes also reduces the virus. The overall effect on the viral particles $v(t)$, evolves according to the following second order differential equation :

$$\frac{d^2v(t)}{dt^2} + \frac{dv(t)}{dt} + (\text{c} - 2.5)v(t) = z(t)$$

where $z(t)$ represents the amount of viral medication taken by John and c is the value from your student number.

(a) If no viral medication was given to John following the first introduction of the virus, determine the time when John will experience the first onset of fever. You may assume that there was no trace of the virus in his body before the virus was introduced.

(3 marks)

(b) If John was not given any medication, when will John die? Give your answer in terms of how many days after the introduction of the virus.

(3 marks)

(c) At the onset of fever, John took two doses of the viral medication which can be modelled as two unit impulses which are 1 day apart. Assume that the time when the first dose was taken is $t = 0$ after the onset of fever.

i. Write down an appropriate $z(t)$ which models the medication which John consumed.

(3 marks)

ii. Write down the initial conditions $v(t)|_{t=0}$ and $\frac{dv(t)}{dt}|_{t=0}$.

(4 marks)

iii. Find the transfer function of the dynamical system which determines the evolution of the virus after John is medicated.

(3 marks)

iv. Will John survive this viral attack after the medication? Explain your answer.

(4 marks)

Q7. (a) The signal $x(t)$ has a half-cosine spectrum $X(f)$ as shown in Figure Q7, where the label F in the figure is given by F = d + 1, and d is the value from your student number.

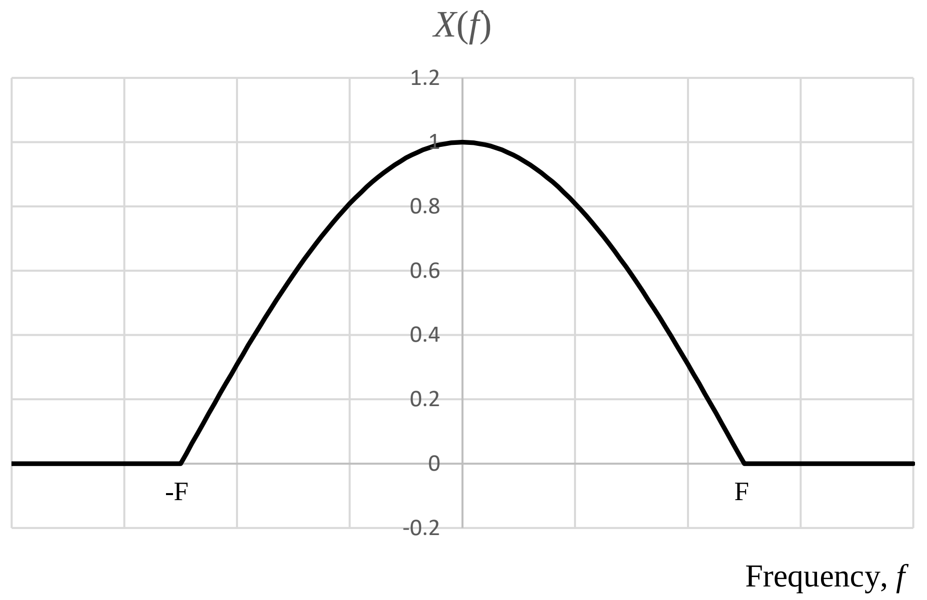

Figure Q7.

i. Obtain the expression for the spectrum $X(f)$.

(2 marks)

ii. Determine the energy spectral density, $E_x(f)$, of the signal $x(t)$.

(2 marks)

iii. Determine the energy of the signal $x(t)$.

(4 marks)

iv. Determine the 3dB bandwidth of the signal $x(t)$.

(3 marks)

(b) Consider the signal $y(t) = \frac{d}{dt}x(t)$.

i. Determine the Fourier transform, $Y(f)$, of the signal $y(t)$.

(3 marks)

ii. Determine the magnitude spectrum, $|Y(f)|$, of the signal $y(t)$.

(3 marks)

iii. Determine the phase spectrum, $\angle Y(f)$, of the signal $y(t)$.

(3 marks)

Q8. The transfer function of an amplifier-speaker system is

$$G(s) = \frac{Y(s)}{X(s)} = \frac{0.025K(s+40)}{\left(\frac{s}{200}+1\right)} \frac{5000^2}{s^2+5000s+5000^2}$$

where K is the amplifier gain.

(a) A set of sinusoidal test tones is used to ascertain if all musical notes in the audible range can be heard clearly, whether loud or soft. One of the test signals in the set is $x(t) = (1+a) + 1.75\cos\omega t$, where $\omega = 200 + 10b$. a and b are values from your student number. What is the steady-state output signal, $\lim_{t\to\infty} y(t)$, if $K = 1$?

(7 marks)

(b) The Bode magnitude plot of the audio amplifier-speaker system when $K = 1$ is shown in Figure Q8.

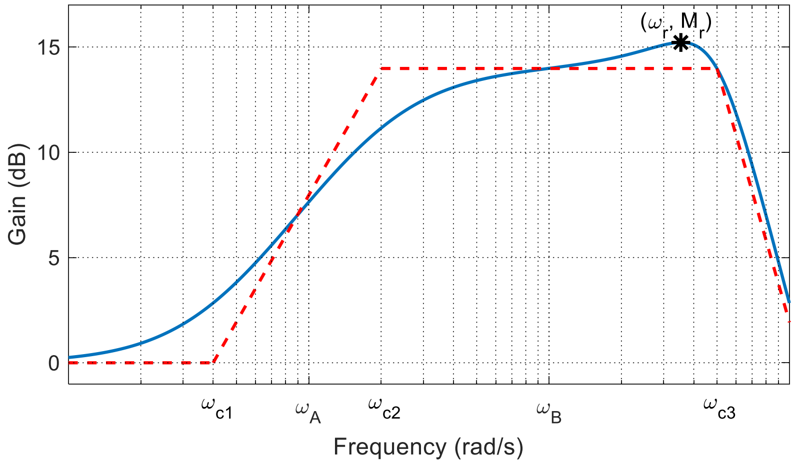

Figure Q8: Bode magnitude plot of $G(s) = \frac{0.025K(s+40)}{\left(\frac{s}{200}+1\right)} \frac{5000^2}{s^2+5000s+5000^2}$ when K=1.

i. Using the transfer function $G(s)$, deduce the values of the 3 corner frequencies ω_{c1}, ω_{c2}, and ω_{c3}. Hence, determine the values of ω_A and ω_B.

(5 marks)

ii. Use the resonant frequency formula for a second order system to compute the value of ω_r. Hence, or otherwise, determine the value of M_r in dB.

(5 marks)

(c) Music produced by the amplifier-speaker system, $G(s)$, is distorted if the moving coil in the speaker hits its limit, and does not move in accordance with the steady-state output signal, $\lim_{t\to\infty} y(t)$. Distortion will occur if any sinusoidal component in $\lim_{t\to\infty} y(t)$ has an amplitude that is larger than $(10 - 0.5c)$, where c is the value from your student number. Suppose the amplitude of all sinusoidal components in a song, $x(t)$, is less than 0.6. Derive the maximum amplifier gain, K, that will not distort the song?

(3 marks)

END OF QUESTIONS

Fourier Series: $$\begin{cases} X_k = \dfrac{1}{T}\displaystyle\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k\,t/T)\,dt \\ x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(j2\pi k\,t/T) \end{cases}$$

Fourier Transform: $$\begin{cases} X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$$

FOURIER TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\operatorname{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\operatorname{rect}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}(fT)$
Triangle	$\operatorname{tri}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}^2(fT)$
Sine Cardinal	$\operatorname{sinc}\left(\frac{t}{T}\right)$	$T\operatorname{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}[\delta(f-f_o)+\delta(f+f_o)]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}[\delta(f-f_o)-\delta(f+f_o)]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp(-\alpha^2\pi^2 f^2)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES

	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi ft_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty}X_1(\zeta)X_2(f-\zeta)d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
		$\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n[th] order Ramp	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/(s^2+\omega_o^2)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/(s^2+\omega_o^2)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES

	Time-domain	*s*-domain
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)u(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\frac{1}{s}X(s)$
Differentiation in the time-domain	$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\frac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\frac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\frac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\frac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

System Type	Transfer Function (Standard Form)	Unit Impulse and Unit Step Responses	Remarks
1st order system	$G(s)=\frac{K}{T}\cdot\frac{1}{s+1/T}$	$y_\delta(t)=\frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$ $y_{step}(t)=K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	T : Time-constant K : DC Gain Real Pole at $s=-\frac{1}{T}$
2nd order system $(\zeta>1)$ **Overdamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}$	$y_\delta(t)=\left[K_1\exp(-p_1t)+K_2\exp(-p_2t)\right]u(t)$ $y_{step}(t)=\left[K-\frac{K_1}{p_1}\exp(-p_1t)-\frac{K_2}{p_2}\exp(-p_2t)\right]u(t)$	K : DC Gain $\left.\begin{array}{l} p_1=\omega_n\zeta+\omega_n\sqrt{\zeta^2-1} \\ p_2=\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\end{array}\right\\}$ $K_1=-K_2=\frac{K\omega_n^2}{p_2-p_1}$ Real Distinct Poles at $s=-p_1$ and $s=-p_2$
2nd order system $(\zeta=1)$ **Critically damped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{(s+\omega_n)^2}$	$y_\delta(t)=K\omega_n^2 t\exp(-\omega_n t)u(t)$ $y_{step}(t)=K\left[1-\exp(-\omega_n t)-\omega_n t\exp(-\omega_n t)\right]u(t)$	K : DC Gain Real Repeated Poles at $s=-\omega_n$
2nd order system $(0<\zeta<1)$ **Underdamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\left(\sigma^2+\omega_d^2\right)}{(s+\sigma)^2+\omega_d^2}$	$y_\delta(t)=K\frac{\omega_n^2}{\omega_d}\exp(-\sigma t)\sin(\omega_d t)u(t)$ $y_{step}(t)=K\left[1-\frac{\omega_n}{\omega_d}\exp(-\sigma t)\sin(\omega_d t+\phi)\right]u(t)$	K : DC Gain ω_n : Undamped Natural Frequency ζ : Damping Ratio ω_d : Damped Natural Frequency $\sigma=\zeta\omega_n$ $\omega_d^2=\omega_n^2\left(1-\zeta^2\right)$ $\omega_n^2=\sigma^2+\omega_d^2$ $\tan(\phi)=\frac{\omega_d}{\sigma}$ Complex Conjugate Poles at $s=-\sigma\pm j\omega_d$
2nd order system $(\zeta=0)$ **Undamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{s^2+\omega_n^2}$	$y_\delta(t)=K\omega_n\sin(\omega_n t)u(t)$ $y_{step}(t)=K(1-\cos\omega_n t)u(t)$	K : DC Gain ω_n : Undamped Natural Frequency Imaginary Conjugate Poles at $s=\pm j\omega_n$

2nd order system RESONANCE $\left(\mathbf{0 \le \zeta < 1/\sqrt{2}}\right)$ ➔ *RESONANCE FREQUENCY :* $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ *RESONANCE PEAK :* $M_r = \left|G(j\omega_r)\right| = \dfrac{K}{2\zeta\sqrt{1-\zeta^2}}$

Trigonometric Identities

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\theta) = 0.5\left(e^{j\theta} + e^{-j\theta}\right)$	$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
$\sin(\theta) = -0.5j\left(e^{j\theta} - e^{-j\theta}\right)$ $\sin^2(\theta) + \cos^2(\theta) = 1$	$\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta) = 0.5\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$	$\cos(\alpha)\cos(\beta) = 0.5\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
$\sin^2(\theta) = 0.5\left[1 - \cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta) = 0.5\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$
$\cos^2(\theta) = 0.5\left[1 + \cos(2\theta)\right]$	$C\cos(\theta) - S\sin(\theta) = \sqrt{C^2+S^2}\cos\left[\theta + \tan^{-1}\left(\frac{S}{C}\right)\right]$

Complex Unit (j) ➔ $\left(j = \sqrt{-1} = e^{j\pi/2} = e^{j90^\circ}\right)$ $\left(-j = \dfrac{1}{j} = e^{-j\pi/2} = e^{-j90^\circ}\right)$ $\left(j^2 = -1\right)$

Definitions of Basic Functions

Rectangle:

$$\operatorname{rect}\left(\frac{t}{T}\right) = \begin{cases} 1; & -T/2 \le t < T/2 \\ 0; & \text{elsewhere} \end{cases}$$

Triangle:

$$\operatorname{tri}\left(\frac{t}{T}\right) = \begin{cases} 1 - |t|/T; & |t| \le T \\ 0; & |t| > T \end{cases}$$

Sine Cardinal:

$$\operatorname{sinc}\left(\frac{t}{T}\right) = \begin{cases} \dfrac{\sin(\pi t/T)}{\pi t/T}; & t \ne 0 \\ 1; & t = 0 \end{cases}$$

Signum:

$$\operatorname{sgn}(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$

Unit Impulse:

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \ne 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

Unit Step:

$$u(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$