National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems
Tutorial 8 Solutions

Section I

1. (a) As the output (transient + steady-state) signal is required, the expression should be found using Laplace Transform or calculus.

Using the Laplace transform method, the output of the first order system, $G(s)$, is

$$Y(s) = \frac{2}{0.2s+1}\mathcal{L}\{\sin(3t)\} = \frac{2}{0.2s+1}\frac{3}{s^2+9} = \frac{10}{s+5}\frac{3}{s^2+9} = \frac{30}{(s+5)(s^2+9)}$$

Let $Y(s) = \frac{A_1}{s+5} + \frac{A_2 s}{s^2+9} + \frac{A_3}{s^2+9} = \frac{A_1(s^2+9) + A_2 s(s+5) + A_3(s+5)}{(s+5)(s^2+9)}$

$$= \frac{\left(A_1 + A_2\right)s^2 + \left(5A_2 + A_3\right)s + \left(9A_1 + 5A_3\right)}{\left(s+5\right)\left(s^2+9\right)}$$

Comparing the numerators:

Let $s = -5$, we have: $30 = A_1(25+9) \quad \Rightarrow \quad A_1 = \frac{15}{17} = 0.88$

Matching constants, we have:

$$9A_1 + 5A_3 = 30 \quad ; \quad A_3 = (30 - 9A_1)/5 = \left(30 - \frac{9\times 15}{17}\right)/5 = 4.41$$

Matching s^2, we have: $A_1 + A_2 = 0 \quad \Rightarrow \quad A_2 = -A_1 = -0.88$

Hence, we have: $Y(s) = \frac{0.88}{s+5} - \frac{0.88s}{s^2+9} + \frac{4.41}{s^2+9} = \frac{0.88}{s+5} - \frac{0.88s}{s^2+9} + \frac{1.47\times 3}{s^2+9}$

$\therefore \quad y(t) = 0.88e^{-5t} - 0.88\cos(3t) + 1.47\sin(3t)$

The output of the first order system at steady-state is

$$\begin{aligned} y_{ss}(t) &= \lim_{t\to\infty}\left[0.88e^{-5t} - 0.88\cos(3t) + 1.47\sin(3t)\right] \\ &= -0.88\cos(3t) + 1.47\sin(3t) \\ &= A\sin(3t + \phi) \end{aligned}$$

where $A = \sqrt{0.88^2 + 1.47^2} = 1.71$ and $\phi = \tan^{-1}\frac{-0.88}{1.47} = -0.54$ rad. Hence,

$$y_{ss}(t) = 1.71\sin\left(3t - 0.54\right)$$

Alternatively, for an input $x(t) = A\sin(\omega t)$, the steady state output is also given by

$$y_{ss}(t) = A.|G(j\omega)|\sin\left[\omega t + \angle G(jw)\right]$$

The magnitude and phase of $G(j\omega)|_{\omega=3}$ are

$$|G(j\omega)|_{\omega=3} = \left|\frac{2}{0.2\times 3j+1}\right| = 1.71$$

$$\angle\left[G(j\omega)\right]|_{\omega=3} = \angle\left[\frac{2}{0.2\times 3j+1}\right] = -0.54 \quad \text{rad}$$

Hence, we have

$$y_{ss}(t) = A.|G(j3)|\sin\left[3t + \angle G(j3)\right] = 1.71\sin(3t-0.54)$$

2. For an input $x(t) = A\sin(\omega t + \theta)$ to a system $G(j\omega)$, the output is

$$y_{ss}(t) = A.|G(j\omega)|\sin\left[\omega t + \theta + \angle G(jw)\right]$$

Given that the steady state output is: $y_{ss}(t) = 4.5\sin\left[5t - 38^o\right]$

and $|G(j\omega)| = 0.75$ and $\angle G(5j) = -68^o$, then considering the amplitude:

$$A(0.75) = 4.5 \quad \Rightarrow \quad A = 4.5/0.75 = 6$$

Considering the phase: $\theta - 68^o = -38^o \quad \Rightarrow \quad \theta = 68^o - 38^o = 30^o$

Hence the input is: $x(t) = 6\sin(5t + 30^o) = 6\sin(5t + \pi/6)$

Since $\cos(\omega t - 90^o) = \sin(\omega t)$, then we also have:

$$x(t) = 6\sin(5t+30^o) = 6\cos(5t+30^o-90^o) = 6\cos(5t-60^o) = 6\cos(5t-\pi/3)$$

3. (a) Observing the magnitude response around 10^1 rad/ to 10^2 rad/s, we note that the slope is -40dB/decade.

(b) We note the following:

(i) Between 10^{-2} rad/s and 10^{-1} rad/s we have an asymptote with a slope of -20dB/decade. This suggests a pole of $1/s$.

(ii) Between 10^{-1} rad/s to 10^0 rad/s the asymptote has a slope of -40dB/decade. This suggests another pole with a corner frequency at $\omega = 10^{-1}$ rad/s.

(iii) Between 10^0 rad/s and 10^1 rad/s the asymptote has a slope of -20 dB/decade. This suggests there is a zero with a corner frequency at $\omega = 10^0$ rad/s.

(iv) Between 10^1 rad/s and 10^2 rad/s the asymptote has a slope of -40 dB/decade. This suggest another pole with a corner frequency of $\omega = 10^1$ rad/s

Hence there are 3 poles and 1 zero. One of the poles is an integrator.

(c) At low frequency, the asymptote has a magnitude given by:

$|G(j\omega)| = \dfrac{K}{\omega^N}$ or in dB, we have:

$$|G(j\omega)|_{dB} = 20\log_{10}\left[\frac{K}{\omega^N}\right] = 20\log_{10} K - 20N\log_{10}\omega$$

Hence when $\omega = 1$ rad/s, then $|G(j1)|_{dB} = 20\log_{10} K$. From the figure, and observing the 1st pole of 1/*s*, when $\omega = 1$ rad/s, the magnitude is ≈ 12 dB. Thus,

$$12 = 20\log_{10} K$$

$$K = 10^{12/20} = 4$$

Section II

1. Transfer function of the simplified suspension system is:

$$\frac{X_o(s)}{X_i(s)} = \frac{bs+k}{ms^2+bs+k}$$

where m = 1kg, k = 1N/m and $b = \sqrt{2}$ N/m/s. The input, $x_i(t)$, may be approximated by the following Fourier Series representation:

$$x_i(t) = \frac{4}{\pi}\left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \dots\right]$$

where ω = 1 rad/s.

Since the input consists of a series of sinusoidal waveforms [sin(t), sin(3t), sin(5t), …] and the system is linear, principle of superposition may be used to determine the solution, i.e.

- Find the outputs when the inputs are the sinusoidal waveforms $\sin(\omega_n t)$ when ω_n = 1, 3, 5, … rad/s.
- The output when the input is the periodic square wave is the sum of the output due to the series of sinusoidal waveforms

Given that m = 1kg, k = 1N/m and $b = \sqrt{2}$ N/m/s, the magnitude and phase of

$$G(j\omega) = \frac{j\sqrt{2}\omega + 1}{(j\omega)^2 + j\sqrt{2}\omega + 1} = \frac{j\sqrt{2}\omega + 1}{1-\omega^2 + j\sqrt{2}\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+\left(\sqrt{2}\omega\right)^2}}{\sqrt{\left(1-\omega^2\right)^2 + \left(\sqrt{2}\omega\right)^2}} = \frac{\sqrt{1+2\omega^2}}{\sqrt{1+\omega^4 - 2\omega^2 + 2\omega^2}} = \frac{\sqrt{1+2\omega^2}}{\sqrt{1+\omega^4}}$$

$$\angle G(j\omega) = \tan^{-1}(\sqrt{2}\omega) - \tan^{-1}\left(\frac{\sqrt{2}\omega}{\left(1-\omega^2\right)}\right)$$

when ω =1 rad/s, 3 rad/s, and 5 rad/s are tabulated in the following table

ω (rad/s)	$\vert G(j\omega)\vert$	$\angle G(j\omega)$ (rad)
1	1.2247	-0.6155
3	0.4814	1.8269
5	0.2854	1.7168

For an input $x(t) = A\sin(\omega t)$, the steady-state output is

$$y_{ss}(t) = A.|G(j\omega)|\sin\left[\omega t + \theta + \angle G(j\omega)\right]$$

Hence, the steady-state output is

$$x_{o,ss}(t) = \frac{4}{\pi}\left[1.2247\sin(t-0.6155)+\frac{0.4814}{3}\sin(3t-1.3147)+\frac{0.2854}{5}\sin(5t-1.4248)+\ldots\right]$$

$$= \frac{4}{\pi}\left[1.2247\sin(t-0.6155)+0.1605\sin(3t-1.3147)+0.005708\sin(5t-1.4248)+\ldots\right]$$

2. In this problem, the output and the transfer function are known and the task is to determine the input.

$$?? \longrightarrow \boxed{G(s)} \longrightarrow y(t) = 50 + 2\sin(2\pi 2t)$$

Question states that the plant (thermocouple and recorder) may be represented by a first order system with unity steady-state gain (K = 1), the time constant (τ) of approximately 1 minute and no dead time, i.e.

$$G(s) = \frac{K}{\tau s+1} = \frac{1}{s+1}$$

At steady state, the recorded temperature oscillates with a frequency of 2 cycles per minute between 50°C and 48°C, i.e.

$y_{ss}(t) = 50 + 2\sin(2\pi ft)$ where $f = 2$ cycles per minute

By principle of superposition, the input will have two components because the output comprises of a dc component and a sinusoidal signal.

- The dc component of the output is 50. Since the transfer function has unity gain when ω =0 rad/s, the dc component of the input must also be 50.
- The frequency of the sinusoidal component is 2 cycles/min. The magnitude and phase of the system, $G(s)$, at that frequency is

$$G(jw) = \frac{1}{2\pi fj+1} = \frac{1}{4\pi j+1}$$

$$|G(j\omega)|_{\omega=4\pi} = \frac{1}{\sqrt{16\pi^2+1}} = 0.0793$$

It has been established that $|G(j\omega)|$ is the ratio of the output to the input. As the sinusoidal component in the output has an amplitude of $2 = A|G(j\omega)|$, the amplitude of the input sinusoidal waveform is

$$A = \frac{\text{Output amplitude}}{|G(j\omega)|} = \frac{2}{0.0793} = 25.2$$

Hence, the input temperature oscillates between 50-25.2 = 24.8°C and 50+25.2 = 75.2°C. Clearly, the results show that the recorder does not have sufficient bandwidth!

3. (a) Method is similar to Q3 in Section I. From the asymptotic magnitude response plot,

- At 4 rad/s, slope changes from 0 to 20 dB/decade

  $\Rightarrow$ Presence of factor $\frac{1}{4}s+1$

- At 10 rad/s, slope changes from 20 dB/decade to 0

  $\Rightarrow$ Presence of factor $\left(\frac{1}{10}s+1\right)^{-1}$

- At 20 rad/s, slope changes from 0 to -40 dB/decade

  $\Rightarrow$ Presence of factor $\left(\frac{1}{20}s+1\right)^{-2}$

- Static gain = $10^{13.9794/20} = 5$

Using $G(s)=\dfrac{A(s+\alpha)}{(s+\beta)(s+\gamma)(s+\lambda)}$, the transfer function is:

$$G(s)=\frac{5\left(\frac{1}{4}s+1\right)}{\left(\frac{1}{10}s+1\right)\left(\frac{1}{20}s+1\right)^2}$$

$$=\frac{5\times\frac{1}{4}(s+4)}{\frac{1}{10}(s+10)\frac{1}{20}(s+20)\frac{1}{20}(s+20)}$$

$$=\frac{5000(s+4)}{(s+10)(s+20)(s+20)}$$

$\therefore$ A = 5000, α= 4, β = 10, $\gamma = \lambda$ = 20.

(b) For other systems to have the magnitude response in the question, it should have factors that only affect the phase response.

One such factor is the transport delay $|e^{-j\omega L}| = 1$ so $|G(jw)|=|G(jw)e^{-j\omega L}|$ .So an example is:

$$G(s)=\frac{5000(s+4)}{(s+10)(s+20)^2}e^{-sL}$$

Another possibility is RHP poles/zeros, because $|j\omega T+1|=|j\omega T-1|$. So examples include:

$$G(s)=\frac{5000(s-4)}{(s+10)(s+20)^2} \quad \text{or} \quad G(s)=\frac{5000(s+4)}{(s-10)(s+20)^2} \quad \text{, etc.}$$

4. (a) From the pole-zero map, we observe that the poles are at s = -2 and s = -5, and the zero is at s = 1.

Also note that at $\omega_2 = 2$ rad/s, the slope of the Bode magnitude diagram changes by -40 dB/decade, which indicates a repeated pole.

Hence, comparing with the given transfer function: $G(s) = \dfrac{K\left(-\dfrac{s}{\alpha}+1\right)}{\left(\dfrac{s}{\beta}+1\right)\left(\dfrac{s}{\gamma}+1\right)^2}$

We obtain $\alpha = 1$, $\beta = 5$ and $\gamma = 2$, and the corner frequencies of the Bode magnitude diagram are $\omega_1 = 1$ rad/s, $\omega_2 = 2$ rad/s and $\omega_3 = 5$ rad/s.

(b) The repeated pole is located at s = -2.

(c) From the Bode magnitude plot, DC gain $K = G(0)$ = 16.9 dB = $10^{16.9/20}$ = 7.

(d) $G(s)$ is stable because all the system poles lie in the LHP. Zeros do not affect stability.

Section III

1. Problem requires the steady-state values of signals when the input signal is a sinusoid so frequency response theorem is concept for formulating the solution. To use the frequency response theorem, a transfer function relating the input and output signals is needed so derive the necessary function:
   - Using Kirchoff current law
   - Express the capacitor and inductor as impedances and applying the current division law.

   We have:

$$I(s) = I_C(s) + I_L(s) + I_R(s) \qquad [1]$$

$$\frac{1}{sC} I_C(s) = sLI_L(s) = RI_R(s) \qquad [2]$$

To evaluate the expression for $i_C(t)$, we change Eqn. [1] to only include terms in $I_C(s)$:

$$\begin{aligned} I(s) &= I_C(s) + I_L(s) + I_R(s) \\ &= I_C(s) + \frac{1}{LCs^2} I_C(s) + \frac{1}{RCs} I_C(s) = I_C(s)\left[\frac{1}{LCs^2} + \frac{1}{RCs} + 1\right] = I_C(s)\left[\frac{R + Ls + LRCs^2}{LRCs^2}\right] \end{aligned}$$

$$\frac{I_C(s)}{I(s)} = G_C(s) = \frac{LRCs^2}{LRCs^2 + Ls + R}$$

Since the input is $x(t) = 5\cos(200t)$, we have A = 5, and ω = 200 rad/s. Hence:

$$\begin{aligned} G_C(j\omega)\big|_{\omega=200} &= \frac{(1)(100)(10^{-4})(j200)^2}{(1)(100)(10^{-4})(j200)^2 + (1)(j200) + 100} \\ &= \frac{-400}{-400 + j200 + 100} = \frac{-400}{j200 - 300} = \frac{-4}{j2 - 3} \end{aligned}$$

And we have: $|G(j200)| = \frac{4}{\sqrt{4+9}} = \frac{4}{\sqrt{13}}$ and

$$\angle G(j200) = -\tan^{-1}\left(-\frac{2}{3}\right) = \tan^{-1}\frac{2}{3} = 33.7^o$$

Hence, the steady-state current through the capacitor is:

$$y_{C-ss}(t) = A\,|G(j\omega)|\cos\left(\omega t + \angle G(j\omega)\right) = 5\frac{4}{\sqrt{13}}\cos\left(\omega t + 33.7^o\right) = \frac{20}{\sqrt{13}}\cos\left(\omega t + 33.7^o\right)$$

Using a similar approach we can obtain the steady-state currents through the inductor & resistor.

2. Method is similar to Section II Q3. Parameters of the first order factor may be identified using the corner frequencies. The second order factor may be identified using the resonant peak and resonant frequency.

Given that $G(s) = \dfrac{K(s+a)}{(s+b)(s^2 + 2\varsigma\omega_n s + \omega_n^2)}$, we note that:

- At 2 rad/s, slope changes from 0 dB/decade to -40 dB/decade.

  $\Rightarrow$ Presence of factor $\left(\dfrac{1}{2}s+1\right)^{-2}$ and we have ω_1 = $\boldsymbol{\omega_n}$ **= 2 rad/s**

- At 9 rad/s, slope changes from -40 dB/decade to -60 dB/decade.

  $\Rightarrow$ Presence of factor $\left(\dfrac{1}{9}s+1\right)^{-1}$ and we have ω_2 = 9 rad/s, ***b* = 9**

- At 30 rad/s, slope changes from -60 dB/decade to -40 dB/decade.

  $\Rightarrow$ Presence of factor $\left(\dfrac{1}{30}s+1\right)$ and we have ω_3 = 30 rad/s, ***a* = 30**

- Static gain is $10^{20/20}$ = 10 = ***K***

Hence the system transfer function is:

$$
\begin{aligned}
G(s) &= \left[\frac{(10)(2^2)}{s^2 \_ 2\varsigma(2)s + (2^2)}\right]\left[\frac{1}{s/9+1}\right]\left[s/30+1\right] \\
&= \frac{40\times 9}{30}\left[\frac{1}{s^2+4\varsigma s+4)}\right]\left[\frac{1}{s+9}\right]\left[s+30\right] \\
&= 12\left[\frac{1}{s^2+4\varsigma s+4)}\right]\left[\frac{1}{s+9}\right]\left[s+30\right]
\end{aligned}
$$

The resonant frequency is:

$$
\begin{aligned}
&\omega_r = \omega_n\sqrt{1-2\varsigma^2} \\
&1.87 = 2\sqrt{1-2\varsigma^2} \\
&\left(\frac{1.87}{2}\right)^2 = 1-2\varsigma^2 \\
&\varsigma = \sqrt{\frac{1}{2}\left[1-\left(\frac{1.87}{2}\right)^2\right]} = 0.25
\end{aligned}
$$