National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems
Tutorial 6 Solutions

Section I

1. (a) Differential equation can be obtained simply by inspecting the transfer function and using the following properties :

- Under the assumption that all initial conditions are zero, the derivative of transform rule reduces to

$$\mathcal{L}\left\{\frac{d^n x(t)}{dt}\right\} = s^n X(s)$$

- Numerator polynomial is formed using coefficients of the input function and its derivative(s).
- Denominator polynomial is formed using coefficients of the output signal and its derivative(s).

Hence, the differential equation is

$$\ddot{y}(t) + 6\dot{y}(t) + 13y(t) = \dot{x}(t) + 9x(t)$$

Transfer function assumes that all initial conditions are zero. As the output signal of a dynamic system must be continuous and the given system is 2nd order, the initial conditions needed are $y(0) = \dot{y}(0) = 0$.

(b) Question provides the input signal and the system transfer function. Solution required is the steady-state output signal of a dynamic system, i.e. $\lim_{t\to\infty} y(t)$.

Hence, concept needed to solve problem is the definition of a transfer function $Y(s) = G(s)X(s)$.

- Given that input is a step function of magnitude 2, $X(s) = \dfrac{2}{s}$.
- $Y(s) = G(s)X(s) = \dfrac{s+9}{s^2+6s+13}\dfrac{2}{s}$
- Performing inverse Laplace Transform, the time-domain expression of the output signal is

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2(s+9)}{s(s^2+6s+13)}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+9)}{s\left[(s+3)^2+4\right]}\right\}$$

Consider: $$\frac{2(s+9)}{s\left[(s+3)^2+4\right]} = \frac{A_1}{s} + \frac{A_2}{\left[(s+3)^2+4\right]} + \frac{A_3(s+3)}{\left[(s+3)^2+4\right]}$$

Consider the numerators on both sides of the equation, we have:

$$2s+18 = A_1\left[(s+3)^2+4\right] + A_2 s + A_3(s+3)s$$

Setting $s=0$: $18 = 13A_1 \quad \Rightarrow \quad A_1 = 18/13$

Substitute A_1 into the numerator equation:

$$2s+18 = \frac{18}{13}\left[(s+3)^2+4\right] + A_2 s + A_3(s+3)s$$

$$2s+18 = \frac{18}{13}\left(s^2+6s+13\right) + A_2 s + A_3 s^2 + A_3 s$$

$$2s+18 = \left(\frac{18}{13}+A_3\right)s^2 + \left(\frac{18\times 6}{13}+A_2+3A_3\right)s+18$$

Matching s^2 term: $\frac{18}{13}+A_3 = 0 \quad \Rightarrow \quad A_3 = -\frac{18}{13}$

Matching s term: $2 = \frac{108}{13}+A_2+3\mathrm{A}_3 \quad \Rightarrow \quad A_2 = 2-\frac{108}{13}+3\left(\frac{18}{13}\right) = 2-\frac{108}{13}+\frac{54}{13} = -\frac{28}{13}$

Hence: $$Y(s) = \frac{18}{13}\frac{1}{s} - \frac{28}{13}\frac{1}{\left[(s+3)^2+4\right]} - \frac{18}{13}\frac{(s+3)}{\left[(s+3)^2+4\right]}$$

$$= \frac{18}{13}\frac{1}{s} - \frac{14}{13}\frac{2}{\left[(s+3)^2+4\right]} - \frac{18}{13}\frac{(s+3)}{\left[(s+3)^2+4\right]}$$

Taking the inverse Laplace Transform:

$$\text{y(t)}=\frac{18}{13}-\frac{14}{13}e^{-3t}\sin(2t)-\frac{18}{13}e^{-3t}\cos(2t)$$

$\therefore$ Steady-state value of $y(t) = \lim_{t\to\infty}\left\{\frac{18}{13}-\frac{14}{13}e^{-3t}\sin(2t)-\frac{18}{13}e^{-3t}\cos(2t)\right\} = \frac{18}{13}$

- Final Value Theorem states that $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$. Substituting

  $Y(s) = G(s)X(s) = \frac{s+9}{s^2+6s+13}\frac{2}{s}$ into the Final Value Theorem,

  $$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\left\{\frac{2(s+9)}{s^2+6s+13}\right\} = \frac{18}{13}$$

2. The problem provides information about the convolution integral. To formulate the solution, a concept that links the functions in a convolution integral to the transfer function is needed. We have :

$$y(t) = g(t)\otimes x(t) = \int_0^t g(\tau)x(t-\tau)d\tau$$

As $x(t)=u(t)$ and we have: $y(t) = \int_0^t \underbrace{150e^{-0.5\tau}\sin(0.5\tau)}_{g(\tau)}.u(t-\tau)d\tau$

Hence, the required transfer function is:

$$G(s) = \mathcal{L}\left\{150e^{0.5\tau}\sin(0.5\tau)\right\} = \frac{150\times 0.5}{(s+0.5)^2+0.5^2} = \frac{75}{s^2+s+0.5}$$

Section II

1. Circuit has two independent input sources but question only requires the transfer function relating $i_1(t)$ and $i(t)$. Hence, the first step is to apply the principle of the superposition and "kill" the voltage source (short the voltage source (so only the input signal needed to solve the problem is left in the circuit.

   Applying Kirchoff current law and Kirchoff voltage law, the following differential equation relating $i_1(t)$ and $i(t)$.

$$v_C(t) = L\frac{di_1(t)}{dt} + R_1.i_1(t)$$

$$v_C(t) = \frac{1}{C}\int i_2(t).dt \quad \Rightarrow \quad \frac{dv_C(t)}{dt} = \frac{1}{C}i_2(t) \quad \Rightarrow \quad i_2(t) = C\frac{dv_C(t)}{dt} = LC\frac{d^2i_1(t)}{dt} + R_1C\frac{di_1(t)}{dt}$$

$$i(t) = i_2(t) + i_1(t) = LC\frac{d^2i_1(t)}{dt} + R_1C\frac{di_1(t)}{dt} + i_1(t)$$

   Applying Laplace Transform and assuming all initial conditions are zero:

$$I(s) = LCs^2I_1(s) + R_1CsI_1(s) + I_1(s)$$
$$= \left[LCs^2 + R_1Cs + 1\right]I_1(s)$$
$$\therefore \quad \frac{I_1(s)}{I(s)} = \frac{1}{LCs^2 + R_1Cs + 1}$$

   Alternatively, using the impedance approach, where:

   - Voltage-Current relationship for the capacitor is $V(s) = {}^1\!/_{sC}\,I(s)$
   - Voltage-Current relationship for an inductor is $V(s) = sLI(s)$

$$V_C(s) = LsI_1(s) + R_1I_1(s)$$

$$V_C(s) = \frac{1}{Cs}I_2(s) \quad \Rightarrow \quad \frac{1}{Cs}I_2(s) = LsI_1(s) + R_1I_1(s) \quad \Rightarrow \quad I_2(s) = \left[LCs^2 + R_1Cs\right]I_1(s)$$

$$I(s) = I_2(s) + I_1(s) = \left[LCs^2 + R_1Cs + 1\right]I_1(s) \quad \Rightarrow \quad \frac{I_1(s)}{I(s)} = \frac{1}{LCs^2 + R_1Cs + 1}$$

   Or, using the current division rule:

$$I_1(s) = \frac{{}^1\!/_{sC}}{{}^1\!/_{sC} + R_1 + sL}I(s) \quad \Rightarrow \quad \frac{I_1(s)}{I(s)} = \frac{1}{LCs^2 + R_1Cs + 1}$$

2. (a) Question states that the temperature reading is allowed to stabilise, ie.e reach steady state. Hence, the question requires the steady-state input signal to be derived using the differential equation and the steady-state output signal.

When the temperature reading stabilizes, $\frac{dy(0)}{dt} = 0$ so the differential equation reduces to: $y(t) = 0.99x(t)$

Given that $y(0) = 24.75$, the temperature of the heat bath is

$$x(0) = \frac{y(0)}{0.99} = \frac{24.75}{0.99} = 25^o\text{C} .$$

(b) Since the bath temperature increases at a steady rate of 1°C/second, the input signal $x(t)$ is a straight line. Using the general form of a straight line $y = mx + c$ where m is the gradient and c is the y-intercept,

$x(t) = [25 + t]u(t)$ and its L.T. is $X(s) = \frac{25}{s} + \frac{1}{s^2}$

Substituting $x(t)$ into the differential equation, the time-domain expression for the measured temperature can be found by solving:

$5\frac{dy(t)}{dt} + y(t) = 0.99x(t)$ where $y(0) = 24.75$

Taking the Laplace Transform:

$$5s(Y(s) - 5y(0) + Y(s) = 0.99X(s)$$

$$Y(s)[5s+1] = 0.99\left[\frac{25}{s} + \frac{1}{s^2}\right] + 5y(0)$$

$$Y(s) = \frac{1}{5s+1}\left[\frac{24.75}{s} + \frac{0.99}{s^2} + 123.75\right]$$

$$= \frac{0.2}{s+0.2}\left[\frac{24.75s + 0.99 + 123.75s^2}{s^2}\right]$$

$$= \frac{4.95s + 0.198 + 24.75s^2}{(s+0.2)s^2}$$

Let $Y(s) = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{(s+0.2)} = \frac{A_1 s(s+0.2) + A_2(s+0.2) + A_3 s^2}{(s+0.2)s^2}$

$$= \frac{0.2A_2 + (0.2A_1 + A_2)s + (A_1 + A_3)s^2}{(s+0.2)s^2}$$

Then for the numerators: $0.198 + 4.95s + 24.75s^2 = 0.2A_2 + (0.2A_1 + A_2)s + (A_1 + A_3)s^2$

Setting $s = 0,$ we have: $0.198 = 0.2A_2 \quad \Rightarrow \quad A_2 = 0.198 / 0.2 = 0.99$

Setting $s = -0.2,$ we have: $0.198 + 4.95(-0.2) + 24.75(-0.2)^2 = A_3(-0.2)^2 \quad \Rightarrow \quad A_3 = 4.95$

Matching s^2 terms, we have: $24.75 = A_1 + A_3 = A_1 + 4.95 \quad \Rightarrow \quad A_1 = 24.75 - 4.95 = 19.8$

Hence: $Y(s) = \dfrac{19.8}{s} + \dfrac{0.99}{s^2} + \dfrac{4.95}{s + 0.2}$

Taking the inverse Laplace transform: $y(t) = 19.8 + 0.99t + 4.95e^{-0.2t}$

(c) We have: $5\dfrac{dy(t)}{dt} + y(t) = 0.99x(t)$

Taking the Laplace transform and assuming all initial conditions are zero:

$$5sY(s) + Y(s) = 0.99X(s) \quad \Rightarrow \quad Y(s)\left[5s + 1\right] = 0.99X(s) \quad \Rightarrow \quad \therefore \quad G(s) = \frac{Y(s)}{X(s)} = \frac{0.99}{5s + 1}$$

(d) Transfer functions are defined under the assumption that the system is initially at rest. In this problem, $y(0) \neq 0$ so the zero initial conditions assumption is violated. As $\dfrac{dy(0)}{dt} = 0$, the system is initially at rest so transfer function is applicable by shifting the input output axes. Let:

$y_1(t) = y(t) - y(0) = y(t) - 24.75$

$x_1(t) = x(t) - x(0) = x(t) - 25 = [t + 25] - 25 = t \quad \text{as} \quad x(t) = 25 + t$

$\therefore \quad X_1(s) = \dfrac{1}{s^2} \quad \text{and} \quad Y_1(s) = \dfrac{0.99}{5s + 1}X_1(s) = \dfrac{0.99}{5s + 1}\dfrac{1}{s^2} = \dfrac{0.198}{(s + 0.2)s^2}$

Let $Y_1(s) = \dfrac{A_1}{s} + \dfrac{A_2}{s^2} + \dfrac{A_3}{s + 0.2} = \dfrac{A_1 s(s + 0.2) + A_2(s + 0.2) + A_3 s^2}{(s + 0.2)s^2}$

Equating the numerators: $0.198 = A_1 s(s + 0.2) + A_2(s + 0.2) + A_3 s^2$

Setting $s = 0,$ we have: $0.198 = 0.2A_2 \quad \Rightarrow \quad A_2 = 0.198 / 0.2 = 0.99$

Setting $s = -0.2,$ we have: $0.198 = A_3(-0.2)^2 = 0.04A_3 \quad \Rightarrow \quad A_3 = 0.198 / 0.04 = 4.95$

Matching s^2 terms, we have: $0 = A_1 + A_3 = A_1 + 4.95 \quad \Rightarrow \quad A_1 = -4.95$

Hence: $Y_1(s) = -\dfrac{4.95}{s} + \dfrac{0.99}{s^2} + \dfrac{4.95}{s + 0.2}$

Taking the inverse Laplace transform: $y_1(t) = -4.95 + 0.99t + 4.95e^{-0.2t}$

By definition, $y(t) = y_1(t) + y(0)$. The time-domain expression for the measured temperature is:

$y(t) = y_1(t) + y(0) = y_1(t) + 24.75 = 19.8 + 0.99t + 4.95e^{-0.2t}$

3. Stability of a dynamic system depends on whether the transient response is bound or on the location of the system poles.

(a) Transient response is $e^{-t} + e^{2t}$ for $t > 0$.
System is unstable because of the presence of e^{2t} causes the transient response to grow without bound.

(b) Transient response is $\sin(2t)$ for $t \geq 0$.
System is marginally stable because the transient response oscillates with constant amplitude.

(c) Transient response is $e^{-t}\sin(2t)$ for $t \geq 0$.
System is stable because the transient response decays to zero when $t \to \infty$.

(d) Given $\ddot{y}(t) - \dot{y}(t) - 6y(t) = 4x(t)$, its Laplace transform is:

$s^2Y(s) - sY(s) - 6(Y(s) = 4X(s)$

The transfer function is: $G(s) = \dfrac{Y(s)}{X(s)} = \dfrac{4}{s^2 - s - 6} = \dfrac{4}{(s-3)(s+2)}$

Hence the poles are s = 3, -2. As, the pole s = 3 is on the RHP, hence the system is unstable.

(e) Given $G(s) = \dfrac{s+3}{s^2+3}$,

System poles are located at $s = \pm j\sqrt{3}$

Since system poles lie on the imaginary axis, transient response is a sinusoid so the system is marginally stable.

(f) Transfer function is $\dfrac{4}{(s^2+4)^2}$. The poles are located at $s = \pm 2j, \pm 2j$.

There is one pair of *repeated* poles on the imaginary axis. To determine if such a system is stable, consider the case where the input is a step function (bounded input signal). The step response is

$$y_{step}(t) = \mathcal{L}^{-1}\left\{\frac{4}{(s^2+4)^2}\frac{1}{s}\right\} = \frac{1}{4} - \frac{1}{4}\cos(2t) - \frac{1}{4}t\sin(2t)$$

When $t \to \infty$, $\lim\limits_{t\to\infty} y_{step}(t)$ is unbounded because of the $\frac{1}{4}t\sin(2t)$ term.

$\therefore$ the system is unstable because a bounded input signal resulted in an unbounded output signal.

(g) Transfer function is $\dfrac{2s-1}{s^2+2s+4}$.

Poles are located at $s=-1\pm j\sqrt{3}$.

Since system poles are in the LHP, system is stable.

Note that system zeros does not influence stability.

(h) Since the input is a ramp, we have:

$x(t)=t.u(t)$ and its Laplace transform is: $X(s)=1/s^2$

$y(t)=2t-\dfrac{2}{5}+\dfrac{2}{5}e^{-5t}$ and its Laplace transform is:

$$\mathrm{Y(s)}=\frac{2}{s^2}-\frac{2}{5s}+\frac{2}{5(s+5)}=\frac{10(s+5)-2s(s+5)+2s^2}{5s^2(s+5)}=\frac{10}{s^2(s+5)},$$

Hence:

$$G(s)=\frac{Y(s)}{X(s)}=\frac{10}{s^2(s+5)}\bigg/\frac{1}{s^2}=\frac{10}{s+5}$$

Hence, the system pole *s* = -5 lies in the LHP, the system is stable.

4. (a) Transfer function of the air heating system is

For $RC\dfrac{d\theta_o(t)}{dt} + \theta_o(t) = Rh(t)$, its Laplace transform, assuming initial conditions are zero is:

$$RCs\theta_o(s) + \theta_o(s) = RH(s)$$

$$\theta_o(s)\left[RCs+1\right] = RH(s)$$

$$\therefore \quad \frac{\theta_o(s)}{H(s)} = \frac{R}{RCs+1}$$

Hence impulse response of the heating system is

$$\theta_o(t) = \mathcal{L}^{-1}\left\{\frac{R}{RCs+1}\right\} = \frac{1}{C}e^{-\frac{t}{RC}}$$

(b) Substitute two points from the graph into $\theta_o(t) = \dfrac{1}{C}e^{-\frac{t}{RC}}$, and solve simultaneously for *R* and *C*. Of the 5 points provided, the simultaneous equations can be solved most easily if the following data points are used to formulate the equations

- At $t = 0$, $\theta_o(t) = \dfrac{1}{C} = 10 \quad \Rightarrow \quad C = 0.1$
- At $t = RC$, $\theta_o(t) = \dfrac{1}{C}e^{-1} = \dfrac{0.36788}{C} = 3.6788$.

  Hence from figure, output is 3.6788 when $t = 3$. Therefore, $3 = R.(0.1)$, hence, $R = 30$.

Section III

1. Method similar to Q1 in Section II.

2. Assuming that there are no system zeros, $Y(s) = \dfrac{K}{D(s)}$

- Given that the poles of $Y(s)$ (roots of $D(s)$ are $s = 0, -3, -7\pm5j$, so $D(s) = s(s+3)(s+7+5j)(s+7-5j) = s(s+3)(s^2+14s+74)$
- Value of *k* can be found from $\lim\limits_{t\to\infty} y(t) = \lim\limits_{s\to 0} sY(s) = \lim\limits_{s\to 0} s\dfrac{K}{s(s+3)(s^2+14s+74)} = 8$

  .
- Since the input signal is a step function of magnitude 4, $Y(s) = G(s)\times\dfrac{4}{s} = \dfrac{K}{D(s)}$.

  The system transfer function, $G(s)$, then can be found.
- Note that the answer given in the tutorial problem sheet is not unique, because the question does not provide any information about zeros. Hence, all answers that satisfy the two given constraints are acceptable.