EE2023 TUTORIAL 1 (SOLUTIONS)

Solution to Q.1

Write z in polar form:

$$z = x + jy = |z|\exp(j\angle z).$$

Since adding integer multiples of 2π to $\angle z$ does not affect the value of z, we may also express z as

$$z = |z|\exp\big(j(\angle z + 2k\pi)\big)$$

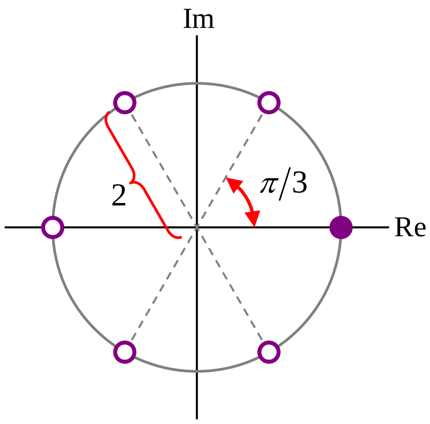
where k is an integer. This leads to

$$\sqrt[N]{z} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right); \quad k = 0,1,\cdots,N-1,$$

which yields the N distinct values of $\sqrt[N]{z}$.

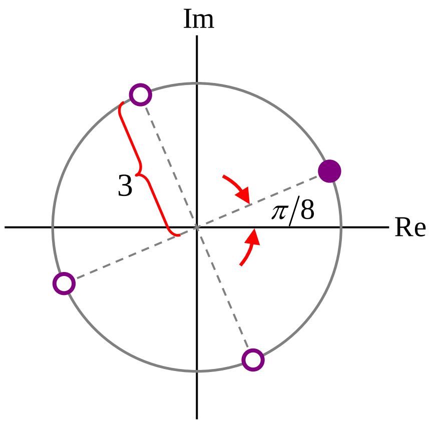
$\sqrt[6]{64}$ **:**

$$\begin{cases} z = 64 \;\rightarrow\; \begin{cases} |z| = 64 \\ \angle z = 0 \end{cases} \\ \sqrt[6]{64} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right)\Big|_{z=64,\ N=6} \\ \quad = 2\exp\left(j\left(\frac{k\pi}{3}\right)\right); \quad k = 0,1,\cdots,5 \\ \quad = \begin{cases} (\ 2),\ 2\exp\left(j\left(\frac{\pi}{3}\right)\right);\ 2\exp\left(j\left(\frac{2\pi}{3}\right)\right); \\ (-2),\ 2\exp\left(j\left(\frac{4\pi}{3}\right)\right);\ 2\exp\left(j\left(\frac{5\pi}{3}\right)\right) \end{cases} \end{cases}$$



$\sqrt[4]{j81}$ **:**

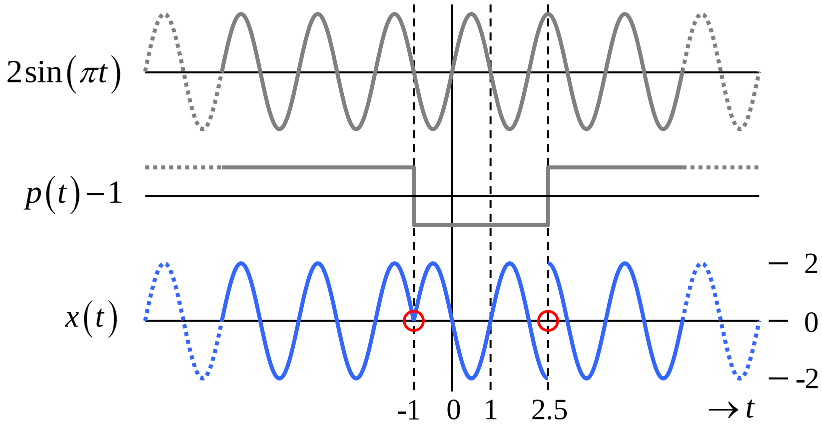
$$\begin{cases} z = j81 \;\rightarrow\; \begin{cases} |z| = 81 \\ \angle z = \frac{\pi}{2} \end{cases} \\ \sqrt[4]{j81} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right)\Big|_{z=81,\ N=4} \\ \quad = 3\exp\left(j\left(\frac{\pi}{8} + \frac{k\pi}{2}\right)\right); \quad k = 0,1,\cdots,3 \\ \quad = \begin{cases} 3\exp\left(j\left(\frac{\pi}{8}\right)\right),\ 3\exp\left(j\left(\frac{5\pi}{8}\right)\right), \\ 3\exp\left(j\left(\frac{9\pi}{8}\right)\right),\ 3\exp\left(j\left(\frac{13\pi}{8}\right)\right) \end{cases} \end{cases}$$



Solution to Q.2

(a) $p(t) = 2 - 2\text{rect}\left(\frac{t-0.75}{3.5}\right)$

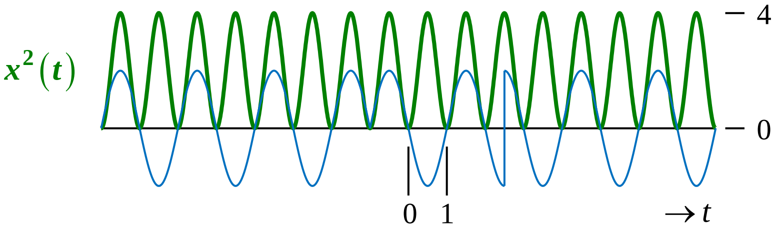
(b) By inspection, $x(t)$ is not periodic.



Notice the π rad $(\text{or } 180°)$ phase jumps in $x(t)$ occurring at the zero crossings of $p(t)-1$.

(c)

$$
\begin{aligned}
x^2(t) &= 4\sin^2(\pi t)\underbrace{\left(p(t)-1\right)^2}_{1} \\
&= 4\sin^2(\pi t) \\
&= 2\left(1-\cos(2\pi t)\right)
\end{aligned}
$$



Note that $x^2(t)$ is periodic with a period of $T = 1$.

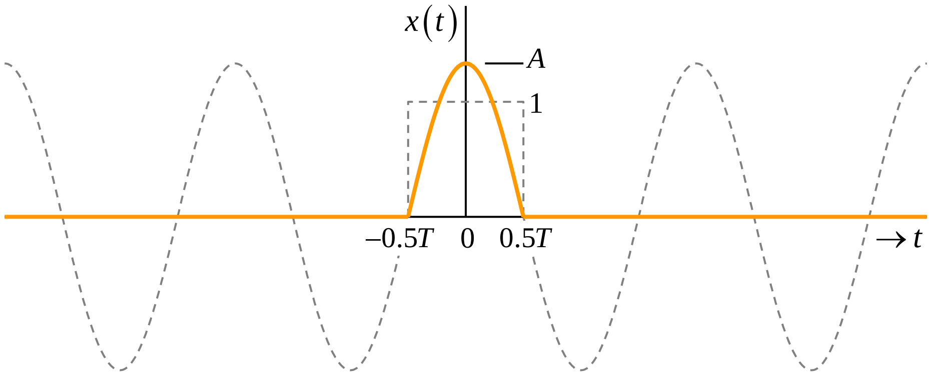
Total energy: $$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \sum_{n=-\infty}^{\infty} \underbrace{\int_{nT}^{(n+1)T} x^2(t)\,dt}_{\int \text{over one period thus independent of } n} = \Bigg(\underbrace{\int_0^T x^2(t)\,dt}_{\text{finite}}\Bigg)\underbrace{\sum_{n=-\infty}^{\infty} 1}_{\infty} = \infty$$

Average Power: $$P = \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt}_{x^2(t) \text{ is periodic. } \therefore\ P \text{ can be obtained by averaging over one period.}} = \int_{-0.5}^{0.5} 2\left(1-\cos(2\pi t)\right)dt = 2$$

Conclusion: $x(t)$ is an aperiodic power signal.

Solution to Q.3

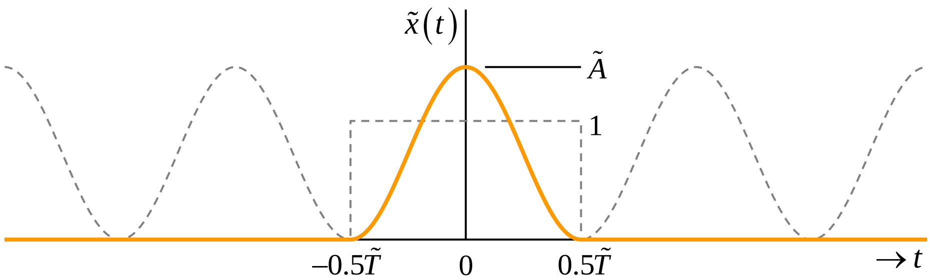
Half-cosine pulse: $x(t) = A\cos\left(\frac{\pi t}{T}\right)\text{rect}\left(\frac{t}{T}\right)$



$$x^2(t) = \frac{A^2}{2}\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]\text{rect}\left(\frac{t}{T}\right)$$

Energy: $E = \frac{A^2}{2}\int_{-0.5T}^{0.5T} 1 + \underbrace{\cos\left(\frac{2\pi t}{T}\right)}_{\substack{\int \text{over one} \\ \text{period } =0}} dt = \frac{1}{2}A^2 T$

Raised-cosine pulse: $\tilde{x}(t) = \frac{\tilde{A}}{2}\left(1+\cos\left(\frac{2\pi t}{\tilde{T}}\right)\right)\text{rect}\left(\frac{t}{\tilde{T}}\right)$



$$\tilde{x}^2(t) = \frac{\tilde{A}^2}{4}\left[\frac{3}{2}+2\cos\left(\frac{2\pi t}{\tilde{T}}\right)+\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)\right]\text{rect}\left(\frac{t}{\tilde{T}}\right)$$

Energy: $\tilde{E} = \frac{\tilde{A}^2}{4}\int_{-0.5\tilde{T}}^{0.5\tilde{T}} \frac{3}{2} + 2\underbrace{\cos\left(\frac{2\pi t}{\tilde{T}}\right)}_{\substack{\int \text{over one} \\ \text{period } =0}} + \frac{1}{2}\underbrace{\cos\left(\frac{4\pi t}{\tilde{T}}\right)}_{\substack{\int \text{over two} \\ \text{periods } =0}} dt = \frac{3}{8}\tilde{A}^2\tilde{T}$

Both $x(t)$ and $\tilde{x}(t)$ will have the same energy if $A^2T = \frac{3}{4}\tilde{A}^2\tilde{T}$.

Solution to Q.4

(a) Let m, n and k be positive integers. Based on the definition of a periodic signal, we have

$$\begin{aligned} x_1(t) &= x_1(t+T_1) = x_1(t+mT_1) \\ x_2(t) &= x_2(t+T_2) = x_2(t+nT_2) \end{aligned}.$$

Hence,

$$x_0(t) = x_1(t) + x_2(t) = x_1(t+mT_1) + x_2(t+nT_2).$$

If $mT_1 = nT_2 = kT_0$, then

$$x_0(t) = x_1(t+kT_0) + x_2(t+kT_0) = x_0(t+kT_0)$$

which shows that $x_0(t)$ is periodic with a period of T_0 and a fundamental frequency of $f_0 = \dfrac{1}{T_0}$.

Under this condition, $(\text{i.e. } mT_1 = nT_2 = kT_0)$,

$$\left.\begin{aligned} \frac{1}{T_1} &= m\frac{1}{kT_0} \\ \frac{1}{T_2} &= n\frac{1}{kT_0} \end{aligned}\right\} \text{ implying that } \begin{cases} \cdots\cdots \ \dfrac{1}{kT_0} \text{ are common factors of } \left\{\dfrac{1}{T_1}, \dfrac{1}{T_2}\right\} \ \cdots\cdots \\ \text{and} \\ \dfrac{1}{T_0} \text{ is the highest common factor (HCF) of } \left\{\dfrac{1}{T_1}, \dfrac{1}{T_2}\right\} \\ \text{or} \\ f_0 \text{ is the highest common factor (HCF) of } \{ f_1, f_2 \} \end{cases}$$

Conclusion: For $x_0(t)$ to be periodic, $\{ f_1, f_2 \}$ must have a HCF. In turn, this HCF is the fundamental frequency of $x_0(t)$

(b) i. $x(t) = \cos(3.2t) + \sin(1.6t) + \exp(j2.8t) \ \cdots \begin{cases} \cos(3.2t) & \text{has a frequency of } 3.2 \ rad/s \\ \sin(1.6t) & \text{has a frequency of } 1.6 \ rad/s \\ \exp(j2.8t) & \text{has a frequency of } 2.8 \ rad/s \end{cases}$

Highest common factor (HCF) of $\{3.2,\ 1.6,\ 2.8\}$ exists and is equal to 0.4. Thus, $x(t)$ is periodic and has a fundamental frequency of $0.4\ rad/s$ $(\text{or } 0.2/\pi\,Hz)$ and a fundamental period of $5\pi\ s$.

REMARKS: *Although $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$, it does not contain the fundamental frequency component itself.*

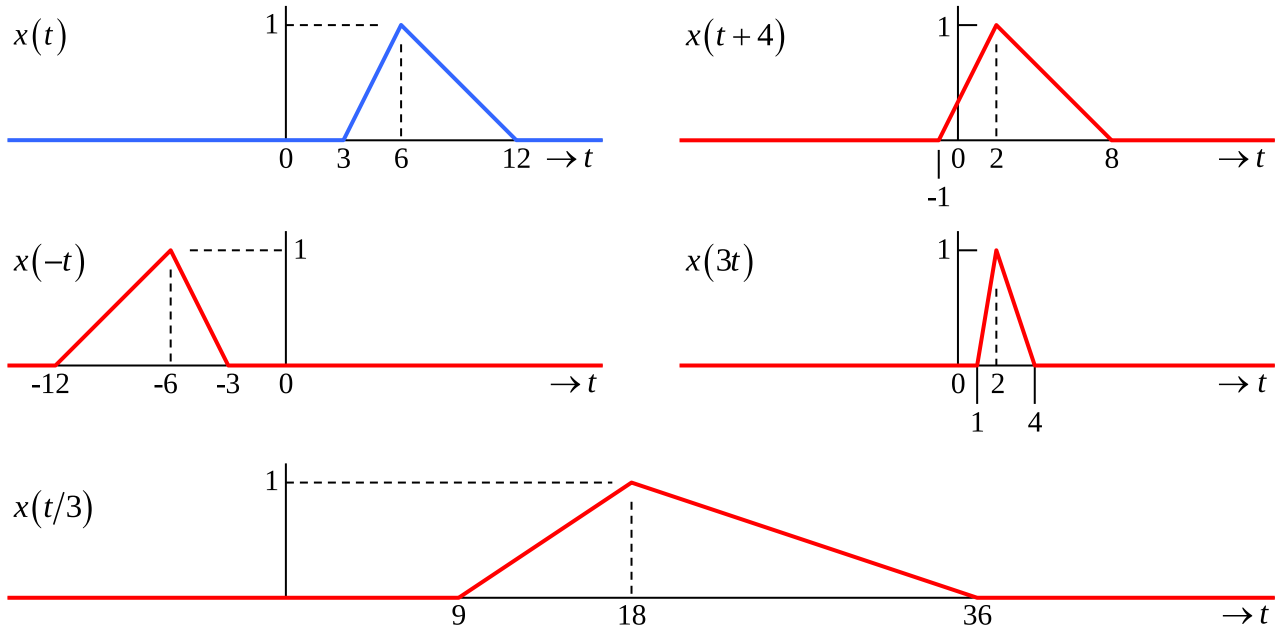
(b) ii. $x(t) = \cos(4t) + \sin(\pi t) \ \cdots \begin{cases} \cos(4t) \text{ has a frequency of } 4 \ rad/s \\ \sin(\pi t) \text{ has a frequency of } \pi \ rad/s \end{cases}$

Highest common factor (HCF) of $\{4,\ \pi\}$ does not exist. Thus, $x(t)$ is not periodic.

REMARKS: *Summing sinusoids does not necessarily lead to a periodic signal unless the frequencies of the sinusoids are harmonics of a common fundamental frequency.*

Solution to Q.5

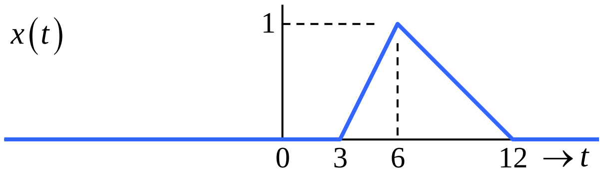
(a)



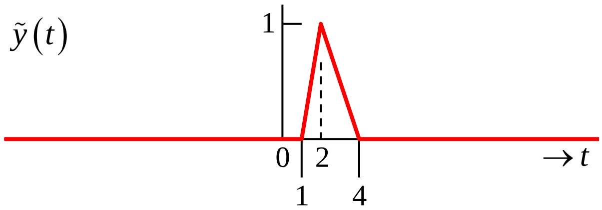
(b) We observe that $y(t)$ is a time-scaled, -reversed and –shifted version of $x(t)$.

For problems of this nature, we should start with time-scaling first since it involves linear warping of the time axis. If we were to start with time-shifting and/or time-reversal, we may have to redo them after time-scaling. However, this sequence of operation need not be followed if we are sketching the signal from the mathematical expression.

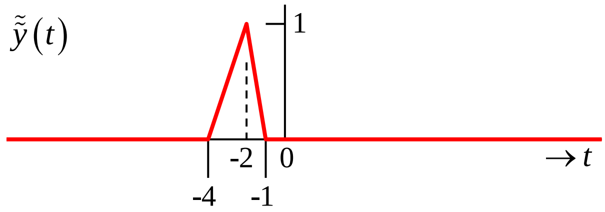
Comparing $x(t)$ and $y(t)$, we note that $y(t)$ involves time-scaling (or contraction) of $x(t)$ by a factor of 3.



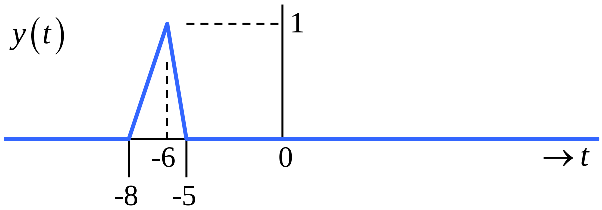
Time-scaling of $x(t)$: $\tilde{y}(t) = x(3t)$



Time-reversal of $\tilde{y}(t)$: $\tilde{\tilde{y}}(t) = \tilde{y}(-t) = x(-3t)$

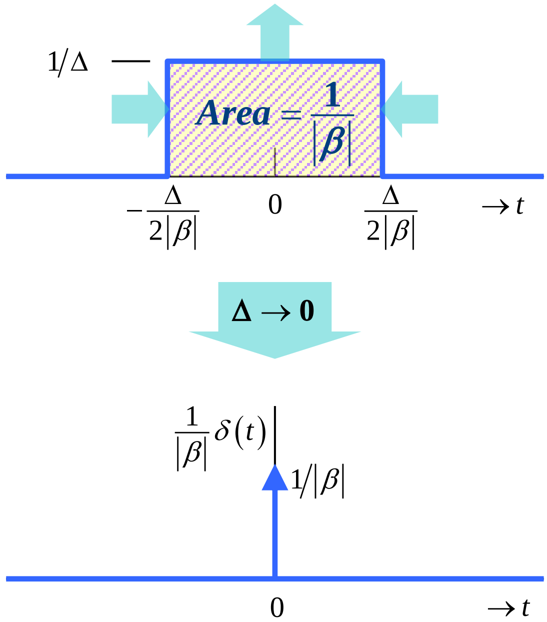


Time shifting of $\tilde{\tilde{y}}(t)$: $\begin{cases} y(t) = \tilde{\tilde{y}}(t+4) \\ \quad\;\; = x(-3(t+4)) \end{cases}$



$\therefore\ \boldsymbol{y(t) = x(-3(t+4))}$

Solution to Q.6



Let

$$\delta(t) = \lim_{\Delta \to 0} x(t) \text{ where } x(t) = \frac{1}{\Delta}\mathrm{rect}\left(\frac{t}{\Delta}\right).$$

Here, we have implicitly assumed that $\Delta > 0$.

Due to symmetry, we have $\delta(\beta t) = \delta(|\beta| t)$.

$$\begin{aligned}\delta(\beta t) &= \delta(|\beta| t) \\ &= \lim_{\Delta \to 0} x(|\beta| t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\mathrm{rect}\left(\frac{t}{\Delta/|\beta|}\right)\end{aligned}$$

But $\left(\begin{aligned} &\lim_{\Delta \to 0} \frac{1}{\Delta}\mathrm{rect}\left(\frac{t}{\Delta/|\beta|}\right) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases} \\ &\lim_{\Delta \to 0} \int_{-\infty}^{\infty} \frac{1}{\Delta}\mathrm{rect}\left(\frac{t}{\Delta/|\beta|}\right) dt = \frac{1}{|\beta|} \end{aligned}\right).$

Hence, $\delta(\beta t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\mathrm{rect}\left(\frac{t}{\Delta/|\beta|}\right) = \frac{1}{|\beta|}\delta(t)$.

Domain-scaling of $\delta(\cdot)$ is often encountered in transforming a spectrum containing $\delta(\cdot)$ between cyclic-frequency (f) and radian frequency (ω) domain:

$$\left[\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)\right] \text{ or } \left[\delta(f) = \delta\left(\frac{\omega}{2\pi}\right) = 2\pi\delta(\omega)\right].$$

Supplementary Questions (Solutions)

S1(a) Given that integration of unit step function, $u(t)$, is a ramp, i.e. $t.u(t)$, then $x(t)$ is made up of:

$$\begin{aligned} x(t) &= \frac{1}{2}\left[t.u(t)\right].u(2-t) \\ &= \left[\int_{-\infty}^{\infty} \frac{1}{2}u(\tau)d\tau\right]u(2-t) \end{aligned}$$

S1(b) The signal $x(t)$ is observed to be made up of various $u(t)$ functions that are shifted in time and/or reversed in time. Hence:

$$x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$$

S2(a) Given: $x(t) = \cos(2t + 0.25\pi)$

$x(t)$ is periodic with an angular frequency of 2 rads/s.

Hence, its frequency is $\dfrac{2}{2\pi} = \dfrac{1}{\pi}$ and period of π.

$$\begin{aligned} P &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |x(t)|^2\, dt \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\left(2t + 0.25\pi\right) dt \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[1 + \cos\left(4t + 0.5\pi\right)\right] dt \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2} dt + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos\left(4t + 0.5\pi\right) dt \\ &= \frac{1}{2\pi}\left[t\right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] \\ &= \frac{1}{2} \end{aligned}$$

Note that $\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos\left(4t + 0.5\pi\right) dt = 0$.

S2(b)

$$\begin{aligned}
P &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|x(t)\right|^2\,dt \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|\frac{1}{2}\left[1+\cos\left(2t\right)\right]\right|^2\,dt \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{4}\left[1+\cos^2\left(2t\right)\right]dt \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}dt+\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}\cos^2\left(2t\right)dt \\
&= \frac{1}{4\pi}\left[t\right]_{-\pi/2}^{\pi/2}+\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{2}\left[1+\cos(4t)\right]dt \\
&= \frac{1}{4}+\frac{1}{8\pi}\int_{-\pi/2}^{\pi/2}dt+\frac{1}{8\pi}\int_{-\pi/2}^{\pi/2}\cos(4t)dt \\
&= \frac{1}{4}+\frac{1}{8\pi}\left[t\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{4}+\frac{1}{8} \\
&= \frac{3}{8}
\end{aligned}$$

Note that $\frac{1}{8\pi}\int_{-\pi/2}^{\pi/2}\cos(4t)dt = 0$.

S2(c) $x(t)=\cos(2\pi t)u(t)$ is not a periodic signal since $x(t) = 0$ for $t < 0$.

S2(d) $x(t)=e^{j\pi t}$; $f = 0.5$; $T = 2$; and $x(t)$ is periodic.

$$\begin{aligned}
P &= \frac{1}{2}\int_{-1}^{1}\left|e^{j\pi t}\right|^2\,dt \\
&= \frac{1}{2}\int_{-1}^{1}1.dt \\
&= \frac{1}{2}\left[t\right]_{-1}^{1} \\
&= 1
\end{aligned}$$

S3(a) $\int_{-\infty}^{t} \cos(\tau)u(\tau)d\tau = \int_{0}^{t} \cos(\tau)d\tau = \sin(t)u(t)$

Note that since $\cos(\tau)u(\tau)$ is zero for negative time, then the integration will yield sin(t) for positive time only.

S3(b) $\int_{-\infty}^{t} \cos(\tau)\delta(\tau)d\tau = \int_{-\infty}^{t} \cos(0)\delta(\tau)d\tau = \int_{-\infty}^{t} 1.\delta(\tau)d\tau = u(t)$

S3(c) $\int_{-\infty}^{\infty} \cos(t)u(t-1)dt = \int_{1}^{\infty} \cos(t)dt = \infty$

Note that integrating cos(t) from 1 to infinity will result in zero since the positive areas will cancel out the negative areas.

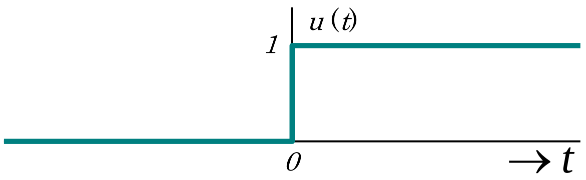
S3(d)

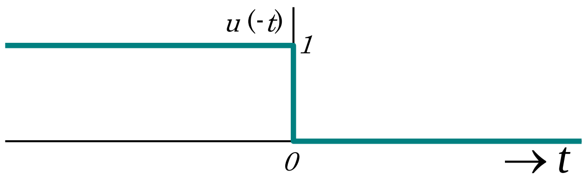
$$\begin{aligned}
\int_{0}^{2\pi} t.\sin\left(\frac{t}{2}\right)\delta\left(\pi - t\right)dt &= \int_{0}^{2\pi} t.\sin\left(\frac{t}{2}\right)\delta\left(-(t-\pi)\right)dt \\
&= \int_{0}^{2\pi} \pi \sin\left(\frac{\pi}{2}\right)\delta\left(-(t-\pi)\right)dt \\
&= \pi\int_{0}^{2\pi} \delta\left(-(t-\pi)\right)dt \\
&= \pi
\end{aligned}$$

Note that $\int_{0}^{2\pi} \delta\left(-(t-\pi)\right)dt = 1$, as it is the area within $\delta\left(-(t-\pi)\right)$

S4 $x(t) = x_e(t) + x_o(t)$; $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$; $x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$; Given: $x(t) = u(t)$

$$x_e(t) = \frac{1}{2}\left[u(t) + u(-t)\right] = \begin{cases} 1, & t = 0 \\ 0.5 & t \neq 0 \end{cases}$$





$$x_o(t) = \frac{1}{2}\left[u(t) - u(-t)\right] = \begin{cases} -0.5, & t < 0 \\ 0.5, & t \geq 0 \end{cases} = \frac{1}{2}\mathrm{sgn}(t)$$