National University of Singapore

Department of Electrical & Computer Engineering

EE2023 Signals & Systems Notes 3

1 Parameters in First and Second Order Systems

From the last chapter, you should have learnt that the transfer function $G(s)$ are of the general form given by :

$$G(s) = K\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\ldots\left(\frac{s}{z_m}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\ldots\left(\frac{s}{p_n}+1\right)}$$

where n is the order of $G(s)$. In this chapter, we will focus on behaviours of systems with transfer functions where $n = 1$ and $n = 2$.

This chapter focuses on two main topics :

- Parameters in first and second order systems. These are special terms given to the parameters of the transfer functions which are important to know.
- Output responses in first and second order systems. Output responses refer to outputs when an input signal is injected into the system.

1.1 Parameters in First Order Systems

General first order ($n = 1$) systems have transfer functions which can be written as

$$G(s) = \frac{K}{sT+1}$$

where K is the d.c./static/steady state gain (gain at $s = 0$), and T is the time constant which is the time taken for the output response to reach 63.2% of its steady state value. The system pole is given by $s = -1/T$.

Example of first order system : $G(s) = \frac{1}{s+2}$. In this case,

$$G(s) = \frac{1}{s+2} = \frac{\frac{1}{2}}{\frac{s}{2}+1}.$$

Thus, the steady state gain is $K = 1/2$ and $T = 0.5$. A physical example of a first order system is the thermocouple or RC or RL circuits. An RC circuit is given in Figure 1:

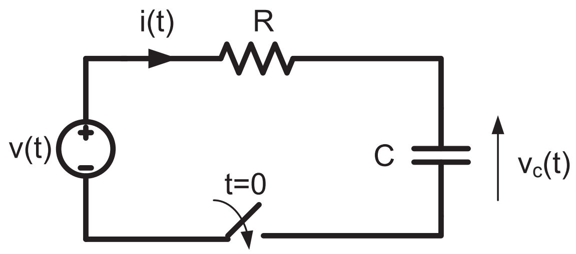

Fig. 1: Simple RC circuit

The switch in the RC circuit closes only at $t = 0$ which implies that the capacitor is not charged ($v_c(t = 0) = 0$) initially. Deriving the model in time domain using differential equation and by applying circuit laws :

$$\begin{aligned} v(t) &= i(t)R + v_c(t) \\ i(t) &= C\frac{dv_c(t)}{dt} \\ v(t) &= RC\frac{dv_c(t)}{dt} + v_c(t) \end{aligned} \tag{1}$$

where $v_c(t)$ which is the voltage across the capacitor is chosen as the output of the model and the input voltage, $v(t)$ serves as the input to the circuit.

To convert this model into a transfer function, take Laplace Transform on both sides of (1) and apply initial condition, $v_c(t = 0) = 0$. Since the input voltage is a general input $v(t)$, the LT of $v(t)$ is written generally as $V(s)$ while the LT of the output voltage $v_c(t)$ is written as $V_c(s)$.

$$\begin{aligned} V(s) &= RCsV_c(s) - RCv_c(0) + V_c(s) \\ &= RCsV_c(s) - RCv_c(0) + V_c(s) \\ G(s) &= \frac{V_c(s)}{V(s)} = \frac{1}{RCs+1} \quad \text{since the assumption is } v_c(0) = 0 \end{aligned} \tag{2}$$

The transfer function of the RC circuit is therefore given by 2 and has a steady state gain of $K = 1$ and a time constant of $T = RC$.

In conclusion, first order systems are therefore very simple and are characterized by its steady state gain K and its time constant T. When both these parameters are known, you can write down the $G(s)$.

Exercise : Write down the transfer function of a first order system with a steady state gain of 2 and a time constant of 3 seconds. Can you convert from the transfer function model to the differential equation (time domain) model?

1.2 Parameters in Standard Second Order Systems

Standard second order ($n = 2$) systems have transfer functions which can be written as

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \text{ compare with } G(s) = K\frac{1}{(\frac{s}{p_1}+1)(\frac{s}{p_2}+1)} \tag{3}$$

where K is the d.c./static/steady stage gain (gain at $s = 0$), ζ is the damping ratio, and ω_n is the undamped natural frequency. ζ and ω_n are functions of the poles $-p_1$ and $-p_2$. Can you find these relationships?

The system poles are given by :

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -p_1, -p_2 \tag{4}$$

An example of a second order system : $G(s) = \frac{2}{s^2+5s+4}$. In this example,

$$\begin{aligned} \omega_n^2 &= 4 \Rightarrow \omega_n = 2 \\ 2\zeta\omega_n &= 5 \Rightarrow \zeta = 1.25 \\ K &= 0.5 \end{aligned}$$

A physical example of a second system is a spring-damper system or any system with 2 energy storages. Second order systems can be categorized as :

(i) underdamped if $\zeta < 1$. Such a system has output responses which are oscillatory (will see this later) and the system poles (from (4)) are complex conjugates given by

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -\zeta\omega_n \pm j\omega_d \end{aligned} \tag{5}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

Poles on the complex plane are shown in Figure 2.

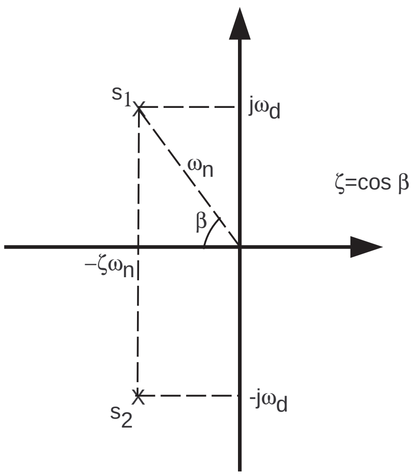

Fig. 2: Complex Conjugate Poles of a Second Order Underdamped System

Question you need to know how to answer : Given a set of complex conjugate poles, you have to know how to determine ζ and ω_n.

For example if the poles are at $s_{1,2} = -a \pm jb$, then we can compare :

$$s^2 + 2\zeta\omega_n + \omega_n^2 = (s + a + jb)(s + a - jb) = s^2 + 2as + (a^2 + b^2).$$

From the above equation, it follows that

$$2\zeta\omega_n = 2a \text{ and } \omega_n^2 = a^2 + b^2 = |-a \pm jb|^2.$$

Thus

$$\omega_n = \sqrt{a^2 + b^2} = |-a \pm jb| = \text{magnitude of poles.}$$

You can see this in Figure 2. At the same time, the real part of the pole $(-a)$ is equal to $-\zeta\omega_n$ ie $-a = -\zeta\omega_n$. From Figure 2, it is also clear that $\zeta = \cos\beta$ where $\beta = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

When the second order system is given in the form of

$$G(s) = \frac{K'}{as^2 + bs + c},$$

then you should make the following manipulation before you can compare with the standard second order system as follows :

$$\begin{aligned} G(s) = \frac{K'}{as^2 + bs + c} &= \frac{K'}{a(s^2 + \frac{b}{a}s + \frac{c}{a})} \\ &= \frac{\frac{K'}{a}}{s^2 + \frac{b}{a}s + \frac{c}{a}} \\ \text{compare} \quad \text{with} &\quad \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \end{aligned}$$

Thus $K = K'/a$, $\omega_n = \sqrt{c/a}$ and $\zeta = \frac{b}{2\sqrt{ac}}$.

(ii) critically damped if $\zeta = 1$. Such a system has output responses which look very much like overdamped or first order systems. The system poles are real and repeated. They are given by $s_{1,2} = -\zeta\omega_n$ if you substitute $\zeta = 1$ into (4). The poles are shown in Figure 3.

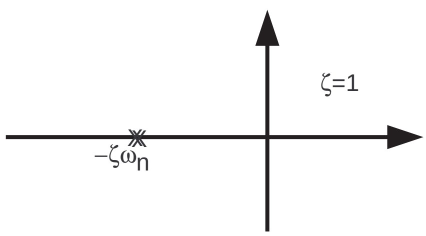

Fig. 3: Repeated Poles of a Second Order Critically Damped System

(iii) overdamped if $\zeta > 1$. Such a system has output responses which are very sluggish or slow and also appear like first order system responses. The system poles are real and distinct and given by $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$. The poles are shown in Figure 4.

Example of second order system is the RLC circuit given in Figure 5.

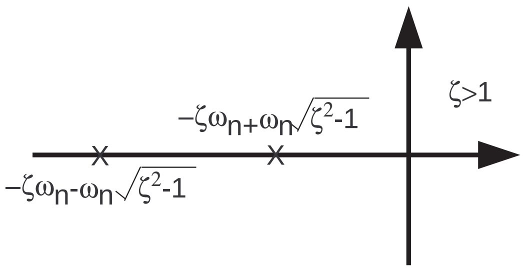

Fig. 4: Real Distinct Poles of a Second Order Overdamped System

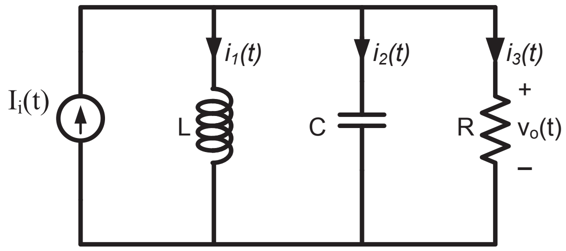

Fig. 5: Second order RLC circuit

The differential equation which models the output $v_0(t)$ can be derived as follows :

$$\begin{aligned}
I_i(t) &= i_1(t) + i_2(t) + i_3(t) \\
&= \frac{1}{L}\int v_0(t)dt + C\frac{dv_0(0)}{dt} + \frac{v_0(t)}{R} \\
RLI_i(t) &= R\int v_0(t)dt + RLC\frac{dv_0(0)}{dt} + Lv_0(t) \\
RL\frac{dI_i(t)}{dt} &= Rv_0(t) + RLC\frac{d^2v_0(0)}{dt^2} + L\frac{dv_0(t)}{dt} \qquad (6)
\end{aligned}$$

Assuming zero initial conditions, (6) can be transformed into a transfer function $G(s)$ relating $I_i(s)$ to $V_0(s)$. Taking Laplace Transform on both sides of (6),

$$\begin{aligned}
RLsI_i(s) &= RV_0(s) + RLCs^2V_0(s) + LsV_0(s) \\
G(s) &= \frac{V_0(s)}{I_i(s)} = \frac{RLs}{RLCs^2 + Ls + R} \\
&= \frac{\frac{1}{C}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \qquad (7)
\end{aligned}$$

$G(s)$ in (7) represents a transfer function model of the RLC circuit in Figure 5. However, it is not of the standard order form given in (3) because it has a zero at $s = 0$. The poles of

$G(s)$ are given by :

$$s_{1,2} = \frac{-L \pm \sqrt{L^2 - 4R^2LC}}{2RLC}.$$

Depending on the relative values of the R, L and C components, different behaviours of the circuit can be obtained :

- If $L^2 - 4R^2LC < 0$, then the circuit is underdamped. The parameters of $G(s)$ will be :

$$\omega_n = \sqrt{\frac{1}{LC}}, \quad \zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}.$$

- If $L^2 - 4R^2LC > 0$, then the circuit is overdamped and $\zeta > 1$.
- If $L^2 - 4R^2LC = 0$ or $L = 4R^2C$, the circuit is critically damped and $\zeta = 1$.
- Can this circuit have zero damping?

2 Output Responses of LTI Systems

LTI systems satisfy the property of superposition. This implies that if the system has a number of independent inputs, then the corresponding output is a sum of the individual outputs due to the individual inputs. This is illustrated as follows :

If $G(s)$ has k independent inputs, then

$$u(t) = u_1(t) + u_2(t) + \ldots + u_k(t).$$

Then the output of $G(s)$ will be a linear combination given by :

$$Y(s) = G(s)U_1(s) + G(s)U_2(s) + \ldots + G(s)U_k(s)$$

where $U_{1\ldots,k}(s)$ are the Laplace transform of $u_{1,\ldots,k}(t)$ and $Y(s)$ is the Laplace transform of $y(t)$.

Examples of systems where superposition can be used are circuits consisting of passive elements such as resistors, inductors and capacitors where there are more than one <u>independent</u> input sources. Examples of independent input sources in a circuit include voltage and current sources.

How do you analyse a circuit when there are more than one independent sources? If the independent sources are current and voltage sources, then to obtain the total response from both the current and voltage sources, you consider the contribution of each source independently. For contribution from current source, you "short circuit" the voltage source. The idea is that you remove the effect of the voltage source ("shorting" implies zero voltage) and only consider the effect of the current source. For contribution from voltage source, you "open circuit" the current source. Then the total response will be the sum of the responses from the individual sources which you have just calculated.

An example of a circuit with 2 independent inputs is given in Figure 6.

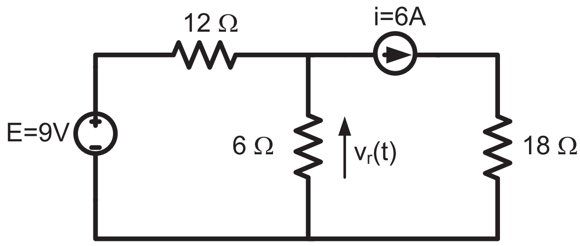

Fig. 6: Circuit with 2 independent inputs.

To obtain the total voltage, $v_t(t)$, across the 6Ω resistor, consider the following steps :

Step 1 : "Kill" the current source by setting $i = 0$. Solve for $v_{r1}(t)$ in Figure 7.

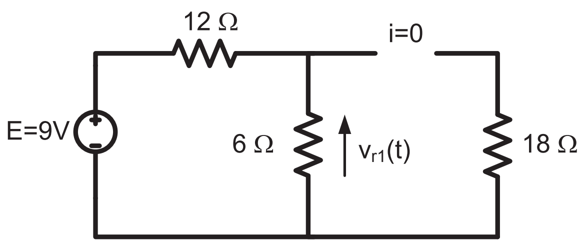

Fig. 7: Circuit with current source "killed".

Step 2 : "Short" the voltage source by setting $E = 0$. Solve for $v_{r2}(t)$ in Figure 8.

Step 3 : Apply superposition. Final solution : $v_r(t) = v_{r1}(t) + v_{r2}(t)$.

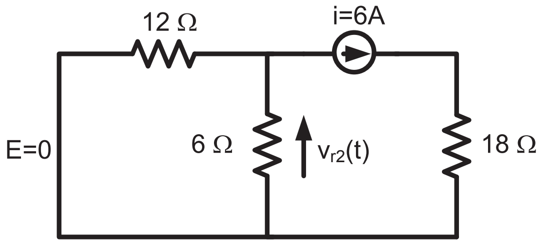

Fig. 8: Circuit with voltage source "killed".

In general, for more complex dynamic systems, for each independent input source, the output $y(t)$ of the system can always be obtained either by solving the differential equation or using $Y(s) = G(s)U(s)$ directly.

For LTI systems, typical output responses which we are interested in are :

- impulse response ie system responding to an impulse (Dirac delta) signal
- step response ie system responding to a constant signal of any magnitude
- sinusoidal response ie system responding to a sinusoidal signal

In obtaining these responses, the assumption is that $G(s)$ is stable where stability is defined by the notion of bounded-input bounded-output. Such a stability implies that if the input is bounded, then the output is guaranteed to be bounded as well. Systems with poles on the left half plane ie poles with negative real parts will result in bounded input bounded output behaviours.

We consider the output $Y(s)$ for a general system $G(s)$ (see Figure 9) with 3 different types of inputs, $U(s)$:

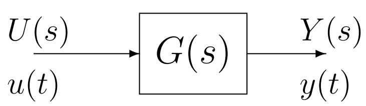

Fig. 9: General Input-Output Model

- $U(s)$ is an impulse signal

- $U(s)$ is a step function
- $U(s)$ is a sinusoid

All transfer functions in this section are assumed to be stable. Hence analysis here only holds for stable systems.

2.1 Impulse Response

If the input is a Dirac delta function : $u(t) = \delta(t)$, then $U(s) = 1$.

Hence $Y(s) = G(s)U(s) = G(s)$ and $y_{imp}(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)\} = g(t)$. Important to note that this means that the transfer function is the Laplace transform of the impulse response and this relationship applies to transfer functions of any order, that is

$$\boxed{y_{imp}(t) = \mathcal{L}^{-1}\{G(s)\} = g(t) \text{ for any } G(s)}$$

For first order systems : $G(s) = \dfrac{K}{sT+1} = \dfrac{K}{T}\dfrac{1}{s+1/T}$

Impulse response : $g(t) = \mathcal{L}^{-1}\{G(s)\} = \dfrac{K}{T}e^{-t/T}$.

The impulse response of a typical first order stable system is given in Figure 10.

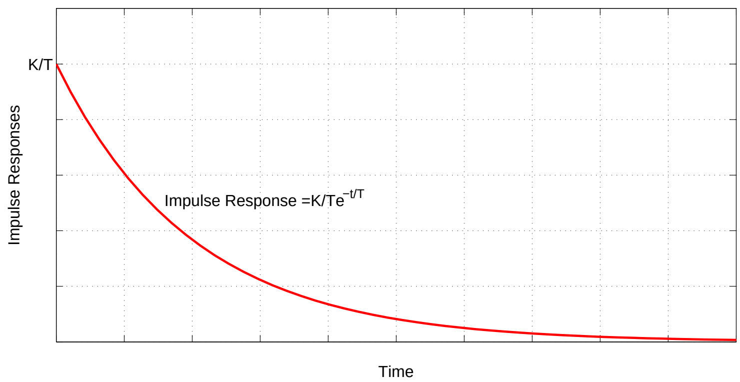

Fig. 10: Impulse Response of a typical first order stable system

For second order systems :

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- If $G(s)$ is overdamped with poles at $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -a, -b$, then the impulse response will be given by
$$g(t) = K_1 e^{-at} + K_2 e^{-bt}$$
where $K_1 = \frac{K\omega_n}{2\sqrt{\zeta^2-1}} = -K_2$.

- If $G(s)$ is underdamped with poles at $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, then the impulse response will be given by
$$g(t) = \frac{K\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

- If $G(s)$ is critically damped with poles at $s_{1,2} = -\zeta\omega_n$, then the impulse response will be given by
$$g(t) = K\omega_n^2 t e^{-\zeta\omega_n t}.$$

- If $G(s)$ is <u>marginally stable</u> with $\zeta = 0$, then
$$G(s) = \frac{K\omega_n^2}{s^2 + \omega_n^2}.$$
The poles of $G(s)$ is at $s_{1,2} = \pm j\omega_n$, ie the poles are strictly on the imaginary axis. The impulse response of $G(s)$ is given by
$$g(t) = K\omega_n \sin \omega_n t.$$
Note that $g(t)$ is a pure sinusoidal function.

Another important concept involving the impulse response is the <u>convolution integral</u> between an input, $u(t)$, and an impulse response, $g(t)$:
$$g(t) * u(t) = \int_0^\infty u(t-\tau)g(\tau)d\tau. \tag{8}$$
Taking Laplace Transform :
$$\begin{aligned}
\mathcal{L}\left\{g(t) * u(t)\right\} &= \int_0^\infty \left(\int_0^\infty u(t-\tau)g(\tau)d\tau\right) e^{-st}dt \\
&= \int_0^\infty \int_0^\infty u(t-\tau)g(\tau)e^{-st}d\tau dt \\
&= \int_0^\infty \int_0^\infty u(t-\tau)g(\tau)e^{-s(t-\tau)}e^{-s\tau}d\tau dt \\
&= \int_0^\infty \int_0^\infty u(\lambda)g(\tau)e^{-s\lambda}e^{-s\tau}d\tau d\lambda \\
&= G(s)U(s) = Y(s) = \mathcal{L}\{y(t)\} \qquad (9) \\
\text{Therefore } g(t) * u(t) &= y(t) \qquad (10)
\end{aligned}$$

Hence (10) tells us that the output $y(t)$ for any arbitrary input $u(t)$ can also be obtained through the convolution integral defined in (8). Obviously, the convolution integral is not easy to compute and thus it is not often used. Instead, we generally use Laplace transform to compute the output $Y(s)$, followed by inverse LT.

2.2 Step Response

If the input is a unit step function : $u(t) = 1$ for all $t \geq 0$, then $U(s) = \frac{1}{s}$.

Hence $Y(s) = G(s)U(s) = \frac{G(s)}{s}$ and $y_{step}(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{G(s)}{s}\}$.

For first order systems : $G(s) = \frac{K}{sT+1}$

Step response : $y_{step}(t) = y_{ss}(t) + y_{tr}(t) = K(1 - e^{-t/T})$.

The steady state output, $y_{step}(t = \infty) = y_{step,ss} = K$.

At $t = T$, $y_{step}(t = T) = K(1 - e^{-1}) = 0.632K$. Thus $t = T$ is the time taken for the step response to reach 63.2% of the final steady state value, K. T is defined to be the system's time constant.

Notice how the steady state output is related to the steady state gain, K, defined in the earlier section. In general, if the step input has a magnitude of r ie $u(t) = r$, then the output will be given by :

$$y_{step}(t) = r \times K(1 - e^{-t/T}), \quad y_{step,ss} = rK.$$

For second order systems :

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- If $G(s)$ is overdamped ($\zeta > 1$) with poles at $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -a, -b$, then the step response will be given by

$$y_{step}(t) = K + K_3 e^{-at} + K_4 e^{-bt}$$

where $K_3 = \frac{K}{2}\left(\frac{\zeta}{\sqrt{\zeta^2-1}} - 1\right)$ and $K_4 = -\frac{K}{2}\left(1 + \frac{\zeta}{\sqrt{\zeta^2-1}}\right)$. Note that $a = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$ and $b = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$. The steady state output of the step response is $y_{step,ss} = K$. This is again related to the steady state gain, K.

- If $G(s)$ is underdamped ($\zeta < 1$) with poles at $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, then the step response will be given by

$$y_{step}(t) = K\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right), \quad y_{step,ss} = K.$$

A typical step response of a second order underdamped system is given in Figure 11. The steady state value will be $y_{step,ss} = K$.

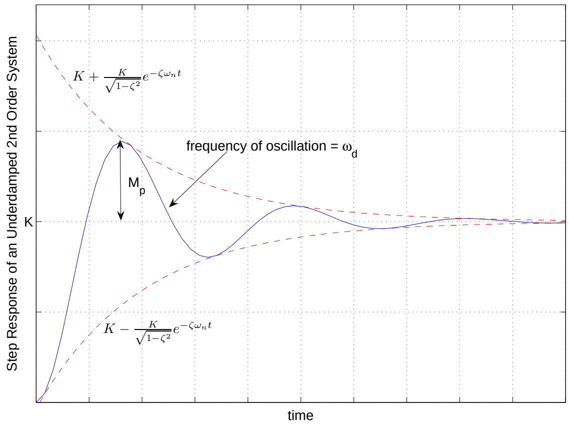

Fig. 11: Typical step response of an underdamped second order system

Furthermore, since the step response is oscillatory, the following time domain criteria specifies the transient response :

- Maximum Overshoot, $M_p = Ke^{-\zeta\pi/\sqrt{1-\zeta^2}}$. Note that this is the absolute overshoot value above the steady state value of K for a unit step input.

  If the step input has a magnitude of r, then the maximum overshoot becomes

$$M_p = Kre^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

  In terms of percentage overshoot, it is

$$M_p(\%) = \frac{Kre^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{Kr} \times 100\% = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\%$$

  This is an important quantity which you need to note clearly.

– The 2%-settling time, t_s, is defined to be the time taken for the oscillatory step response to reach within 2% of the final steady state value. This is given by the expression :

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{real\ part\ of\ pole}.$$

Hence the further the real part of the pole is from the origin or the imaginary axis, the smaller is the t_s and therefore the faster is the response.

– The peak time, t_p is defined to be the time taken for the oscillatory response to reach its first highest peak. It is given by

$$t_p = \frac{\pi}{\omega_d}.$$

– The rise time, t_r, is defined to be the time taken for the oscillatory response to cross the steady state value for the first time. It is given by

$$t_r = \frac{1.8}{\omega_n}.$$

- If $G(s)$ is critically damped ($\zeta = 1$) with poles at $s_{1,2} = -\zeta\omega_n = -\omega_n$, then the step response will be given by

$$y_{step}(t) = K\left(1 - e^{-\zeta\omega_n t} - \zeta\omega_n t e^{-\zeta\omega_n t}\right).$$

There will be NO oscillations and hence no overshoots in this response. $y_{step,ss} = K$.

- If $G(s)$ is marginally stable with $\zeta = 0$, $G(s)$ becomes

$$G(s) = \frac{K\omega_n^2}{s^2 + \omega_n^2}.$$

The poles of $G(s)$ are at $s_{1,2} = \pm j\omega_n$, ie the poles are strictly on the imaginary axis.

The step response is given by :

$$y_{step}(t) = K(1 - \cos\omega_n t)$$

which does not have a fixed steady state value. The step response is a sinusoidal function which implies steady state oscillation even though the input is a step function. This is a result of the sinusoidal impulse response $g(t)$.

In general, if the transfer function is of the form $G(s) = \frac{K}{s^2+\omega^2}$, then $G(s)$ has poles at $s_{1,2} = \pm j\omega$. It is marginally stable and both step and impulse responses will always have steady state oscillations with frequency corresponding to the pole at ω.

In summary, for first and second order systems, the impulse and step responses are given in Figures (12) to (16).

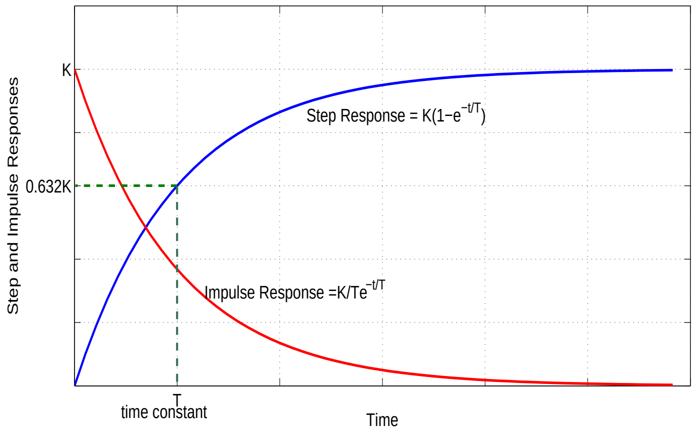

Fig. 12: Impulse and step responses of a first order system

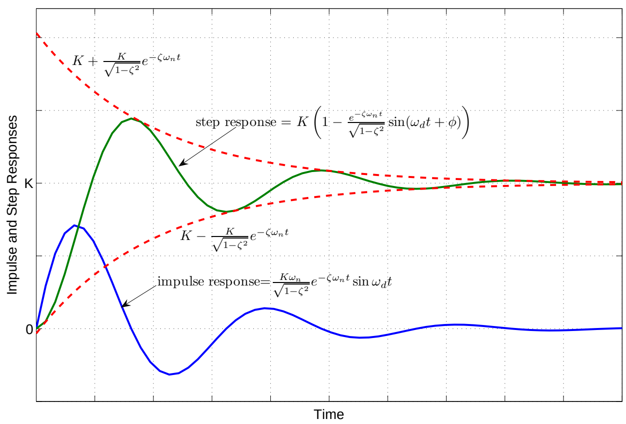

Fig. 13: Impulse and step responses of an underdamped second order system

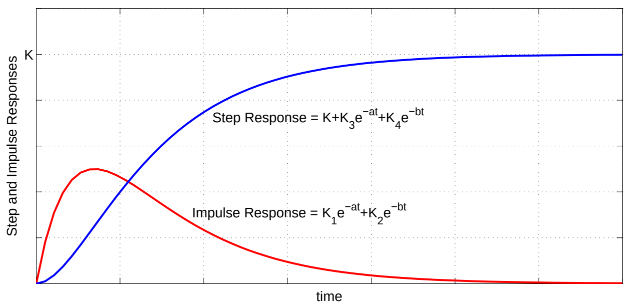

Fig. 14: Impulse and step responses of an overdamped second order system

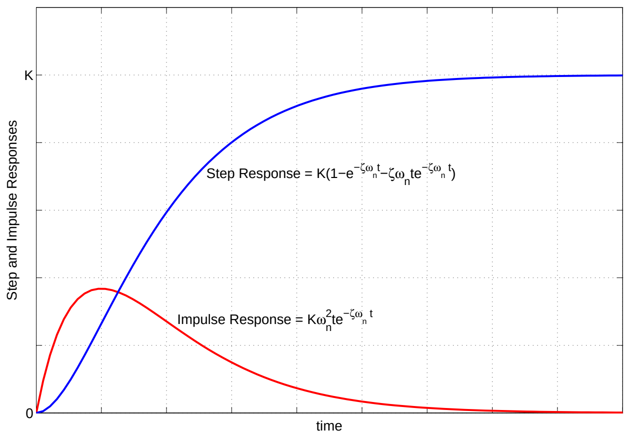

Fig. 15: Impulse and step responses of a critically damped second order system

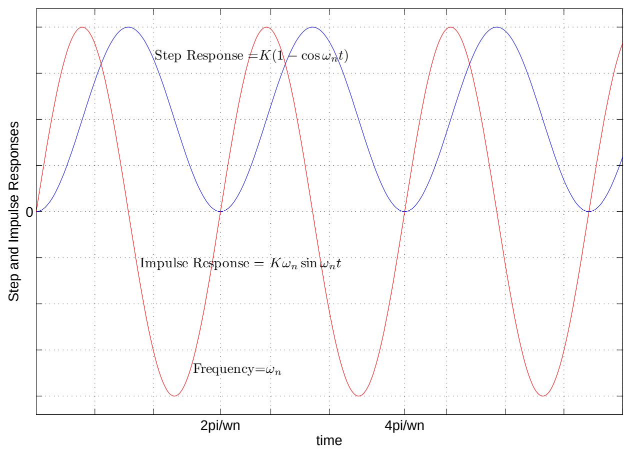

Fig. 16: Impulse and step responses of a marginally stable second order system

2.3 Sinusoidal Response

For any stable $G(s)$, the steady state response (ie after the transient part has decayed away) of $G(s)$ to a sinusoidal input, $u(t) = A\sin(\omega t + \theta_0)$ is always given by

$$y_{sinusoidal,ss}(t) = AM\sin(\omega t + \theta_0 + \phi)$$

where θ_0 is the initial phase of the $u(t)$, M = gain of $G(s)$ at $s = j\omega$ ie $M = |G(j\omega)|$ and ϕ = phase of $G(s)$ at $s = j\omega$ ie $\phi = \angle G(j\omega)$. You must be careful with the computation of the phase.

If the input is a bunch of sinusoids, $u(t) = \Sigma_{k=1,N}^{\infty} A_k \sin\omega_k t$, then the output is also a sum of sinusoids given by :

$$y(t) = \Sigma_{k=1,N}^{\infty} A_k M_k \sin(\omega_k t + \phi_k)$$

where $M_k = |G(j\omega_k)|$ and $\phi_k = \angle G(j\omega_k)$. This is a result of superposition.

Do not forget that you CANNOT use final value theorem for sinusoidal responses because its steady state output is not a constant.

It is also not advisable to use the Laplace transform approach given below

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= G(s)A\frac{\omega}{s^2 + \omega^2} \end{aligned}$$

because it will be too difficult to solve for $y(t)$ from $Y(s)$. If you do use this approach, then the solution $y(t)$ which you get consists of both the transient response and the steady state response. This will be the complete response.