EE2023/TEE2023 TUTORIAL 2 (SOLUTIONS)

Solution to Q.1

Description of $x(t)$:

- $x(t)$ is a REAL & EVEN function of t $\because$ Spectrum is REAL and SYMMETRIC
- $x(t)$ has an average (or DC) value of 2 $\because$ Zero-frequency component has value 2
- $x(t)$ is APERIODIC $\because$ $\{\pi,\ \pi^2, \pi^3\}\ \cdots$ has no common factor
- $x(t)$ is a POWER SIGNAL $\because$ $\begin{cases}\text{Spectrum is defined only at discrete} \\ \text{frequency points (sum of sinusoids)}\end{cases}$

Since $x(t)$ is non-periodic, it does not have a Fourier series expansion.

Solution to Q.2

(a) The fundamental frequency of $x(t) = 6\sin(12\pi t) + 4\exp\left(j\left(8\pi t + \frac{\pi}{4}\right)\right) + 2$ is $\begin{cases}\boldsymbol{f_p = HCF\{6,4\} = 2} \\ \boldsymbol{T_p = 0.5}\end{cases}$.

Re-write $x(t)$ as a sum of complex exponentials:

$$\begin{aligned} x(t) &= \frac{6}{j2}\left[\exp(j12\pi t) - \exp(-j12\pi t)\right] + 4\exp(j\pi/4)\exp(j8\pi t) + 2 \\ &= j3\exp(-j12\pi t) + 2 + 4\exp(j\pi/4)\exp(j8\pi t) - j3\exp(j12\pi t) \end{aligned} \tag{1}$$

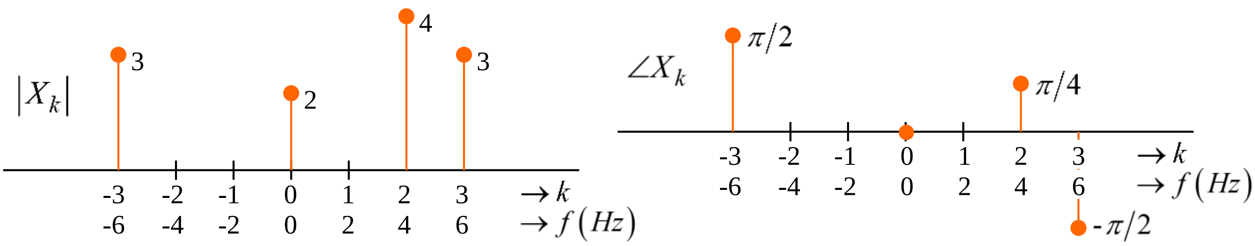
Express $x(t)$ as a complex exponential Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T_p}t\right) = \sum_{k=-\infty}^{\infty} X_k \exp(j4\pi kt)$$

$$= \begin{pmatrix} \cdots + X_{-3}\exp(-j12\pi t) + X_{-2}\exp(-j8\pi t) + X_{-1}\exp(-j4\pi t) \\ +X_0 \\ +X_1\exp(j4\pi t) + X_2\exp(j8\pi t) + X_3\exp(j12\pi t) + \cdots\cdots\cdots \end{pmatrix} \tag{2}$$

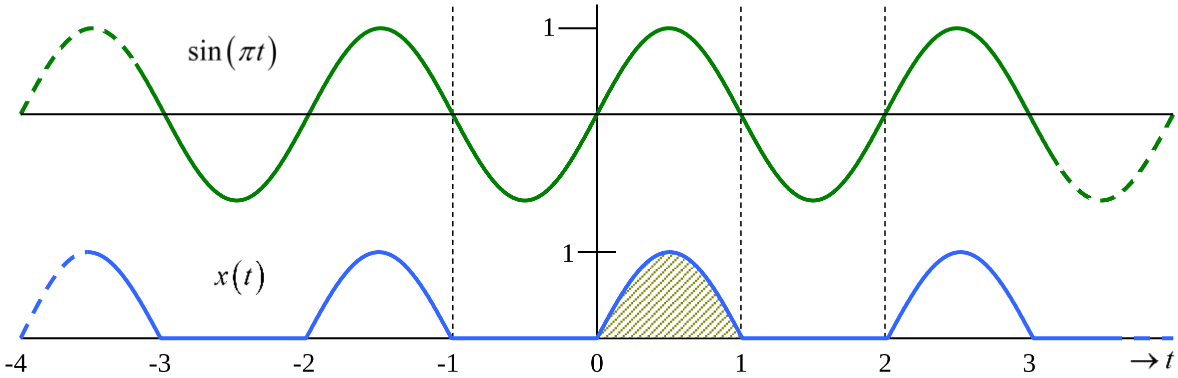
Comparing coefficients of complex exponential terms in (1) and (2), we conclude that:

$$\boldsymbol{X_{-3} = j3,\quad X_0 = 2,\quad X_2 = 4\exp\left(j\frac{\pi}{4}\right),\quad X_3 = -j3} \text{ and } \boldsymbol{\left[X_k = 0;\ k \neq 0,\ 2,\ \pm 3\right]}.$$



Remarks: *If a periodic signal is given as a sum of sinusoids, then its Fourier series coefficients can be evaluated using the above method without the need to perform any integration.*

(b) $x(t)=\frac{1}{2}\left(\left|\sin(\pi t)\right|+\sin(\pi t)\right)$: Half-wave rectification of $\sin(\pi t)$.



Period of $x(t)$: $T=2$; Fundamental frequency = 0.5

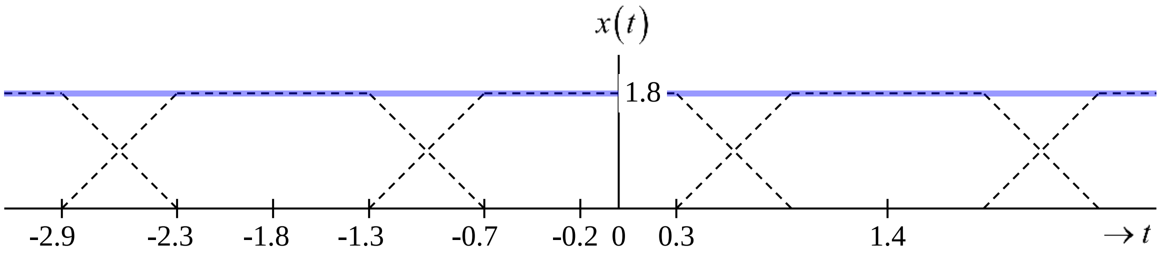
Frequency of 4th harmonic = 4 x 0.5 = 2 Hz

The Fourier series coefficients, X_k, can also be calculated as follows, though this is not required in the question:

$$
\begin{aligned}
X_k &= \frac{1}{T}\int_0^T x(t)\exp(-j2\pi kt/T)\,dt = \frac{1}{2}\int_0^2 x(t)\exp(-j\pi kt)\,dt \\
&= \frac{1}{2}\int_0^1 \sin(\pi t)\exp(-j\pi kt)\,dt = \frac{1}{2}\int_0^1 \frac{1}{j2}\left[\exp(j\pi t)-\exp(-j\pi t)\right]\exp(-j\pi kt)\,dt \\
&= \frac{1}{j4}\int_0^1 \exp(-j\pi(k-1)t)-\exp(-j\pi(k+1)t)\,dt \\
&= \frac{1}{j4}\left[\frac{\exp(-j\pi(k-1)t)}{-j\pi(k-1)}-\frac{\exp(-j\pi(k+1)t)}{-j\pi(k+1)}\right]_0^1 \\
&= \frac{1}{j4}\left[\exp(-j\pi k)\left(\frac{\exp(j\pi)}{-j\pi(k-1)}-\frac{\exp(-j\pi)}{-j\pi(k+1)}\right)-\left(\frac{1}{-j\pi(k-1)}-\frac{1}{-j\pi(k+1)}\right)\right] \\
&= \frac{1}{j4}\left[(-1)^k\left(\frac{1}{j\pi(k-1)}-\frac{1}{j\pi(k+1)}\right)+\left(\frac{1}{j\pi(k-1)}-\frac{1}{j\pi(k+1)}\right)\right] \\
&= \frac{(-1)^k}{4\pi}\left[-\frac{1}{(k-1)}+\frac{1}{(k+1)}\right]+\frac{1}{4\pi}\left[-\frac{1}{(k-1)}+\frac{1}{(k+1)}\right] \\
&= \frac{(-1)^k}{4\pi}\left[-\frac{(k+1)}{(k^2-1)}+\frac{(k-1)}{(k^2+1)}\right]+\frac{1}{4\pi}\left[-\frac{(k+1)}{(k^2-1)}+\frac{(k-1)}{(k^2+1)}\right] \\
&= \frac{(-1)^k}{4\pi}\left[\frac{2}{(1-k^2)}\right]+\frac{1}{4\pi}\left[\frac{2}{(1-k^2)}\right] \\
&= \begin{cases} \dfrac{1}{\pi(1-k^2)}; & k = even \\ 0; & k = \ odd \end{cases} \\
&= \frac{1+(-1)^k}{2\pi(1-k^2)}
\end{aligned}
$$

Solution to Q.3

Graphically, we observe that $x(t) = \sum_{n=-\infty}^{\infty} 2p(t-1.6n) = 1.8$.



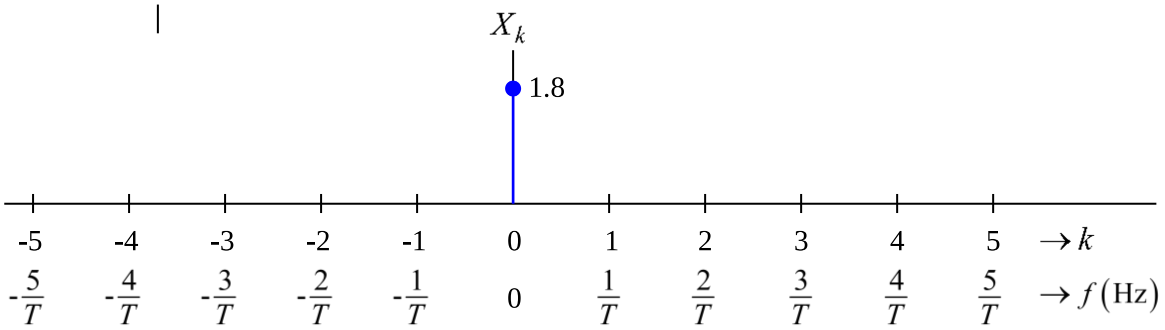
By Deduction:

- $x(t)$ has a *zero-frequency* component of value 1.8, which implies that $X_0 = 1.8$.
- $x(t)$ has no *non-zero frequency* components, which implies that $X_k = 0;\ \ k \neq 0$.

By Derivation:

Since $x(t)$ is a constant (or a DC signal), it may be treated as a periodic signal of arbitrary period *T*, where $0 < T < \infty$. Its Fourier series coefficients can thus be computed as

$$
\begin{aligned}
X_k &= \frac{1}{T}\int_{-T/2}^{T/2} 1.8\exp\left(-j2\pi\frac{k}{T}t\right)dt \\
&= \frac{1.8}{T}\left[\frac{\exp(-j2\pi kt/T)}{-j2\pi k/T}\right]_{-T/2}^{T/2} \\
&= \frac{1.8}{T}\left[\frac{\exp(-j\pi k)}{-j2\pi k/T} - \frac{\exp(j\pi k)}{-j2\pi k/T}\right] \\
&= 1.8\frac{\sin(\pi k)}{\pi k} \\
&= 1.8\ \mathrm{sinc}(k) \\
&= \begin{cases} 1.8; & k = 0 \\ 0; & k \neq 0 \end{cases}
\end{aligned}
$$



Solution to Q.4

This is the continuous frequency spectrum of a periodic signal. Recall that for a periodic signal, $x_p(t)$, we have following representations :

Fourier series (discrete frequency spectrum) : $x_p(t) = \sum_{k=-\infty}^{\infty} c_k \; e^{j2\pi k f_p t}$

Fourier transform (continuous frequency spectrum) : $X_p(f) = \sum_{k=-\infty}^{\infty} c_k \, \delta(f - kf_p)$

In both representations, c_k are the Fourier series coefficients.

Given the spectrum in the figure, we recognize that $X(f)$ is the continuous frequency spectrum and hence $X(f)$ can be written as :

$$X(f) = 2e^{j0.25\pi}\delta(f+32) + 8e^{j0.5\pi}\delta(f+24) + 5\delta(f+16) + 5\delta(f-16) + 8e^{-j0.5\pi}\delta(f-24) + 2e^{-j0.25\pi}\delta(f-32)$$

Recall that by applying the frequency shifting property of the Fourier Transform, you get

$\mathcal{F}\left\{e^{j2\pi f_0 t}\right\} = \delta(f - f_0)$ since $\mathcal{F}\{1\} = \delta(f)$.

Hence applying this property to $X(f)$, we get

$$\begin{aligned} x(t) &= 2e^{j0.25\pi}e^{-j2\pi 32t} + 8e^{j0.5\pi}e^{-j2\pi 24t} + 5e^{-j2\pi 16t} + 5e^{j2\pi 16t} + 8e^{-j0.5\pi}e^{j2\pi 24t} + 2e^{-j0.25\pi}e^{j2\pi 32t} \\ &= 4\cos(64\pi t - 0.25\pi) + 16\cos(48\pi t - 0.5\pi) + 10\cos(32\pi t) \\ &= 4\cos(64\pi t - 0.25\pi) + 16\sin(48\pi t) + 10\cos(32\pi t) \end{aligned}$$

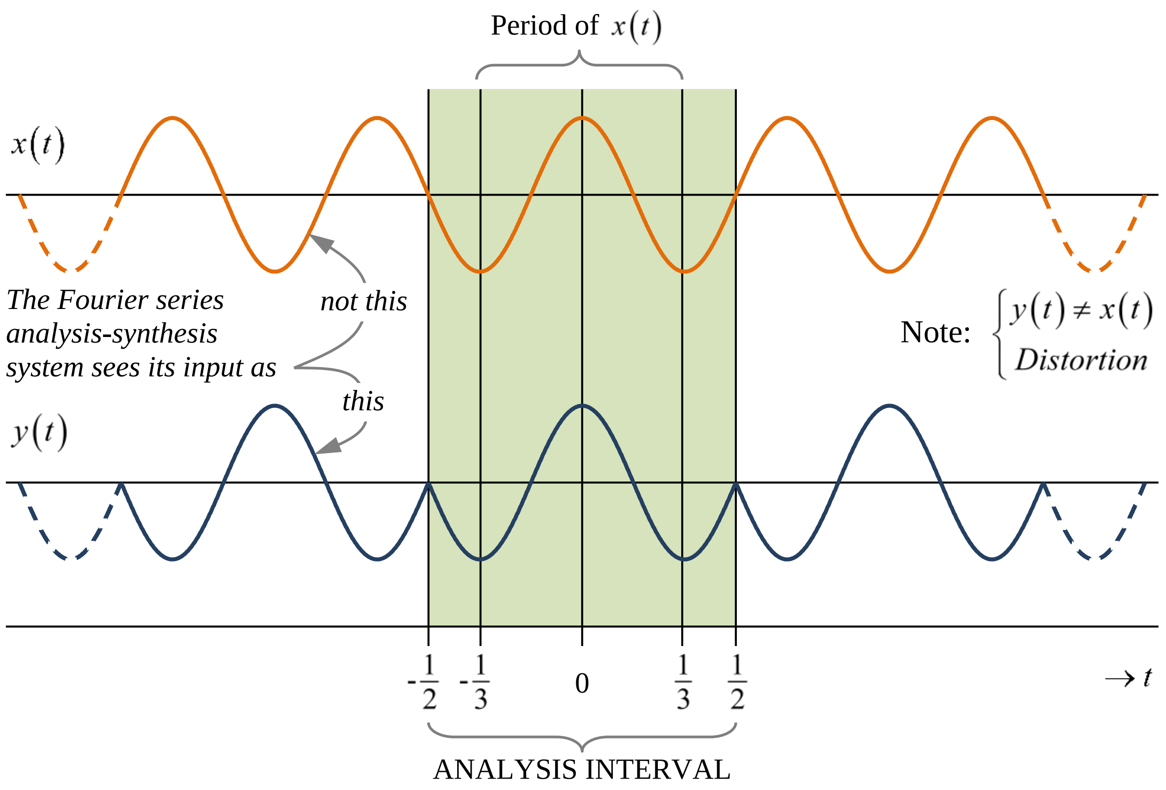
Hence $x(t)$ consists of only real sinusoids.

Solution to Q.5

(a) The computation of the Fourier series coefficients, c_k, assumes that $x(t)$ has a period of 1 over the interval $[-0.5, 0.5]$.

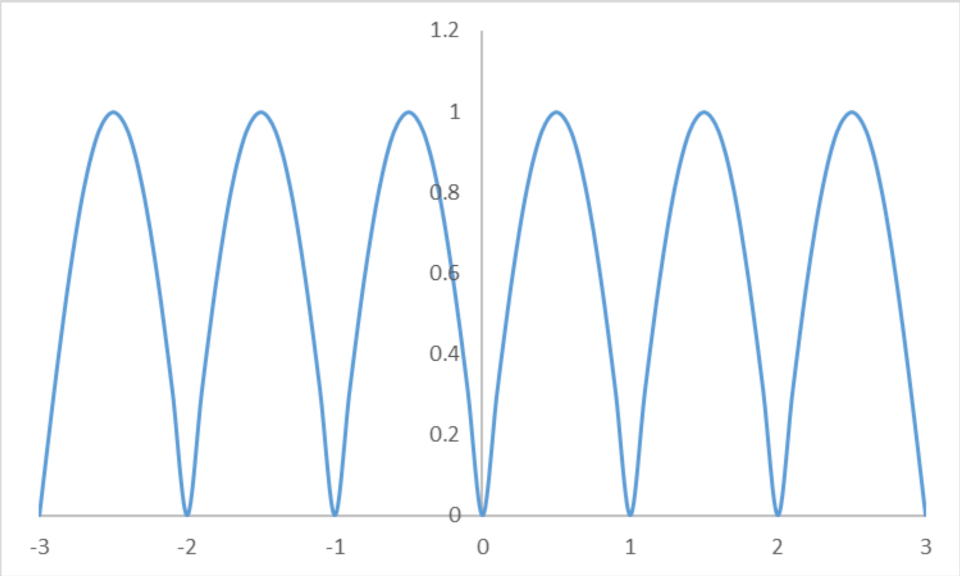
(b) The synthesis of $y(t)$ uses c_k as Fourier series coefficients to synthesize a periodic signal of period equal to 1.

(c) The first graph illustrates the signal $x(t) = \cos(3\pi t)$, which has a period of 2/3 seconds. However, since the analysis and synthesis systems are based on a periodic signal with period of 1 second, then the portion of $x(t)$ of duration 1 second is analysed and synthesized to give the output shown in the lower graph.



Supplementary Questions (Solutions)

S1 The signal $x(t)=|\sin(\pi t)|$ is as follows:



Hence it has a period of 1 second.

$$c_0=\int_0^1 \sin(\pi t)dt=\frac{1}{\pi}[\cos(\pi t)]_0^1=-\frac{2}{\pi}$$

$$\begin{aligned}
c_k &= \int_0^1 \sin(\pi t)e^{-j2\pi kt}dt \\
&= \int_0^1 \frac{1}{2j}\left[e^{j\pi t}-e^{-j\pi t}\right]e^{-j2\pi kt}dt \\
&= \frac{1}{2j}\int_0^1\left[e^{j\pi t(1-2k)}-e^{-j\pi t(1+2k)}\right]dt \\
&= \frac{1}{2j}\left[\frac{e^{j\pi t(1-2k)}}{j\pi(1-2k)}+\frac{e^{-j\pi t(1+2k)}}{-j\pi(1+2k)}\right]_0^1 \\
&= -\frac{1}{2\pi}\left[\frac{e^{j\pi(1-2k)}-1}{(1-2k)}+\frac{e^{-j\pi(1+2k)}-1}{(1+2k)}\right] \\
&= -\frac{1}{2\pi}\left[\frac{-1-1}{(1-2k)}+\frac{-1-1}{(1+2k)}\right] \\
&= -\frac{1}{2\pi}\left[\frac{-2}{(1-2k)}+\frac{-2}{(1+2k)}\right] \\
&= -\frac{1}{2\pi}\left[\frac{-2(1+2k)-2(1-2k)}{1-4k^2}\right] \\
&= -\frac{1}{2\pi}\cdot\frac{4}{4k^2-1}
\end{aligned}$$

Hence, we have: $x(t)=\sum_{k=-\infty}^{\infty}c_k e^{j2\pi kt}=-\frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{4k^2-1}e^{j2\pi kt}$

We have:

$$c_0 = a_0 = \frac{2}{\pi}$$

$$A_k = |c_k| = \frac{2}{\pi}\frac{1}{4k^2-1}$$

$$\angle c_k = \theta_k = -\pi$$

$$x(t) = a_0 + 2\sum A_k \cos\left(2\pi kt + \theta_k\right)$$

$$= \frac{2}{\pi} + 2\sum_{k=1}^{\infty}\frac{2}{\pi}\frac{1}{4k^2-1}\cos\left(2\pi kt - \pi\right)$$

$$= \frac{2}{\pi} - \sum_{k=1}^{\infty}\frac{4}{\pi}\frac{1}{4k^2-1}\cos\left(2\pi kt - \pi\right)$$

S2 Given $x(t) = t^2 \quad ; \quad -\pi < t < \pi$ and $x(t+2\pi) = x(t)$, we have a periodic signal with period of 2π.

$$c_0 = \frac{1}{2\pi}\int_{-\vartheta}^{\pi} t^2 dt = \frac{1}{2\pi}\left[\frac{1}{3}t^3\right]_{-\pi}^{\pi} = \frac{1}{6\pi}\left[\pi^3 + \pi^3\right] = \frac{\pi^2}{3}$$

$$c_k = \frac{1}{2\pi}\int_{=\pi}^{\pi} t^2 e^{-j2\pi\left(\frac{k}{2\pi}\right)t} dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-jkt} dt$$

Consider $\int t^2 e^{-jkt} dt$, we have:

$$\int t^2 e^{-jkt} dt = t^2\left(\frac{e^{-jkt}}{-jk}\right) - \int \frac{e^{-jkt}}{jk}.2t.dt$$

$$= \frac{jt^2}{k}e^{-jkt} - \frac{2j}{k}\int te^{-jkt}$$

$$= \frac{jt^2}{k}e^{-jkt} - \frac{2j}{k}\left[t\frac{e^{-jkt}}{-jk} - \int \frac{e^{-jkt}}{-jk}dt\right]$$

$$= \frac{jt^2}{k}e^{-jkt} - \frac{2j}{k}\left[-\frac{te^{-jkt}}{jk} + \frac{1}{jk}\frac{e^{-jkt}}{-jk}\right]$$

$$= \frac{jt^2}{k}e^{-jkt} + \frac{2}{k^2}te^{-jkt} - \frac{2j}{k^3}e^{-jkt}$$

Hence:

$$c_k = \frac{1}{2\pi}\left[\frac{jt^2}{k}e^{-jkt} + \frac{2}{k^2}te^{-jkt} - \frac{2j}{k^3}e^{-jkt}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{j\pi^2}{k}e^{-j\pi k} + \frac{2\pi}{k^2}e^{-j\pi k} - \frac{2j}{k^3}e^{-j\pi k} - \frac{j\pi^2}{k}e^{j\pi k} + \frac{2\pi}{k^2}e^{j\pi k} - \frac{2j}{k^3}e^{j\pi k}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{k^2}e^{-j\pi k} + \frac{2\pi}{k^2}e^{j\pi k}\right] = \frac{1}{2\pi}\left[\frac{4\pi}{k^2}(-1)^k\right] = \frac{2}{k^2}(-1)^k$$

Hence we have: $x(t) = \dfrac{\pi^2}{3} + \sum_k \dfrac{2}{k^2}(-1)^k e^{jkt}$

S3 $x(t) = c_0 + \sum c_k \cos\left(2\pi \frac{k}{T_0} t - \theta_k\right)$

Hence: $X_0 = c_0$ and $\left|X_k = \frac{1}{2} c_k\right|$ and $\angle X_k = \tan(-\theta_k) = -\tan(\theta_k)$