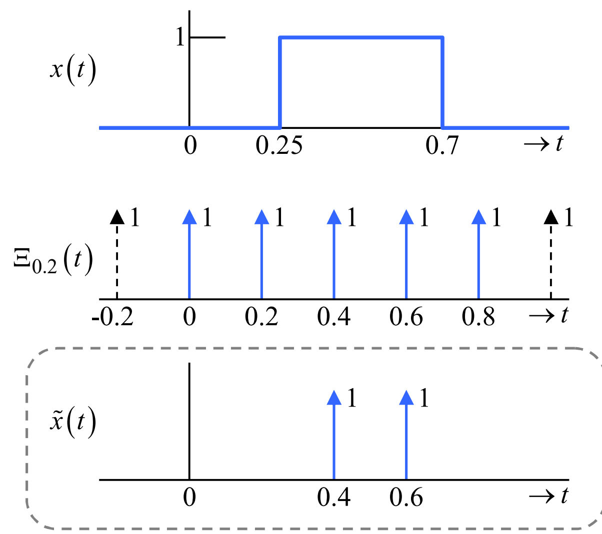
EE2023 TUTORIAL 4 (SOLUTIONS)

Solution to Q.1



$$\tilde{x}(t) = \text{rect}\left(\frac{t-0.475}{0.45}\right).\sum_{n=-\infty}^{\infty}\delta(t-0.2n)$$
$$= \delta(t-0.4)+\delta(t-0.6)$$

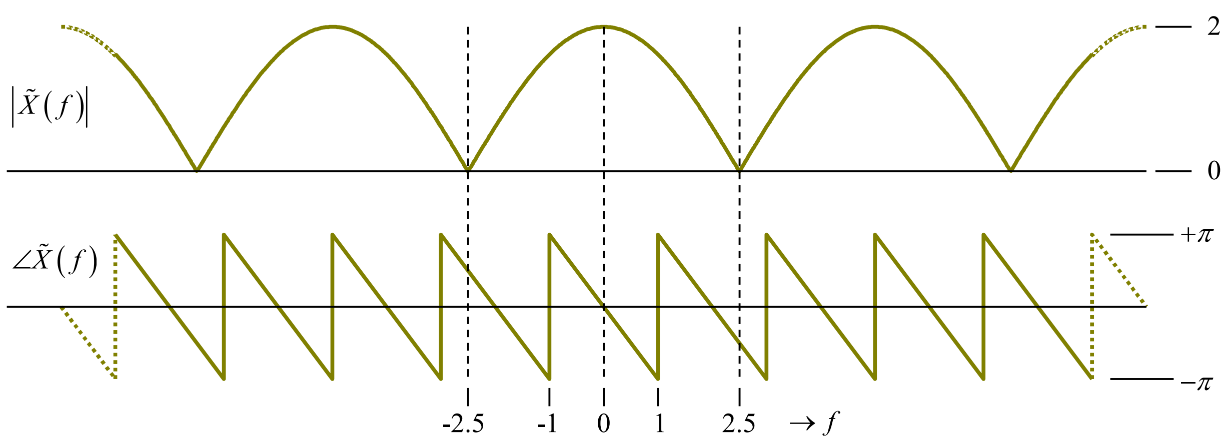
$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$

$$\tilde{X}(f) = \exp(-j2\pi f 0.4)+\exp(-j2\pi f 0.6)$$
$$= \begin{cases} \cos(0.8\pi f) - j\sin(0.8\pi f) + \\ \quad \cos(1.2\pi f) - j\sin(1.2\pi f) \end{cases}$$
$$= \begin{cases} \left[\cos(0.8\pi f)+\cos(1.2\pi f)\right] - \\ \quad j\left[\sin(0.8\pi f)+\sin(1.2\pi f)\right] \end{cases}$$
$$= 2\cos(\pi f)\cos(0.2\pi f) - j2\sin(\pi f)\cos(0.2\pi f)$$

$$\left|\tilde{X}(f)\right|^2 = 4\cos^2(\pi f)\cos^2(0.2\pi f)+4\sin^2(\pi f)\cos^2(0.2\pi f)$$
$$= 4\cos^2(0.2\pi f)\left[\cos^2(\pi f)+\sin^2(\pi f)\right]$$
$$= 4\cos^2(0.2\pi f)$$

$$\left|\tilde{X}(f)\right| = 2\left|\cos(0.2\pi f)\right|$$

$$\angle\tilde{X}(f) = \angle[2\cos(\pi f)\cos(0.2\pi f) - j2\sin(\pi f)\cos(0.2\pi f)]$$
$$= -\tan^{-1}\left(\frac{2\sin(\pi f)\cos(0.2\pi f)}{2\cos(\pi f)\cos(0.2\pi f)}\right) = -\tan^{-1}(\tan(\pi f)) = -\pi f$$

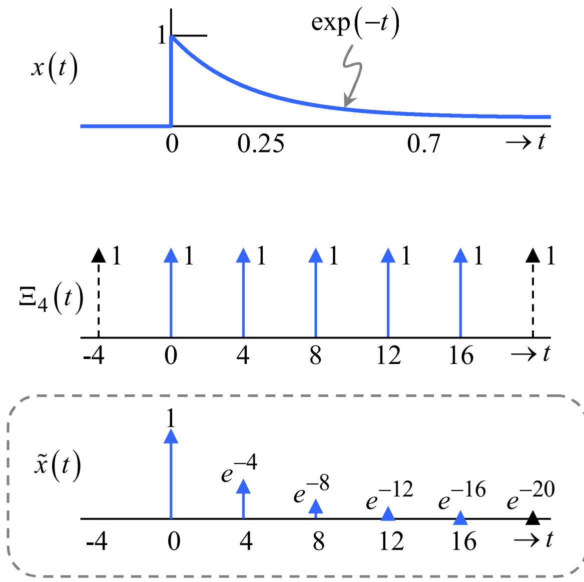


Solution to Q.2

$$x(t) = \exp(-t)u(t)$$

$$\tilde{x}(t) = \underbrace{\exp(-t)u(t)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-4n)}_{\Xi_4(t)}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\delta(t-4n)$$



$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$

$$\tilde{X}(f) = \sum_{n=0}^{\infty} \exp(-4n)\Im\{\delta(t-4n)\}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j2\pi f 4n)$$

$$= \underbrace{\sum_{n=0}^{\infty} \exp(-4n(j2\pi f + 1))}_{\text{Geometric Series}}$$

$$= \frac{1}{1-\exp(-4(j2\pi f+1))} = \frac{1}{1-\exp(-4)\exp(-j8\pi f)}$$

Using $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, where $-1 < r < 1$

When a signal is sampled in the time domain, the spectrum of the sampled signal is periodic with a period equal to the sampling frequency, which in this case is 0.25 Hz. Thus, to show that $\tilde{X}(f)$ is periodic, all we need to show is $\tilde{X}(f+0.25) = \tilde{X}(f)$ as follows:

$$\tilde{X}(f+0.25) = \frac{1}{1-\exp(-4)\exp(-j8\pi(f+0.25))}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)\underbrace{\exp(-j2\pi)}_{1}}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)} = \tilde{X}(f)$$

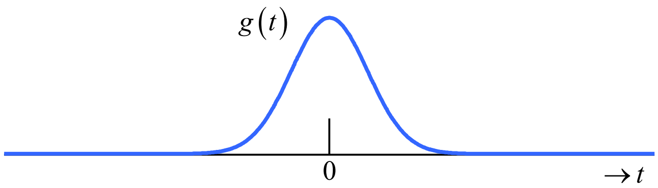
or:

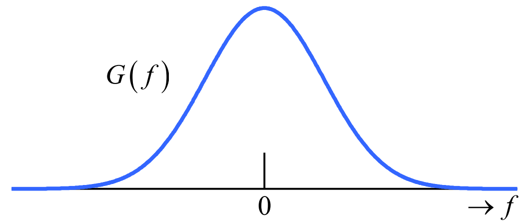
$$\tilde{X}(f+0.25) = \sum_{n=0}^{\infty} \exp(-4n)\exp\left[-j8\pi n(f+0.25)\right]$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j8\pi nf)\underbrace{\exp(-j2\pi n)}_{1}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j8\pi nf) = \tilde{X}(f)$$

Solution to Q.3





Rewrite $x(t)=\sum_{n=-\infty}^{\infty} g(t-nT_p)$ in convolution form:

$$x(t)=g(t)*\sum_{n=-\infty}^{\infty}\delta(t-nT_p) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (*)$$

Applying the 'Convolution' property of the Fourier transform to $(*)$:

$$X(f)=G(f)\cdot\frac{1}{T_p}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T_p}\right) \quad \cdots\cdots\cdots\cdots \quad (**)$$

Conclusion:

$X(f)$ can be obtained by sampling $G(f)/T_p$ in the frequency-domain at regular spacings of $1/T_p$ Hz.

Relationship between X_k and $G(f)$

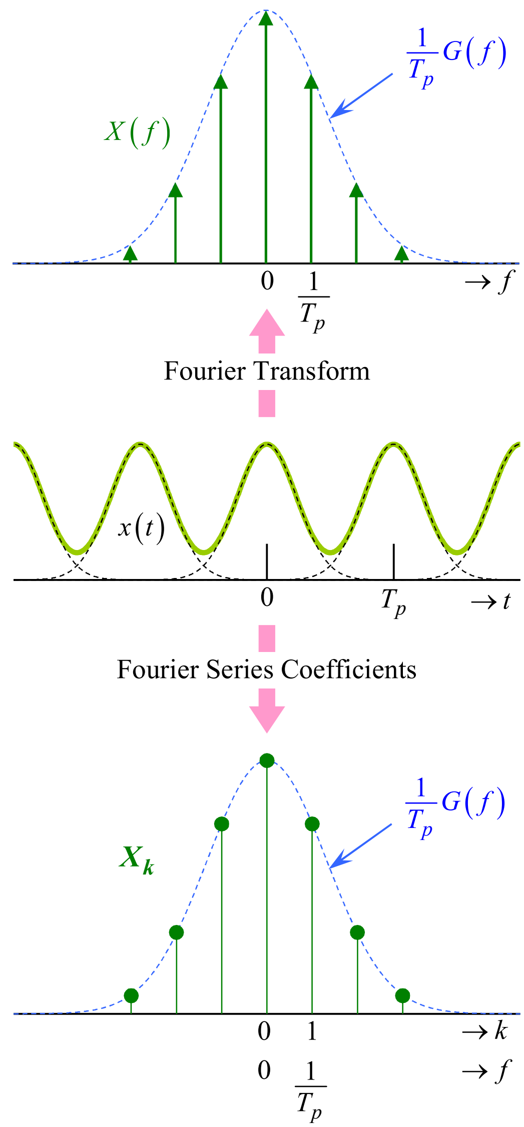
Rewrite $(**)$ as: $X(f)=\sum_{k=-\infty}^{\infty}\frac{1}{T_p}G\left(\frac{k}{T_p}\right)\delta\left(f-\frac{k}{T_p}\right)$

In terms of X_k: $\quad X(f)=\sum_{k=-\infty}^{\infty}X_k\delta\left(f-\frac{k}{T_p}\right)$

Hence, $X_k=\frac{1}{T_p}G(f)\Big|_{f=k/T_p}=\frac{1}{T_p}G\left(\frac{k}{T_p}\right)$

Uniqueness of a generating function

The generating function of a periodic signal is NOT unique. For instance, one period of a periodic signal can be used as its generating function.



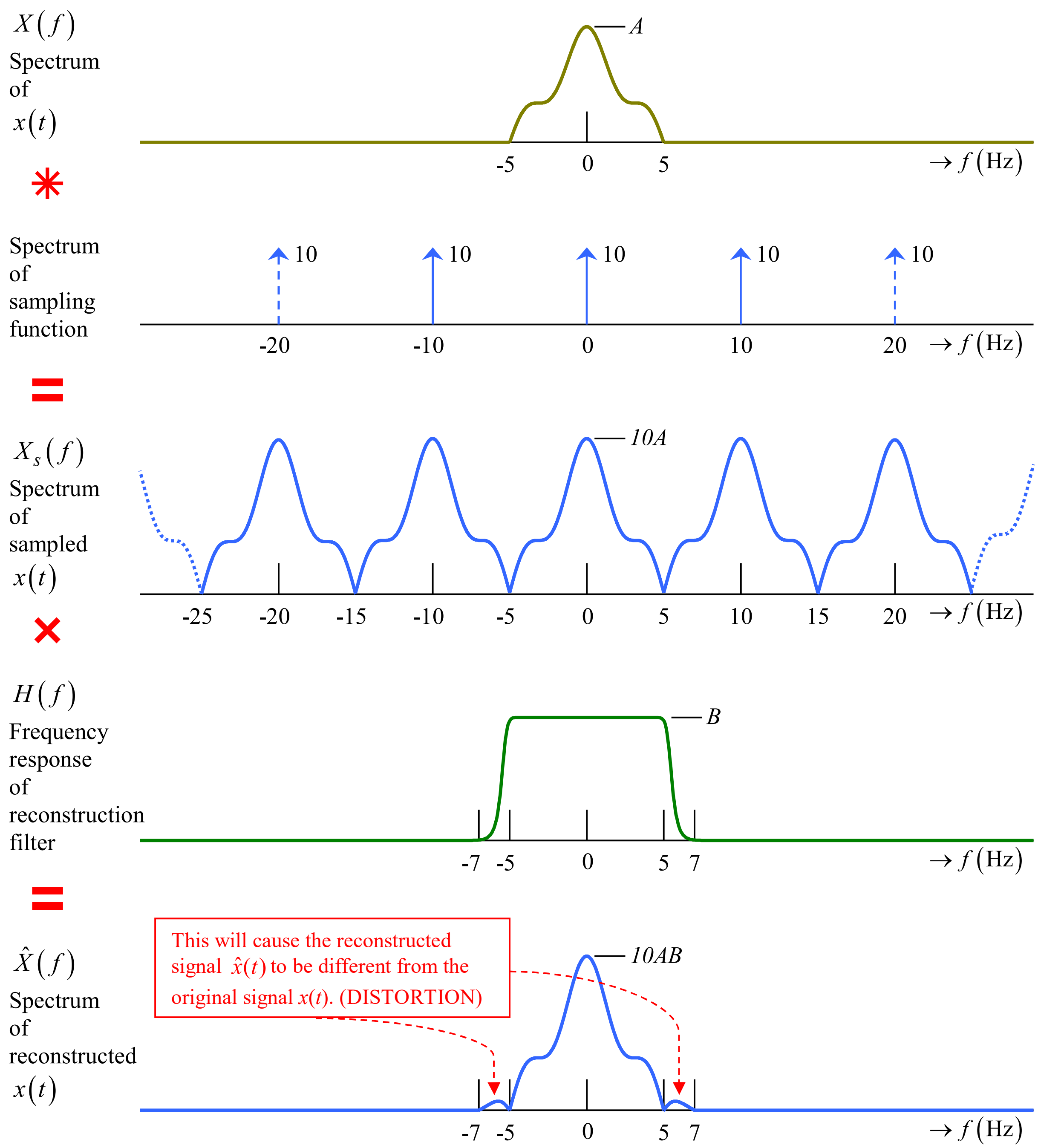
Remarks: From the sampling theorem, we observed that when a signal is sampled at regular time-intervals of T_s (sec), the spectrum corresponding to the sampled signal is periodic with a period of $f_s=1/T_s$ (Hz). In this exercise, we showed that when a spectrum is sampled at regular frequency-intervals of f_p (Hz), the signal corresponding to the sampled spectrum is periodic with a period of $T_p=1/f_p$ (sec). Hence, sampling a signal leads to a periodic spectrum and sampling a spectrum leads to periodic signal. This phenomenon is due to the convolution property of the Fourier transform and the fact that $\left[comb(t)\rightleftarrows COMB(f)\right]$.

Solution to Q.4

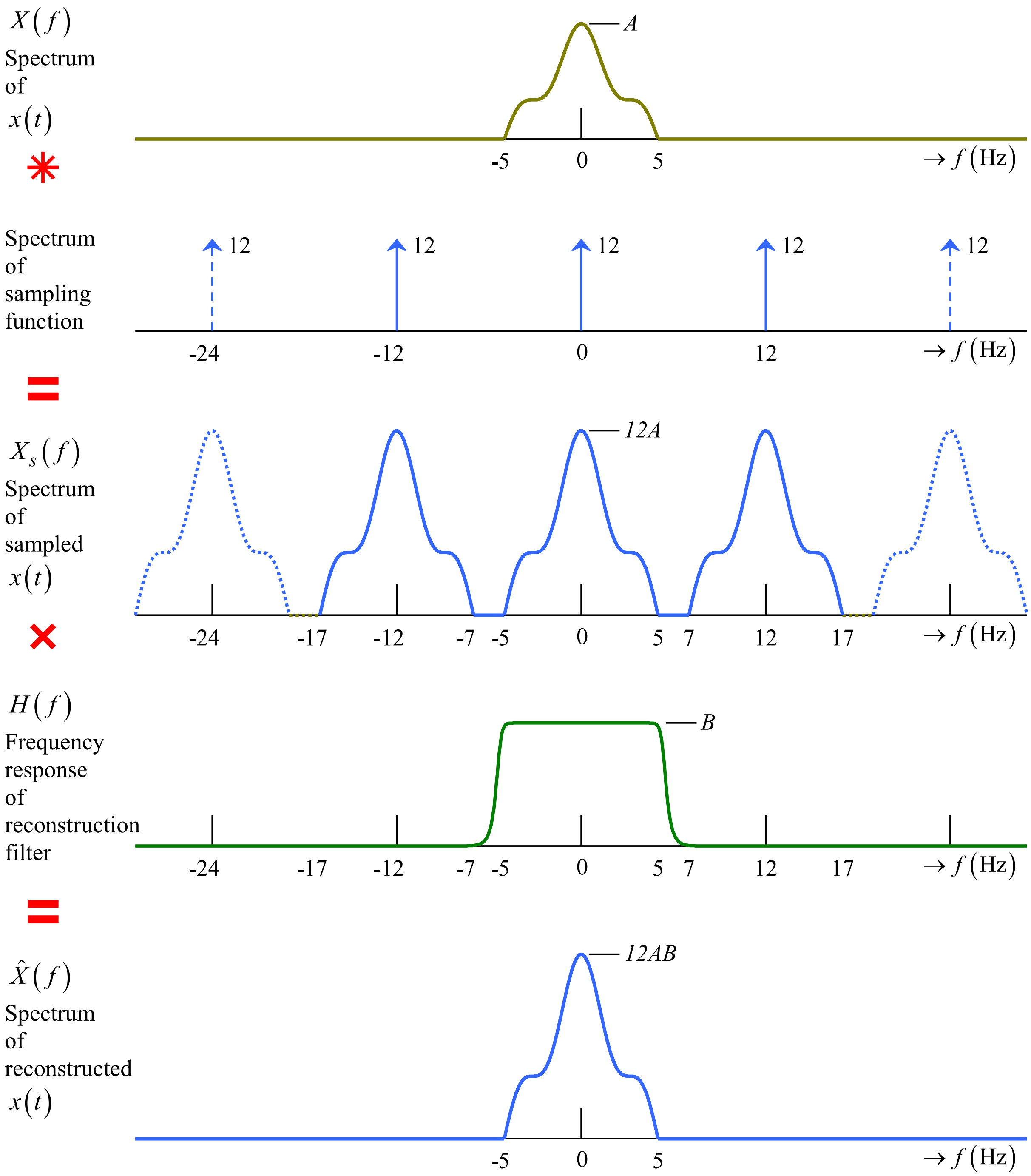
- Nyquist sampling frequency: $2 \times 5 = 10$ Hz
- Recommended sampling frequency: 12 Hz

  The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.

Sampling frequency: **10 Hz (Nyquist Rate)**



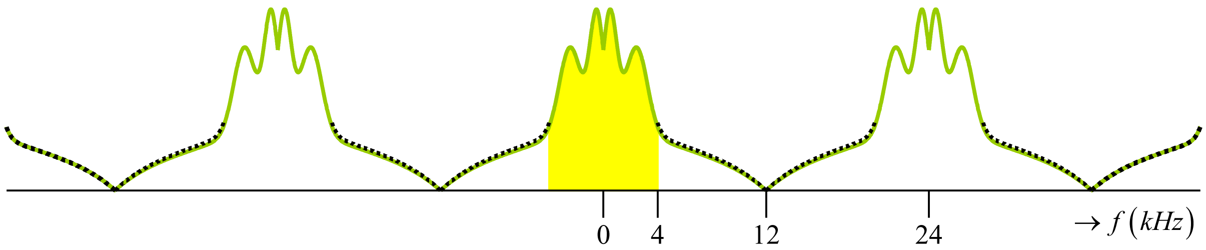
Sampling frequency: **12 Hz**



Remarks: *Sampling frequencies higher that 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.*

Solution to Q.5

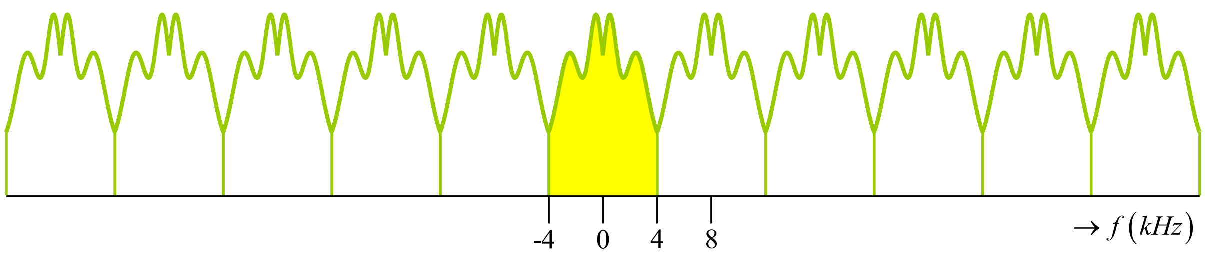
Situation A: $\left(\begin{array}{c}\textbf{\textit{No anti-aliasing lowpass filter}}\\ \textbf{\textit{and no frequency aliasing}}\end{array}\right)$: $\left\{\textbf{Sampling frequency} = \mathbf{2\times 12 = 24}\ \boldsymbol{kHz}\right.$



Advantage: No anti-aliasing LPF needed.

Disadvantage: Sampling frequency is higher than necessary to preserve the *TQ-Band*.

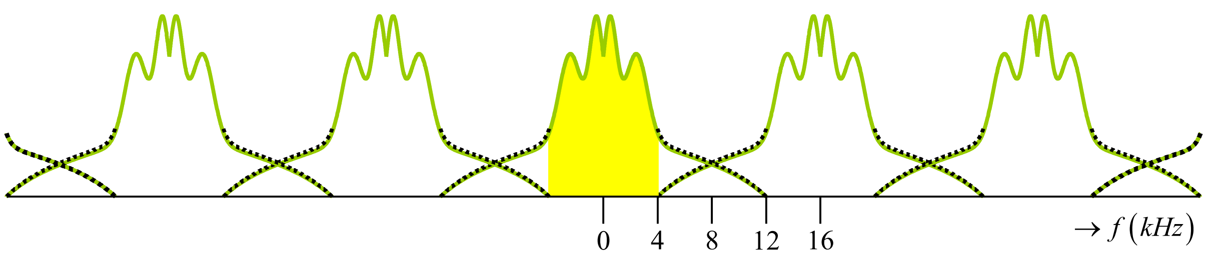
Situation B: ***Lowest sampling frequency***: $\left\{\begin{array}{l}\textbf{Ideal anti-aliasing filter of bandwidth 4}\ \boldsymbol{kHz}\\ \textbf{Sampling frequency} = \mathbf{2\times 4 = 8}\ \boldsymbol{kHz}\end{array}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band.*

Disadvantage: Require anti-aliasing LPF with sharp cutoff.

Situation C: ***No anti-aliasing lowpass filter***: $\left\{\textbf{Sampling frequency} = \mathbf{2}\times\left(\dfrac{\mathbf{12+4}}{\mathbf{2}}\right) = \mathbf{16}\ \boldsymbol{kHz}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

Disadvantage: Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.

Supplementary Questions (Solutions)

S1 Given: f_{max} = 20 kHz

Nyquist rate = 2 x f_{max} = 40 kHz

Note that the sampled signal is:

$$x_s(t) = x(t).\sum_{k=-\infty}^{\infty} \delta\left(t - \frac{k}{40000}\right) \quad ; \quad X_s f) = X(f) \otimes 40000 \sum_{k=-\infty}^{\infty} \delta\left(f - k40000\right)$$

Hence the spectrum of the sampled signal comprises repetitions of $X(f)$ at multiples of the sampling frequency of 40 KHz.

To recover the original signal we will need an ideal low pass filter of bandwidth 20 kHz, which is described by $\text{rect}\left(\frac{f}{40000}\right)$ and we also need to take care of the magnitude of 40000 in $X_s(f)$. Hence the reconstruction filter is:

$$H(f) = \frac{1}{40000}\text{rect}\left(\frac{f}{40000}\right)$$

S2 Given: $x_s(t) = \sum_{n=-5}^{5} x(5n)\delta\left(t - 5n\right)$

(a) Sampling period, $T_s = 5$; $f_s = 1/5 = 0.2$

(b) Given : $x(t) = \text{tri}(t)$
Since T_s is 5 seconds, then the sampling frequency is too low to be able to recover $x(t)$

S3 The maximum frequency, f_{max} = 20.5 Hz.
Hence the Nyquist frequency = 2 x f_{max} = 2 x 20.5 = 41 Hz.

The sampled signal and its spectrum are:

$$x_s(t) = x(t).\sum_{k=-\infty}^{\infty} \delta\left(t - kT_s\right) \quad ; \quad X_s f) = X(f) \otimes \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right)\right] = X(f) \otimes \left[f_s \sum_{k=-\infty}^{\infty} \delta\left(f - kf_s\right)\right]$$

$\therefore$ $X(f)$ is replicated at multiples of the sampling frequency, f_s. As long as the replicated $X(f)$ do not overlap with each other, then the original signal can be recovered with a suitable bandpass filter.

In this case, the minimum sampling frequency is 7 Hz as replication of $X(f)$ at multiples of 7 Hz will not result in overlaps and thus allow for recovery of $X(f)$ from $X_s(f)$.
