NUS CONFIDENTIAL

NATIONAL UNIVERSITY OF SINGAPORE

EE2023/EE2023E/TEE2023 – SIGNALS AND SYSTEMS

(Semester II : 2017/2018)

Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. Do not write your name.

2. Students should write the answers for each question on a new page.

3. This paper contains **EIGHT (8)** questions and comprises **ELEVEN (11)** printed pages.

4. Answer **ALL** questions in **Section A** and **ANY THREE (3)** questions in **Section B**.

5. This is a **CLOSED BOOK** examination. However you are allowed to bring one self-prepared handwritten A4-size crib sheet to the examination hall.

6. Programmable and/or graphic calculators are not allowed.

7. Tables of formulas are provided on Pages 8 to 11.

SECTION A : Answer ALL questions in this section

Q.1 Given the periodic signal $x(t) = 1 + 4\cos(0.9\pi t)\cos(0.3\pi t) - 2\sin(1.8\pi t - \frac{\pi}{6})$.

(a) Determine the fundamental frequency of $x(t)$.

(2 marks)

(b) Find the Fourier series coefficients c_k and Fourier transform $X(f)$ of $x(t)$.

(6 Marks)

(c) Find the average power of $x(t)$.

(2 marks)

Q2. Consider a signal $x(t) = \begin{cases} 0.5 + \cos(\pi t), & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$.

(a) Find the Fourier transform $X(f)$ of $x(t)$.

(6 marks)

(b) Indicate the first-null bandwidth of $x(t)$.

(2 marks)

(c) Find the energy of $x(t)$.

(2 marks)

Q3. The second order differential equation that describes a temperature measurement system is:

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = x(t)$$

where $y(t)$ is the measured temperature and $x(t)$ is the input stimulus.

(a) Derive the transfer function of this temperature measurement system. Is the system stable? Justify your answer.

(4 marks)

(b) Find the steady state measured temperature given an input stimulus of $x(t) = 4u(t)$, where $u(t)$ is the unit step function, and the initial conditions are $y(0) = 0$ and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$.

(3 marks)

(c) Provide an alternative method to find the steady state measured temperature of the same input conditions in part (b) above.

(3 marks)

Q4. A motorized system has a transfer function of

$$G(s) = \frac{\alpha}{3s^2 + 4s + \beta}$$

where α and β are system design parameters.

(a) If the undamped natural frequency is 2 rad/s, find β.

(2 marks)

(b) Find the range of values of β for which the system is underdamped.

(2 marks)

(c) Determine the values of α and ς, the damping ratio, for an underdamped system with a system gain of 5 when the input is a unit step function. Sketch the output response of this system.

(6 marks)

SECTION B : Answer 3 out of the 4 questions in this section

Q.5 The Dirac-δ function usually cannot be implemented in a practical circuit. Instead of sampling a signal $x(t)$ with a comb function, natural sampling gets the samples of $x(t)$ by multiplying $x(t)$ with signal $y(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t-nT_s}{T_0}\right)$. That is, the obtained sampled signal $x_s(t) = x(t) \times y(t) = x(t) \times \sum_{n=-\infty}^{\infty} \text{rect}(\frac{t-nT_s}{T_0})$ where T_s and T_0 are both positive constants and T_s is much greater than T_0. Assume $x(t)$ is a real baseband signal with bandwidth B Hz and $X(f)$ is its spectrum. The sampling frequency $f_s = \frac{1}{T_s}$.

(a) Find the spectrum $Y(f)$ of $y(t)$.

(5 marks)

(b) Find the spectrum $X_s(f)$ of $x_s(t)$ in terms of $X(f)$.

(4 marks)

(c) Find the minimal f_s such that $x(t)$ can be reconstructed from $x_s(t)$.

(5 marks)

(d) If minimal value of f_s is applied, indicate the ideal reconstruction filter gain and minimal cut-off frequency such that $x(t)$ is reconstructed from $x_s(t)$.

(6 marks)

Q6. The Fourier transform $X(f)$ of real baseband signal $x(t)$ is shown in Figure Q6.1. The single sideband (SSB) signal $y(t)$ of $x(t)$ is obtained through the processing shown in Figure Q6.2.

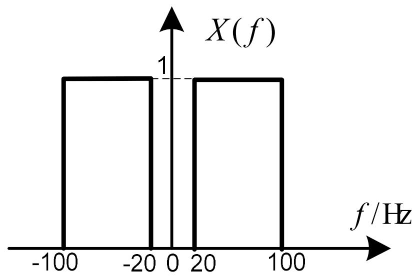

Figure Q6.1

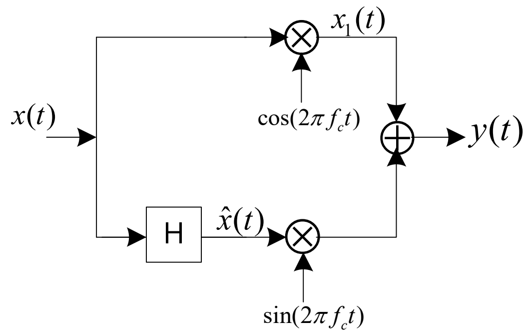

Figure Q6.2

In Figure Q6.2, $f_c = 2000$ Hz and the block H is an LTI system which has frequency response

$$H(f) = \begin{cases} j, & f < 0 \\ 0, & f = 0 \\ -j, & f > 0 \end{cases} \quad \text{where } j = \sqrt{-1} \ .$$

(a) Find the time domain expression of $x(t)$ and the time domain expression of signal $\hat{x}(t)$ in Figure Q6.2.

(8 marks)

(b) Determine the spectrum $Y(f)$ of the SSB signal $y(t)$ and indicate its bandwidth.

(8 marks)

(c) How can $x(t)$ be recovered from the SSB signal $y(t)$?

(4 marks)

Q7. Figure Q7 shows the Bode magnitude plot of $G(s)=\dfrac{Y(s)}{X(s)}$, where $X(s)$ and $Y(s)$ are the Laplace transforms of the input signal $x(t)$ and output signal $y(t)$, respectively.

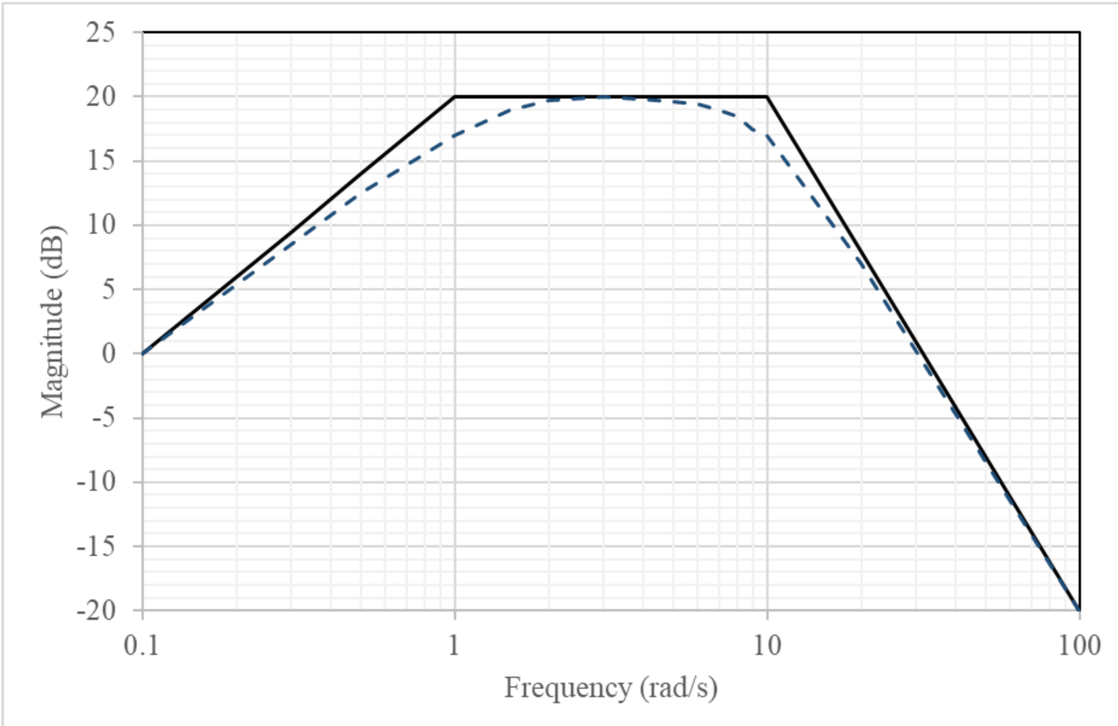

Figure Q7

(a) Determine the transfer function $G(s)$.

(6 marks)

(b) Sketch the Bode phase plot of $G(s)$.

(4 marks)

(c) If the system encounters a delay of 50 ms, determine the phase of the system at a frequency of 300 rad/s.

(4 marks)

(d) When the input signal is $x(t)=10\sin\left(200t+\dfrac{\pi}{8}\right)$, determine the steady state output signal.

(6 marks)

Q8. The unit step response of a second order system, $G(s)$, is shown in Figure Q8.

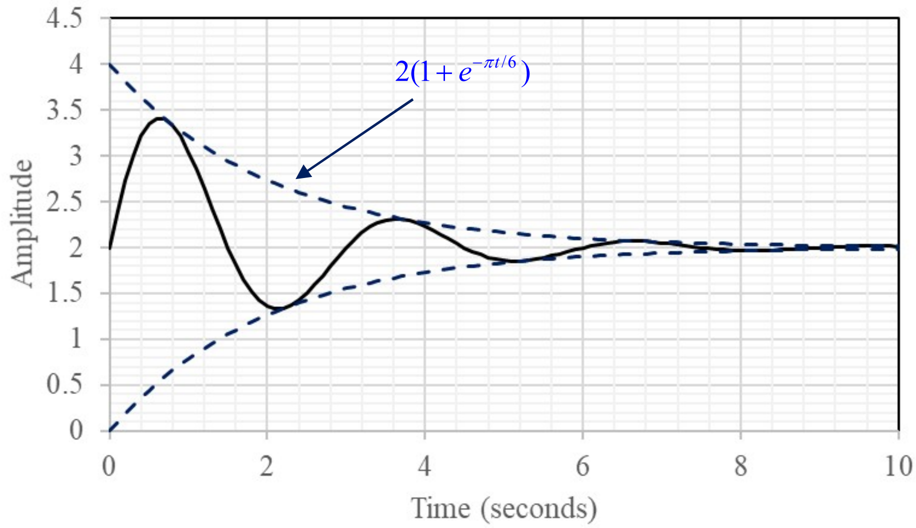

Figure Q8

(a) Determine the transfer function $G(s)$.

(10 marks)

(b) Determine the poles of $G(s)$.

(3 marks)

(c) Determine the expression for the output $y(t)$ given that the input is a unit step signal.

(5 marks)

(d) If the system is to be modified into a critically damped system with the same undamped natural frequency, ω_n, what will be the poles of this critically damped system.

(2 marks)

END OF QUESTIONS

Fourier Series: $$\begin{cases} X_k = \dfrac{1}{T}\displaystyle\int_{\tilde{t}}^{\tilde{t}+T} x(t)\exp(-j2\pi k\,t/T)\,dt \\ x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(j2\pi k\,t/T) \end{cases}$$

Fourier Transform: $$\begin{cases} X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \end{cases}$$

FOURIER TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\operatorname{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\operatorname{rect}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}(fT)$
Triangle	$\operatorname{tri}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}^2(fT)$
Sine Cardinal	$\operatorname{sinc}\left(\frac{t}{T}\right)$	$T\operatorname{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp\left(-\alpha^2\pi^2 f^2\right)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

FOURIER TRANSFORM PROPERTIES

	Time-domain	**Frequency-domain**
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(f)+\beta X_2(f)$
Time scaling	$x(\beta t)$	$\frac{1}{\lvert\beta\rvert}X\left(\frac{f}{\beta}\right)$
Duality	$X(t)$	$x(-f)$
Time shifting	$x(t-t_o)$	$X(f)\exp(-j2\pi f t_o)$
Frequency shifting (Modulation)	$x(t)\exp(j2\pi f_o t)$	$X(f-f_o)$
Differentiation in the time-domain	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Multiplication in the time-domain	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty}X_1(\zeta)X_2(f-\zeta)\,d\zeta$ or $X_1(f)\ast X_2(f)$
Convolution in the time-domain	$\int_{-\infty}^{\infty}x_1(\zeta)x_2(t-\zeta)\,d\zeta$ or $x_1(t)\ast x_2(t)$	$X_1(f)X_2(f)$
Integration in the time-domain	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
		$\frac{1}{j2\pi f}X(f)$ **if** $X(0)=0$

Unilateral Laplace Transform: $X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)\,dt$

LAPLACE TRANSFORMS OF BASIC FUNCTIONS

	$x(t)$	$X(s)$
Unit Impulse	$\delta(t)$	**1**
Unit Step	$u(t)$	$1/s$
Ramp	$t\,u(t)$	$1/s^2$
n[th] order Ramp	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
Damped Ramp	$t\exp(-\alpha t)u(t)$	$1/(s+\alpha)^2$
Exponential	$\exp(-\alpha t)u(t)$	$1/(s+\alpha)$
Cosine	$\cos(\omega_o t)u(t)$	$s/\left(s^2+\omega_o^2\right)$
Sine	$\sin(\omega_o t)u(t)$	$\omega_o/\left(s^2+\omega_o^2\right)$
Damped Cosine	$\exp(-\alpha t)\cos(\omega_o t)u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$
Damped Sine	$\exp(-\alpha t)\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+\alpha)^2+\omega_o^2}$

LAPLACE TRANSFORM PROPERTIES

	Time-domain	*s*-domain
Linearity	$\alpha x_1(t)+\beta x_2(t)$	$\alpha X_1(s)+\beta X_2(s)$
Time shifting	$x(t-t_o)u(t-t_o)$	$\exp(-st_o)X(s)$
Shifting in the s-domain	$\exp(s_o t)x(t)$	$X(s-s_o)$
Time scaling	$x(\alpha t)$	$\dfrac{1}{\lvert\alpha\rvert}X\left(\dfrac{s}{\alpha}\right)$
Integration in the time-domain	$\int_{0^-}^{t} x(\zeta)\,d\zeta$	$\dfrac{1}{s}X(s)$
Differentiation in the time-domain	$\dfrac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\dfrac{d^n x(t)}{dt^n}$	$s^n X(s)-\sum_{k=0}^{n-1} s^{n-1-k}\left.\dfrac{d^k x(t)}{dt^k}\right\rvert_{t=0^-}$
Differentiation in the s-domain	$-tx(t)$	$\dfrac{dX(s)}{ds}$
	$(-t)^n x(t)$	$\dfrac{d^n X(s)}{ds^n}$
Convolution in the time-domain	$\int_{-\infty}^{\infty} x_1(\zeta)x_2(t-\zeta)\,d\zeta$	$X_1(s)X_2(s)$
Initial value theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$	
Final value theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

System Type	Transfer Function (Standard Form)	Unit Impulse and Unit Step Responses	Remarks
1st order system	$G(s)=\frac{K}{T}\cdot\frac{1}{s+1/T}$	$y_\delta(t)=\frac{K}{T}\exp\left(-\frac{t}{T}\right)u(t)$ $y_{step}(t)=K\left[1-\exp\left(-\frac{t}{T}\right)\right]u(t)$	T : Time-constant K : DC Gain Real Pole at $s=-\frac{1}{T}$
2nd order system $(\zeta>1)$ **Overdamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K_1}{s+p_1}+\frac{K_2}{s+p_2}$	$y_\delta(t)=\left[K_1\exp(-p_1t)+K_2\exp(-p_2t)\right]u(t)$ $y_{step}(t)=\left[K-\frac{K_1}{p_1}\exp(-p_1t)-\frac{K_2}{p_2}\exp(-p_2t)\right]u(t)$	K : DC Gain $\left.\begin{array}{l}p_1=\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\\ p_2=\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\end{array}\right\\}\ K_1=-K_2=\frac{K\omega_n^2}{p_2-p_1}$ Real Distinct Poles at $s=-p_1$ and $s=-p_2$
2nd order system $(\zeta=1)$ **Critically damped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{(s+\omega_n)^2}$	$y_\delta(t)=K\omega_n^2 t\exp(-\omega_n t)u(t)$ $y_{step}(t)=K\left[1-\exp(-\omega_n t)-\omega_n t\exp(-\omega_n t)\right]u(t)$	K : DC Gain Real Repeated Poles at $s=-\omega_n$
2nd order system $(0<\zeta<1)$ **Underdamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K(\sigma^2+\omega_d^2)}{(s+\sigma)^2+\omega_d^2}$	$y_\delta(t)=K\frac{\omega_n^2}{\omega_d}\exp(-\sigma t)\sin(\omega_d t)u(t)$ $y_{step}(t)=K\left[1-\frac{\omega_n}{\omega_d}\exp(-\sigma t)\sin(\omega_d t+\phi)\right]u(t)$	K : DC Gain ω_n : Undamped Natural Frequency ζ : Damping Ratio ω_d : Damped Natural Frequency $\sigma=\zeta\omega_n$ $\quad\omega_d^2=\omega_n^2(1-\zeta^2)$ $\quad\omega_n^2=\sigma^2+\omega_d^2$ $\quad\tan(\phi)=\frac{\omega_d}{\sigma}$ Complex Conjugate Poles at $s=-\sigma\pm j\omega_d$
2nd order system $(\zeta=0)$ **Undamped**	$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $=\frac{K\omega_n^2}{s^2+\omega_n^2}$	$y_\delta(t)=K\omega_n\sin(\omega_n t)u(t)$ $y_{step}(t)=K(1-\cos\omega_n t)u(t)$	K : DC Gain ω_n : Undamped Natural Frequency Imaginary Conjugate Poles at $s=\pm j\omega_n$

2nd order system RESONANCE $\left(0 \le \zeta < 1/\sqrt{2}\right)$ ➔ *RESONANCE FREQUENCY :* $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ *RESONANCE PEAK :* $M_r = \left|G(j\omega_r)\right| = \dfrac{K}{2\zeta\sqrt{1-\zeta^2}}$

Trigonometric Identities

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\theta) = 0.5\left(e^{j\theta} + e^{-j\theta}\right)$	$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
$\sin(\theta) = -0.5j\left(e^{j\theta} - e^{-j\theta}\right)$	$\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$
$\sin^2(\theta) + \cos^2(\theta) = 1$	
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	$\sin(\alpha)\sin(\beta) = 0.5\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$	$\cos(\alpha)\cos(\beta) = 0.5\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
$\sin^2(\theta) = 0.5\left[1 - \cos(2\theta)\right]$	$\sin(\alpha)\cos(\beta) = 0.5\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$
$\cos^2(\theta) = 0.5\left[1 + \cos(2\theta)\right]$	$C\cos(\theta) - S\sin(\theta) = \sqrt{C^2+S^2}\cos\left[\theta + \tan^{-1}\left(\dfrac{S}{C}\right)\right]$

Complex Unit (j) ➔ $\left(j = \sqrt{-1} = e^{j\pi/2} = e^{j90^\circ}\right)$ $\left(-j = \dfrac{1}{j} = e^{-j\pi/2} = e^{-j90^\circ}\right)$ $\left(j^2 = -1\right)$

Definitions of Basic Functions

Rectangle:

$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1; & -T/2 \le t < T/2 \\ 0; & \text{elsewhere} \end{cases}$$

Triangle:

$$\text{tri}\left(\frac{t}{T}\right) = \begin{cases} 1 - |t|/T; & |t| \le T \\ 0; & |t| > T \end{cases}$$

Sine Cardinal:

$$\text{sinc}\left(\frac{t}{T}\right) = \begin{cases} \dfrac{\sin(\pi t/T)}{\pi t/T}; & t \ne 0 \\ 1; & t = 0 \end{cases}$$

Signum:

$$\text{sgn}(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$

Unit Impulse:

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \ne 0 \end{cases} \qquad \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

Unit Step:

$$u(t) = \begin{cases} 1; & t \ge 0 \\ 0; & t < 0 \end{cases}$$