National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems
Tutorial 7 Solutions

Section I

1. (a) By definition, unit step response is the output of the system when the input is the unit step function. Since the system transfer function is $G(s)=\dfrac{Y(s)}{X(s)}=\dfrac{1}{\tau s+1}$ and Laplace transform of the input signal is $X(s)=1/s$,

$$Y_{step}(s)=\frac{1}{s(\tau s+1)}$$

Let: $Y(s)=\dfrac{A_1}{s}+\dfrac{A_2}{\tau s+1}=\dfrac{A_1(\tau s+1)+A_2 s}{s(\tau s+1)}=\dfrac{s(\tau A_1+A_2)+A_1}{s(\tau s+1)}$

Comparing the numerators:

Let $s=0$ on both sides, we have $\Rightarrow\ A_1=1$

Let $s=-1/\tau$ on both sides, we have $1=A_2(-1/\tau)\ \Rightarrow\ A_2=-\tau$

Hence, $Y(s)=\dfrac{1}{s}-\dfrac{\tau}{\tau s+1}=\dfrac{1}{s}-\dfrac{1}{s+1/\tau}$

$$\therefore\quad y_{step}(t)=1-e^{-\frac{t}{\tau}}$$

(b) Unit impulse response is the output of the system when the input is the unit impulse function, i.e. $x(t)=\delta(t)$ and $X(s)=1$.

$$Y_{impulse}(s)=\frac{1}{\tau s+1}$$

$$y_{impulse}(t)=\frac{1}{\tau}e^{-t/\tau}$$

(c) Differentiate the step response and integrate the impulse response.

$$\int_0^t\frac{1}{\tau}e^{-\gamma/\tau}d\gamma=\frac{1}{\tau}\int_0^t e^{-\gamma/\tau}d\gamma=\frac{1}{\tau}\left[-\frac{1}{1/\tau}e^{-\gamma/\tau}\right]_0^t=1-e^{-t/\tau}$$

$$\frac{d}{dt}\left[y_{step}(t)\right]=\frac{d}{dt}\left[1-e^{-t/\tau}\right]=\frac{1}{\tau}e^{-t/\tau}$$

(d) Since $y_{step}(t) = 1 - e^{-t/\tau}$, then $y_{step}(t) = 1 - e^{-1} = 0.632$, i.e. when $t/\tau = 1$.

When $\tau = 1$, then $t = 1$.
When $\tau = 2$, then $t = 2$.
When $\tau = -1$, then $y_{step}(t)$ never reaches a final value.

Step responses when $\tau = 1$, 2 and when $\tau = -1$ are shown in Figure 1.

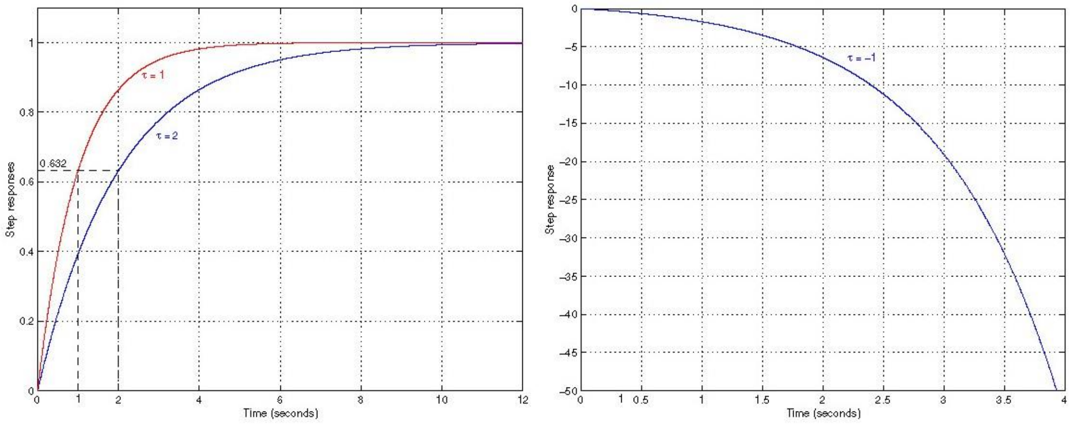

Figure 1: Step responses when $\tau = 1, 2$ and -1

Figure 1 and the mathematical expressions for the step responses indicate that

- System is stable if the system poles lie in the LHP.
- Step response reaches steady-state relatively more quickly if real part of the system pole is more negative.

2. Information provided is the steady-state gain, the damping ratio and the undamped natural frequency of a second order system and requires the convolution integral. Steps involved are:

- Construct the second order transfer function, $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$.
- Derive the impulse response, $\mathcal{L}^{-1}\{G(s)\}$ and substitute into the convolution integral.

Given $K = 0.75$; $\zeta = 0.6$; $\omega_n = 2$

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n + \omega_n^2} = \frac{(0.75)(2^2)}{s^2 + 2(0.6)(2)s + 2^2}$$

$$= \frac{3}{s^2 + 2.4s + 4}$$

$$= \frac{3}{(s+1.2)^2 + 4 - 1.44}$$

$$= \frac{3}{(s+1.2)^2 + 2.56}$$

$$= \frac{3}{(s+1.2)^2 + 1.6^2}$$

$$= \frac{15}{8}\frac{1.6}{(s+1.2)^2 + 1.6^2}$$

$$\therefore \quad g(t) = \frac{15}{8}e^{-1.2t}\sin(1.6t)$$

$$\therefore \quad y(t) = g(t) \otimes f(t)$$

$$= \int_0^t g(\tau) f(t-\tau) d\tau$$

$$= \int_0^t \frac{15}{8} e^{-1.2\tau} \sin(1.6\tau) f(t-\tau) d\tau$$

$$= \int_0^t \frac{15}{8} e^{-1.2(t-\tau)} \sin\left[1.6(t-\tau)\right] f(\tau) d\tau$$

Alternatively, we can also have:

$$y(t) = g(t) \otimes f(t)$$

$$= \int_0^t g(t-\tau) f(\tau) d\tau$$

$$= \int_0^t \frac{15}{8} e^{-1.2(t-\tau)} \sin\left[1.6(t-\tau)\right] f(\tau) d\tau$$

Section II

1. (a) As verified in Section I Q1(c), the relationship between impulse response and step response is

$$\int_0^t y_{impulse}(\gamma)d\gamma = y_{step}(t)$$

Hence, $y_{step}(t)$ may be obtained by performing graphical integration, i.e. summing the area under the impulse responses from 0 to t. The result is shown in Figure 2. Alternatively, from the information that the system has first order dynamics, plots (i), (ii) and (iv) should be decaying or growing exponential functions. Perform numerical integration to derive the step responses.

(i) $y_{impulse}(t) = e^{-t/\tau}$; From Figure 1, $0.37 = e^{-1/\tau} \Rightarrow \tau = 1$

$$y_{step}(t) = \int_0^t y_{impulse}(\gamma)d\gamma = \int_0^t e^{-\gamma}d\gamma = 1 - e^{-t}$$

(ii) $y_{impulse}(t) = e^{-t/\tau}$; From Figure 1, $0.61 = e^{-1/\tau} \Rightarrow \tau = 2$

$$y_{step}(t) = \int_0^t y_{impulse}(\gamma)d\gamma = \int_0^t e^{-\gamma/2}d\gamma = 2(1 - e^{-t/2})$$

(iii) $y_{impulse}(t) = 1$

$$y_{step}(t) = \int_0^t y_{impulse}(\gamma)d\gamma = \int_0^t 1.d\gamma = t$$

(iv) $y_{impulse}(t) = e^{t/\tau}$; From Figure 1, $1.2 = e^{2/\tau} \Rightarrow \tau = 11$

$$y_{step}(t) = \int_0^t y_{impulse}(\gamma)d\gamma = \int_0^t e^{\gamma/11}d\gamma = 11(e^{t/11} - 1)$$

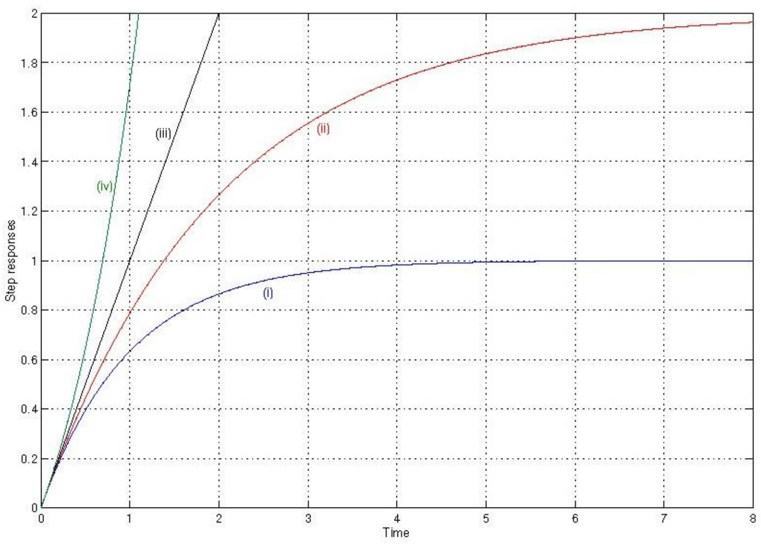

Figure 2: Step responses of processes (i) – (iv)

(b) System poles are the roots of the transfer function $\frac{K}{s\tau+1}$, i.e. $s=-\frac{1}{\tau}$. The required transfer function may be inferred by taking the Laplace transform of the impulse responses : (i) Ae^{-at}; (ii) Be^{-bt} , $|b| < |a|$; (iii) $Cu(t)$; (iv) De^{dt}. The solution is shown in Figure 3.

(i) $y_{impulse}(t)=e^{-t}$; $Y(s)=\frac{1}{s+1}$; System pole is $s=-1$

(ii) $y_{impulse}(t)=e^{-t/2}$; $Y(s)=\frac{1}{s+1/2}$; System pole is $s=-1/2$

(iii) $y_{impulse}(t)=1$; $Y(s)=\frac{1}{s}$; System pole is $s=0$

(iv) $y_{impulse}(t)=e^{t/11}$; $Y(s)=\frac{1}{s-1/11}$; System pole is $s=1/11$

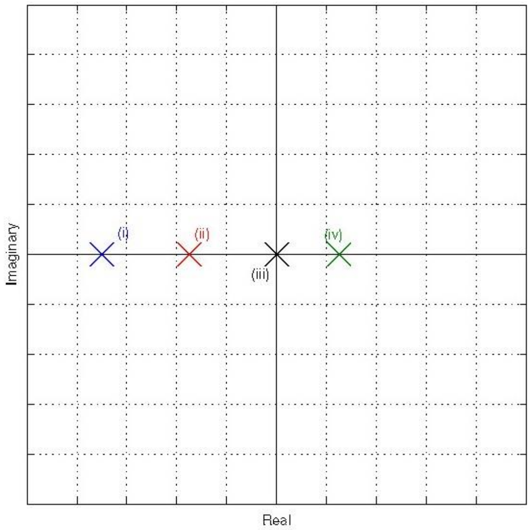

Figure 3: Pole location of the processes (i) – (iv)

2. Objective is to determine parameters of the transfer function, $G_i(s)=\frac{A}{as^2+bs+c}e^{-sL}$, ($i$ = 1, 2, 3, 4), using information about the unit impulse responses and unit step responses. Concept needed to formulate solution are:

- G(s) = Laplace Transform of the impulse response
- $\frac{G(s)}{s}$ = Laplace Transform of the step response

Hence, procedure for solving problem is to write a mathematical equation representing the impulse/step response, perform Laplace Transform and then compare the result with $G(s)$.

(a) Impulse response, $y_{impulse}(t) = 1.5u(t-1)$. So:

$$Y_{impulse}(s) = \frac{1.5}{s}e^{-s} \quad \Rightarrow \quad G(s) = Y_{impulse}(s) = \frac{1.5}{s}e^{-s} = \frac{A}{as^2+bs+c}e^{-sL}.$$

Hence, $A = 1.5$, $a = 0$, $b = 1$, $c = 0$, $L = 1$.

(b) Same method as part (a). In this case, unit impulse response, $y_{impulse}(t) = 4(t-0.5)u(t-0.5)$. So:

$$Y_{impulse}(s) = \frac{4}{s^2}e^{-0.5s} \quad \Rightarrow \quad G(s) = Y_{impulse}(s) = \frac{4}{s^2}e^{-0.5s} = \frac{A}{as^2+bs+c}e^{-sL}$$

Hence, $A = 4$, $a = 1$, $b = 0$, $c = 0$, $L = 0.5$.

(c) Step response, $y_{step}(t) = 4(t-0.5)u(t-0.5)$. So:

$$Y_{step}(s) = \frac{4}{s^2}e^{-0.5s} \quad \Rightarrow \quad G(s) = sY_{step}(s) = \frac{4}{s}e^{-0.5s} = \frac{A}{(as^2+bs+c)}e^{-sL}$$

Hence, $A = 4$, $a = 0$, $b = 1$, $c = 0$, $L = 0.5$.

(d) As the step response provided is a curve, it is not easy to derive a mathematical equation. A simpler approach is to match the step response in the problem with the step responses of common systems shown in the lecture notes.

Step response of $G_4(s)$ increases monotonically to the steady-state value (i.e. no oscillations and $\left.\frac{dy(t)}{dt}\right|_{t=0} \neq 0$). Hence, $G_4(s)$ is a first-order plus dead-time system, i.e. $\frac{Ke^{sL}}{\tau s+1} = \frac{Ae^{sL}}{s(as^2+bs+c)}$. Problem is solved by using the following characteristics of the unit step response :

- Input value is unity and final output value is 3, so DC gain $G(0) = K = 3/1 = 3$.
- Dead-time, $L = 0.25$ because the output signal starts to change 0.25 time units after the input is applied.
- Time constant, $\tau = 0.5$, is the time taken to reach 63.2% of the final output value.

Step response, $y_{step}(t) = K\left[1-e^{-(t-L)/\tau}\right]u(t-L) = 3\left[1-e^{-(t-0.25)/0.5}\right]u(t-0.25)$. So:

$$Y_{step}(s) = 3\left[\frac{1}{s} - \frac{1}{s+1/0.5}\right]e^{-0.25s} = 3\left[\frac{1}{s} - \frac{0.5}{0.5s+1}\right]e^{-0.25s} = 3\left[\frac{0.5s+1-0.5s}{s(0.5s+1)}\right]e^{-0.25s}$$

$$= \frac{1}{s}\left[\frac{3}{0.5s+1}e^{-0.25s}\right]$$

Hence: $G(s) = sY_{step}(s) = \frac{3}{0.5s+1}e^{-0.25s} = \frac{A}{as^2+bs+c}e^{-sL}$

Hence, $A = 3$, $a = 0$, $b = 0.5$, $c = 1$, L =0.25

3. (a) The system unit step response is

$$y(t) = 1 - 0.49e^{-15.1t} - 0.51\cos(1.31t) - 0.97\sin(1.31t)$$

Performing Laplace Transform on both sides of the equation

$$Y(s) = \frac{1}{s} - 0.49\frac{1}{s+15.1} - 0.51\frac{s}{s^2+1.31^2} - 0.97\frac{1.31}{s^2+1.31^2}$$

$$Y(s) = G(s)X(s) = G(s)\frac{1}{s} = \frac{N(s)}{s(s+15.1)(s^2+1.31^2)}$$

Hence: $G(s) = \dfrac{N(s)}{(s+15.1)(s^2+1.31^2)}$

Hence, system poles are s = -15.1, ±1.31j.

(b) Comparing the denominators of $G(s)$ we have:

$$s^3 + (9+A)s^2 + (20-3A)s + 4.25A = (s+15.1)(s^2+1.31^2)$$

Comparing the constants, we have:

$$4.25A = 15.1\times 1.31^2$$

$$\therefore \; A = 4.25/(15.1\times 1.31^2) = 6.1$$

4. (a) As the steady-state gain is 1, then K = 1, and the transfer function is:

$G(s) = \dfrac{1}{\tau s+1}$ then:

$Y_{step}(s) = G(s)X(s) = \dfrac{1}{\tau s+1}\dfrac{1}{s}$ and its inverse Laplace tranform is: $y_{step}(t) = 1 - e^{-t/\tau}$

$$\frac{dy_{step}(t)}{dt} = \frac{1}{\tau}e^{-t/\tau}$$

Given that $\dfrac{dy_{step}(0)}{dt} = 0.025$, then: $0.025 = \dfrac{1}{\tau}e^0 \quad \Rightarrow \quad \tau = 1/0.025 = 40$

(b) To reach 0.99 of the steady-state, we have:

$$0.99 = 1 - e^{-t/40}$$

$$e^{-t/40} = 1 - 0.99 = 0.01$$

$$e^{t/40} = 100$$

$$\frac{t}{40} = \ln(100) = 4.605$$

$$\therefore t = 40\times 4.605 = 184.2$$

Section III

1. The system differential equation is

$$m\frac{d^2x_o(t)}{dt}+b\frac{dx_o(t)}{dt}+kx_o(t)=kx_i(t)+b\frac{dx_i(t)}{dt}$$

Performing Laplace transform and assuming that initial conditions are zero, we have:

$$ms^2X_o(s)+bsX_o(s)+kX_o(s)=kX_i(s)+bsX_i(s)$$

$$X_o(s)\left[ms^2+bs+k\right]=X_i(s)\left[bs+k\right]$$

$$\therefore \quad G(s)=\frac{X_o(s)}{X_i(s)}=\frac{bs+k}{ms^2+bs+k}$$

Car height, $x_o(t)$, when it encounters a curb of unit height is:

$$x_o(t)=\mathcal{L}^{-1}\left\{G(s)\times\frac{1}{s}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+3s+2}\times\frac{1}{s}\right\}$$

Let:

$$\frac{3s+2}{s(s+1)(s+2)}=\frac{A_1}{s}+\frac{A_2}{s+1}+\frac{A_3}{s+2}=\frac{A_1(s+1)(s+2+A_2s(s+2)+A_3s(s+1)}{s(s+1)(s+2)} \quad ,$$

then comparing numerators we have:

$$3s+2=A_1(s+1)(s+2+A_2s(s+2)+A_3s(s+1)$$

Set $s=0 \quad : \quad 2=2A_1 \quad \Rightarrow \quad A_1=1$

Set $s=-1: \quad -3+2=A_2(-1)(-1+2) \quad \Rightarrow \quad A_2=1$

Set $s=-2: \quad -6+2=A_3(-2)(-2+1) \quad \Rightarrow A_3=-2$

Hence:

$$X_o(s)=\frac{1}{s}+\frac{1}{s+1}-\frac{2}{s+2}$$

Taking the inverse Laplace transform:

$$x_o(t)=1+e^{-t}-2e^{-2t}$$

2. Problem requires the unit step response of $\frac{6(-s+30)}{(s+4)(s+13)}$. From the definition of transfer function. The unit step response may be found by inverse Laplace transforming

$$\frac{6(-s+30)}{(s+4)(s+13)}\times\frac{1}{s}$$

However, the method would be tedious mathematically and does not make use of the information provided by the question.

By examining the two transfer functions in the problem, it may be concluded that

$$\frac{6(-s+30)}{(s+4)(s+13)}=\frac{6(-s+30)}{30}\times\frac{30}{(s+4)(s+13)}$$
$$=-\frac{6s}{30}\frac{30}{(s+4)(s+13)}+\frac{30}{(s+4)(s+13)}$$

The unit step response of $\frac{6(-s+30)}{(s+4)(s+13)}$ may be obtained using the Transform of derivative rule and the unit step response of $\frac{30}{(s+4)(S+13)}$ provided by the question.

$$Y(s)=G(s)X(s)=\frac{G(s)}{s}$$
$$Y_1(s)=\frac{G_1(s)}{s}$$

Note that:

$$G_1(s)=\frac{6(-s+30)}{(s+4)(s+13)}=-\frac{6s}{30}\frac{30}{(s+4)(s+30)}+6\frac{30}{(s+4)(s+30)}=-\frac{6s}{30}G(s)+6G(s)$$

Hence:

$$Y_1(s)=\frac{1}{s}\left[-\frac{6s}{30}G(s)+6G(s)\right]=-\frac{6s}{30}\frac{G(s)}{s}+6\frac{G(s)}{s}=-\frac{6s}{30}Y(s)+6Y(s)$$

Taking the inverse Laplace transform:

$$y_1(t)=-\frac{6}{30}\frac{d}{dt}y(t)+6y(t)$$
$$=-\frac{1}{5}\frac{d}{dt}\left[\frac{15}{26}+\frac{10}{39}e^{-13t}-\frac{5}{6}e^{-4t}\right]+6\left[\frac{15}{26}+\frac{10}{39}e^{-13t}-\frac{5}{6}e^{-4t}\right]$$
$$=\left[\frac{2}{3}e^{-13t}-\frac{2}{3}e^{-4t}\right]+\left[\frac{45}{13}+\frac{20}{13}e^{-13t}-5e^{-4t}\right]$$
$$=\frac{45}{13}+\frac{86}{39}e^{-13t}-\frac{17}{3}e^{-4t}$$