National University of Singapore

Department of Electrical & Computer Engineering

EE2023 Signals & Systems Revision Notes

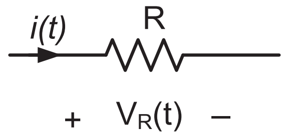
1 Circuit Elements and their Models

1. Resistors, Capacitors and Inductors

Resistors

$$\text{Time Domain} : i(t) = \frac{V_R(t)}{R}$$

$$\text{Frequency Domain} : I(s) = \frac{V_R(s)}{R}$$



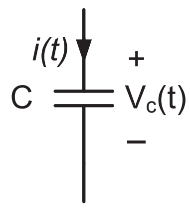
Capacitors

$$\text{Time Domain} : i(t) = C\frac{dV_c(t)}{dt}$$

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$\text{Frequency Domain} : V_c(s) = \frac{I(s)}{sC}$$

$$Z_c(s) = \frac{V_c(s)}{I(s)} = \frac{1}{sC}$$



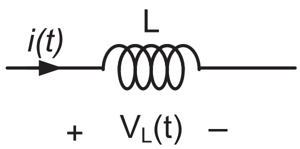
Inductors

$$\text{Time Domain} : V_L(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(\tau)d\tau$$

$$\text{Frequency Domain} : V_L(s) = sLI(s)$$

$$Z_L(s) = \frac{V_L(s)}{I(s)} = sL$$



2. Parallel and series combinations of resistors, capacitors and inductors

Series Combination :

$$\begin{aligned}\text{Resistance : } R_{total} &= R_1 + R_2 + \ldots + R_n \\ \text{Impedances : } Z_{c,total}(s) &= \frac{1}{sC_1} + \frac{1}{sC_2} + \ldots + \frac{1}{sC_n} \\ Z_{L,total}(s) &= sL_1 + sL_2 + \ldots + sL_n\end{aligned}$$

Parallel Combination :

$$\begin{aligned}\text{Resistance : } \frac{1}{R_{total}} &= \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \\ \text{Impedances : } Z_c(s) &= \frac{1}{sC_1 + sC_2 + \ldots + sC_n} \\ \frac{1}{Z_L(s)} &= \frac{1}{sL_1} + \frac{1}{sL_2} + \ldots + \frac{1}{sL_n}\end{aligned}$$

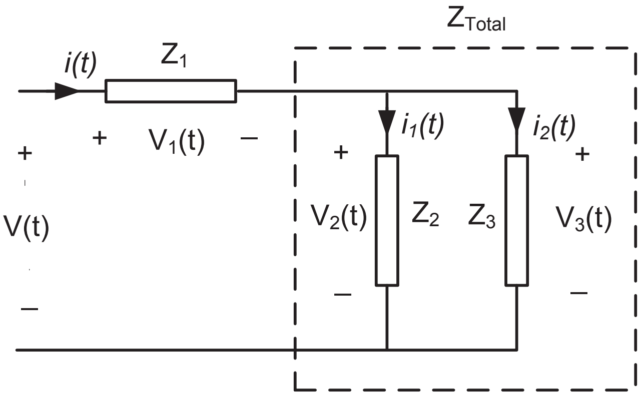
$Z_c(s)$ and $Z_L(s)$ are complex impedances ie they are complex numbers which behave like resistances except that their impedance values change with frequency when $s = j\omega$ where ω is the frequency of operation. For example, in the case of a capacitor with capacitance C Farad, its impedance value is $Z_c(s = j\omega) = \frac{1}{j\omega C}$ Ω and therefore, this impedance value, $Z_c(j\omega)$ will decrease as ω increases. At $\omega = 0$ or at DC, the capacitor operates like an open circuit since $Z_c(j0) = \infty$. A similar behaviour can be said of an inductor with inductance L Henry except that the impedance of an inductor is proportional to ω. Hence an inductor has zero impedance at DC. This is in contrast to resistances, R, which have real values which are not affected very much by frequencies of operation.

3. Current and Voltage Division Laws

Applying Kirchoff's Current and Voltage Laws in Figure 1, and working in the frequency domain, we have :

$$\begin{aligned}I(s) &= I_1(s) + I_2(s) \\ I_1(s) &= \frac{Z_3(s)}{Z_2(s) + Z_3(s)} I(s) \\ I_2(s) &= \frac{Z_2(s)}{Z_2(s) + Z_3(s)} I(s)\end{aligned}$$

$$\begin{aligned}V_2(s) &= V_3(s) \\ V(s) &= V_1(s) + V_2(s) \\ V_1(s) &= \frac{Z_1(s)}{Z_1(s) + Z_{Total}(s)} V(s) \\ V_2(s) &= \frac{Z_{Total}(s)}{Z_1(s) + Z_{Total}(s)} V(s)\end{aligned}$$



$$\frac{1}{Z_{Total}} = \frac{1}{Z_2} + \frac{1}{Z_3}$$

Fig. 1: Current and Voltage Division

These principles are useful in analysing circuits containing resistors, inductors and capacitors. The above equations, however, are only valid in the frequency domain. In time domain, differential equations are needed to model the currents and voltages in such circuits.

2 Complex Numbers

Complex numbers, z, are of the form

$$z = x + jy = Re^{j\theta} = R\angle\theta$$

where x and y are real quantities. $R > 0$ is also the magnitude of z, denoted as $|z| = R$ and θ is the phase of z in radians. Unless otherwise stated, x and y are assumed positive in the following sections. I trust that you are able to prove the above relationships.

Complex numbers play a very important part in the analysis of many electrical systems as you may already have seen in EG1108 and elsewhere. Hence it is very important that the manipulation of complex numbers are done correctly. In particular, you have to be fully familiar in converting between the different forms of the complex numbers. This is especially crucial when you deal with frequency response plots involving complex numbers and other calculations involving magnitudes and phases.

Some tips for ensuring the correct calculation of phase of a complex quantity.

- $z = j : |z| = 1$ and $\angle z = 0.5\pi$.
- $z = -j : |z| = 1$ and $\angle z = -0.5\pi$.
- $z = 1/j = -j : |z| = 1$ and $\angle z = -0.5\pi$
- $z = -1/j = j : |z| = 1$ and $\angle z = 0.5\pi$
- $z = j \times j = -1 : |z| = 1$ and $\angle z = -\pi$ or π.
- $z = x + jy : z = \sqrt{x^2 + y^2} e^{j \tan^{-1} \frac{y}{x}}$. Thus

$$|z| = \sqrt{x^2 + y^2} \text{ and } \angle z = \tan^{-1} \frac{y}{x}.$$

See Figure 2 for illustration.

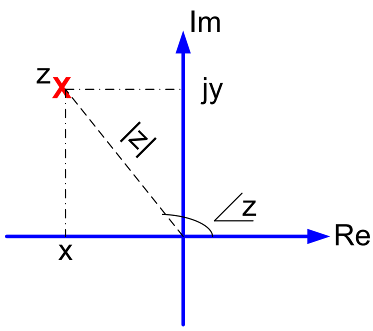


Fig. 2: General form of a complex number

- $z = x + j0 : z = xe^{j0} = x\angle 0$. Thus $|z| = x$ and $\angle z = 0$.
- $z = 0 + jy : z = ye^{j0.5\pi}$. Thus $|z| = y$ and $\angle z = 0.5\pi$.
- $z = -x + j0 : z = xe^{j\pi} = xe^{-j\pi}$. Thus $|z| = x$ and $\angle z = \pi$ or $-\pi$.
- $z = 0 - jy : z = ye^{-j0.5\pi}$. Thus $|z| = y$ and $\angle z = -0.5\pi$.
- $z = (x + jy)^2 : |z| = |x + jy|^2 = x^2 + y^2$ and $\angle z = 2 \times \angle(x + jy) = 2 \tan^{-1} \frac{y}{x}$.
- $z = \frac{1}{x+jy} : |z| = \frac{1}{|x+jy|} = \frac{1}{\sqrt{x^2+y^2}}$ and $\angle z = \angle 1 - \angle(x + jy) = 0 - \tan^{-1} \frac{y}{x} = -\tan^{-1} \frac{y}{x}$.

- $z = -x + jy$: $|z| = \sqrt{x^2 + y^2}$ and $\angle z = \tan^{-1} \frac{y}{-x} = -\tan^{-1} \frac{y}{x}$. Calculators will generally give arctan values between $-0.5\pi < \theta < 0.5\pi$ which means that the phase angles given by calculators are only in the first or fourth quadrant. However, in this case when $z = -x + jy$, the complex number is in the second quadrant since the real part is negative while the imaginary part is positive. Therefore, the value obtained from the calculator will not be the correct phase and has to be adjusted as follows :

$$\angle z = \pi - \tan^{-1} \frac{y}{x} \text{ or } -(\pi + \tan^{-1} \frac{y}{x}).$$

- $z = x - jy$: $|z| = \sqrt{x^2 + y^2}$ and

$$\angle z = \tan^{-1} \frac{-y}{x} = -\tan^{-1} \frac{y}{x} \tag{1}$$

The phase given by the calculator will be correct since this complex number lies in the fourth quadrant.

- $z = \frac{1}{x-jy}$: $|z| = \frac{1}{\sqrt{x^2+y^2}}$ and $\angle z = -\left\{\tan^{-1} \frac{-y}{x}\right\}$. It is not easy to see what the phase should be without further manipulation as follows :

$$\begin{aligned} z &= \frac{1}{x - jy} \\ &= \frac{1}{x - jy} \times \frac{x + jy}{x + jy} \\ &= \frac{x + jy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + j\frac{y}{x^2 + y^2} \end{aligned} \tag{2}$$

Hence in this case, although $z = \frac{1}{x-jy}$, z is actually a complex number in the first quadrant, deducing from (2) where both the real and imaginary parts are positive. Thus

$$\angle z = -\left\{\tan^{-1} \frac{-y}{x}\right\} = \tan^{-1} \frac{y}{x}.$$

Another way to determine the phase is to consider :

$$\begin{aligned} z &= \frac{1}{x - jy} \\ &= \frac{1}{Re^{-j\theta}} = \frac{1}{R}e^{j\theta} \end{aligned}$$

where $-\theta$ is the angle in the fourth quadrant as in (1) above. Thus $\angle z = \theta$ which is in the first quadrant : same conclusion earlier.

3 Laplace Transforms

Laplace Transforms (LT) are an integral part of systems and control. It is used widely in solving ordinary differential equations (ODE) by transforming them into algebraic equations involving the complex variable s. Via the LT, the time domain ODEs are converted into algebraic equations in frequency domain where s is the frequency variable. This also gives the fundamental link between time and frequency domains of signals and systems. Via the frequency domain, many properties of linear time invariant systems can be described and characterised without the need to solve the original ODEs. This leads to a generalization of system behaviour for this class of systems.

In mathematics, the Laplace Transform is an integral transform defined by :

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

where $F(s)$ and $f(t)$ are Laplace Transform pair. The lower limit of 0^- is required in cases where there are non-zero initial conditions. Example on how the Laplace integral is used :

$$\begin{aligned}
\mathcal{L}\{\sin\omega t\} &= \int_{0^-}^{\infty} \sin\omega t e^{-st}dt \\
&= \int_{0^-}^{\infty} \frac{1}{2j}\{e^{j\omega t} - e^{-j\omega t}\}e^{-st}dt \\
&= \int_{0^-}^{\infty} \frac{1}{2j}\{e^{-(s-j\omega)t} - e^{-(s+j\omega t)}\}dt \\
&= \frac{1}{2j}\left[\frac{1}{-s+j\omega}e^{-(s-j\omega)t} + \frac{1}{s+j\omega}e^{-(s+j\omega)t}\right]\Bigg|_{0^-}^{\infty} \\
&= \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

△ Can you find $\mathcal{L}\{\cos\omega t\}$?

Laplace transforms are linear operators which has the following properties. Note that $F(s)$ and $G(s)$ are used to denote the Laplace Transforms of $f(t)$ and $g(t)$.

- Linearity : $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$, α and β are constants.
- Transform of derivatives : $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$ where $f(0^-)$ denotes the initial

condition of $f(t)$. In general,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^k(0^-)$$

An example :

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0^-) - \left.\frac{df}{dt}\right|_{t=0^-}$$

- Transform of an Integral : $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$
- Derivatives of Transforms : $\mathcal{L}\{-tf(t)\} = \frac{dF(s)}{ds}$. In general,

$$(-1)^n \mathcal{L}\{t^n f(t)\} = \frac{d^n F(s)}{ds^n}$$

Example : Since $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$, then according to the formula :

$$\mathcal{L}\{t\sin\omega t\} = -\frac{d}{ds}\{\frac{\omega}{s^2+\omega^2}\} = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

- Shift in Time Domain : $\mathcal{L}\{f(t-t_0)\} = e^{-st_0}F(s)$. An example of a signal shifted in time is given in Figure 3.

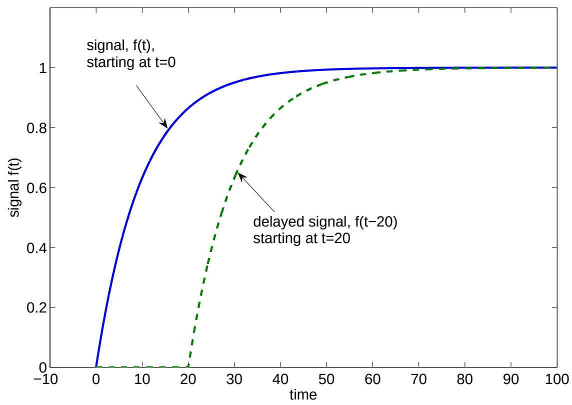


Fig. 3: Signal delayed by 20 time units

Notice how a shifted time function is written. Its Laplace transform has a simple relationship with the original LT of $f(t)$. This can be shown as follows :

$$\begin{aligned}
\mathcal{L}\{f(t-t_0)\} &= \int_0^\infty f(t-t_0)e^{-st}dt \\
&= \int_0^\infty f(t-t_0)e^{-s(t-t_0)}e^{-st_0}dt \\
&= e^{-st_0}\int_0^\infty f(t-t_0)e^{-s(t-t_0)}dt \\
&= e^{-st_0}F(s) \quad \text{after making the substitution } \lambda = t-t_0
\end{aligned}$$

where $F(s) = \mathcal{L}\{f(t)\}$.

- Shift in the Frequency Domain : $\mathcal{L}\{e^{-\alpha t}f(t)\} = F(s+\alpha)$.

  You should be able to prove this following the steps in the delayed signal case.

- Final Value Theorem : For a time domain function which has a finite steady state value, the final value theorem is given as :

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

  Proof :

$$\begin{aligned}
\mathcal{L}\{\frac{df(t)}{dt}\} &= \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = sF(s) - f(0) \\
\lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}e^{-st}dt &= \lim_{s\to 0}[sF(s) - f(0)] \\
\lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}dt &= \lim_{s\to 0}[sF(s) - f(0)] \\
f(\infty) - f(0) &= \lim_{s\to 0}[sF(s) - f(0)] \\
\lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s)
\end{aligned}$$

  This is a convenient formula to use when you want to determine the steady state value of a system's output response. You can obtain this steady state value from $F(s)$ instead of from $f(t)$. Applicable only when the $f(t)$ has a constant steady state value. Not applicable when $f(t)$ is always changing with time. Examples where the final value theorem fails is when $f(t) = \sin\omega t$, $f(t) = t$, etc.

Fortunately, we do not need to evaluate the LT integral each time we want to find the Laplace Transform of a function $f(t)$, at least for the kind of $f(t)$ that are generally encountered in systems and control. We rely on tables to determine them. Some examples are given in Table 1.

Table 1: Some examples of Laplace Transform pairs

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{-at}t$	$\frac{1}{(s+a)^2}$

4 Inverse Laplace Transform and Partial Factorization

Inverse Laplace Transform is used to recover the time domain signal from its s-domain equivalent. The inverse Laplace Transform is defined as :

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2j\pi}\int_{-\infty}^{\infty} F(s)e^{st}ds$$

It can be seen that this is a complex integral. In most cases, we do not need to find the inverse Laplace Transform using this integral. We should be able to use some well known Laplace Transform pairs such as those in Table 1 to help us find the inverse.

Example 1 : Find the inverse Laplace Transform of $\dfrac{1}{s^2+2s+4}$.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+3}\right\} \\
&= \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}\right\} \\
&= \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s_1^2+(\sqrt{3})^2}\right\} \text{ where } s_1 = s+1
\end{aligned}$$

Since

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2} \text{ and } \mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a),$$

it follows that

$$\mathcal{L}\left\{\frac{1}{s^2+2s+4}\right\} = \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t$$

Laplace Transforms which are encountered in linear systems theory are of the form :

$$F(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in the s-variable. Examples are $F(s) = 1/s,\ F(s) = \frac{s+1}{(s+1)^2+3},\ F(s) = \frac{1}{s+2}$, etc. For simple numerator and denominator polynomials of order 2 or lower, the inverse Laplace Transform of $F(s)$ can be obtained by inspection from Table 1. However, when $N(s)$ and $D(s)$ are polynomials of higher order, the inverse may not be so easily obtainable from the table.

The approach to obtain the inverse LT of a general $F(s)$ is to first partial factorize $F(s)$ so that the partial factors consists of only polynomials of orders at most 2. The inverse LT of $F(s)$ is then the inverse LT of each partial factor which can be easily deduced from Table 1. This approach works because of the linearity property of the LT.

There are 3 different forms of partial factors :

- Functions involving only <u>distinct</u> linear factors in the $D(s)$:

$$F(s) = \frac{N(s)}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_n)} = \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2} + \ldots + \frac{A_n}{s+\alpha_n}$$

where $\alpha_i \neq \alpha_j,\ i \neq j$.

For these distinct linear factors, the inverse LT is given by :

$$f(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} + \ldots + A_n e^{-\alpha_n t}.$$

- Functions involving repeated linear factors in the $D(s)$:

$$F(s) = \frac{N(s)}{(s+\alpha)^n} = \frac{A_1}{s+\alpha} + \frac{A_2}{(s+\alpha)^2} + \ldots + \frac{A_n}{(s+\alpha)^n}$$

For these repeated linear factors, the inverse LT is given by :

$$f(t) = \left(A_1 + A_2 t + \frac{1}{2} A_3 t^2 + \ldots + \frac{1}{n!} A_n t^{n-1} \right) e^{-\alpha t}.$$

- Functions involving quadratic factors :

$$F(s) = \frac{N(s)}{(s^2 + 2\beta_1 s + \gamma_1^2)(s^2 + 2\beta_2 s + \gamma_2^2)} = \frac{A_1 s + B_1}{(s^2 + 2\beta_1 s + \gamma_1^2)} + \frac{A_2 s + B_2}{(s^2 + 2\beta_2 s + \gamma_2^2)}$$

For such linear factors, the general form of the inverse LT is given by :

$$f(t) = R_1 e^{-\beta_1 t} \sin\left(\sqrt{\gamma_1^2 - \beta_1^2}\, t + \phi_1 \right) + R_2 e^{-\beta_2 t} \sin\left(\sqrt{\gamma_2^2 - \beta_2^2}\, t + \phi_2 \right).$$

The R_i are functions of β_i and γ_i. In this quadratic form, it should be presumed that each quadratic factor admits complex roots.

Each of these different forms of partial factors require a different approach to determine A_i. These methods are briefly given as follows :

- Functions with distinct linear factors :

$$F(s) = \frac{N(s)}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_n)} = \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2} + \ldots + \frac{A_n}{s+\alpha_n}$$

To find A_k, use the following formula :

$$A_k = \left. \frac{\cancel{(s+\alpha_k)} N(s)}{(s+\alpha_1)(s+\alpha_2)\cancel{(s+\alpha_k)}\ldots(s+\alpha_n)} \right|_{s=-\alpha_k}$$

Example : Find $f(t)$ from its LT $F(s) = \frac{2}{(s+1)(s+2)}$.

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Using the above method :

$$\begin{aligned}
A_1 &= \left.\frac{\cancel{(s+1)}2}{\cancel{(s+1)}(s+2)}\right|_{s=-1} = 2 \\
A_2 &= \left.\frac{\cancel{(s+2)}2}{(s+1)\cancel{(s+2)}}\right|_{s=-2} = -2 \\
F(s) &= \frac{2}{s+1} - \frac{2}{s+2} \\
f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} = 2e^{-t} - 2e^{-2t}
\end{aligned}$$

- Functions with repeated factors

$$F(s) = \frac{N(s)}{(s+\alpha)^n} = \frac{A_1}{s+\alpha} + \frac{A_2}{(s+\alpha)^2} + \ldots + \frac{A_n}{(s+\alpha)^n}$$

To find A_k, use the following formula :

$$A_k = \frac{1}{(n-k)!}\left.\left[\frac{d^{n-k}}{ds^{n-k}}\left((s+\alpha)^n F(s)\right)\right]\right|_{s=-\alpha}$$

where $\frac{d^{n-k}}{ds^{n-k}}(.)$ denotes differentiating with respect to s, $(n-k)$ times.

Example : Find $f(t)$ from its LT $F(s) = \frac{2}{(s+1)(s+2)^2}$.

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad = \quad \frac{A_1}{s+1} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)}$$

Using the above method :

$$\begin{aligned}
A_1 &= \left.\frac{\cancel{(s+1)}2}{\cancel{(s+1)}(s+2)^2}\right|_{s=-1} = 2 \\
A_2 &= \left.\frac{\cancel{(s+2)^2}2}{(s+1)\cancel{(s+2)^2}}\right|_{s=-2} = -2 \\
A_3 &= \frac{1}{(1)!}\left.\left[\frac{d}{ds}\left((s+2)^2 F(s)\right)\right]\right|_{s=-2} = -2 \\
F(s) &= \frac{2}{s+1} - \frac{2}{(s+2)^2} - \frac{2}{s+1} \\
f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} = 2e^{-t} - 2te^{-2t} - 2e^{-2t}
\end{aligned}$$

- Functions with quadratic factors

$$F(s) = \frac{N(s)}{(s^2+2\beta_1 s+\gamma_1^2)(s^2+2\beta_2 s+\gamma_2^2)} = \frac{A_1 s + B_1}{(s^2+2\beta_1 s+\gamma_1^2)} + \frac{A_2 s + B_2}{(s^2+2\beta_2 s+\gamma_2^2)}$$

Each of the quadratic factors can be further factorized as :

$$\begin{aligned}\frac{As+B}{(s^2+2\beta s+\gamma^2)} &= \frac{As+B}{(s+\beta+j\sqrt{\gamma^2-\beta^2})(s+\beta-j\sqrt{\gamma^2-\beta^2})}\\ &= \frac{P}{(s+\beta+j\sqrt{\gamma^2-\beta^2})}+\frac{Q}{(s+\beta-j\sqrt{\gamma^2-\beta^2})}\end{aligned}$$

P and Q can be obtained in the same way as those with distinct linear factors.

△ Find the inverse Laplace Transform of $\dfrac{10}{s(s+2)(s+3)^2}$ and $\dfrac{1}{s^2+s+1}$.

5 Using Laplace Transform to solve Ordinary Differential Equation

Consider the first order RC circuit given in Figure 4. Assume a general input voltage $v(t)$

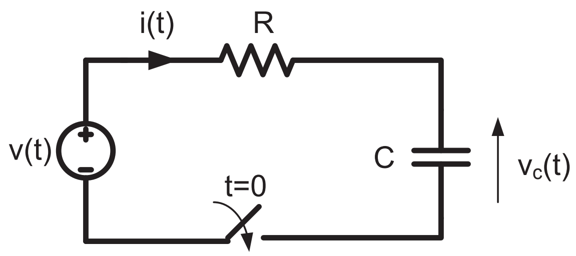


Fig. 4: First order RC circuit.

and an output voltage $v_c(t)$ across the capacitor. Deriving the model in time domain using differential equation and by applying circuit laws :

$$\begin{aligned}v(t) &= i(t)R+v_c(t)\\ i(t) &= C\frac{dv_c(t)}{dt}\\ v(t) &= RC\frac{dv_c(t)}{dt}+v_c(t) \qquad (3)\end{aligned}$$

Equation (3) may be solved using mathematical techniques and it can also be solved using Laplace Transform. Using the LT approach, since the input voltage is a general input $v(t)$, the LT of $v(t)$ is written generally as $V(s)$ while the LT of the output voltage $v_c(t)$ is written as $V_c(s)$. Taking the LT of (3),

$$\begin{aligned} V(s) &= RCsV_c(s) - RCv_c(0) + V_c(s) \\ &= RCsV_c(s) - RCv_c(0) + V_c(s) \\ V_c(s) &= \frac{V(s)}{RCs+1} + \frac{RCv_c(0)}{RCs+1} \end{aligned} \tag{4}$$

If the input voltage $v(t)$ is a constant DC source with value $v(t) = V$, then $V(s) = \frac{V}{s}$ and (4) becomes

$$V_c(s) = \frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}$$

Then $v_c(t) = \mathcal{L}^{-1}\{V_c(s)\}$ can be found as follows :

$$\begin{aligned} v_c(t) &= \mathcal{L}^{-1}\left\{\frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{V}{s} - \frac{VRC}{RCs+1}\right\} + \mathcal{L}^{-1}\left\{\frac{v_c(0)}{s+\frac{1}{RC}}\right\} \\ &= V - Ve^{-\frac{t}{RC}} + v_c(0)e^{-\frac{t}{RC}} \\ &= V + [v_c(0) - V]\, e^{-\frac{t}{RC}} \end{aligned} \tag{5}$$

From (5), it is clear that $\lim_{t\to\infty} v_c(t) = V$ which is the final value which the capacitor will be charged up to when a DC voltage of V is applied. This same result can be obtained by applying the Final Value Theorem to $V_c(s)$ as follows :

$$\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} s\left\{\frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}\right\} = V.$$

The output voltage in (5) can also be rewritten as :

$$v_c(t) = v_c(0)e^{-\frac{t}{RC}} + V\left(1 - e^{-\frac{t}{RC}}\right)$$

The different components in $v_c(t)$ can be seen in Figure 5.

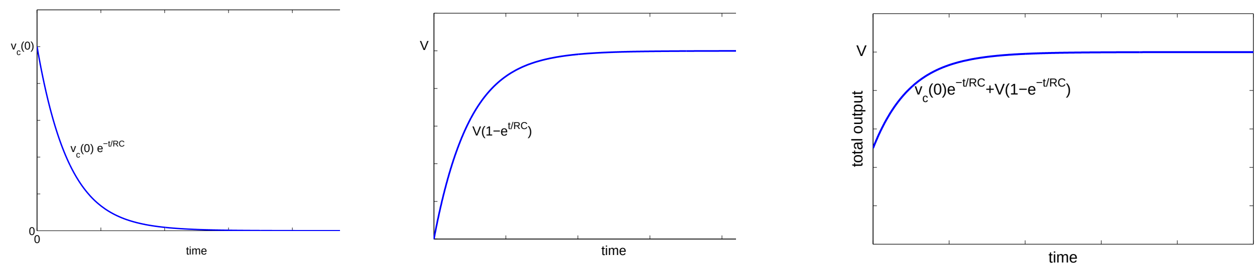


Fig. 5: Plotting individual components of $v_c(t)$.

Notice how the initial condition $v_c(0)$ decays over time. The plot in the middle is the typical charging characteristic of a capacitor with zero initial condition. Finally, the sum total of the first two plots gives the actual charging trajectory of the capacitor, including the non-zero initial condition.