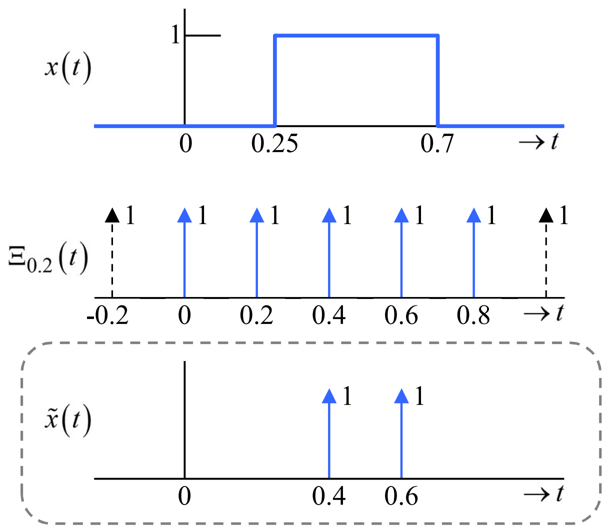
EE2023/TEE2023 TUTORIAL 4 (SOLUTIONS)

Solution to Q.1



$$\tilde{x}(t) = \text{rect}\left(\frac{t-0.475}{0.45}\right).\sum_{n=-\infty}^{\infty}\delta(t-0.2n)$$
$$= \delta(t-0.4)+\delta(t-0.6)$$

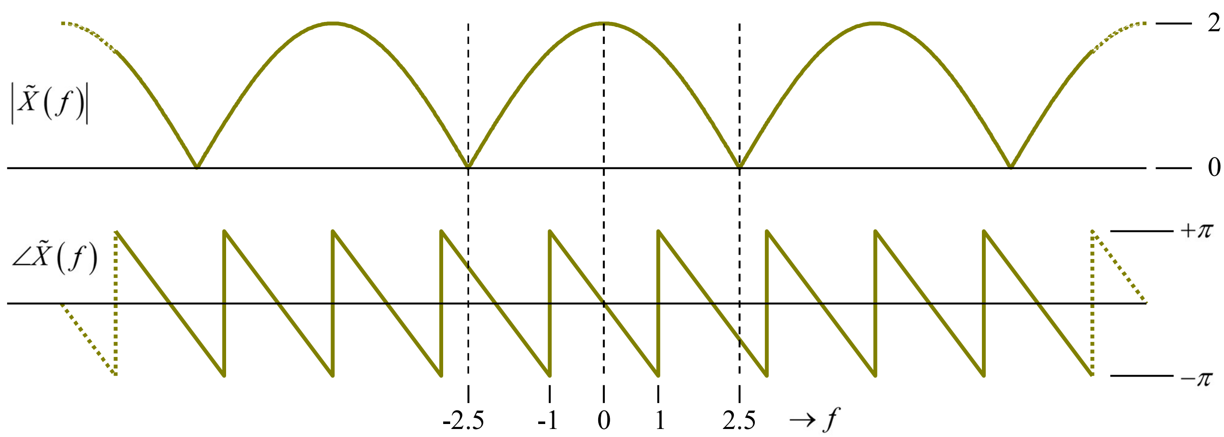
$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$

$$\tilde{X}(f) = \exp(-j2\pi f0.4)+\exp(-j2\pi f0.6)$$
$$= \begin{cases}\cos(0.8\pi f)-j\sin(0.8\pi f)+ \\ \quad\cos(1.2\pi f)-j\sin(1.2\pi f)\end{cases}$$
$$= \begin{cases}\left[\cos(0.8\pi f)+\cos(1.2\pi f)\right]- \\ \quad j\left[\sin(0.8\pi f)+\sin(1.2\pi f)\right]\end{cases}$$
$$= 2\cos(\pi f)\cos(0.2\pi f)-j2\sin(\pi f)\cos(0.2\pi f)$$

$$\left|\tilde{X}(f)\right|^2 = 4\cos^2(\pi f)\cos^2(0.2\pi f)+4\sin^2(\pi f)\cos^2(0.2\pi f)$$
$$= 4\cos^2(0.2\pi f)\left[\cos^2(\pi f)+\sin^2(\pi f)\right]$$
$$= 4\cos^2(0.2\pi f)$$

$$\left|\tilde{X}(f)\right| = 2\left|\cos(0.2\pi f)\right|$$

$$\angle\tilde{X}(f) = \angle[2\cos(\pi f)\cos(0.2\pi f)-j2\sin(\pi f)\cos(0.2\pi f)]$$
$$= -\tan^{-1}\left(\frac{2\sin(\pi f)\cos(0.2\pi f)}{2\cos(\pi f)\cos(0.2\pi f)}\right) = -\tan^{-1}\left(\tan(\pi f)\right) = -\pi f$$

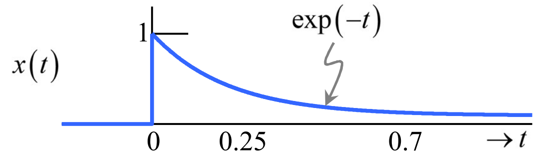


Solution to Q.2

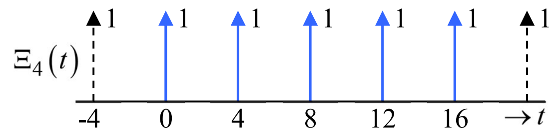
$$x(t) = \exp(-t)u(t)$$

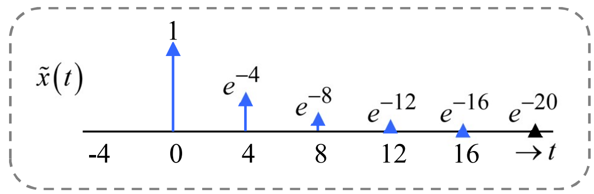
$$\tilde{x}(t) = \underbrace{\exp(-t)u(t)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-4n)}_{\Xi_4(t)}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\delta(t-4n)$$



$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$





$$\tilde{X}(f) = \sum_{n=0}^{\infty} \exp(-4n)\Im\{\delta(t-4n)\}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j2\pi f 4n)$$

$$= \underbrace{\sum_{n=0}^{\infty} \exp\left(-4n(j2\pi f + 1)\right)}_{\text{Geometric Series}}$$

$$= \frac{1}{1-\exp\left(-4(j2\pi f + 1)\right)} = \frac{1}{1-\exp(-4)\exp(-j8\pi f)}$$

Using $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$, where $-1 < r < 1$

When a signal is sampled in the time domain, the spectrum of the sampled signal is periodic with a period equal to the sampling frequency, which in this case is 0.25 Hz. Thus, to show that $\tilde{X}(f)$ is periodic, all we need to show is $\tilde{X}(f+0.25) = \tilde{X}(f)$ as follows:

$$\tilde{X}(f+0.25) = \frac{1}{1-\exp(-4)\exp\left(-j8\pi(f+0.25)\right)}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)\underbrace{\exp(-j2\pi)}_{1}}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)} = \tilde{X}(f)$$

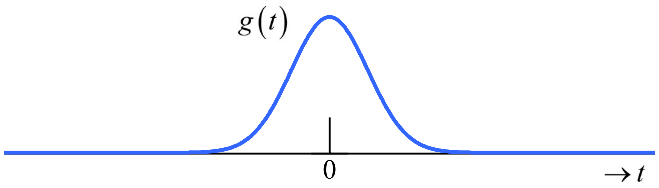
or:

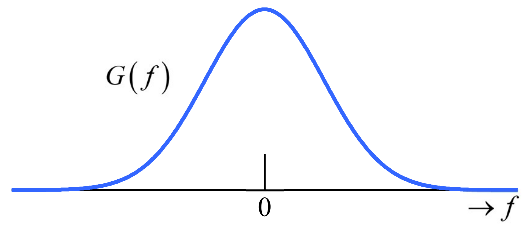
$$\tilde{X}(f+0.25) = \sum_{n=0}^{\infty} \exp(-4n)\exp\left[-j8\pi n(f+0.25)\right]$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j8\pi nf)\underbrace{\exp(-j2\pi n)}_{1}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j8\pi nf) = \tilde{X}(f)$$

Solution to Q.3





Rewrite $x(t)=\sum_{n=-\infty}^{\infty} g\left(t-nT_p\right)$ in convolution form:

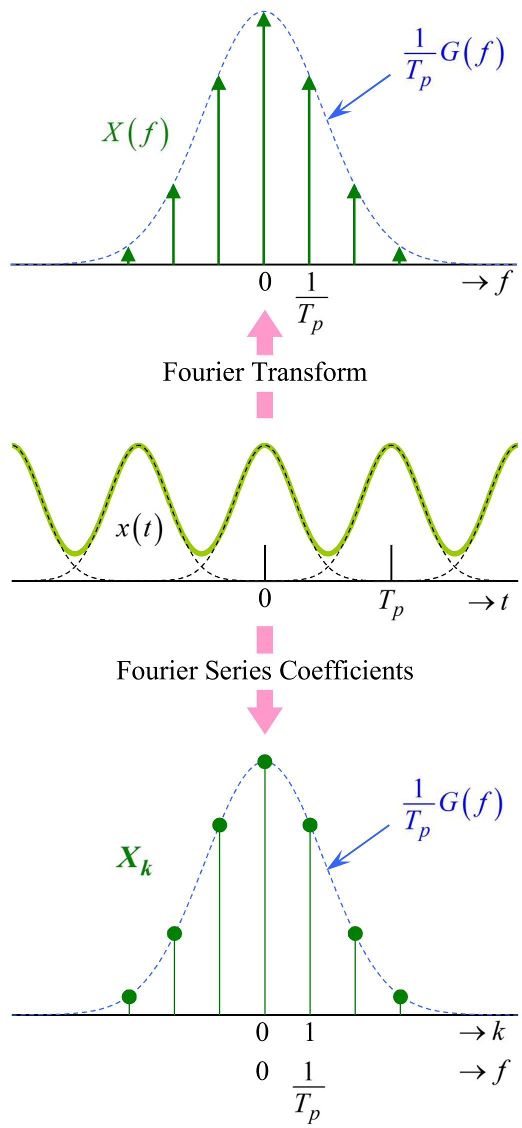
$$x(t)=g(t)*\sum_{n=-\infty}^{\infty}\delta\left(t-nT_p\right) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (*)$$

Applying the 'Convolution' property of the Fourier transform to $(*)$:

$$X(f)=G(f)\cdot\frac{1}{T_p}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T_p}\right) \quad \cdots\cdots\cdots\cdots (**)$$

Conclusion:

$X(f)$ can be obtained by sampling $G(f)/T_p$ in the frequency-domain at regular spacings of $1/T_p$ Hz.



Relationship between X_k and $G(f)$

Rewrite $(**)$ as: $X(f)=\sum_{k=-\infty}^{\infty}\frac{1}{T_p}G\left(\frac{k}{T_p}\right)\delta\left(f-\frac{k}{T_p}\right)$

In terms of X_k: $\quad X(f)=\sum_{k=-\infty}^{\infty}X_k\delta\left(f-\frac{k}{T_p}\right)$

Hence, $X_k=\frac{1}{T_p}G(f)\Big|_{f=k/T_p}=\frac{1}{T_p}G\left(\frac{k}{T_p}\right)$

Uniqueness of a generating function

The generating function of a periodic signal is NOT unique. For instance, one period of a periodic signal can be used as its generating function.

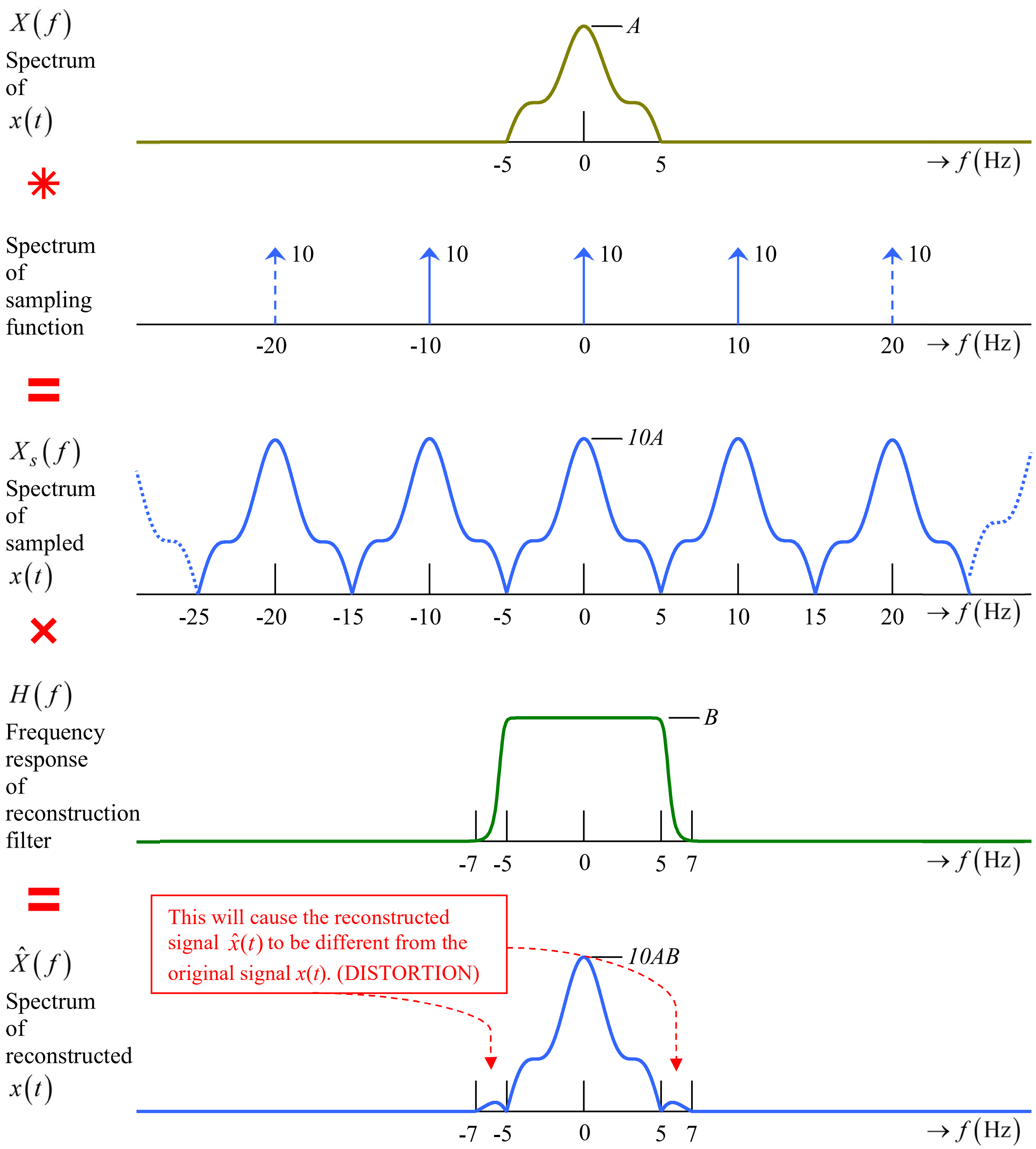
Remarks: From the sampling theorem, we observed that when a signal is sampled at regular time-intervals of $T_s(\mathrm{sec})$, *the spectrum corresponding to the sampled signal is periodic with a period of* $f_s=1/T_s(\mathrm{Hz})$. *In this exercise, we showed that when a spectrum is sampled at regular frequency-intervals of* $f_p(\mathrm{Hz})$, *the signal corresponding to the sampled spectrum is periodic with a period of* $T_p=1/f_p(\mathrm{sec})$. *Hence, sampling a signal leads to a periodic spectrum and sampling a spectrum leads to periodic signal. This phenomenon is due to the convolution property of the Fourier transform and the fact that* $\left[comb(t)\rightleftarrows COMB(f)\right]$.

Solution to Q.4

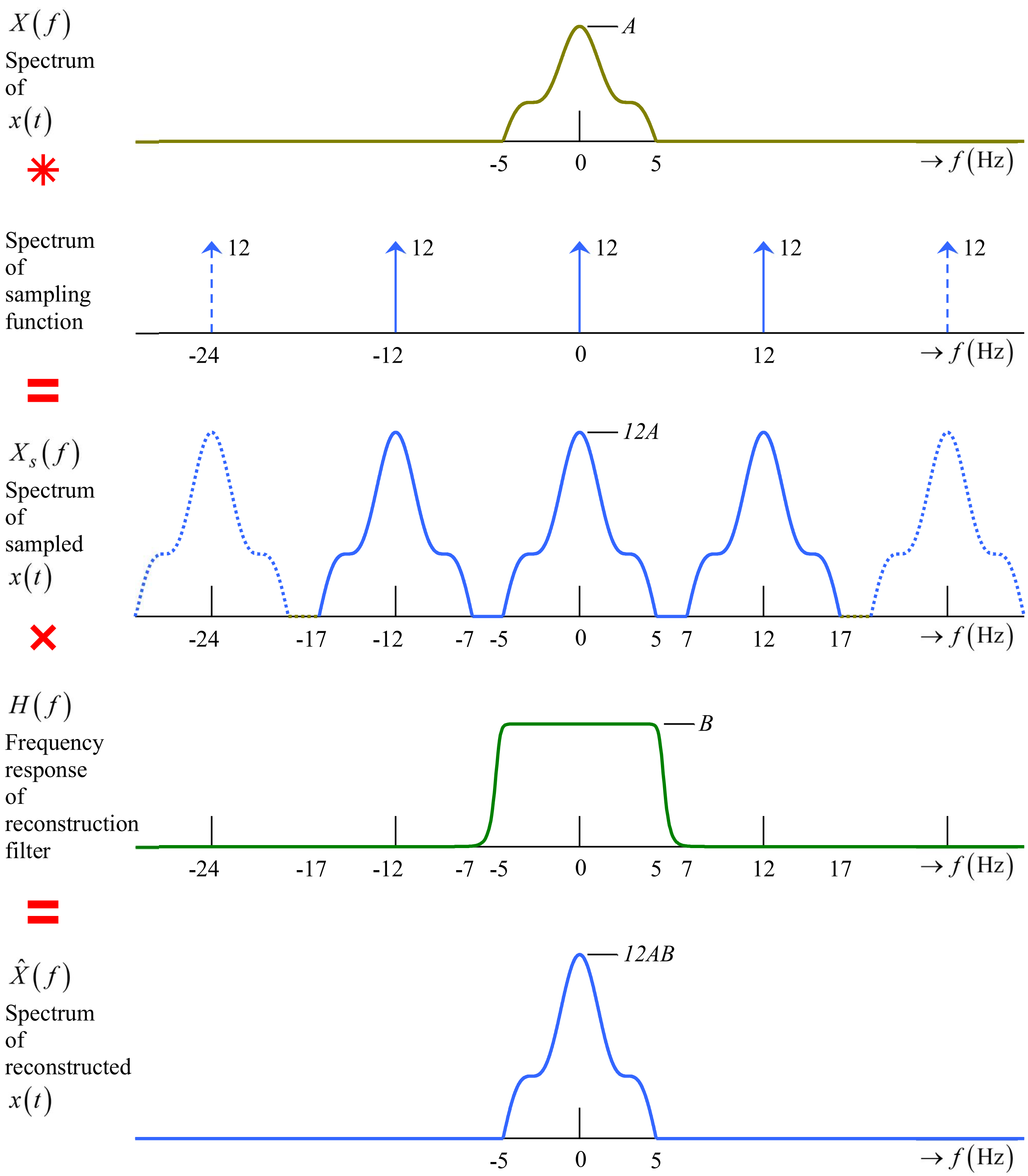
- Nyquist sampling frequency: $2 \times 5 = 10$ Hz
- Recommended sampling frequency: 12 Hz

  The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.

Sampling frequency: **10 Hz (Nyquist Rate)**



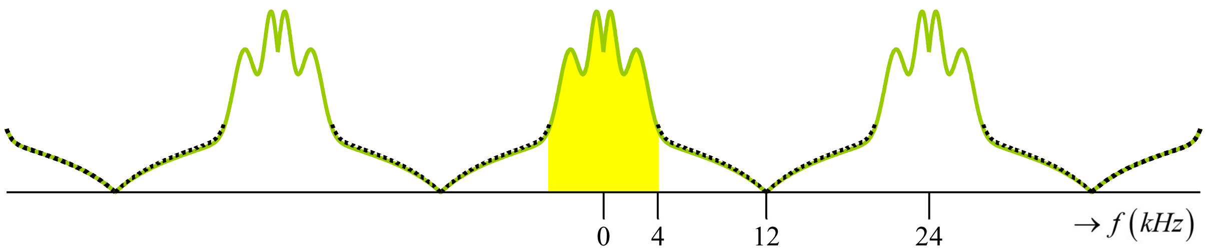
Sampling frequency: **12 Hz**



Remarks: Sampling frequencies higher that 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.

Solution to Q.5

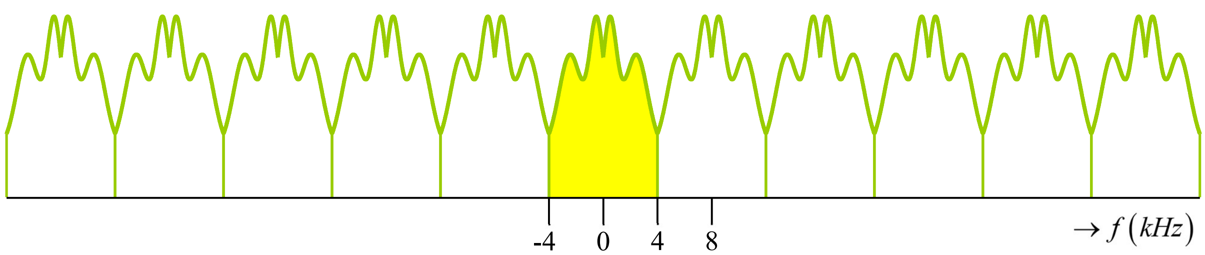
Situation A: $\left(\begin{array}{c}\textbf{\textit{No anti-aliasing lowpass filter}}\\ \textbf{\textit{and no frequency aliasing}}\end{array}\right)$**:** $\left\{\textbf{Sampling frequency} = \mathbf{2 \times 12 = 24}\ \boldsymbol{kHz}\right.$



Advantage: No anti-aliasing LPF needed.

Disadvantage: Sampling frequency is higher than necessary to preserve the *TQ-Band*.

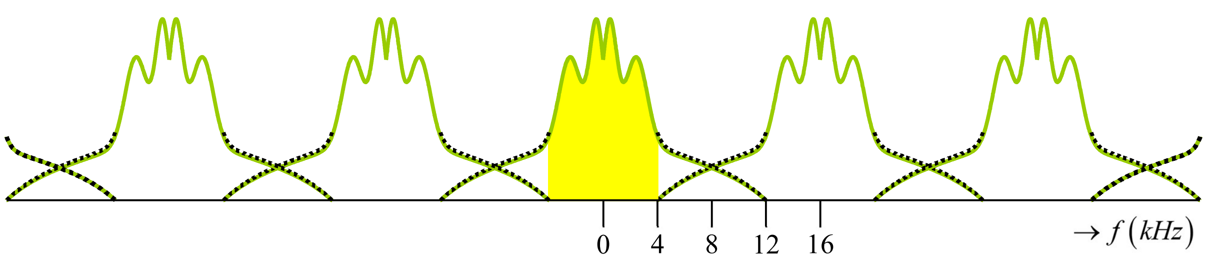
Situation B: ***Lowest sampling frequency*:** $\left\{\begin{array}{l}\textbf{Ideal anti-aliasing filter of bandwidth } \mathbf{4}\ \boldsymbol{kHz}\\ \textbf{Sampling frequency} = \mathbf{2 \times 4 = 8}\ \boldsymbol{kHz}\end{array}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band.*

Disadvantage: Require anti-aliasing LPF with sharp cutoff.

Situation C: ***No anti-aliasing lowpass filter*:** $\left\{\textbf{Sampling frequency} = \mathbf{2} \times \left(\dfrac{\mathbf{12+4}}{\mathbf{2}}\right) = \mathbf{16}\ \boldsymbol{kHz}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

Disadvantage: Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.

Supplementary Questions (Solutions)

S1 Given: f_{max} = 20 kHz

Nyquist rate = 2 x f_{max} = 40 kHz

Note that the sampled signal is:

$$x_s(t) = x(t).\sum_{k=-\infty}^{\infty} \delta\left(t - \frac{k}{40000}\right) \quad ; \quad X_s f) = X(f) \otimes 40000 \sum_{k=-\infty}^{\infty} \delta\left(f - k40000\right)$$

Hence the spectrum of the sampled signal comprises repetitions of $X(f)$ at multiples of the sampling frequency of 40 KHz.

To recover the original signal we will need an ideal low pass filter of bandwidth 20 kHz, which is described by $\text{rect}\left(\frac{f}{40000}\right)$ and we also need to take care of the magnitude of 40000 in $X_s(f)$. Hence the reconstruction filter is:

$$H(f) = \frac{1}{40000}\text{rect}\left(\frac{f}{40000}\right)$$

S2 Given: $x_s(t) = \sum_{n=-5}^{5} x(5n)\delta\left(t - 5n\right)$

(a) Sampling period, T_s = 5 ; f_s = 1/5 = 0.2

(b) Given : $x(t)$ = tri(t)
Since T_s is 5 seconds, then the sampling frequency is too low to be able to recover $x(t)$

S3 The maximum frequency, f_{max} = 20.5 Hz.
Hence the Nyquist frequency = 2 x f_{max} = 2 x 20.5 = 41 Hz.

The sampled signal and its spectrum are:

$$x_s(t) = x(t).\sum_{k=-\infty}^{\infty} \delta\left(t - kT_s\right) \quad ; \quad X_s f) = X(f) \otimes \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right)\right] = X(f) \otimes \left[f_s \sum_{k=-\infty}^{\infty} \delta\left(f - kf_s\right)\right]$$

$\therefore$ $X(f)$ is replicated at multiples of the sampling frequency, f_s. As long as the replicated $X(f)$ do not overlap with each other, then the original signal can be recovered with a suitable bandpass filter.

In this case, the minimum sampling frequency is 7 Hz as replication of $X(f)$ at multiples of 7 Hz will not result in overlaps and thus allow for recovery of $X(f)$ from $X_s(f)$.
