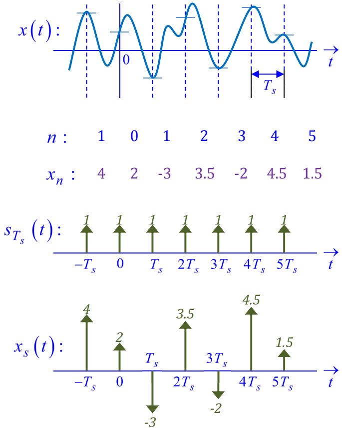
Outline of Lecture

1. Sampling and Reconstruction of Signals
    - Impulse Sampling
2. Properties of the Dirac-δ Function
3. Ideal Reconstruction Filters
4. Continuous-Time Sampling & Reconstruction
5. Conditions for Reconstruction & Nyquist Sampling Theorem

Sampling and Reconstruction of Signals

1. Impulse sampling

   Sampling is a process for converting an analog signal into a discrete-time sequence, which (after quantization) can be digitally stored and processed. This is what is done in CD, DVD and Blue Ray recording, and it is also done to your voice signal for transmission over a radio channel by your digital cell phone.



When we want to sample a signal, we need to think about an appropriate sampling frequency which has to be chosen carefully so that no information is lost during the sampling process and neither do we over-sample because we risk sampling the noise that exists in the signal.

In practice, when we sample an analog signal $x(t)$, we capture the value of the signal at every T_s seconds to produce a discrete-time sequence, $x_n = x(nT_s)$.

Now suppose we sample $x(t)$ by multiplying it with an impulse train :

$$\underbrace{x_s(t) = x(t).\overbrace{\sum_n \delta(t - nT_s)}^{s_{Ts}(t)}}_{\text{impulse sampling}} = \sum_n x(nT_s)\delta(t - nT_s)$$

$$= \sum_n x(n)\delta(t - nT_s)$$

where

- $x_s(t)$ is the sampled version of $x(t)$
- $x(nT_s) = x(n) = x(t)|_{t=nT_s}$ or the value of $x(t)$ at $t = nT_s$
- T_s is the sampling period and samping frequency $f_s = \frac{1}{T_s}$
- The important question we ask is what is the spectrum of $x_s(t)$ and how does it compare with the spectrum of the original signal $x(t)$?

Properties of the Dirac-δ function

Recall that the Dirac-δ function is defined as

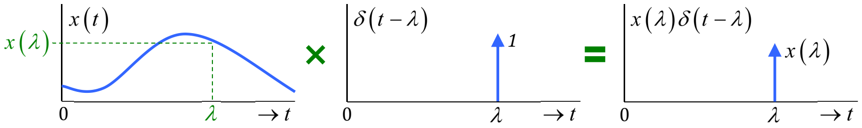
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\epsilon}^{\epsilon} \delta(t)dt = 1 \quad \forall\ \epsilon > 0$$

Property A (Symmertry)

$\delta(t)$ *is symmetrical :* $\delta(t) = \delta(-t)$

Property B (Sampling)

$\delta(t)$ *is used in sampling :* $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$



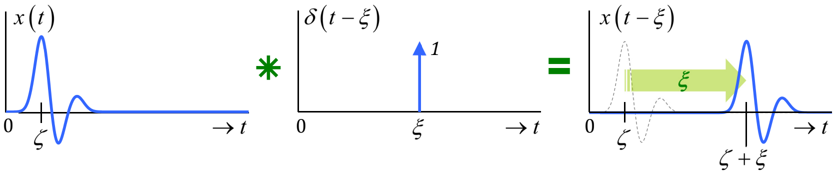
Property C (Sifting - NOT Shifting)

Sifting : $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)\int_{-\infty}^{\infty} \delta(t-\lambda)dt = x(\lambda)$

Property D (Replication - similar idea as duplication)

$$
\begin{aligned}
x(t) * \delta(t-\xi) &= \overbrace{\int_{-\infty}^{\infty} x(\zeta)\delta(t-\zeta-\xi)d\zeta = \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta-(t-\xi))d\zeta}^{\textit{apply symmetry property}} \\
&= x(t-\xi) \qquad \ldots \textit{apply sifting in the second integral}
\end{aligned}
$$

Note that $x(t) * \delta(t) = x(t)$ *ie no change to* $x(t)$.

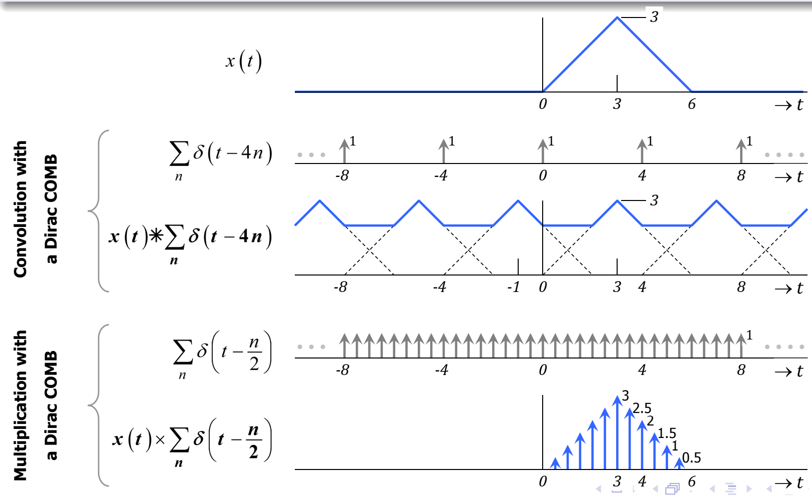


Notice that $x(t)$ is replicated or duplicated at $t = \xi$ after convolution with $\delta(t-\xi)$. Another interpretation is $x(t)$ is SHIFTED in this process!

Example 1 (Convolution & multiplication with Dirac comb function)

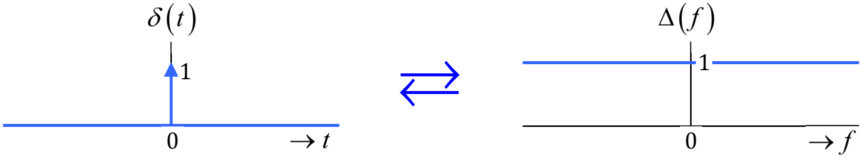
For the signal $x(t)$ *shown below, sketch :*

- $x(t) * \sum_n \delta(t - n4)$
- $x(t). \sum_n \delta(t - n/2)$



Property E (White Spectrum)

$$\Delta(f) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = 1 \ldots \textit{Fourier transform of } \delta(t)$$



This shows that the spectrum of $\delta(t)$ ($\Delta(f)$) consists of frequencies from $-\infty$ to ∞ with a magnitude of 1 across all frequencies. This is analogous to white light which contains all frequencies in the optical range (7 colours in the rainbow). Hence the spectrum of $\delta(t)$ is also known as the white spectrum.

Before going into sampling and re-construction, we shall first introduce the idea of ideal reconstruction filters.

Ideal Reconstruction Filters

- A filter is a system (can be a circuit) which alters the characteristics of a signal passing thru it. Typically it is a linear time invariant (LTI) system which alters the spectrum of the signal.
- A LTI system is completely characterized by its frequency response $H(f)$ in the frequency-domain or its impulse response $h(t)$ in the time-domain where $h(t) = \mathcal{F}^{-1}[H(f)]$.

$$\left.\begin{array}{c} \textit{System} \\ \boldsymbol{Input/Output} \\ \textit{relationship} \end{array}\right\} \rightarrow \left\{ \begin{array}{l} \textit{Frequency-domain}: X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = H(f) \times X(f) \\ \\ \textit{Time-domain}: \quad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) \ast x(t) \end{array}\right.$$

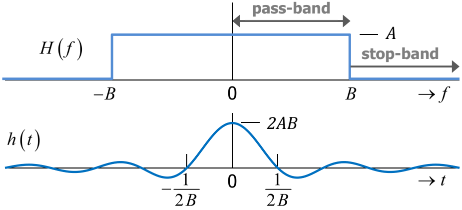
- The band of frequencies passed by a filter is referred to as the pass-band, and the band of frequencies rejected by a filter is called the stop-band.
- An ideal filter is one that has full transmission in the pass-band, and complete attenuation in the stop-band, with abrupt transitions. Two such filters are defined below.

- Ideal Lowpass Filter (LPF)

Frequency response : $H(f) = A\,\mathrm{rect}\left(\frac{f}{2B}\right)$

Impulse response : $h(t) = 2AB\,\mathrm{sinc}(2Bt)$

Cutoff frequency = Bandwidth = B



- Ideal Bandpass Filter (BPF)

Frequency response :

$$H(f) = A\left[\mathrm{rect}\left(\frac{f+f_o}{B}\right) + \mathrm{rect}\left(\frac{f-f_o}{B}\right)\right]$$

Impulse response :

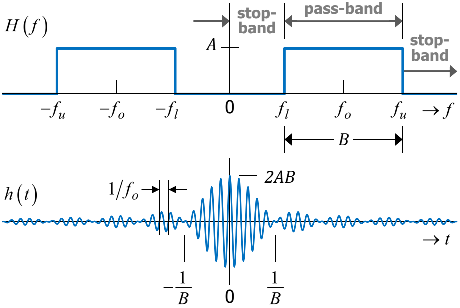
$$h(t) = 2AB\,\mathrm{sinc}(Bt)\cos(2\pi f_o t)$$

Upper Cutoff frequency = f_u

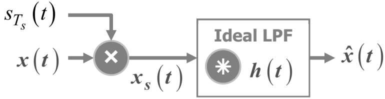
Lower Cutoff frequency = f_l

Center frequency = $f_o = 0.5(f_u + f_l)$

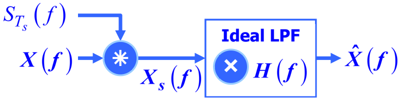
Bandwidth = $B = f_u - f_l$



Continuous-time Sampling & Reconstruction

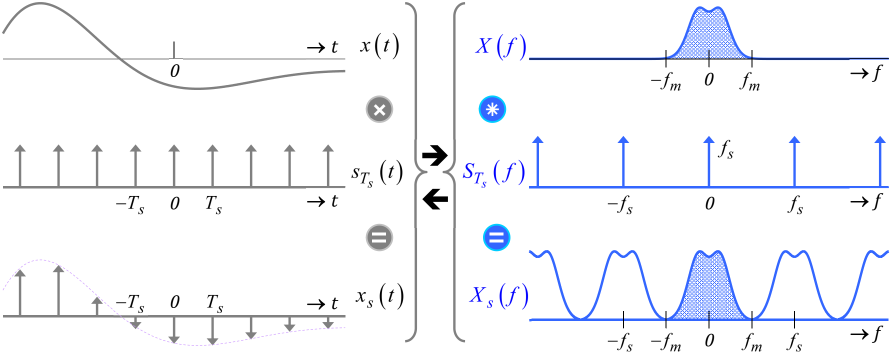


Sampling & reconstruction in t-domain



Equivalent in f-domain

- Sampling with sampling frequency $f_s = \frac{1}{T_s} = 2f_m$ where f_m is the maximum frequency component in $x(t)$ or equivalently $X(f)$.



Mathematically, the spectrum of $x_s(t)$ can be derived as follows :

$$x_s(t) = x(t).\sum_n \delta(t - nT_s)$$

$$\text{In frequency domain } X_s(f) = \overbrace{X(f) * S_{T_s}(f)}^{\text{multiplication property}}$$

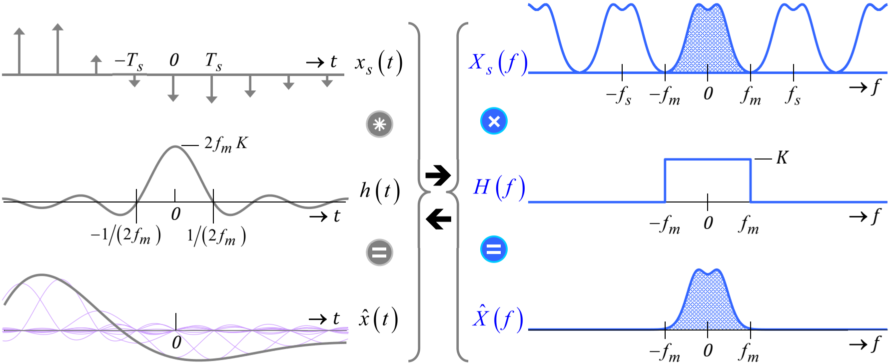
$$\text{Recall that } S_{T_s}(f) = \mathcal{F}[s_{T_s}(t)] = \underbrace{\frac{1}{T_s}\sum_n \delta(f - nf_s)}_{\text{Fourier transform of } s_{T_s}(t)}$$

$$\therefore \quad X_s(f) = X(f) * \frac{1}{T_s}\sum_n \delta(f - nf_s)$$

$$= \underbrace{\frac{1}{T_s}\sum_n X(f - nf_s)}_{\text{replication property}} \qquad (1)$$

Equation (1) means that the spectrum of $x(t)$ has been replicated infinite times at $f = nf_s$, n from $-\infty$ to ∞. Therefore $x_s(t)$ now contains infinite frequencies and recovery of $x(t)$ from $x_s(t)$ requires lowpass filtering.

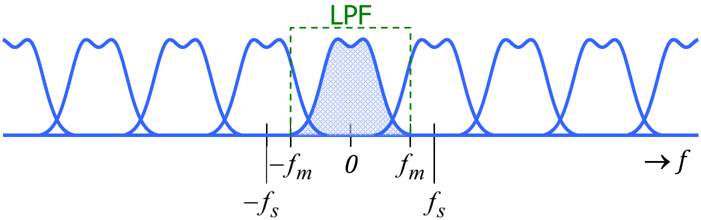
- Reconstruction of $x(t)$ from $x_s(t)$. Reconstruction of $x(t)$ is achieved by filtering $x_s(t)$ thru a LPF.



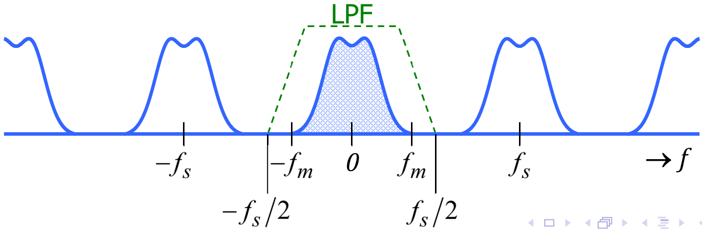
Note that the sampling frequency f_s was deliberately chosen as $f_s = 2f_m$.

Question : What happens when $f_s \neq 2f_m$?

- Sampling frequency, $f_s < 2f_m$
  - Spectral images overlap. This is called frequency aliasing.
  - Perfect reconstruction is not possible.



- Sampling frequency, $f_s > 2f_m$
  - Gaps between spectral images appear due to oversampling.
  - Perfect reconstruction is possible.
  - Do not require an ideal LPF.



Conditions for Reconstruction & Nyquist Sampling Thm

- Conditions for Perfect Reconstruction after Sampling

$$\begin{aligned} x(t) \text{ must be bandlimited} &: \quad X(f) = 0 \quad |f| > f_m \\ \text{Sampling frequency of } x(t) &: \quad f_s \geq 2f_m \\ \text{Reconstruction LPF} &: \quad \begin{cases} |H(f)| = \begin{cases} K & |f| < f_m \\ 0 & |f| > f_m \end{cases} \\ \angle H(f) \ldots \text{linear} \end{cases} \end{aligned}$$

- Nyquist Sampling Theorem
  A band-limited signal of finite energy, which has no frequency components higher than f_m Hz, may be completely recovered from a knowledge of its samples taken at the rate of $2f_m$ samples/sec.

  $2f_m$ is called the Nyquist sampling frequency or Nyquist rate.

Example 2

A signal $x(t) = \text{sinc}^2(2t)$ is sampled at 8 Hz to produce the sampled signal $x_s(t)$. Sketch the spectra of $x(t)$ and $x_s(t)$. Can $x(t)$ be perfectly reconstructed from $x_s(t)$ using an ideal low-pass filter? If "yes", specify the filter. What is the Nyquist sampling frequency for $x(t)$?

From Fourier transform table : $\mathcal{F}[A\,\text{tri}(t/T)] = AT\text{sinc}^2(fT)$.

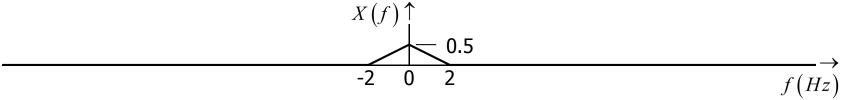
Letting $T = 2$ and $A = 0.5$, we get $\mathcal{F}[0.5\,\text{tri}(t/2)] = \text{sinc}^2(2f)$.

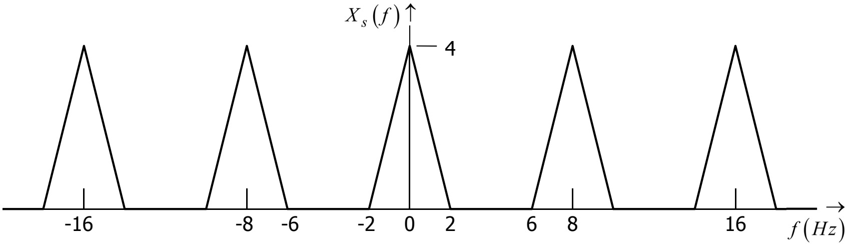
Applying the duality property of the Fourier transform :
$X(f) = \mathcal{F}[\text{sinc}^2(2t)] = 0.5\,\text{tri}(-f/2) = 0.5\,\text{tri}(f/2)\ \ldots f_m = 2$ Hz

The sampled signal and its Fourier transform are given by :

$$\begin{aligned} x_s(t) &= x(t).\textstyle\sum_n \delta(t - \frac{n}{8}) \\ &= \text{sinc}^2(2t).\textstyle\sum_n \delta(t - \frac{n}{8}) \\ &= \textstyle\sum_n \text{sinc}^2(2\frac{n}{8})\delta(t - \frac{n}{8}) \end{aligned}$$

$$\begin{aligned} X_s(f) &= X(f) * 8\textstyle\sum_n \delta(f - 8n) \\ &= 0.5\,\text{tri}(\frac{f}{2}) * 8\textstyle\sum_n \delta(f - 8n) \\ &= 4\textstyle\sum_n \text{tri}(\frac{f-8n}{2}) \end{aligned}$$

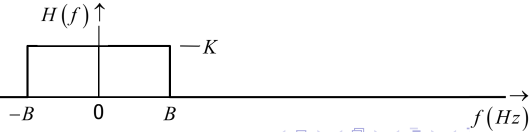




The Nyquist frequency for $x(t)$ is $2 \times f_m = 4$ Hz. However $x(t)$ has been oversampled at 8 Hz which satisfies the Nyquist sampling theorem. Hence $x(t)$ can be recovered from $x_s(t)$ using an ideal lowpass filter. Choose B of the LPF to be anywhere between 2 and 6 Hz.

Reconstruction filter

$2 \leq B \leq 6 \qquad K = 1/8$



Exercise 1

Consider $x(t) = \text{rect}(t/T_w)$ *and* $y(t) = \sum_n \delta(t - nT_p)$.

- *Construct another signal* $v(t) = x(t) * y(t)$.
- *Assume* $T_p > T_w > 0$, *sketch the spectra of* $x(t),\ y(t)$ *and* $v(t)$.
- *What is* $v(t)$ *if* $T_p = T_w$?

Exercise 2

The signal $x(t) =$ sinc$(2t)$ is sampled at 4 Hz to obtain the sampled signal, $x_s(t)$.

- *Derive the Fourier transform, $X_s(f)$ of the sampled signal $x_s(t)$ and sketch its spectrum.*
- *What is the Nyquist sampling frequency?*
- *If $x(t)$ is sampled at a frequency of 2 Hz, sketch the sampled signal.*

Exercise 3

A bandpass signal $x(t)$ has a spectrum $X(f)$ given in the figure below.

- *What is the Nyquist sampling frequency for $x(t)$?*
- *Can $x(t)$ be perfectly reconstructed if the sampling frequency is 20 Hz? Show your answer graphically.*