EE2023 Signals & Systems
Chapter 3 – Continuous-Frequency Spectrum (Fourier Transform)

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Fourier Transform : Introduction

- Fourier series analysis is useful as it decomposes a **periodic** signal, $x_p(t)$, into an infinite number of harmonically related complex sinusoids (complex exponential functions). Hence, a periodic time-domain signal may be represented by discrete-frequency spectra plots.
- Likewise, an **aperiodic** time-domain signal may be represented in the frequency domain using a technique known as Fourier transform.
  - The **continuous-frequency spectral**, $X(f)$, of a time-domain signal, $x(t)$, is given by the Fourier transform analysis (a.k.a. forward Fourier transform) operation

    $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt \qquad \ldots \text{Compute } X(f) \text{ from } x(t)$$

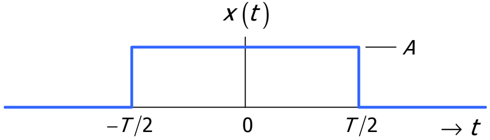
    In general, $X(f)$ is a continuous-frequency complex function represented graphically by the magnitude ($|X(f)|$ vs f) and phase ($\angle X(f)$ vs f) spectra plots.
  - The Fourier transform synthesis (a.k.a. inverse Fourier transform) process is defined as

    $$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df \qquad \ldots \text{Construct } x(t) \text{ from } X(f)$$

  - $X(f)$ is the limiting form of c_k with $T_P \to \infty$, $k \to \infty$ and $\frac{k}{T_p} \to f$.

Example Derive the spectrum of a rectangular pulse, $x(t)$.

$$x(t) = A \cdot \mathrm{rect}(\frac{t}{T}) = \begin{cases} A; & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0; & \text{otherwise} \end{cases}$$



The Fourier Transform of $x(t)$ is

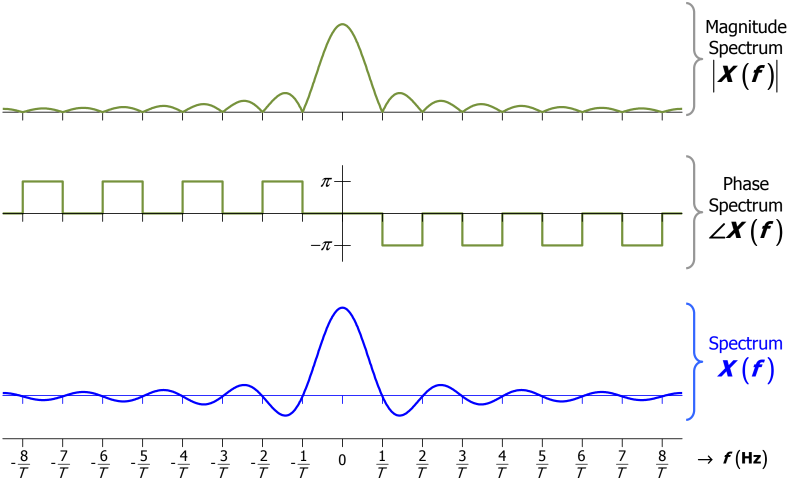
$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)\, e^{(-j2\pi ft)}\, dt \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} A \cdot e^{(-j2\pi ft)}\, dt \\ &= \begin{cases} AT\frac{\sin(\pi fT)}{\pi fT} & f \neq 0 \\ AT & f = 0 \end{cases} \\ &= AT\,\mathrm{sinc}\,(fT) \end{aligned}$$

$A \cdot \mathrm{rect}\left(\frac{t}{T}\right)$ and $AT\,\mathrm{sinc}\,(fT)$ are known as **Fourier transform pairs** i.e.

$$A \cdot \mathrm{rect}\left(\frac{t}{T}\right) \rightleftarrows AT\,\mathrm{sinc}\,(fT)$$

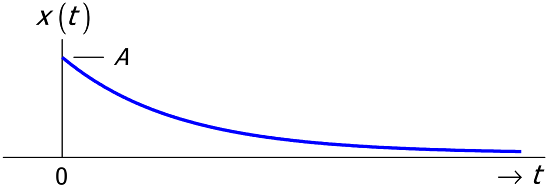
$X(f)$ is a real function, which may be presented as two plots (magnitude and phase spectra) OR as a single plot (spectrum).

$$|X(f)| = AT|\text{sinc}\,(fT)| \qquad \angle X(f) = \begin{cases} 0; & X(f) \geq 0 \\ \pm\pi; & X(f) < 0 \end{cases}$$



Example Derive the spectrum of an exponentially decaying pulse, $x(t)$.

$$x(t) = Ae^{-\alpha t}u(t) = \begin{cases} Ae^{-\alpha t}; & t \geq 0 \\ 0; & t < 0 \end{cases}$$

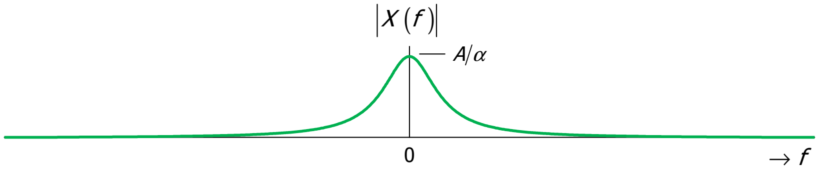


where $\alpha > 0$.

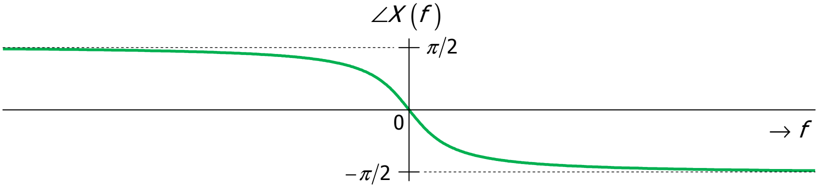
The Fourier Transform of $x(t)$ is

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)\, e^{(-j2\pi ft)} dt \\ &= \int_{0}^{\infty} Ae^{(-\alpha t)} e^{(-j2\pi ft)dt} \\ &= A\left[\frac{e^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)}\right]_0^{\infty} = \frac{A}{\alpha + j2\pi f} \end{aligned}$$

- Magnitude spectrum : $|X(f)| = \sqrt{Re[X(f)]^2 + Im[X(f)]^2} = \dfrac{A}{\sqrt{\alpha^2 + 4\pi^2 f^2}}$



- Phase spectrum : $\angle X(f) = \tan^{-1}\left(\dfrac{Im[X(f)]}{Re[X(f)]}\right) = -\tan^{-1}\left(\dfrac{2\pi f}{\alpha}\right)$



Note that $X(f)$ is a complex function. Hence, the magnitude and phase spectra cannot be combined into a single spectrum.

Fourier Transforms of Basic Functions

Basic time-domain signals, $x(t)$		Spectrum, $X(f)$
Constant	K	$K\delta(f)$
Unit Impulse	$\delta(t)$	$\mathbf{1}$
Unit Step	$u(t)$	$\frac{1}{2}\left[\delta(f)+\frac{1}{j\pi f}\right]$
Sign (or Signum)	$\operatorname{sgn}(t)$	$\frac{1}{j\pi f}$
Rectangle	$\operatorname{rect}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}(fT)$
Triangle	$\operatorname{tri}\left(\frac{t}{T}\right)$	$T\operatorname{sinc}^2(fT)$
Sine Cardinal	$\operatorname{sinc}\left(\frac{t}{T}\right)$	$T\operatorname{rect}(fT)$
Complex Exponential	$\exp(j2\pi f_o t)$	$\delta(f-f_o)$
Cosine	$\cos(2\pi f_o t)$	$\frac{1}{2}\left[\delta(f-f_o)+\delta(f+f_o)\right]$
Sine	$\sin(2\pi f_o t)$	$-\frac{j}{2}\left[\delta(f-f_o)-\delta(f+f_o)\right]$
Gaussian	$\exp\left(-\frac{t^2}{\alpha^2}\right)$	$\alpha\pi^{0.5}\exp\left(-\alpha^2\pi^2 f^2\right)$
Comb	$\sum_{m=-\infty}^{\infty}\delta(t-mT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

Fourier transform Properties

- Much of the usefulness of Fourier transform stems directly from its properties.
- The Fourier transform properties might appear to be abstract mathematical manipulations. However, they have important interpretation and meaning in signal and image processing. The applications of the properties should become clearer when sampling and filtering are discussed in Chapter 5.
- Fourier transform properties will be presented using the following syntax:

$$X(f) = \mathcal{F}\{x(t)\} \qquad \ldots \text{Forward Fourier transform}$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} \qquad \ldots \text{Inverse Fourier transform}$$

$$x(t) \rightleftarrows X(f) \qquad \ldots \text{Fourier transform pair}$$

FT Properties – Linearity

$$\alpha x_1(t) + \beta x_2(t) \rightleftarrows \alpha X_1(f) + \beta X_2(f)$$

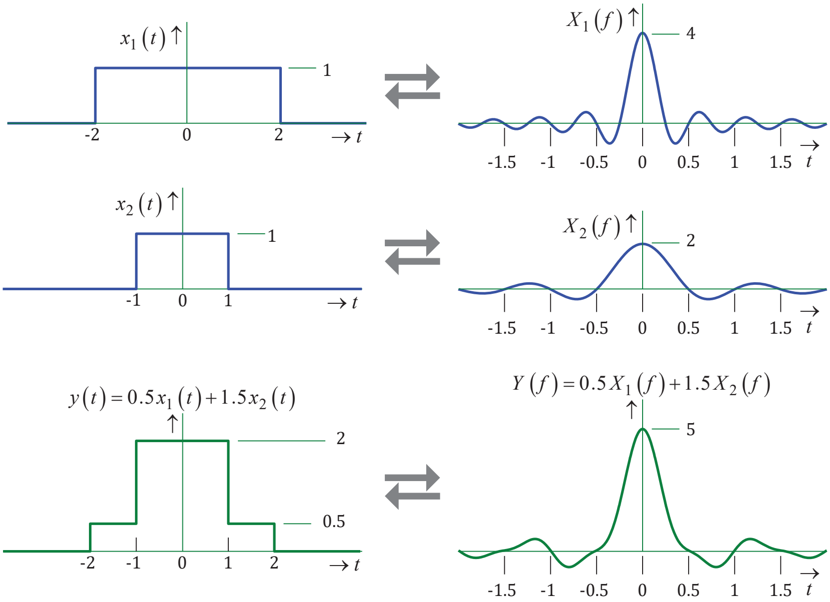
$\boxed{\text{Example}}$ Derive the Fourier transform of $y(t) = 0.5x_1(t) + 1.5x_2(t)$ where $x_1(t) = \text{rect}\left(\frac{t}{4}\right)$ and $x_2(t) = \text{rect}\left(\frac{t}{2}\right)$

- The Fourier transfrom table contains the Fourier transform pair $\text{rect}\left(\frac{t}{T}\right) \rightleftarrows T\text{sinc}(fT)$

$$\therefore \quad x_1(t) = \text{rect}\left(\frac{t}{4}\right) \rightleftarrows X_1(f) = 4\text{sinc}(4f) \qquad x_2(t) = \text{rect}\left(\frac{t}{2}\right) \rightleftarrows X_2(f) = 2\text{sinc}(2f)$$

- Applying the **Linearity** FT property,

$$\begin{aligned} Y(f) &= 0.5X_1(f) + 1.5X_2(f) = 0.5[4\text{sinc}(4f)] + 1.5[2\text{sinc}(2f)] \\ &= 2\text{sinc}(4f) + 3\text{sinc}(2f) \end{aligned}$$


$x_1(t)$
1
-2
0
2
$\to t$
$X_1(f)$
4
-1.5
-1
-0.5
0
0.5
1
1.5
$\to t$
$x_2(t)$
1
-1
0
1
$\to t$
$X_2(f)$
2
-1.5
-1
-0.5
0
0.5
1
1.5
$\to t$
$y(t) = 0.5x_1(t) + 1.5x_2(t)$
2
0.5
-2
-1
0
1
2
$\to t$
$Y(f) = 0.5X_1(f) + 1.5X_2(f)$
5
-1.5
-1
-0.5
0
0.5
1
1.5
$\to t$

FT Properties – Time Scaling

$$x(\beta t) \rightleftarrows \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right)$$

Example

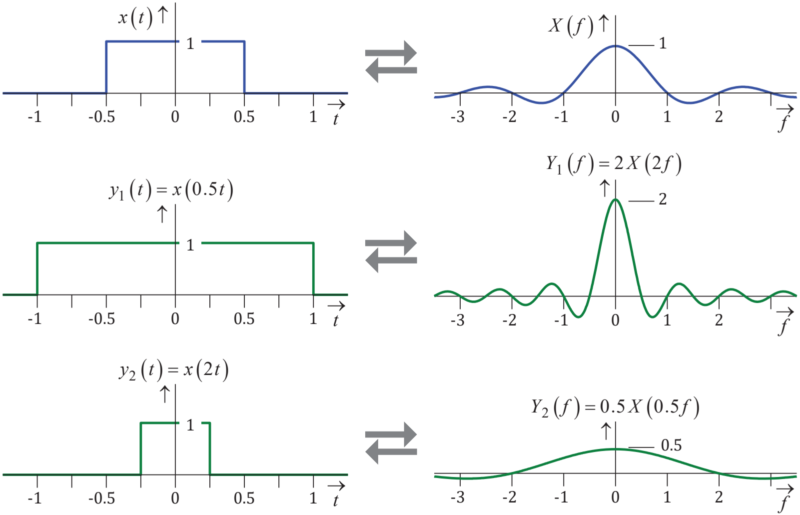
Given the Fourier transform pair $x(t) = \text{rect}\left(\frac{t}{1}\right) \rightleftarrows X(f) = 1 \cdot \text{sinc}\,(1.f)$, find the Fourier transform of

$$\begin{aligned} y_1(t) &= x(0.5t) = \text{rect}\left(\frac{0.5t}{1}\right) && \ldots \text{Expansion in time domain} \\ y_2(t) &= x(2t) = \text{rect}\left(\frac{2t}{1}\right) && \ldots \text{Compression in time domain} \end{aligned}$$

Applying the **time scaling** FT property,

$$\begin{aligned} Y_1(f) &= 2X(2f) = 2\,\text{sinc}(2f) && \ldots \text{Compression in freq domain} \\ Y_2(f) &= 0.5X(0.5f) = 0.5\,\text{sinc}(0.5f) && \ldots \text{Expansion in freq domain} \end{aligned}$$

Linear scaling in time is reflected as an inverse scaling in frequency



FT Properties – Duality

$$X(t) \rightleftarrows x(-f) \quad \text{or} \quad X(-t) \rightleftarrows x(f)$$

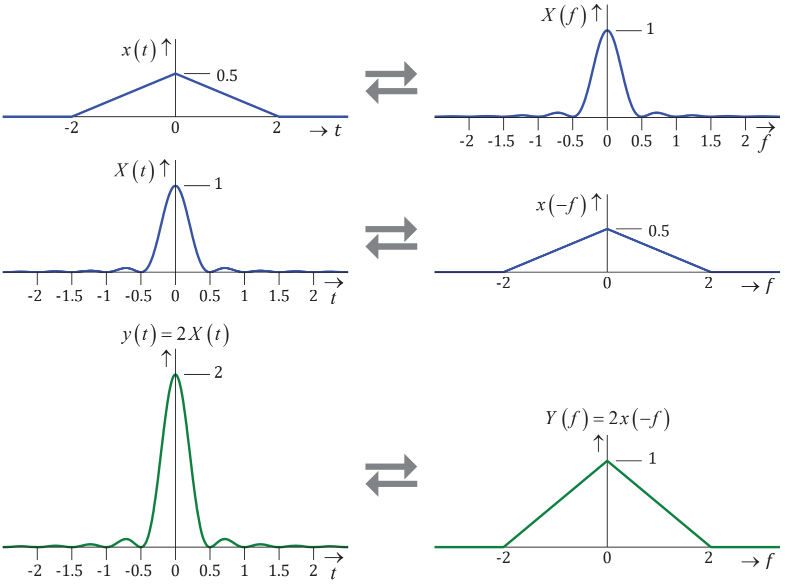
Example

Given the Fourier transform pair $x(t) = \frac{1}{2}\,\text{tri}\left(\frac{t}{2}\right) \rightleftarrows X(f) = \text{sinc}^2(2f)$, find the Fourier transform of $y(t) = 2\,\text{sinc}^2(2t)$.

- Conclusion from comparing $y(t)$ and $X(f)$ is $y(t) = 2X(t)$.
- Applying the **duality** FT property,

$$\underbrace{\frac{1}{2}\,\text{tri}\left(\frac{t}{2}\right)}_{x(t)} \rightleftarrows \underbrace{\text{sinc}^2(2f)}_{X(f)} \xrightarrow{\text{Duality Property}} \underbrace{\text{sinc}^2(2t)}_{X(t)} \rightleftarrows \underbrace{\frac{1}{2}\,\text{tri}\left(\frac{-f}{2}\right)}_{x(-f)}$$

- Hence, $Y(f) = \mathcal{F}\{2X(t)\} = 2\mathcal{F}\{\text{sinc}^2(2t)\} = \text{tri}\left(\frac{-f}{2}\right) = \text{tri}\left(\frac{f}{2}\right)$


$x(t)$
0.5
-2
0
2
$\rightarrow t$
$X(f)$
1
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
$\rightarrow f$
$X(t)$
1
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
$\rightarrow t$
$x(-f)$
0.5
-2
0
2
$\rightarrow f$
$y(t) = 2X(t)$
2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
$\rightarrow t$
$Y(f) = 2x(-f)$
1
-2
0
2
$\rightarrow f$

FT Properties – Time shifting

$$x(t-t_0) \rightleftarrows X(f)\, e^{-j2\pi t_0 f}$$

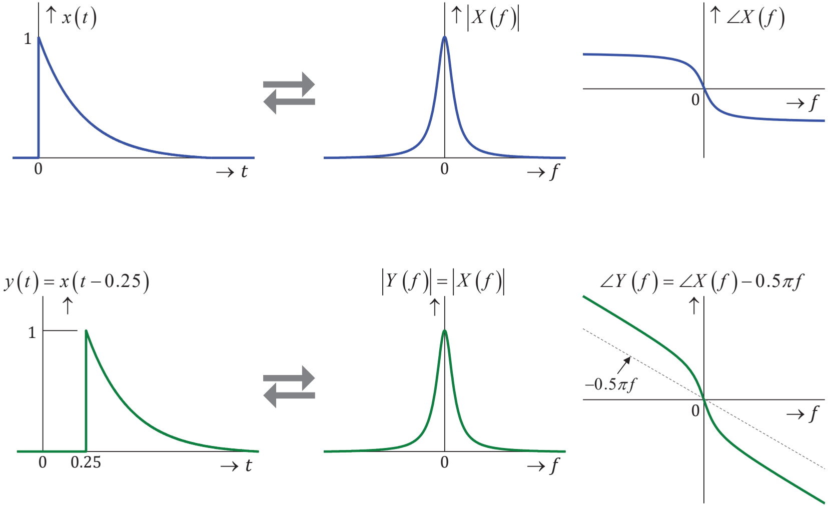
$\boxed{\text{Example}}$ Given the Fourier transform pair $x(t) = e^{-t}\, u(t) \rightleftarrows X(f) = \dfrac{1}{1+j2\pi f}$, find the Fourier transform of $y(t) = x(t-0.25)$.

Applying the **time shifting** FT property,

$$Y(f) = \mathcal{F}\{x(t-0.25)\} = X(f)e^{-j2\pi(0.25)f} = \frac{1}{1+j2\pi f}e^{-j2\pi(0.25)f}$$

$$\begin{aligned} |Y(f)| &= |X(f)| \cdot \underbrace{\left|e^{-j2\pi(0.25)f}\right|}_{=1} = |X(f)| = \frac{1}{\sqrt{1+4\pi^2 f^2}} \\ \angle Y(f) &= \angle X(f) + \angle e^{-j2\pi(0.25)f} \\ &= \angle X(f) - 0.5\pi f = -\tan^{-1}(2\pi f) - 0.5\pi f \end{aligned}$$

Magnitude spectra plots for $x(t)$ and $x(t-1)$ are identical.



FT Properties – Frequency shifting (Modulation)

$$x(t)\, e^{j2\pi f_o t} \rightleftarrows X(f - f_0)$$

Example

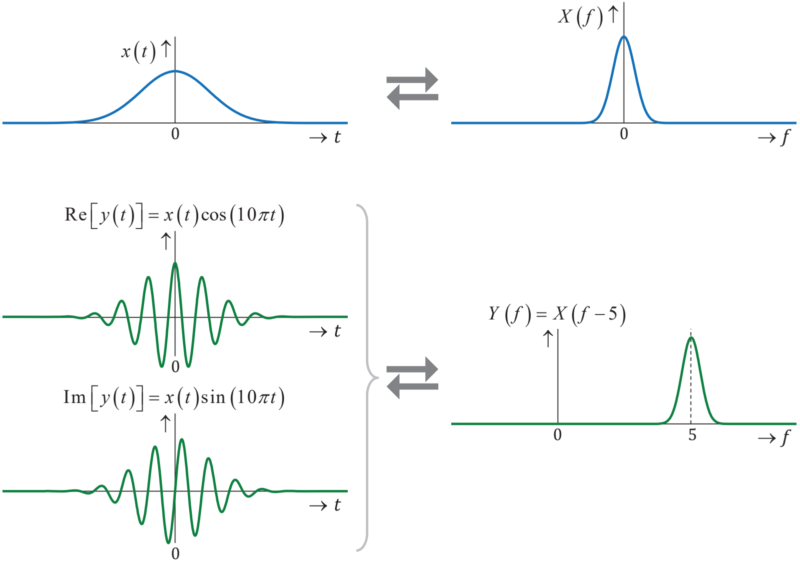
Given the Fourier transform pair $x(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \rightleftarrows X(f) = e^{-2\pi^2 f^2}$, find the Fourier transform of $y(t) = x(t)e^{j2\pi(5)t}$.

- Applying the Euler's Identity,

$$y(t) = x(t)e^{j2\pi(5)t} = \underbrace{x(t)\cos[2\pi(5)t]}_{\text{Real part}} + \underbrace{j\,x(t)\sin[2\pi(5)t]}_{\text{Imaginary part}}$$

- Applying the **frequency shifting** FT property,

$$\begin{aligned} Y(f) = \mathcal{F}\{x(t)e^{j2\pi(5)t}\} &= X(f-5) \\ &= e^{-2\pi^2(f-5)^2} \end{aligned}$$



$x(t)$
0
$\rightarrow t$
$X(f)$
0
$\rightarrow f$
$\mathrm{Re}[y(t)] = x(t)\cos(10\pi t)$
0
$\rightarrow t$
$\mathrm{Im}[y(t)] = x(t)\sin(10\pi t)$
0
$\rightarrow t$
$Y(f) = X(f-5)$
0
5
$\rightarrow f$

FT Properties – Differentiation in time domain

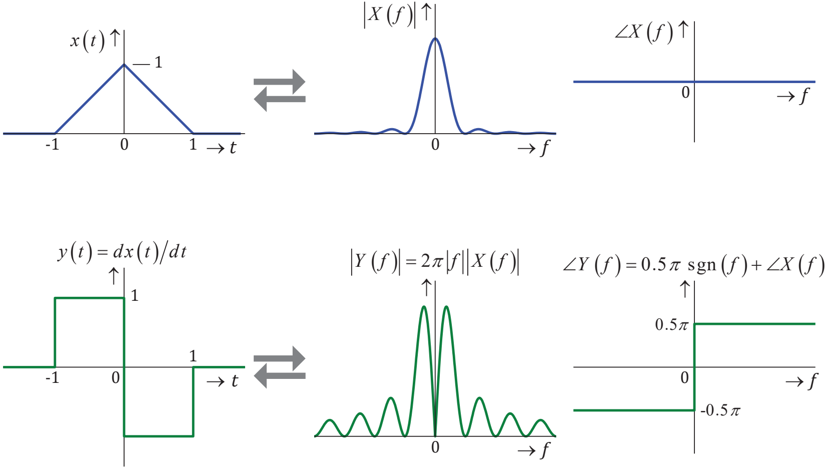
$$\frac{d}{dt}x(t) \rightleftarrows j2\pi f \cdot X(f)$$

Example

Given the Fourier transform pair $x(t) = \text{tri}(t) \rightleftarrows X(f) = \text{sinc}^2(f)$, find the Fourier transform of $\frac{dx(t)}{dt}$

Applying the **differentiation in time domain** FT property,

$$\begin{aligned}
Y(f) &= \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f \cdot X(f) \\
|Y(f)| &= 2\pi|f| \cdot |X(f)| = 2\pi|f| \cdot \text{sinc}^2(f) \\
\angle Y(f) &= \angle j2\pi f + \angle X(f) \\
&= 0.5\pi \,\text{sgn}(f) + \underbrace{\angle X(f)}_{=0} = 0.5\pi \,\text{sgn}(f)
\end{aligned}$$



Differentiating a signal in the time domain has the effect of “emphasizing” the high frequency components in the magnitude spectrum.

Application of differentiating in "time" – Edge detection in image processing

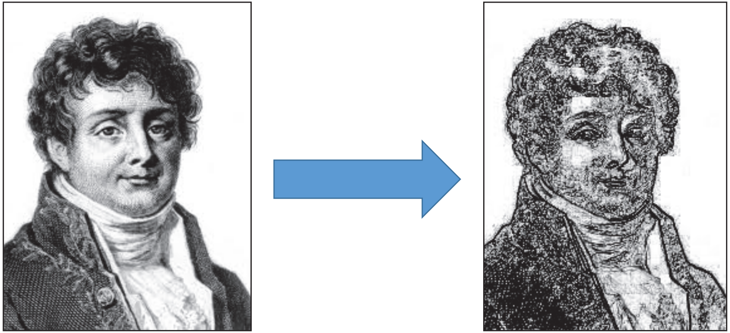


Image of Joseph Fourier

"Derivative" of the original image
Zero gradient mapped to white
Large gradient mapped to black

- "Differentiation" emphasizes abrupt changes.
  - For example, the derivative of the step discontinuity in the square wave is infinitely large.
  - "Differentiating" the orignal image results in more distinct edges.

FT Properties – Integration in time domain

$$\int_{-\infty}^{t} x(\tau)\, d\tau \rightleftarrows \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\delta(f)$$

where $\delta(f)$ is the unit impulse function defined in the frequency domain and $X(0) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi(0)t}\, dt = \int_{-\infty}^{\infty} x(t)\, dt$ is the net area under $x(t)$.

Example

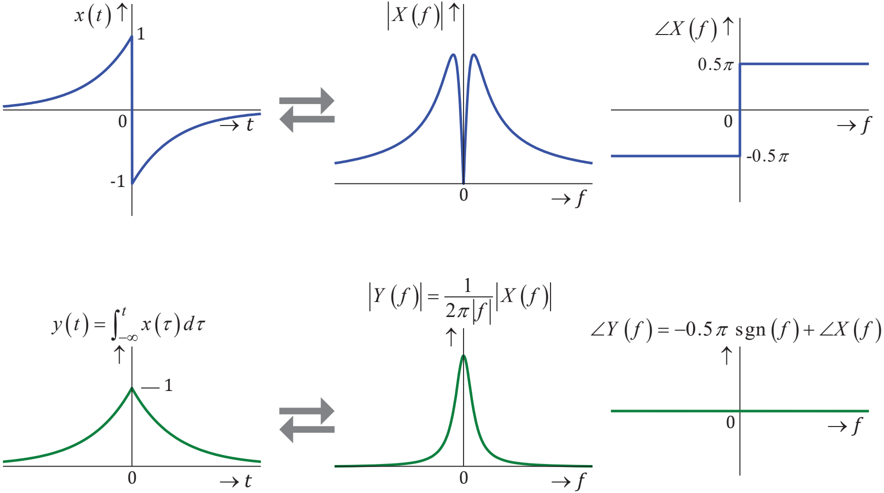
Given the Fourier transform pair

$$x(t) = e^{t}u(-t) - e^{-t}u(t) \rightleftarrows X(f) = \frac{j4\pi f}{1 + 4\pi^2 f^2} = \begin{cases} |X(f)| = \frac{j4\pi|f|}{1+4\pi^2 f^2} \\ \angle X(f) = 0.5\pi \operatorname{sgn}(f) \end{cases}$$

Find the Fourier transform of $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$.

- Since $X(f) = \dfrac{j4\pi f}{1+4\pi^2 f^2}$, $X(0) = \dfrac{j4\pi(0)}{1+4\pi^2(0)^2} = 0$
- Applying the **integration in time domain** FT property,

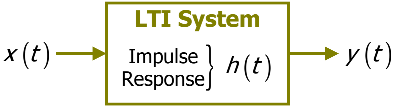
$$\begin{aligned}
Y(f) &= \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} \\
&= \frac{1}{j2\pi f}\cdot X(f) + \frac{1}{2}\underbrace{X(0)}_{=0}\delta(f) \\
&= \frac{1}{j2\pi f}\frac{j4\pi f}{1+4\pi^2 f^2} = \frac{2}{1+4\pi^2 f^2} \\
|Y(f)| &= \frac{1}{j2\pi|f|}\cdot|X(f)| = \frac{2}{1+4\pi^2 f^2} \\
\angle Y(f) &= \angle\frac{1}{j2\pi f} + \angle X(f) \\
&= -0.5\pi\,\mathrm{sgn}(f) + 0.5\pi\,\mathrm{sgn}(f) = 0
\end{aligned}$$


$x(t)$
1
0
-1
$\to t$
$|X(f)|$
0
$\to f$
$\angle X(f)$
0.5π
0
-0.5π
$\to f$
$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$
1
0
$\to t$
$|Y(f)| = \frac{1}{2\pi|f|}|X(f)|$
0
$\to f$
$\angle Y(f) = -0.5\pi\ \mathrm{sgn}(f) + \angle X(f)$
0
$\to f$

FT Properties – Convolution in time domain

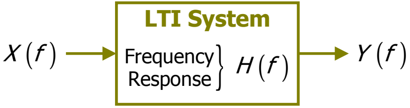
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\alpha)\, x_2(t-\alpha)\, d\alpha \rightleftarrows X_1(f)\, X_2(f)$$

- Convolution in time domain property is also known as Multiplication in frequency domain property.
- This property is central to the study of linear time-invariant systems.



TIME-DOMAIN

$$y(t) = x(t) * h(t)$$



FREQUENCY-DOMAIN

$$Y(f) = X(f) \cdot H(f)$$

$x(t)$ is the input signal, $y(t)$ is the output signal and $h(t)$, known as the **impulse response**, represents the system characteristics.

Convolution

- Convolution of two functions, $v(t)$ and $w(t)$, is defined as

$$\begin{aligned} v(t) * w(t) &= \int_{-\infty}^{\infty} v(\alpha)\, w(t-\alpha)\, d\alpha \\ &= \int_{-\infty}^{\infty} v(t-\alpha)\, w(\alpha)\, d\alpha \end{aligned}$$

  where $*$ denotes the convolution operator.

- Convolution is **commutative**

$$v(t) * w(t) = w(t) * v(t)$$

- Convolution is **associative**

$$[v(t) * w(t)] * z(t) = v(t) * [w(t) * z(t)]$$

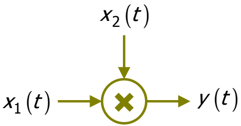
- Convolution is **distributive**

$$v(t) * [w(t) + z(t)] = [v(t) * w(t)] + [v(t) * z(t)]$$

FT Properties – Multiplication in time domain

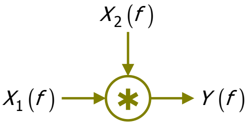
$$x_1(t)x_2(t) \rightleftarrows X_1(f) * X_2(f) = \int_{-\infty}^{\infty} X_1(\alpha)\, X_2(f-\alpha)\, d\alpha$$

- Multiplication in time domain property is also known as Convolution in frequency domain property.
- This property is central to the study of communication systems, where mixer circuits often requires the multiplication of 2 signals.



TIME-DOMAIN

$$y(t) = x_1(t)\, x_2(t)$$



FREQUENCY-DOMAIN

$$Y(f) = X_1(f) * X_2(f)$$

In a modulation circuit, $x_1(t)$ may be the message signal and $x_2(t)$ is the carrier signal (a sinusoidal function).

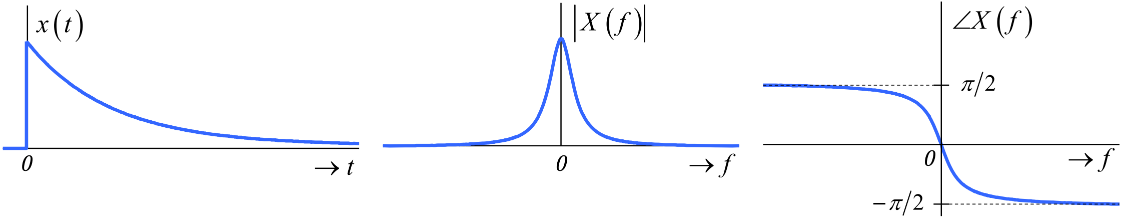
FT Properties – Spectra of real signals

- The time domain signal, $x(t)$ is real (i.e. no imaginary component) if $x(t) = x^*(t)$.
- If $x(t)$ is a real signal, then its spectum is conjugate symmetric i.e.

$$X^*(f) = X(-f)$$

$$\underbrace{|X(f)| = |X(-f)|}_{\text{Even Symmetry}} \qquad \textit{and} \qquad \underbrace{\angle X(f) = -\angle X(f)}_{\text{Odd Symmetry}}$$

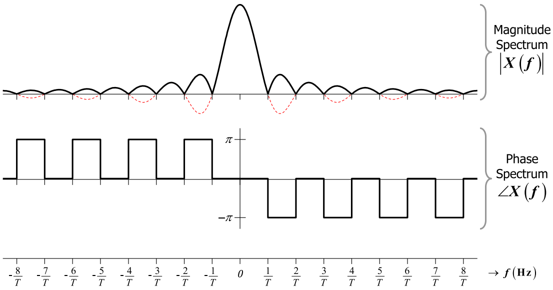
$\boxed{\text{Example}}$ Given $x(t) = e^{-4t}u(t) \rightleftarrows X(f) = \dfrac{1}{4 + j2\pi f}$



Example Given $A\,\mathrm{rect}\left(\frac{t}{T}\right) \rightleftarrows AT\,\mathrm{sinc}\,Tf$

$$|X(f)| = AT|\mathrm{sinc}\,(fT)|$$

$$\angle X(f) = \begin{cases} 0; & X(f) \geq 0 \\ \pm\pi; & X(f) < 0 \end{cases}$$



- Fourier series coefficients of real periodic signal, $x_p(t)$, are also conjugate symmetric

$$\underbrace{c_k^* = c_{-k}}_{\text{Conjugate symmetric}} \qquad \underbrace{|c_k| = |c_{-k}|}_{\text{Even symmetry}} \qquad \underbrace{\angle c_k = -\angle c_{-k}}_{\text{Odd symmetry}}$$

- Due to the conjugate symmetric property, either the positive frequency portion or the negative frequency portion of a spectrum would suffice to specify a real signal completely. Hence, **spectrum analyzers** only display the positive frequency portion of a spectrum.

Continuous-frequency spectrum of DC and Complex Exponential signal

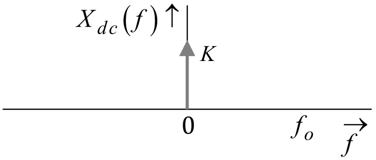
- **DC Signal**, $x_{dc}(t) = K$
  A dc signal may be viewed as a periodic signal of arbitrary period.

$$\mathcal{F}\{K\delta(t)\} = K$$

$$\Big\downarrow \text{ Duality Property}$$

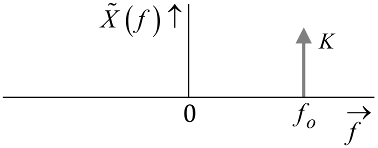
$$\mathcal{F}\{K\} = K\delta(f)$$

$$X_{dc}(f) = \mathcal{F}\{K\} = K\delta(f)$$



- **Complex Exponential Signal**, $\tilde{x}(t) = Ke^{j2\pi f_o t}$
  Using the frequency-shifting property,

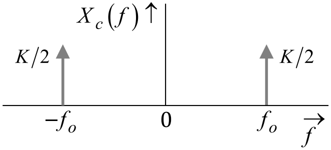
$$\begin{aligned}\tilde{X}(f) &= \mathcal{F}\{\tilde{x}(t)\} \\ &= K\delta(f - f_o)\end{aligned}$$



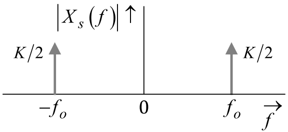
Continuous-frequency spectrum of Real Sinusoidal Signals

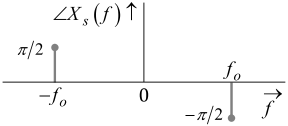
- **Cosine signal**, $x_c(t) = K\cos(2\pi f_o t)$

$$
\begin{aligned}
x_c(t) &= \frac{K}{2}e^{j2\pi f_o t} + \frac{K}{2}e^{-j2\pi f_o t} \\
X_c(f) &= \mathcal{F}\{x_c(t)\} = \frac{K}{2}\delta(f - f_o) + \frac{K}{2}\delta(f + f_o)
\end{aligned}
$$



- **Sine signal**, $x_s(t) = K\sin(2\pi f_o t)$





$$
\begin{aligned}
x_s(t) &= \frac{K}{j2}e^{j2\pi f_o t} - \frac{K}{j2}e^{-j2\pi f_o t} \\
&= \frac{K}{2}e^{-j\frac{\pi}{2}}e^{j2\pi f_o t} + \frac{K}{2}e^{j\frac{\pi}{2}}e^{-j2\pi f_o t} \\
X_s(f) &= \mathcal{F}\{x_s(t)\} \\
&= \frac{K}{2}e^{-j\frac{\pi}{2}}\delta(f - f_o) + \frac{K}{2}e^{j\frac{\pi}{2}}\delta(f + f_o)
\end{aligned}
$$

Continuous-frequency spectrum of Arbitrary Periodic Signals

A periodic signal $x_p(t)$, with fundamental period T_p, may be expressed as

$$x_p\left(t\right) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_p} t} \quad \text{where } c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t) e^{-j2\pi \frac{k}{T_p} t}\, dt$$

Applying Fourier Transform to $x_p(t)$,

$$X_p(f) = \mathcal{F}\{x_p(t)\} = \underbrace{\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_p} t}\right\} = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left\{e^{j2\pi \frac{k}{T_p} t}\right\}}_{\text{Linearity property of Fourier Transform}}$$

Since $\mathcal{F}\left\{e^{j2\pi \frac{k}{T_p} t}\right\} = \delta\left(f - \frac{k}{T_p}\right)$,

$$\boxed{X_p\left(f\right) = \sum_{k=-\infty}^{\infty} c_k\, \delta\left(f - \frac{k}{T_p}\right)}$$

Implication: Continuous-frequency spectrum, $X_p(f)$ of any periodic signal, $x_p(t)$, can be obtained by first computing c_k then substituting into $X_p(f) = \sum_{k=-\infty}^{\infty} c_k\, \delta\left(f - \frac{k}{T_p}\right)$.

Example

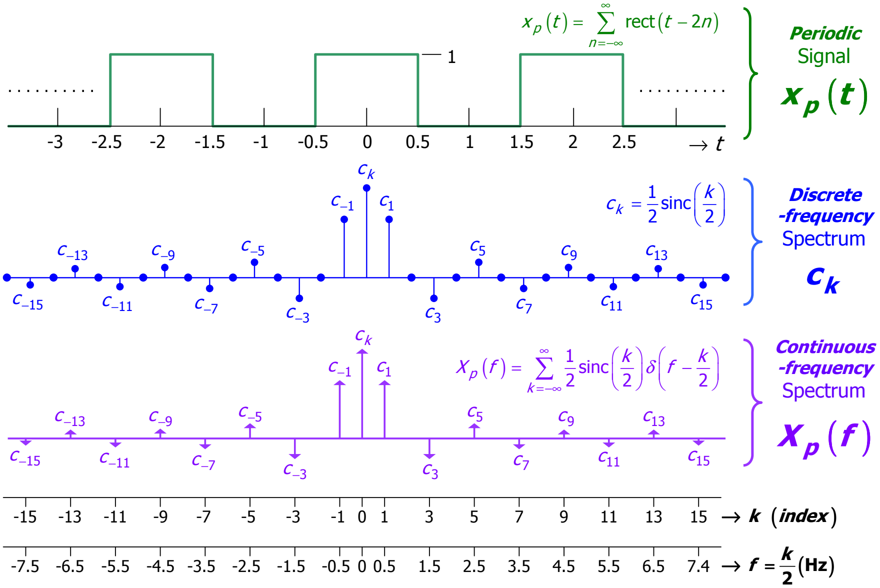
Consider the periodic pulse train, $x_p(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\,(t - 2n)$.

- Period of $x_p(t)$ is $T_p = 2$.
- As shown in Chapter 2, Fourier Series Coefficients for x_p are

$$c_k = \frac{1}{2}\int_{-0.5}^{1.5} x_p(t)\, e^{-jk\pi t}\, dt = \frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right)$$

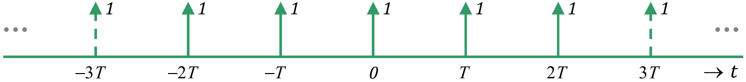
- Hence,

$$X_p(f) = \sum_{k=-\infty}^{\infty} \frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right)\delta\left(f - \frac{k}{2}\right)$$

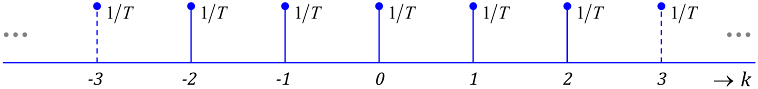

$x_p(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(t - 2n)$
Periodic Signal
$x_p(t)$
$c_k = \frac{1}{2}\mathrm{sinc}\left(\frac{k}{2}\right)$
Discrete -frequency Spectrum
c_k
$X_p(f) = \sum_{k=-\infty}^{\infty} \frac{1}{2}\mathrm{sinc}\left(\frac{k}{2}\right)\delta\left(f - \frac{k}{2}\right)$
Continuous -frequency Spectrum
$X_p(f)$
$\rightarrow k$ (*index*)
$\rightarrow f = \frac{k}{2}$(Hz)

Example

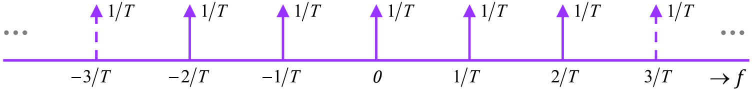
- **Dirac Comb** function (a.k.a impulse train) is $\xi_T(t) = \displaystyle\sum_{n=-\infty}^{\infty} \delta(t - nT)$



- Discrete-frequency spectrum, $c_k = \dfrac{1}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} \xi_T(t) e^{-j2\pi\frac{k}{T}t}\, dt = \dfrac{1}{T}\displaystyle\int_{-\infty}^{\infty} \delta(t) e^{-j\frac{2\pi k}{T}t}\, dt = \dfrac{1}{T}$



- Continuous-frequency spectrum, $\Xi_t(f) = \displaystyle\sum_{k=-\infty}^{\infty} c_k\, \delta\left(f - \frac{k}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$

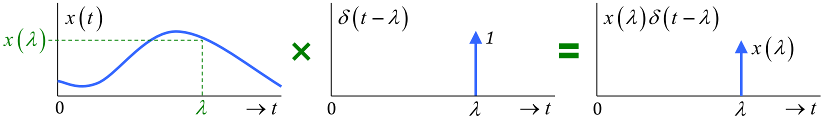


Properties of Dirac-δ function

As introduced in Chapter 1, the **unit impulse** function, also known as the **Dirac Delta**function, is defined as

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases} \qquad \text{and} \qquad \int_{-\epsilon}^{\epsilon} \delta(t)\, dt = 1; \quad \forall \epsilon > 0$$

- **Symmetry**: $\delta(t) = \delta(-t)$
- **Sampling**: $\boxed{x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)}$



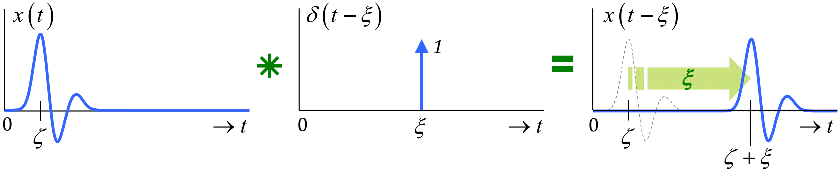
- **Sifting**: $\displaystyle\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)\, dt = \int_{-\infty}^{\infty} \underbrace{x(\lambda)\delta(t-\lambda)}_{\text{from sampling property}}\, dt = x(\lambda)$
- **δ-Square** : Applying sampling property, $\delta^2(t) = \infty \cdot \delta(t)$

- **Replication**: $\boxed{x(t) * \delta(t-\xi) = x(t-\xi)}$

$$x(t) \quad * \quad \delta(t-\xi) = \int_{-\infty}^{\infty} x(\alpha)\,\delta(t-\alpha-\xi)\,d\alpha$$

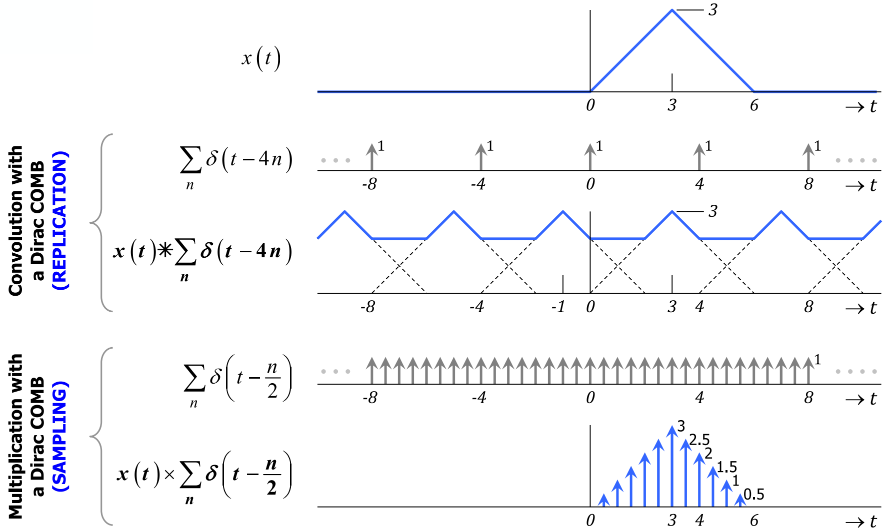
$$= \underbrace{\int_{-\infty}^{\infty} x(\alpha)\,\delta(-(\alpha-t+\xi))\,d\alpha = \int_{-\infty}^{\infty} (\alpha)\,\delta(\alpha-t+\xi)\,d\alpha}_{\text{Symmetry of } \delta(t)}$$

$$= \underbrace{\int_{-\infty}^{\infty} x(\alpha)\,\delta(\alpha-(t-\xi))\,d\alpha = x(t-\xi)}_{\text{Sifting Property}}$$



If $\xi = 0$, then the replication property reduces to $x(t) * \delta(t) = x(t)$

Example


$x(t)$
$\sum_n \delta(t-4n)$
$x(t) * \sum_n \delta(t-4n)$
$\sum_n \delta\left(t-\frac{n}{2}\right)$
$x(t) \times \sum_n \delta\left(t-\frac{n}{2}\right)$
Convolution with a Dirac COMB (REPLICATION)
Multiplication with a Dirac COMB (SAMPLING)
3
0
3
6
$\rightarrow t$
1
-8
-4
-1
4
8
2.5
2
1.5
0.5