ent 2

Assignment 5

$\frac{1}{2}$

$\Rightarrow$

1a. Let $u = \frac{x}{a}$, $v = \frac{y}{b}$

$x = au$ $\quad$ $y = bv$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

when $z = 0$, $u^2 + v^2 = 1$

$$\therefore \text{area} = \iint dx\,dy$$

$$= \iint ab\; du\,dv$$

$$= ab \int_0^{2\pi} \int_0^1 r\,dr\,d\theta$$

$$= ab \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$= ab \int_0^{2\pi} \frac{1}{2}\, d\theta$$

$$= ab \left[\frac{\theta}{2}\right]_0^{2\pi}$$

$$= ab\pi$$

b. Let $x = au$, $y = bv$, $z = cw$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow u^2 + v^2 + w^2 = 1 \Rightarrow r = 1$$

$$\therefore \text{Volume} = \iiint_V dV = \iiint abc \, du\,dv\,dw$$

$$= abc\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{4}{3} abc\pi$$

c. $I_{ij} = \iiint \rho(\underline{r}) \left( \delta_{ij} \sum_{k=1}^{3} x_k^2 - x_i x_j \right) dV, \quad i,j = 1,2,3$

Since only $i=j$ terms exist, solve for $I_{11}, I_{22}, I_{33}$

$I_{11} = \iiint \rho(\underline{r}) (x_1^2 + x_2^2 + x_3^2 - x_1^2)\, dV$
$= \iiint \rho(\underline{r}) (x_2^2 + x_3^2)\, dV$

Let $x_1 = au, \; x_2 = bv, \; x_3 = cw$

$I_{11} = \rho(\underline{r}) \iiint (b^2 v^2 + c^2 w^2)\, abc \; du\,dv\,dw$

Let $u = r\sin\theta\cos\phi$
$v = r\sin\theta\sin\phi$
$w = r\cos\theta$

$$J = \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$I_{11} = \rho(\underline{r}) \iiint (b^2 v^2 + c^2 w^2)\, abc \; du\,dv\,dw$

$= \rho(\underline{r})abc \int_0^{2\pi}\int_0^{\pi}\int_0^1 \left[ b^2(r\sin\theta\sin\phi)^2 + c^2(r\cos\theta)^2 \right] r^2\sin\theta \; dr\,d\theta\,d\phi$

$= \rho(\underline{r})abc \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^4 (b^2\sin^3\theta\sin^2\phi + c^2\cos^2\theta\sin\theta)\, dr\,d\theta\,d\phi$

$= \rho(\underline{r})abc \int_0^{2\pi}\int_0^{\pi} \left[ \frac{r^5}{5}(b^2\sin^3\theta\sin^2\phi + c^2\cos^2\theta\sin\theta) \right]_0^1 dr\,d\theta\,d\phi$

$= \rho(\underline{r})abc\left(\frac{1}{5}\right) \int_0^{2\pi}\int_0^{\pi} b^2\sin^3\theta\sin^2\phi + c^2\cos^2\theta\sin\theta \; d\theta\,d\phi$

$= \frac{1}{5}\rho(\underline{r})\,abc \int_0^{2\pi} \left[ b^2\sin^2\phi\left(\frac{1}{3}\cos^3\theta - \cos\theta\right) - c^2\left(\frac{1}{3}\cos^3\theta\right) \right]_0^{\pi} d\phi$

$= \frac{1}{5}\rho(\underline{r})\,abc \int_0^{2\pi} b^2\sin^2\phi\left(\frac{4}{3}\right) - c^2\left(-\frac{2}{3}\right) d\phi$

$$= \frac{1}{5}\rho(\underline{r})\,abc\int_0^{2\pi} \frac{4}{3}b^2\sin^2\phi + \frac{2}{3}c^2 \, d\phi$$

$$= \frac{1}{15}\rho(\underline{r})\,abc\left[ b^2\left[2\phi - \sin(2\phi)\right]_0^{2\pi} + \left[2c^2\phi\right]_0^{2\pi}\right]$$

$$= \frac{1}{15}\rho(\underline{r})\,abc\left[b^2(4\pi) + 4\pi c^2\right]$$

$$= \frac{4}{15}\pi\rho(\underline{r})\,abc\,(b^2+c^2)$$

Since homogeneous ellipsoid,

$$I_{22} = \frac{4}{15}\pi\,abc\,\rho(\underline{r})\,(a^2+c^2)$$

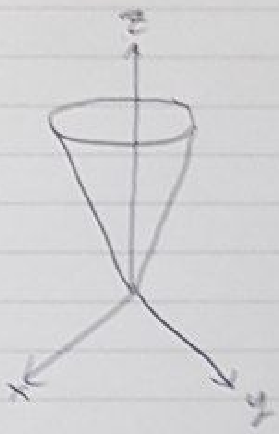
$$I_{33} = \frac{4}{15}\pi\,abc\,\rho(\underline{r})\,(a^2+b^2)$$

$$\Rightarrow I = \frac{4}{15}\pi\,abc\,\rho(\underline{r})\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

Assignment 6

1a. $(1-x^2)\frac{d^2y(x)}{dx^2} - 2x$

2. Let position vector of $P = \vec{P}(x, y, z)$
$= (0, 0, z)$




when P inside cone, $z <$ height of cone,
radius $< a \Rightarrow$ radius $= z\tan\alpha$

$m = \rho \cdot$ vol of cone formed by P
$= \rho \cdot \frac{1}{3}\pi (z\tan\alpha)^3$
$= \frac{\rho}{3}\pi z^3 \tan^3\alpha$

$$g(r) = -\frac{Gm}{|r|^2}\hat{e}_r = -\frac{G(\frac{\rho}{3}\pi z^3\tan^3\alpha)}{z^2}\hat{e}_r$$
$$= -\frac{G\rho}{3}\pi z\tan\alpha\,\hat{e}_r$$

when P outside cone, $z >$ height of cone,
radius $> a \Rightarrow$ radius $= \frac{a}{\tan\alpha}$

$m = \rho \cdot$ vol of cone formed by P
$= \rho \cdot \frac{1}{3}\pi\left(\frac{a}{\tan\alpha}\right)^3$
$= \frac{\rho}{3}\pi\frac{a^3}{\tan^3\alpha} = \frac{\rho}{3\tan^3\alpha}\pi a^3$

$$g(r) = -\frac{Gm}{|r|^2}\hat{e}_r = -\frac{G\left(\frac{\rho}{3\tan^3\alpha}\pi a^3\right)}{z^2}\hat{e}_r$$
$$= -\frac{G\rho}{3\tan^3\alpha\, z^2}\pi a^3\,\hat{e}_r$$