Assignment 9

Date No.

1a.

$$\Phi(\vec{p},t) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \iiint \left[e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\right] \Psi(\vec{r},t)\, d^3\vec{r}$$

x-components:

$$\Phi^*(P_x,0)\Phi(P_x,0) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\left[\frac{P_x y}{\hbar} - \frac{P_x x}{\hbar}\right]} \Psi(x,0)\Psi^*(y,0)\,dy\,dx$$

$$\int_{-\infty}^{\infty} \Phi^*(P_x,0)\Phi(P_x,0)\,dP_x$$

$$= \frac{1}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(x,0)\Psi^*(y,0)\left[\frac{1}{2\pi}\right]_{-\infty}^{\infty} e^{i\left(\frac{y}{\hbar}-\frac{x}{\hbar}\right)P_x}\, dP_x\,dy\,dx$$

$$= \frac{1}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(x,0)\Psi^*(y,0)\,\delta\!\left(\frac{y}{\hbar}-\frac{x}{\hbar}\right)dy\,dx$$

Let $A = \frac{x}{\hbar}$, $B = \frac{y}{\hbar}$,

$$\Rightarrow \int_{-\infty}^{\infty} \Phi^*(P_x,0)\Phi(P_x,0)\,dP_x = \hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(A\hbar,0)\Psi^*(B\hbar,0)\,\delta(B-A)\,dB\,dA$$

$$= \hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(A\hbar,0)\Psi^*(A\hbar,0)\,dA$$

$$= \int_{-\infty}^{\infty} \Psi(A,0)\Psi^*(A,0)\,dA = 1$$ (shown)

1. $$i\hbar \frac{d\Phi}{dt} = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int e^{-\frac{i\vec{p}\cdot\vec{r}}{\hbar}} \left[ -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t) \right] d^3\vec{r}$$

$$= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \left[ -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r},t) \right] dx$$

$$+ \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} V(\vec{x}) \Psi(\vec{x},t)\, dx \quad \text{— (1)}$$

$$\Rightarrow e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \nabla^2 \Psi = \nabla \cdot \left[ e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \nabla \Psi \right] - \nabla e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \cdot \nabla \Psi$$

Evaluating $\frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \left[ -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x},t) \right] d\vec{x}$

$$= -\frac{\hbar^2}{2m\sqrt{2\pi\hbar}} \left[ \int e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \frac{d\Phi}{dx} ds \right] - \int \nabla e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \cdot \nabla \Psi \, d\vec{x}$$

$$= -\frac{i\hbar}{2m\sqrt{2\pi\hbar}} \int p_x \vec{x}\, e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \frac{d\Psi}{dx} \vec{x}\, d\vec{x}$$

$$= \frac{p^2}{2m\sqrt{2\pi\hbar}} \int e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \Psi(\vec{x},t)\, d\vec{x}$$

$$= \frac{p^2}{2m} \Phi(\vec{x},t) \quad \text{— (2)}$$

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$$\Rightarrow V(x) = \sum_{n=0}^{\infty} C_n x^n$$

Evaluating $e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} x^n = \left(i\hbar\frac{\partial}{\partial p_x}\right) e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$

$$e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} V(\vec{x}) = \sum_{n=0}^{\infty} C_n \left(i\hbar\frac{\partial}{\partial p_x}\right)^n e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$$

$$= V(i\hbar\nabla_x) e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \quad \text{— (3)}$$

sub (2), (3) into (1)

$$i\hbar \frac{\partial \Phi(x,t)}{\partial t} = \frac{p^2}{2m}\Phi(x,t).$$

$$+ \int V\left(i\hbar\frac{\partial}{\partial p_x}\right)\delta(p-p')\Phi(p',t)\, d^3p'$$

$$= \frac{p^2}{2m}\Phi(\vec{p},t) + V\left(i\hbar\frac{\partial}{\partial p_x}\right)\Phi(\vec{p},t)$$

$$i\hbar\frac{\partial \Phi(p_x,t)}{\partial t} = \frac{p_x^2}{2m}\Phi(p_x,t) + V\left(i\hbar\frac{\partial}{\partial p_x},t\right)\Phi(p_x,t)$$

(shown)

2. $f(x)=\begin{cases}\pi, & |x|<1\\ 0, & |x|>1\end{cases}$

$I_1=\int_{-\infty}^{\infty}\operatorname{sinc}x\,dx$

$I_2=\int_{-\infty}^{\infty}(\operatorname{sinc}x)^2\,dx$

$$\tilde{f}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f(x)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{-ikx}\,\pi\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\pi\left(-\frac{1}{ik}\right)e^{-ikx}\right]_{-1}^{1}$$

$$=\frac{\pi}{\sqrt{2\pi}}\left[-\frac{1}{ik}e^{-ik}+\frac{1}{ik}e^{ik}\right]$$

$$=\frac{\pi}{\sqrt{2\pi}}\frac{e^{-ik}-e^{ik}}{-ik}$$

$$=\frac{\pi}{\sqrt{2\pi}}\frac{\sin k}{k}=\frac{\pi}{\sqrt{2\pi}}\operatorname{sinc}k$$

$$\therefore\ I_1=\int_{-\infty}^{\infty}\operatorname{sinc}x\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{2\pi}{\pi}\right)\int_{-\infty}^{\infty}e^{-ikx}\operatorname{sinc}x\,dx$$

$$=\frac{\sqrt{2\pi}}{\pi}(-\pi)$$

$$=-\sqrt{2\pi}$$

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$$\Rightarrow I_2 = \int_{-\infty}^{\infty} (\operatorname{sinc} x)^2 \, dx$$

$$= \int_{-\infty}^{\infty} [f(x)]^2 \, dx = \int_{-\infty}^{\infty} [\hat{f}(x)]^2 \, dx$$

$$= \frac{2\pi}{\pi^2} (\pi^2) = 2\pi$$

$g(x) = \operatorname{sinc} x$

$$F\{g(x)\} = F\left\{\frac{\sqrt{2\pi}}{\pi} \hat{f}(x)\right\}$$

$$= \frac{\sqrt{2\pi}}{\pi} F\{\hat{f}(x)\}$$

$$= \frac{\sqrt{2\pi}}{\pi} (\pi) = \sqrt{2\pi}$$

convolution $\Rightarrow F\{g(x+a) * g(x-a)\} = F\{\cos k_0 x\} * F\{g(x)\}$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{\frac{\pi}{2}} \left[\delta(k'-k_0) + \delta(k'+k_0)\right] \operatorname{sinc}(x-x') \, dx$$

$$= \sqrt{\frac{\frac{\pi}{2}}{2\pi}} \left[\sin x' - x + \sin x' + x\right]$$

$$= \frac{1}{2} \left[(\sin x' - x) + (\sin x' + x)\right]$$

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3. $-\frac{d^2\psi(x)}{dx^2}+k^2\psi(x)=Q\delta(x),\quad -\infty<x<\infty$

$F\{y''(x)\}=(ik)^2\tilde{y}(k)$

$-\frac{d^2\Psi(x)}{dx^2}=-(ik)^2\tilde{\Psi}(k)$ , $k^2\Psi(x)=k^2\tilde{\Psi},\ Q\delta(x)=\frac{Q}{\sqrt{2\pi}}\tilde{\delta}(k)$

$=k^2\tilde{\Psi}(k)$

$\Rightarrow(k^2+k^2)\tilde{\Psi}(k)=\frac{Q}{\sqrt{2\pi}}\tilde{\delta}(k)$

$\tilde{\Psi}(k)=\frac{Q}{\sqrt{2\pi}(k^2+k^2)}\tilde{\delta}(k)$

$\Rightarrow\Psi(-x)=\frac{Q}{k^2+k^2}\delta(-x)$