Assignment 1

1. $a_y(t) = \frac{dv_y}{dt} = -g - \alpha v_y$

$\int dt = \int (-g - \alpha v_y)^{-1} dv_y$

$t = -\frac{1}{\alpha} \ln(g + \alpha v_y) + C$

when $t = 0$, $v_y = V_0 \sin\theta$

$\Rightarrow 0 = -\frac{1}{\alpha} \ln(g + \alpha V_0 \sin\theta) + C$

$C = \frac{1}{\alpha} \ln(g + \alpha V_0 \sin\theta)$

$\Rightarrow t = \frac{1}{\alpha} \ln(g + \alpha V_0 \sin\theta - \frac{1}{\alpha} \ln(g + \alpha V_y)$

$t = \frac{1}{\alpha} \ln\left(\frac{g + \alpha V_0 \sin\theta}{g + \alpha V_y}\right)$

$\ln\left(\frac{g + \alpha V_0 \sin\theta}{g + \alpha V_y}\right) = \alpha t$

$\frac{g + \alpha V_0 \sin\theta}{g + \alpha V_y} = e^{\alpha t}$

$V_y = \frac{g + \alpha V_0 \sin\theta}{\alpha} e^{-\alpha t} - \frac{g}{a}$

when $V_y = \max$,

$\Rightarrow V_y = \frac{g + \alpha V_0 \sin\theta}{\alpha} e^{-\alpha t} = 0$

$e^{-\alpha t} = \frac{g}{g + \alpha V_0 \sin\theta}$

Since $m\alpha V_0 = mg \Rightarrow \alpha V_0 = g$,

$e^{-\alpha t} = \frac{g}{g + g\sin\theta} = \frac{1}{1 + \sin\theta}$ —— (1)

$$a_x(t) = \frac{dv_x}{dt} = -\alpha v_x$$

$$\int dt = -\int \alpha v_x \, dv_x$$

$$t = -\frac{1}{\alpha}\ln(\alpha v_x) + D$$

when $t = 0$, $v_x = v_0 \cos\theta$

$$\Rightarrow 0 = -\frac{1}{\alpha}\ln(\alpha v_0 \cos\theta) + D$$

$$D = \frac{1}{\alpha}\ln(\alpha v_0 \cos\theta)$$

$$\Rightarrow t = \frac{1}{\alpha}\ln(\alpha v_0 \cos\theta) - \frac{1}{\alpha}\ln(\alpha v_x)$$

$$\alpha t = \ln\left(\frac{\alpha v_0 \cos\theta}{\alpha v_x}\right)$$

$$e^{\alpha t} = \frac{v_0 \cos\theta}{v_x}$$

$$v_x = v_0 \cos\theta \, e^{-\alpha t}$$

$$x = \int v_x \, dt = \int v_0 \cos\theta \, e^{-\alpha t} \, dt = -\frac{v_0}{\alpha}\cos\theta \, e^{-\alpha t} + \varepsilon$$

when $t = 0$, $x = 0$,

$$\Rightarrow -\frac{v_0}{\alpha}\cos\theta \, e^{-\alpha t} + \varepsilon = 0$$

$$\varepsilon = \frac{v_0}{\alpha}\cos\theta$$

$$\Rightarrow x = \frac{v_0}{\alpha}\cos\theta - \frac{v_0}{\alpha}\cos\theta \, e^{-\alpha t}$$

$$= \frac{v_0}{\alpha}\cos\theta\left(1 - e^{-\alpha t}\right) \quad \text{—— (2)}$$

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sub (1) into (2)

$$x = \frac{V_0}{\alpha} \cos\theta \left(1 - \frac{1}{1+\sin\theta}\right)$$

$$= \frac{V_0}{\alpha} \left(\frac{\cos\theta + \cos\theta \sin\theta - \cos\theta}{1+\sin\theta} \right)$$

$$= \frac{V_0}{\alpha} \left(\frac{\cos\theta \sin\theta}{1+\sin\theta} \right)$$

$$\frac{dx}{d\theta} = \frac{V_0}{\alpha} \left[\frac{(1+\sin\theta)(-\sin^2\theta + \cos^2\theta) - \cos^2\theta \sin\theta}{(1+\sin\theta)^2} \right]$$

$$= \frac{V_0}{\alpha} \left[\frac{\cos^2\theta - \sin^2\theta + \sin\theta\cos^2\theta - \sin^3\theta - \cos^2\theta \sin\theta}{(1+\sin\theta)^2} \right]$$

$$= \frac{V_0}{\alpha} \left[\frac{\cos^2\theta - \sin^3\theta - \sin^2\theta}{(1+\sin\theta)^2} \right]$$

$$= \frac{V_0}{\alpha} \left[\frac{1 - \sin^2\theta - \sin^3\theta - \sin^2\theta}{(1+\sin\theta)^2} \right]$$

$$= -\frac{V_0}{\alpha} \left[\frac{\sin^3\theta + 2\sin^2\theta - 1}{(1+\sin\theta)^2} \right]$$

for $x = max$, $\frac{dx}{d\theta} = 0$

$$\Rightarrow -\frac{V_0}{\alpha} \left[\frac{\sin^3\theta + 2\sin^2\theta - 1}{(1+\sin\theta)^2} \right] = 0$$

$$\sin^3\theta + 2\sin^2\theta - 1 = 0$$

$\sin\theta \approx 0.618$ or $\sin\theta = -1$ (rej, $\theta \geq 0$) or $\sin\theta = -1.62$ (rej)

$\Rightarrow \theta = 38.2°$ (3sf)

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1. $\mathcal{E}_p(r,t) = \sum_{n}^{N}$

2a. $$\text{Area} = \int_0^{2\pi}\int_0^{a(1-\sin\phi)} \rho \, d\rho \, d\phi$$

$$= \int_0^{2\pi} \left[\frac{\rho^2}{2}\right]_0^{a(1-\sin\phi)} d\phi$$

$$= \int_0^{2\pi} \frac{a^2(1-\sin\phi)^2}{2} \, d\phi$$

$$= \int_0^{2\pi} \frac{a^2}{2}(1 - 2\sin\phi + \sin^2\phi) \, d\phi$$

$$= \frac{a^2}{2}\left[\phi + 2\cos\phi\right]_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi} \frac{1-\cos 2\phi}{2} \, d\phi$$

$$= \frac{a^2}{2}\left[2\pi + 2\cos(2\pi) - 2\right] + \frac{a^2}{2}\left[\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\right]_0^{2\pi}$$

$$= \pi a^2 + \frac{a^2}{2}(\pi)$$

$$= \frac{3}{2}\pi a^2$$

$$\text{Volume} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3}\pi \rho^3 \sin\phi \, d\phi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3}\pi a^3(1-\sin\phi)^3 \sin\phi \, d\phi$$

$$= \int_0^{\pi} \frac{2}{3}\pi a^3(1-\cos\phi)^3 \sin\phi \, d\phi$$

$$= \frac{2}{3}\pi a^3\left[\frac{1}{4}(1-\cos\phi)^4\right]_0^{\pi}$$

$$= \frac{2}{3}\pi a^3 (4)$$

$$= \frac{8}{3}\pi a^3$$

30. $xy''(x)+(1-x)y'(x)+ny(x)=0$

$P_2(x)=x$

$P_1(x)=1-x$

$$\int^x \frac{P_1(\varepsilon)}{P_2(\varepsilon)}\,d\varepsilon = \int^x \frac{1-\varepsilon}{\varepsilon}\,d\varepsilon = \int^x \varepsilon^{-1}-1\,d\varepsilon = \ln x - x$$

$$w(x) = \frac{1}{x}e^{\ln x - x}$$
$$= \frac{1}{x}e^{-x} = e^{-x}$$

$$y_n(x) = \frac{1}{w(x)}\left(\frac{d}{dx}\right)^n\left[w(x)P_2^n(x)\right]$$
$$= \frac{1}{e^{-x}}\left(\frac{d}{dx}\right)^n\left(e^{-x}x^n\right) = e^x\left(\frac{d}{dx}\right)^n\left(e^{-x}x^n\right)$$

By Leibnitz theorem,

$$e^x\left(\frac{d}{dx}\right)^n\left(e^{-x}x^n\right) = e^x\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left[\left(\frac{d}{dx}\right)^k e^{-x}\right]\left[\left(\frac{d}{dx}\right)^{n-k}x^n\right]$$

$$\left(\frac{d}{dx}\right)^k e^{-x} = \left(\frac{d}{dx}\right)^{k-1}\left[\frac{d}{dx}(e^{-x})\right] = \left(\frac{d}{dx}\right)^{k-1}\left[(-1)e^{-x}\right]$$
$$= \left(\frac{d}{dx}\right)^{k-2}\left[\frac{d}{dx}(-1)e^{-x}\right] = \left(\frac{d}{dx}\right)^{k-2}\left[(-1)^2e^{-x}\right]$$

$$= (-1)^k e^{-x}$$

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$$\left(\frac{d}{dx}\right)^{n-k} x^n = \left(\frac{d}{dx}\right)^{n-k-1}\left[\frac{d}{dx}(x^n)\right] = \left(\frac{d}{dx}\right)^{n-k-1}\left[n x^{n-1}\right]$$

$$= \left(\frac{d}{dx}\right)^{n-k-2}\left[\frac{d}{dx}(n x^{n-1})\right] = \left(\frac{d}{dx}\right)^{n-k-2}\left[n(n-1)x^{n-2}\right]$$

$$\vdots$$

$$= n(n-1)(n-2)\dots(k+1)x^k$$

$$= n(n-1)(n-2)..(k+1)\,\frac{k(k-1)\dots 3\cdot 2\cdot 1}{k(k-1)\dots 3\cdot 2\cdot 1}\,x^k$$

$$= \frac{n!}{k!}x^k$$

$$L_n(0) = C_n y(0) = 1$$

$$\Rightarrow C_n\left(\frac{n!}{n!}\right)(-1)^0\left(\frac{n!}{0!}\right) = 1$$

$$C_n = \frac{1}{n!}$$

$$\Rightarrow L_n(x) = C_n y(x) = \frac{1}{n!}e^x \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\left[(-1)^k e^{-x}\right]\left[\frac{n!}{k!}x^k\right]$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2(n-k)!}x^k \quad \text{(shown)}$$

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b. $L_n(x) = \frac{1}{n!} e^x \left(\frac{d}{dx}\right)^n (x^n e^{-x})$

$$\int_0^\infty e^{-x} L_n(x) L_n(x)\,dx = \int_0^\infty e^{-x} L_n(x)\left[\frac{1}{n!} e^x \left(\frac{d}{dx}\right)^n (x^n e^{-x})\right]dx$$

$$= \frac{1}{n!}\int_0^\infty L_n(x)\left(\frac{d}{dx}\right)^n (x^n e^{-x})\,dx$$

$$= \frac{1}{n!}\left\{\int_0^\infty \frac{d}{dx}\left[L_n(x)\left(\frac{d}{dx}\right)^{n-1}(x^n e^{-x})\right]dx - \int_0^\infty \frac{d}{dx} L_n(x)\left(\frac{d}{dx}\right)^{n-1}(x^n e^{-x})\,dx\right\}$$

$$= \frac{1}{n!}\left\{\left[L_n(x)\left(\frac{d}{dx}\right)^{n-1}(x^n e^{-x})\right]_0^\infty - \int_0^\infty \frac{d}{dx}\left[\frac{d}{dx}(L_n(x))\left(\frac{d}{dx}\right)^{n-2} x^n e^{-x}\right]dx + \int_0^\infty \left(\frac{d}{dx}\right)^2 L_n(x)\left(\frac{d}{dx}\right)^{n-2} x^n e^{-x}\,dx\right\}$$

$$= \frac{1}{n!}\left\{\left[L_n(x)\left(\frac{d}{dx}\right)^{n-1}(x^n e^{-x})\right]_0^\infty - \left[\left(\frac{d}{dx}\right)(L_n(x))\left(\frac{d}{dx}\right)^{n-2}(x^n e^{-x})\right]_0^\infty + \int_0^\infty \frac{d}{dx}\left(\frac{d}{dx}\right)^2 (L_n(x))\left(\frac{d}{dx}\right)^{n-3}(x^n e^{-x})\,dx - \int_0^\infty \left(\frac{d}{dx}\right)^3 (L_n(x))\left(\frac{d}{dx}\right)^{n-3}(x^n e^{-x})\,dx\right\}$$

n times integagation by parts,

$$\Rightarrow \int_0^\infty e^{-x} L_n(x) L_n(x)\,dx$$

$$= \frac{1}{n!}\left\{\left[\sum_{k=0}^{n-1}\left(\frac{d}{dx}\right)^k L_n(x)\left(\frac{d}{dx}\right)^{n-1-k}(x^n e^{-x})(-1)^k\right]_0^\infty + (-1)^n\int_0^\infty \left(\frac{d}{dx}\right)^n L_n(x)\left(\frac{d}{dx}\right)^0 (x^n e^{-x})\,dx\right\}$$

$$= \frac{1}{n!}\left[0 + (-1)^n\int_0^\infty \left(\frac{d}{dx}\right)^n L_n(x)(x^n e^{-x})\,dx\right]$$

$$= \frac{1}{n!}(-1)^n\int_0^\infty \left(\frac{d}{dx}\right)^n L_n(x)(x^n e^{-x})\,dx$$

Assignment 3

1. $\mathcal{E}_P(r,t) = \sum_{n=0}^{N-1} \mathcal{E}_0 \cos(kr - \omega t + n\phi)$

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$$\Rightarrow L_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} x^k$$

$\Rightarrow$ Only $k=n$ term exist

$$\therefore \int_0^\infty e^{-x} L_n(x) L_n(x)\, dx = \frac{1}{n!}\left[(-1)^n \int_0^\infty (-1)^n \frac{n!}{(n!)^2 (n-n)!} (n!)(x^n e^{-x}) dx\right]$$

$$= \frac{(-1)^{2n}}{n!} \int_0^\infty x^n e^{-x} dx$$

$$= \frac{1}{n!} \int_0^\infty x^n e^{-x}\, dx$$

$$= \frac{1}{n!}\left[-\int_0^\infty x^n \frac{d}{dx} e^{-x}\, dx\right]$$

$$= \frac{1}{n!}\left[-[x^n e^{-x}]_0^\infty - \int_0^\infty \left(\frac{d}{dx}\right)(x^n) e^{-x}\, dx\right]$$

$$= \frac{1}{n!}\left\{-[x^n e^{-x}]_0^\infty + \int_0^\infty \left(\frac{d}{dx}\right) x^n e^{-x}\, dx\right\}$$

$$= \frac{1}{n!}\left\{-[x^n e^{-x}]_0^\infty - \int_0^\infty \frac{d}{dx} x^n \frac{d}{dx} e^{-x} dx\right\}$$

$$= \frac{1}{n!}\left\{-[x^n e^{-x}]_0^\infty - \left[\frac{d}{dx} x^n e^{-x}\right]_0^\infty + \int_0^\infty \left(\frac{d}{dx}\right)^2 x^n e^{-x} dx\right\}$$

$$= \frac{1}{n!}\left\{-[x^n e^{-x}]_0^\infty - \left[\frac{d}{dx} x^n e^{-x}\right]_0^\infty - \int_0^\infty \left(\frac{d}{dx}\right)^2 x^n \frac{d}{dx} e^{-x} dx\right\}$$

n times integration by parts:

$$\int_0^\infty e^{-x} L_n(x) L_n(x) dx = \frac{1}{n!}\left\{-\left[\sum_{k=0}^{n-1} \left(\frac{d}{dx}\right)^k x^n e^{-x}\right]_0^\infty - \int_0^\infty \left(\frac{d}{dx}\right)^n (x^n) \frac{d}{dx} e^{-x}\, dx\right\}$$

$$= \frac{1}{n!}\left[-\int_0^\infty n! \frac{d}{dx} e^{-x}\, dx\right]$$

$$= [-e^{-x}]_0^\infty$$

$$= 0 + 1 = 1$$ (shown)