Assignment6

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1a. $(1-x^2)\dfrac{d^2y(x)}{dx^2}-2x\dfrac{dy(x)}{dx}+l(l+1)y(x)=0$

$P_2(x)=(1-x^2),\ P_1(x)=2x,\ P_0(x)=l(l+1)$

$\lim_{x\to 0}\dfrac{P_1(x)}{P_2(x)}=\lim_{x\to 0}\dfrac{2x}{1-x^2}=0\longrightarrow$ finite

$\lim_{x\to 0}\dfrac{P_0(x)}{P_2(x)}=\lim_{x\to 0}\dfrac{l(l+1)}{1-x^2}=l(l+1)\longrightarrow$ finite

$\Rightarrow x=0$ is an ordinary point

$\lim_{x\to\pm 1}\dfrac{P_1(x)}{P_2(x)}=\lim_{x\to\pm1}\dfrac{2x}{1-x^2}=\infty\rightarrow x=\pm1$ are singular points

$$\lim_{x\to\pm1}[x-(\pm1)]\frac{P_1(x)}{P_2(x)}\Rightarrow\begin{cases}\lim_{x\to 1}(x-1)\dfrac{2x}{1-x^2}=-\lim_{x\to1}\dfrac{2x}{1+x}=-1\rightarrow \text{finite}\\[2ex] \lim_{x\to -1}(x+1)\dfrac{2x}{1-x^2}=\lim_{x\to-1}\dfrac{2x}{1-x}=1\longrightarrow\text{finite}\end{cases}$$

$\therefore x=\pm1$ are regular singular points

let $u=\frac{1}{x}\Rightarrow\dfrac{du}{dx}=-\dfrac{1}{x^2}$

$$\begin{cases}\dfrac{dy(x)}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}=-u^2\dfrac{dy}{du}\\[2ex]\dfrac{d^2y(x)}{dx^2}=\dfrac{du}{dx}\dfrac{d}{du}\left(\dfrac{dy}{dx}\right)=-u^2\left(-2u\dfrac{dy}{du}-u^2\dfrac{d^2y}{du^2}\right)=u^3\left(2\dfrac{dy}{du}+u\dfrac{d^2y}{du^2}\right)\end{cases}$$

$\Rightarrow(1-x^2)\dfrac{d^2y(x)}{dx^2}-2x\dfrac{dy(x)}{dx}+l(l+1)y(x)=0$

$\Rightarrow\left(1-\dfrac{1}{u^2}\right)u^3\left(2\dfrac{dy}{du}+u\dfrac{d^2y}{du^2}\right)-\dfrac{2}{u}\left(u^2\dfrac{dy}{du}\right)+l(l+1)y(x)=0$

$\Rightarrow(u^4-u^2)\dfrac{d^2y}{du^2}+2u^3\dfrac{dy}{dx}+l(l+1)y(x)=0$

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$P_2(u) = u^4 - u^2$

$P_1(u) = 2u^3$

$P_0(u) = l(l+1)$

As $x \to \pm\infty$, $u \to 0$

$\lim_{u \to 0} \frac{P_1(u)}{P_2(u)} = \lim_{u \to 0} \frac{2u^3}{u^4 - u^2} = 0 \to$ finite

$\lim_{u \to 0} \frac{P_0(u)}{P_2(u)} = \lim_{u \to 0} \frac{l(l+1)}{u^4 - u^2} = \infty$

$\therefore$ $u = 0 \Rightarrow x = \pm\infty$ are singular

$\lim_{u \to 0} u \frac{2u^3}{u^4 - u^2} = 0 \to$ finite

$\lim_{u \to 0} u^2 \frac{l(l+1)}{u^4 - u^2} =$ finite

$\therefore$ $u = 0 \Rightarrow x = \pm\infty$ are regular singular ment 3

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2. Since $x=0$ is ordinary,

$y=\sum_{n=0}^{\infty} a_n x^n$

$\Rightarrow (1-x^2)\left[\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}\right] - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - n(n-1)a_n x^n - 2na_n x^n + l(l+1)a_n x^n = 0$

$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - [n(n+1) - l(l+1)]a_n\right]x^n = 0$

$\Rightarrow a_{n+2} = \dfrac{n(n+1) - l(l+1)}{(n+2)(n+1)} a_n$

$a_2 = -\dfrac{l(l+1)}{2} a_0$ $\qquad a_3 = \dfrac{1(2) - l(l+1)}{3(2)} a_1$

$a_4 = \dfrac{2(3) - l(l+1)}{(4)(3)} a_2$ $\qquad a_5 = \dfrac{3(4) - l(l+1)}{5(4)} a_3$

$= \dfrac{l(l+1)[2(3) - l(l+1)]}{4(3)(2)} a_0$ $\qquad = \dfrac{[(1)(2) - l(l+1)][(3)(4) - l(l+1)]}{5(4)(3)(2)} a_1$

$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n$

$= a_0 + a_1 x + \left[-\frac{1}{2}l(l+1)a_0\right]x^2 + \dfrac{2 - l(l+1)}{3!} a_1 x^3 + \ldots$

$= a_0\left[1 - \frac{1}{2}l(l+1)x^2 + \ldots\right] + a_1\left[x + \dfrac{2 - l(l+1)}{3!}x^3 + \ldots\right]$

C.

Even: $y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{[(l-2n+2)\cdots(l-2)l][l(l+1)(l+3)\cdots(l+2n-1)]}{(2n)!} x^{2n}$

Odd: $y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{[(l-2n+1)\cdots(l-3)(l-1)][(l+2)(l+4)\cdots(l+2n)]}{(2n+1)!} x^{2n+1}$

When l is even,

$\Rightarrow y_1(x)$ converge to polynomial of order l and powers of x are even,

$\Rightarrow y_2(x)$ diverge

when l is odd,

$\Rightarrow y_1(x)$ converge to polynomial of order l and powes of x are odd

$\Rightarrow y_2(x)$ diverge

$\therefore$ Converging $y(x) \Rightarrow$ Legendre polynomial

diverging $y(x) \Rightarrow$ Legendre function of the second kind gnment

1a.

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2a. $$-\frac{\hbar^2}{2m}\frac{d^2R(r)}{dr^2}+\frac{l(l+1)\hbar^2}{2mr^2}R(r)+V(r)R(r)=\mathcal{E}R(r)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2R(r)}{dr^2}+\left[\frac{l(l+1)\hbar^2}{2mr^2}+V(r)-\mathcal{E}\right]R(r)=0$$

$$P_2(r)=-\frac{\hbar^2}{2m}$$

$$P_1(r)=0$$

$$P_0(r)=\frac{l(l+1)\hbar^2}{2mr^2}+V(r)-\mathcal{E}$$

$$\lim_{r\to 0}\frac{P_1(r)}{P_2(r)}=0\ ;\ \lim_{r\to 0}\frac{P_0(r)}{P_2(r)}=\lim_{r\to 0}\frac{r^2\left[\frac{l(l+1)\hbar^2}{2mr^2}+V(r)-\mathcal{E}\right]}{-\frac{\hbar^2}{2m}}$$

$$=-l(l+1)$$

$\Rightarrow r=0$ is a regular point

let $R(r)=r^\sigma\sum_{n=0}^{\infty}a_nr^n$

$$\Rightarrow -\frac{\hbar^2}{2m}\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nr^{n+\sigma-2}+\frac{l(l+1)}{2m}\left(\frac{1}{r^2}\right)\sum_{n=0}^{\infty}a_nr^{n+\sigma}$$

$$+\sum_{m=-1}^{\infty}b_mr^m\sum_{n=0}^{\infty}a_nr^{n+\sigma}-\mathcal{E}\sum_{n=0}^{\infty}a_nr^{n+\sigma}=0$$

$$\Rightarrow -\frac{\hbar^2}{2m}\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nr^{n+\sigma-2}+\frac{l(l+1)\hbar^2}{2m}\sum_{n=0}^{\infty}a_nr^{n+\sigma-2}$$

$$+\sum_{m=-1}^{\infty}\sum_{n=0}^{\infty}a_nb_mr^{n+\sigma-m}-\mathcal{E}\sum_{n=0}^{\infty}a_nr^{n+\sigma}=0$$

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r^0:

$$\Rightarrow -\frac{\hbar^2}{2m}(\sigma)(\sigma-1)a_0 + \frac{l(l+1)\hbar^2}{2m}a_0 = 0$$

$$a_0(-\sigma^2 + \sigma + l^2 + l) = 0$$

$$a = 0 \quad \text{or} \quad \sigma = \frac{1 \pm (1+2l)}{2}$$

$$\Rightarrow \sigma_1 = \frac{1+(1+2l)}{2} = l+1,$$

$$\sigma_2 = \frac{1-(1+2l)}{2} = -l$$

~~As~~ By Fuch's theorem,
there is one regular solution $R_1(r) = r^{l+1}\sum_{n=0}^{\infty} a_n r^n$

b. $R_2(r) = C r^{l+1} \ln r \sum_{n=0}^{\infty} a_n r^n + r^{-l}\sum_{n=0}^{\infty} b_n r^n$

$$= R_1(r)\int^r \frac{1}{[r_1(\varepsilon)]^2}\left[e^{-\int^{\varepsilon}\frac{P_1(\varepsilon')}{P_2(\varepsilon')}d\varepsilon'}\right]d\varepsilon$$

$$= R_1(r)\int^r \frac{1}{[r_1(\varepsilon)]^2}\left[e^0\right]d\varepsilon$$

$$= R_1(r)\int^r \frac{1}{\varepsilon^{2l+2}\left[\sum_{n=0}^{\infty} a_n \varepsilon^n\right]^2}d\varepsilon$$

$$= R_1(r)\int^r \left[\sum_{n=0}^{\infty} a_n \varepsilon^{n+l+1}\right]^{-2}d\varepsilon$$

As $\varepsilon \to 0$, $\left(\sum_{n=0}^{\infty} a_n \varepsilon^{n+l+1}\right)^{-2} \to \infty$

as $r \to 0$, $\int^r \left(\sum_{n=0}^{\infty} a_n \varepsilon^{n+l+1}\right)^{-2} d\varepsilon \to \infty$

$\Rightarrow R_2(r)$ diverges at the origin

3. $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (\ln x)^2$

Euler-Cauchy Eqn $\Rightarrow x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

characteristic eqn $\Rightarrow \lambda^2 - 2\lambda + 1 = 0$

$$\lambda^2 - 1 = 0$$

$$\lambda = 1$$

Let solution be $y = (C_1 + C_2 \ln|x|)x$

$$= C_1 x + C_2 x \ln x$$

$$y_p(x) = \int^x (x-\varepsilon)e^{x-\varepsilon} - \varepsilon e^{x-\varepsilon}\left(\frac{\ln x}{x}\right)^2 d\varepsilon$$

$$\vdots$$

$$= [\ln(x)]^2 + 4\ln x + 6$$

$$\Rightarrow y = C_1 + C_2 x \ln x + (\ln x)^2 + 4\ln x + 6$$