Assignment 7

1a. $f(x) = x, \quad 0 \leq x \leq 2$

odd extension of $f(x) \Rightarrow f(x) = \begin{cases} x, & 0<x<2 \\ x, & -2<x<0 \end{cases}$

$L = 2 \qquad \langle f(x) \rangle = 0$

$$f_{FS}(x) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \cancel{a_n}^{\,0} \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} \sin\left(\frac{n\pi x}{2}\right) f(x)\,dx = \int_0^2 \sin\left(\frac{n\pi x}{2}\right) \cdot x\,dx$$

Integrating by parts,

$$b_n = \left[-\frac{x\cos\left(\frac{n\pi x}{2}\right)}{n\pi/2}\right]_0^2 - \int_0^2 -\frac{\cos\left(\frac{n\pi x}{2}\right)}{n\pi/2}\,dx$$

$$= (-1)^n\left(-\frac{4}{n\pi}\right)$$

$$\therefore f_{FS}(x) = \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{2}\right) :$$

$$x \approx f_{FS}(x) = \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{2}\right)$$

$$\int_0^2 x\,dx = \int_0^2 \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n \sin\frac{n\pi x}{2}\,dx$$

$$\left[\frac{x^2}{2}\right]_0^2 = 2 = \left[\sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^n \frac{\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)}\right]_0^2$$

$$2 = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^n \left[\frac{\cos(n\pi)}{n\pi/2} - \frac{1}{n\pi/2}\right]$$

$$2 = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^n \left(\frac{2}{n\pi}\right)\left[(-1)^n - 1\right]$$

$$2 = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\left[1 - (-1)^n\right]$$

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{\pi^2}{4} = \frac{\pi^2}{6} - \frac{\pi^2}{4} = -\frac{\pi^2}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = (-1)\left(-\frac{\pi^2}{12}\right) = \frac{\pi^2}{12}$$

ment 3

b. $f(x) = x, \quad 0 \leqslant x \leqslant 2$

Even extension of $f(x)$:

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ -x, & -2 < x < 0 \end{cases}$$

$L = 2$

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \overset{0}{\cancel{b_n}} \sin\left(\frac{n\pi x}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= 2 \cdot \frac{1}{2}\int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2 = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} \cos\left(\frac{n\pi x}{2}\right) f(x)\,dx$$

$$= \int_0^2 \cos\left(\frac{n\pi x}{2}\right)\cdot x\,dx$$

Integrating by parts, $a_n = \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 - \int_0^2 \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx$

$$= \frac{4}{n\pi}\sin(n\pi) + \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 1\right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow f_{FS}(x) = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{2}\right)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{2}\right)$$

Using Parseval's theorem,

$$\langle [f(x)]^2 \rangle = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + \cancel{b_n^2}^{\,0}\right)$$

$$\frac{1}{4}\int_{-2}^{2} x^2\,dx = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\right)^2\left[(-1)^n - 1\right]^2$$

$$\frac{1}{2}\int_{0}^{2} x^2\,dx = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}\left[(-1)^{2n} - 2(-1)^n + 1\right]$$

$$\frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}\left[2 - 2(-1)^n\right]$$

$$2\left(\frac{1}{2}\left(\frac{8}{3}\right) - 1\right) = \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}\left[2 - 2(-1)^n\right]$$

$$\frac{1}{3}\left(\frac{\pi^4}{16}\right)\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n^4}\left[1 - (-1)^n\right]$$

$$\frac{\pi^4}{48} = \sum_{n=1}^{\infty}\frac{1}{n^4}\left[1 - (-1)^n\right] = \frac{2}{1^4} + \frac{0}{2^4} + \frac{2}{3^4} + \dots$$

$$= \sum_{n=0}^{\infty}\frac{2}{(2n+1)^4}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{2}{(2n+1)^4}$$

$$= \frac{1}{2}\left(\frac{\pi^4}{48}\right)$$

$$= \frac{1}{96}\pi^4$$

2. Let input signal be $U(x) = \sin x, \; -\pi \leq x \leq \pi$

output $= V(x) = \begin{cases} \sin x, & 0 \leq x < \pi \\ 0, & -\pi \leq x < 0 \end{cases}$

$$V_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$
$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left[1-(-1)\right]$$
$$= \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx) f(x)\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)\sin x\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}\left[\sin(x-nx) + \sin(x+nx)\right]dx$$
$$= \frac{1}{\pi}\cdot\frac{1}{2}\left[-\frac{1}{1-n}\cos(x-nx) - \frac{1}{1+n}\cos(x+nx)\right]_0^{\pi}$$
$$= \frac{1}{2\pi}\left[-\frac{1}{1-n}\cos(\pi-n\pi) - \frac{1}{1+n}\cos(x+nx) + \frac{1}{1-n} + \frac{1}{1+n}\right]$$
$$= \frac{1}{2\pi}\left[\frac{1+n+1-n}{1-n^2} - \cos(\pi-n\pi)\left(\frac{1}{1-n} + \frac{1}{1+n}\right)\right]$$
$$= \frac{1}{2\pi}\left[\frac{2}{1-n^2} - (-1)^{n+1}\left(\frac{2}{1-n^2}\right)\right]$$
$$= \frac{1}{\pi(1-n^2)}\left[1+(-1)^n\right]$$

Assignm

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\, f(x)\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\cdot \sin x\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}\left[\cos(nx-x) - \cos(nx+x)\right] dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{n-1}\sin(nx-x) - \frac{1}{n+1}\sin(nx+x)\right]_0^{\pi}$$

$$= 0$$

$$\therefore V_{FS}(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi(1-n^2)}\left[1+(-1)^n\right]\cos(nx)$$

Using Parseval's theorem,

$$\text{output power} = \langle [V(x)]^2 \rangle = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$= \frac{1}{\pi^2} + \frac{1}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{\pi(1-n^2)}\left[1+(-1)^n\right]\right\}^2$$

$$= \frac{1}{\pi^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\pi^2(1-n^2)^2}\left[1+2(-1)^n+(-1)^{2n}\right]$$

$$= \frac{1}{\pi^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\pi^2(1-n^2)^2}\left[2+2(-1)^n\right]$$

$$= \frac{1}{\pi^2} + \sum_{n=1}^{\infty}\frac{1}{\pi^2(1-n^2)^2}\left[1+(-1)^n\right]$$

$$\langle [V(x)]^2 \rangle_{DC} = \frac{1}{\pi^2}$$

Input power $= \langle [U(x)]^2 \rangle = \langle (\sin x)^2 \rangle = \frac{1}{2}$

Ratio DC output to input

$$= \frac{\langle [V(x)]^2 \rangle_{DC}}{\langle [U(x)]^2 \rangle} = \frac{\frac{1}{\pi^2}}{\frac{1}{2}}$$

$$= 0.203 \text{ (3s.f.)}$$

Date No.

3. Even extension of $f(x)$

$$f(x) = \begin{cases} y_0 + x, & -L \le x < 0 \\ y_0 - x & 0 < x \le L \end{cases}$$

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)$$

When $t=0$, $y=y_0$,

$$\Rightarrow y(x,0) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v(0)}{L}\right)$$

$$= \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)(y_0 - x)\,dx$$

$$= \frac{2y_0}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx - \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)x\,dx$$

$$= \frac{2y_0}{L}\left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L - \frac{2}{L}\left[x\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \int\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)dx\right]_0^L$$

$$= -\frac{2}{L}\left[x\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= -\frac{2L}{n^2\pi^2}\cos(n\pi) + \frac{2L}{n^2\pi^2}$$

$$= \frac{2L}{n^2\pi^2}\left[1-(-1)^n\right]$$

when $n=2k$, $f_{FS}(x)=0$

$\Rightarrow n = 2k+1,\ k \in R$

$\therefore$

$\Rightarrow$ only odd harmonics are present