Assignment 4

1. $\hat{L} = \sum_{k=1}^{3} \hat{e}_k \hat{L}_k$, $\hat{L}_k = -i\hbar \sum_{m=1}^{3} \sum_{n=1}^{3} \epsilon_{mnk} x_m \frac{\partial}{\partial x_n}$,

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2+z^2}}{z}\right) \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

$$\hat{L}_x = -i\hbar \sum_{m=1}^{3} \sum_{n=1}^{3} \epsilon_{mn1} x_m \frac{\partial}{\partial x_n}$$

$$= -i\hbar\left(\epsilon_{231}\, y \frac{\partial}{\partial z} + \epsilon_{321}\, z \frac{\partial}{\partial y}\right) = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar \sum_{m=1}^{3} \sum_{n=1}^{3} \epsilon_{mn2} x_m \frac{\partial}{\partial x_n}$$

$$= -i\hbar\left(\epsilon_{132}\, x \frac{\partial}{\partial z} + \epsilon_{312}\, z \frac{\partial}{\partial x}\right) = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$dr = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$$

$$= \sin\theta\cos\phi\, dx + \sin\theta\cos\phi\, dy + \cos\theta\, dz$$

$$d\theta = \frac{1}{\sqrt{x^2+y^2}} \frac{x^2}{r^2} dx + \frac{1}{\sqrt{x^2+y^2}} \frac{y^2}{r^2} dy - \frac{\sqrt{x^2+y^2}}{r^2} dz$$

$$= \frac{1}{r\sin\theta} \sin\theta\cos\theta\cos\phi\, dx + \frac{1}{r\sin\theta} \sin\theta\cos\theta\sin\phi - \frac{\sin\theta}{r} dz$$

$$= \frac{1}{r}\cos\theta\cos\phi\, dx + \frac{1}{r}\cos\theta\sin\phi\, dy - \frac{\sin\theta}{r} dz$$

$$d\phi = \frac{x}{x^2+y^2} dy - \frac{y}{x^2+y^2} dx = -\frac{\sin\phi}{r\sin\theta} dx + \frac{\cos\phi}{r\sin\theta} dy$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}$$

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y$$

$$= -i\hbar\left[-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right] - \hbar\left[\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$$

$$= \hbar\left[i\sin\phi\frac{\partial}{\partial\theta} + i\cot\theta\cos\phi\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$$

$$= \hbar\left[(-\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta} - (-\cos\phi + i\sin\phi)\left(i\cot\theta\frac{\partial}{\partial\phi}\right)\right]$$

$$= \hbar\left[(\cos\phi - i\sin\phi)\left(i\cot\theta\frac{\partial}{\partial\phi}\right) - (\cos\phi - i\sin\phi)\frac{\partial}{\partial\theta}\right]$$

$$= \hbar(\cos\phi - i\sin\phi)\left(i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right)$$

$$= \hbar e^{-i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right)$$

2. $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \ldots, N$

$$[F, G]_{PB} = \sum_{i=1}^{N} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

(a)
$$[q_i, q_j]_{PB} = \sum_{i=1}^{N} \frac{\partial q_i}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial q_j}{\partial q_i}$$
$$= \sum_{i=1}^{N} \frac{\partial q_i}{\partial q_i}(\emptyset) - (\emptyset)\frac{\partial q_j}{\partial q_i}$$
$$= \emptyset \quad \text{(proven)}$$

$$[p_i, p_j]_{PB} = \sum_{i=1}^{N} \frac{\partial p_i}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial p_i}{\partial p_i}\frac{\partial p_j}{\partial q_i}$$
$$= \sum_{i=1}^{N} (\emptyset)\frac{\partial p_j}{\partial p_i} - \frac{\partial p_i}{\partial p_i}(\emptyset)$$
$$= \emptyset \quad \text{(proven)}$$

$$[q_i, p_j]_{PB} = \sum_{i=1}^{N} \frac{\partial q_i}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial p_j}{\partial q_i}$$
$$= \sum_{i=1}^{N} (1)\frac{\partial p_j}{\partial p_i} - (\emptyset)\frac{\partial p_j}{\partial q_i}$$
$$= \sum_{i=1}^{N} \frac{\partial p_j}{\partial p_i}$$
$$= \delta_{ij} \quad \text{(proven)}$$

(b) $$\frac{dF}{dt} = \frac{\partial F}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial F}{\partial t}$$

$$= \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial F}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial F}{\partial t}$$

$$= \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial F}{\partial t}$$

$$= [F, H]_{PB} + \frac{\partial F}{\partial t}$$ (shown)

$$\frac{dH}{dt} = [H, H]_{PB} + \frac{\partial H}{\partial t}$$

Since, $H(p,q)$ does not depend on time, $\frac{\partial H}{\partial t} = \phi$

$$\Rightarrow \frac{dH}{dt} = [H, H]_{PB} = \sum_{i=1}^{N} \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= \phi$$

Since $\frac{dH}{dt} = \phi$, $H(p,q)$ is a constant of motion

3a. $\begin{cases} x = a\cosh\mu\cos\nu \\ y = a\sinh\mu\sin\nu \\ z = z \end{cases}$, $\quad \begin{matrix} 0 \le u < \infty \\ 0 \le \nu < 2\pi \\ -\infty < z < \infty \end{matrix}$

$$\vec{r} = \alpha\cosh\mu\cos\nu\,\hat{e}_x\ a\sinh\mu\sin\nu\,\hat{e}_y + z\,\hat{e}_z$$

$$\frac{\partial\vec{r}}{\partial u} = a\sinh\mu\cos\nu\,\hat{e}_x + a\cosh\mu\sin\nu\,\hat{e}_y$$

$$\frac{\partial\vec{r}}{\partial \nu} = -a\cosh\mu\sin\nu\,\hat{e}_x + a\sinh\mu\cos\nu\,\hat{e}_y$$

$$\frac{\partial\vec{r}}{\partial z} = \hat{e}_z$$

$$\begin{cases} h_\mu = \left|\frac{\partial\vec{r}}{\partial\mu}\right| = a\sqrt{\sinh^2\mu + \sin^2\nu} \\ h_\nu = \left|\frac{\partial\vec{r}}{\partial\nu}\right| = a\sqrt{\sinh^2\mu + \sin^2\nu} \\ h_z = \left|\frac{\partial\vec{r}}{\partial z}\right| = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{e}_\mu = \frac{1}{a\sqrt{\sinh^2\mu + \sin^2\nu}}(\sinh\mu\cos\nu\,\hat{e}_x + \cosh\mu\sin\nu\,\hat{e}_y) \\ \hat{e}_\nu = \frac{1}{a\sqrt{\sinh^2\mu + \sin^2\nu}}(-\cosh\mu\sin\nu\,\hat{e}_x + \sinh\mu\cos\nu\,\hat{e}_y) \\ \hat{e}_z = \hat{e}_z \end{cases}$$

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b. $r(t) = \vec{r} = \alpha \cosh\mu \cos v\, \hat{e}_x + \alpha \sinh\mu \sin v\, \hat{e}_y + \hat{e}_z$

$$\vec{v}(t) = \frac{\partial}{\partial t}\alpha\cosh\mu\cos v\,\hat{e}_\mu + \frac{\partial}{\partial t}\alpha\sinh\mu\sin v\,\hat{e}_v + \frac{\partial}{\partial t}\hat{e}_z$$

$$\vec{a}(t) = \frac{\partial^2 x}{\partial t^2}\alpha\cosh\mu\cos v\,\hat{e}_\mu + \frac{\partial^2 y}{\partial t^2}\alpha\sinh\mu\sin v\,\hat{e}_v + \frac{\partial^2 z}{\partial t^2}\hat{e}_z$$