Assignment 3

Date No.

1. $$\mathcal{E}_p(r,t) = \sum_{n=0}^{N-1} \mathcal{E}_0 \cos(kr - \omega t + n\phi)$$

$$= \mathcal{E}_0 \sum_{n=0}^{N-1} \cos(kr - \omega t + n\phi)$$

$$= \mathcal{E}_0 \sum_{n=0}^{N-1} e^{i(kr-\omega t+n\phi)}$$

$$= \mathcal{E}_0 e^{i(kr-\omega t)} \sum_{n=0}^{N-1} e^{in\phi}$$

$$= \mathcal{E}_0 e^{i(kr-\omega t)} (e^0 + e^{i\phi} + e^{2i\phi} + \dots)$$

$$= \mathcal{E}_0 e^{i(kr-\omega t)} \left[\frac{1-e^{i\phi N}}{1-e^{i\phi}}\right]$$

$$= \mathcal{E}_0 \frac{e^{i(kr-\omega t)} - e^{i(kr-\omega t)} e^{iN\phi}}{1-e^{i\phi}}$$

$$= \mathcal{E}_0 \frac{\cos(kr-\omega t) - \cos(kr-\omega t+N\phi)}{1-\cos\phi}$$

$$= \mathcal{E}_0 \frac{2\sin\left(\frac{2kr-2\omega t+\phi N}{2}\right)\sin\left(\frac{N\phi}{2}\right)}{2\sin^2\left(\frac{\phi}{2}\right)}$$

$$= \left[\mathcal{E}_0 \frac{\sin(N\phi/2)}{\sin(\phi/2)}\right]\left[\frac{\sin(kr-\omega t+N\phi)}{\sin\left(\frac{\phi}{2}\right)}\right]$$

$$= \left[\mathcal{E}_0 \frac{\sin(N\phi/2)}{\sin(\phi/2)}\right] e^{i(kr-\omega t+\frac{\phi}{2}N-\frac{\phi}{2})}$$

$$= \mathcal{E}_0 \left[\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right] \cos\left[kr-\omega t+(N-1)\frac{\phi}{2}\right] \text{ (shown)}$$

2a. $[\vec{S}_m, \vec{S}_n] = \vec{S}_m \vec{S}_n - \vec{S}_n \vec{S}_m$

$$= \delta_{mn} \vec{I} + i \sum_{k=1}^{3} \epsilon_{mnk} \vec{S}_k - \delta_{nm} \vec{I} - i \sum_{k=1}^{3} \epsilon_{nmk} \vec{S}_k$$

$$= 2i \sum_{k=1}^{3} \epsilon_{mnk} \vec{S}_k$$

$$\{\vec{S}_m, \vec{S}_n\} = \vec{S}_m \vec{S}_n + \vec{S}_n \vec{S}_m$$

$$= \delta_{mn} \vec{I} + i \sum_{k=1}^{3} \epsilon_{mnk} \vec{S}_k + \delta_{nm} \vec{I} + i \sum_{k=1}^{3} \epsilon_{nmk} \vec{S}_k$$

$$= 2\delta_{mn} \vec{I} + i \sum_{k=1}^{3} \epsilon_{mnk} \vec{S}_k - i \sum_{k=1}^{3} \epsilon_{mnk} \vec{S}_k$$

$$= 2\delta_{mn} \vec{I}$$

b. $\vec{S}(\vec{a}) \vec{S}(\vec{b}) = \sum_{i=1}^{3} a_i \vec{S}_i \sum_{j=1}^{3} b_j \vec{S}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \vec{S}_i \vec{S}_j$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \left(\delta_{ij} \vec{I} + i \sum_{k=1}^{3} \epsilon_{ijk} \vec{S}_k \right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{ij} \vec{I} + i \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a_i b_j \vec{S}_k$$

$$= \sum_{i=1}^{3} a_i b_i \vec{I} + i \vec{S}(\vec{a} \times \vec{b})$$

$$= (\vec{a} \cdot \vec{b}) \vec{I} + i \vec{S}(\vec{a} \times \vec{b}) \quad \text{(shown)}$$

condition for $\vec{S}(\vec{a})$ and $\vec{S}(\vec{b})$ to commute

$\Rightarrow \vec{a} \times \vec{b} = 0$

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let $u = \frac{x}{a}$

x

(c) let $x = i\phi \vec{S}(\hat{n})$,

$$e^x = e^{i\phi\vec{S}(\hat{n})} = \sum_{r=0}^{\infty} \frac{[i\phi\vec{S}(\hat{n})]^r}{r!}$$

$$= \sum_{\text{even}}^{\infty} \frac{[i\phi\vec{S}(\hat{n})]^r}{r!} + \sum_{\text{odd}}^{\infty} \frac{[i\phi\vec{S}(\hat{n})]^r}{r!}$$

$$= \sum_{k=0}^{\infty} \frac{[i\phi\vec{S}(\hat{n})]^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{[i\phi\vec{S}(\hat{n})]^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k[\phi\vec{S}(\hat{n})]^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k[\phi\vec{S}(\hat{n})]^{2k+1}}{(2k+1)!}$$

$$= \cos\theta \sum_{k=0}^{\infty} [\vec{S}(\hat{n})\vec{S}(\hat{n})]^k + i\sin\theta \sum_{k=0}^{\infty} [\vec{S}(\hat{n})\vec{S}(\hat{n})]^k \vec{S}(\hat{n})$$

$$= \cos\theta \sum_{k=0}^{\infty} [(\hat{n}\cdot\hat{n})\vec{I} + i\vec{S}(\vec{n}\times\vec{n})]^k + i\sin\theta \sum_{k=0}^{\infty} [(\vec{n}\cdot\vec{n})\vec{I} + i\vec{S}(\vec{n}\times\vec{n})]^k \vec{S}(\hat{n})$$

$$= \cos\theta\,\vec{I} + i\sin\theta\,\vec{S}(\hat{n}) \quad \text{(proven)}$$