Assignment 2

1a. $\frac{1}{2} l \left[\frac{d\theta(t)}{dt}\right]^2 = g[\cos\theta(t) - \cos\theta_0]$

$$\frac{d\theta(t)}{dt} = \sqrt{\frac{2g}{l}[\cos\theta(t) - \cos\theta_0]} \quad (\text{ig -ve; when } t=0, \tfrac{d\theta}{dt} > 0)$$

$$\Rightarrow d\theta(t) = \sqrt{\frac{2g}{l}[\cos\theta(t) - \cos\theta_0]}\, dt$$

$$\int dt = \sqrt{\frac{l}{2g}} \int \frac{1}{\sqrt{\cos\theta(t) - \cos\theta_0}}\, d\theta(t)$$

when $t=0, \theta=0$; when $t = \frac{T}{4}, \theta = \theta_0$

$$\Rightarrow \int_0^{\frac{T}{4}} dt = \sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta$$

$$\frac{T}{4} = \sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2\sin^2\left(\frac{\theta_0}{2}\right) - 2\sin^2\left[\frac{\theta}{2}\right]}}$$

$$= \sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2\sin^2\left(\frac{\theta_0}{2}\right)}\sqrt{1 - \frac{1}{\sin^2\left(\frac{\theta_0}{2}\right)}\sin^2\left(\frac{\theta}{2}\right)}}$$

Let $\sin(t') = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_0}{2}\right)} \Rightarrow \cos(t')\,dt' = \frac{1}{\sin\left(\frac{\theta_0}{2}\right)} \frac{1}{2}\cos\left(\frac{\theta}{2}\right) d\theta$

$$d\theta = \frac{2\sin\left(\frac{\theta_0}{2}\right)\cos(t')}{\cos\left(\frac{\theta}{2}\right)}\, dt'$$

when $\theta = \theta_0$, $\sin(t') = 1$, $t' = \frac{\pi}{2}$; when $\theta = 0$, $\sin(t') = 0$, $t' = 0$

$$\Rightarrow \frac{T}{4} = \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{1}{\sin\left(\frac{\theta_0}{2}\right)\sqrt{1-\operatorname{cosec}^2\left(\frac{\theta_0}{2}\right)\sin^2\left(\frac{\theta}{2}\right)}}\left[\frac{2\sin\left(\frac{\theta_0}{2}\right)\cos(t')}{\cos\left(\frac{\theta}{2}\right)}\right]dt'$$

$$= \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{2\cos(t')}{\cos\left(\frac{\theta}{2}\right)\sqrt{1-\operatorname{cosec}^2\left(\frac{\theta_0}{2}\right)\sin^2\left(\frac{\theta}{2}\right)}}\,dt'$$

$$= \sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{\cos(t')}{\sqrt{1-\sin^2\left(\frac{\theta}{2}\right)}\sqrt{1-\operatorname{cosec}^2\left(\frac{\theta_0}{2}\right)\sin^2(t')\sin^2\left(\frac{\theta_0}{2}\right)}}\,dt'$$

$$= \sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{\cos(t')}{\sqrt{1-\sin^2(t')\sin^2\left(\frac{\theta_0}{2}\right)}\sqrt{1-\sin^2(t')}}\,dt'$$

$$= \sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{\cos(t')}{\sqrt{1-\sin^2(t')\sin^2\left(\frac{\theta_0}{2}\right)}\cos(t')}\,dt'$$

$$= \sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{dt'}{\sqrt{1-\sin^2(t')\sin^2\left(\frac{\theta_0}{2}\right)}}$$

$$= \sqrt{\frac{l}{g}}\,F\left(\frac{\pi}{2}, \sin\frac{\theta_0}{2}\right)$$

$$\therefore T = 4\sqrt{\frac{l}{g}}\,F\left(\frac{\pi}{2}, \sin\frac{\theta_0}{2}\right) \quad \text{(shown)}$$

b. $T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\left[1-\sin^2\left(\frac{\theta_0}{2}\right)\sin^2(t)\right]^{-\frac{1}{2}} dt$

$= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\left[1+\frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\sin^2(t)+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}\sin^4\left(\frac{\theta_0}{2}\right)\sin^4(t)+\dots\right]dt$

$\approx 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} 1+\frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\left[\frac{1}{2}(1-\cos(2t))\right]+\frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\sin^4(t)\,dt$

$= 4\sqrt{\frac{l}{g}}\left\{\left[t+\frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\cdot\frac{1}{2}\left(t-\frac{1}{2}\sin(2t)\right)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\left[\frac{1}{2}(1-\cos(2t))\right]^2 dt\right\}$

$= 4\sqrt{\frac{l}{g}}\left\{\left[\frac{\pi}{2}+\frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right)\left(\frac{\pi}{2}\right)\right] + \frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\int_0^{\frac{\pi}{2}}\frac{1}{4}\left[1-2\cos(2t)+\frac{1}{2}+\frac{1}{2}\cos(4t)\right]dt\right\}$

$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+\frac{\pi}{8}\sin^2\left(\frac{\theta_0}{2}\right)+\frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\int_0^{\frac{\pi}{2}}\frac{3}{8}-\frac{1}{2}\cos(2t)+\frac{1}{8}\cos(4t)\,dt\right]$

$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+\frac{\pi}{8}\sin^2\left(\frac{\theta_0}{2}\right)+\frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\left[\frac{3}{8}t-\frac{1}{4}\sin(2t)+\frac{1}{32}\sin(4t)\right]_0^{\frac{\pi}{2}}\right]$

$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+\frac{\pi}{8}\sin^2\left(\frac{\theta_0}{2}\right)+\frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\cdot\frac{3}{16}\pi\right]$

$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+\frac{\pi}{8}\sin^2\left(\frac{\theta_0}{2}\right)+\frac{9}{128}\pi\sin^4\left(\frac{\theta_0}{2}\right)\right]$

2a. Let $S_1 = \sum_{n=0}^{\infty} n(\hbar\omega) e^{-n\hbar\omega/kT}$ (arithmetico-geometric series, $|r|<1$)

$$= \frac{a}{1-r} + \frac{rd}{(1-r)^2}, \text{ where } \begin{cases} a = 0 \\ r = e^{-\hbar\omega/kT} \\ d = \hbar\omega \end{cases}$$

$$= \frac{\hbar\omega e^{-\hbar\omega/kT}}{(1-e^{-\hbar\omega/kT})^2}$$

Let $S_2 = \sum_{n=0}^{\infty} e^{-n\hbar\omega/kT}$ (geometric series)

$$= \frac{a}{1-r} \quad \text{where } a=1,\ r = e^{-\hbar\omega/kT}$$

$$= \frac{1}{1-e^{-\hbar\omega/kT}}$$

$$\therefore \langle \varepsilon \rangle = \frac{S_1}{S_2} = \frac{\left[\frac{\hbar\omega e^{-\hbar\omega/kT}}{(1-e^{-\hbar\omega/kT})^2}\right]}{\left(\frac{1}{1-e^{-\hbar\omega/kT}}\right)}$$

$$= \frac{\hbar\omega e^{-\hbar\omega/kT}}{1-e^{-\hbar\omega/kT}}$$

$$= \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}$$

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b. $C = N_A \frac{d\langle\varepsilon\rangle}{dT}$

$$= N_A \hbar\omega \left[\frac{\frac{\hbar\omega}{kT^2} e^{-\hbar\omega/kT}}{(1-e^{-\hbar\omega/kT})^2} \right]$$

$$= N_A k \left[\frac{\frac{\hbar\omega}{kT^2} e^{-\hbar\omega/kT}}{(1-e^{-\hbar\omega/kT})^2} \right]$$

Let $x = \frac{\hbar\omega}{kT}$,

High temperature limits : as $T \to \infty$, $x \to 0$

$$\Rightarrow \lim_{x\to 0} C = \lim_{x\to 0} N_A k \left[\frac{x^2 e^{-x}}{(1-e^{-x})^2} \right]$$

$$= \lim_{x\to 0} N_A k \left[\frac{e^{-x}}{\frac{1}{x^2}\left(1-1+x-\frac{x^2}{2}+\frac{x^3}{3!}+\ldots\right)^2} \right]$$

$$\approx \lim_{x\to 0} N_A k \frac{e^{-x}}{\left(1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^3}{24}\right)}$$

$$= N_A k$$

Low temperature limits: as $T \to 0$, $x \to \infty$

$$\Rightarrow \lim_{x\to\infty} C \approx \lim_{x\to\infty} N_A k \frac{e^{-x}}{\left(1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^4}{24}\right)}$$

$$= 0$$

3. $\text{RHS} = |\vec{A}|^2|\vec{B}|^2 - (\vec{A}\cdot\vec{B})^2$

$$= \sum_{i=1}^{3}\sum_{m=1}^{3}\delta_{im}A_iA_m \sum_{j=1}^{3}\sum_{n=1}^{3}\delta_{jn}B_jB_n$$

$$- \sum_{i=1}^{3}\sum_{n=1}^{3}\delta_{in}A_iB_n \sum_{j=1}^{3}\sum_{m=1}^{3}\delta_{jm}A_jB_m$$

$$= \delta_{im}\delta_{jn}\left(\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{j=1}^{3}\sum_{n=1}^{3}A_iA_mB_jB_n\right) - \delta_{in}\delta_{jm}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}A_iB_nA_jB_m\right)$$

$$= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}A_iB_jA_mB_n$$

$$= \epsilon_{ijk}\epsilon_{mnk}\sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_j\sum_{m=1}^{3}\sum_{n=1}^{3}A_mB_n$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_j\epsilon_{ijk}\sum_{m=1}^{3}\sum_{n=1}^{3}A_mB_n\epsilon_{mnk}$$

$$= (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B})$$

$$= \text{LHS}$$

(proven)