NATIONAL UNIVERSITY OF SINGAPORE
Department of Statistics and Applied Probability

(2020/21) Semester 1 ST2334 Probability and Statistics Tutorial 5

1. The random variable X, representing the number of errors per 100 lines of software code, has the following probability function (or probability mass function):

X	2	3	4	5	6
$f_X(x)$	0.01	0.25	0.40	0.30	0.04

(a) Find the first and second moment of X.
(b) Find the variance of X using (i) the definition of variance and (ii) $V(X) = E(X^2) - [E(X)]^2$.
(c) Find the mean and variance of the discrete variable $Z = 3X - 2$.
(d) Find the probability (mass) function of the random variable Z. Hence, find the mean and variance of Z directly from its probability (mass) function.
(e) Suppose that $W = aZ + b$. Find the mean and variance of W in terms of a and b.

2. Suppose that a grocery store purchases 5 cartons of skim milk at the wholesale price of $1.20 per carton and retails the milk at $1.65 per carton. After the expiration date, the unsold milk is removed from the shelf and the grocer receives a credit from the distributor equal to three-fourths of the wholesale price. Find the expected profit if the probability distribution of the random variable X, the number of cartons that are sold from this lot is

X	0	1	2	3	4	5
$f_X(x)$	1/15	2/15	2/15	3/15	4/15	3/15

3. (a) Let X be a positive integer-valued (excluding 0) random variable. Show that
$$E(X) = \sum_{k=1}^{\infty} \Pr(X \geq k).$$
(b) Suppose that 3 fair dice are rolled. Let M be the minimum of 3 numbers rolled. Find $\Pr(M \geq 1), \Pr(M \geq 2), \cdots, \Pr(M \geq 6)$. Hence, find $E(M)$.
[Hint: What can we say about X_1, X_2 and X_3 if $M \geq k$, where X_i is the outcome of the i-th die.]

4. On a laboratory assignment, if the equipment is working, the probability density function (pdf) of the observed outcome is $Y = 3X - 2$, where X has the probability density function
$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the mean and variance of the random variable X.
(b) Find the mean and variance of the random variable Y.

5. The probability density function of random variable X is of the form

$$f_X(x) = \begin{cases} a + bx^2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $E(X) = 3/5$, find a and b.

6. If a random variable X satisfies $E[(X-1)^2] = 10$ and $E[(X-2)^2] = 6$, find the mean and the variance of X.

7. A random variable X has a mean $\mu = 10$ and a variance $\sigma^2 = 4$. Find an upper bound or a lower bound for the probabilities in (a) to (d).

(a) $\Pr(5 < X < 15)$
(b) $\Pr(5 < X < 14)$
(c) $\Pr(|X - 10| < 3)$
(d) $\Pr(|X - 10| \geq 3)$
(e) Determine a constant c such that $\Pr(|X - 10| \geq c) \leq 0.04$.

8. Let X be a random variable with the probability density function

$$f_X(x) = \begin{cases} 6x(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the mean, μ, and the standard deviation, σ, of the random variable X.
(b) Compute the exact value of $\Pr(\mu - 2\sigma < X < \mu + 2\sigma)$.
(c) Apply Chebyshev's inequality to give a lower bound of $\Pr(\mu - 2\sigma < X < \mu + 2\sigma)$.
(d) Comment on the answers you obtain in (b) and (c).

9. A firm manufactures a 100-watt light bulb, which, according to specifications written on the package, has a mean life of 900 hours with a standard deviation of 50 hours. At most, what percentage of the bulbs fails to last even 700 hours? Assume that the distribution is symmetric about the mean.

Answers to selected problems

1. (a) 4.11, 17.63
   (b) 0.7379
   (c) 10.33, 6.6411
   (d)

Z	4	7	10	13	16
$f_Z(z)$	0.01	0.25	0.40	0.30	0.04

   (e) mean $= 10.33a + b$; variance $= 6.6411a^2$

2. Profit = $0.75X - 1.5$
   E(Profit) = \$0.80

3. (b) $E(M) = \frac{1^3+2^3+\cdots+6^3}{6^3} = 2.0417$

4. (a) 1/3, 1/18
   (b) -1, 1/2

5. (a) $a = 3/5$, $b = 6/5$

6. $\mu = 7/2, \sigma^2 = 15/4$

7. (a) $k = 5/2$, prob $\geq 21/25$
   (b) $k = 2$, prob $\geq 3/4$
   (c) $k = 3/2$, prob $\geq 5/9$
   (d) $k = 3/2$, prob $\leq 4/9$
   (e) $k = 5$, $c = 10$

8. (a) $\mu = 0.5$, $\sigma = \sqrt{0.05} = 0.2236$
   (b) 0.9839
   (c) prob ≥ 0.75

9. 0.03125