NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF STATISTICS & APPLIED PROBABILITY

ST2334 PROBABILITY AND STATISTICS

SEMESTER I, AY 2022/2023

Tutorial 10: Solution

This set of questions will be discussed by your tutors during the tutorial in Week 13.

Please work on the questions before attending the tutorial.

1. In 64 randomly selected hours of production, the mean and the standard deviation of the number of acceptable pieces produced by a automatic stamping machine are $\bar{x} = 1,038$ and $s = 146$. At the 0.05 level of significance, does this enable us to reject the null hypothesis $\mu = 1,000$ against the alternative hypothesis $\mu > 1,000$?

   SOLUTION

   Step 1. Hypothesis
   $$H_0 : \mu = 1000 \quad \text{vs} \quad H_1 : \mu > 1000.$$

   Step 2. Level of significance: $\alpha = 0.05$.

   Step 3. Criterion: Since the sample is large, we will use the normal approximation to the distribution of the mean substituting S for σ. We reject the null hypothesis when
   $$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} > z_\alpha.$$
   Since $z_\alpha = 1.645$, the null hypothesis must be rejected if $Z > 1.645$.

   Step 4. Calculations: $\mu_0 = 1000$, $\bar{x} = 1038$, $s = 146$, and $n = 64$, so
   $$z = \frac{1038 - 1000}{146/\sqrt{64}} = 2.08.$$

   Step 5. Decision: Because $2.08 > 1.645$, the null hypothesis that $\mu = 1000$ is rejected at level 0.05. The p-value is $P(Z > 2.08) = 0.019$. The evidence against the null hypothesis, $\mu = 1000$, is quite strong.

2. A manufacturer claims that the average tar content of a certain kind of cigarette is $\mu = 14.0$. In an attempt to show that it differs from this value, five measurements are made:

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   14.5 14.2 14.4 14.3 14.6
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   Show that the difference between the mean of this sample, $\bar{x} = 14.4$, and the average tar claimed by the manufacturer, $\mu = 14.0$, is significant at $\alpha = 0.05$. Assume normality.

   SOLUTION

   Step 1. Hypothesis
   $$H_0 : \mu = 14.0 \quad \text{vs} \quad H_1 : \mu \neq 14.0.$$

   Step 2. Level of significance: $\alpha = 0.05$.

Step 3. Criterion: Assuming the population is normal, we can use the t statistic.

$$t = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}.$$

Since the alternative hypothesis is two-sided, the critical region is defined by $t < -t_{0.025}$ or $t > t_{0.025}$ where $t_{0.025}$ with 4 degrees of freedom is 2.776.

Step 4. Calculations: In this case, $\mu_0 = 14.0$, $n = 5$, $\bar{x} = 14.4$ and $s = 0.158$ so

$$t = \frac{14.4 - 14.0}{0.158/\sqrt{5}} = 5.66.$$

Step 5. Decision: Because $5.66 > 2.776$, we reject the null hypothesis in favor of the alternative hypothesis $\mu \neq 14.0$ at the 0.05 level of significance. The p-value is $2P(t_4 > 5.66) = 0.0048$.

3. Suppose that in the last question, the first measurement is recorded incorrectly as 16.0 instead of 14.5. Show that, even though the mean of the sample increases to $\bar{x} = 14.7$, the null hypothesis $H_0 : \mu = 14.0$ is not rejected at level $\alpha = 0.05$.

   Explain the apparent paradox that even though the difference between observed $\bar{x}$ and μ has increased, the null hypothesis is no longer rejected.

   SOLUTION

   We are testing the null hypothesis $H_0 : \mu = 14.0$ against the alternative $H_1 : \mu \neq 14.0$ at the 0.05 level of significance. The critical region is defined by $t < -t_{0.025}$ or $t > t_{0.025}$ where $t_{0.025}$ with 4 degrees of freedom is 2.776. With the first value changed, $\bar{x} = 14.7$ and $s = 0.74162$ so

$$t = \frac{14.7 - 14.0}{0.74162/\sqrt{5}} = 2.11$$

   and we cannot reject the null hypothesis. The p-value is $2P(t_4 > 2.11) = 0.1025$.

   The "paradox" is explained by the standard deviation, which has greatly increased.

4. A study based on a sample size of 36 reported a mean of 87 with a margin of error of 10 for 95% confidence.

   (a) Give the 95% confidence interval for the population mean μ.

   (b) You desire a margin of error of 2.5 with the same confidence level. What is the sample size that will give you that kind of accuracy? Assume that we know the population variance.

   (c) You are asked to test the hypothesis that $\mu = 80$ against a two sided alternative at $\alpha = 0.05$. What is your conclusion?

   SOLUTION

   (a) The 95% confidence interval for μ is given as

$$\bar{x} \pm E = 87 \pm 10 = (77, 97).$$

   (b) The margin of error is given as

$$m = z_{\alpha/2}\sigma/\sqrt{n}$$

   So to decrease the accuracy by a quarter, we need to increase the sample size 16 times. A sample size of $36 \times 16 = 576$ is required.

(c) The 95% CI contains the the value 80 so there is no evidence to reject the null at $\alpha = 0.05$.

5. In a study, the mean CAP (cumulative average point) of a random sample of 49 final year students is calculated to be 4.5. The standard deviation for this sample is given as 0.75.

   (a) Find a 95% confidence interval for the mean CAP of the entire final year class.

   (b) The university administration claims that the mean CAP for the entire final year class is 4.3. Does our study offer evidence against this claim? Explain.

   SOLUTION

   (a) We are in the large sample case. The 95% CI is given as

   $$\bar{x} \pm z_{0.025}\frac{s}{\sqrt{n}} = 4.5 \pm 1.96 \times \frac{0.75}{7} = (4.29, 4.70).$$

   (b) No, the 95% CI contains the alleged value of 4.3. So the study does not offer evidence against the claim.

6. Suppose we wish to test the hypothesis

   $$H_0 : \mu = 2 \text{ vs } H_1 : \mu \neq 2$$

   and found a two-sided p-value of 0.03. Separately, a 95% confidence interval for μ is computed to be $(1.5, 4.0)$. Are these two results compatible? Why or why not?

   SOLUTION
   No; a p-value of 0.03 suggests that we will reject the null hypothesis at 0.05 level. On the other hand, if the 95% CI contains the null value of 2, then we should not reject the null hypothesis at 0.05 level. So these two statements are not compatible.

7. The dynamic modulus of concrete is obtained for two different concrete mixes. For the first mix, $n_1 = 33$, $\bar{x} = 115.1$, and $s_1 = 0.47$ psi. For the second mix, $n_2 = 31, \bar{y} = 114.6$, and $s_2 = 0.38$ psi. Test, with $\alpha = 0.05$, the null hypothesis of equality of mean dynamic modulus versus the two-sided alternative.

   SOLUTION
   Let μ_1 and μ_2 be the means of the two populations.

   Step 1. Hypothesis

   $$H_0 : \mu_1 - \mu_2 = 0 \quad \text{vs} \quad H_1 : \mu_1 - \mu_2 \neq 0$$

   Step 2. Level of significance: $\alpha = 0.05$.

   Step 3. Criterion: The null hypothesis specifies $\delta = \mu_1 - \mu_2 = 0$. Since the samples are large, we use the large sample statistic where we estimate each population variance by the sample variance.

   $$Z = \frac{\overline{X} - \overline{Y} - \delta}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}.$$

   The alternative is two-sided so we reject the null hypothesis for $Z > z_{0.025} = 1.96$ or $Z < -z_{0.025} = -1.96$.

   Step 4. Calculations: Since $n_1 = 33$, $n_2 = 31$, $\bar{x} = 115.1$, $\bar{y} = 114.6$, $s_1 = 0.47$, and $s_2 = 0.38$,

   $$z = \frac{115.1 - 114.6}{0.10655} = 4.69 > 1.96.$$

Step 5. Decision: Because $4.69 > 1.96$, we reject the null hypothesis at the 0.05 level of significance. The p-value $2P(Z > 4.69)$ rounds to 0.00000 and gives extremely strong support for rejecting the null hypothesis.

8. Obtain a 95% confidence interval for the difference in mean dynamic modulus in Question 7.

SOLUTION

The sample sizes are large so we use the large samples confidence interval with $z_{0.025} = 1.96$.

$$(\bar{x} - \bar{y}) + z_{0.025}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = (115.1 - 114.6) \pm 1.96 \times 0.10655$$

or

$$0.29 < \mu_1 - \mu_2 < 0.71.$$

We are 95% confident that the mean time to repair is 0.29 to 0.71 hour higher for the first kind of equipment.

9. Two procedures for etching integrated circuits are to be compared. Given 10 units, five are prepared using etching procedure A and five are prepared using etching procedure B.

(a) The response is the percent of area on the integrated circuit where the etching was inadequate. Suppose the results are

Prodedure A	Prodedure B
5	1
2	3
9	4
6	0
3	2

Find a 95% confidence interval for the difference in means.

(Summary statistics are given as $\bar{x} = 5$, $s_1 = 2.73$ and $\bar{y} = 2$, $s_2 = 1.58$.)

(b) What assumptions did you make for your answer to part (b)?

SOLUTION

(a) We have $n_1 = n_2 = 5$ and we find $\bar{x} = 25/5 = 5$, $s_1 = 2.73$ and $\bar{y} = 10/5 = 2$, $s_2 = 1.58$. Note that $\frac{1}{2} < s_1/s_2 < 2$, so the equal variance assumption applies. The small sample confidence interval will be used. The pooled variance is

$$s_p^2 = \frac{\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_i - \bar{y})^2}{n_1 + n_2 - 2} = 40/8 = 5.$$

Since $t_{8,0.025} = 2.306$, the 95% confidence interval is given by

$$(\bar{x} - \bar{y}) \pm t_{8,0.025} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 5 - 2 \pm 2.306\sqrt{5}\sqrt{0.4} = 3 \pm 3.26$$

or $-0.26 < \mu_1 - \mu_2 < 6.26$.

(b) We assumed both populations were normal. Of course the usual assumption of independent random samples must also hold.

10. A manager is considering instituting an additional 15-minute coffee break if it can be shown to decrease the number of errors that employees commit. The manager divided a sample of 20 employees into two groups of 10 each. Members of one group followed the same work schedule as before, but the members of the other group were given a 15-minute coffee break in the middle of the day. The following data give the total number of errors committed by each of the 20 workers over the next 20 working days.

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Coffee break group:      8, 7,  5,  8, 10, 9, 7, 8,  4, 5
No-break group:          7, 6, 14, 12, 13, 8, 9, 6, 10, 9
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(Summary statistics are given as $\bar{x} = 7.1$, $s_1 = 1.91$, $\bar{y} = 9.4$ and $s_2 = 2.84$.)

(i) Using a suitable test, at the 5 percent level of significance, test the hypothesis that instituting a coffee break reduces the mean number of errors. What is your conclusion?

(ii) What assumptions have you made?

(iii) What is the p-value of your test?

SOLUTION

(a) Let μ_1 be the mean number of errors employees with a coffee break commits over 20 days and μ_2 be the mean for those without.

We have $\bar{x} = 7.1$, $s_1 = 1.91$, $\bar{y} = 9.4$ and $s_2 = 2.84$. Further, since $s_1/s_2 = 0.67$, we can assume that the population variances of the two groups are the same. The pooled estimator for the variance is

$$s_p^2 = \frac{9 \times 1.91^2 + 9 \times 2.84^2}{10 + 10 - 2} = 5.86 = 2.41^2.$$

Step 1. We are interested to test

$$H_0 : \mu_1 - \mu_2 = 0 \quad \text{vs} \quad H_1 : \mu_1 - \mu_2 < 0$$

Step 2. Level of significance: $\alpha = 0.05$.

Step 3. Criterion: The null hypothesis specifies $\mu_1 - \mu_0 = 0$. Since the samples are small, we assume that the populations are normal with the same variance, we use the two-sample t statistic

$$t = \frac{\overline{X} - \overline{Y}}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

Since the alternative hypothesis is one-sided, we reject the null hypothesis when $t < -t_{0.05} = -1.734$ for 18 degrees of freedom.

Step 4. Calculations:

$$t = \frac{7.1 - 9.4}{2.41 \sqrt{\frac{1}{10} + \frac{1}{10}}} = -2.13.$$

Step 5. Decision: Since $-2.13 < -t_{0.05} = -1.734$, we reject the null hypothesis at level of significance $\alpha = 0.05$.

We conclude that instituting a coffee break reduces the mean number of errors.

(b) Since the sample sizes are small, we assume that the two populations follow normal distributions.

(c) From the t-table, we see that

$$t_{0.025} = 2.101 < 2.13 < 2.552 = t_{0.01}.$$

So the p-value is between 0.01 to 0.025.

Alternatively, the exact p-value is 0.02378 from a software.

11. A taxi company manager is trying to decide whether the use of radial tires instead of regular belted tires improves fuel economy. Twelve cars were equipped with radial tires and driven over a prescribed test course. Without changing drivers, the same cars were then equipped with regular belted tires and driven once again over the test course. The gasoline consumption, in kilometers per liter, was recorded as follows:

Car	1	2	3	4	5	6	7	8	9	10	11	12
Radial Tires	4.2	4.7	6.6	7	6.7	4.5	5.7	6	7.4	4.9	6.1	5.2
Belted Tires	4.1	4.9	6.2	6.9	6.8	4.4	5.7	5.8	6.9	4.7	6	4.9

(a) Find a 95% confidence interval for the difference of the true mean gasoline consumption between cars equipped with radial tires and cars equipped with belted tires.

(b) Can we conclude that cars equipped with radial tires give better fuel economy than those equipped with belted tires? Assume the populations to be normally distributed. Use a p-value in your conclusion.

SOLUTION

X_R = gasoline consumption by radial tires $\sim$ Normal;

X_B = gasoline consumption by belted tires $\sim$ Normal.

From the data, $n_d = 12$; $\bar{x}_d = 0.1417$, $s_d = 0.1975$.

(a) 95% confidence interval for $\mu_d = \bar{x}_d \pm t_{11.0.025} \dfrac{s_d}{\sqrt{n_d}} = 0.1417 \pm 2.201 \dfrac{0.1975}{\sqrt{12}} = (0.0162, 0.2672)$.

(b) $H_0 : \mu_d = 0$ versus $H_1 : \mu_d > 0$.

$$t_{obs} = \frac{\bar{x}_d}{s_d/\sqrt{n}} = \frac{0.14167}{0.1975/\sqrt{12}} = 2.485 > t_{11,0.05} (= 1.796).$$

Therefore we reject H_0.
Alternatively p-value$= P(t_{11} > 2.485) = 0.01515$.