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Null and Alternative Hypotheses (Continued)

Null hypothesis:

- Hypothesis that we formulate with the hope of rejecting, denoted by H_0.
- A null hypothesis concerning a population parameter will always be stated to specify an exact value of the parameter.

Alternative hypothesis:

- The rejection of H_0 leads to the acceptance of an alternative hypothesis, denoted by H_1.
- It allows for the possibility of several values.

✓ For a valid set of hypotheses (i.e., H_0 versus H_1), they need to be disjoint. For example "$H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$" is a valid set; "$H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$" is another valid set.

✓ By convention (and theoretically supported), we usually write the null hypothesis in the "equal" form, i.e., $H_0 : \theta = \theta_0$, with θ_0 being a given value. So the hypotheses have three possible forms:

★ $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$; in this case $H_0 : \theta_0 = \theta_0$ in fact means $\theta \leq \theta_0$;

★ $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$; in this case $H_0 : \theta_0 = \theta_0$ in fact means $\theta \geq \theta_0$;

★ $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

So we need to check the form of H_1 to ensure the meaning of $H_0 : \theta = \theta_0$ in practice.

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7.1.2 Types of Error

- Two types of errors in the hypothesis testing:

	State of Nature	
Decision	H_0 is true	H_0 is false
Reject H_0	**Type I error** Pr(Reject H_0 given that H_0 is true) = α	Correct decision Pr(Reject H_0 given that H_0 is false) = 1 − β
Do not reject H_0	Correct decision Pr(Do not reject H_0 given that H_0 is true) = 1 − α	**Type II error** Pr(Do not reject H_0 given that H_0 is false) = β

This page gives us a fundamental explanation of the outcomes of the hypothesis testing.

✓ Type I error is usually treated more seriously; we need to control it in the first place. We set an α, called the significant level, and require the decision rule satisfy:

$$Pr \left(\text{Reject } H_0 \middle| H_0 \text{ is true} \right) \leq \alpha. \tag{1}$$

Note that "given H_0 is true" in the statement essentially tells that "given $\theta = \theta_0$ is true". This is important when we are to derive the distribution of the test statistic, and it is why we write $H_0 : \theta = \theta_0$.

✓ When type I error is controlled by α, i.e., (1) is satisfied, we try to minimize type II error. So, we typically require

$$Pr \left(\text{Reject } H_0 \middle| H_0 \text{ is true} \right) = \alpha,$$
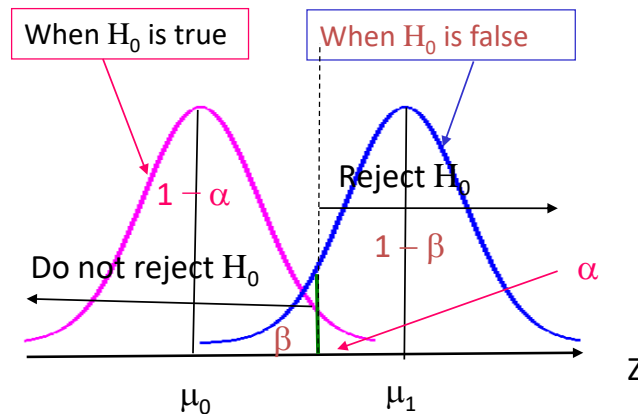
so that the type I error is controlled but maximized to reduce the type II error.

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✓ When establishing a testing rule, type I and type II errors trade off each other. Page 7-13 gives a clear explanation of the idea:

Types I and II Error (Continued)

Test $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$

When H_0 is true

When H_0 is false

$1 - \alpha$

Reject H_0

Do not reject H_0

$1 - \beta$

α

β

μ_0

μ_1

Z

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Think of how type I and type II errors may change if we push the green line, which defines the rejection/acceptance region of the decision rule, to the left or the right.

✓ When a test is performed, one can make only one type of error, but not both:

★ If H_0 is rejected, type I error is possible.

★ If H_0 is not rejected, type II error is possible.

Test statistic plays a key role in performing the hypothesis testing. Test statistic must be a function of the sample, e.g., $X_1, X_2, \ldots, X_n$, and does not rely on any unknown parameter.

Similarly to the construction of the confidence interval, we can summarize the procedure for constructing the test statistic for mean related hypothesis tests as follows. Denote by θ the parameter of interest. We consider the hypotheses:

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \ldots$$

where θ_0 is a given value, e.g., 0, which is assumed to be the true value of θ in developing the distribution of the test statistic. H_1 could be $\theta < \theta_0$, $\theta > \theta_0$, or $\theta \neq \theta_0$.

✓ Step 1: Look for an estimator $\widehat{\theta}$ for θ, e.g., $\bar{X}$ for μ.

✓ Step 2: Derive the formula for $V(\widehat{\theta})$.

✓ Step 3: The test statistic is constructed to be

$$T = \frac{\widehat{\theta} - \theta_0}{\sqrt{V}}. \tag{2}$$

★ If $V(\widehat{\theta})$ does not depend on any unknown parameter, e.g., when σ^2 is known, $V(\bar{X}) = \sigma^2/n$, we set $V = V(\widehat{\theta})$. The statistic T given in (2) (approximately) follows the $N(0,1)$ distribution, when the data are normal or the sample size is sufficiently large.

Sections 7.2.1 and 7.3.1 belong to this situation.

★ If $V(\widehat{\theta})$ contains some other unknown parameters, e.g., σ^2, we replace the parameter with its estimator, e.g., S^2 can be used to replace σ^2, and result in $\widehat{V}(\widehat{\theta})$. We set $V = \widehat{V}(\widehat{\theta})$. The distribution of T given in (2) has two possibilities:

(1) The sample size is sufficiently large; then $T \sim N(0,1)$ approximately.

Section 7.3.2 belongs to this situation. The test problem in Sections 7.2.2 and 7.3.4 can also be solved by this situation if their corresponding sample sizes are suffi-

ciently large, and if so we don't need to require the distribution of the observations are normal.

(2) If the sample size is small, but the observations are normally distributed, then $T \sim t(df)$, where df is the degrees of the freedom of the parameter estimated in $V(\widehat{\theta})$.

The testing problems discussed in Sections 7.2.2, 7.3.3, and 7.3.4 belong to this situation.

The above strategy is not applicable to construct test statistic for the variance related tests.

7.1.3 Acceptance and Rejection Regions

- To test a hypothesis about a population parameter, we first select a suitable test statistic for the parameter under the hypothesis.
- Once the significance level, α, is given, a decision rule can be found such that it divides the set of all possible values of the test statistic into two regions,
- one being the **rejection region** (or critical region) and the other the **acceptance region**.

Acceptance and Rejection Regions (Continued)

- Once a sample is taken, the value of the test statistic is obtained.
- If the test statistic assumes a value in the rejection region, the null hypothesis is rejected; otherwise it is not rejected.
- The value that separates the rejection and acceptance regions is called the **critical value**.

✓ Acceptance and rejection regions are defined based on the test statistic so that the type I error is controlled and type II error is minimized.

✓ They are disjoint, and they together contain all the possible values of the test statistic.

✓ When constructing the the acceptance and rejection regions, one should jointly view the hypotheses (mainly the alternative) and how the corresponding test statistic will perform when H_0 or H_1 is true.

For example,

★ Suppose we are to test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$, and use $Z = \frac{\bar{X} - \mu_0}{\sigma^2/\sqrt{n}} \sim N(0,1)$ as the test statistic, where σ^2 is assumed to be known.

★ When H_1 is true, Z should be "stay away" from 0 as $\bar{X}$ should be stay away from μ_0 when $\mu \neq \mu_0$. So the rejection region must be $Z < z_1$ and $Z > z_2$, with $z_1 < 0$ and $z_2 > 0$. Because $N(0,1)$ is symmetric about 0, therefore, we set $z_1 = -z_2 \equiv -z$ with $z > 0$.

★ To get type I error controlled by α, it is equivalent to

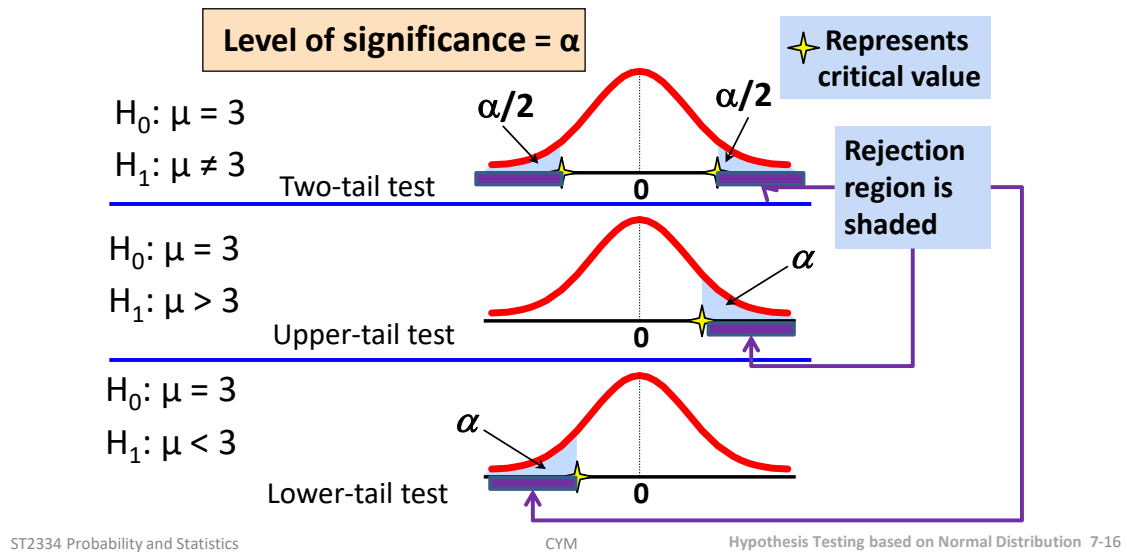$$\alpha \geq Pr(Z < -z \quad or \quad Z > z) = 2Pr(Z > z).$$

Therefore we need $z = z_{\alpha/2}$.

★ The rejection region should be $|z_{\text{obs}}| > z_{\alpha/2}$, where z_{obs} denotes the computed value of Z when observations are obtained.

With the similar strategy, we can obtain the rejection region if "$H_1 : \mu > \mu_0$": $z_{\text{obs}} > z_\alpha$; and if "$H_1 : \mu < \mu_0$": $z_{\text{obs}} < -z_\alpha$.

The idea has been well summarized as a figure in page 7-16 of the lecture slide.

Level of Significance and the Rejection Region

Level of significance = α

H_0: μ = 3

H_1: μ ≠ 3

Two-tail test

α/2   α/2

0

Represents critical value

Rejection region is shaded

H_0: μ = 3

H_1: μ > 3

Upper-tail test

α

0

H_0: μ = 3

H_1: μ < 3

Lower-tail test

α

0

Think of how the rejection/acceptance region might be changed if we use $Z_1 = \frac{\mu_0 - \bar{X}}{\sigma^2/\sqrt{n}} \sim N(0,1)$ as the test statistic instead.

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p-Value Approach to Testing

- *p*-value: Probability of obtaining a test statistic more extreme ($\leq$ or $\geq$) than the observed sample value given H_0 is true
  - Also called **observed level of significance**

✓ p-value is the probability that we may obtain a test statistic value that is as extreme or more extreme than the observed value of the test statistic, when H_0 is true.

✓ So the smaller the p-value, the better that we reject H_0, as less likely we shall make such an extreme observation given H_0 is true.

✓ Given a significant level α, we reject H_0 when p-value$< \alpha$; and do not reject H_0 otherwise.

✓ Using p-value or the rejection region approach to perform the test must result in exactly the same conclusion (in terms of reject or do not reject H_0).

★ p-value$< \alpha$ if and only if the test statistic is in the rejection region.

★ p-value$\geq \alpha$ if and only if the test statistics is in the acceptance region.

Example 1 (Continued)

- How likely is it to see a sample mean of 34.5 (or something further from the mean, in either direction) if the true mean is 35?   ($\sigma = 1.5$ and $n = 49$)
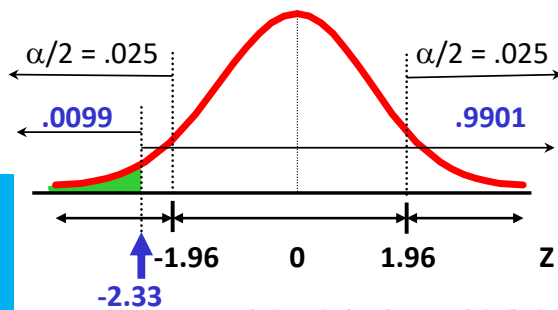
$\bar{X}$ = 34.5 is translated to a Z score of Z = -2.33

Pr(Z < -2.33) = 0.0099

Pr(Z > -2.33) = 0.9901

p-value

= 2 min{Pr(Z < -2.33), Pr(Z >-2.33) }

= 2(0.0099) = .0198

$\alpha/2 = .025$ $\alpha/2 = .025$

.0099   .9901

-2.33   -1.96   0   1.96   Z

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Relationship between two-sided test and confidence interval

- The two-sided test procedure just described is equivalent to finding a $(1-\alpha)100\%$ confidence interval for μ
- H_0 is accepted if the confidence interval covers μ_0.
- If the C.I. does not cover μ_0, we reject $\mu = \mu_0$ in favour of the alternative H_1: $\mu \neq \mu_0$ since

$$\Pr\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

$$\Leftrightarrow \quad \Pr\left(\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

Confidence interval is (approximately; see the example in the next slide) equivalent to the hypothesis testing if and only if the latter is two sided; this is applicable not only for the mean related inference but also for the variance related inference. The theoretical foundation is given in the formulation at the bottom of the slide above. Use $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ as an example:

✓ If the confidence interval contains μ_0, we do not reject H_0.

✓ If the confidence interval does not contain μ_0, we reject H_0.

Furthermore, note that

$$\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

is equivalent to $|Z| > z_{\alpha/2}$ with $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$.

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One example: solve the following problem.

✓ The observation X follows a binomial distribution: $X \sim Bin(200, p)$.

✓ Establish a testing procedure for the following hypotheses:

$$H_0 : p = 0.5 \qquad H_1 : p \neq 0.5.$$

✓ If we observe $X = 90$, perform the above test.