ST2334 (2020/21 Semester 2)
Solution to Tutorial 4

Question 1

(a) The possible values for X are 3 (3 one-cent coins = {1, 1, 1}), 7 (2 one-cent coins and 1 five-cent coin = {1, 1, 5}), and 11 (1 one-cent coin and 2 five-cent coins = {1, 5, 5}).

(b) $\Pr(X = 3) = \Pr(3 \text{ one-cent coins in the sample}) = ({}_4C_3 \times {}_2C_0)/{}_6C_3 = 1/5.$

$\Pr(X = 7) = \Pr(2 \text{ one-cent coins and 2 five-cent coin in the sample})$
$= ({}_4C_2 \times {}_2C_1)/{}_6C_3 = 3/5.$

$\Pr(X = 11) = \Pr(1 \text{ one-cent coin and 2 five-cent coins in the sample})$
$= ({}_4C_1 \times {}_2C_2)/{}_6C_3 = 1/5.$

Therefore

x	3	7	11
$f_X(x)$	1/5	3/5	1/5

Question 2

Outcome	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
W	3	1	1	1	–1	–1	–1	–3
Probability for (a)	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
Probability for (b)	8/27	4/27	4/27	4/27	2/27	2/27	2/27	1/27

(a)

w	3	1	–1	–3
$f_W(w)$	1/8	3/8	3/8	1/8

(b)

w	3	1	–1	–3
$f_W(w)$	8/27	12/27	6/27	1/27

Question 3

$$\sum_{x=0}^{3} c(x^2+4) = 1 \iff 4c + 5c + 8c + 13c = 1 \iff 30c = 1 \iff c = 1/30.$$

Question 4

(a) $\sum_{y=1}^{5} ky = 1 \iff k + 2k + 3k + 4k + 5k = 1 \iff 15k = 1 \iff k = 1/15.$

(b) $\Pr(Y \le 3) = f_Y(1) + f_Y(2) + f_Y(3) = 6/15 = 0.4.$

(c) $\Pr(2 \le Y \le 4) = f_Y(2) + f_Y(3) + f_Y(4) = 9/15 = 0.6.$

(d) The c.d.f. of Y is given as follows

$$F_Y(y) = \begin{cases} 0, & y < 1, \\ 1/15, & 1 \le y < 2, \\ 3/15, & 2 \le y < 3, \\ 6/15, & 3 \le y < 4, \\ 10/15, & 4 \le y < 5, \\ 1, & 5 \le y. \end{cases}$$

Question 5

(a) Possible X values are those values at which $F_X(x)$ jumps and the probability of any possible values is the size of the jump at that value. Thus we have

x	1	3	4	6	12
$f_X(x)$	0.3	0.1	0.05	0.15	0.4

(b) $\Pr(3 \le X \le 6) = F_X(6) - F_X(3-) = 0.6 - 0.3 = 0.3.$
$\Pr(4 \le X) = 1 - \Pr(X < 4) = 1 - F_X(4-) = 1 - 0.4 = 0.6.$

Question 6

(a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Leftrightarrow \int_0^1 k\sqrt{x}\,dx = 1 \Leftrightarrow k\left[\frac{2}{3}x^{3/2}\right]_0^1 = 1 \Leftrightarrow \frac{2k}{3} = 1 \Leftrightarrow k = \frac{3}{2}.$

(b) For $x < 0$, $F_X(x) = \int_{-\infty}^{x} 0dt = 0.$
For $0 \le x < 1$, $F_X(x) = \int_{-\infty}^{0} 0dt + \int_0^x \frac{3}{2}\sqrt{t}\,dt = 0 + \left[t^{3/2}\right]_0^x = x^{3/2}.$
For $x \ge 1$, $F_X(x) = \int_{-\infty}^{0} 0dt + \int_0^1 \frac{3}{2}\sqrt{t}\,dt + \int_0^x 0dt = 0 + \left[t^{3/2}\right]_0^1 + 0 = 1.$
Hence,

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^{3/2}, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

(c) $\Pr(0.3 < X < 0.6) = F_X(0.6) - F_X(0.3) = 0.6^{1.5} - 0.3^{1.5} = 0.3004.$

Question 7

(a) $\Pr\left(-\frac{1}{2} < X < \frac{1}{2}\right) = \int_{-1/2}^{1/2} \frac{3}{4}(1 - x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1/2}^{1/2} = \frac{33}{48} = 0.6875.$

(b) $\Pr\left(X < -\frac{1}{4} \text{ or } X > \frac{1}{4}\right) = 1 - \Pr\left(-\frac{1}{4} < X < \frac{1}{4}\right) = 1 - \int_{-1/4}^{1/4} \frac{3}{4}(1 - x^2)\,dx = 1 - \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1/4}^{1/4} = 1 - \frac{47}{128} = \frac{81}{128} = 0.6328.$

(c) For $x < -1$, $F_X(x) = \int_{-1}^{x} 0\,dt = 0.$
For $-1 \le x \le 1$, $F_X(x) = \int_{-1}^{x} \frac{3}{4}(1 - t^2)\,dt = \frac{3}{4}\left[t - \frac{t^3}{3}\right]_{-1}^{x} = \frac{1}{4}(2 + 3x - x^3).$
For $x > 1$, $F_X(x) = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{1} \frac{3}{4}(1 - t^2)\,dt + \int_1^x 0\,dt = 1.$
Hence,

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4}(2 + 3x - x^3), & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Note: Answer for (a) = $F_X(0.5) - F_X(-0.5) = 0.84375 - 0.15625 = 0.6875$ and answer for (b) = $1 - \left(F_X(0.25) - F_X(-0.25)\right) = 1 - (0.6836 - 0.3164) = 0.6328.$

Question 8

(a) $\Pr\left(X < \frac{1}{5}\right) = F_X\left(\frac{1}{5}\right) = 1 - e^{-8(1/5)} = 0.7981.$

(b) $f_X(x) = \frac{\partial F_X(x)}{\partial x} = \frac{\partial}{\partial x}(1 - e^{-8x}) = 8e^{-8x}$, for $x \ge 0$, and $f_X(x) = 0$ for $x < 0$.