Chapter 3: Joint Distributions

1 Joint Distributions for Multiple Random Variables

- Very often, we are interested in more than one random variables simultaneously.
- For example, an investigator might be interested in both the height (H) and the weight (W) of an individual from a certain population.
- Another investigator could be interested in both the hardness (H) and the tensile strength (T) of a piece of cold-drawn copper.

DEFINITION 1

- *Let E be an experiment and S be a corresponding sample space.*
- *Let X and Y be two functions each assigning a real number to each $s \in S$.*
- *We call (X,Y) a* ***two-dimensional random vector****, or a* ***two-dimensional random variable****.*

Similarly to one-dimensional situation, we can denote the **range space** of (X,Y) by

$$R_{X,Y} = \left\{(x,y) \middle| x = X(s), y = Y(s), s \in S\right\}.$$

The definition above can be extended to more than two random variables.

DEFINITION 2

Let $X_1, X_2, \ldots, X_n$ be n functions each assigning a real number to ***every outcome*** *$s \in S$. We call $(X_1, X_2, \ldots, X_n)$ an* ***n-dimensional random variable*** *(or an* ***n-dimensional random vector****).*

We define the discrete and continuous two-dimensional RVs as follows.

DEFINITION 3

1 *(X,Y) is a* ***discrete*** *two-dimensional RV if the number of possible values of $(X(s),Y(s))$ are finite or countable.*

That is the possible values of $(X(s),Y(s))$ may be represented by

$$(x_i, y_j), i = 1,2,3,\ldots; j = 1,2,3,\ldots$$

2 *(X,Y) is a* ***continuous*** *two-dimensional RV if the possible values of $(X(s),Y(s))$ can assume any value in some region of the Euclidean space $\mathbb{R}^2$.*

Remark

we can view X and Y separately to judge whether (X,Y) is discrete or continuous.

- If both X and Y are discrete RVs, then (X,Y) is a discrete RV.
- Likewise, if both X and Y are continuous random variables, then (X,Y) is a continuous RV.
- Clearly, there are other cases. For example, X is discrete, but Y is continuous. These are not our focus in this module.

Example 3.1 (Discrete Random Vector)

- Consider a TV set to be serviced.
- Let
$$X = \{\text{age to the nearest year of the set}\};$$
$$Y = \{\text{\# of defective components in the set}\}.$$
- (X,Y) is a discrete 2-dimensional RV.
- $R_{X,Y} = \{(x,y)|x = 0,1,2,\ldots;y = 0,1,2,\ldots,n\}$, where n is the total number of components in the TV.
- $(X,Y) = (5,3)$ means that the TV is 5 years old and has 3 defective components.

L–example 3.1

- A fast food restaurant operates a **drive-up facility** and a **walk-up window**.

- On a day, Let

$$
\begin{aligned}
X &= \text{the proportion of time that the } \textbf{drive-up facility} \text{ is in use;} \\
Y &= \text{the proportion of time that the } \textbf{walk-up window} \text{ is in use.}
\end{aligned}
$$

$0 \le x \le 1$

- Then $R_{X,Y} = \{(x, y) | 0 \le x, 0 \le y \le 1\}$.

- (X, Y) is a continuous 2-dimensional RV.

Joint Probability Function

- We introduce the probability function for the discrete and continuous RVs separately.
- For discrete random vector, similar to the one-dimensional case, we define its probability function by associate a number with each possible value of the RV.

DEFINITION 4 (JOINT PROBABILITY FUNCTION FOR DISCRETE RV) *Let (X,Y) be a 2-dimensional **discrete** RV, the **joint probability (mass) function** is defined by*

$$f_{X,Y}(x,y) = P(X = x, Y = y),$$

for x, y being possible values of X and Y, or in the other words $(x,y) \in R_{X,Y}$.

The joint probability mass function has the following properties:

(1) $f_{X,Y}(x,y) \geq 0$ for any $(x,y) \in R_{X,Y}$.

(2) $f_{X,Y}(x,y) = 0$ for any $(x,y) \notin R_{X,Y}$.

(3) $\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{X,Y}(x_i,y_j) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(X = x_i, Y = y_j) = 1;$

or equivalently $\sum\sum_{(x,y)\in R_{X,Y}} f(x,y) = 1$.

(4) Let A be any subset of $R_{X,Y}$, then

$$P((X,Y) \in A) = \sum\sum_{(x,y)\in A} f_{X,Y}(x,y).$$

$\left\{\begin{array}{l} f(x,y) \geqslant 0 \\ \sum\sum f(x,y) = 1 \end{array}\right.$

Example 3.2 Find the value of k such that $f(x,y) = kxy$ for $x = 1,2,3$ and $y = 1,2,3$ can serve as a joint probability function.
Solution: $R_{X,Y} = \{(x,y) | x = 1,2,3; y = 1,2,3\}$.

$$
\begin{aligned}
f(1,1) &= k, & f(1,2) &= 2k, & f(1,3) &= 3k,\\
f(2,1) &= 2k, & f(2,2) &= 4k, & f(2,3) &= 6k,\\
f(3,1) &= 3k, & f(3,2) &= 6k, & f(3,3) &= 9k.
\end{aligned}
$$

Based on property (3), we have

$$
\begin{aligned}
1 &= \sum\sum_{(x,y)\in R_{X,Y}} f(x,y)\\
&= 1k + 2k + 3k + 2k + 4k + 6k + 3k + 6k + 9k,
\end{aligned}
$$

which results in $k = {}^{1}/_{36}$.

L–example 3.2

- A company has 2 production lines, A and B, which produce at most 5 and 3 machines respectively.
- Let

$$
\begin{aligned}
X &= \text{number of machines produced by line } A \\
Y &= \text{number of machines produced by line } B.
\end{aligned}
$$

- The joint probability function $f(x, y)$ for (X, Y) is given in the table, where each entry represents $f(x_i, y_j) = P(X = x_i, Y = y_j)$.
- What is the probability that in a day line A produces more machines than line B?

Table for the joint probability function $f(x,y)$

y	x						Row Total
	0	1	2	3	4	5	
0	0	0.01	0.02	0.05	0.06	0.08	0.22
1	0.01	0.03	0.04	0.05	0.05	0.07	0.25
2	0.02	0.03	0.05	0.06	0.06	0.07	0.29
3	0.02	0.04	0.03	0.04	0.06	0.05	0.24
Column Total	0.05	0.11	0.14	0.20	0.23	0.27	1

$x = 2, \quad y = 3$

$f_{X,Y}(2,3) = 0.03$

Consider the event

$$A = \{\text{line } A \text{ produces more machines than line } B\} = \{X > Y\}.$$

Then we have

$$\begin{aligned}
P(A) &= P(X > Y) \\
&= P\Big((X,Y) = (1,0) \text{ or } (X,Y) = (2,0) \text{ or} \\
&\qquad (X,Y) = (2,1) \text{ or } \ldots \text{ or } (X,Y) = (5,3)\Big) \\
&= P\Big((X,Y) = (1,0)\Big) + \ldots + P\Big((X,Y) = (5,3)\Big) \\
&= f(1,0) + f(2,0) + \ldots + f(5,3) = 0.73.
\end{aligned}$$

L–example 3.3

- A company has 9 executives; 4 are married, 3 have never married, and 2 are divorced.

  G_3 G_1 G_2

- Three executives are to be randomly selected for promotion.
- Among the selective executives, let

$$
\begin{aligned}
X &= \{\text{number of married executives}\} \\
Y &= \{\text{number of never married executives}\}.
\end{aligned}
$$

- Find the joint probability function of X and Y.

Solution: Note that the executives are selected randomly; so every possible selection of the executives are equally likely.

- The total number of ways to select 3 executives out of 9 is $\binom{9}{3}$.
- The possible values of x and y are constrained by $x, y = 0, 1, 2, 3$ and $1 \leq x + y \leq 3$. The number of ways to select x married and y never married is given by $\binom{4}{x} \times \binom{3}{y} \times \binom{2}{3-x-y}$.

- Therefore, the joint probability function of (X,Y) is given by

$$
\begin{aligned}
f_{X,Y}(x,y) &= P(X=x, Y=y) \\
&= \frac{\binom{4}{x}\binom{3}{y}\binom{2}{3-x-y}}{\binom{9}{3}},
\end{aligned}
\qquad \frac{\binom{4}{1}\binom{3}{0}\binom{2}{2}}{\binom{9}{3}}
$$

for $x,y = 0,1,2,3$ such that $1 \le x+y \le 3$ and $f_{X,Y}(x,y) = 0$ otherwise.

- This joint p.f. can be summarized as a table.

x	y				Row Total
	0	1	2	3	
0	0	$3/84$	$6/84$	$1/84$	$10/84$
1	$4/84$	$24/84$	$12/84$	0	$40/84$
2	$12/84$	$18/84$	0	0	$30/84$
3	$4/84$	0	0	0	$4/84$
Column Total	$20/84$	$45/84$	$18/84$	$1/84$	1

DEFINITION 5 (JOINT PROBABILITY FUNCTION FOR CONTINUOUS RV) *Let (X,Y) be a 2-dimensional* ***continuous*** *RV; its* ***joint probability (density) function*** *is a function $f_{X,Y}(x,y)$ such that*

$$P((X,Y) \in D) = \int\int_{(x,y)\in D} f_{X,Y}(x,y)dydx,$$

for any $D \subset \mathbb{R}^2$. More specifically,

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)dydx.$$

The joint probability density function has the following properties:

(1) $f_{X,Y}(x,y) \geq 0$, for any $(x,y) \in R_{X,Y}$.

(2) $f_{X,Y}(x,y) = 0$, for any $(x,y) \notin R_{X,Y}$.

(3) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$;

or equivalently $\int\int_{(x,y)\in R_{X,Y}} f_{X,Y}(x,y)dxdy = 1$.

$f(x,y) \geq 0$

$\leftarrow \int f(x,y)dxdy = 1$

Example 3.3 Find the value c such that $f(x,y)$ below can serve as a joint p.d.f. for a RV (X,Y):

$$f(x,y) = \begin{cases} cx(x+y), & 0 \le x \le 1; 1 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Solution: In order for $f(x,y)$ to be a p.d.f., we need

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^1\int_1^2 cx(x+y)dydx = c\int_0^1 x\left(x+\frac{1}{2}y^2\Big|_1^2\right)dx \\ &= c\int_0^1 x(x+1.5)dx = c\left(\frac{1}{3}x^3 + 1.5\cdot\frac{1}{2}x^2\right)\Big|_0^1 = c\cdot\boxed{\frac{13}{12}}, \end{aligned}$$

which implies $\boxed{c = 12/13}$.

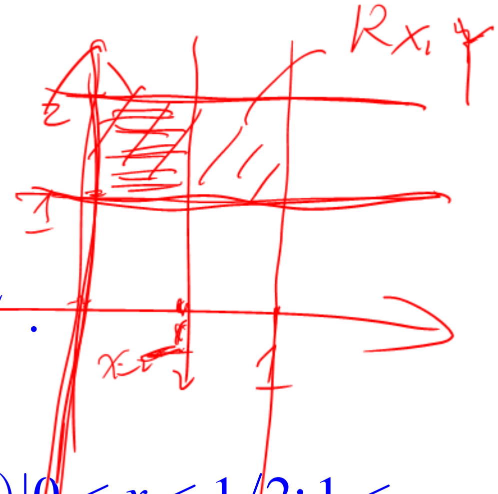
L–example 3.4

Reuse the p.d.f. of Example 3.3:

$x=0 \quad y=1$

$x=1 \quad y=2$

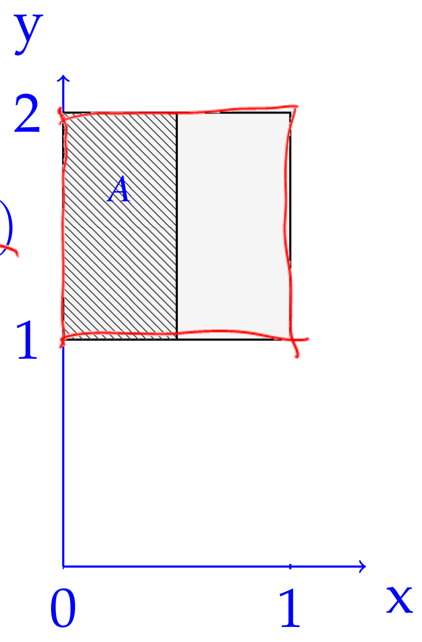
$$f(x,y) = \begin{cases} \frac{12}{13}x(x+y), & 0 \le x \le 1; 1 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$



Assume that it is the joint p.d.f. of (X,Y). Let $A = \{(x,y)|0 < x < 1/2; 1 < y < 2\}$. Compute $P((X,Y) \in A)$.

- Set A corresponds to the shaped area in the figure on the right.

- We have



$$
\begin{aligned}
P((X,Y)\in A) &= \underline{P(0<X<1/2; 1<Y<2)} \\
&= \int_0^{1/2}\int_1^2 \boxed{\frac{12}{13}} x(x+y)\,dy\,dx \\
&= \frac{12}{13}\int_0^{1/2} x(x+1.5)\,dx \\
&= \frac{12}{13}\left(\frac{1}{3}x^3+1.5\cdot\frac{1}{2}x^2\right)\Big|_0^{1/2} \\
&= {}^{11}/_{52}.
\end{aligned}
$$

$\int_1^2 (x+y)dy$

$= x+\int_1^2 y\,dy$

Definition 6 (Marginal Probability Distribution) 100

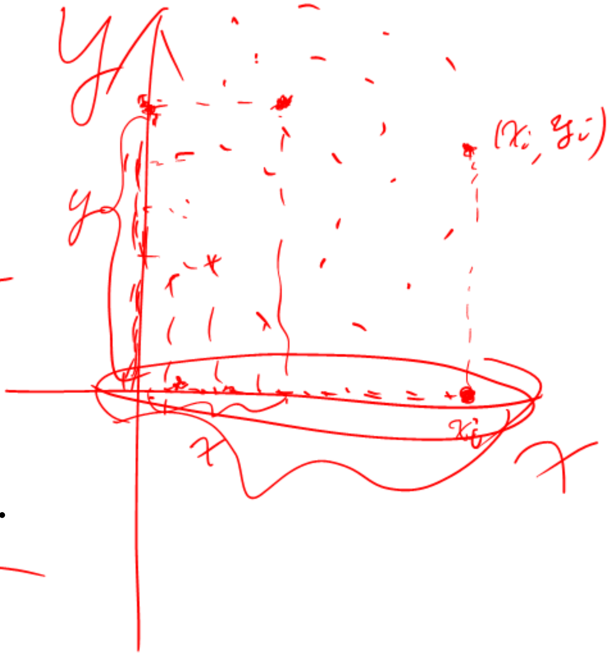
Let (X,Y) be a two-dimensional RV with joint p.f. $f_{X,Y}(x,y)$. We define the marginal distribution for X as follows.

- *If Y is a discrete RV, then for any x,*

$$f_X(x) = \sum_y f_{X,Y}(x,y).$$

- *If Y is a continuous RV, then for any x,*

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy.$$



REMARK

- $f_Y(y)$ for Y is defined in the same way as that of X.
- We can view the marginal distribution as the "projection" of the 2D function $f_{X,Y}(x,y)$ to the 1D function.
- More intuitively, it is the distribution of X by ignoring the presence of Y.

  For example, consider a person of a certain community,

  - suppose $X =$ body weight, $Y =$ height. (X, Y) has a joint distribution $f_{X,Y}(x,y)$.
  - the marginal distribution $f_X(x)$ of X is the **distribution of body weights for all people in the community**.

- $f_X(x)$ should not involve the variable y; this can be viewed from its definition: y is either summed out or integrated over.

- $f_X(x)$ is a **probability function** so it satisfies all the properties of the probability function.

Example 3.4

- Revisit Example 3.2. The joint p.f. is given by $f(x,y) = \frac{1}{36}xy$ for $x = 1,2,3$ and $y = 1,2,3$.
- Note that X has three possible values: 1, 2, and 3. The marginal distribution for X is given by
  - for $x = 1$, $f_X(1) = f(1,1) + f(1,2) + f(1,3) = 6/36 = 1/6$.
  - for $x = 2$, $f_X(2) = f(2,1) + f(2,2) + f(2,3) = 12/36 = 1/3$.
  - for $x = 3$, $f_X(3) = f(3,1) + f(3,2) + f(3,3) = 18/36 = 1/2$.
  - for other values of x, $f_X(x) = 0$.

- Alternatively, for each $x \in \{1, 2, 3\}$,

$$\begin{aligned} f_X(x) &= \sum_y f(x, y) = \sum_{y=1}^{3} \frac{1}{36} xy \\ &= \frac{1}{36} x \sum_{y=1}^{3} y = \frac{1}{6} x. \end{aligned}$$

L–example 3.5

We reuse the joint p.f. of (X, Y) derived in L–Example 1:

x	y				Row Total
	0	1	2	3	
0	0	$3/84$	$6/84$	$1/84$	$10/84$
1	$4/84$	$24/84$	$12/84$	0	$40/84$
2	$12/84$	$18/84$	0	0	$30/84$
3	$4/84$	0	0	0	$4/84$
Column Total	$20/84$	$45/84$	$18/84$	$1/84$	1

$f_X(x)$

$f_Y(y)$

Can we read out the marginal p.f. of X and Y from the table directly?

L–example 3.6

Reuse the p.d.f. of Example 3.3:

$$f(x,y) = \begin{cases} \frac{12}{13}x(x+y), & 0 \le x \le 1; 1 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

Assume that it is the joint p.d.f. of (X,Y). Find the marginal distribution of X.

Solution: (X,Y) is a continuous RV. For each $x \in [0,1]$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy = \int_1^2 \frac{12}{13}x(x+y)dy \\ &= \frac{12}{13}x\left(x+\int_1^2 ydy\right) \\ &= \frac{12}{13}x(x+1.5); \end{aligned}$$

and for $x \notin [0,1]$, $f_X(x) = 0$.

Definition 7 (Conditional Distribution)

Let (X,Y) be a RV with joint p.f. $f_{X,Y}(x,y)$. Let $f_X(x)$ be the marginal p.f. for X. Then for any x such that $f_X(x) > 0$, the ***conditional probability function of*** *Y* ***given*** *$X = x$ is defined to be*

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

REMARK

- For any y such that $f_Y(y) > 0$, we can similarly define the **conditional distribution of X given $Y = y$**:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

- $f_{Y|X}(y|x)$ is defined only for x such that $f_X(x) > 0$; likewise $f_{X|Y}(x|y)$ is defined only for y such that $f_Y(y) > 0$.

- The practical meaning of $f_{Y|X}(y|x)$: the distribution of Y given that the random variable X is observed to take the value x.

- Considering y as the variable (x as a fixed value), $f_{Y|X}(y|x)$ is a p.f., so it must satisfy all the properties of p.f.. $\sum_y f_{Y|X}(y|x) = 1$; $\int = 1$

- But $f_{Y|X}(y|x)$ is not a p.f. for x; this means that there is **NO** requirement $\int_{-\infty}^{\infty} f_{Y|X}(y|x)dx = 1$ for X continuous or $\sum_x f_{Y|X}(y|x) = 1$ for X discrete.

- With the definition, we immediately have
  - If $f_X(x) > 0$, $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x)$.
  - If $f_Y(y) > 0$, $f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y)$.

- One immediate application of the conditional distribution is to compute, for continuous RV,

$P(Y \le y | X) = \boxed{G(X)}$

$$P(Y \le y | X = x) = \int_{-\infty}^{y} f_{Y|X}(y|x)\,dy; \quad G(x)$$

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy. \quad \tilde{G}(x)$$

$E(Y|X) = \tilde{G}(X)$

Their practical meanings are clear: the former is the probability that $Y \le y$, given $X = x$; the latter is the average value of Y given $X = x$.

For discrete case, the computation is similarly established based on $f_{Y|X}(y|x)$; please fill in the details on your own.

Example 3.5 Revisit Examples 3.2 and 3.4.

- The joint p.f. for (X, Y) is given by $f(x, y) = \frac{1}{36}xy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$.
- The marginal p.f. for X is $f_X(x) = \frac{1}{6}x$ for $x = 1, 2, 3$.
- Therefore, $f_{Y|X}(y|x)$ is defined for any $x = 1, 2,$ or 3:

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{(1/36)xy}{(1/6)x} = \frac{1}{6}y,$$

for $y = 1, 2, 3$.

- We can compute

$$
\begin{aligned}
P(Y=2|X=1) &= f_{Y|X}(2|1) = \frac{1}{6}\cdot 2 = 1/3; \\
P(Y\le 2|X=1) &= P(Y=1|X=1)+P(Y=2|X=1) \\
&= f_{Y|X}(1|1)+f_{Y|X}(2|1) = 1/6+1/3 = 1/2; \\
E(Y|X=2) &= 1\cdot f_{Y|X}(1|2)+2\cdot f_{Y|X}(2|2)+3\cdot f_{Y|X}(3|2) \\
&= 1\cdot(1/6)+2\cdot(2/6)+3\cdot(3/6) = 7/3.
\end{aligned}
$$

L–example 3.7

We reuse the joint p.f. of (X,Y) derived in L–Example 1:

$p(Y=0|X=1) = \frac{4/84}{40/84}$

$E(Y|X=1)$

x	y = 0	1	2	3	Row Total
0	0	$3/84$	$6/84$	$1/84$	$10/84$
1	$4/84$	$24/84$	$12/84$	0	$40/84$
2	$12/84$	$18/84$	0	0	$30/84$
3	$4/84$	0	0	0	$4/84$
Column Total	$20/84$	$45/84$	$18/84$	$1/84$	1

Can we read out the conditional p.f. $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ from the table directly? How to compute $E(Y|X=x)$?

L–example 3.8 Reuse Examples 3.3 and L–Example 2.

- The joint p.f. for (X,Y) is given by

$$f(x,y) = \begin{cases} \frac{12}{13}x(x+y), & 0 \le x \le 1; 1 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

- The marginal p.f. for X is given by

$$f_X(x) = \frac{12}{13}x(x+1.5),$$

for $x \in [0,1]$.

- For each $x \in [0,1]$, the conditional p.f. $f_{Y|X}(y|x)$,

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} = \frac{(12/13)x(x+y)}{(12/13)x(x+1.5)} \\
&= \frac{x+y}{x+1.5},
\end{aligned}
$$

for $y \in [1,2]$.

- We can compute

$$
P(Y \le 1.5 | X = 0.5) = \int_1^{1.5} \boxed{\frac{0.5+y}{0.5+1.5}} dy = 0.5625.
$$

$f_{Y|X}(y|0.5)$

- Furthermore

$$
\begin{aligned}
E(Y|X=0.5) &= \int_1^2 y \boxed{\frac{0.5+y}{0.5+1.5}} dy \\
&= \frac{1}{2}\int_1^2 (0.5y+y^2)dy \\
&= \frac{1}{2}\left(\frac{3}{4}+\frac{7}{3}\right) = 37/24.
\end{aligned}
$$

3 Independent Random Variables

Definition 8 (Independent Random Variables)

- *Random variables X and Y are **independent** if and only if for **any** x and y,*

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

- *Random variables $X_1, X_2, \ldots, X_n$ are **independent** if and only if for any $x_1, x_2, \ldots, x_n$,*

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n).$$

REMARK

- The above definition is applicable no matter whether (X, Y) is continuous or discrete.

- The "product feature" in the definition implies one necessary condition for independence: $R_{X,Y}$ needs to be a product space. In the sense that if X and Y are independent, for any $x \in R_X$ and any $y \in R_Y$, we have

  $R_{X,Y} = \{(x,y) : f(x,y) > 0\}$

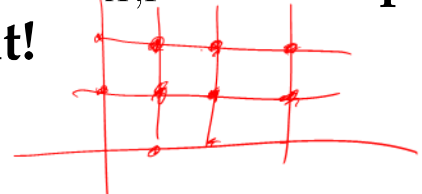
  $R_X = \{x : f_X(x) > 0\}$

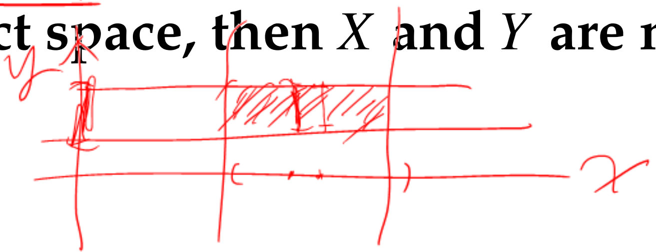
  $R_Y = \{y : f_Y(y) > 0\}$

$$f_{X,Y}(x,y) = \overset{>0}{f_X(x)}\,\overset{>0}{f_Y(y)} > 0,$$

implying $R_{X,Y} = \{(x,y) | x \in R_X; y \in R_y\} = R_X \times R_Y$.

Conclusion: if $R_{X,Y}$ is not a product space, then X and Y are not independent!





① P.S. $\Leftarrow X \perp Y$

② P.S. ✗ $\Rightarrow X \not\perp Y$

Properties of Independent Random Variables

Suppose X, Y are independent RVs.

(1) If A and B are arbitrary subsets of $\mathbb{R}$, the events $X \in A$ and $Y \in B$ are independent events in S. Thus

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B).$$

In particular, for any real numbers x, y,

$$P(X \leq x; Y \leq y) = P(X \leq x)P(Y \leq y).$$

$F(x,y) = F_X(x)F_Y(y)$

(2) For arbitrary functions $g_1(\cdot)$ and $g_2(\cdot)$, $g_1(X)$ and $g_2(Y)$ are independent. For example,

$E[g_1(X)g_2(Y)] = E(g_1(X))E(g_2(Y))$

- X^2 and Y are independent. $E(X^2Y) = EX^2 \cdot EY$
- $\sin(X)$ and $\cos(Y)$ are independent.
- e^X and $\log(Y)$ are independent. $E(e^{X+Y}) = Ee^X \cdot Ee^Y$

(3) Independence is connected with conditional distribution.

- If $f_X(x) > 0$, then $f_{Y|X}(y|x) = f_Y(y)$.
- Likewise, if $f_Y(y) > 0$, then $f_{X|Y}(x|y) = f_X(x)$.

Example 3.6 The joint p.f. of (X, Y) is given below.

x	y			$f_X(x)$
	1	3	5	
2	0.1	0.2	0.1	0.4
4	0.15	0.3	0.15	0.6
$f_Y(y)$	0.25	0.5	0.25	1

Are X and Y independent?

Solution:

- We need to check that for every x and y combination, whether we have
$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$
For example, from the table, we have $f_{X,Y}(2,1) = 0.1$; $f_X(2) = 0.4$, $f_Y(1) = 0.25$. Therefore
$$f_{X,Y}(2,1) = 0.1 = 0.4 \times 0.25 = f_X(2)f_Y(1).$$

- In fact, we can check for each $x \in \{2,4\}$ and $y \in \{1,3,5\}$ combination, the equality holds.

- We conclude that X and Y are independent.

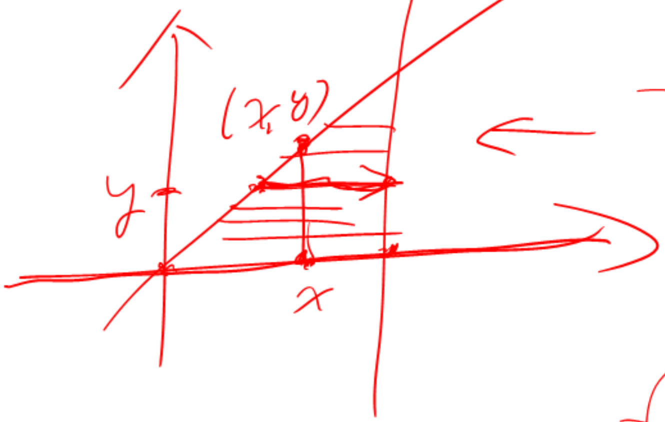
$-1 < y < 1$ $\quad x > 0$

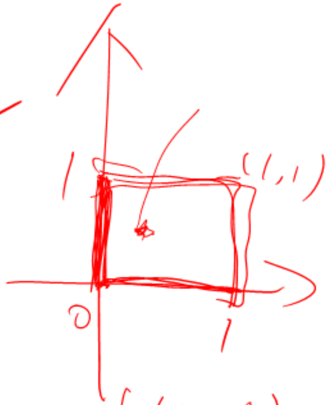
L–example 3.9 Given that

$$f_{X,Y}(x,y) = \begin{cases} 2(x+y), & \text{for } 0 \le x \le 1, 0 < y < x \\ 0 & \text{elsewhere} \end{cases}$$

Are X and Y independent?

$$\begin{cases} f_X(x) = \int_0^x f(x,y)\,dy \\ f_Y(y) = \int_y^1 f(x,y)\,dx \end{cases}$$





$f(x,y) = xy$

$f(0,0) = 0$

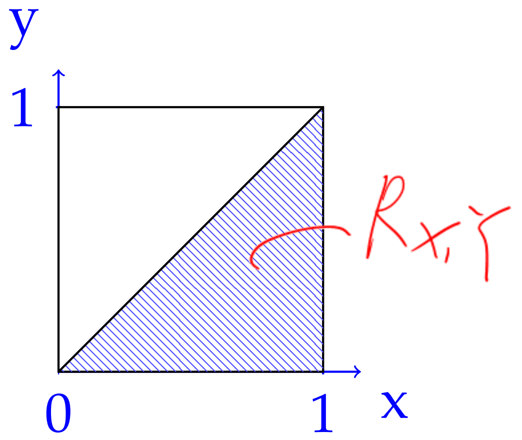
Solution:

- The direct way of checking the independence is to check whether

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

holds for every (x, y) combination. The detail of this method is left as an exercise.

- For this question, we can immediately conclude that X and Y are not independent by checking that $R_{X,Y}$ is not a product space.



L–example 3.10 Suppose that (X, Y) is a discrete RV. The joint p.f. is given by

x	y				$f_X(x)$
	0	1	2	3	
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
$f_Y(y)$	1/8	3/8	3/8	1/8	1

Are X and Y independent?

Solution:
The zero entries in the table indicate that $R_{X,Y}$ is not a product space. Therefore, X and Y are not independent.

L–example 3.11 We have a handy way to check independence when $f_{X,Y}(x,y)$ has an explicit formula in $R_{X,Y}$.

X and Y are independent if and only if both of the following hold:

$\Longleftrightarrow X \perp Y$

- $R_{X,Y}$, the range that the p.f. is positive, is a product space.
- For any $(x,y) \in R_{X,Y}$, we have $f_{X,Y}(x,y) = C \cdot g_1(x) g_2(y)$; that is, it can be "factorized" as the product of two functions g_1 and g_2, where the former **depends on** x **only**, the latter **depends on** y **only**, and C is a constant not depending on both x and y.

Note: $g_1(x)$ and $g_2(y)$ on their own are NOT necessarily p.f.s.

- We use the joint p.d. in Example 3.2 to illustrate: $f(x,y) = \frac{1}{36}xy$ for $x = 1,2,3$ and $y = 1,2,3$.
- $A_1 = \{1,2,3\}$ and $A_2 = \{1,2,3\}$, so the $R_{X,Y}$ is a product space.
- $f_{X,Y}(x,y) = \frac{1}{36} \cdot (x) \cdot (y)$: $C = 1/36$, $g_1(x) = x$, $g_2(y) = y$.
- We conclude that X and Y are independent.
- The advantage of this method is that we don't need to find the marginal distributions $f_X(x)$ and $f_Y(y)$ and check $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

$= C g_1(x) \cdot g_2(y)$

Following this strategy, we can get $f_X(x)$ and $f_Y(y)$ by standardizing $g_1(x)$ and $g_2(y)$. Consider $f_X(x)$ for illustration; $f_Y(y)$ is obtained similarly.

- If X is a discrete RV, its p.m.f. is given by

$$f_X(x) = \frac{g_1(x)}{\sum_{t \in R_X} g_1(t)}.$$

- If X is a continuous RV, its p.d.f. is given by

$$f_X(x) = \frac{g_1(x)}{\int_{t \in R_X} g_1(t)} dt.$$

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- We continue to use the example above to illustrate. Here X is a discrete RV, $R_X = A_1 = \{1, 2, 3\}$. We obtain its p.m.f.:

$$f_X(x) = \frac{g_1(x)}{\sum_{x \in R_X} g_1(x)} = \frac{x}{\sum_{x=1}^{3} x} = x/6.$$

- Similarly, we get $f_Y(y) = y/6$.

L–example 3.12 Given that

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}x(1+y), & \text{for } 0 < x < 2, 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Are X and Y independent?

$\not\equiv c \cdot g_1(x) g_2(y)$

$$f(x,y) = \begin{cases} x+y & x \in A_1,\ y \in A_2 \\ 0 & \text{elsewhere} \end{cases}$$

$X \perp Y$?

Solution:

- Set $A_1 = (0,2)$ and $A_2 = (0,1)$, then $R_{X,Y} = A_1 \times A_2$ is a product space.

- $f_{X,Y}(x,y)$ in $R_{X,Y}$ can be factorized by $C = 1/3$, $g_1(x) = x$, $g_2(y) = 1+y$. Therefore, we conclude that X and Y are independent.

- Furthermore,

$$f_X(x) = \frac{g_1(x)}{\int_{x\in A_1} g_1(x)dx} = \frac{x}{\int_0^2 xdx} = x/2;$$

$$f_Y(y) = \frac{g_2(y)}{\int_{y\in A_2} g_2(y)dy} = \frac{1+y}{\int_0^1 (1+y)dy} = \frac{2}{3}(1+y).$$

4 Expectation and Covariance

Definition 9 (Expectation)
For any two variable function $g(x,y)$,

- *if (X,Y) is a discrete RV,*

$$E(g(X,Y)) = \sum_x \sum_y g(x,y) f_{X,Y}(x,y);$$

- *if (X,Y) is a continuous RV,*

$$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dydx.$$

If we let

$$g(X,Y) = (X - E(X))(Y - E(Y)) = (X - \mu_X)(Y - \mu_y),$$

the expectation $E[g(X,Y)]$ leads to the covariance of X and Y.

DEFINITION 10 (COVARIANCE)
*The **covariance** of X and Y is defined to be*

$$cov(X,Y) = E[(X - E(X))(Y - E(Y))]$$

- If X and Y are discrete RVs,

$$\mathrm{cov}(X,Y) = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f_{X,Y}(x,y).$$

- If X and Y are continuous RVs,

$$\mathrm{cov}(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f_{X,Y}(x,y) dx dy.$$

The covariance has the following properties.

(1) $\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$.

$E(XY) = EX\,EY$

(2) If X and Y are independent, then $\operatorname{cov}(X,Y) = 0$. However, $\operatorname{cov}(X,Y) = 0$ does not imply that X and Y are independent.

$\operatorname{cov}(g_1(X)\,g_2(Y)) = E g_1(X)\, E g_2(Y)$

(3) $\operatorname{cov}(aX+b, cY+d) = ac \cdot \operatorname{cov}(X,Y)$.

-1

$\operatorname{cov}(X, -Y) = (-1)\operatorname{cov}(X,Y) = -\operatorname{cov}(X,Y)$

(4) $V(aX+bY) = a^2V(X) + b^2V(Y) + 2ab \cdot \operatorname{cov}(X,Y)$.

Example 3.7 Given the joint distribution for (X,Y):

x	y = 0	1	2	3	$f_X(x)$
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
$f_Y(y)$	1/8	3/8	3/8	1/8	1

(a) Find $E(Y-X)$.

(b) Find $\mathrm{cov}(X,Y)$.

Solution:

(a) Method 1:

$$\begin{aligned} E(Y-X) &= (0-0)(1/8)+(1-0)(1/4)+(2-0)(1/8) \\ &\quad +\ldots+(3-1)(1/8)=1. \end{aligned}$$

Method 2:

$$E(Y-X)=\boxed{E(Y)}-\boxed{E(X)}=1.5-0.5=1,$$

where

$$\begin{aligned} E(Y) &= 0\cdot(1/8)+1\cdot(3/8)+2\cdot(3/8)+3\cdot(1/8)=1.5 \\ E(X) &= 0\cdot(1/2)+1\cdot(1/2)=0.5. \end{aligned}$$

(b) We use $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$ to compute. Note that we have computed $E(X)$ and $E(Y)$ in Part (a).

$$\begin{aligned} E(XY) &= (0)(0)(1/8) + (0)(1)(1/4) + (0)(2)(1/8) \\ &\quad + \ldots + (1)(3)(1/8) = 1. \end{aligned}$$

Therefore

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y) = 1 - (0.5)(1.5) = 0.25.$$

L–example 3.13 Suppose that (X,Y) has the p.f.

$= x(x+\frac{1}{3}y) \neq g_1(x)\, g_2(y)$

$$f_{X,Y}(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & \text{for } 0 \le x \le 1, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find $f_X(x)$, $f_Y(y)$ and $f_{Y|X}(y|x)$.

(b) Find $\mathrm{cov}(X,Y)$.

Solution:

(a) We first find the marginal density of X.

For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_0^2 \left(x^2 + \frac{xy}{3}\right) \mathrm{d}y$$
$$= \left(x^2 y + \frac{xy^2}{6}\right)\Big|_{y=0}^{2} = 2x^2 + \frac{2x}{3}.$$

It is clear that $f_X(x) = 0$ for $x < 0$ or $x > 1$. Thus

$$f_X(x) = \begin{cases} 2x^2 + \dfrac{2x}{3}, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Similarly, the marginal density of Y is given as

$$f_Y(y) = \begin{cases} \dfrac{1}{3} + \dfrac{y}{6}, & \text{for } 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

The conditional probability density function of Y given $X = x$ when $0 \le x \le 1$ is then given as

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f(y)} =$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{x^2 + xy/3}{2x^2 + 2x/3}, & \text{for } 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{3x+y}{2(3x+1)}, & \text{for } 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

$$\int_0^1 \frac{3x+y}{2(3x+1)}\,dx \neq 1$$

$$\int_0^2 \frac{3x+y}{2(3x+1)}\,dy = 1$$

(b) We shall use the expression $\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$.

Now

$$
\begin{aligned}
E(XY) &= \int_0^2 \int_0^1 xy\left(x^2 + \frac{xy}{3}\right) \mathrm{dx}\,\mathrm{dy} \\
&= \int_0^2 \int_0^1 \left(yx^3 + \frac{y^2x^2}{3}\right) \mathrm{dx}\,\mathrm{dy} \\
&= \int_0^2 \left(y\frac{x^4}{4} + \frac{y^2x^3}{9}\right)\Bigg|_{x=0}^1 \mathrm{dy} \\
&= \int_0^2 \left(\frac{y}{4} + \frac{y^2}{9}\right) \mathrm{dy} \\
&= \frac{43}{54}.
\end{aligned}
$$

We have computed the marginal distributions for X and Y in Part (a). Thus

$$E(X) = \int_0^1 x\left(2x^2 + \frac{2x}{3}\right) \mathrm{dx} = \left(\frac{2x^4}{4} + \frac{2x^3}{9}\right)\Big|_{x=0}^{1} = \frac{13}{18},$$

and

$$E(Y) = \int_0^2 y\left(\frac{1}{3} + \frac{y}{6}\right) \mathrm{dy} = \left(\frac{y^2}{6} + \frac{y^3}{18}\right)\Big|_{y=0}^{2} = \frac{10}{9}.$$

This gives

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{43}{54} - \frac{13}{18} \times \frac{10}{9} = -\frac{1}{162}.$$

L–example 3.14

$V(X+Y) = E(X+Y)^2 - [E(X+Y)]^2 = EX^2 + 2EXY + EY^2 - [EX]^2 - 2EXEY - [EY]^2$

$= V(X) + 2\,\mathrm{cov}(X,Y) + V(Y)$

- Start from $V(X+Y) = V(X) + V(Y) + 2\,\mathrm{cov}(X,Y)$, we can have some interesting results.

  $V(X-Y) = V(X+(-Y)) = V(X) + V(-Y) + 2\,\mathrm{cov}(X,-Y)$

- By induction, we have for any random variables $X_1, X_2, \ldots, X_n$,

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \ldots + V(X_n) + 2\sum_{j>i} \mathrm{cov}(X_i, X_j).$$

- If X and Y are independent, we have

$V(X \pm Y) = V(X) \oplus V(Y)$

$V(X - Y) = V(X + (-Y))$

$= V(X) + V(-Y)$

$= V(X) + V(Y)$

$$V(X \pm Y) = V(X) + V(Y).$$

- By induction, we have if $X_1, X_2, \ldots, X_n$ are independent,

$$V(X_1 \pm X_2 \pm \ldots \pm X_n) = V(X_1) + V(X_2) + \ldots + V(X_n).$$

$V(\bar{X})$ $V\left(\frac{X_1 + \cdots + X_n}{n}\right)$