ST2334 (2020/2021 Semester 1) Solutions to Questions in Tutorial 8

Question 1

X = number of pipework failures caused by operator error out of 20 pipework.
$X \sim Binomial\ (20, 0.30)$

(a) $\Pr(X \geq 10) = 1 - \Pr(X < 9) = 1 - 0.9520 = 0.0480$
(b) $\Pr(X \leq 4) = 0.2375$
(c) $\Pr(X = 5) = 0.1789$

The probability is not very small, so it is not a rare event. Thus $p = 0.30$ is reasonable.

Some discussion on extreme event.

It is better to check if $X = 5$ is an extreme event. That is, to check if $\Pr(X \leq 5)$ is very small when $p = 0.30$. Note: $\Pr(X \geq 5) > \Pr(X = 5) = 0.1789$

What event is considered an extreme event?

Let Y have the following probability function $f_Y(y) = \frac{1}{100}$ for $y = 1, 2, \cdots, 100$; and 0 otherwise. Is $Y = 70$ an extreme event? No, it is because $\Pr(Y \geq 70) = \frac{31}{100}$. Also note that $\Pr(Y = 70) = \frac{1}{100}$.

Question 2

X = number of trucks out of 15 trucks with blowout. $X \sim Binomial(15, 0.25)$

(a) $\Pr(X = 0) = 0.0134$
(b) $\Pr(X \geq 8) = 1 - 0.9824 = 0.0173$
(c) $E(X) = np = (15)(0.25) = 3.75$
(d) $V(X) = np(1 - p) = (15)(0.25)(0.75) = 2.8125$

For $k = 2$, $(\mu \pm 2\sigma) = 3.75 \pm 2\left(\sqrt{2.8125}\right) = (0.4, 7.1)$. Hence, $\Pr(0.4 < X < 7.1) = \Pr(|X - \mu| < 2\sigma) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$. Since X is a discrete random variable, therefore $\Pr(0.4 < X < 7.1) = \Pr(1 \leq X \leq 7)$.

Note: With the knowledge of the distribution of X, the exact probability can be computed and it equals $\Pr(X \leq 7) - \Pr(X \leq 0) = 0.9693$, which is much bigger than 0.75.

Question 3

$X =$ number of forms with error in 10000 forms. $X \sim Binomial(n = 10{,}000, p = 0.001)$
As n is large, p is small, $X \dot{\sim} Poisson\ (\lambda = np = 10)$

(a) $\Pr(X = 6, 7, 8) = \Pr(X \leq 8) - \Pr(X \leq 5) = 0.2657$.
(b) $E(X) = np = 10, V(X) = npq = 9.99$
(c) For $k = 3$, $(\mu \pm 3\sigma) = 10 \pm 3\left(\sqrt{9.99}\right) = (0.52, 19.48)$ $\therefore 1 \leq X \leq 19$

Question 4

$X =$ number of persons interviewed to get the fifth person to own a dog.
$X \sim Negative\ Binomial\ (k = 5, p = 0.3)$. $\Pr(X = 10) = \binom{9}{4}(0.7)^5(0.3)^5 = 0.0515$.

Question 5

$X =$ number of children until two sons. $X \sim Negative\ Binomial\ (k = 2, p = 0.5)$

(a) $\Pr(X = 7) = \binom{6}{1}(0.5)^7 = 0.0469$.
(b) $E(X) = \frac{k}{p} = 4$

Question 6

$\Pr(HHH, TTT) = (1/2)^3 + (1/2)^3 = 1/4$

$X \sim Geometric\ (p = 3/4)$

(a) $\Pr(X < 4) = (3/4) + (1/4)(3/4) + (1/4)^2(3/4) = 63/64 = 0.9844$

(b) $\Pr(X \le x) = \sum_{n=1}^{x}(3/4)(1/4)^{n-1} = (3/4)\frac{1-(1/4)^x}{1-(1/4)} = 1 - (1/4)^x$.

Note: $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$

Question 7

$X =$ number of errors in one page. $X \sim Poisson\ (\lambda = 2)$

(a) $V(X) = \lambda = 2$

(b) $\Pr(X \ge 4) = 1 - \Pr(X \le 3) = 0.1429$. $\Pr(X = 0) = 0.1353$.

Question 8

$X \sim Poisson\ (\lambda = 5\ per\ hour)$

(a) $\Pr(X = 0) = 0.00673$

(b) $\Pr(X > 10) = 1 - \Pr(X \le 10) = 1 - 0.9863 = 0.0137$

(c) $Y \sim Poisson\ (\lambda = 15\ per\ 3\text{-}hour)$. $\Pr(X > 20) = 0.0830$

Question 9

(a) $X =$ number of cars in the sample that have defects. $X \sim B(10000, 0.0005)$. So $\mu = np = 5$ and $\sigma = \sqrt{np(1-p)} = 2.2355$.

(b) Use Poisson approximation since n is large and p is small. $X\ approx \sim Poisson(5)$.

$\Pr(X \ge 10) \approx \sum_{x=5}^{\infty}\frac{e^{-5}5^x}{x!} = 1 - \sum_{x=0}^{4}\frac{e^{-5}5^x}{x!} = 0.0318$.

(c) As in (b), $\Pr(X = 0) \approx e^{-5} = 0.0067$.

Exact probability $= (1-p)^{10000} = 0.9995^{10000} = 0.006729527023$ ...

Question 10

$X \sim$ Continuous uniform$(0, 4)$

(a) $f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4, \\ 0, & \text{otherwise} \end{cases}$

(b) $\Pr(X \ge 3) = \int_3^4 \frac{1}{4}\, dx = \frac{1}{4} = 0.25$

(c) $E(X) = \frac{0+4}{2} = 2, V(X) = \frac{(4)^2}{12} = \frac{4}{3} = 1.3333$.

Question 11

$X =$ length of time to be served, in minutes

$X \sim Exponential\ (1/\mu)$, where $\mu = 4$

(a) $\Pr(X > 3) = e^{-\left(\frac{1}{4}\right)(3)} = 0.4724$

(b) $\Pr(X < 3) = 1 - e^{-\left(\frac{1}{4}\right)(3)} = 0.5276$

(c) $Y =$ number of days being served in less than 3 minutes. $Y \sim Binomial(6, 0.5276)$

$\Pr(Y \ge 4) = \Pr(Y = 4) + \Pr(Y = 5) + \Pr(Y = 6) = 0.3968$