Chapter 3

Two-dimensional Random Variables and Conditional Probability Distributions

Overview

- 2 dimensional random variables
- Joint probability functions for discrete random variables
- Joint probability density functions for continuous random variables
- Marginal distributions
- Conditional distributions
- Independent random variables
- Expectation

3.1 Two Dimensional Random Variables

- There are many experiment situations in which more than one random variable will be of interest to an investigator.

- For example, the investigator may be interested in studying the height (H) and weight (W) of a person chosen from a certain population.

- Another researcher may be interested in the hardness (H) and tensile strength (T) of a piece of cold-drawn copper

Two Dimensional Random Variables (Continued)

Definition 3.1

- Let E be an experiment and S a sample space associated with E.

- Let X and Y be two functions each assigning a real number to each $s \in S$.

- We call (X, Y) a **two-dimensional random variable**. (Sometimes called a **random vector**).

Two Dimensional Random Variables (Continued)

Range Space

$$R_{X,Y} = \{(x, y) \mid x = X(s), y = Y(s), s \in S\}.$$

The above definition can be extended to more than two random variables.

Definition 3.2

- Let $X_1, X_2, \cdots, X_n$ be n functions each assigning a real number to **every** outcome $s \in S$. We call $(X_1, X_2, \cdots, X_n)$ an ***n*-dimensional random variable.** (or an ***n*-dimensional random vector**).

Two Dimensional Random Variables (Continued)

Definition 3.3

1. (X, Y) is a two-dimensional **discrete** random variable if the possible values of $(X(s), Y(s))$ are finite or countable infinite.

   i.e. the possible values of $(X(s), Y(s))$ may be represented as
   $$\left(x_i, y_j\right), i = 1, 2, 3, \cdots; j = 1, 2, 3, \cdots$$

2. (X, Y) is a two-dimensional **continuous** random variable if the possible values of $(X(s), Y(s))$ can assume all values in some region of the Euclidean plane $\mathbb{R}^2$.

Example 1

- Consider a television set to be serviced.
- Let X represent the age to the nearest year of the set and Y represent the number of defective components in the set.
- (X, Y) is a discrete 2-dimensional random variable.
- Then the set of possible values for (X, Y) is $R_{X,Y} = \{(x, y): x = 0, 1, 2, \cdots; y = 0, 1, 2, \cdots, n\}$, where n is the total number of components in the television set.
- $(X, Y) = (5, 3)$ means the television set is 5 years old and has 3 defective components.

Example 2

- A fast food restaurant operates a drive-up facility and a walk-up window.
- On a randomly selected day, let X = the proportion of time that the **drive-up facility** is in use (at least one customer is being served or waiting to be served) and Y = the proportion of the time that the **walk-up window** is in use.
- Then the set of possible values for (X, Y) is

$$R_{X,Y} = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

- (X, Y) is a **continuous** 2-dimensional random variable.

3.2 Joint Probability Density Function

- As in the one-dimensional random variable case, we would like to have a number associated to the probability or probability density of a 2-dimensional random variable to take on a certain value.

3.2.1 Joint Probability Function for Discrete RVs

Definition 3.4

- Let (X, Y) be a 2-dimensional **discrete** random variable defined on the sample space of an experiment. With each possible value (x_i, y_j), we associate a number $f_{X,Y}(x_i, y_j)$ representing $\Pr(X = x_i, Y = y_j)$ and satisfying the following conditions:

1. $f_{X,Y}(x_i, y_j) \geq 0$ for all $(x_i, y_j) \in R_{X,Y}$.
2. $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{X,Y}(x_i, y_j) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Pr(X = x_i, Y = y_j) = 1$ (3.1)

Joint Probability Function (Continued)

- The function $f_{X,Y}(x,y)$ defined for all pairs of values $(x_i, y_j) \in R_{X,Y}$ is called the joint probability function of (X, Y).
- Let A be any set consisting of pairs of (x, y) values. Then the probability $\Pr((X,Y) \in A)$ is defined by summing the joint probability function over pairs in A :

$$\Pr\big((X,Y) \in A\big) = \underbrace{\sum\sum}_{(x,y)\in A} f_{X,Y}(x,y)$$

Example 1

- Find the value of k so that the function given by $f_{X,Y}(x,y) = kxy$ for $x = 1,2,3,$ and $y = 1,2,3,$ can serve as a **joint probability function.**

Solution

$$R_{X,Y} = \{(x,y) \mid x = 1,2,3, \text{and } y = 1,2,3\}.$$

$$f(1,1) = k, f(1,2) = 2k, f(1,3) = 3k,$$
$$f(2,1) = 2k, f(2,2) = 4k, f(2,3) = 6k,$$
$$f(3,1) = 3k, f(3,2) = 6k, f(3,3) = 9k.$$

Example 1 (Continued)

By (3.1) on p3-10, we obtain

$$\sum_{x=1}^{3}\sum_{y=1}^{3} f_{X,Y}(x,y) = 1$$

$$\Leftrightarrow \quad 1k + 2k + 3k + 2k + 4k + 6k + 3k + 6k + 9k = 1$$

$$\Leftrightarrow \quad k = \frac{1}{36}.$$

Example 2

- A company has 2 production lines, A and B, which produces at most 5 and 3 machines, respectively.
- Assume that the number of machines produced is a random variable.
- Let (X, Y) represent the 2-dimensional random variable yielding the numbers of machines produced by Line A and Line B respectively on a given day.
- The joint probability function, $f_{X,Y}(x, y)$, of (X, Y) is given on next slide.
- What is the probability that more chips are produced by Line A than by Line B on a given day?

Example 2 (Continued)

The table below gives the joint probability function for (X, Y).

y	x						Row Total
	0	1	2	3	4	5	
0	0	0.01	0.02	0.05	0.06	0.08	0.22
1	0.01	0.03	0.04	0.05	0.05	0.07	0.25
2	0.02	0.03	0.05	0.06	0.06	0.07	0.29
3	0.02	0.04	0.03	0.04	0.06	0.05	0.24
Column Total	0.05	0.11	0.14	0.20	0.23	0.27	1

- Each entry represents $f_{X,Y}(x_i, y_j) = \Pr(X = x_i, Y = y_j)$.
- For example, $f_{X,Y}(2, 3) = \Pr(X = 2, Y = 3) = 0.03$.

Solution to Example 2

Let $B = \{X > Y\}$.

$$\begin{aligned}
\Pr(B) &= \Pr(X > Y) \\
&= \Pr[(X,Y) = (1,0) \text{ or } (X,Y) = (2,0) \text{ or } \\
&\qquad (X,Y) = (2,1) \text{ or } \cdots \text{ or } (X,Y) = (5,3)] \\
&= \Pr[(X,Y) = (1,0)] + \Pr[(X,Y) = (2,0)] \\
&\qquad + \cdots + \Pr[(X,Y) = (5,3)] \\
&= f_{X,Y}(1,0) + f_{X,Y}(2,0) + f_{X,Y}(2,1) + \cdots + f_{X,Y}(5,3) \\
&= 0.01 + 0.02 + 0.04 + 0.05 + \cdots + 0.06 + 0.05 \\
&= 0.73.
\end{aligned}$$

Example 3

- In a group of 9 executives of a certain company, 4 are married, 3 have never married and 2 are divorced.
- Three of the executives are to be randomly selected for promotion.
- Let X denote the number of married executives and Y the number of never married executives among the three selected for promotion.
- Find the joint probability function of X and Y.

Solution to Example 3

- The number of ways to select 3 executives out of 9 executives for promotion is

$$\binom{9}{3}$$

- For $x, y = 0, 1, 2, 3$ such that $1 \leq x + y \leq 3$,

  the number of ways to select x executives from 4 married executives, y executives from 3 never married executives and the rest from 2 divorced executives is

$$\binom{4}{x}\binom{3}{y}\binom{2}{3-x-y}$$

Solution to Example 3 (Continued)

- Therefore

$$f_{X,Y}(x,y) = \Pr(X = x, Y = y)$$
$$= \frac{\binom{4}{x}\binom{3}{y}\binom{2}{3-x-y}}{\binom{9}{3}}$$

for $x, y = 0, 1, 2, 3$ such that $1 \le x + y \le 3$

and $f_{X,Y}(x,y) = 0$ otherwise.

Solution to Example 3 (Continued)

The above p.f. are given explicitly in the following table.

x	*y*				Row Total
	0	**1**	**2**	**3**	
0	0	3/84	6/84	1/84	10/84
1	4/84	24/84	12/84	0	40/84
2	12/84	18/84	0	0	30/84
3	4/84	0	0	0	4/84
Column Total	20/84	45/84	18/84	1/84	1

3.2.2 Joint pdf for Continuous RVs

- Let (X, Y) be a 2-dimensional **continuous** random variable assuming all values in some region R of the Euclidean plane, $\mathbb{R}^{\mathbf{2}}$.

- $f_{X,Y}(x, y)$ is called a **joint probability density function** if it satisfies the following conditions:

Joint pdf for Continuous RVs (Continued)

1. $f_{X,Y}(x,y) \geq 0$ for all $(x,y) \in R_{X,Y}$.
2.

$$\iint_{(x,y)\in R_{X,Y}} f_{X,Y}(x,y)dx\,dy = 1$$

or

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\,dy = 1.$$

Example 1

Suppose that the two-dimensional continuous random variable (X, Y) have the joint p.d.f. given by

$$f_{X,Y}(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & \text{for } 0 \le x \le 1, 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\Pr(X + Y \ge 1)$.

Solution to Example 1

- First check if $f_{X,Y}(x,y)$ is a joint p.d.f.
- It is obvious that $f_{X,Y}(x,y) \geq 0$ for all (x,y).
- Check that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\,dy = 1.$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\,dy = \int_{0}^{2}\int_{0}^{1}\left(x^2 + \frac{xy}{3}\right)dx\,dy$$

$$= \int_{0}^{2}\left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_{x=0}^{1} dy = \int_{0}^{2}\left(\frac{1}{3} + \frac{y}{6}\right)dy = \left[\frac{y}{3} + \frac{y^2}{12}\right]_{y=0}^{2}$$

$$= 2/3 + 4/12 = 1.$$

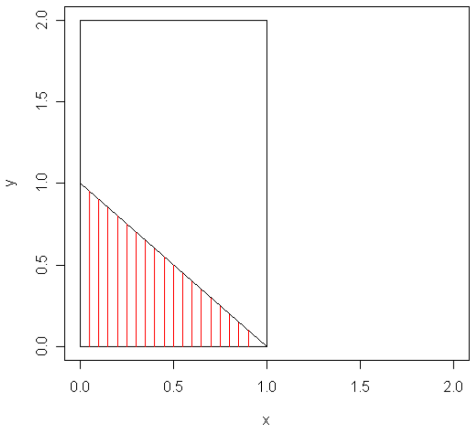
Solution to Example 1 (Continued)

- Let $A = \{X + Y \geq 1\}$. Then $A' = \{X + Y < 1\}$.

$$\Pr(A) = 1 - \Pr(A')$$

$$= 1 - \iint_{x+y<1} f_{X,Y}(x,y)dx\,dy$$

$$= 1 - \int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3}\right) dy\ dx$$

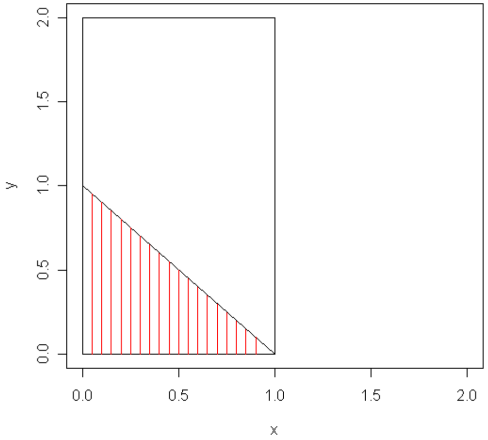


Solution to Example 1 (Continued)

$$\begin{aligned}\Pr(A) &= 1 - \Pr(A') \\ &= \cdots \\ &= 1 - \int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3}\right) dy \; dx\end{aligned}$$

- The integration limits of y are based on the facts that $0 \le y \le 2$ and

  $0 < y < 1 - x$ for a fixed x with $0 \le x \le 1$.



Solution to Example 1 (Continued)

- Hence

$$\Pr(A) = 1 - \int_0^1 \left[x^2 y + \frac{xy^2}{6}\right]_{y=0}^{1-x} dx$$

$$= 1 - \int_0^1 x^2(1-x) + \frac{1}{6}x(1-x)^2\, dx$$

$$= 1 - \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{6}\left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right)\right]_{x=0}^{1}$$

$$= 1 - \frac{7}{72} = \frac{65}{72}.$$

Example 2

- If the joint p.d.f. of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{12}{13}x(x+y), & \text{for } 0 \le x \le 1, 1 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Define $A = \{(x,y) : 0 < x < 1/2, 1 < y < 2\}$.
- Find $\Pr((X,Y) \in A)$.

Solution to Example 2

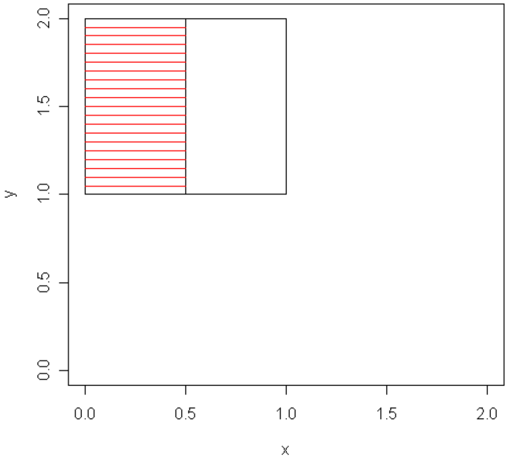
$$\Pr\big((X,Y) \in A\big) = \Pr(0 < X < 1/2, 1 < Y < 2)$$

$$= \int_1^2 \int_0^{1/2} \frac{12}{13}\, x\,(x+y)dx\ dy$$

$$= \frac{12}{13}\int_1^2 \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_{x=0}^{1/2} dy$$

$$= \frac{12}{13}\int_1^2 \frac{1}{24} + \frac{y}{8}\ dy$$

$$= \frac{1}{26}\left[y + \frac{3y^2}{2}\right]_{y=1}^{2} = \frac{11}{52}.$$



3.3 Marginal and Conditional Probability Distributions

3.3.1 Marginal probability distributions

Definition 3.6

- Let (X, Y) be a 2-dimensional discrete (or continuous) random variable with joint probability function (or joint probability density function) $f_{X,Y}(x, y)$.
- The **marginal probability distributions** of X and Y are respectively given by:

Marginal Distributions (Continued)

- For **discrete** case,

$$f_X(x) = \sum_y f_{X,Y}(x,y) \quad \text{and} \quad f_Y(y) = \sum_x f_{X,Y}(x,y)$$

- For **continuous** case,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Example 1

- Refer to the example 2 in Section 3.2.1 on p3-15.
- Find the marginal probability distributions of X and Y.

<u>Solution</u>

To find the marginal distribution of X, $f_X(x)$

$$f_X(0) = \sum_{y=0}^{3} f_{X,Y}(0, y) = 0 + 0.01 + 0.02 + 0.02 = 0.05.$$

$$f_X(1) = \sum_{y=0}^{3} f_{X,Y}(1, y) = 0.01 + 0.03 + 0.03 + 0.04 = 0.11.$$

Example 1 (Continued)

- Similarly we obtain the values of the marginal distribution of $f_X(x)$ for $x = 2, 3, 4$ and 5.

x	0	1	2	3	4	5
$f_X(x) = \Pr(X = x)$	0.05	0.11	0.14	0.20	0.23	0.27

Note:

1. $f_X(x) \geq 0$ for $x = 0, 1, 2, 3, 4, 5$
2. $\sum_{x=0}^{5} f_X(x) = 1$

Example 1 (Continued)

To find the marginal distribution of Y, $f_Y(y)$

$$f_Y(0) = \sum_{x=0}^{5} f_{X,Y}(x, 0) = 0 + 0.01 + 0.02 + 0.05 + 0.06 + 0.08$$
$$= 0.22.$$

$$f_Y(1) = \sum_{x=0}^{5} f_{X,Y}(x, 1) = 0.01 + 0.03 + 0.04 + 0.05 + 0.07$$
$$= 0.25.$$

Example 1 (Continued)

- Similarly we obtain the other values of the marginal distribution of $f_Y(y)$ for $y = 2$ and 3.

y	0	1	2	3
$f_Y(y) = \Pr(Y = y)$	0.22	0.25	0.29	0.24

Example 1 (Continued)

- $f_{X,Y}(x,y), f_X(x)$ and $f_Y(y)$ are displayed in the following table

y	x						$f_Y(y)$
	0	1	2	3	4	5	
0	0	0.01	0.02	0.05	0.06	0.08	0.22
1	0.01	0.03	0.04	0.05	0.05	0.07	0.25
2	0.02	0.03	0.05	0.06	0.06	0.07	0.29
3	0.02	0.04	0.03	0.04	0.06	0.05	0.24
$f_X(x)$	0.05	0.11	0.14	0.20	0.23	0.27	1

Example 2

- Refer to example 2 in Section 3.2.2 on p3-28.
- The joint p.d.f. of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{12}{13}x(x+y), & \text{for } 0 \le x \le 1, 1 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the marginal distributions of X and Y.

Solution to Example 2

- For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_1^2 \frac{12}{13} x(x+y)dy$$

$$= \frac{12}{13}\left[x^2 y + \frac{1}{2} x y^2\right]_{y=1}^{2}$$

$$= \frac{6}{13} x(2x+3).$$

- For $x < 0$ or $x > 1$, $f_X(x) = 0$

since $f_{X,Y}(x,y) = 0$ and $\int_{-\infty}^{\infty} 0 \, dy = 0$.

Solution to Example 2 (Continued)

- For $1 \le y \le 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^1 \frac{12}{13}x(x+y)dx$$

$$= \frac{12}{13}\left[\frac{x^3}{3} + \frac{1}{2}x^2y\right]_{x=0}^{1}$$

$$= \frac{2}{13}(2+3y).$$

- For $y < 1$ or $y > 2$, $f_Y(y) = 0$

since $f_{X,Y}(x,y) = 0$ and $\int_{-\infty}^{\infty} 0\, dx = 0$.

3.3.2 Conditional Distribution

Definition 3.7

- Let (X, Y) be a discrete (or continuous) 2-dimensional random variable with joint probability function (or p.d.f.) $f_{X,Y}(x, y)$.
- Let $f_X(x)$ and $f_Y(y)$ be the marginal probability functions of X and Y respectively.

Conditional Distribution (Continued)

Definition 3.7 (Continued)

- Then **the conditional distribution of Y given that $X = x$** is given by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \qquad \text{if } f_X(x) > 0,$$

for each x within the range of X.

Conditional Distribution (Continued)

Definition 3.7 (Continued)

- Similarly, the **conditional probability distribution of X given $Y = y$** is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad \text{if } f_Y(y) > 0,$$

for each y within the range of Y.

Remarks

1. The conditional p.f.'s (p.d.f.'s) satisfy all the requirements for a 1-dimensional p.f. (p.d.f.). Thus, we have

   (a) For a fixed y,

   $$f_{X|Y}(x|y) \geq 0$$

   and for a fixed x,

   $$f_{Y|X}(y|x) \geq 0.$$

Remarks (Continued)

1. (b)

For discrete r.v.'s,

$$\sum_x f_{X|Y}(x|y) = 1 \quad \text{and} \quad \sum_y f_{Y|X}(y|x) = 1.$$

For continuous r.v.'s

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = 1.$$

Remarks (Continued)

2. For $f_X(x) > 0$,
$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x).$$

For $f_Y(y) > 0$,
$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y).$$

Example 1

- Suppose an experiment consists of 3 tosses of a fair coin with each outcome being equally likely.

- Let X be the number of head on the last flip and Y the total number of heads for the 3 tosses.

- Find the conditional distribution of Y given $X = 1$.

Example 1 (Continued)

Outcome	HHH	THH	HTH	HHT	TTH	THT	HTT	TTT
(x, y)	(1,3)	(1,2)	(1,2)	(0,2)	(1,1)	(0,1)	(0,1)	(0,0)
$f_{X,Y}(x, y)$	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

- The joint probability distribution of (X, Y) is given in the following table:

x	y				$f_X(x)$
	0	1	2	3	
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
$f_Y(y)$	1/8	3/8	3/8	1/8	1

Example 1 (Continued)

Note: Summing across the rows gives $f_X(x)$ and summing across the columns gives $f_Y(y)$.

$$f_{Y|X}(0|1) = \frac{f_{X,Y}(1,0)}{f_X(1)} = \frac{0}{1/2} = 0.$$

$$f_{Y|X}(1|1) = \frac{f_{X,Y}(1,1)}{f_X(1)} = \frac{1/8}{1/2} = \frac{1}{4}.$$

$$f_{Y|X}(2|1) = \frac{f_{X,Y}(1,2)}{f_X(1)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

$$f_{Y|X}(3|1) = \frac{f_{X,Y}(1,3)}{f_X(1)} = \frac{1/8}{1/2} = \frac{1}{4}.$$

Example 1 (Continued)

Therefore the conditional distribution of Y given $X = 1$ is

y	0	1	2	3
$f_{Y\|X}(y\|1)$	0	1/4	1/2	1/4

Note:

$$\sum_{y=0}^{3} f_{Y|X}(y|1) = 1.$$

Example 2

- Refer to Example 1 in Section 3.2.1 on p3-12.

$$f_{X,Y}(x, y) = \frac{1}{36}xy, \qquad \text{for } x = 1, 2, 3, \text{ and } y = 1, 2, 3.$$

- Find $f_X(x)$ and $f_{Y|X}(y|x)$.

Example 2 (Continued)

<u>Solution</u>

$$f_X(x) = \sum_{y=1}^{3} \frac{1}{36} xy = \frac{x}{36}\left(\sum_{y=1}^{3} y\right)$$

$$= \frac{x}{36}(1+2+3) = \frac{x}{6}, \qquad \text{for } x = 1, 2, 3.$$

and $f_X(x) = 0$ for other values of X.

Example 2 (Continued)

For $x = 1, 2 \text{ or } 3$,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{xy/36}{x/6} = \frac{y}{6}, \qquad \text{for } y = 1, 2, 3,$$

and 0 otherwise.

Example 3

Suppose (X, Y) has the joint p.d.f

$$f_{X,Y}(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & \text{for } 0 \le x \le 1, 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $f_X(x)$ and $f_Y(y)$.

(b) Find $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$.

Solution to Example 3

(a) For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= \left[x^2 y + \frac{xy^2}{6}\right]_{y=0}^{2} = 2x^2 + \frac{2}{3}x.$$

For $x < 0$ and $x > 1$, $f_X(x) = 0$,

since $f_{X,Y}(x,y) = 0$ and $\int_{-\infty}^{\infty} 0 \, dy = 0$.

Solution to Example 3 (Continued)

(a) (Continued)

Hence

$$f_X(x) = \begin{cases} 2x^2 + \dfrac{2}{3}x, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 3 (Continued)

(a) (Continued)

For $0 \le y \le 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_{x=0}^{1} = \frac{1}{3} + \frac{1}{6}y.$$

For $y < 0$ and $y > 2$, $f_Y(y) = 0$,

since $f_{X,Y}(x,y) = 0$ and $\int_{-\infty}^{\infty} 0 \, dx = 0$.

Solution to Example 3 (Continued)

(a) (Continued)

Hence

$$f_Y(y) = \begin{cases} \dfrac{1}{3} + \dfrac{1}{6}y, & \text{for } 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 3 (Continued)

(b) For $0 \leq x \leq 1$,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \begin{cases} \dfrac{x^2 + xy/3}{2x^2 + 2x/3}, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{3x + y}{2(3x+1)}, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 3 (Continued)

For example

- When $x = 1$, then

  $f_{Y|X}(y|1) = (3 + y)/8$, for $0 \leq y \leq 2$ and

  $f_{Y|X}(y|1) = 0$, otherwise.

- When $x = 0.5$, then

  $f_{Y|X}(y|0.5) = (3 + 2y)/10$, for $0 \leq y \leq 2$ and

  $f_{Y|X}(y|0.5) = 0$, otherwise.

Solution to Example 3 (Continued)

(b) For $0 \le y \le 2$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{x^2 + xy/3}{(2+y)/6}, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{2x(3x+y)}{2+y}, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 3 (Continued)

For example

- When $y = 1$, then

  $f_{X|Y}(x|1) = [2x(3x+1)]/3$, for $0 \le x \le 1$ and

  $f_{X|Y}(x|1) = 0$, otherwise.

- When $y = 0.5$, then

  $f_{X|Y}(x|0.5) = [2x(6x+1)]/5$, for $0 \le x \le 1$ and

  $f_{X|Y}(x|0.5) = 0$, otherwise.

Example 4

- A fast food restaurant operates a drive-up facility and a walk-up window.
- On a randomly selected day, let X = the proportion of time that the drive-up facility is in use (at least one customer is being served or waiting to be served) and Y = the proportion of the time that the walk-up window is in use.
- Suppose that the joint p.d.f. of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & \text{for } 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 4 (Continued)

(a) Find the probability that neither facility is busy more than one-quarter of the time.

i.e. Find $\Pr(0 < X < 1/4, 0 < Y < 1/4)$.

(b) Find the **probability distribution** of busy time for the drive-up facility without reference to the walk-up window. i.e. Find $f_X(x)$.

Hence find the **probability** that the drive-up facility is busy more than one-quarter of the time but less than three quarters of the time.

i.e. Find $\Pr(1/4 < X < 3/4)$.

Example 4 (Continued)

(c) Given that the drive-up facility is busy 80% of the time, what is the probability that the walk-in facility is busy at most half the time?

i.e. Find $\Pr(Y \leq 1/2 \mid X = 4/5)$.

(d) Given that the drive-up facility is busy 80% of the time, what is the expected proportion of time that the walk-in facility is busy?

i.e. Find $E(Y \mid X = 4/5)$.

Solution to Example 4

(a) $\Pr\left(0 \le X \le \frac{1}{4}, 0 \le Y \le \frac{1}{4}\right)$

$$= \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x + y^2)\, dx\ dy$$

$$= \frac{6}{5}\left[\int_0^{1/4}\int_0^{1/4} x\, dx\ dy + \int_0^{1/4}\int_0^{1/4} y^2\, dx\ dy\right]$$

$$= \frac{6}{5}\left[\int_0^{1/4}\left[\frac{x^2}{2}\right]_{x=0}^{1/4} dy + \int_0^{1/4}[xy^2]_{x=0}^{1/4} dy\right]$$

Solution to Example 4

(a) (Continued)

$$= \frac{6}{5}\left[\int_0^{1/4} \frac{1}{2(4)^2}\, dy + \int_0^{1/4} \frac{y^2}{4}\, dy\right]$$

$$= \frac{6}{5}\left\{\left[\frac{y}{32}\right]_{y=0}^{1/4} + \left[\frac{y^3}{12}\right]_{y=0}^{1/4}\right\}$$

$$= \frac{7}{640} = 0.0109.$$

Solution to Example 4 (Continued)

(b)

$$f_X(x) = \int_0^1 \frac{6}{5}(x + y^2)dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_{y=0}^{1}$$

$$= \frac{6}{5}x + \frac{2}{5}.$$

for $0 \leq x \leq 1$

and 0 otherwise.

Solution to Example 4 (Continued)

(b) (Continued)

$$\Pr\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right) = \int_{1/4}^{3/4} \left(\frac{6}{5}x + \frac{2}{5}\right) dx$$

$$= \left[\frac{3}{5}x^2 + \frac{2}{5}x\right]_{x=1/4}^{3/4} = \frac{1}{2}.$$

Solution to Example 4 (Continued)

(c)

$$f_{Y|X}\left(y \middle| \frac{4}{5}\right) = \frac{f_{X,Y}(4/5, y)}{f_X(4/5)}$$

$$= \frac{6[(4/5) + y^2]/5}{(6/5)(4/5) + (2/5)}$$

$$= \frac{3(4 + 5y^2)}{17}.$$

for $0 < y < 1$ and 0 otherwise.

Solution to Example 4 (Continued)

(c) (Continued)

Hence

$$\Pr\left(Y \le \frac{1}{2} \middle| X = \frac{4}{5}\right) = \int_{-\infty}^{1/2} f_{Y|X}\left(y \middle| \frac{4}{5}\right) dy$$

$$= \int_{0}^{1/2} \frac{3}{17}(4 + 5y^2)dy$$

$$= \frac{3}{17}\left[4y + \frac{5}{3}y^3\right]_{y=0}^{1/2} = \frac{53}{136} = 0.3897.$$

Solution to Example 4 (Continued)

(d)

$$E\left(Y \middle| X = \frac{4}{5}\right) = \int_{-\infty}^{\infty} y\, f_{Y|X}\left(y \middle| \frac{4}{5}\right) dy$$

$$= \int_0^1 \frac{6}{34} y(4 + 5y^2) dy$$

$$= \frac{6}{34}\left[2y^2 + \frac{5}{4}y^4\right]_{y=0}^{1} = \frac{39}{68} = 0.5735.$$

Example 5

Let X and Y be **uniformly distributed** over the triangle with the boundaries: $0 \leq x \leq y, 0 \leq y \leq 2$.

(a) Find the joint p.d.f. of (X, Y),

(b) Find $f_X(x)$ and $f_Y(y)$.

(c) Find $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$.

(d) Find $\Pr(X \leq 1/2 \mid Y = 1)$

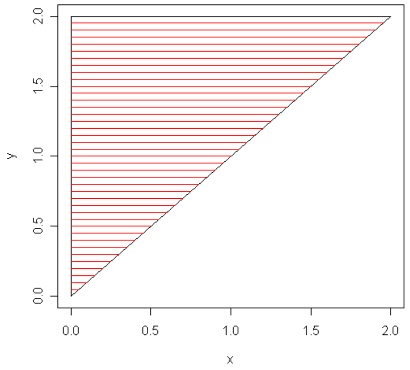
(e) Find $\Pr(X \leq 1, Y \leq 1)$.

Solution to Example 5

- Since $f_{X,Y}(x, y)$ is uniform over the triangle bounded by $0 \le x \le y, 0 \le y \le 2$, therefore

$$f_{X,Y}(x, y) = k \quad \text{for } 0 \le x \le y, 0 \le y \le 2.$$

- Note: The area bounded by $0 \le x \le y, 0 \le y \le 2$ is $(1/2)(2)(2) = 2$.



Solution to Example 5 (Continued)

(a) (Continued)

$$\int_0^2 \int_0^y k\,dx\;\,dy = \int_0^2 [kx]_{x=0}^{y}\,dy$$

$$= \int_0^2 ky\,dy = \left[\frac{ky^2}{2}\right]_0^2 = 2k.$$

Hence

$$\int_0^2 \int_0^y k\,dx\,dy = 1 \Leftrightarrow 2k = 1 \Leftrightarrow k = \frac{1}{2}.$$

Solution to Example 5 (Continued)

(a) (Continued)

Therefore,

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{2}, & \text{for } 0 \le x \le y, 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 5 (Continued)

(b)

$$f_X(x) = \begin{cases} \displaystyle\int_x^2 \frac{1}{2}\, dy, & \text{for } 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left[\dfrac{y}{2}\right]_{y=x}^{2}, & \text{for } 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{1}{2}(2-x), & \text{for } 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 5 (Continued)

(b) (Continued)

$$f_Y(y) = \begin{cases} \int_0^y \frac{1}{2}\, dx, & \text{for } 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{y}{2}, & \text{for } 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 5 (Continued)

(c) For $0 \le x \le 2$,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{1/2}{(2-x)/2}, & \text{for } x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{1}{2-x}, & \text{for } x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 5 (Continued)

(c) (Continued)

For $0 \le x \le 2$,

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{1}{2-x}, & \text{for } x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

For example, when $x = 1.5$,

$f_{Y|X}(y|1.5) = 2, \;$ for $1.5 \le y \le 2$ and

$f_{Y|X}(y|1.5) = 0, \;$ otherwise.

Solution to Example 5 (Continued)

(c) (Continued)

For $0 \le y \le 2$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1/2}{y/2}, & \text{for } 0 \le x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1/y, & \text{for } 0 \le x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 5 (Continued)

(c) (Continued)

For $0 \le y \le 2$,

$$f_{X|Y}(x|y) = \begin{cases} 1/y, & \text{for } 0 \le x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

For example, when $y = 0.5$,

$f_{X|Y}(x|0.5) = 2 \quad \text{for } 0 \le x \le 0.5$ and

$f_{X|Y}(x|0.5) = 0,\ \text{otherwise.}$

Solution to Example 5 (Continued)

(d) From (c), we have $f_{X|Y}(x|1) = 1$ for $0 \le x \le 1$ and 0 otherwise.

Therefore

$$\Pr\left(X \le \frac{1}{2} \middle| Y = 1\right) = \int_{-\infty}^{1/2} f_{X|Y}(x|1)dx$$

$$= \int_{0}^{1/2} 1\,dx = \frac{1}{2}.$$

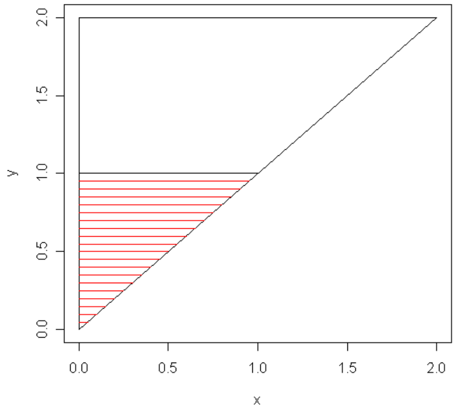
Solution to Example 5 (Continued)

(e)

$$\Pr(X \le 1, Y \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1} f_{X,Y}(x,y)dx\ dy$$

$$= \int_0^1\int_0^y \frac{1}{2}\, dx\ dy = \int_0^1 \frac{y}{2}\, dy$$

$$= \left[\frac{1}{2}\left(\frac{y^2}{2}\right)\right]_0^1 = \frac{1}{4}.$$



3.4 Independent Random Variables

3.4.1 Definition of Independent RVs

Definition

- Random variables X and Y are **independent** if and only if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y), \quad \textbf{for all } \boldsymbol{x}, \boldsymbol{y}.$$

Extension:

- Random variables $X_1, X_2, \cdots, X_n$ are independent if and only if

$$f_{X_1,X_2,\cdots,X_n}(x_1, x_2, \cdots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

for all $\boldsymbol{x_i}$, $\boldsymbol{i = 1, \cdots, n}$.

Remark

- The product of 2 positive functions $f_X(x)$ and $f_Y(y)$ means a function which is positive on a **product space**.

- That is, if

$$f_X(x) > 0, \quad \text{for } x \in A_1 \quad \text{and}$$
$$f_Y(y) > 0, \quad \text{for } y \in A_2$$

then $$f_X(x)f_Y(y) > 0, \text{ for } (x, y) \in A_1 \times A_2.$$

Example 1

1. The joint p.d.f. $f_{X,Y}(x, y)$ is given as follows.

x	y			$f_X(x)$
	1	3	5	
2	0.1	0.2	0.1	0.4
4	0.15	0.3	0.15	0.6
$f_Y(y)$	0.25	0.5	0.25	1

Are X and Y independent?

Solution to Example 1

$$f_X(2)f_Y(1) = 0.4(0.25) = 0.1 = f_{X,Y}(2,1).$$

Similarly, we have

$$f_X(2)f_Y(3) = 0.4(0.5) = 0.2 = f_{X,Y}(2,3).$$

$$f_X(2)f_Y(5) = 0.4(0.25) = 0.1 = f_{X,Y}(2,5).$$

$$f_X(4)f_Y(1) = 0.6(0.25) = 0.15 = f_{X,Y}(4,1).$$

$$f_X(4)f_Y(3) = 0.6(0.5) = 0.3 = f_{X,Y}(4,3).$$

$$f_X(4)f_Y(5) = 0.6(0.25) = 0.15 = f_{X,Y}(4,5).$$

Since $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for **all** (x,y), hence X and Y are independent.

Example 2

- Refer to example 1 in Section 3.2.1 on p3-12.

$$f_{X,Y}(x,y) = \frac{xy}{36}$$

for $x = 1, 2, 3,$ and $y = 1, 2, 3.$

- Are X and Y independent?

Example 2 (Continued)

Solution

$$f_X(x) = \sum_{y=1}^{3} \frac{1}{36} xy = \frac{x}{36} \sum_{y=1}^{3} y$$

$$= \frac{x}{36}(1 + 2 + 3) = \frac{1}{6}x \quad \text{for } x = 1, 2, 3,$$

and 0 otherwise.

Example 2 (Continued)

- Similarly

$$f_Y(y) = \sum_{x=1}^{3} \frac{1}{36} xy = \frac{y}{36} \sum_{x=1}^{3} x$$

$$= \frac{y}{36}(1 + 2 + 3) = \frac{1}{6} y \quad \text{for } y = 1, 2, 3,$$

and 0 otherwise.

Example 2 (Continued)

- Hence

$$f_{X,Y}(x,y) = \frac{1}{36}\,xy = f_X(x)f_Y(y) = \left(\frac{x}{6}\right)\left(\frac{y}{6}\right)$$

for **all** $\boldsymbol{x}, \boldsymbol{y} = \mathbf{1}, \mathbf{2}, \mathbf{3}$.

- Therefore X and Y are independent.

Example 3

- X and Y are 2 **independent** random variables with

$$f_X(x) = e^{-x}, \qquad \text{for } x \geq 0 \text{ and}$$
$$f_Y(y) = e^{-y}, \qquad \text{for } y \geq 0.$$

- What is $f_{X,Y}(x, y)$?

Example 3 (Continued)

Solution

- Since X and Y are independent, therefore

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} e^{-x}e^{-y}, & \text{for } x \geq 0 \text{ and } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

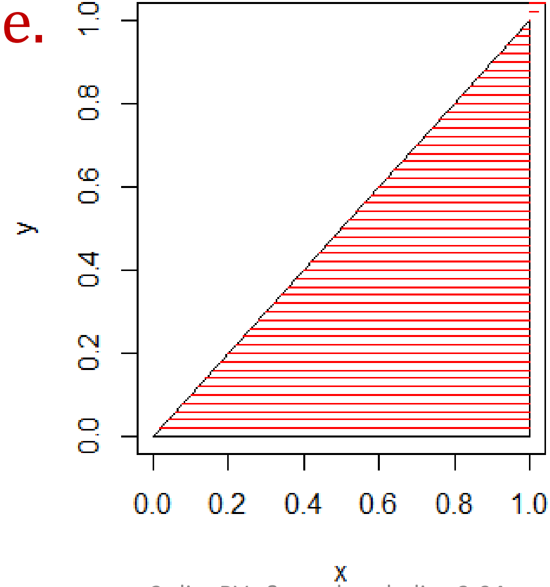
$$= \begin{cases} e^{-(x+y)}, & \text{for } x \geq 0 \text{ and } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 4

- Given that

$$f_{X,Y}(x,y) = \begin{cases} 2\,(x+y), & \text{for } 0 \le x \le 1, 0 < y < x, \\ 0, & \text{otherwise.} \end{cases}$$

- are X and Y independent?



Solution to Example 4

- $f_X(x)$ is given by

$$f_X(x) = \begin{cases} \int_0^x 2(x+y)dy, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 2\left[xy + \frac{y^2}{2}\right]_{y=0}^{x}, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 3x^2, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 4 (Continued)

- $f_Y(y)$ is given by

$$f_Y(y) = \begin{cases} \displaystyle\int_y^1 2(x+y)\,dx, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 2\left[\dfrac{x^2}{2} + yx\right]_{x=y}^{1}, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 + 2y - 3y^2, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 4 (Continued)

- Since
  - $f_{X,Y}(x,y) = 2(x+y)$ for $0 < x < 1$ and $0 < y < x$
  - $f_X(x) = 3x^2$ for $0 < x < 1$
  - $f_Y(y) = 1 + 2y - 3y^2$ for $0 < y < 1$.
- Therefore $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$ for some x and y
- Hence X and Y are not independent.
- Note that the region for which $f_{X,Y}(x,y) > 0$ is not a rectangle and cannot be expressed as the product of 2 intervals.

Solution to Example 4 (Continued)

Alternatively, if the region for which $f_{X,Y}(x,y) > 0$ is not a rectangle, then we look for a point

1. in the product space of the interval for which $f_X(x) > 0$ (i.e. $0 < x < 1$) and
2. the interval for which $f_Y(y) > 0$ (i.e. $0 < y < 1$)
3. But not in the region for which $f_{X,Y}(x,y) > 0$

e.g. Consider $(x,y) = (0.6, 0.8)$. Since $x(= 0.6) < y(= 0.8)$, therefore $(0.6, 0.8)$ lies outside the region for which $f_{X,Y}(x,y) > 0$. On the other hand, $x = 0.6$ lies in the interval $0 < x < 1$ and $y = 0.8$ lies in the interval $0 < y < 1$.

Solution to Example 4 (Continued)

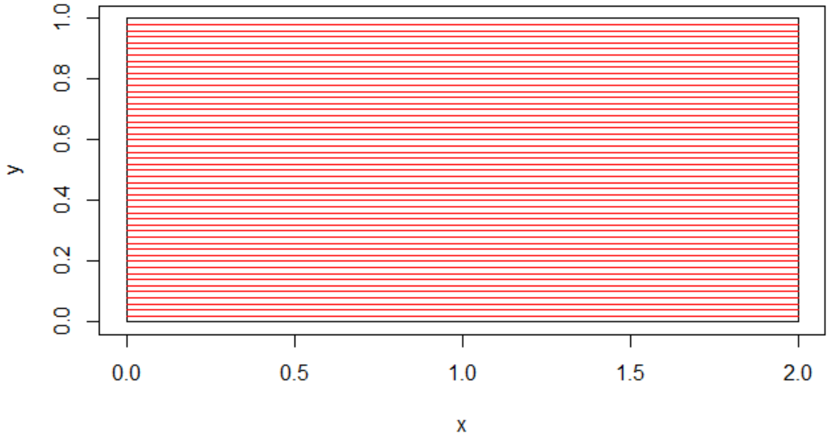
- Consider the point $(x, y) = (0.6, 0.8)$ (note: $y > x$)
  - $f_X(0.6) = 3(0.6)^2 = 1.08 > 0$ (Refer to p3.95)
  - $f_Y(0.8) = 1 + 2(0.8) - 3(0.8)^2 = 0.68 > 0$ (Refer to p3.96)
  - $f_{X,Y}(0.6, 0.8) = 0$ (Refer to p3.94)
- Therefore $f_{X,Y}(x, y) \neq f_X(x) f_Y(y)$ for $(x, y) = (0.6, 0.8)$
- Hence X and Y are not independent

Example 5

- Given that

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}\, x(1+y), & \text{for } 0 < x < 2, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- are X and Y independent?



Solution to Example 5

- $f_X(x)$ is given by

$$f_X(x) = \begin{cases} \displaystyle\int_0^1 \frac{x}{3}(1+y)dy\,, & \text{for } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{x}{3}\left[y + \frac{y^2}{2}\right]_{y=0}^{1}, & \text{for } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{2}x, & \text{for } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 5 (continued)

- $f_Y(y)$ is given by

$$f_Y(y) = \begin{cases} \displaystyle\int_0^2 \frac{x}{3}(1+y)dx\,, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{(1+y)}{3}\left[\frac{x^2}{2}\right]_{x=0}^{2}, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{2}{3}(1+y), & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 5 (Continued)

- Since

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for $0 < x < 2, 0 < y < 1,$

- therefore X and Y are **independent.**

Example 6

- Given that

$$f_{X,Y}(x,y) = \begin{cases} x + \frac{3}{2}y^2, & \text{for } 0 < x < 1, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- are X and Y independent?

Solution to Example 6

- $f_X(x)$ is given by

$$f_X(x) = \begin{cases} \int_0^1 \left(x + \frac{3}{2}y^2\right) dy, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left[xy + \frac{3}{2}\frac{y^3}{3}\right]_{y=0}^1, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} x + \frac{1}{2}, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 6 (Continued)

- $f_Y(y)$ is given by

$$f_Y(y) = \begin{cases} \displaystyle\int_0^1 \left(x + \frac{3}{2}y^2\right) dx, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left[\dfrac{x^2}{2} + \dfrac{3}{2}y^2 x\right]_{x=0}^{1}, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{1}{2}(1 + 3y^2), & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Example 6 (Continued)

- Since

$$f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$$

for $0 < x < 1, 0 < y < 1$,

- therefore X and Y are **not** independent.

3.5 Expectation

Definition 3.5.1

- The expectation of $g(X, Y)$ is defined as

$$E[g(X,Y)] = \begin{cases} \displaystyle\sum_x \sum_y g(x,y) f_{X,Y}(x,y), & \text{for Discrete RV's,} \\ \displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy, & \text{for Cont. RV's} \end{cases}$$

A Special Case

- Let $g(X, Y) = (X - \mu_X)(Y - \mu_Y)$. This leads to the definition of covariance between two random variables.

Definition 3.5.2

- Let (X, Y) be a bivariate random vector with joint p.f. (or p.d.f.) $f_{X,Y}(x, y)$, then the **covariance** of (X, Y) is defined as
$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

A Special Case (Continued)

- For **discrete** case

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f_{X,Y}(x,y)$$

- For **continuous** case

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f_{X,Y}(x,y)\, dx\, dy$$

Remarks

1. $Cov(X, Y) = E(XY) - \mu_X \mu_Y$.

2. If X and Y are independent, then $Cov(X, Y) = 0$. However $Cov(X, Y) = 0$ does not imply X and Y are independent.

3. $Cov(aX + b, cY + d) = ac\ Cov(X, Y)$

4. $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab\ Cov(X, Y)$

Correlation coefficient

Definition 3.5.2

- The **correlation coefficient** of X and Y, denoted by $Cor(X,Y)$, $\rho_{X,Y}$ or ρ, is defined by

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}}$$

Remarks on Correlation coefficient

1. $-1 \leq \rho_{X,Y} \leq 1$.

2. $\rho_{X,Y}$ is a measure of the degree of **linear** relationship between X and Y.

3. If X and Y are independent, then $\rho_{X,Y} = 0$.

   On the other hand, $\rho_{X,Y} = 0$ does **not** imply independence.

Example 1

- Refer to Example 1 in Section 3.3.2 on p3-47. The joint distribution of (X, Y) is given by

x	y				$f_X(x)$
	0	1	2	3	
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
$f_Y(y)$	1/8	3/8	3/8	1/8	1

Example 1 (Continued)

(a) Find $E(Y - X)$.

(b) Find $Cov(X, Y)$.

(c) Find $\rho_{X,Y}$.

(d) Find $E(Y \mid X = 1)$.

Solution to Example 1

(a)

$$E(Y-X) = (0-0)(1/8) + (1-0)(1/4) + (2-0)(1/8) + \cdots + (3-1)(1/8) = 1.$$

Or

$$E(Y-X) = E(Y) - E(X) = 1.5 - 0.5 = 1.$$

(See part (b))

Solution to Example 1 (Continued)

(b)

$$E(XY) = (0)(0)(1/8) + (0)(1)(1/4) + (0)(2)(1/8) + \cdots + (1)(3)(1/8) = 1.$$

$$E(X) = 0(1/2) + 1(1/2) = 0.5.$$

$$E(Y) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 1.5.$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= 1 - (0.5)(1.5) = 0.25.$$

Solution to Example 1 (Continued)

(c) $V(X) = [0^2(1/2) + 1^2(1/2)] - (0.5)^2 = 0.25.$

$V(Y) = [0^2(1/8) + 1^2(3/8) + 2^2(3/8) + 3^2(1/8)] - \ 1.5^2$

$= \ 3 - 2.25 = 0.75.$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{0.25}{\sqrt{(0.25)(0.75)}}$$

$$= \frac{1}{\sqrt{3}} = 0.5774.$$

Solution to Example 1 (Continued)

(d) The conditional distribution of Y given $X = 1$ is

y	1	2	3
$f_{Y\|X}(y \mid 1)$	1/4	1/2	1/4

$$E(Y \mid X = 1) = 1(1/4) + 2(1/2) + 3(1/4) = 2.$$

(Refer to the conditional distribution on p3-49)

Example 2

Refer to Example 3 in Section 3.3.2 on p3-53. The joint p.d.f. of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} x^2 + \dfrac{xy}{3}, & \text{for } 0 \le x \le 1, 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $Cov(X, Y)$.

(b) Find $E(Y \mid X = 1/2)$.

Solution to Example 2

(a)

$$E(XY) = \int_0^2 \int_0^1 xy\left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^2 \left[y\frac{x^4}{4} + \frac{y^2x^3}{9}\right]_{x=0}^1 dy = \int_0^2 \left(\frac{y}{4} + \frac{y^2}{9}\right) dy$$

$$= \left[\frac{y^2}{8} + \frac{y^3}{27}\right]_{y=0}^2 = \frac{43}{54}.$$

Solution to Example 2 (Continued)

(a)

$$E(X) = \int_0^1 x\left(2x^2 + \frac{2}{3}x\right)\,dx \quad \text{(Refer to p3−55)}$$

$$= \left[\frac{2x^4}{4} + \frac{2x^3}{9}\right]_{x=0}^{1} = \frac{13}{18}.$$

$$E(Y) = \int_0^2 y\left(\frac{1}{3} + \frac{y}{6}\right)\,dy \quad \text{(Refer to p3−57)}$$

$$= \left[\frac{y^2}{6} + \frac{y^3}{18}\right]_{y=0}^{2} = \frac{10}{9}.$$

Solution to Example 2 (Continued)

Hence

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{43}{54} - \left(\frac{13}{18}\right)\left(\frac{10}{9}\right)$$

$$= -\frac{1}{162}.$$

Solution to Example 2 (Continued)

(b) From p3-58, the conditional distribution of Y given $X = 1/2$ is given by

$$f_{(Y|X)}\left(y\middle|\frac{1}{2}\right) = \begin{cases} \dfrac{3+2y}{10}, & \text{for } 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$E\left(Y\middle|X = \frac{1}{2}\right) = \int_0^2 y\left(\frac{3+2y}{10}\right) dy = \frac{1}{10}\left[3\frac{y^2}{2} + 2\frac{y^3}{3}\right]_{y=0}^{2} = \frac{17}{15}.$$