ST2334 (2020/2021 Semester 2) Solutions to Questions in Tutorial 7

Question 1

$f_{(X,Y)}(x,y)$		$x=2$	$x=4$	$f_Y(y)$
y	1	0.10	0.15	**0.25**
	3	0.20	0.30	**0.50**
	5	0.10	0.15	**0.25**
$f_X(x)$		**0.40**	**0.60**	**1**

(a) It can be verified directly that
$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \text{ for } x = 2,4 \text{ and } y = 1,3,5.$$
Hence, X and Y are independent

(b)

y	1	3	5
$f_{Y\|X}(y\|2)$	0.10/0.40 = 1/4	0.20/0.40 = 2/4	0.10/0.40 = 1/4

$E(Y|X = 2) = 1(1/4) + 3(2/4) + 5(1/4) = 3.$
Alternatively, as X and Y are independent, then $E(Y|X = 2) = E(Y) = 1(0.25) + 3(0.5) + 5(0.25) = 3.$

(c)

x	2	4
$f_{X\|Y}(x\|3)$	0.20/0.50 = 2/5	0.30/0.50 = 3/5

$E(X|Y = 3) = 2(2/5) + 4(3/5) = 16/5 = 3.2$
Alternatively, as X and Y are independent, then $E(X|Y = 3) = E(X) = 2(0.4) + 4(0.6) = 3.2.$

(d) $E(X) = 2(0.40) + 4(0.60) = 3.2$
$E(Y) = 1(0.25) + 3(0.50) + 5(0.25) = 3$
$E(2X - 3Y) = 2E(X) - 3E(Y) = 2(3.2) - 3(3) = -2.6.$ We used the results in parts (b) and (c) in the computation.

(e) $E(XY) = (2)(1)(0.10) + (2)(3)(0.10) + (2)(5)(0.10) + (4)(1)(0.10) + +(4)(3)(0.10) + (4)(5)(0.10) = 9.6.$
Alternatively, as X and Y are independent, then $E(XY) = E(X)E(Y) = 3.2(3) = 9.6.$

(f) $E(X^2) = 2^2(0.4) + 4^2(0.6) = 11.2.$ Hence, $V(X) = E(X^2) - [E(X)]^2 = 11.2 - (3.2)^2 = 0.96$
Similarly, $E(Y^2) = 1^2(0.25) + 3^2(0.5) + 5^2(0.25) = 11.$ Hence, $V(Y) = E(Y^2) - [E(Y)]^2 = 11 - (3)^2 = 2$

(g) $\sigma_{X,Y} = 0$ as X and Y are independent.
Alternatively, $Cov(X,Y) = E(XY) - E(X)E(Y) = 9.6 - 3.2(3) = 0$
$\rho_{X,Y} = 0$ as $\sigma_{X,Y} = 0$. Alternatively, $\rho_{X,Y} = 0$ as X and Y are independent.

Question 2

By the definition of X and Y, Profit $= 8X + 3Y - 10$; and the marginal distributions of X and Y are given in the following table.

$f_{(X,Y)}(x,y)$		x = 0	1	2	$\boldsymbol{f_Y(y)}$
y	0	0.01	0.01	0.03	**0.05**
	1	0.03	0.08	0.07	**0.18**
	2	0.03	0.06	0.06	**0.15**
	3	0.07	0.07	0.13	**0.27**
	4	0.12	0.04	0.03	**0.19**
	5	0.08	0.06	0.02	**0.16**
$\boldsymbol{f_X(x)}$		**0.34**	**0.32**	**0.34**	**1**

$E(X) = \sum x f_X(x) = 1$. $E(X^2) = \sum x^2 f_X(x) = 1.68$. $V(X) = E(X^2) - [E(X)]^2 = 0.68$.
$E(Y) = \sum y f_Y(y) = 2.85$. $E(Y^2) = \sum y^2 f_Y(y) = 10.25$. $V(Y) = E(Y^2) - [E(Y)]^2 = 2.1275$.
$\sigma_{X,Y} = E(XY) - E(X)E(Y) = (2.47) - (1)(2.85) = -0.38$
Profit = $8X + 3Y - 10$. $E(\text{Profit}) = 8E(X) + 3E(Y) - 10 = 6.55$
$V(\text{profit}) = V[8X + 3Y - 10] = 8^2V(X) + 3^2E(Y) + 2(8)(3)Cov(X,Y) = 44.4275$.

<u>Question 3</u>

We first compute the marginal probability density functions of X and Y. They are also needed for parts (b) to (d).

$f_X(x) = \int_0^1 \frac{2}{3}(x+2y)\,dy = \frac{2}{3}\left[xy + \frac{2y^2}{2}\right]_0^1 = \frac{2}{3}(x+1)$, for $0 \le x \le 1$; and 0 otherwise

$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1 = \frac{2}{3}\left(\frac{1}{2} + 2y\right)$, for $0 \le y \le 1$; and 0 otherwise,

(a) Since $f_{X,Y}(x,y) \ne f_X(x)f_Y(y)$, X and Y are dependent.

(b) $E(X) = \frac{2}{3}\int_0^1 x(x+1)\,dx = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{2}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{9} = 0.55556.$

$E(X^2) = \frac{2}{3}\int_0^1 x^2(x+1)\,dx = \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{3}\right) = \frac{7}{18} = 0.38889.$

$V(X) = E(X^2) - [E(X)]^2 = 13/162 = 0.08024.$

(c) $E(Y) = \frac{2}{3}\int_0^1 y\left(\frac{1}{2} + 2y\right)\,dy = \frac{2}{3}\left[\frac{y^2}{4} + \frac{2y^3}{3}\right]_0^1 = \frac{2}{3}\left(\frac{1}{4} + \frac{2}{3}\right) = \frac{11}{18} = 0.61111.$

$E(Y^2) = \frac{2}{3}\int_0^1 y^2\left(\frac{1}{2} + 2y\right)\,dx = \frac{2}{3}\left[\frac{y^3}{6} + \frac{y^4}{2}\right]_0^1 = \frac{2}{3}\left(\frac{1}{6} + \frac{1}{2}\right) = \frac{4}{9} = 0.44444$.

$V(Y) = E(Y^2) - [E(Y)]^2 = 23/324 = 0.07099.$

(d) $E(XY) = \frac{2}{3}\int_0^1\int_0^1 xy(x+2y)\,dxdy = \frac{2}{3}\int_0^1\int_0^1 x^2y + 2xy^2\,dxdy$

$= \frac{2}{3}\int_0^1 \left[\frac{x^3y}{3} + \frac{2x^2y^2}{2}\right]_0^1 dy = \frac{2}{3}\int_0^1\left(\frac{y}{3} + y^2\right)dy = \frac{2}{3}\left[\frac{y^2}{6} + \frac{y^3}{3}\right]_0^1 = \frac{2}{3}\left(\frac{1}{6} + \frac{1}{3}\right) = \frac{1}{3} = 0.33333.$

Hence, $\sigma_{X,Y} = E(XY) - E(X)E(Y) = 1/3 - (5/9)(11/18) = -1/162 = -0.00617.$

<u>Question 4</u>

$f_X(x) = \int_0^1 \frac{3}{2}(x^2+y^2)\,dy = \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}(x^2 + \frac{1}{3})$, for $0 \le x \le 1$; and 0 otherwise.

$f_Y(y) = \int_0^1 \frac{3}{2}(x^2+y^2)\,dx = \frac{3}{2}\left[\frac{x^3}{3} + xy^2\right]_0^1 = \frac{3}{2}(\frac{1}{3} + y^2)$, for $0 \le y \le 1$; and 0 otherwise.

(a) Since $f_{X,Y}(x,y) \ne f_X(x)f_Y(y)$, X and Y are dependent

(b) $E(X) = \frac{3}{2}\int_0^1 x\left(x^2+\frac{1}{3}\right)\,dx = \frac{3}{2}\left[\frac{x^4}{4}+\frac{x^2}{6}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4}+\frac{1}{6}\right) = \frac{5}{8} = 0.625.$

$E(X^2) = \frac{3}{2}\int_0^1 x^2\left(x^2+\frac{1}{3}\right)\,dx = \frac{3}{2}\left[\frac{x^5}{5}+\frac{x^3}{9}\right]_0^1 = \frac{3}{2}\left(\frac{1}{5}+\frac{1}{9}\right) = \frac{7}{15} = 0.46667.$

$V(X) = E(X^2) - [E(X)]^2 = 73/960 = 0.07604.$

(c) Method 1: Repeat the work in part (b)

$E(Y) = \frac{3}{2}\int_0^1 y\left(y^2+\frac{1}{3}\right)\,dy = \frac{3}{2}\left[\frac{y^4}{4}+\frac{y^2}{6}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4}+\frac{1}{6}\right) = \frac{5}{8} = 0.625.$

$E(Y^2) = \frac{3}{2}\int_0^1 y^2\left(y^2+\frac{1}{3}\right)\,dy = \frac{3}{2}\left[\frac{y^5}{5}+\frac{y^3}{9}\right]_0^1 = \frac{3}{2}\left(\frac{1}{5}+\frac{1}{9}\right) = \frac{7}{15} = 0.46667.$

$V(Y) = E(Y^2) - [E(Y)]^2 = 73/960 = 0.07604.$

Method 2: Observe that the joint p.d.f. is symmetric in x and y (i.e. $f_{(X,Y)}(x,y) = f_{X,Y}(y,x)$), so $E(Y) = E(X) = 5/8 = 0.625$ and $V(Y) = V(X) = 73/960 = 0.07604$.

(d) $E(XY) = \frac{3}{2}\int_0^1\int_0^1 xy(x^2+y^2)\,dxdy = \frac{3}{2}\int_0^1\int_0^1 x^3y + xy^3\,dxdy$

$= \frac{3}{2}\int_0^1\left[\frac{x^4y}{4}+\frac{x^2y^3}{2}\right]_0^1 dy = \frac{3}{2}\int_0^1\left(\frac{y}{4}+\frac{y^3}{2}\right)dy = \frac{3}{2}\left[\frac{y^2}{8}+\frac{y^4}{8}\right]_0^1 = \frac{3}{2}\left(\frac{1}{8}+\frac{1}{8}\right) = \frac{3}{8} = 0.375.$

$\sigma_{X,Y} = E(XY) - E(X)E(Y) = (3/8) - (5/8)(5/8 = -1/64 = -0.01563.$

(e) $E(X+Y) = E(X) + E(Y) = 5/8 + 5/8 = 5/4 = 1.25.$

(f) $V(X+Y) = V(X) + V(Y) + 2(\sigma_{X,Y}) = \frac{73}{960} + \frac{73}{960} + 2\left(\frac{-1}{64}\right) = \frac{29}{240} = 0.12083.$

<u>Question 5</u>

(a) $f_X(x) = \int_0^1 (x+y)\,dy = \left[xy+\frac{y^2}{2}\right]_0^1 = x+\frac{1}{2}$, for $0 \le x \le 1$; and $f_X(x) = 0$ otherwise.

$E(X) = \int_0^1 x\left(x+\frac{1}{2}\right)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{4}\right]_0^1 = \left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{12} = 0.58333.$

$f_Y(y) = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2}+xy\right]_0^1 = y+\frac{1}{2}$, for $0 \le y \le 1$; and $f_Y(y) = 0$ otherwise.

$E(Y) = \int_0^1 y\left(y+\frac{1}{2}\right)\,dy = \left[\frac{y^3}{3}+\frac{y^2}{4}\right]_0^1 = \left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{12} = 0.58333.$

$E(XY) = \int_0^1\int_0^1 xy(x+y,)\,dxdy = \int_0^1\int_0^1 x^2y + xy^2\,dxdy = \int_0^1\left[\frac{x^3y}{3}+\frac{x^2y^2}{2}\right]_0^1 dy$

$= \int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \left[\frac{y^2}{6}+\frac{y^3}{6}\right]_0^1 = \frac{1}{3} = 0.33333.$

$\sigma_{X,Y} = E(XY) - E(X)E(Y) = 1/3 - (7/12)(7/12) = -1/144 = -0.00694.$

(b) To compute $E(Y|X=0.2)$, we need to compute $f_{Y|X}(y|0.2)$. We shall restrict ourselves to $0 \le y \le 1$ because the conditional probability density function is 0 for $y \notin [0,1]$. We have $f_{Y|X}(y|x=0.2) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+1/2} = \frac{0.2+y}{0.7} = \frac{2+10y}{7}$, for $0 \le y \le 1$; and $f_{Y|X}(y|x = 0.2) = 0$ otherwise.

$E(Y|X=0.2) = \int_0^1 y\left(\frac{2+10y}{7}\right)\,dy = \frac{1}{7}\left[y^2+\frac{10y^3}{3}\right]_0^1 = \frac{1}{7}\left(1+\frac{10}{3}\right) = \frac{13}{21} = 0.61905.$

(c) Similarly, $f_{X|Y}(x|y=0.5) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x+y}{y+1/2} = \frac{0.5+x}{1} = x+\frac{1}{2}$, for $0 \le x \le 1$; and $f_{X|Y}(x|y=0.5) = 0$ otherwise.

$E(X|Y=0.5) = \int_0^1 x\left(x+\frac{1}{2}\right)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{4}\right]_0^1 = \left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{12} = 0.58333.$

Question 6

$V(Z) = V(-2X + 4Y) = (-2)^2V(X) + 4^2V(Y) + 2(-2)(4)Cov(X,Y)$

(a) If X and Y are independent, then $Cov(X,Y) = 0$, hence,
$V(Z) = V[-2X + 4Y - 3] = (-2)^2V(X) + (4)^2V(Y) = 4(5) + 16(3) = 68.$

(b) If $Cov(X,Y) = 1$, then
$V(Z) = V[-2X + 4Y - 3] = (-2)^2V(X) + (4)^2V(Y) + 2(-2)(4)Cov(X,Y)$
$= 4(5) + 16(3) + 2(-2)(4)(1) = 52.$

(c) From part (b), $\rho_{X,Y} = \frac{Cov\,(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{1}{\sqrt{5}\sqrt{3}} = 0.2582.$

Question 7

$X \sim discrete\ uniform$

(a) You can give your answer either in a table

x	1	2	3	4	5	6	7	8	9	10
$f_X(x)$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

or

$$f_X(x) = \begin{cases} \frac{1}{10}, & x = 1, 2, \cdots, 10\,; \\ 0, & \text{otherwise}\,. \end{cases}$$

(b) $\Pr(X < 4) = \sum_{x=1}^{3} f(x) = \frac{3}{10} = 0.3$

(c) $\mu = \sum_{x=1}^{10} x\left(\frac{1}{10}\right) = 5.5$. $\sigma^2 = \sum_{x=1}^{10}(x - 5.5)^2\,\frac{1}{10} = 8.25$
Alternatively, $E(X^2) = \sum_{x=1}^{10} x^2\left(\frac{1}{10}\right) = 38.5$. $V(X) = 38.5 - 5.5^2 = 8.25$.