NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF STATISTICS & APPLIED PROBABILITY

ST2334 PROBABILITY AND STATISTICS

SEMESTER I, AY 2022/2023

Tutorial 04: Solution

This set of questions will be discussed by your tutors during the tutorial in Week 6.

Please work on the questions before attending the tutorial.

1. An insurance company offers its policyholders a number of different premium payment options. For a randomly selected policyholder, let X = the number of months between successive payments. The c.d.f. of X is as follow.

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 0.3, & 1 \le x < 3 \\ 0.4, & 3 \le x < 4 \\ 0.45, & 4 \le x < 6 \\ 0.6, & 6 \le x < 12 \\ 1, & 12 \le x \end{cases}$$

(a) What is the probability function of X?

(b) Using the c.d.f., compute $P(3 \le X \le 6)$ and $P(X \ge 4)$.

SOLUTION

(a) Possible X values are those values at which $F_X(x)$ jumps, and the probability of any possible values is the size of the jump at that value. Thus we have

x	1	3	4	6	12
$f_X(x)$	0.3	0.1	0.05	0.15	0.4

(b) $P(3 \le X \le 6) = F(6) - F(3-) = 0.6 - 0.3 = 0.3$;
$P(X \ge 4) = 1 - P(X < 4) = 1 - F_X(4-) = 1 - 0.4 = 0.6$.

2. Determine the value c, such that the following function can serve as a probability function of a discrete random variable X.

$$f_X(x) = \begin{cases} c(x^2+4), & x = 0, 1, 2, 3; \\ 0, & \text{elsewhere.} \end{cases}$$

SOLUTION
In order for $f_X(x)$ to be a p.m.f., we need

$$1 = \sum_{x=0}^{3} f_X(x) = c[(0^2+4) + (1^2+4) + (2^2+4) + (3^2+4)] = 30c,$$

which implies $c = 1/30$.

3. Consider the probability density function

$$f_X(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the value of the constant k.

(b) Find the cumulative distribution function $F_X(x)$, and use it to evaluate $P(0.3 < X < 0.6)$.

SOLUTION

(a) For $f_X(x)$ to be valid p.d.f., we need

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 kx^{1/2}dx = k\frac{1}{1+1/2}\,x^{1+1/2}\Big|_0^1 = \frac{2}{3}k,$$

which indicates $k = 3/2$.

(b) Clearly, when $x \le 0$, $F_X(x) = 0$; and when $x \ge 1$, $F_X(x) = 1$. When $0 < x < 1$, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_0^x \frac{3}{2}t^{1/2}dt = t^{3/2}|_0^x = x^{3/2}.$$

In summary,

$$F_X(x) = \begin{cases} 0, & x \le 0; \\ x^{3/2}, & 0 < x < 1; \\ 1, & x \ge 1. \end{cases}$$

As a consequence

$$P(0.3 < X < 0.6) = F_X(0.6) - F_X(0.3) = 0.6^{1.5} - 0.3^{1.5} = 0.3004.$$

4. Suppose the distance X between a point target and a shot aimed at the point in a coin-operated target game is a continuous random variable with p.d.f.

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2), & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Compute $P(-1/2 < X < 1/2)$.

(b) Compute $P(X < -1/4 \text{ or } X > 1/4)$.

(c) Find the c.d.f. of X.

SOLUTION

(a)

$$P(-1/2 < X < 1/2) = \int_{-1/2}^{1/2} \frac{3}{4}(1-x^2)dx = \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Bigg|_{-1/2}^{1/2} = \frac{11}{16}.$$

(b)

$$\begin{aligned} P\left(X < -\frac{1}{4} \text{ or } X > \frac{1}{4}\right) &= 1 - P\left(-\frac{1}{4} < X < \frac{1}{4}\right) = 1 - \int_{-1/4}^{1/4} \frac{3}{4}(1-x^2)dx \\ &= 1 - \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Bigg|_{-1/4}^{1/4} = 1 - \frac{47}{128} = 81/128. \end{aligned}$$

(c) For $x < -1$, $F_X(x) = 0$; for $x > 1$, $F_X(x) = 1$; for $-1 \le x \le 1$,

$$F_X(x) = \int_{-1}^{x} \frac{3}{4}(1-t^2)dt = \frac{3}{4}\left(t - \frac{t^3}{3}\right)\Big|_{-1}^{x} = \frac{1}{4}(2+3x-x^3).$$

In summary,

$$F_X(x) = \begin{cases} 0, & x < -1; \\ \frac{1}{4}(2+3x-x^3), & -1 \le x \le 1; \\ 1, & x > 1. \end{cases}$$

5. The waiting time, in hours, between successive speeders spotted by a radar unit is a continuous random variable with cumulative distribution

$$F_X(x) = \begin{cases} 0, & x \le 0; \\ 1 - e^{-8x}, & x > 0. \end{cases}$$

(a) Find the probability of waiting less than 12 minutes between successive speeders.

(b) Find the probability density function of X.

SOLUTION

(a) 12 minutes is equivalent to $1/5$ hours, so

$$P\left(X < \frac{1}{5}\right) = F_X\left(\frac{1}{5}\right) = 1 - e^{-8(1/5)} = 0.7981.$$

(b)

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d}{dx}\left(1 - e^{-8x}\right) = 8e^{-8x},$$

for $x \ge 0$, and $f_X(x) = 0$ for $x < 0$.

6. The random variable X, representing the number of errors per 100 lines of software code, has the following probability function (or probability mass function):

x	2	3	4	5	6
$f_X(x)$	0.01	0.25	0.40	0.30	0.04

(a) Find $E(X)$ and $E(X^2)$.

(b) Find the variance of X using (i) the definition of variance and (ii) $V(X) = E(X^2) - [E(X)]^2$.

(c) Find the mean and variance of the discrete variable $Z = 3X - 2$.

(d) Find the probability function of the random variable Z. Hence, find the mean and variance of Z directly from its probability function).

(e) Suppose that $W = aZ + b$. Find the mean and variance of W in terms of a and b.

SOLUTION

(a) $E(X) = \sum x f_X(x) = 2(0.01) + \ldots + 6(0.04) = 4.11.$
$E(X^2) = \sum x^2 f_X(x) = 2^2(0.01) + \ldots + 6^2(0.04) = 17.63.$

(b) (i) By definition:

$$\begin{aligned} V(X) &= \sum_x (x-\mu)^2 f_X(x) \\ &= (2-4.11)^2(0.01)+\ldots+(6-4.11)^2(0.04)=0.7379. \end{aligned}$$

(ii) By the alternative formula:

$$V(X)=E(X^2)-[E(X)]^2=17.63-4.11^2=0.7379.$$

(c) $E(Z)=E(3X-2)=3E(X)-2=10.33$;
$V(Z)=V(3X-2)=3^2V(X)=6.6411$.

(d) The probability function Z is given by

x	2	3	4	5	6
$z=3x-2$	4	7	10	13	16
$f_Z(z)$	0.01	0.25	0.40	0.30	0.04

Mean: $E(Z)=\sum z f_Z(z)=4(0.01)+\ldots+16(0.04)=10.33=\mu$;
Variance: $V(Z)=\sum(z-\mu)^2 f_Z(z)=(4-10.33)^2 0.01+\ldots+(16-10.33)^2 0.04=6,6411$.

(e) $W=aZ+b$.
Mean: $E(W)=aE(Z)+b=10.33a+b$;
Variance: $V(W)=a^2V(Z)=6.6411a^2$.

7. If the probability density of a random variable X is given by

$$f(x)=\begin{cases} x, & \text{for } 0<x<1 \\ 2-x, & \text{for } 1\le x<2 \\ 0, & \text{elsewhere} \end{cases}.$$

(a) Find the probability that the random variable will take on a value between 0.6 and 1.2.

(b) Find $E(X)$ and $V(X)$.

SOLUTION

(a)

$$P(0.6<X<1.2)=\int_{0.6}^{1.2} f(x)\,\mathrm{d}x=\int_{0.6}^{1} x\,\mathrm{d}x+\int_{1}^{1.2}(2-x)\,\mathrm{d}x=\left(\frac{x^2}{2}\right)\Big|_{0.6}^{1}+\left(2x-\frac{x^2}{2}\right)\Big|_{1}^{1.2}=0.50.$$

(b) In this case,

$$\begin{aligned} E(X) &= \int_0^2 x f(x)\,\mathrm{d}x=\int_0^1 x^2\,\mathrm{d}x+\int_1^2 x(2-x)\,\mathrm{d}x \\ &= \left(x^3/3\right)\big|_0^1+\left(x^2-x^3/3\right)\big|_1^2=1/3+4-1-8/3+1/3=1, \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_0^2 x^2 f(x)\,\mathrm{d}x=\int_0^1 x^3 f(x)\,\mathrm{d}x+\int_1^2 x^2(2-x)\,\mathrm{d}x \\ &= \left(x^4/4\right)\big|_0^1+\left(2x^3/3-x^4/4\right)\big|_1^2=7/6. \end{aligned}$$

Thus,

$$V(X)=7/6-1^2=1/6.$$

8. (a) Let X be a positive integer-valued (excluding 0) random variable. Show that

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k).$$

(b) Suppose that 3 fair dice are rolled. Let M be the minimum of 3 numbers rolled. Find

$$P(M \geq 1),\ P(M \geq 2),\ \ldots,\ P(M \geq 6).$$

Hence, find $E(M)$.

SOLUTION

(a) We have

$$\begin{array}{rcl} P(X \geq 1) & = & P(X=1) + P(X=2) + P(X=3) + P(X=4) + \cdots \\ P(X \geq 2) & = & \qquad\qquad\quad + P(X=2) + P(X=3) + P(X=4) + \cdots \\ P(X \geq 3) & = & \qquad\qquad\qquad\qquad\qquad\quad + P(X=3) + P(X=4) + \cdots \\ \cdots & = & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \end{array}$$

Adding up these quantities, we have

$$\begin{aligned} \sum_{k=1}^{\infty} P(X \geq k) &= 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) + \cdots \\ &= \sum_{k=1}^{\infty} k \cdot P(X=k) = E(X). \end{aligned}$$

(b) Let X_1, X_2, X_3 denote respectively the number obtained in the first, second and third die. Then $M = \min\{X_1, X_2, X_3\}$. For $k = 1, 2, \ldots, 6$,

$$\begin{aligned} P(M \geq k) &= P(X_1 \geq k, X_2 \geq k, X_3 \geq k) \\ &= P(X_1 \geq k) P(X_2 \geq k) P(X_3 \geq k) \\ &= \frac{6-(k-1)}{6} \cdot \frac{6-(k-1)}{6} \cdot \frac{6-(k-1)}{6} \\ &= \left(\frac{7-k}{6}\right)^3. \end{aligned}$$

In the other words,

$$P(M \geq 1) = 1, \quad P(M \geq 2) = \frac{5^3}{216}, \quad P(M \geq 3) = \frac{4^3}{216}$$
$$P(M \geq 4) = \frac{3^3}{216}, \quad P(M \geq 5) = \frac{2^3}{216}, \quad P(M \geq 6) = \frac{1}{216},$$

and $P(M \geq k) = 0$ for $k \geq 7$. As a consequence,

$$\begin{aligned} E(M) &= \sum_{k=1}^{\infty} P(M \geq k) = \sum_{k=1}^{6} P(M \geq k) \\ &= \sum_{k=1}^{6} \left(\frac{7-k}{6}\right)^3 = \frac{1^3 + 2^3 + \ldots + 6^3}{6^3} = 2.0417. \end{aligned}$$