NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF STATISTICS & APPLIED PROBABILITY

ST2334 PROBABILITY AND STATISTICS

SEMESTER I, AY 2022/2023

Tutorial 05: Solution

This set of questions will be discussed by your tutors during the tutorial in Week 8.

Please work on the questions before attending the tutorial.

1. Let X denote the number of times a certain numerical control machine will malfunction: 1, 2, or 3 times on any given day. Let Y denote the number of times a technician is called on an emergency call. Their joint probability distribution is given below.

$f_{X,Y}(x,y)$		$x=1$	$x=2$	$x=3$
y	1	0.05	0.05	0.1
	2	0.05	0.10	0.35
	3	0	0.2	0.1

(a) Evaluate the marginal distributions of X and Y.

(b) Find $P(Y=3|X=2)$.

(c) Find the conditional distribution of Y given $X=2$.

(d) Determine whether X and Y are dependent or independent.

SOLUTION

(a)

$f_{X,Y}(x,y)$		$x=1$	$x=2$	$x=3$	$f_Y(y)$
y	1	0.05	0.05	0.1	0.20
	2	0.05	0.10	0.35	0.50
	3	0	0.2	0.1	0.30
$f_X(x)$		0.10	0.35	0.55	1

(b)

$$P(Y=3|X=2) = f_{Y|X}(y=3|x=2) = \frac{f_{X,y}(2,3)}{f_X(2)} = \frac{0.20}{0.35} = 4/7.$$

(c) $f_{Y|X}(y|x=2) = \dfrac{f_{X,Y}(2,y)}{f_X(2)}$, we have

- $f_{Y|X}(y=1|x=2) = 0.05/0.35 = 1/7;$
- $f_{Y|X}(y=2|x=2) = 0.1/0.35 = 2/7;$
- $f_{Y|X}(y=3|x=2) = 0.2/0.35 = 4/7.$

(d) Since $f_{X,Y}(1,1) \neq f_X(1)f_Y(1)$, so X and Y are dependent.

2. From a sack of fruit containing 3 oranges, 2 apples, and 3 bananas, a random sample of 4 pieces of fruit is selected. If X is the number of oranges and Y is the number of apples in the sample, find

(a) the joint probability distribution of X and Y;

(b) $P(X=1, Y=1)$;

(c) $P(X+Y \leq 2)$;

(d) $f_X(x)$;

(e) $f_{Y|X}(y|2)$ and hence $P(Y=0|X=2)$.

SOLUTION

(a) First, random variable X can only take values in 0; 1; 2; 3; Y in 0; 1; 2. As only 4 pieces of fruit is selected, therefore $x+y \leq 4$. Since there are only three bananas, one piece of the selected fruit must be either an orange or an apple, that is, $x+y \geq 1$.

$$f(x,y) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}, & x=0,1,2,3; y=0,1,2; 1 \leq x+y \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

(b) $P(X=1, Y=1) = f(1,1) = \dfrac{\binom{3}{1}\binom{2}{1}\binom{3}{2}}{\binom{8}{4}} = 0.2571.$

(c) $P(X+Y \leq 2) = f(0,1)+f(0,2)+f(1,0)+f(1,1)+f(2,0) = 0.5.$

(d) Recall the possible values of X are 0; 1; 2; 3. Since 4 pieces of fruit are selected, $(4-X)$ pieces of fruit must be selected from 5 pieces of apples and bananas. That is,

$$f_X(x) = \begin{cases} \dfrac{\binom{3}{x}\binom{5}{4-x}}{\binom{8}{4}}, & x=0,1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

(e) For $x=2$,

$$f_{Y|X}(y|2) = \begin{cases} \dfrac{\binom{2}{y}\binom{3}{4-2-y}}{\binom{5}{4-2}} = \dfrac{1}{10}\dbinom{2}{y}\dbinom{3}{2-y}, & y=0,1,2 \\ 0, & \text{elsewhere} \end{cases}$$

$$P(Y=0|X=2) = \frac{1}{10}\binom{2}{0}\binom{3}{2} = 0.3.$$

3. Consider an experiment that consists of two rolls of a balanced die. If X is the number of fours and Y is the number of fives obtained in the two rolls of the die, find

(a) the joint probability distribution of X and Y;

(b) $P(2X+Y<3)$;

(c) Determine whether X and Y are dependent or independent.

SOLUTION

Let D_1 and D_2 denote the number obtained by the first die and the second die respectively. The entries of the table below correspond to the values of (x,y) defined in the question:

d_2	d_1					
	1	2	3	4	5	6
1	(0,0)	(0,0)	(0,0)	(1,0)	(0,1)	(0,0)
2	(0,0)	(0,0)	(0,0)	(1,0)	(0,1)	(0,0)
3	(0,0)	(0,0)	(0,0)	(1,0)	(0,1)	(0,0)
4	(1,0)	(1,0)	(1,0)	(2,0)	(1,1)	(1,0)
5	(0,1)	(0,1)	(0,1)	(1,1)	(0,2)	(0,1)
6	(0,0)	(0,0)	(0,0)	(1,0)	(0,1)	(0,0)

(a) From the table above, we have

$f_{X,Y}(x,y)$		x			$f_Y(y)$
		0	1	2	
y	0	$16/36 = 4/9$	$8/36 = 2/9$	$1/36$	$25/36$
	1	$8/36 = 2/9$	$2/36 = 1/18$	0	$5/18$
	2	$1/36$	0	0	$1/36$
$f_X(x)$		$25/36$	$5/18$	$1/36$	1

(b) Based on Part (a), we have

$$\begin{aligned} P(2X+Y<3) &= f_{X,Y}(0,0)+f_{X,Y}(0,1)+f_{X,Y}(0,2)+f_{X,Y}(1,0) \\ &= 4/9+2/9+1/36+2/9 = 11/12. \end{aligned}$$

(c) X and Y are dependent since $f_{X,Y}(2,2) \neq f_X(2)f_Y(2)$.

4. Each rear tire on an experimental airplane is supposed to be filled to a pressure of 40 pound per square inch (psi). Let X denote the actual air pressure (in 10 pound per square inch) for the right tire and Y denote the actual air pressure (in 10 pound per square inch) for the left tire. Suppose that X and Y are random variables with the joint density

$$f_{X,Y}(x,y) = \begin{cases} k(x^2+y^2), & 3 \le x \le 5; 3 \le y \le 5; \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine k;

(b) Compute $P(3 \le X \le 4 \text{ and } 4 \le Y < 5)$;

(c) Find $f_X(x)$ and hence $P(3.5 < X < 4)$.

SOLUTION

(a) By the definition of the joint p.d.f.,

$$\begin{aligned} 1 &= \int_3^5 \int_3^5 k(x^2+y^2)dydx = k\int_3^5 \int_3^5 \left(yx^2+\frac{y^3}{3}\right)\Big|_{y=3}^5 dx = \frac{2}{3}k\int_3^5 (3x^2+49)dx \\ &= \frac{2}{3}k(x^3+49x)|_{x=3}^5 = \frac{392}{3}k, \end{aligned}$$

which implies $k = 3/392$.

(b)

$$\begin{aligned} P(3 \le X \le 4 \text{ and } 4 \le Y \le 5) &= \frac{3}{392}\int_3^4 \int_4^5 (x^2+y^2)dydx = \frac{3}{392}\int_3^4 \left(yx^2+\frac{y^3}{3}\right)\Big|_{y=4}^5 \\ &= \frac{3}{392}\int_3^4 \left(x^2+\frac{61}{3}\right)dx = \frac{1}{392}(x^3+61x)|_3^4 = \frac{1}{392}(98) = 1/4. \end{aligned}$$

(c) For $3 \le x \le 5$,

$$f_X(x) = \frac{3}{392}\int_3^5 (x^2+y^2)dy = \frac{3}{392}\left(x^2 y + \frac{y^3}{3}\right)\Big|_{y=3}^{5} = \frac{3}{392}\left(2x^2+\frac{98}{3}\right) = \frac{1}{196}(3x^2+49),$$

$$P(3.5<X<4) = \frac{1}{196}\int_{3.5}^4 (3x^2+49)dx = \frac{1}{196}(x^3+49x)|_{3.5}^4 = 0.2328.$$

5. Two random variables have the joint density

$$f(x_1,x_2) = \begin{cases} x_1x_2, & \text{for } 0<x_1<2, 0<x_2<1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the probability that both random variables will take on values less than 1.

(b) Find the marginal densities of the two random variables, and check whether the two random variables are independent.

(c) Find the expected value of the random variable whose values are given by $g(x_1,x_2) = x_1+x_2$.

SOLUTION

(a)

$$P(X_1<1, X_2<1) = \int_0^1\int_0^1 x_1x_2\,\mathrm{dx}_2\,\mathrm{dx}_1 = \frac{1}{2}\int_0^1 x_1\,\mathrm{dx}_1 = \frac{1}{4}.$$

(b) X and Y are independent with $g_1(x_1)=x_1$ and $g_2(x_2)=x_2$.

$$f_1(x_1) = \frac{g_1(x_1)}{\int_0^2 g_1(x_1)dx_1} = \frac{x_1}{\int_0^2 x_1dx_1} = \frac{1}{2}x_1; \quad 0\le x_1\le 2;$$

$$f_2(x_2) = \frac{g_2(x_2)}{\int_0^1 g_2(x_2)dx_2} = \frac{x_2}{\int_0^1 x_2dx_2} = 2x_2; \quad 0\le x_2\le 1;.$$

(c) The expected value of $g(X_1,X_2)$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1,x_2)f(x_1,x_2)\,\mathrm{dx}_2\,\mathrm{dx}_1 = \int_0^1\int_0^2 (x_1+x_2)x_1x_2\,\mathrm{dx}_2\,\mathrm{dx}_1 = \int_0^1 (2x_1^2+8x_1/3)\,\mathrm{dx}_1 = 2.$$

6. Consider the random variables X and Y that have a joint probability density function given by

$$f(x,y) = x^2e^{-x}, \quad \text{for} \quad x>0, \quad -1/4<y<1/4.$$

(a) Compute the probability $P(X<1, Y>0)$.

(b) Find the marginal distributions of X and Y. Are X and Y independent?

SOLUTION

(a)

$$\begin{aligned}
P(X<1,Y>0) &= \int_0^1\int_0^{\frac{1}{4}} x^2e^{-x}\,\mathrm{dy}\,\mathrm{dx} = \frac{1}{4}\int_0^1 x^2e^{-x}\,\mathrm{dx} \\
&= \frac{1}{4}\left(\left[-x^2e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,\mathrm{dx}\right) \\
&= \frac{1}{4}\left(-e^{-1}+2\left(\left[-xe^{-x}\right]_0^1+\int_0^1 e^{-x}\,\mathrm{dx}\right)\right) \\
&= \frac{1}{4}\left(-e^{-1}+2\left(-e^{-1}+1-e^{-1}\right)\right) = \frac{1}{4}\left(2-\frac{5}{e}\right).
\end{aligned}$$

(b) The marginal distribution of X is

$$f(x) = \int_{-\frac{1}{4}}^{\frac{1}{4}} f(x,y)\,\mathrm{dy} = \int_{-\frac{1}{4}}^{\frac{1}{4}} x^2 e^{-x}\,\mathrm{dy} = \frac{1}{2}x^2 e^{-x}, \quad \text{for } x > 0.$$

The marginal distribution of Y is

$$g(y) = \int_0^{\infty} f(x,y)\,\mathrm{dx} = 2\int_0^{\infty} \frac{1}{2}x^2 e^{-x}\,\mathrm{dy} = \ldots = 2, \quad \text{for } -\frac{1}{4} < y < \frac{1}{4},$$

using integration by parts twice.

As $f(x)g(x) = f(x,y)$ for $x > 0$ and $-1/4 < y < 1/4$, we say that X and Y are independent.