NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF STATISTICS & APPLIED PROBABILITY

ST2334 PROBABILITY AND STATISTICS

SEMESTER I, AY 2022/2023

Tutorial 08: Solution

This set of questions will be discussed by your tutors during the tutorial in Week 11.

Please work on the questions before attending the tutorial.

1. The length of time for one individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 4 minutes. Find the probability that

   (a) A person is served in more than 3 minutes.

   (b) A person is served in less than 3 minutes.

   (c) A person is served in less than 3 minutes on at least 4 of the next 6 days.

   SOLUTION
   Let X = length of the time for one individual to be served. Then $X \sim \text{Exp}(1/\mu)$ with $\mu = 4$.

   (a) $P(X > 3) = e^{-3/4} = 0.4724$;

   (b) $P(X < 3) = 1 - e^{-3/4} = 0.5276$;

   (c) Let Y = number of days being served in less than 3 mins out of 6 days. Then $Y \sim B(6, 0.5276)$. Therefore

   $$P(Y \geq 4) = P(Y = 4) + P(Y = 5) + P(Y = 6) = 0.3968.$$

2. Extensive experience with fans of a certain type used in diesel engines has suggested that the exponential distribution provides a good model for time until failure. Suppose the mean time until failure is 25000 hours.

   (a) What is the probability that a randomly selected fan will last 20000 hours? At most 30000 hours? Between 20000 and 30000 hours?

   (b) What is the probability that the lifetime of a fan exceeds the mean value by more than 2 standard deviations?

   SOLUTION
   Let X be the time until failure for the fan. Then $X \sim \text{Exp}(1/25,000)$.

   (a) $P(X > 20000) = e^{-20000/25000} = e^{-0.8} = 0.4493$;
   $P(X \leq 30000) = 1 - e^{-30000/250000} = 1 - e^{-1.2} = 0.6988$;
   $P(20000 \leq X \leq 30000) = F_X(30000) - F_x(20000) = e^{-20000/25000} - e^{-30000/25000} = e^{-0.8} - e^{-1.2} = 0.1481$.

   (b) $\sigma = 1/\lambda = 25000$. Therefore $P(X > \mu + 2\sigma) = P(X > 75000) = e^{-75000/25000} = e^{-3} = 0.0498$.

3. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is

   (a) the probability that a repair time exceeds 2 hours?

(b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

SOLUTION

(a) Let T denote the repair time. So we need to compute $P(T>2)$ which is given by

$$P(T>2)=e^{-\lambda x}=e^{-1/2\times 2}=e^{-1}.$$

(b) The probability we are after is given by $P(T>10|T>9)$ which equals $P(T>1)$ by the memoryless property of the exponential distribution. This is given by

$$P(T>1)=e^{-1/2\times 1}=e^{-1/2}.$$

4. Let $X_1,X_2,\ldots,X_n$ are i.i.d. and follow the $\mathrm{Exp}(\lambda)$ distribution. Let $X=\min\{X_1,X_2,\ldots,X_n\}$. What is the distribution of X?

SOLUTION

For any real number $x>0$, we have

$$\begin{aligned} P(X>x) &= P(\min\{X_1,X_2,\ldots,X_n\}>x)=P(X_1>x,X_2>x,\ldots,X_n>x) \\ &= \prod_{i=1}^{n}P(X_i>x)=\prod_{i=1}^{n}e^{-\lambda x}=e^{-n\lambda x}. \end{aligned}$$

which implies $X\sim\mathrm{Exp}(n\lambda)$.

5. Let $Z\sim N(0,1)$ and $z\geq 0$, show that

$$P(|Z|<z)=2\Phi(z)-1,$$

where $\Phi(\cdot)$ denotes the c.d.f. for the standard normal distribution.

SOLUTION

$$\begin{aligned} P(|Z|<z) &= P(-z<Z<z)=P(Z<z)-P(Z\leq -z) \\ &= P(Z<z)-P(Z\geq z) \\ &= P(Z<z)-(1-P(Z<z)) \\ &= 2P(Z<z)-1=2\Phi(z)-1. \end{aligned}$$

6. A soft-drink machine is regulated so that it discharges an average of 200 ml per cup. Suppose that the amount of drink is normally distributed with a standard deviation equal to 15 ml.

(a) Find the probability that the cups will contain more than 224 ml.

(b) Find the probability that a cup contains between 191 and 209 ml.

(c) How many cups will probably overflow if 230 ml cups are used for the next 1000 drinks?

(d) Below what value do we get the smallest 25% of the drinks?

SOLUTION

Let X = amount of the soft drink; then $X\sim N(\mu=200,\sigma^2=15^2)$.

(a) $P(X>224)=P(Z>1.60)=1-\Phi(1.60)=0.548$, where $Z\sim N(0,1)$.

(b) $P(191<X<209)=P(-0.60<Z<0.60)=2P(Z<0.60)-1=2\Phi(0.60)-1=0.4514$.

(c) $P(X > 230) = P(Z > 2.0) = 1 - \Phi(2.0) = 0.02275 = p$.
Let Y = number of cups out of 1000 that will overflow; then $Y \sim B(n = 1000, p = 0.02275)$.
$E(Y) = np = 1000(0.02275) = 22.75 \approx 23$.

(d) Let z is the value such that $P(Z < z) = 0.25$; then $z = z_{0.25} = -0.6745$. On the other hand,

$$Z < z_{0.25} \iff \frac{X-\mu}{\sigma} < z_{0.25} \iff X < \mu + \sigma z_{0.25} = 189.88.$$

So the "smallest 25% drinks" are below 189.88 ml.

7. A lawyer commutes daily from his suburban home to his midtown office. The average time for a one-way trip is 24 minutes, with a standard deviation of 3.8 minutes. Assume the distribution of trip times to be normally distributed.

(a) What is the probability that a trip will take at least half an hour?

(b) If the office opens at 9.00 a.m. and he leaves his house at 8.45 a.m. daily, what percentage of the time is he late for work?

(c) Find the probability that 2 of the next 3 trips will take at least 1/2 hour.

SOLUTION
Let X = commute time from home to office of a single trip; then $X \sim N(\mu = 24, \sigma^2 = 3.8^2)$.

(a) $P(X > 30) = P(Z > 1.58) = 1 - \Phi(1.58) = 0.0571$.

(b) $P(X > 15) = P(Z > -2.37) = 1 - \Phi(-2.37) = 0.9911 = 99.11\%$.

(c) Let Y = number of trips out of 3 that will take at least half an hour; then $Y \sim B(n = 3, p = 0.0571)$.

$$P(Y = 2) = \binom{3}{2}(0.0571)^2(1 - 0.0571)^1 = 0.00922.$$

8. A coin is tossed 400 times, Use the normal approximation to find the probability of obtaining

(a) between 185 and 210 heads inclusive;

(b) exactly 205 heads;

(c) less than 176 or more than 227 heads.

SOLUTION
Let Y = number of heads in 400 tosses of a coin; then $Y \sim B(n = 400, p = 0.5)$. $E(Y) = np = 400(0.5) = 200$; $V(Y) = np(1-p) = 400(0.5)(0.5) = 100$. $Y \approx N(\mu = 200, \sigma^2 = 10^2)$.

(a) $P(185 \le Y \le 210) = P(184.5 < Y < 210.5) = P(-1.55 < Z < 1.05) = \Phi(1.05) + \Phi(1.55) - 1 = 0.7925$;

(b) $P(Y = 205) = P(204.5 < Y < 205.5) = P(0.45 < Z < 0.55) = \Phi(0.55) - \Phi(0.45) = 0.352$;

(c) $P(Y < 176 \text{ or } Y > 227) = P(Y < 175.5) + P(Y > 227.5) = P(Z < -2.45) + P(Z > 2.75) = 2 - \Phi(2.45) - \Phi(2.75) = 0.01012$.

9. Physicians recommend that children with type-I (insulin dependent) diabetes keep up with their insulin shots to minimize the chance of long-term complications. In addition, some diabetes researchers have observed that growth rate of weight during adolescence among diabetic patients is affected by level of compliance with insulin therapy.

Suppose 12-year-old type-I diabetic boys who comply with their insulin shots have a weight gain over 1 year that is normally distributed, with mean 12 lb and variance 12 lb.

Conversely, 12-year-old type-I diabetic boys who do not take their insulin shots have a weight gain over 1 year that is normally distributed with mean 8 lb and variance 12 lb.

It is generally assumed that 75% of type-I diabetics comply with their insulin regimen. Suppose that a 12-year-old type-I diabetic boy comes to clinic and shows a 5-lb weight gain over 1 year (actually, because of measurement error, assume this is an actual weight gain from 4.5 to 5.5 lb). The boy claims to be taking his insulin medication. What is the probability that he is telling the truth?

SOLUTION

Let C denote compliant, and D denote a weight gain of between 4.5 to 5.5 lb over 1 year. Then $P(C) = 0.75$ and what we want is $P(C|D)$.

Note that

$$\begin{aligned} P(D|C) &= P(4.5 < X < 5.5), \text{ where } X \sim N(12, 12) \\ &= P\left(\frac{4.5-12}{\sqrt{12}} < X < \frac{5.5-12}{\sqrt{12}}\right) \\ &= P(-2.165 < Z < -1.876) \\ &= 0.015, \end{aligned}$$

and

$$\begin{aligned} P(D|C^c) &= P(4.5 < Y < 5.5), \text{ where } Y \sim N(8, 12) \\ &= P\left(\frac{4.5-8}{\sqrt{12}} < Y < \frac{5.5-8}{\sqrt{12}}\right) \\ &= P(-1.010 < Z < -0.722) \\ &= 0.079. \end{aligned}$$

By Baye's Theorem,

$$\begin{aligned} P(C|D) &= \frac{P(D|C)P(C)}{P(D|C)P(C) + P(D|C^c)P(C^c)} \\ &= \frac{0.015 \times 0.75}{0.015 \times 0.75 + 0.079 \times 0.25} \\ &= 0.364. \end{aligned}$$