Chapter 7

Hypotheses Testing based on Normal Distribution

Overview

- Hypotheses testing based on Normal distribution
- Types I and II Error
- Level of significance
- Hypotheses testing concerning mean
- Critical value approach and p-value approach
- Hypotheses testing concerning difference between two means
- Hypotheses testing concerning variances

7.1 Null and Alternative Hypotheses

7.1.1 Statistical Hypothesis

- A **statistical hypothesis** is an assertion or conjecture concerning one or more populations.
- We shall use the terms accept and reject frequently throughout this chapter.

Null and Alternative Hypotheses (Continued)

7.1.1 Statistical Hypothesis (Continued)

- It is important to understand that the rejection of a hypothesis is to conclude that it is false, while the acceptance of a hypothesis merely implies that we have insufficient evidence to believe otherwise.
- Because of this terminology, the statistician or experimenter will often choose to state the hypothesis in a form that hopefully will be rejected.

Null and Alternative Hypotheses (Continued)

Null hypothesis:

- Hypothesis that we formulate with the hope of rejecting, denoted by H_0.
- A null hypothesis concerning a population parameter will always be stated to specify an exact value of the parameter.

Alternative hypothesis:

- The rejection of H_0 leads to the acceptance of an alternative hypothesis, denoted by H_1.
- It allows for the possibility of several values.

Example 1

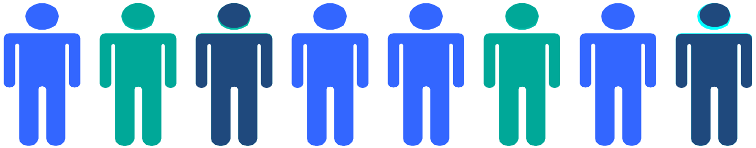
- We may wish to determine whether the mean IQ of the pupils of a certain school is different from 100.
- Then we have $\mathrm{H}_0: \mu = 100$ against $\mathrm{H}_1: \mu \neq 100$.
- This is called a two-sided alternative. The test used is called a two-sided (or two-tailed) test
- We may like to test whether the mean IQ of the pupils is greater than 100 (or less than 100). This is called a one-sided alternative. The test is called a one-tailed (or one-sided) test.
- That is,

  $\mathrm{H}_0: \mu = 100$ against $\mathrm{H}_1: \mu > 100$ or

  $\mathrm{H}_0: \mu = 100$ against $\mathrm{H}_1: \mu < 100$.

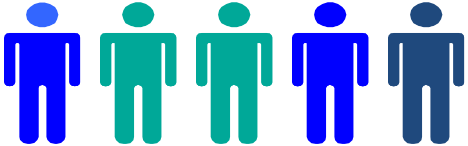
Example 1

Claim: **the population mean IQ is 100.**
(Null hypothesis: $H_0 : \mu = 100$)



Population

Now select a random sample



Sample

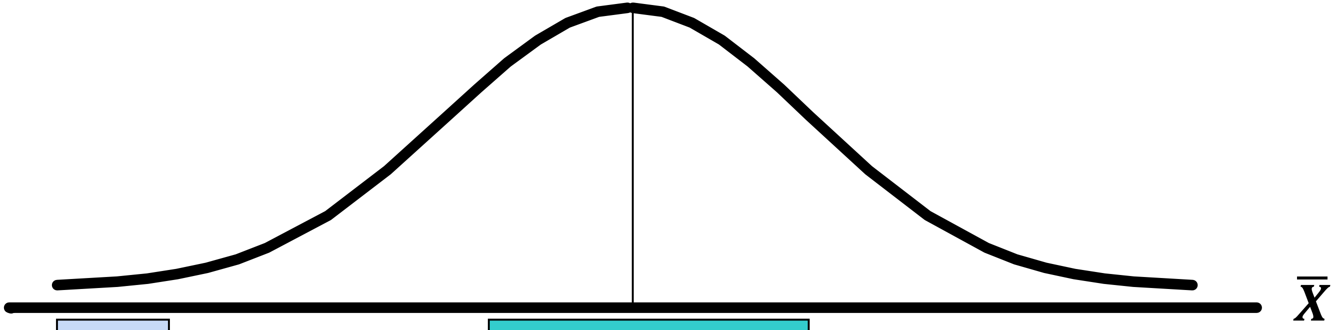
Is $\bar{X} = 70$ likely if $\mu = 100$?

Suppose the sample mean IQ is 70:
$\bar{X} = 70$

If not likely,
REJECT
Null Hypothesis

Reason for Rejecting H_0

Sampling Distribution of $\bar{X}$



70

$\mu = 100$

If H_0 is true

If it is unlikely that we would get a sample mean of this value ...

... if in fact this were the population mean...

... then we reject the null hypothesis that $\mu = 100$.

7.1.2 Types of Error

- Two types of errors in the hypothesis testing:

	State of Nature	
Decision	H_0 is true	H_0 is false
Reject H_0	**Type I error** Pr(Reject H_0 given that H_0 is true) = α	Correct decision Pr(Reject H_0 given that H_0 is false) = $1 - \beta$
Do not reject H_0	Correct decision Pr(Do not reject H_0 given that H_0 is true) = $1 - \alpha$	**Type II error** Pr(Do not reject H_0 given that H_0 is false) = β

Types I and II Error

Type I error

- Rejection of H_0 when H_0 is true is called a type I error.
- It is considered as a serious type of error

Type II Error

- Not rejecting H_0 when H_0 is false is called a type II error.

Types I and II Error (Continued)

- α = level of significance

  = Pr(type I error)

  = Pr(reject H_0 when H_0 is true)

  = Pr(reject H_0 | H_0).
- α is set by the researcher in advance
- α is usually set at 5% or 1%

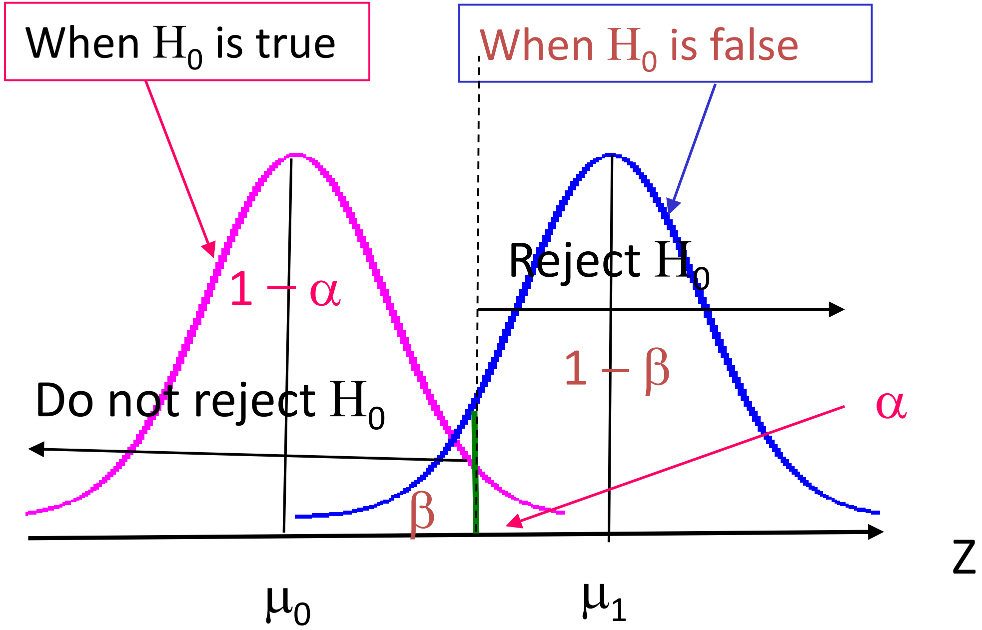
Types I and II Error (Continued)

- β = Pr(type II error)
  = Pr(do not reject H_0 when H_0 is false)
  = Pr(do not reject $H_0 \mid H_1$).

- $1 - \beta$ = Power of a test = Pr(reject $H_0 \mid H_1$)

Types I and II Error (Continued)

Test $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$



7.1.3 Acceptance and Rejection Regions

- To test a hypothesis about a population parameter, we first select a suitable test statistic for the parameter under the hypothesis.
- Once the significance level, α, is given, a decision rule can be found such that it divides the set of all possible values of the test statistic into two regions,
- one being the **rejection region** (or critical region) and the other the **acceptance region**.

Acceptance and Rejection Regions (Continued)

- Once a sample is taken, the value of the test statistic is obtained.
- If the test statistic assumes a value in the rejection region, the null hypothesis is rejected; otherwise it is not rejected.
- The value that separates the rejection and acceptance regions is called the **critical value**.

Level of Significance and the Rejection Region

Level of significance = α

$H_0: \mu = 3$

$H_1: \mu \neq 3$

Two-tail test

$H_0: \mu = 3$

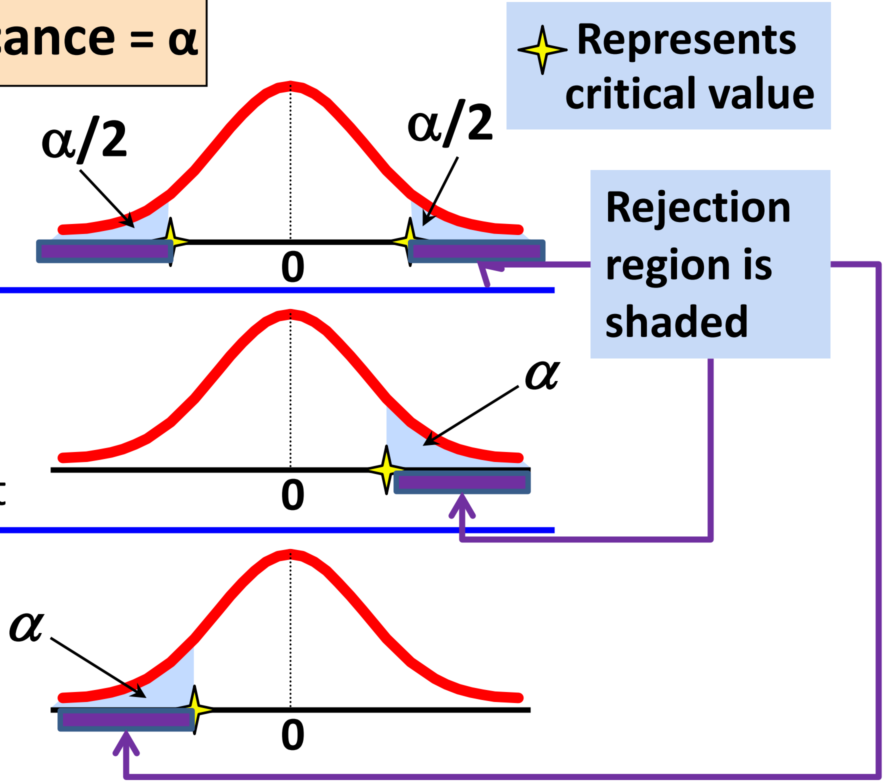
$H_1: \mu > 3$

Upper-tail test

$H_0: \mu = 3$

$H_1: \mu < 3$

Lower-tail test



Example 1

- A certain type of cold vaccine is known to be only 25% effective after a period of 2 years.
- In order to determine if a new and somewhat more expensive vaccine is superior in providing protection against the same virus for a longer period of time.
- 20 people are chosen at random and inoculated with the new vaccine.
- If more than 8 of those receiving the new vaccine surpass the 2-year period without contracting the virus, the new vaccine will be considered superior to the one presently in use.

Example 1 (Continued)

- This is equivalent to testing the hypothesis that the binomial parameter for the probability of a success on a given trial is $p = 1/4$ against the alternative that $p > 1/4$. Or
- $\mathrm{H}_0: p = 1/4$ against $\mathrm{H}_1: p > 1/4$.

$$X \quad \underbrace{0\ 1\ 2\ 3\ 4 \cdots 7\ 8}_{\text{Acceptance Region}}\ \underbrace{9\ 10\ 11 \cdots 19\ 20}_{\text{Rejection Region}}$$

where X is the number of individuals who remain free of the virus for at least 2 years

Example 1 (Continued)

- The above decision rule has the level of significance given by

$$
\begin{aligned}
\alpha &= \Pr(\text{Type I error}) \\
&= \Pr(\text{Reject } H_0 \mid H_0) \\
&= \Pr(X > 8 \text{ when } p = 1/4) \\
&= \sum_{i=9}^{20} \binom{20}{i} \left(\frac{1}{4}\right)^{i} \left(\frac{3}{4}\right)^{20-i} \\
&= 0.0409.
\end{aligned}
$$

Example 1 (Continued)

- The probability of committing a type II error, denoted by β, is impossible to compute unless we have a specific alternative hypothesis.
- Consider testing

  H_0: $p = 1/4$ against H_1: $p = 1/2$ (Note $1/2 > 1/4$).

Example 1 (Continued)

- Then

$$\beta = \Pr(\text{Type II error}) = \Pr(\text{Accept } H_0 \mid H_1)$$

$$= \Pr(X \leq 8 \text{ when } p = 1/2)$$

$$= \sum_{i=0}^{8} \binom{20}{i} \left(\frac{1}{2}\right)^{i} \left(\frac{1}{2}\right)^{20-i} = 1 - \sum_{i=9}^{20} \binom{20}{i} \left(\frac{1}{2}\right)^{20}$$

$$= 1 - 0.7483 = 0.2517.$$

7.2 Hypotheses Testing Concerning Mean

7.2.1 Hypo. Testing on Mean with Known Variance

Consider the problem of testing the hypothesis concerning the mean, μ, of a population with

1. **Variance, $\boldsymbol{\sigma^2}$, known and**
2. **Underlying distribution is normal or $\boldsymbol{n}$ is sufficiently large (say $\boldsymbol{n > 30}$)**

   Refer to Section 6.3.1

7.2.1.1 Two-sided Test

- Test $\mathrm{H}_0: \mu = \mu_0$ against $\mathrm{H}_1: \mu \neq \mu_0$.
- When the population is normal or the sample size is large (then by the Central Limit Theorem), we can expect that

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

- Hence under $\mathrm{H}_0: \mu = \mu_0$, we have

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right).$$

Two-sided Test (Continued)

Critical Value Approach

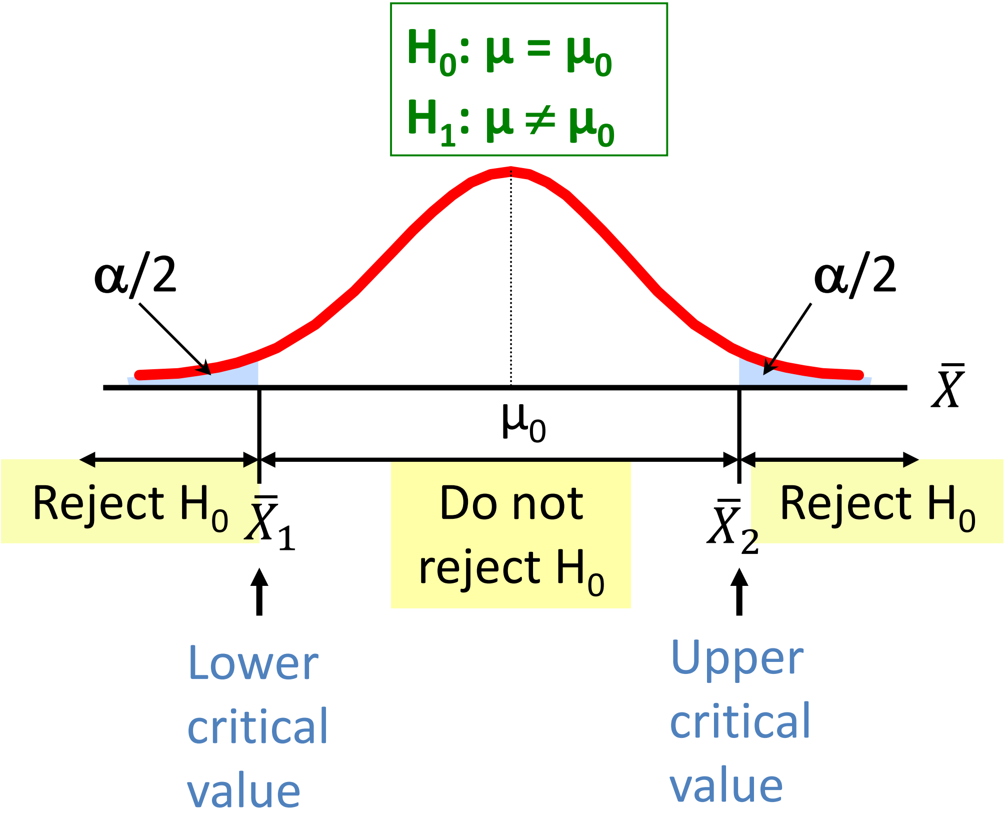
- By using a significance level of α, it is possible to find two critical values $\bar{x}_1$ and $\bar{x}_2$ such that

- the interval $\bar{x}_1 < \bar{X} < \bar{x}_2$ defines the acceptance region and

- the two tails of the distribution, $\bar{X} < \bar{x}_1$ and $\bar{X} > \bar{x}_2$ constitute the critical (or rejection) region.

Two-sided Test (Continued)

- There are two cutoff values (critical values), defining the regions of rejection



Finding critical values

The critical region can be given in terms of z values by means of the transformation

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

Note: μ_0 is the value of μ under H_0.

Finding critical values (Continued)

Therefore

$$\Pr\left(-z_{\alpha/2} < \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1-\alpha$$

or

$$\Pr\left(\mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha.$$

Hence $\bar{x}_1 = \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and $\bar{x}_2 = \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

Hypothesis testing process

- From the population we select a random sample of size n and compute the sample mean.
- If $\bar{X}$ falls in the acceptance region $\bar{x}_1 < \bar{X} < \bar{x}_2$, we conclude that $\mu = \mu_0$; otherwise we reject H_0 and accept the H_1: $\mu \neq \mu_0$.
- Since $Z = (\bar{X} - \mu_0)/(\sigma/\sqrt{n})$, therefore $\bar{x}_1 < \bar{X} < \bar{x}_2$ is equivalent to $-z_{\alpha/2} < Z < z_{\alpha/2}$.
- The critical region is usually stated in terms of Z rather then $\bar{X}$.

Example 1

- The director of a factory wants to determine if a new machine A is producing cloths with a breaking strength of 35 kg with a standard deviation of 1.5 kg.
- A random sample of 49 pieces of cloths is tested and found to have a mean breaking strength of 34.5 kg.
- Is there evidence that the machine is not meeting the specifications for mean breaking strength?
- Use $\alpha = 0.05$

Solution to Example 1

Step 1

- Let μ be the mean breaking strength of cloths manufactured by the new machine.
- Test $\mathrm{H}_0: \mu = 35 \text{ kg}$ vs $\mathrm{H}_1: \mu \neq 35 \text{ kg}$. (Why?)

Step 2

- Set $\alpha = 0.05$.

Solution to Example 1 (Continued)

Step 3

- Since σ is known, the test statistic

$$Z = \frac{(\bar{X} - \mu_0)}{\sigma/\sqrt{n}}$$

  is used.
- $z_{\alpha/2} = z_{0.025} = 1.96.$
- Critical region $z < -1.96$ or $z > 1.96$, where

$$z = \frac{(\bar{x} - \mu_0)}{\sigma/\sqrt{n}}$$

Solution to Example 1

Step 4

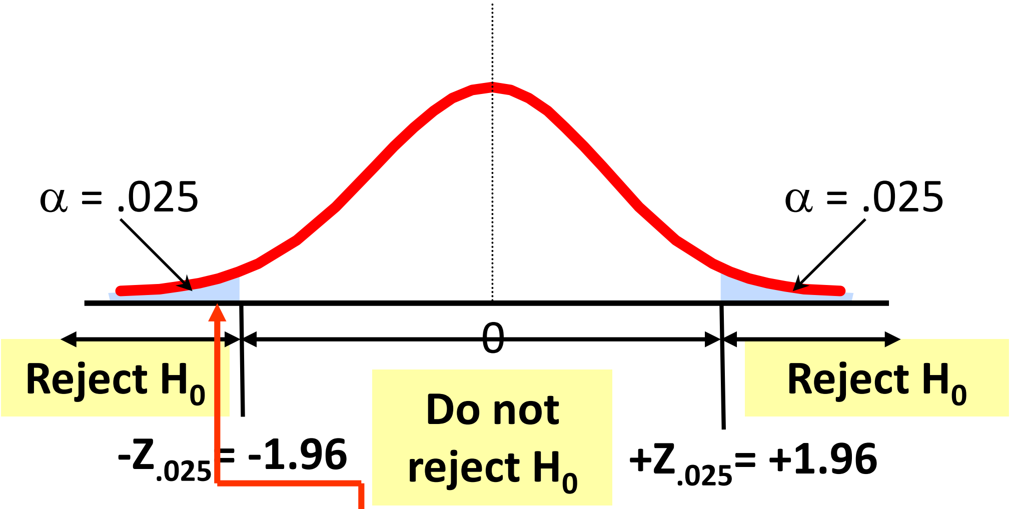
- Computations: $\bar{x} = 34.5$ kg, $n = 49$, and hence

$$z = \frac{34.5 - 35}{1.5/\sqrt{49}} = -2.3333$$

Step 5

- Conclusion: Since the observed z value = –2.3333 falls inside the critical region (i.e. $z < z_{0.025} = -1.96$), hence H_0: $\mu = 35$ kg is rejected at the 5% level of significance.

Solution to Example 1 (Continued)



Reject H_0 if $Z < -1.96$ or $Z > 1.96$; otherwise do not reject H_0

Here, $Z = -2.3333 < -1.96$, so the test statistic is in the rejection region

Relationship between two-sided test and confidence interval

- The two-sided test procedure just described is equivalent to finding a $(1-\alpha)100\%$ confidence interval for μ
- H_0 is accepted if the confidence interval covers μ_0.
- If the C.I. does not cover μ_0, we reject $\mu = \mu_0$ in favour of the alternative H_1: $\mu \neq \mu_0$ since

$$\Pr\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$\Leftrightarrow \quad \Pr\left(\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha.$$

Example 1 (Continued)

- For $\bar{x} = 34.5$, $\sigma = 1.5$ and $n = 49$, the 95% confidence interval is:

$$34.5 - (1.96)\frac{1.5}{\sqrt{49}} < \mu < 34.5 + (1.96)\frac{1.5}{\sqrt{49}}$$

$$34.08 \leq \mu \leq 34.92$$

- Since this interval does not contain the hypothesized mean, μ_0 (= 35), we reject the null hypothesis at $\alpha = 0.05$.

p-Value Approach to Testing

- p-value: Probability of obtaining a test statistic more extreme ($\leq$ or $\geq$) than the observed sample value given H_0 is true
  - Also called **observed level of significance**

p-Value Approach to Testing (Continued)

- Convert a sample statistic (e.g., $\bar{X}$) to a test statistic (e.g., Z statistic)
- Obtain the *p*-value
- Compare the *p*-value with $\boldsymbol{\alpha}$

$$\boxed{\begin{array}{l} -\text{ If } p\text{-value} < \alpha\text{, reject } H_0 \\ -\text{ If } p\text{-value} \geq \alpha\text{, do not reject } H_0 \end{array}}$$

Example 1 (Continued)

- How likely is it to see a sample mean of 34.5 (or something further from the mean, in either direction) if the true mean is 35? ($\sigma = 1.5$ and $n = 49$)

$\bar{X}$ = 34.5 is translated to a Z score of Z = -2.33

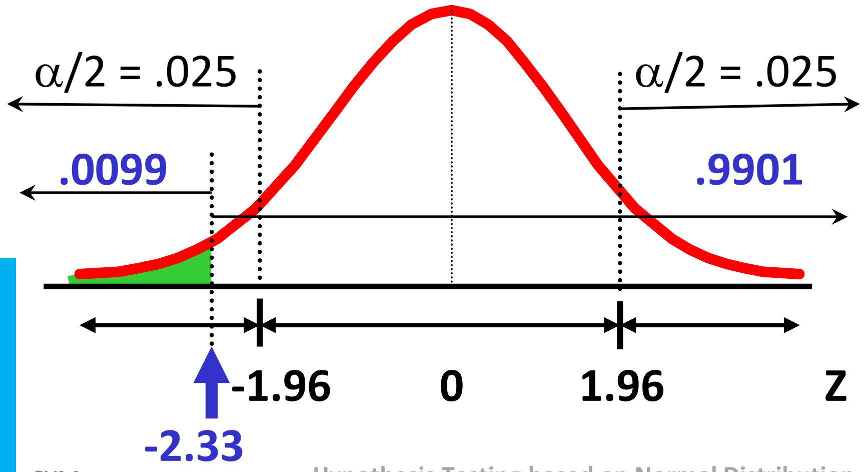
Pr(Z < -2.33) = 0.0099

Pr(Z > -2.33) = 0.9901

***p*-value**

= 2 min{Pr(Z < -2.33), Pr(Z >-2.33) }

= 2(0.0099) = .0198



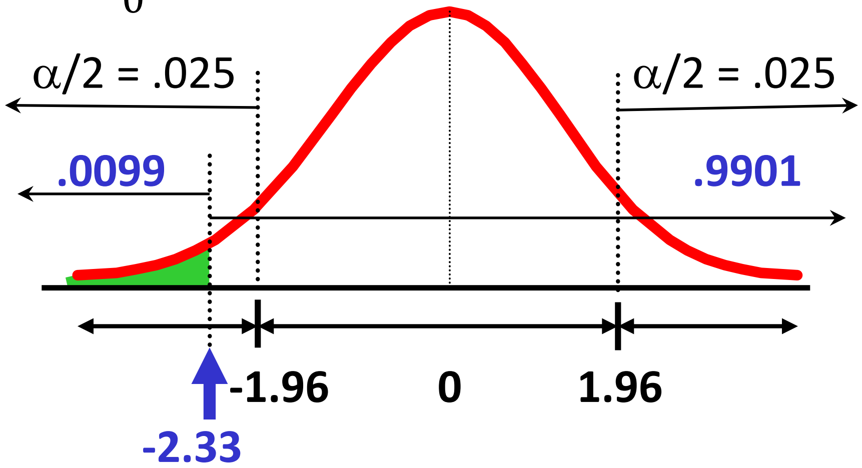
p-Value Approach to Testing (Continued)

- Compare the *p*-value with α
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0

Here: *p*-value = .0198

α = .05

Since .0198 < .05, we reject the null hypothesis

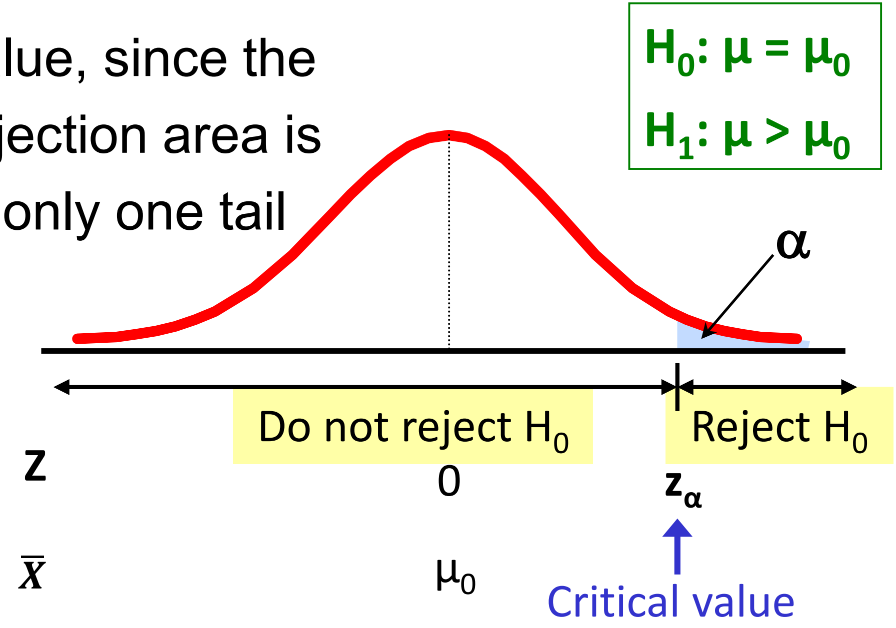


7.2.1.2 One sided test

(a) Test $H_0: \mu = \mu_0$ against $H_1: \mu > \mu_0$.

- Let $Z = \frac{(\bar{X}-\mu_0)}{\sigma/\sqrt{n}}$.
- Then H_0 is rejected if the observed values of Z, say z, is greater than z_α.

There is only one critical value, since the rejection area is in only one tail

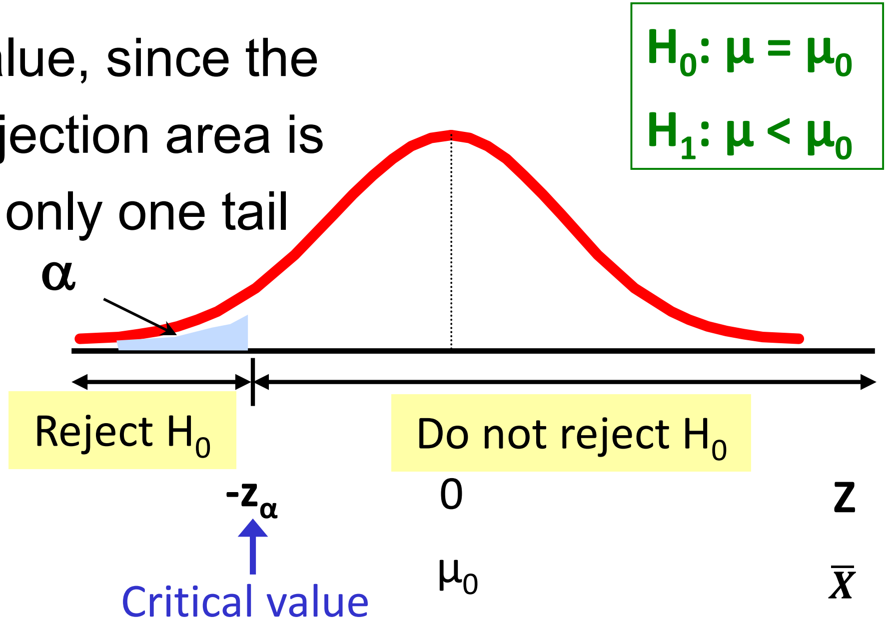


7.2.1.2 One sided test

(b) Test H_0: $\mu = \mu_0$ against H_1: $\mu < \mu_0$.

- Let $Z = \frac{(\bar{X}-\mu_0)}{\sigma/\sqrt{n}}$.
- Then H_0 is rejected if the observed values of Z, say z, is less than $-z_\alpha$.

There is only one critical value, since the rejection area is in only one tail



Example 2

- A manufacturer of sports equipment has developed a new synthetic fishing line that he claims has a mean breaking strength better than the market average strength of 8 kilograms.
- Suppose the breaking strength of this type of fishing lines has a standard deviation of 0.5 kg.
- A random sample of 50 lines is tested and found to have a mean breaking strength of 8.2 kg.
- Test the manufacturer's claim.
- Use a 0.01 level of significance.

Solution to Example 2

Step 1

- Let μ be the mean breaking strength of the new type of fishing lines.
- Test $\mathrm{H}_0: \mu = 8$ against $\mathrm{H}_1: \mu > 8$. (Why?)

Step 2

- Set $\alpha = 0.01$.

Solution to Example 2 (Continued)

Step 3

- Since σ is known, the test statistic
$$Z = \frac{(\bar{X} - 8)}{0.5/\sqrt{50}}$$
is used.
- $z_\alpha = z_{0.01} = 2.326.$
- Critical region $z > 2.326$, where
$$z = \frac{(\bar{x} - 8)}{0.5/\sqrt{50}}$$

Solution to Example 2 (Continued)

Step 4

- Computations: $\bar{x} = 8.2$, hence

$$z = \frac{8.2 - 8}{0.5/\sqrt{50}} = 2.828.$$

- p-value $= \Pr(Z > 2.828) \approx 0.00233.$

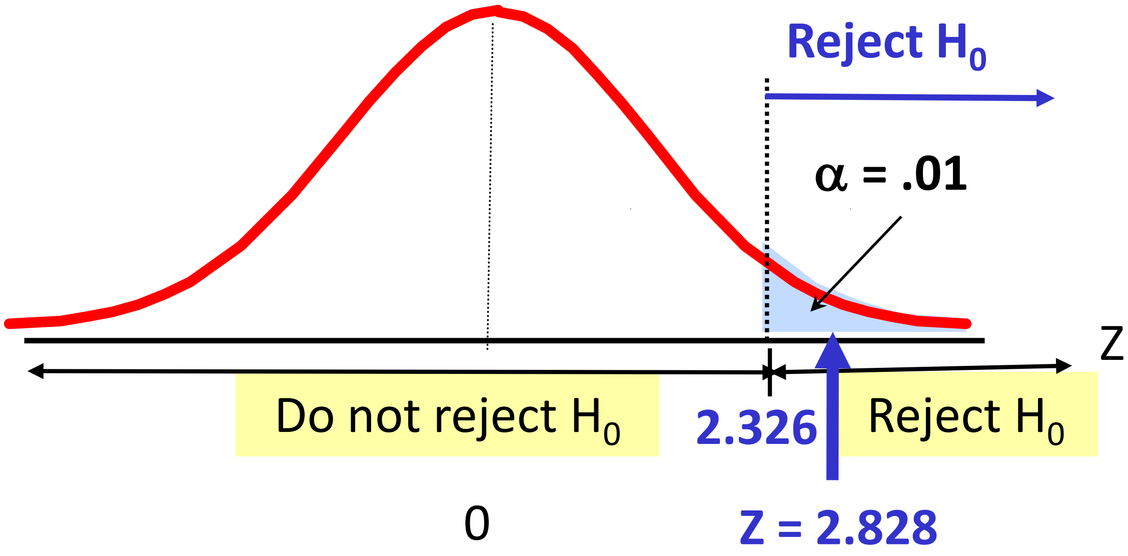
Solution to Example 2 (Continued)

Step 5

- Conclusion: Since the observed z value $= 2.828$ falls inside the critical region (i.e. $z > z_{0.01} = 2.326$), hence H_0: $\mu = 8$ kg is rejected at the 1% level of significance.
- Conclusion based on p-value: Since p-value ≈ 0.00233 is less than 0.01, hence H_0 is rejected at the 1% level of significance.

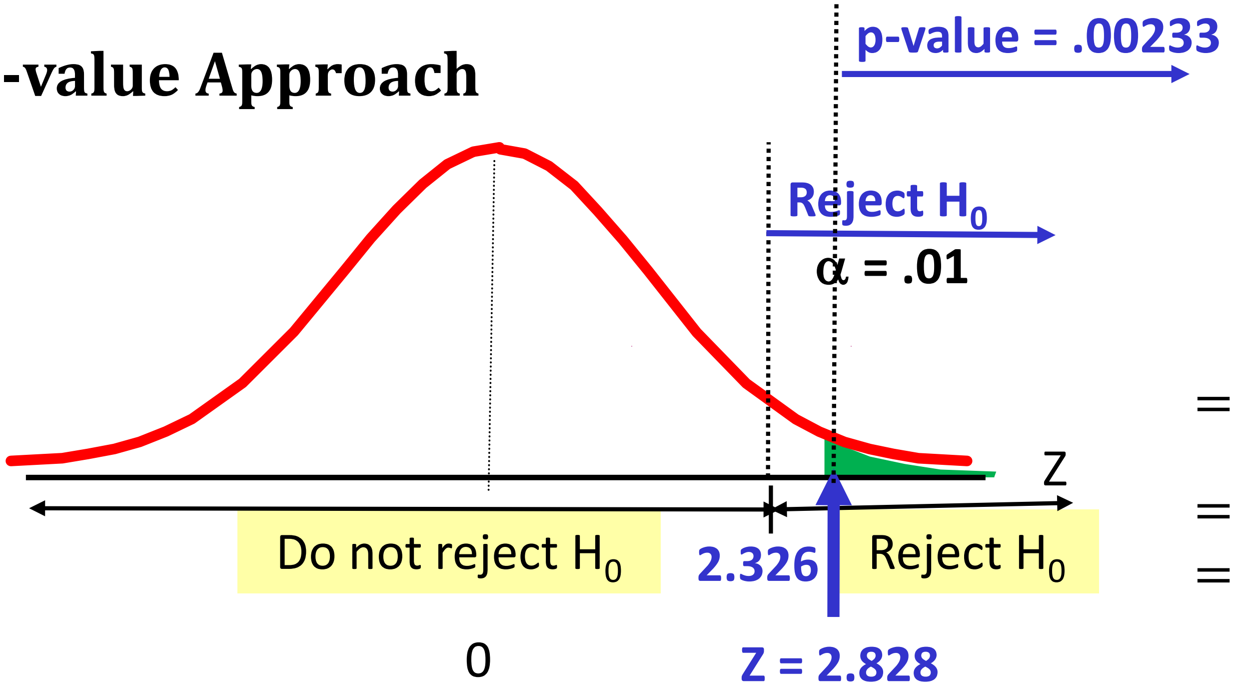
Solution to Example 2 (Continued)

Critical Value Approach



Solution to Example 2 (Continued)

p-value Approach



$$\Pr(\bar{X} > 8.2)$$
$$= \Pr\left(Z \geq \frac{8.2 - 8}{0.5/\sqrt{50}}\right)$$
$$= \Pr(Z \geq 2.828)$$
$$= 0.00233$$

Reject H_0 since *p*-value = .00233 < α = .01

7.2.2 Hypothesis Testing on Mean with Variance Unknown

Consider the problem of testing the hypothesis concerning the mean, μ, of a population with

1. **Variance unknown and**
2. **Underlying distribution is normal**

   Refer to Section 6.3.2

Test for mean with unknown variance

(1) Two sided test

- Test $\mathrm{H}_0: \mu = \mu_0$ against $\mathrm{H}_1: \mu \neq \mu_0$.
- Let

$$T = \frac{(\bar{X} - \mu_0)}{S/\sqrt{n}}$$

where S^2 is the sample variance.

- Then H_0 is rejected if the observed value of T, say t, $> t_{n-1;\alpha/2}$ or $< -t_{n-1;\alpha/2}$.

Test for mean with unknown variance (Continued)

(2) One sided test

- Test $H_0: \mu = \mu_0$ against $H_1: \mu > \mu_0$.
- Then H_0 is rejected if $t > t_{n-1;\alpha}$.
- Test $H_0: \mu = \mu_0$ against $H_1: \mu < \mu_0$.
- Then H_0 is rejected if $t < -t_{n-1;\alpha}$.

Example 3

- The average length of time for students to register for summer classes at a certain college has been 50 minutes.
- A new registration procedure is being tried.
- A random sample of 12 students had an average registration time of 42 minutes with a standard deviation of 11.9 minutes under the new system.
- Test the hypothesis that the population mean is now less than 50, using a level of significance of 0.05.
- Assume the population of times to be normal.

Solution to Example 3

Step 1

- Let μ be the mean registration time.
- Test $\mathrm{H}_0: \mu = 50$ against $\mathrm{H}_1: \mu < 50$. (Why?)

Step 2

- Set $\alpha = 0.05$.

Solution to Example 3 (Continued)

Step 3

- Since σ is unknown, the test statistic

$$T = \frac{(\bar{X} - \mu_0)}{S/\sqrt{n}}$$

  is used.
- $n = 12$ implies that $t_{11;0.05} = 1.796$
- **Critical region** $t < -1.796$, where

$$t = \frac{(\bar{x} - \mu_0)}{s/\sqrt{n}}$$

Solution to Example 3 (Continued)

Step 4

- **Computations:** $\bar{x} = 42, s = 11.9, n = 12$, and hence

$$t = \frac{42 - 50}{11.9/\sqrt{12}} = -2.329$$

- p-value $= \Pr(T < -2.329) = 0.0199$.

  [or p-value is between 0.025 and 0.01 since 2.329 is between $t_{11;0.025} = 2.201$ and $t_{11;0.01} = 2.718$ if statistical table is used.]

Solution to Example 2 (Continued)

Step 5

- **Conclusion:** Since the observed $t = -2.329$ falls inside the critical region (i.e. $t < t_{0.05} = -1.796$), hence H_0: $\mu = 50$ minutes is rejected at the 5% level of significance and we conclude that the true mean is likely to be less than 50 minutes.
- **Conclusion based on *p*-value:** Since *p*-value = 0.0199 is less than 0.05, hence H_0 is rejected at the 5% level of significance and we conclude that the true mean is likely to be less than 50 minutes.

7.3 Hypotheses Testing Concerning Difference Between Two Means

7.3.1 Known Variances

1. **Variances σ_1^2 and σ_2^2 are known and**
2. **Underlying distribution is normal or both n_1 and n_2 are sufficiently large**

   (say $n_1 \geq 30, n_2 \geq 30$)

   Refer to Section 6.4.1

Example 1

- Analysis of a random sample consisting of $n_1 = 20$ specimens of cold-rolled steel to determine yield strengths resulted in a sample average strength of $\bar{x}_1 = 29.8$ ksi.
- A second random sample of $n_2 = 25$ two-side galvanized steel specimens gave a sample average strength of $\overline{x}_2 = 34.7$ ksi.
- Assuming that the two yield strength distributions are normal with $\sigma_1 = 4.0$ and $\sigma_2 = 5.0$,
- does the data indicate that the corresponding true average yield strengths μ_1 and μ_2 are different?
- Use $\alpha = 0.01$.

Solution to Example 1

Step 1

- Let μ_1 and μ_2 be the mean strength of cold-rolled steel and two-side galvanized steel respectively.
- Test $\mathrm{H}_0: \mu_1 - \mu_2 = 0$ against $\mathrm{H}_1: \mu_1 - \mu_2 \neq 0$.

Step 2

- Set $\alpha = 0.01$.

Solution to Example 1 (Continued)

Step 3

- Since σ_1^2 and σ_2^2 are known, therefore the test statistic

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

  is used.
- $\alpha = 0.01$ implies $z_{\alpha/2} = z_{0.005} = 2.5728$.

Solution to Example 1 (Continued)

Step 3 (Continued)

- Critical region: $z < -2.5758$ or $z > 2.5758$, where

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

Solution to Example 1 (Continued)

Step 4

- **Computations:** $\bar{x}_1 = 29.8, \bar{x}_2 = 34.7, \sigma_1^2 = 16, \sigma_2^2 = 25,$ $n_1 = 20$ and $n_2 = 25$, so

$$z = \frac{[(29.8 - 34.7) - 0]}{\sqrt{16/20 + 25/25}} = -3.652.$$

- ***p*-value** $= 2 \times \min\{\Pr(Z > -3.652), \Pr(Z < -3.652)\} = 2(0.00013) = 0.00026.$

Solution to Example 1 (Continued)

Step 5

- Conclusion: Since $z = -3.652$ falls inside the critical region, hence H_0: $\mu_1 = \mu_2$ is rejected at the 1% level of significance and conclude that the sample data strongly suggest that the true average yield strength for cold-rolled steel differs from that for galvanized steel.
- Conclusion based on p-value: Since p-value = 0.00026 is less than the level of significance 0.01, hence H_0 is rejected at the 1% level of significance.

7.3.2 Large Sample Testing with Unknown Variances

1. **Variances $\boldsymbol{\sigma_1^2}$ and $\boldsymbol{\sigma_2^2}$ are unknown and**
2. **both $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ are sufficiently large (say $\boldsymbol{n_1 \geq 30, n_2 \geq 30}$)**

   Refer to Section 6.4.2

Example 2

- In selecting a sulfur concrete for roadway construction in regions that experience heavy frost,
- it is important that the chosen concrete have a low value of thermal conductivity in order to minimize subsequent damage due to changing temperatures.
- Suppose two types of concrete, a graded aggregate and a no-fines aggregate, are being considered for a certain road.

Example 2 (Continued)

- The following table summarizes data from an experiment carried out to compare the two types of concrete.

Type	Sample size	Sample average conductivity	Sample s.d.
Graded	35	0.497	0.187
No-fines	35	0.359	0.158

- Does this information suggest that the true conductivity for the graded concrete exceeds that for the no-fines concrete?
- Use $\alpha = 0.01$.

Solution to Example 2

Step 1

- Let μ_1 and μ_2 be the mean conductivity of graded and no-fines concretes respectively.
- Test $\mathrm{H}_0: \mu_1 - \mu_2 = 0$ against $\mathrm{H}_1: \mu_1 - \mu_2 > 0$.

Step 2

- Set $\alpha = 0.01$.

Solution to Example 2 (Continued)

Step 3

- Since σ_1^2 and σ_2^2 are unknown and the sample sizes are large, therefore the test statistic

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}$$

  is used.
- $\alpha = 0.01$ implies $z_\alpha = z_{0.01} = 2.3263$.

Solution to Example 2 (Continued)

Step 3 (Continued)

- **Critical region:** $z > 2.3263$, where

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

Solution to Example 2 (Continued)

Step 4

- **Computations:** $\bar{x}_1 = 0.497, \bar{x}_2 = 0.359, s_1^2 = 0.187^2, s_2^2 = 0.158^2, n_1 = n_2 = 35$, so

$$z = \frac{[(0.497 - 0.359) - 0]}{\sqrt{0.187^2/35 + 0.158^2/35}} = 3.335.$$

- p-value $= \Pr(Z > 3.335) = 0.00043.$

Solution to Example 2 (Continued)

Step 5

- **Conclusion:** Since $z = 3.335$ falls inside the critical region, hence H_0: $\mu_1 = \mu_2$ is rejected at the 1% level of significance and conclude that the sample data argue strongly that the true average thermal conductivity for the graded concrete does exceed that for the no-fines concrete.
- **Conclusion based on *p*-value**: Since p-value = 0.00043 is less than the level of significance 0.01, hence H_0 is rejected at the 1% level of significance.

7.3.3 Unknown but Equal Variances

1. $\boldsymbol{\sigma_1^2}$ **and** $\boldsymbol{\sigma_2^2}$ **are unknown but equal and**
2. **the populations are normal**
3. **Small sample sizes (say** $\boldsymbol{n_1 \leq 30, n_2 \leq 30}$**)**

   Refer to Section 6.4.3

Example 3

- A course in mathematics is taught to 12 students by the conventional classroom procedure.
- A second group of 10 students was given the same course by means of programmed materials.
- At the end of the semester the same examination was given to each group.
- The 12 students meeting in the classroom made an average grade of 85 with a standard deviation of 4,

Example 3 (Continued)

- while the 10 students using programmed materials made an average of 81 with a standard deviation of 5.
- Test the hypothesis that the two methods of learning are equal using a 0.10 level of significance.
- Assume the populations to be approximately normal with equal variances.

Solution to Example 3

Step 1

- Let μ_1 and μ_2 be the average grades students taking this course by the classroom and programmed presentations, respectively.
- Test $\mathrm{H}_0\colon \mu_1 - \mu_2 = 0$ against $\mathrm{H}_1\colon \mu_1 - \mu_2 \neq 0$.

Step 2

- Set $\alpha = 0.1$.

Solution to Example 3 (Continued)

Step 3

- $n_1 = 12$ and $n_2 = 10$ implies $t_{n_1+n_2-2;\alpha} = t_{20;0.05} = 1.725$.
- **Critical region :** $t < -1.725$ or $t > 1.725$, where

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}},$$

with

$$S_p^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2].$$

Solution to Example 3 (Continued)

Step 4

- **Computations:**

$\bar{x}_1 = 85$ and $\bar{x}_2 = 81$,

$s_1^2 = 16, s_2^2 = 25$,

$n_1 = 12$ and $n_2 = 10$, so

$$s_p^2 = \frac{[11(16) + 9(25)]}{(12 + 10 - 2)} = 20.05$$

and $s_p = 4.478$.

Solution to Example 3 (Continued)

Step 4 (Continued)

- Hence

$$t = \frac{[(85 - 81) - 0]}{\sqrt{20.05(1/12 + 1/10)}} = 2.086$$

- p-value $= 2 \times \min\{\Pr(T_{20} > 2.086)\,, \Pr(T_{20} < 2.806)\}$

$$= \; 2(0.025) \; = \; 0.05.$$

Solution to Example 2 (Continued)

Step 5

- **Conclusion**: Since the observed t-value = 2.086 which falls inside the critical region, hence H_0: $\mu_1 = \mu_2$ is rejected at the 10% level of significance

  and conclude that the two methods of learning are not equal.
- Since p-value = 0.05 is less than 0.10, therefore we reject H_0 at the 10% level of significance and conclude that the two methods of learning are not equal.

7.3.4 Paired Data

Refer to Section 6.4.4

Example 4

- We wish to compare two methods for determining the percentage of iron ore in ore samples.
- Each of 12 ore samples was split into two parts.
- One-half of each sample was randomly selected and subjected to Method 1;
- The other half was subject to Method 2.
- The results are given in next slide.

Example 4 (Continued)

Sample	1	2	3	4	5	6
Method 1	38.25	31.68	26.24	41.29	44.81	46.37
Method 2	38.27	31.71	26.22	41.33	44.80	46.39
$d_i = X_1 - X_2$	–0.02	–0.03	0.02	–0.04	0.01	–0.02

Sample	7	8	9	10	11	12
Method 1	35.42	38.41	42.68	46.71	29.20	30.76
Method 2	35.46	38.39	42.72	46.76	29.18	30.79
$d_i = X_1 - X_2$	–0.04	0.02	–0.04	–0.05	0.02	–0.03

Example 4 (Continued)

- Do the data provide sufficient evidence that Method 2 yields a higher average percentage than Method 1?
- Assume the differences are normally distributed.
- Use $\alpha = 0.05$.

Solution to Example 4

Step 1

- Let μ_d be the average difference in percentage between methods 1 and 2.
- Test $\mathrm{H}_0: \mu_d = 0$ against $\mathrm{H}_1: \mu_d < 0$. (why?)

Step 2

- Set $\alpha = 0.05$.

Solution to Example 4 (Continued)

Step 3

- $n = 12$ implies $t_{n-1;\alpha} = t_{11;0.05} = 1.796$.
- Critical region $t < -1.796$, where

$$t = \frac{\bar{d} - \mu_d}{s_d/\sqrt{n}}, \qquad \text{with } d_i = X_{1i} - X_{2i}$$

Solution to Example 4 (Continued)

Step 4

- **Computations:** From the data, we have $\sum_i d_i = -0.2$ and $\sum_i d_i^2 = 0.0112$. Hence $\bar{d} = -0.0167$ and $s_d^2 = [0.0112 - 12(-0.0167)^2]/11 = 0.00072.$
- Therefore

$$t = [(-0.0167) - 0]/\sqrt{0.00072/12} = -2.156.$$

- p-value $= \Pr(\boldsymbol{T_{11}} < -2.156) = 0.027$.

  [or p-value is between 0.05 and 0.025 since 2.156 is between $t_{11;0.05} = 1.796$ and $t_{11;0.025} = 2.201$ if statistical table is used.]

Solution to Example 4 (Continued)

Step 5

- Since the observed t-value = –2.156 falls in the critical region, hence H_0: $\mu_d = 0$ is rejected at the 5% level of significance and conclude that sufficient evidence exists to permit us to conclude that Method 2 yields a higher average percentage than does Method 1.
- Since p-value = 0.027 is less than 0.05, therefore we reject H_0 and conclude that sufficient evidence exists to permit us to conclude that Method 2 yields a higher average percentage than does Method 1.

7.4 Hypotheses Testing Concerning Variance

7.4.1 One Variance case

Assumption: Underlying distribution is normal

- Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a (approximate) $N(\mu, \sigma^2)$ distribution, where σ^2 is unknown.
- We wish to test null hypothesis

$$\mathrm{H_0}: \sigma^2 = \sigma_0^2.$$

- We know that

$$\chi^2 = \frac{(n_1 - 1)S^2}{\sigma_0^2} \sim \chi^2(n-1).$$

Hypothesis Testing for σ^2 (Continuous)

Hence

H_0	Test Statistic
$\sigma^2 = \sigma_0^2$	$\chi^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$

Hypothesis Testing for σ^2 (Continuous)

- $\mathrm{H}_0: \sigma^2 = \sigma_0^2$ is rejected if the observed χ^2-value

$\mathbf{H_1}$	**Critical Region**
$\sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{n-1;\alpha}$
$\sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{n-1;1-\alpha}$
$\sigma^2 \neq \sigma_0^2$	$\chi^2 < \chi^2_{n-1;1-\alpha/2}$ or $\chi^2 > \chi^2_{n-1;\alpha/2}$

where $\Pr\left(W > \chi^2_{n-1;\alpha}\right) = \alpha$ with $W \sim \chi^2(n-1)$

Example 1

- A manufacturer of car batteries claims that the life of his batteries is approximately normally distributed with a standard deviation equal to 0.9 year.
- If a random sample of 10 of these batteries has a standard deviation of 1.2 years,
- do you think that $\sigma > 0.9$ year?
- Use a 0.05 level of significance.

Solution to Example 1

Step 1

- Let σ^2 be the variance of the battery life.
- Test H_0: $\sigma^2 = 0.81$. H_1: $\sigma^2 > 0.81$.

Step 2

- Set $\alpha = 0.05$.

Solution to Example 1 (Continued)

Step 3

- $n = 10$ implies $\chi^2_{n-1;\,\alpha} = \chi^2_{9;0.05} = 16.919$.
- Critical region $\chi^2 > 16.919$, where

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2},$$

with $n = 10$ and $\sigma_0^2 = 0.81$.

Solution to Example 1 (Continued)

Step 4

- **Computations:**

  $s^2 = 11.44$, and $n = 10$, so

$$\chi^2 = \frac{9(1.44)}{0.81} = 16.0.$$

- p-value $= \Pr(\chi_9^2 > 16) = 0.0669$. [or it is between 0.05 and 0.10 from the statistical table]

Solution to Example 1 (Continued)

Step 5

- **Conclusion:** Since the observed χ^2-value = 16, which falls outside the critical region, hence H_0: $\sigma^2 = 0.81$ is not rejected at the 5% level of significance and conclude that there is no reason to doubt that the standard deviation is 0.9 year. Or
- Since p-value is greater than 0.05, we do not reject H_0.

7.4.2 H.T. Concerning Ratio of Variances

Assumption:

1. **Underlying distributions is normal**
2. **Means are unknown**

H.T. Concerning Ratio of Variances (Continued)

Examples

- When we are comparing the precision of one measuring device with that of another,
- the variability in grading practices of one teacher with that of another, and
- the consistency of one production process with that of another,
- we are testing about the difference between two population variances (or standard deviations).

H.T. Concerning Ratio of Variances (Continued)

- We know that when two independent samples of sizes n_1 and n_2 are randomly selected from two normal populations then

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(n_1 - 1, n_2 - 1).$$

- Under H_0: $\sigma_1^2 = \sigma_2^2$,

$$F = \frac{S_1^2}{S_2^2} \sim F(n_1 - 1, n_2 - 1).$$

H.T. Concerning Ratio of Variances (Continued)

- Hence

H_0	Test Statistic
$\sigma_1^2 = \sigma_2^2$	$F = \dfrac{S_1^2}{S_2^2}$

H.T. Concerning Ratio of Variances (Continued)

- $H_0: \sigma_1^2 = \sigma_2^2$ is rejected if the observed F-value falls in the critical region

H_1	Critical Region
$\sigma_1^2 > \sigma_2^2$	$F > F_{(n_1-1, n_2-1\,; \alpha)}$
$\sigma_1^2 > \sigma_2^2$	$F < F_{(n_1-1, n_2-1\,; 1-\alpha)}$
$\sigma_1^2 \neq \sigma_2^2$	$F < F_{(n_1-1, n_2-1\,; 1-\alpha/2)}$ or $F > F_{(n_1-1, n_2-1\,; \alpha/2)}$

where $\Pr\left(W > F_{v_1, v_2; \alpha}\right) = \alpha$ with $W \sim F(v_1, v_2)$

Example 2

- An experiment was performed to compare the abrasive wear of two different laminated materials.
- Eleven pieces of Material 1 were tested, by exposing each piece to a machine measuring wear.
- Nine pieces of Material 2 were similarly tested.
- In each case, the depth of wear was observed.
- The samples of Material 1 gave an average (coded) wear of 85 units with a standard deviation of 4,

Example 2 (Continued)

- while the samples of Material 2 gave an average of 81 and a standard deviation of 5.
- Assume that the two unknown populations to be approximately normal,
- test the two variances are equal.
- Use a 0.10 level of significance.

Solution to Example 2

Step 1

- Let σ_1^2 and σ_2^2 be the variances of the abrasive wear made from Materials 1 and 2 respectively.
- Test: $\sigma_1^2 = \sigma_2^2$ against H_1: $\sigma_1^2 \neq \sigma_2^2$.

Step 2

- Set $\alpha = 0.1$.

Solution to Example 2 (Continued)

Step 3

- $n_1 = 11, n_2 = 9$ implies $F_{n_1-1\,,n_2-1\,;\alpha/2} = F_{10,8;0.05} = 3.35$ and
- $F_{n_1-1,n_2-1;1-\alpha/2} = F_{10,8;0.95} = 1/F_{8,10;0.05} = 1/3.07 = 0.326.$
- Critical region: $F > 3.35$ or $F < 0.326$, where $F = s_1^2/S_2^2$

Solution to Example 2 (Continued)

Step 4

- **Computations:**
  $s_1^2 = 16, s_2^2 = 25$, so $F = 16/25 = 0.64$.

Step 5

- **Conclusion:** Since the observed F-value = 0.64 which falls outside the critical region, hence H_0: $\sigma_1^2 = \sigma_2^2$ is not rejected at the 10% level of significance and we conclude that we were justified in assuming the unknown variances equal.