NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF STATISTICS & APPLIED PROBABILITY

ST2334 PROBABILITY AND STATISTICS

SEMESTER I, AY 2022/2023

Tutorial 07: Solution

This set of questions will be discussed by your tutors during the tutorial in Week 10.

Please work on the questions before attending the tutorial.

1. A box contains 2 red marbles and 98 blue ones. Draws are made at random with replacement. In n draws from the box, there is better than a 50% chance for a red marble to appear at least once. What is the smallest possible value for n?

   SOLUTION
   Let X be the number of red marbles drawn in n draws. Then $X \sim \text{Bin}(n, 0.02)$. We want the least n so that $P(X \geq 1) > 0.5$ or
   $$P(X = 0) = (1 - 0.02)^n \leq 0.5$$
   which means
   $$n \geq \log(0.5)/\log(0.98) = 34.31.$$
   So we need n to be at least 35.

2. Suppose that, on average, 1 person in 1000 makes a numerical error in preparing his or her income tax return. 10,000 forms are selected at random and examined.

   (a) Find the probability that 6, 7, or 8 of the forms contain an error.

   (b) Find the mean and variance of the number of persons among 10,000 who make an error in preparing their tax returns.

   SOLUTION
   Let X = number of forms with error in 10,000 forms. $X \sim B(n = 10,000, p = 0.001)$. Since n is large and p is small, $X \approx \text{Poisson}(\lambda = np = 10)$.

   (a) $P(X = 6, 7, \text{ or } 8) = P(X \leq 8) - P(X \leq 5) = 0.2657$.

   (b) $E(X) = np = 10$, $V(X) = npq = 9.99$.

3. A couple decides they will continue to have children until they have two males. Assuming that $P(\text{male}) = 0.5$.

   (a) What is the probability that their second male is their seventh child?

   (b) What is the expected number of children for the couple?

   SOLUTION
   Let X = number of children until two sons. $X \sim NB(k = 2, p = 0.5)$.

   (a) $P(X = 7) = \binom{7-1}{2-1} p^2 (1-p)^{7-2} = 0.0469$;

   (b) $E(X) = k/p = 2/0.5 = 4$.

4. Three people toss a fair coin and the odd man pays for coffee. If the coins all turn up the same, they are tossed again.

(a) Find the probability that fewer than 4 tosses are needed.

(b) Provide a general formula for the probability of at most x tosses are needed.

SOLUTION

Treat getting an "odd" man as the success; then the possibility of failure is to get HHH or TTT. Therefore

$$P(\text{failure}) = P(HHHorTTT) = P(HHH) + P(TTT) = 1/8 + 1/8 = 1/4.$$

Therefore the probability of success is $p = 1 - 1/4 = 3/4$.

Let X = number of tosses needed to get the "odd man". $X \sim G(p)$.

(a) $P(X < 4) = 3/4 + (1/4)(3/4) + (1/4)^2(3/4) = 63/64$.

(b) For any positive integer x,

$$P(X \le x) = \sum_{n=1}^{x}(1-p)^{n-1}p = p\frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x.$$

In particular, when $p = 3/4$, $P(X \le x) = 1 - (1/4)^x$.

Here, we have used the fact:

$$1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}.$$

5. Hospital administrators in large cities anguish about problems with traffic in emergency rooms in hospitals. For a particular hospital in a large city, the staff on hand cannot accommodate the patient traffic if there are more than 10 emergency cases in a given hour. It is assumed that patient arrival follows a Poisson process and historical data suggest that, on the average, 5 emergencies arrive per hour. Find the probability that

(a) In a given hour, there is no emergency.

(b) In a given hour the staff can no longer accommodate the traffic?

(c) More than 20 emergencies arrive during a 3-hour shift of personnel?

SOLUTION

Let X = number of emergencies arrive in a certain hour. Then $X \sim \text{Poisson}(5)$.

(a) $P(X = 0) = 0.00673$.

(b) $P(X > 10) = 1 - P(X \le 10) = 0.0137$.

(c) Let Y = number of emergencies arrive in 3 hours. Then $Y \sim \text{Poisson}(3 \times 5 = 15)$. So $P(Y > 20) = 0.0830$.

6. A notice is sent to all owners of a certain type of automobile, asking them to bring their cars to a dealer to check for the presence of a particular type of defect. Suppose that only 0.05% of the cars have the defect. Consider a random sample of 10,000 cars.

(a) What are the expected value and variance of the number of cars in the sample that have the defect?

(b) What is the (approximate) probability that at least 10 sampled cars have the defect?

(c) What is the (approximate) probability that no sampled cars have the defect?

SOLUTION

Let X = number of cars in the sample that have defects. Then $X \sim B(10,000, 0.0005)$.

(a) $E(X) = 5$ and $V(X) = np(1-p) \approx 5$.

(b) Use Poisson approximation as n is large and p is small. $X \approx \text{Poisson}(5)$.

$$P(X \geq 10) \approx \sum_{x=10}^{\infty} \frac{e^{-5}5^x}{x!} = 0.0318.$$

(c) Similarly to Part (b),

$$P(X = 0) \approx e^{-5} = 0.0067.$$

Exact probability based on Binomial distribution is $(1-p)^{10,000} = 0.9995^{10,000} = 0.006729527023$.

7. A company rents time on a computer for periods of t hours, for which it receives \$600 an hour. The number of times the computer breaks down during t hours is a random variable having the Poisson distribution with $\lambda = 0.8t$, and if the computer breaks down x times during t hours, it costs $50x^2$ dollars to fix it. How should the company select t in order to maximize its expected profit?

SOLUTION

Let X denote the number of times the computer breaks down. Then $X \sim \text{Poisson}(0.8t)$.

The profit is given as

$$\text{profit} = 600t - 50X^2,$$

and the expected profit given as

$$E(\text{profit}) = 600t - 50E(X^2).$$

Now, $E(X^2) = V(X) + [E(X)]^2$, but $E(X) = V(X) = 0.8t$. So

$$E(\text{profit}) = 560t - 32t^2.$$

Taking the derivative, setting it equal to 0, and solving for t gives $t = 8.75$. So the expected profit is maximized when $t = 8.75$ hours.

8. Compute the following:

(a) $\displaystyle\sum_{x=1}^{\infty} \frac{x}{2^x}$;

(b) $\displaystyle\sum_{x=1}^{\infty} \frac{x^2}{2^x}$.

SOLUTION

Consider $X \sim G(p = 0.5)$, then $f_X(x) = 1/2^x$ for $x = 1, 2, 3, \ldots,$.

(a) $E(X) = \displaystyle\sum_{x=1}^{\infty} x f_X(x) = \sum_{x=1}^{\infty} \frac{x}{2^x}$; on the other hand, for geometric distribution, we know that $E(X) = 1/p = 2$. Therefore $\displaystyle\sum_{x=1}^{\infty} \frac{x}{2^x} = 2$.

(b) $E(X^2) = \sum_{x=1}^{\infty} x^2 f_X(x) = \sum_{x=1}^{\infty} \frac{x^2}{2^x}$. On the other hand, $E(X^2) = V(X) + [E(X)]^2 = (1-p)/p^2 + 1/p^2 = 2+4 = 6$. So, $\sum_{x=1}^{\infty} \frac{x^2}{2^x} = 6$.

9. You arrive at the bus stop at 10 a.m., knowing that the bus will arrive at some time uniformly distributed between 10 a.m. and 10:30 a.m.

   (i) What is the probability that you will have to wait longer than 10 minutes?

   (ii) If the bus has not yet arrived at 10:15 a.m., what is the probability that you will have to wait at least an additional 10 minutes?

   SOLUTION

   All times are measured in minutes and start from 10 a.m.

   Let $X :=$ arrival of the bus. Then $X \sim U(0,30)$.

   (i)

$$P(X \geq 10) = \int_{10}^{30} \frac{1}{30}\,\mathrm{dx} = \frac{2}{3}$$

   (ii)

$$\begin{aligned} P(X \geq 25 | X \geq 15) &= \frac{P(X \geq 25; X \geq 15)}{P(X \geq 15)} \\ &= \frac{P(X \geq 25)}{P(X \geq 15)} \\ &= \frac{\frac{5}{30}}{\frac{15}{30}} \\ &= \frac{1}{3} \end{aligned}$$