NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF STATISTICS & APPLIED PROBABILITY

ST2334 PROBABILITY AND STATISTICS

SEMESTER I, AY 2022/2023

Tutorial 03: Solution

This set of questions will be discussed by your tutors during the tutorial in Week 5.

Please work on the questions before attending the tutorial.

1. An East Coast manufacturer of printed circuit boards exposes all finished boards to an online automated verification test. During one period, 900 boards were completed and 890 passed the test. The test is not infallible. Of 30 boards intentionally made to have noticeable defects, 25 were detected by the test.

   (a) Approximate $P(\text{Pass test}|\text{board has defects})$. Explain why your answer may be too small.

   (b) Give an approximate value for the probability that a manufactured board will have defects. In order to answer the question, you need information about the conditional probability that a good board will fail the test. This is important to know but was not available at the time an answer was required. To proceed, you can assume that this probability is zero.

   (c) Approximate the probability that a board has defects given that it passed the automated test.

   SOLUTION
   Define events $A = \{\text{a circuit board passes the automated test}\}$; $D = \{\text{the board is defective}\}$. We approximate $P(A'|D) = 25/30$ and $P(A) = 890/900$.

   (a) We then approximate $P(\text{Pass test}|\text{board has defects}) = P(A|D)$ using the relation

   $$25/30 = P(A'|D) = 1 - P(A|D)$$

   or $P(A|D) = 5/30$.

   The approximation $P(A|D) = 5/30$ may be too small because the boards were intentionally made to have noticeable defects. Likely, many defects are not very noticeable.

   (b) To proceed, we assume that $P(A'|D') = 0$; then $P(A|D') = 1$. By the law of total probability

   $$P(A) = P(A|D)P(D) + P(A|D')P(D')$$

   so

   $$\frac{890}{900} = \frac{5}{30} \times P(D) + 1 \times (1 - P(D))$$

   or

   $$\frac{25}{30}P(D) = \frac{10}{900} \quad \text{so} \quad P(D) = \frac{1}{75} = 0.013.$$

   (c)

   $$P(D|A) = \frac{P(A|D)P(D)}{P(A|D)P(D) + P(A|D')P(D')} = \frac{\frac{5}{30} \times \frac{1}{75}}{\frac{5}{30} \times \frac{1}{75} + 1 \times \frac{74}{75}} = 5/2225 = 0.00225.$$

2. For customers purchasing a full set of tires at a particular tire store, consider the events

   $A = \{\text{tires purchased were made in the United States}\}$,

B = {purchaser has tires balanced immediately},

C = {purchaser requests front-end alignment}.

Denote the compliments of A, B, and C by A', B', C' respectively. Assume the following unconditional and conditional probabilities:

$$P(A) = 0.75, \quad P(B|A) = 0.9, \quad P(B|A') = 0.8, \quad P(C|A \cap B) = 0.8, \quad P(C|A' \cap B) = 0.7.$$

(a) Compute $P(A \cap B \cap C)$;

(b) Compute $P(B)$;

(c) Compute $P(A|B)$, the probability of a purchase of U.S. tires given that balancing was requested;

(d) Compute $P(B \cap C)$;

(e) Compute $P(A|B \cap C)$, the probability of a purchase of U.S. tires given that both balancing and an alignment were requested.

SOLUTION

(a) $P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B) = 0.75(0.9)(0.8) = 0.54$;

(b) $P(B) = P(A \cap B) + P(A' \cap B) = P(A)P(B|A) + P(A')P(B|A') = (0.75)(0.9) + (0.25)(0.8) = 0.875$;

(c) $P(A|B) = P(A \cap B)/P(B) = [(0.75)(0.9)]/0.875 = 0.7714$;

(d) $P(B \cap C) = P(A \cap (B \cap C)) = P(A' \cap (B \cap C))$. But $P(A' \cap (B \cap C) = P(A')P(B|A')P(C|A' \cap B) = 0.25(0.8)(0.7) = 0.14$. Therefore $P(B \cap C) = 0.54 + 0.14 = 0.68$.

(e) $P(A|B \cap C) = P(A \cap B \cap C)/P(B \cap C) = 0.54/0.68 = 0.7941$.

3. Total quality management (TQM) is a management philosophy and system of management techniques to improve product and service quality and worker productivity. TQM involves such techniques as teamwork, empowerment of workers, improved communication with customers, evaluation of work processes, and statistical analysis of processes and their output. One hundred Singapore companies were surveyed and it was found that 30 had implemented TQM. Among the 100 companies surveyed, 60 reported an increase in sales last year. Of those 60, 20 had implemented TQM. Suppose one of the 100 surveyed companies is to be selected for additional analysis.

   (a) What is the probability that a firm that implemented TQM is selected? That a firm whose sales increased is selected?

   (b) Are the two events {TQM implemented} and {Sales increased} independent or dependent? Explain.

   (c) Suppose that instead of 20 TQM-implementers among the 60 firms reporting sales increases, there were 18. Now are the events {TQM implemented} and {Sales increased} independent or dependent? Explain.

SOLUTION

Let $A = \{\text{TMQ implemented}\}$ and $B = \{\text{sales increased}\}$.

(a) $P(A) = 0.3$; $P(B) = 0.6$;

(b) Since $P(A|B) = 20/60$, therefore, $P(A \cap B) = P(A|B)P(B) = (1/3)0.6 = 0.2$. As $P(A \cap B) \neq P(A)P(B) = 0.18$, therefore A and B are not independent events.

(c) Since $P(A|B) = 18/60$, therefore, $P(A \cap B) = P(A|B)P(B) = (0.3)0.6 = 0.18$. As $P(A \cap B) = P(A)P(B)$, therefore A and B are independent events.

4. A company uses three different assembly lines, A_1, A_2, and A_3, to manufacture a particular component. Of those manufactured by line A_1, 5% need rework to remedy a defect, whereas 8% of A_2's components need rework, and 10% of A_3's components need rework. Suppose that 50% of all components are produced by line A_1 while 30% are produced by line A_2, and 20% come from line A_3. If a randomly selected component needs rework, what is the probability that it came

   (a) from line A_1?

   (b) from line A_2?

   (c) from line A_3?

   SOLUTION
   Let B be the event that a component needs rework. Then $P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) = 0.5(0.05) + (0.3)(0.08) + (0.2)(0.1) = 0.069$.

   (a) $P(A_1|B) = [(0.5)(0.05)/0.069 = 0.3623$.

   (b) $P(A_2|B) = [(0.3)(0.08)/0.069 = 0.3478$.

   (c) $P(A_3|B) = [(0.2)(0.1)/0.069 = 0.2899$.

   Note that $P(A_1|B) + P(A_2|B) + P(A_3|B) = 1$.

5. A real estate agent has 8 keys to open several new homes. For any given house, there is only 1 key which will open the house. If 40% of these homes are usually left unlocked, what is the probability that the real estate agent can get into a specific home if the agent selects 3 keys at random before leaving the office?

   SOLUTION
   Let $A = \{\text{Get into a house}\}$, $B = \{\text{the house is unlocked}\}$ and $C = \{\text{Agent gets the correct key}\}$.

   It is given that $P(B) = 0.4$. Note that $A \cap B' = \{\text{Get into a house and the house is locked}\} = \{\text{Agents gets the correct key and the house is locked}\} = C \cap B'$.

$$P(C) = 1/8 + (7/8)(1/7) + (7/8)(6/7)(1/6) = 3/8.$$

   Alternatively,

$$P(C) = \frac{\binom{1}{1}\binom{7}{2}}{\binom{8}{3}} = 3/8.$$

   As a consequence,

$$P(A) = P(B)P(A|B) + P(A \cap B') = 0.4(1) + P(C \cap B') = 0.4 + P(C)P(B') = 0.4 + 3/8(0.6) = 0.625.$$

6. Check whether the following can define probability distributions, and explain your answers.

   (a) $f(x) = {}^{x}/_{14}$ for $x = 0, 1, 2, 3, 4$.

   (b) $f(x) = {}^{3-x^2}/_{4}$ for $x = 0, 1, 2$.

   (c) $f(x) = {}^{1}/_{5}$ for $x = 5, 6, 7, 8, 9$.

   (d) $f(x) = {}^{2x+1}/_{50}$ for $x = 1, 2, 3, 4, 5$.

   SOLUTION

(a) No. $\sum f(i) = 10/14 < 1$.

(b) No. $f(2) = -1/4 < 0$.

(c) Yes. $0 \leq f(i) \leq 1$, and $\sum f(i) = 1$.

(d) No. $\sum f(i) = 35/50 < 1$.

7. From a box containing 4 one-cent coins and 2 five-cent coins, 3 coins are selected at random without replacement. Let X denote the total amount of the selected coins.

   (a) Write down R_X, the set of possible values for X.

   (b) Find the probability function of X.

   SOLUTION

   (a) The possible values for X are 3 (3 one-cent coins = $\{1, 1, 1\}$), 7 (2 one-cent coins and 1 five-cent coin = $\{1, 1, 5\}$), and 11 (1 one-cent coin and 2 five-cent coins = $\{1, 5, 5\}$). Therefore $R_X = \{3, 7, 11\}$.

   (b) We have

$$
\begin{aligned}
P(X=3) &= P(\{\text{3 one cent coins in the sample}\}) = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}; \\
P(X=7) &= P(\{\text{ 2 one-cent coins and 2 five-cent coin in the sample}\}) \\
&= \frac{\binom{4}{2}\binom{2}{1}}{\binom{6}{3}} = 3/5; \\
P(X=11) &= P(\{\text{1 one-cent coin and 2 five-cent coins in the sample}\}) \\
&= \frac{\binom{4}{1}\binom{2}{2}}{\binom{6}{3}} = 1/5.
\end{aligned}
$$

   Therefore, the p.f. for X can be summarized as a table given below.

x	3	7	11
$f_X(x)$	$1/5$	$3/5$	$1/5$

8. A contractor is required by the city planning department to submit 1, 2, 3, 4, or 5 forms (depending on the nature of the project) in applying for a building permit. Let Y = the number of forms required of the next application. The probability that y forms are required is known to be proportional to y. That is $f_Y(y) = ky$, for $y = 1, 2, \ldots, 5$.

   (a) What is the value of k?

   (b) What is the probability that at most three forms are required?

   (c) What is the probability that between two and four forms (inclusive) are required?

   (d) Find the cumulative distribution function (c.d.f.) of Y.

   SOLUTION

   (a) $1 = \sum_{y=1}^{5} ky = k \sum_{y=1}^{5} y = 15k$, which leads to $k = 1/15$.

   (b) $P(Y \leq 3) = f_Y(1) + f_Y(2) + f_Y(3) = 6/15 = 0.4$.

   (c) $P(2 \leq Y \leq 4) = f_Y(2) + f_Y(3) + f_Y(4) = 9/15 = 0.6$.

(d) The c.d.f. for Y is given by

$$F_Y(y) = \begin{cases} 0, & y < 1 \\ 1/15, & 1 \leq y < 2 \\ 3/15, & 2 \leq y < 3 \\ 6/15, & 3 \leq y < 4 \\ 10/15, & 4 \leq y < 5 \\ 1, & 5 \leq 5 \end{cases}$$