ST2334 (2020/21 Semester 1
Solution to Tutorial 9

Question 1
Let X be the time between two successive arrivals at the drive-up window of a fast-food restaurant. Then $X \sim Exp(1)$.

(a) $E(X) = 1/\lambda = 1$.
(b) $\sigma = 1/\lambda = 1$.
(c) $\Pr(X \leq 4) = 1 - e^{-1(4)} = 0.9817$.
$\Pr(2 \leq X \leq 5) = 1 - e^{-5} - (1 - e^{-2}) = 0.1286$.

Question 2
Let X be the time until failure for the fan. Then $X \sim Exp(1/25000)$

(a) $\Pr(X > 20000) = e^{-20000/25000} = 0.4493$.
$\Pr(X \leq 30000) = 1 - e^{-30000/25000} = 0.6988$.
$\Pr(20000 \leq X \leq 30000) = 0.6988 - (1 - 0.4493) = 0.1481$.
(b) $\sigma = 1/\lambda = 25000$. Therefore $\Pr(X > \mu + 2\sigma) = \Pr(X > 75000) = e^{-75000/25000} = 0.0498$.

Question 3
$X =$ length of time to fail, in years
$X \sim Exp(1/2)$

(a) $V(X) = [E(X)]^2 = [2]^2 = 4$
(b) $\Pr(X < 1) = 1 - e^{-(1/2)(1)} = 0.39347$
$Y =$ number of electrical switch that fail during the first year.
$Y \sim Binomial(n = 100, p = 0.39347)$
$E(Y) = np = 39.35, \quad V(Y) = np(1-p) = 23.87$

$$\Pr(Y \leq 30) = \Pr(Y \leq 30.5) = \Pr\left(Z \leq \frac{30.5 - 39.35}{\sqrt{23.87}}\right)$$
$$= \Pr(Z < -1.81) = 0.0351$$

[Exact probability: $\Pr(Y \leq 30) = 0.03347$.]

Question 4

$$\Pr(\mu - 3\sigma < X < \mu + 3\sigma) = \Pr(-3 < Z < 3) = 1 - 2\Pr(Z > 3)$$
$$= 1 - 2(0.00135) = 0.9973$$

[Compare with $\Pr(\mu - 3\sigma < X < \mu + 3\sigma) \geq 8/9$ using Chebyshev's Inequality]

Question 5
$X =$ amount of the soft drink
$X \sim Normal\ (\mu = 200;\ \sigma^2 = 15^2)$

(a) $\Pr(X > 224) = \Pr(Z > 1.60) = 0.0548$, where $Z \sim N(0,1)$
(b) $\Pr(191 < X < 209) = \Pr(-0.60 < Z < 0.60) = 1 - 2(0.2743) = 0.4514$
(c) $\Pr(X > 230) = \Pr(Z > 2.00) = 0.02275 = p$
$Y =$ number of cups that overflow
$Y \sim Binomial\ (n = 1000, p = 0.02275)$
$E(Y) = np = 1000(0.02275) = 22.75 = 23$
(d) $\Pr(Z < z_{0.25}) = 0.25; z_{0.25} = -0.6745$
$Z = \dfrac{X - \mu}{\sigma}$ or $X = \mu + \sigma Z$

$x_{0.25} = \mu + z_{0.25}\sigma = 200 + (-0.6745)(15) = 189.88\, ml$

Question 6

$X =$ commute time from home to office

$X \sim Normal\ (\mu = 24;\ \sigma^2 = 3.8^2)$

(a) $\Pr(X > 30) = \Pr(Z > 1.58) = 0.0571$

(b) $\Pr(X > 15) = \Pr(Z > -2.37) = 1 - 0.00889 = 0.99111 = 99.11\%$

(c) Y= number of trips that take at least half an hour

$Y \sim Binomial\ (n = 3, p = 0.0571)$

$$\Pr(Y = 2) = \binom{3}{2}(0.0571)^2(1 - 0.0571)^1 = 0.00922$$

Question 7

$Y =$ number of head in 400 tosses of a coin

$Y \sim Binomial\ (n = 400,\ \ p = 0.5)$

$E(Y) = np = 400(0.5) = 200.\ \ V(Y) = np(1 - p) = 400(0.5)(0.5) = 100$

$Y \dot\sim Normal\ (\mu = 200;\ \sigma^2 = 100)$

(a) $\Pr(185 \le Y \le 210) = \Pr(184.5 < Y < 210.5) = \Pr(-1.55 < Z < 1.05)$
$= 1 - 0.0606 - 0.1469 = 0.7925$

(b) $\Pr(Y = 205) = \Pr(204.5 < Y < 205.5) = \Pr(0.45 < Z < 0.55)$
$= 0.3261 - 0.2912 = 0.0352$

(c) $\Pr(Y < 176 \text{ or } Y > 227) = \Pr(Y < 175.5) + \Pr(Y > 227.5)$
$= \Pr(Z < -2.45) + \Pr(Z > 2.75) = 0.00714 + 0.00298 = 0.01012$

Question 8

$Y =$ number of drunk driver

$Y \sim Binomial\ (n = 400,\ \ p = 0.1)$

$E(Y) = np = 400(0.1) = 40.\ V(Y) = np(1 - p) = 400(0.1)(0.9) = 36$

$Y \dot\sim Normal\ (\mu = 40;\ \sigma^2 = 6^2)$

(a) $\Pr(Y < 32) = \Pr(Y < 31.5) = \Pr(Z < -1.42) = 0.0778$

(b) $\Pr(Y > 49) = \Pr(Y > 49.5) = \Pr(Z > 1.58) = 0.0571$

(c) $\Pr(35 \le Y < 47) = \Pr(34.5 < Y < 46.5) = \Pr(-0.92 < Z < 1.08)$
$= 1 - 0.1788 - 0.1401 = 0.6811$

Question 9

$Y =$ number of defective parts

$Y \sim Binomial\ (n = 100,\ \ p = 0.05)$

$E(Y) = np = 100(0.05) = 5.\ \ V(Y) = np(1 - p) = 100(0.05)(0.95) = 4.75$

$Y \dot\sim Normal\ (\mu = 5;\ \sigma^2 = 4.75)$

(a) $\Pr(Y > 2) = \Pr(Y > 2.5) = \Pr(Z > -1.1471) \approx \Pr(Z > -1.15) = 1 - 0.1251 =$
0.8749. (From Statistical table)

(b) $\Pr(Y > 10) = \Pr(Y > 10.5) = \Pr(Z > 2.52) = 0.00587$

Question 10

(a) $\mu = \sum x f_X(x) = 4(0.2) + 5(0.4) + 6(0.3) + 7(0.1) = 5.3$

$$\sigma^2 = \sum (x - \mu)^2 f(x) = (4 - 5.3)^2(0.2) + (5 - 5.3)^2(0.4) + (6 - 5.3)^2(0.3) + (7 - 5.3)^2(0.1) = 0.81$$

(b) With $n = 36$, $\mu_{\bar{x}} = \mu = 5.3$; $\sigma_{\bar{x}}{}^2 = \frac{\sigma^2}{n} = \frac{0.81}{36} = 0.0225$

(c) Applying the Central Limit Theorem, $\bar{X}\ approx \sim N(5.3, 0.0255)$

$$\Pr(\bar{X} < 5.5) = \Pr\left(Z < \frac{5.5 - 5.3}{\sqrt{0.0225}}\right) = \Pr(Z < 1.33) = 1 - 0.0918 = 0.9082$$

Question 11

$X =$ amount of benzene. $E(X) = \mu$ and $V(X) = 100^2$

(a) $n = 25$. By the CLT, $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$. $\Pr(\bar{X} > 7950 | \mu = 7950) = \Pr(\bar{X} > \mu) = 0.5$

(b) $X \sim N(\mu, 100^2)$. Hence $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$ approximately.

$\Pr(\bar{X} \geq 7960 | \mu = 7950) = \Pr\left(Z > \frac{7960-7950}{100/\sqrt{25}}\right) = \Pr(Z > 0.5) = 0.3085$

No, there is no strong evidence that the population mean exceeds the government limit as it is likely to see a sample mean is equal to or larger than 7960 if the population mean equals to the government limit 7950.