Matrices

Edward Jex

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Transformations using Matrices

Any $n \times n$ matrix can be used to represent a transformation in n dimensions. Given a matrix M and point p, the transformed point would be Mp. Translations can not be represented this way and other transforms must be about the origin. To transform multiple points at the same time, they can be written as a matrix with each column corresponding to a different point.

Example 1

To transform a single point by a matrix.

$$\begin{aligned}
\mathbf{p} &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\
\mathbf{M} &= \begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix} \\
\mathbf{Mp} &= \begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} -13 - 12 \end{bmatrix} \\
&\therefore (2, -3) \mapsto (-13, -12)
\end{aligned}$$

Finding A Matrix For A Transformation

Not all transformations have matrices.

If there is a matrix $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ this would transform $(1, 0) \mapsto (a, c)$ and $(0, 1) \mapsto (b, d)$. Therefore, if we know how the basis vectors are transformed we can write the matrix that causes the transformation with each column containing the point that a basis vector is transformed to.

Inverses of Matrices

For a square matrix $\mathbf{M}$, the inverse is written $\mathbf{M}^{-1}$ and has the property $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$. Not all matrices have inverses. If a matrix has no inverse we call it singular.

Inverses And Transformations

If $\mathbf{M}$ represents a transform T then $\mathbf{M}^{-1}$ represents the inverse transformation. If T is a reflection $\mathbf{M} = \mathbf{M}^{-1}$ or $\mathbf{M}^2 = \mathbf{I}$
$|\det \mathbf{M}|$ represents the area scale factor of T. If $\det \mathbf{M} < 0$ the sense is reversed meaning that the points are labelled the other way around and the shape have been flipped over e.g. a rotation.

2x2

The determinant of $\mathbf{M}$ is $\det \mathbf{M} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. The adjugate of $\mathbf{M}$ is $adj\mathbf{M} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $\mathbf{M}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
If $\det \mathbf{M} = 0$ then the matrix would have no inverse as it would mean dividing by 0.

3x3

We still use $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} adj\mathbf{M}$ but the determinant and adjugate are calculated differently.

Determinant Of a 3x3 Matrix

To calculate the determinant, chose a row or column of values and for each value ignore its row and column and find the determinant of the remaining values. Multiply this by the value and then the sign is given by the following pattern.
$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

the matrix of minors is found. For each value in the matrix, the row and column that the value is in are ignored and the determinant of the remaining values is calculated. This is multiplied by the value.