Calculus

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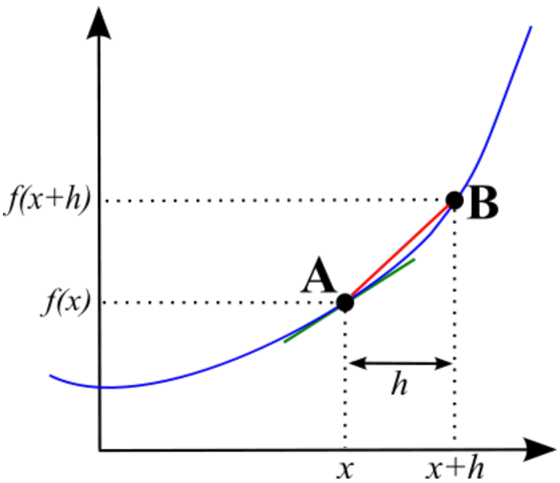
Calculus is the mathematics of change and has two distinct parts. Differentiation and Integration.

Integration

- Rates of change
- Finding the gradient of a tangent to a curve at a point.

Differentiating from First Principles



The gradient of the tangent of the curve at A can be approximated by the gradient of the chord AB. As the distance h gets smaller, the approximation of the gradient gets better.
$f'(a)$ - The gradient of the tangent to the curve $f(x)$ at $x = a$.

$$
\begin{aligned}
f'(A) &= \frac{f(B) - f(A)}{B - A} \\
&\simeq \frac{f(A + h) - f(A)}{h} \\
f'(a) &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\end{aligned}
$$

Differentiating $y = x^2$ from First Principles

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\
&= \lim_{h \to 0} 2a + h \\
&= 2a
\end{aligned}
$$

Tangents and Normals

Tangent - Intersects the curve once locally.
Normal - Perpendicular to the tangent.
$m_1 \times m_2 = -1$ for perpendicular lines.

Stationary Points

Stationary points are when $\frac{dy}{dx} = 0$.
Types:

- Local minimum - Turning point and Stationary point
- Local maximum - Turning point and Stationary point
- Points of inflection - Stationary point

To tell which type of stationary point we have, we can look at the second derivative ($\frac{d^2y}{dx''}$):

- At a minimum - $\frac{d^2y}{dx^2} > 0$

- At a maximum - $\frac{d^2y}{dx^2} < 0$
- At a point of inflection - $\frac{d^2y}{dx^2} = 0$

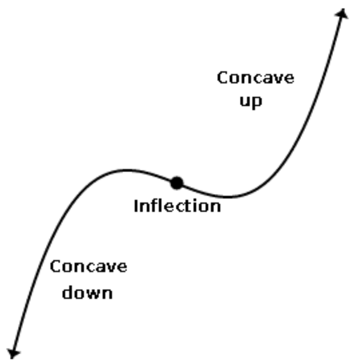
If $\frac{d^2y}{dx^2} = 0$, the point could also be a turning point, more investigation is required.

Shapes of Curves

Points of infection ($\frac{d^2y}{dx^2} = 0$) have two types:

- Stationary (where $\frac{dy}{dx} = 0$)
- Non-stationary

Points can be either concave up, concave down or points of infection.



- Concave down - $\frac{d^2}{dx^2} < 0$, Cord is below the curve.
- Concave up - $\frac{d^2}{dx^2} > 0$, Cord is above the curve.

Optimisation

Differentiation can be used to find minimum of maximum points for a problem which can be the optimum for given conditions.

Differentiation Rules

Trig Functions

Note that these rules only work in radians.

$f(x)$	$f'(x)$
$\sin f(x)$	$f'(x)\cos f(x)$
$\cos f(x)$	$-f'(x)\sin f(x)$
$\tan f(x)$	$\frac{f'(x)}{\cos^2 f(x)} = f'(x)\sec^2 f(x)$

Exponentials and Logarithms

$f(x)$	$f'(x)$
$a^{f(x)}$	$f'(x)a^{f(x)}\ln a$
$\log_a f(x)$	$\frac{f'(x)}{f(x)\ln a}$

Chain Rule

Rates of change can be connected. We can calculate new rates of change by cancelling terms like fractions, although they are not fractions. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. We also see fraction like behaviour when taking reciprocals as $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$. Note that you should not write this as $\frac{dx}{dy}^{-1}$.
The general rule is: $(f \circ g)' = (f' \circ g).g'$

Product Rule

The general rule is: $(f.g)' = f'.g + f.g'$
Note than sometimes an expression may be a fraction. You can treat the denominator as a negative power, or you could use the quotient rule.

Quotient Rule

The general rule is: $y = \frac{f}{g} \Rightarrow y' = \frac{f'.g - f.g'}{g^2}$. This can be easily derived from the product rule. I recommend always using the product rule because it is much nicer.

Implicit Differentiation

Implicit differentiation is used when there is no clear subject e.g. a circle. It relies on the principle than $\frac{d}{dx}f(y) = \frac{d}{dy}f(y) \times \frac{dy}{dx} = f'(y)\frac{dy}{dx}$.

Example 1

$$\begin{aligned}
x^2 + y^2 &= 25 \\
\frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}25 \\
2x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$