A Level Further Maths: Pure Notes

Contents

Chapter 1

Binomial Expansion

Polynomial Binomial Expansion

Binomial Expansion is a way of easily expanding brackets with two terms to a power. The expansion is given by the general formula:

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^{n-r} \cdot y^r$$

where ${}^nC_r$ is equivalent to $\binom{n}{c}$ and defined as:

$${}^nC_r = \frac{n!}{(n-r)! \cdot r!}$$

Note that since pascals triangle is symmetrical it doesn't matter which term way around the powers are.

Example 1:

$$\begin{aligned}(2+x)^4 &= ({}^4C_0 \cdot 2^4 \cdot x^0) + ({}^4C_1 \cdot 2^3 \cdot x^1) + ({}^4C_2 \cdot 2^2 \cdot x^2) + ({}^4C_3 \cdot 2^1 \cdot x^3) + ({}^4C_4 \cdot 2^0 \cdot x^4) \\ &= (1 \cdot 16 \cdot 1) + (4 \cdot 8 \cdot x) + (6 \cdot 4 \cdot x^2) + (4 \cdot 2 \cdot x^3) + (1 \cdot 1 \cdot x^4) \\ &= x^4 + 8x^3 + 24x^2 + 32x + 16\end{aligned}$$

Example 2:

Find the x^3 coefficient in $(3+2x)^5$:
let n = 5, r = 3

$$\begin{aligned} 3^{rd} &=^5 C_3 \cdot 3^{5-3} \cdot (2x)^3 \\ &= 10 \cdot 3^2 \cdot 8x^3 \\ &= 720x^3 \end{aligned}$$

$$\therefore coefficent = 720$$

Example 3:

Find the constant term in $(x^2 + \frac{1}{x})^9$:
For the two terms to cancel out, the power of $\frac{1}{x}$ must be twice the power of x^2 so that when they are multiplied together they produce $x^0 = 1$, meaning the term is constant or independent $\implies$ We want the term where x^2 is to the power of 3 and $\frac{1}{x}$ is to the power of 6 $\implies$ constant term is:

$$\begin{aligned} t = {}& {}^9C_3 \cdot (x^2)^3 \cdot \frac{1}{x}^6 \\ &= 84 \cdot x^6 \cdot x^{-6} \\ &= 84 \end{aligned}$$

General Binomial Expansion

This is a more general version of binomial expansion that doesn't require the power to be a positive integer.

$$\begin{aligned} (1+x)^n &\simeq 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} \dots \\ &= 1 + \sum_{i=1}^{\infty} (\prod_{j=0}^{i-1} .x^i.i!) \end{aligned}$$

Where $n \in \mathbb{R}, |x| < 1$.

Example 1

Expand $(1+2x)^{-\frac{1}{2}}$ up to and including the x^3 term.

$$\begin{aligned} (1+(2x))^{-\frac{1}{2}} &\simeq 1 + (\frac{1}{2})(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})(2x)^3}{2!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(2x)^3}{3!} \\ &\simeq 1 - x + \frac{3x^2}{2} - \frac{5x^3}{2} \end{aligned}$$

Valued for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

Chapter 2

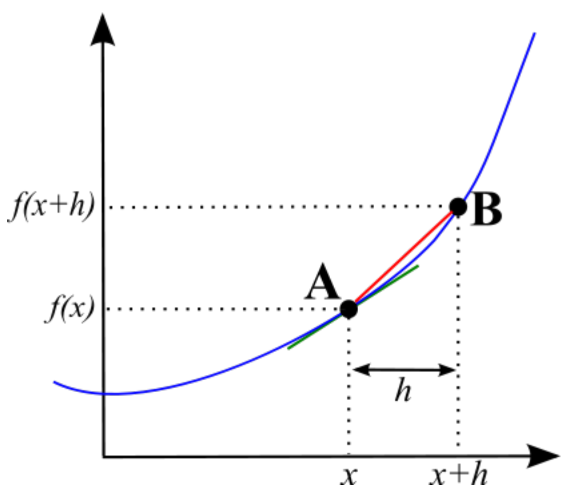
Calculus

Calculus is the mathematics of change and has two distinct parts. Differentiation and Integration.

Integration

- Rates of change
- Finding the gradient of a tangent to a curve at a point.

Differentiating from First Principles



The gradient of the tangent of the curve at A can be approximated by the gradient of the chord AB. As the distance h gets smaller, the approximation of the gradient gets better.
$f'(a)$ - The gradient of the tangent to the curve $f(x)$ at $x = a$.

$$
\begin{aligned}
f'(A) &= \frac{f(B)-f(A)}{B-A} \\
&\simeq \frac{f(A+h)-f(A)}{h} \\
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}
\end{aligned}
$$

Differentiating $y = x^2$ from First Principles

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} \\
&= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\
&= \lim_{h\to 0} 2a + h \\
&= 2a
\end{aligned}
$$

Tangents and Normals

Tangent - Intersects the curve once locally.
Normal - Perpendicular to the tangent.
$m_1 \times m_2 = -1$ for perpendicular lines.

Stationary Points

Stationary points are when $\frac{dy}{dx} = 0$.
Types:

- Local minimum - Turning point and Stationary point
- Local maximum - Turning point and Stationary point
- Points of inflection - Stationary point

To tell which type of stationary point we have, we can look at the second derivative ($\frac{d^2y}{dx"}$):

- At a minimum - $\frac{d^2y}{dx^2} > 0$
- At a maximum - $\frac{d^2y}{dx^2} < 0$
- At a point of inflection - $\frac{d^2y}{dx^2} = 0$

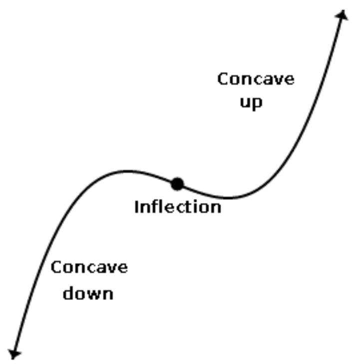
If $\frac{d^2y}{dx^2} = 0$, the point could also be a turning point, more investigation is required.

Shapes of Curves

Points of infection ($\frac{d^2y}{dx^2} = 0$) have two types:

- Stationary (where $\frac{dy}{dx} = 0$)
- Non-stationary

Points can be either concave up, concave down or points of infection.



- Concave down - $\frac{d^2}{dx^2} < 0$, Cord is below the curve.
- Concave up - $\frac{d^2}{dx^2} > 0$, Cord is above the curve.

Optimisation

Differentiation can be used to find minimum of maximum points for a problem which can be the optimum for given conditions.

Differentiation Rules

Trig Functions

Note that these rules only work in radians.

$f(x)$	$f'(x)$
$\sin f(x)$	$f'(x)\cos f(x)$
$\cos f(x)$	$-f'(x)\sin f(x)$
$\tan f(x)$	$\frac{f'(x)}{\cos^2 f(x)} = f'(x)\sec^2 f(x)$

Exponentials and Logarithms

$f(x)$	$f'(x)$
$a^{f(x)}$	$f'(x)a^{f(x)}\ln a$
$\log_a f(x)$	$\frac{f'(x)}{f(x)\ln a}$

Chain Rule

Rates of change can be connected. We can calculate new rates of change by cancelling terms like fractions, although they are not fractions. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. We also see fraction like behaviour when taking reciprocals as $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$. Note that you should not write this as $\frac{dx}{dy}^{-1}$.
The general rule is: $(f \circ g)' = (f' \circ g).g'$

Product Rule

The general rule is: $(f.g)' = f'.g + f.g'$
Note than sometimes an expression may be a fraction. You can treat the denominator as a negative power, or you could use the quotient rule.

Quotient Rule

The general rule is: $y = \frac{f}{g} \Rightarrow y' = \frac{f'.g - f.g'}{g^2}$. This can be easily derived from the product rule. I recommend always using the product rule because it is much nicer.

Implicit Differentiation

Implicit differentiation is used when there is no clear subject e.g. a circle. It relies on the principle than $\frac{d}{dx}f(y) = \frac{d}{dy}f(y) \times \frac{dy}{dx} = f'(y)\frac{dy}{dx}$.

Example 1

$$\begin{aligned} x^2 + y^2 &= 25 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}25 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Chapter 3

Complex Numbers

Definition

$\mathbb{R}$ is the set of real numbers which we have been using before. This can be thought of as a straight number line. The complex numbers ($\mathbb{C}$) can be thought of as a plane of numbers of which the real numbers is one axis. $\mathbb{R} \subset \mathbb{C}$.

$x^2 = -1$ has no real roots. We can imagine that this equation has a root. We define i such that $i^2 = -1$. i is imaginary.

Complex numbers are made up of the form $a + bi$, $a, b \in \mathbb{R}$. We often use z to represent complex numbers rather than x.

$z = a + bi \Rightarrow \mathrm{Re}(z) = a, \mathrm{Im}(z) = b$

We say two complex numbers are equal if both the real and imaginary parts are the same. We also assume that the same rules of algebra apply to complex numbers as real numbers. We can not find roots for any polynomial with real coefficients.

Example 1

$$\begin{aligned} x^2 + 2x + 8 &= 0 \\ \Rightarrow x &= \frac{-2 \pm \sqrt{4 - 32}}{2} \\ &= \frac{-2 \pm \sqrt{4}\sqrt{-7}}{2} \\ &= -1 \pm \sqrt{7}i \end{aligned}$$

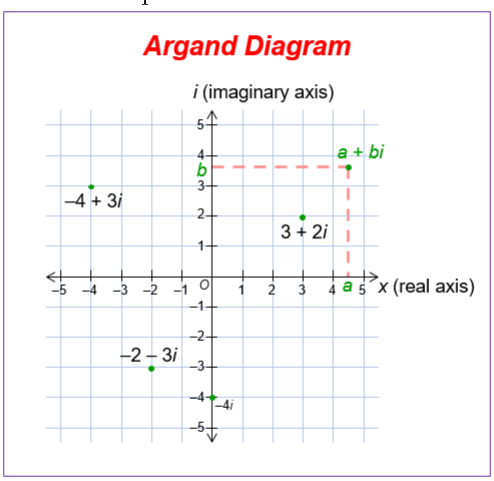
The conjugate of a complex number $z = a + bi$ is $z^* = a - bi$. This is useful for eliminating a complex denominator on a fraction.

Example 2

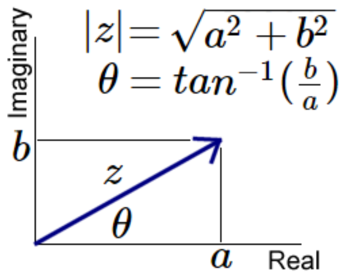
$$\begin{aligned}
\frac{3+5i}{2-3i} &= \frac{3+5i}{2-3i} * \frac{2+3i}{2+3i} \\
&= \frac{6+19i+15i^2}{4-9i^2} \\
&= \frac{-9+19i}{13} \text{ or } \frac{-9}{13} + \frac{19}{13}i
\end{aligned}$$

The Argand Diagram

The Argand diagram is a graphical representation of $\mathbb{C}$, much like a number line is used to represent $\mathbb{R}$.



Modulus and Argument



The argument of z (written $\arg z$) is the angle from the real axis. The principle argument is $-\pi < \arg z \leqslant \pi$. There may be two solutions so it is best to think about where the point would be on the Argand diagram to find which one is correct.

The modulus of z (written $|z|$) is the distance from $(0,0)$ to (a,b) and can be used to compare the size of complex numbers.

Modulus - Argument Form

Complex numbers can be thought of as an angle and distance from the origin.
$|z| = r$, $\arg z = \theta \Rightarrow z = r(\cos\theta + i\sin\theta)$
$\mathrm{Re}(z) = |z|\cos\arg z$, $\mathrm{Im}(z) = |z|\sin\arg z$
Modulus - Argument Form is useful as it makes multiplication of complex numbers easier and allows us to think of them as transforms.

$$\begin{aligned}
z_1 &= r_1(\cos\theta_1 + i\sin\theta_1)\\
z_2 &= r_2(\cos\theta_2 + i\sin\theta_2)\\
\Rightarrow z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)\\
&= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\cos\theta_2\sin\theta_1)\\
&= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\\
\therefore |z_1z_2| &= |z_1| \times |z_2|\\
\arg(z_1z_2) &= \arg z_1 + \arg z_2
\end{aligned}$$

In the context of the Argand diagram multiplying by a complex number is equivalent to:

- Rotation by $\arg z$ around $(0,0)$
- Scale by $|z|$ about $(0,0)$

Dividing is equivalent to:

- Rotation by $-\arg z$ around $(0,0)$
- Scale by $\frac{1}{|z|}$ about $(0,0)$

Chapter 4

Coordinate Geometry

To find the midpoint of a line, take the mean of the endpoints.

Gradient $= \frac{\delta y}{\delta x}$

For perpendicular lines with gradients m_1, m_2: $m_1 = \frac{-1}{m_2}$

General formula of a straight line: $y = mx + c$ or $(y - y_0) = m(x - x_0)$, latter preferred as easier.

The intersection of lines can be found by solving simultaneously. In 2D this gives 0,1,∞ solutions

General formula of a circle: $r^2 = (x - a)^2 + (y - b)^2$ where radius is r, center is (a,b)

Chapter 5

Exponentials and Logarithms

Consider than $100 = 10^2$. In logarithmic form, we say $\log_{10} 100 = 2$.

The laws of logarithms

$$\log_n x + \log_n y = \log_n xy \tag{5.1}$$

$$\log_n x - \log_n y = \log_n \frac{x}{y} \tag{5.2}$$

$$\log_n x^k = klog_n x \tag{5.3}$$

$$\frac{\log_a x}{\log_a b} = \log_b x \tag{5.4}$$

Rules 1, 2 and 3 must be learn for A-Level.

Natural Logarithms

$e \simeq 2.718\ldots$. e is irrational and transcendental. The exponential function is defined as $y = e^x$. It is special as its derivative is itself. The natural logarithm function is defined as $y = \log_e x = \ln x$. The derivative of $\ln x$ is $\frac{d}{dx} \ln x = \frac{1}{x}$.

Modelling curves with Logarithms

The relationship between x and y is believed to be of the form $y = kx^n$ where k and n are constant. By taking logarithms of each side, the power can be moved down. This means a simple linear regression can be used to fit coefficients to the data by taking the logarithms of the given data and then putting the found coefficients to the power of whichever base logarithm you used.

Chapter 6

Factorials

n factorial $(n!) = n \times (n-1) \times (n-2) \cdots \times 1 = \prod_{k=0}^{n} n-k$
In general, the number of ways of placing n different objects in a line is $n!$

Example 1

There are 5 flags in a line, find the probability of the flags being a alphabetical order.
$5! = 120 \Rightarrow p = \frac{1}{120}$

Note: factorials only work if we are ordering objects. If you are choosing objects from a group we need something different.
${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$

Example 2

What are the possible ways of placing 8 flags for a line of 5.
${}_8C_5 = 56$

Chapter 7

Function Notation

What is a function?

$$\begin{aligned} f : A &\to B \\ x &\mapsto f(x) \end{aligned}$$

Where A is the domain (set of the possible inputs to the function), B is the range or co-domain (set of possible outputs from the function) and $x \mapsto f(x)$ is the mapping rule or operation.
The domain can be smaller than it needs to be and likewise the co-domain can be larger than it needs to be.

Example 1:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

$$\begin{aligned} g : \mathbb{Z} &\to \mathbb{Z}^+ \\ x &\mapsto x^2 \end{aligned}$$

$$\begin{aligned} h : \mathbb{R}\backslash\{0\} &\to \mathbb{R} \\ x &\mapsto \frac{1}{x} \end{aligned}$$

$$k : D \to \mathbb{R}$$

D doesn't include the number 0 in the set of reals.

$$x \mapsto 4x^2 + 12x + 73$$

$$D = \{x \in \mathbb{R} \mid \mod x, 2 = 0\}$$

D is the set of even real numbers.

$$\begin{aligned} l : [0, \infty) &\to \mathbb{R} \\ x &\mapsto \sqrt{x} \end{aligned}$$

$[0, \infty)$ represents numbers larger than or equal 0 to smaller than infinity.

Intervals

$$\begin{aligned} [a, b] &= \{x \in \mathbb{R} | a \leqslant x \leqslant b\} \\ (a, b) &= \{x \in \mathbb{R} | a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} | a \leqslant x < b\} \\ (a, b] &= \{x \in \mathbb{R} | a < x \leqslant b\} \end{aligned}$$

Example 2:

$$\begin{aligned} \text{Sine :} & \mathbb{R} \to [-1, 1] \\ & x \mapsto \sin x \\ \text{Cosine :} & \mathbb{R} \to [-1, 1] \\ & x \mapsto \cos x \\ \text{Tangent :} & \{x \in \mathbb{R} | \mod x + \frac{\pi}{2}, \pi \neq 0 \to [-1, 1] \\ & x \mapsto \tan x \end{aligned}$$

Chapter 8

Graphs and Transformations

Notation

See [Function Notation]

Operations

Suppose we have two functions:
$f : A \to B \qquad g : B \to C$
We can compose them using the $\circ$ operator

Example 1:

$$g \circ f : A \to C$$

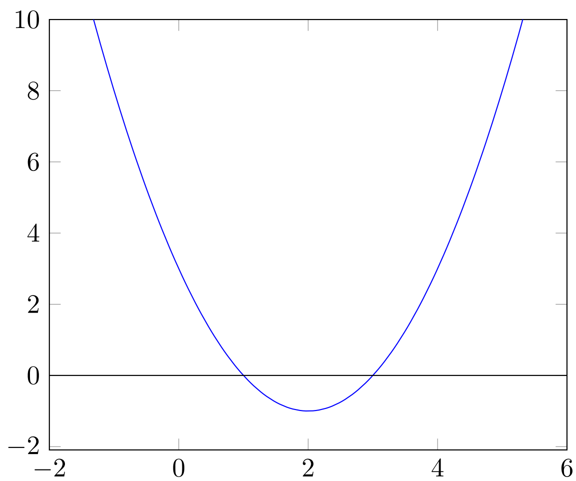
Chapter 9

Inequalities

When multiplying inequalities with negative reals then the relation must swap sign. e.g.: $-x > -4 \iff x < 4$
With quadratic inequalities it is always worth sketching a graph so that you can visualise the area that you are trying to get.

Example 1:

$$\begin{aligned} x^2 - 4x + 3 &< 0 \\ (x-3)(x-1) &< 0 \\ \therefore 1 \leqslant x &\leqslant 3 \end{aligned}$$



Thinking:

$$\begin{aligned} ab > 0 &\iff ((a > 0) \,\&\&\, (b > 0)) \,||\, ((a < 0) \,\&\&\, (b < 0)) \\ a, b &\in \mathbb{R} \end{aligned}$$

Chapter 10

Matrices

Transformations using Matrices

Any $n \times n$ matrix can be used to represent a transformation in n dimensions. Given a matrix M and point p, the transformed point would be Mp. Translations can not be represented this way and other transforms must be about the origin. To transform multiple points at the same time, they can be written as a matrix with each column corresponding to a different point.

Example 1

To transform a single point by a matrix.

$$\begin{aligned}
\mathbf{p} &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\
\mathbf{M} &= \begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix} \\
\mathbf{Mp} &= \begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} -13 - 12 \end{bmatrix} \\
&\therefore (2, -3) \mapsto (-13, -12)
\end{aligned}$$

Finding A Matrix For A Transformation

Not all transformations have matrices.

If there is a matrix $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ this would transform $(1, 0) \mapsto (a, c)$ and $(0, 1) \mapsto (b, d)$. Therefore, if we know how the basis vectors are transformed we can write the matrix that causes the transformation with each column containing the point that a basis vector is transformed to.

Inverses of Matrices

For a square matrix $\mathbf{M}$, the inverse is written $\mathbf{M}^{-1}$ and has the property $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$. Not all matrices have inverses. If a matrix has no inverse we call it singular.

Inverses And Transformations

If $\mathbf{M}$ represents a transform T then $\mathbf{M}^{-1}$ represents the inverse transformation.
If T is a reflection $\mathbf{M} = \mathbf{M}^{-1}$ or $\mathbf{M}^2 = \mathbf{I}$
$|\det \mathbf{M}|$ represents the area scale factor of T. If $\det \mathbf{M} < 0$ the sense is reversed meaning that the points are labelled the other way around and the shape have been flipped over e.g. a rotation.

2x2

The determinant of $\mathbf{M}$ is $\det \mathbf{M} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. The adjugate of $\mathbf{M}$ is $adj\mathbf{M} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $\mathbf{M}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
If $\det \mathbf{M} = 0$ then the matrix would have no inverse as it would mean dividing by 0.

3x3

We still use $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} adj\mathbf{M}$ but the determinant and adjugate are calculated differently.

Determinant Of a 3x3 Matrix

To calculate the determinant, chose a row or column of values and for each value ignore its row and column and find the determinant of the remaining values. Multiply this by the value and then the sign given by the following pattern.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Sum each of the values found for the row or column to get the determinant. This can be made easier if you use a row or column with one or more zeros in it.

Adjuagte Of a 3x3 Matrix

First the matrix of cofactors (written as $\mathbf{C}$) is found. For each value in the matrix the same calculation is done as for finding the determinant up to and including the sign. Next, the transpose of this is found ($\mathbf{C}^T$) to give the adjugate. This is done by turning all the rows to columns or the other way around, effectively reflecting the matrix over the leading diagonal.

Chapter 11

Modelling

Sometimes we have a real life situation that we want to create a mathematical model for.

Chapter 12

Necessary and Sufficient Conditions

The symbol $\Rightarrow$ means implies.
It helps us to write arguments in a logical and step by step fashion. We can write it in one of three ways:

- A $\Rightarrow$ B A implies B - A is a sufficient condition for B
- A $\Leftarrow$ B B implies A - A is a necessary condition for B
- A $\iff$ B A and B imply each other - Necessary and Sufficient.

Converse Statements

The word converse is used to talk about a statement with the implication sign switched. For a triangle ABC:
Angle $ABC = 90\deg \Rightarrow AB^2 + BC^2 = AC^2$
and
$AB^2 + BC^2 = AC^2 \Rightarrow ABC = 90\deg$

The converse isn't always true.

Chapter 13

Planes and Lines

Planes

Planes are represented using either the vector or Cartesian form:

Cartesian Form

$ax + by + cz = d$

Vector Form

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{r} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = d$

In this example, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the normal vector to the plane. This means that it is perpendicular to the plane. d is a constant and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $\mathbf{r}$ is the general point on the plane. A point is on the plane if it satisfies the equation.

The equation of the plane can be found given the normal vector and one point on the plane. From this information, the dot product can be used to find the constant.

Intersection of Planes

An example of question that could be asked would be given 3 planes, find how and where they intersect. In this question you would get the equations of the planes. These form simultaneous equations that can be solved algebraically or by using matrices. If the determinant of the matrix is not 0, there is no single solution. There are several different scenarios:

- If some of the normal vectors are parallel then the planes are parallel. If all 3 are parallel then there are no solutions. If only two are parallel then there are no solutions unless the planes are the same.
- If there are no parallel planes then you could have a sheaf where the planes meet at a line or no solutions if the equations are inconsistent.

Angle Between Planes

To find the angle between two planes you can use the dot product on the two normal vectors. $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \times |\mathbf{n}_2|}$

Example 1

$$\mathbf{r}_1 \cdot \begin{bmatrix} 13 \\ 3 \\ 6 \end{bmatrix} = 4$$

$$\mathbf{r}_2 \cdot \begin{bmatrix} 7 \\ 15 \\ 12 \end{bmatrix} = 9$$

$$\therefore \cos\theta = \frac{\begin{bmatrix} 13 \\ 3 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 15 \\ 12 \end{bmatrix}}{\begin{vmatrix} 13 \\ 3 \\ 6 \end{vmatrix} \cdot \begin{vmatrix} 7 \\ 15 \\ 12 \end{vmatrix}}$$

$$= \frac{208}{\sqrt{214} \cdot \sqrt{418}}$$

$$\theta = 45.9 \deg$$

Vector Equations of Lines

To specify a line in n dimensions, you need to specify a starting point (any point on the line) and a direction vector. Let a particular point on the line be $A \Rightarrow \vec{OA} = \mathbf{a}$, the general point be $P \Rightarrow \vec{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{r}$ and $\mathbf{d}$ be a vector parallel to the line. The equation of the line can be written $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, $\lambda \in \mathbb{R}$.

This can also be written in Cartesian form as $\begin{cases} x = a_1 + \lambda d_1 \\ y = a_2 + \lambda d_2 \\ z = a_3 + \lambda d_3 \end{cases}$. This can be rearranged to $\frac{x-a_1}{d_1} = \frac{x-y_2}{d_2} = \frac{z-a_3}{d_3}$, $d_1, d_2, d_3 \neq 0$.

There is a special case if there are any zeros in $\mathbf{d}$ as it means that one of the dimensions is constant. In this case, that dimension is left out the Cartesian equation and stated at the end e.g. $\frac{x-2}{5} = \frac{z+1}{8}$, $y = 7$.

Intersection of Lines

There are several cases here:

- Lines are parallel
- Lines not parallel and do not meet
- Lines meet

Lines Are Parallel

To tell if lines are parallel you can look at the direction vectors. If one is multiple of the other then they are parallel. This means that the lines either do not meet or are the same line.

Lines Not Parallel and Do Not Meet

3 simultaneous equations can be made for λ and μ. 2 of them can be used to solve and the third to check. If they are not parallel and do not meet then the first two will be solvable but the solutions will not work in the third equation.

Lines Meet

If the solution works in the third equation then they do meet. To find the point that they meet, λ or μ can be substituted back into their equations.

Angle Between Lines

To find the angle between lines you can use the dot product between the two direction vectors, much like planes. $\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1| \times |\mathbf{d}_2|}$

See method for planes above but use the direction vectors instead of normal vectors.

Lines and Planes

Intersection of Lines and Planes

If the direction vector of the line and the normal vector of the plane are perpendicular (the scalar product of the two is zero) then the line is either parallel to the plane or is in the plane. Otherwise, they will always intersect at exactly one point. To find the intersection you need the general point on the line $(\mathbf{a}_1 + \lambda\mathbf{d}_1, \mathbf{a}_2 + \lambda\mathbf{d}_2, \mathbf{a}_3 + \lambda\mathbf{d}_3)$ and substitute this into the equation of the plane $(n_1(\mathbf{a}_1 + \lambda\mathbf{d}_1) + n_2(\mathbf{a}_2 + \lambda\mathbf{d}_2) + n_3(\mathbf{a}_3 + \lambda\mathbf{d}_3) = d)$, solving for lambda. Then substitute lambda back into the equation of the line to find the point.

Angle Between Lines and Planes

We can use a similar method to finding the angle between two lines and the angle between two planes but must remember that the normal vector of the plane is perpendicular to the plane, so we must make an adjustment after finding the angle which is $\theta - 90$ if $\theta > 90$ or $90 = \theta$ if $\theta < 90$.

Example 1

Find the angle between the plane $6x - 5y + z = 15$ and the line $\frac{x-2}{3} = \frac{y}{7} = 1 - z$.

$$\begin{aligned}
\mathbf{n} &= \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix} \\
|\mathbf{n}| &= \sqrt{36 + 25 + 1} = \sqrt{62} \\
\mathbf{d} &= \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix} \\
|\mathbf{n}| &= \sqrt{9 + 49 + 1} = \sqrt{59} \\
\mathbf{n} \cdot \mathbf{d} &= -18 \\
\cos\theta &= \frac{\mathbf{n} \cdot \mathbf{d}}{|\mathbf{n}| \times |\mathbf{d}|} \\
&= \frac{-18}{\sqrt{62} \times \sqrt{59}} \\
\theta &= 107.3\,\mathrm{deg} \\
\therefore \text{ angle is } \theta - 90 &= 17.3
\end{aligned}$$

Chapter 14

Polynomials

The order / degree of a polynomial is the highest power of the variable it contains. For example, a polynomial is order 3.

Division

Long division is the best method when it is not known if there is a remainder or not. Otherwise it can be done by inspection.

Example 1

Divide $2x^3 - 3x^2 + x - 6$ by $x - 2$

$$
\begin{array}{rl}
 & \;2x^2 + x + 3 \\
x - 2 & \overline{)\,2x^3 - 3x^2 + x - 6} \\
 & -(2x^3 - 4x^2) \\
 & \qquad\qquad x^2 \\
 & \qquad\quad -(x^2 - 2x) \\
 & \qquad\qquad\qquad 3x - 6 \\
 & \qquad\qquad\quad -(3x - 6) \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

$\Rightarrow 2x^3 - 3x^2 + x - 6 = (2x^2 + x + 3)(x - 2)$

The Factor Theorem

We can use the factor theorem to help us to solve algebraic equations of order greater than 2.
The factor theorem is as follows:

If $(x-\alpha)$ is a factor of $f(x)$ then $f(\alpha)=0$ and α is the root of the equation of $f(x)=0$.

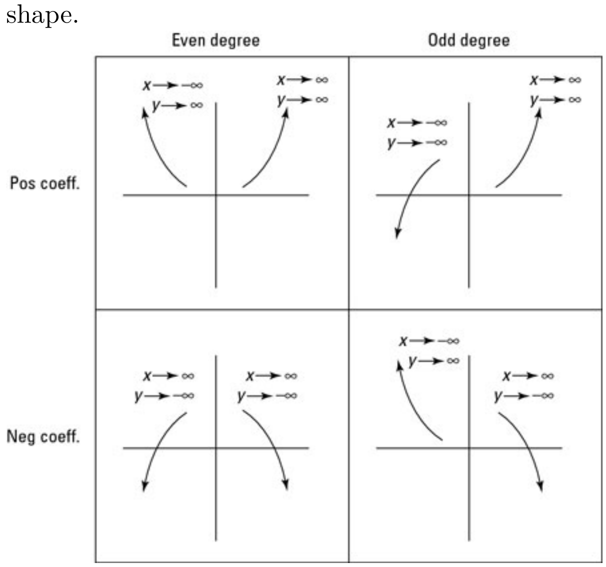
Example 2

Show that $(x-1)$ is a linear factor of $2x^3-5x^2-6x+9$

$$\begin{aligned} f(1)=0 &\Rightarrow (x-1)\text{is a factor by the factor theorum} \\ 2x^3-5x^2-6x+9 &= (x-1)(2x^2-3x-9) \\ &= (x-1)(2x+3)(x-3) \\ &\Rightarrow x = 1, -\frac{3}{2}, 3 \end{aligned}$$

Sketching Polynomials

To sketch polynomials, we must find where it crosses the x-axis and y-axis. We also need to know what order the polynomial is so that we know the general shape.



Turning Points

A point where the gradient is 0. If a polynomial is of order n, it can at most n-1 turning points.

Roots of Polynomials

We denote the root of polynomials using Greek letters starting from α.

Notation

We write:

$$\begin{aligned}\sum \alpha &= \alpha + \beta + \gamma \\ \sum \alpha\beta &= \alpha\beta + \beta\gamma + \gamma\alpha \\ \sum \alpha^2\beta &= \alpha^2\beta + \beta^2\alpha + \alpha^2\gamma + \gamma^2\alpha + \gamma^2\beta + \beta^2\gamma\end{aligned}$$

Cubics

The general equation of a cubic can be written $ax^3 + bx^2 + cx + d = 0 \Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$, $a \neq 0$. If we know the roots α, β, γ it can also be written $(x - \alpha)(x - \beta)(x - \gamma) = 0$. If this is expanded we get $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$. Therefore:

$$\begin{aligned}\sum \alpha &= \frac{-b}{a} \\ \sum \alpha\beta &= \frac{c}{a} \\ \alpha\beta\gamma &= \frac{-d}{a}\end{aligned}$$

From this we can find the coefficients given information about the roots.

Quartics

The general equation of a quartic can be written $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{3}{a} = 0$ with roots α, β, γ, δ. As before, it can also be expressed in the form $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0$. Therefore, we get that:

$$\begin{aligned}\sum \alpha &= \frac{-b}{a} \\ \sum \alpha\beta &= \frac{c}{a} \\ \sum \alpha\beta\gamma &= \frac{-d}{a} \\ \alpha\beta\gamma\delta &= \frac{c}{a}\end{aligned}$$

Related Roots

If you are given equations for the roots in terms of the roots of a different polynomial you can substitute them in to find the new polynomial.

Example 1

$5x^3 - x^2 + 4x + 1 = 0$ has roots α, β, γ. Find a cubic with the roots $2\alpha + 3$, $2\beta + 3$, $2\gamma + 3$.

$$\begin{aligned} y &= 2x + 3 \\ \Rightarrow x &= \frac{y-3}{2} \end{aligned}$$

$$\begin{aligned} 5(\frac{y-3}{2})^3 - (\frac{y-3}{2})^2 + 4(\frac{y-3}{2}) + 1 &= 0 \\ 5(y-3)^3 - 2(y-3)^2 + 16(y-3) + 1 &= 0 \\ 5(y^3 - 9y^2 + 27y - 27) - 2(y^2 - 6y + 9) + 16(y-3) + 1 &= 0 \\ 5y^3 - 47y^2 + 163y - 193 &= 0 \end{aligned}$$

Chapter 15

Proof

Proof by Deduction

This is the most direct proof - generally using algebra - to prove it for all cases.

Example 1

Prove than the sum of four consecutive integer is even.
$(n) + (n + 1) + (n + 2) + (n + 3) = 4n + 6 = 2(2n + 3) \Rightarrow$ The sum is even.

Proof by Exhaustion

Example 2

No square number end in 2.
Only need to look at unit digit.
None of their squares end in 2 $\Rightarrow$ No square number ends in 2.

Disproof by Counter Example

We find an example that breaks the rule.
The values of $n^2 + n + 41$ is prime.
Let $n = 41$

$= 41^2 + 2 \times 41$
$= 41(41 + 2)$
$= 41 \times 43$
$\Rightarrow$ Not prime

Chapter 16

Sequences and Series

A sequence is an ordered list of terms. A series is the sum of a sequence.

Notation

We can write the terms of a sequence as $a_1, a_2, a_3, \ldots, a_r$
The sum of the terms $a_1 + a_2 + a_3 + \cdots + a_n$ can be written as $\sum_{r=1}^{n} a_r$

Summation Using Standard Results

There are several standard results that can be used to more easily evaluate series.

$$\sum_{1}^{n} r = \frac{(n+1)n}{2}$$
$$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{1}^{n} r^3 = \frac{(n+1)^2 n^2}{4}$$

Example 1

Find $\sum_{r=1}^{n}(2r+1)(3r+4)$

$$\begin{aligned}\sum_{r=1}^{n}(2r+1)(3r+4) &= \sum_{r=1}^{n}6r^2+11r+4\\ &= 6(\frac{n(n+1)(2n+1)}{6})+11(\frac{(n+1)n}{2})+4n\\ &= 2n^3+\frac{17}{2}n^2+\frac{21}{2}n\\ &= \frac{n}{2}(4n^2+17n+21)\end{aligned}$$

Method of Differences

This can be used to find series without using standard results.

Example 1

Find $\sum_{r=1}^{n} r(r+1)$ without quoting standard results.

$$\begin{aligned}&r(r+1)(r+2)-(r-1)(r)(r+1)\\ &= r(r+1)((r+2)-(r-1))\\ &= 3r(r+1)\end{aligned}$$

$$\sum_{r=1}^{n}3r(r+1) = \sum_{r=1}^{n}r(r+1)(r+2)-(r-1)(r)(r+1)$$

$$\begin{aligned}&= (1\times 2\times 3)-(0\times 1\times 2)\\ &+ (2\times 3\times 4)-(1\times 2\times 3)\\ &+(3\times 4\times 5)-(2\times 3\times 4)\\ &+\dots\\ &+ (n-2)(n-1)(n)-(n-3)(n-2)(n-1)\\ &+ (n-1)(n)(n+1)-(n-2)(n-1)(n)\\ &+(n)(n+1)(n+2)-(n-1)(n)(n+1)\\ &=n(n+1)(n+2)-(0\times 1\times 2)\\ &=n(n+1)(n+2)\\ &\therefore\sum_{r=1}^{n}r(r+1)=\frac{1}{3}n(n+1)(n+2)\end{aligned}$$

Chapter 17

Simultaneous Equations

Simultaneous equations are set of equations that are all true at the same time. This is often shown using curly braces.

Example 1:

Note in this example that we have one quadratic and one linear equation. The linear equation should always be substituted in to the quadratic as it makes it much easier to solve.

$$\begin{cases} y &= 4 - 2x \\ y &= x^2 + x \end{cases}$$

$$\begin{aligned} 4 - 2x &= x^2 + x \\ 0 &= x^2 + 3x - 4 \\ x &= -4,\ x = 1 \end{aligned}$$

Solving With Matrices

Linear simulations equations can be solved by representing them with matrices and vectors. The equations $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ can be written $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ Therefore, x, y and z can be found by rearranging the equation and finding the inverse of the matrix. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Chapter 18

Solving Equations

Equations with Indices

If we can write an equation in terms of the same bases, we can equate the powers to help us solve them.

Example 1

$$\begin{cases} 2^{x-y} = 64^2 \\ 3^{x+y} = 1 \end{cases}$$

Find x, y

$$\begin{cases} 2^{x-y} = 2^{1}2 \Rightarrow x - y = 12 \\ 3^{x+y} = 3^0 \Rightarrow x + y = 0 \end{cases}$$

$2x = 12$

$$\begin{cases} x = 6 \\ y = -6 \end{cases}$$

Quadratics

Factorising

$(x - \alpha)$ is a factor $\iff x = \alpha$
Given an equation of the form x^2+bx+c, the roots should add to b and multiply to c.

Difference of two squares $x^2 - y^2 = (x + y)(x - y)$

Completing the square Given an equation $x^2 + bx + c$, complete square form is $(x + \frac{b}{2}) - \frac{b^2}{4} + c$.
This can easily be rearranged to get x as the subject.
Given the equation $ax^2 + bx + c$, the completed square form can be rearranged to produce the quadratic formula.

The Quadratic Formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for an equation $ax^2 + bx + c = 0$.

The Discriminant

Looking at the quadratic formula, you can see that there would be no real solutions if $b^2 - 4ac < 0$.
$> 0 \iff$ 2 real roots
$= 0 \iff$ 1 real root (repeated)

Hidden Quadratics

Polynomials of higher power can be solved like quadratics if they fit the form $ax^{2d} + bx^d + c$.

Example 2

For which values of k, does the following equation have repeated roots?

$$\begin{aligned} x^2 + kx + 2k &= 0 \\ k^2 - 8k &= 0 \\ k(k-8) &= 0 \\ k = 0 \qquad & \qquad k = -8 \end{aligned}$$

Example 3

$$\begin{aligned} x^4 - 3x^2 - 4 &= 0 \\ (x^2 - 4)(x^2 + 1) &= 0 \\ x &= \pm 2 \end{aligned}$$

Chapter 19

Trigonometry

The Radian

π radians = 180 degrees

Arc length

$$\begin{aligned} l &= \frac{\theta 2\pi r}{2\pi} \\ &= r\theta \end{aligned}$$

Sector area

$$\begin{aligned} A &= \frac{\theta}{2\pi}\pi r^2 \\ &= \frac{r^2\theta}{2} \end{aligned}$$

Trigonometry

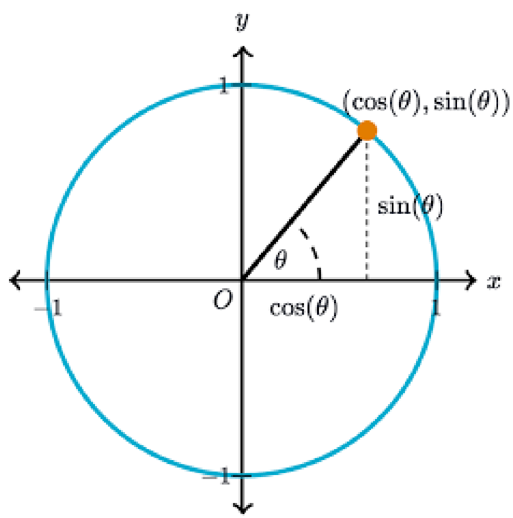
Values of sin, cos and tan can be worked out by using triangles.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	NaN

The unit circle

Identities

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta \equiv 1$$



sin and cos graphs have a period of 2π, tan has a period of π.
sin and tan have a rotational symmetry about the origin.
cos has a line of symmetry on the y axis

$$\cos -\theta = \cos\theta$$
$$\sin -\theta = -\sin\theta$$
$$\tan -\theta = -\tan\theta$$

Solving Equations

Be careful not to divide by an expression that may be 0 as you may lose solutions. Also note that there may be many solutions in a given range. Drawing a CAST diagram or graph sketch may be useful.

Example 1

$$
\begin{aligned}
\text{Solve} \sin\theta - 2\cos\theta &= 0 \qquad\qquad \text{for } 0 \leqslant \theta < 2\pi \\
\sin\theta &= 2\cos\theta \\
\frac{\sin\theta}{\cos\theta} &= 2 \\
\tan\theta &= 2 \\
\theta &= \arctan 2 \\
\theta &= 1.107, 4.249
\end{aligned}
$$

Note, two solutions.

Example 2

$$
\begin{aligned}
\text{Solve } 2\cos\theta\sin\theta &= \cos\theta \qquad\qquad \text{for } 0 \leqslant \theta < 2\pi \\
2\cos\theta\sin\theta - \cos\theta &= 0 \\
\cos\theta(2\sin\theta - 1) &= 0 \\
\cos\theta = 0 \qquad\qquad\qquad & \sin\theta = \frac{1}{2} \\
\theta &= \frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{3\pi}{2}
\end{aligned}
$$

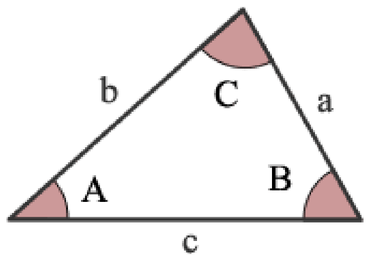
Example 3

$$
\begin{aligned}
\text{Solve } \sin^2\theta + \sin\theta &= \cos^2\theta \qquad\qquad \text{for } 0 \leqslant \theta < 2\pi \\
\sin^2\theta + \sin\theta &= 1 - \sin^2\theta \\
2\sin^2\theta + \sin\theta &= 1 \\
2\sin^2\theta + sin\theta - 1 &= 0 \\
(\sin\theta + 1)(2\sin\theta - 1) &= 0 \\
\sin\theta = -1 \qquad\qquad\qquad & \sin\theta = \frac{1}{2} \\
\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} &
\end{aligned}
$$

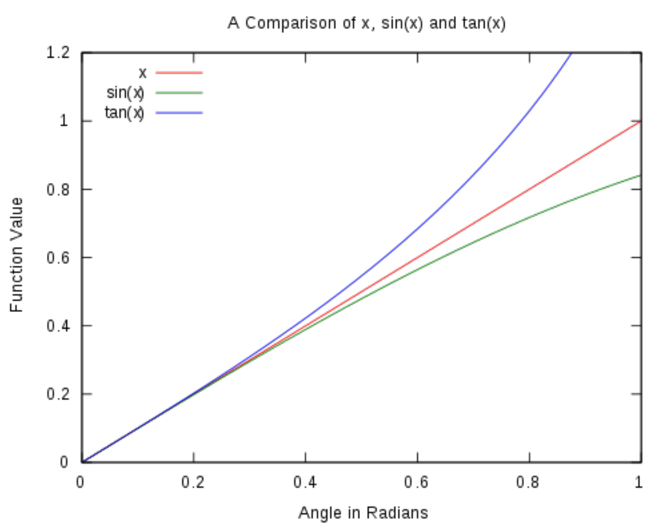
Sine and Cosine rules

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Area of a triangle $= \frac{1}{2}ab\sin C$



Small Angle Approximations



At small angles, trigonometric functions can be approximated. For small θ:

- $\sin\theta \simeq \theta$
- $\tan\theta \simeq \theta$
- $\cos\theta \simeq 1 - \frac{\theta^2}{2}$

These approximations can be found in the first few terms of the Taylor series expansions of the trigonometric functions. They can be used to more easily solve functions with small angles.

Further Trig Functions

The reciprocals of trig functions have their own notations.

$f(x)$	$\frac{1}{f(x)}$
$\sin x$	$\csc x$
$\cos x$	$\sec x$
$\tan x$	$\cot x$

This means that there are some more identities.

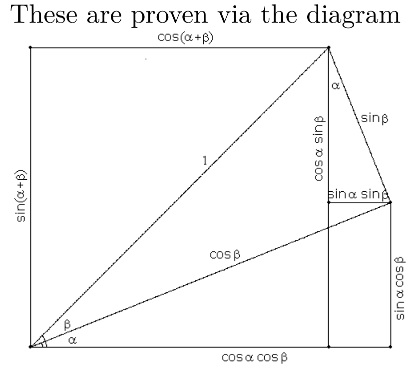
$$
\begin{aligned}
1 + \tan^2 \theta &= \sec^2 \theta \\
\cot^2 \theta + 1 &= \csc^2 \theta \\
\frac{\cos \theta}{\sin \theta} &= \cot \theta
\end{aligned}
$$

Compound Angle Formulae

These are given in the formulae book but it is quicker if you just learn them.

$$
\begin{aligned}
\sin \alpha \pm \beta &= \sin \alpha \cos \beta \pm \sin \beta \cos \alpha \\
\cos \alpha \pm \beta &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\
\tan \alpha \pm \beta &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}
\end{aligned}
$$

These are proven via the diagram below.



Double Angle Formulae

These come from the compound angle formulae but we just make both the angles the same.

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
&= 1 - 2\sin^2\alpha \\
&= 2\cos^2\alpha - 1 \\
\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

Tripe Angle Formulae

These can be derived from the compound angle formulae as well.

$$\begin{aligned}
\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha \\
\cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha
\end{aligned}$$

The form $R\cos x \pm \alpha$ and $R\sin x \pm \alpha$

To make them easier to solve, trigonometric expressions can be written in terms of sin or cosine entirely.

Example 1

Write $2\sin x + 3\cos x$ in the form $R\cos x - \alpha$ where $R > 0$ and $0 < \alpha < 90$

$$\begin{aligned}
R\cos x - \alpha &\equiv R(\cos x \cos alpha + \sin x \sin\alpha) \\
2\sin x + 3\cos x &= R\cos x\cos\alpha + R\sin x\sin\alpha \\
&\Rightarrow \begin{cases} R\sin\alpha = 2 \\ R\cos\alpha = 3 \end{cases} \\
\Rightarrow \tan\alpha &= \frac{2}{3} \\
\alpha &= 33.69 \\
&\Rightarrow \begin{cases} R^2\sin^2\alpha = 4 \\ R^2\cos^2\alpha = 9 \end{cases} \\
\Rightarrow R^2(\sin^2\alpha + \cos^2\alpha) &= 4 + 9 \\
\Rightarrow R &= \sqrt{13} \\
\therefore 2\sin x + 3\cos x &= \sqrt{13}\cos(x - 33.69)
\end{aligned}$$

The output is scaled from between -1 and 1 to $-sqrt13$ and $sqrt13$

There is a short cut to find R as it is just square root of the sum of the two coefficients squared.

Chapter 20

Vectors

Idea

- (physics) Linear displacement of objects.
- (maths) Do geometric algebra.
- Two vectors are the same if they have the same direction and magnitude.
- Vectors represent directed line segments.

Representation

A vector of reals can be represented by $\mathbb{R}^n$ where n is the number of items in the vector.

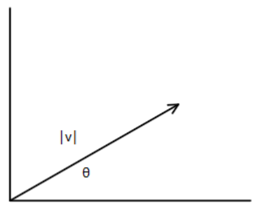
Component representation:

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Basis representation:

$\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_n\vec{e_n}$
In text book $\vec{e_1}, \vec{e_2}, \vec{e_3}$ etc. may be written as $\vec{i}, \vec{j}, \vec{k}$
$\vec{e_n}$ represents a unit vector along one of the axes. E.g. in 2 dimensions, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Polar representation:



Where $\vec{v} = |\vec{v}| \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$

Operations

Addition

Vector addition is commutative (order doesn't matter) and associative (bracketing doesn't matter).

Example 1:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 + q_1 \\ p_2 + q_2 \end{bmatrix}$$

$$\begin{aligned} \vec{u} + \vec{w} + \vec{v} &= \vec{u} + (\vec{w} + \vec{v}) \\ &= \vec{u} + (\vec{v} + \vec{w}) \end{aligned}$$

Scalar Multiplication

$\lambda\vec{v} \qquad \lambda \in \mathbb{R}$

λ Scales the vector but keeps the direction if positive, reverses if negative.

Example 2:

$$\lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{bmatrix}$$

$$|\lambda\vec{v}| = \lambda|\vec{v}|$$

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 \cdots + v_n^2}$$

What happens when $\lambda = 0$? We need to define a zero vector:

$$\vec{o} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Zero vector is the additive identity

Scalar product

Scalar product of $\vec{a}$,$\vec{b}$ is written $\vec{a} \cdot \vec{b}$ where:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}|\left|\vec{b}\right| \cos\theta \\ \therefore \mathbb{R}^n \cdot \mathbb{R}^n &\in \mathbb{R} \end{aligned}$$

and θ is the angle between the two vectors.
This has several interesting properties:

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$
- Dot product is commutative e.g. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a}(\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $(\lambda\vec{a}) \cdot (\mu\vec{b}) = \lambda\mu(\vec{a} \cdot \vec{b})$

$$\begin{aligned} \text{If } \vec{a} &= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\ \vec{b} &= \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ \vec{a} \cdot \vec{b} &= (a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n}) \cdot (b_1\vec{e_1} + b_2\vec{e_2} + \cdots + b_n\vec{e_n}) \\ &= a_1\vec{e_1} \cdot b_1\vec{e_1} + a_1\vec{e_1} \cdot b_2\vec{e_2} + \cdots + a_n\vec{e_n} \cdot b_n\vec{e_n} \text{ } (n^2 \text{ terms}) \\ &= a_1b_1 + 0 + \cdots + a_nb_n \text{ (terms are 0 as } e_n \perp e_h \text{ } h \neq n) \\ &= a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n \end{aligned}$$

Example 3:

Find the angle between $\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$ and $2\vec{e_1} - 3\vec{e_2} + \vec{e_3}$

$$2\vec{e_1} - 3\vec{e_2} + \vec{e_3} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\cos\theta &= \frac{\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}}{\left\|\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}\right\| \cdot \left\|\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right\|} \\
&= \frac{-11}{\sqrt{3^2 + 5^2 + 2^2} \cdot \sqrt{2^2 + (-3)^2 + 1^2}} \\
&= \frac{-11}{2\sqrt{133}} \\
\theta &= \arccos \frac{-11}{2\sqrt{133}} = 118.5\,\text{deg}
\end{aligned}$$

Vectors and coordinates

Vector space is any type that the same algebraic rule as a vector (commutativity, associativity and distributivity).

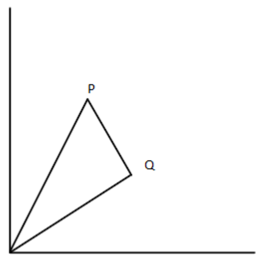
How can we link vectors back to coordinates?

A vector can represent the displacement from the origin of a point. $\vec{v} = \overrightarrow{OP}$ Represents the displacement of point P relative to the origin as we always need a frame of reference. This is called a position vector.

Applications of vectors

We can use vectors to represent shapes with position vectors to vertices. From there you can then do calculations with the vectors.
They can also be used to represent positional and movement data.

Example 3:



$$\overrightarrow{PQ} = \overrightarrow{OP} - \overrightarrow{OP}$$

You can also use vectors to represent the equation of a line:

$$\begin{aligned} v &\in \mathbb{R}^n \\ \vec{x} : \mathbb{R} &\to \mathbb{R}^n \\ t &\mapsto \vec{v}t + \vec{x_0} \end{aligned}$$