Binomial Expansion

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Polynomial Binomial Expansion

Binomial Expansion is a way of easily expanding brackets with two terms to a power. The expansion is given by the general formula:

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^{n-r} \cdot y^r$$

where ${}^nC_r$ is equivalent to $\binom{n}{c}$ and defined as:

$${}^nC_r = \frac{n!}{(n-r)! \cdot r!}$$

Note that since pascals triangle is symmetrical it doesn't matter which term way around the powers are.

Example 1:

$$\begin{aligned}(2+x)^4 &= ({}^4C_0 \cdot 2^4 \cdot x^0) + ({}^4C_1 \cdot 2^3 \cdot x^1) + ({}^4C_2 \cdot 2^2 \cdot x^2) + ({}^4C_3 \cdot 2^1 \cdot x^3) + ({}^4C_4 \cdot 2^0 \cdot x^4) \\ &= (1 \cdot 16 \cdot 1) + (4 \cdot 8 \cdot x) + (6 \cdot 4 \cdot x^2) + (4 \cdot 2 \cdot x^3) + (1 \cdot 1 \cdot x^4) \\ &= x^4 + 8x^3 + 24x^2 + 32x + 16\end{aligned}$$

Example 2:

Find the x^3 coefficient in $(3+2x)^5$:
let n = 5, r = 3

$$\begin{aligned} 3^{rd} &= ^5C_3 \cdot 3^{5-3} \cdot (2x)^3 \\ &= 10 \cdot 3^2 \cdot 8x^3 \\ &= 720x^3 \end{aligned}$$

$\therefore coefficent = 720$

Example 3:

Find the constant term in $(x^2 + \frac{1}{x})^9$:
For the two terms to cancel out, the power of $\frac{1}{x}$ must be twice the power of x^2 so that when they are multiplied together they produce $x^0 = 1$, meaning the term is constant or independent $\implies$ We want the term where x^2 is to the power of 3 and $\frac{1}{x}$ is to the power of 6 $\implies$ constant term is:

$$\begin{aligned} t &= {}^9C_3 \cdot (x^2)^3 \cdot \frac{1}{x}^6 \\ &= 84 \cdot x^6 \cdot x^{-6} \\ &= 84 \end{aligned}$$

General Binomial Expansion

This is a more general version of binomial expansion that doesn't require the power to be a positive integer.

$$\begin{aligned} (1+x)^n &\simeq 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} \dots \\ &= 1 + \sum_{i=1}^{\infty} (\prod_{j=0}^{i-1} .x^i.i!) \end{aligned}$$

Where $n \in \mathbb{R}, |x| < 1$.

Example 1

Expand $(1+2x)^{-\frac{1}{2}}$ up to and including the x^3 term.

$$
\begin{aligned}
(1+(2x))^{-\frac{1}{2}} &\simeq 1+(\frac{1}{2})(2x)+\frac{(-\frac{1}{2})(-\frac{3}{2})(2x)^3}{2!}+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(2x)^3}{3!} \\
&\simeq 1-x+\frac{3x^2}{2}-\frac{5x^3}{2}
\end{aligned}
$$

Valued for $|2x|<1 \Rightarrow |x|<\frac{1}{2}$