Polynomials

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April 16, 2020

The order / degree of a polynomial is the highest power of the variable it contains. For example, a polynomial is order 3.

Division

Long division is the best method when it is not known if there is a remainder or not. Otherwise it can be done by inspection.

Example 1

Divide $2x^3 - 3x^2 + x - 6$ by $x - 2$

$$
\begin{array}{r|l}
 & 2x^2 + x + 3 \\
\hline
x - 2 & 2x^3 - 3x^2 + x - 6 \\
 & -(2x^3 - 4x^2) \\
 & \qquad\quad x^2 \\
 & \qquad -(x^2 - 2x) \\
 & \qquad\qquad\quad 3x - 6 \\
 & \qquad\qquad -(3x - 6) \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

$\Rightarrow 2x^3 - 3x^2 + x - 6 = (2x^2 + x + 3)(x - 2)$

The Factor Theorem

We can use the factor theorem to help us to solve algebraic equations of order greater than 2.

The factor theorem is as follows:
If $(x-\alpha)$ is a factor of $f(x)$ then $f(\alpha)=0$ and α is the root of the equation of $f(x)=0$.

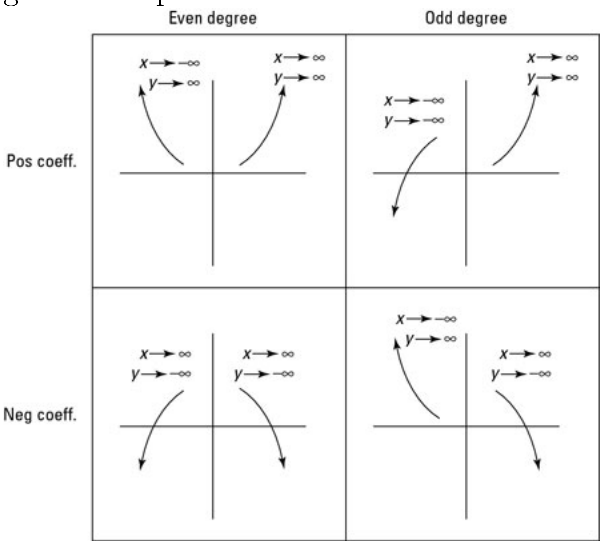
Example 2

Show that $(x-1)$ is a linear factor of $2x^3-5x^2-6x+9$

$$\begin{aligned} f(1)=0 &\Rightarrow (x-1)\text{is a factor by the factor theorum} \\ 2x^3-5x^2-6x+9 &= (x-1)(2x^2-3x-9) \\ &= (x-1)(2x+3)(x-3) \\ &\Rightarrow x=1,-\frac{3}{2},3 \end{aligned}$$

Sketching Polynomials

To sketch polynomials, we must find where it crosses the x-axis and y-axis. We also need to know what order the polynomial is so that we know the general shape.



Turning Points

A point where the gradient is 0. If a polynomial is of order n, it can at most n-1 turning points.

Roots of Polynomials

We denote the root of polynomials using Greek letters starting from α.

Notation

We write:

$$\sum \alpha = \alpha + \beta + \gamma$$
$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha$$
$$\sum \alpha^2\beta = \alpha^2\beta + \beta^2\alpha + \alpha^2\gamma + \gamma^2\alpha + \gamma^2\beta + \beta^2\gamma$$

Cubics

The general equation of a cubic can be written $ax^3 + bx^2 + cx + d = 0 \Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$, $a \neq 0$. If we know the roots α, β, γ it can also be written $(x-\alpha)(x-\beta)(x-\gamma) = 0$. If this is expanded we get $x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$. Therefore:

$$\sum \alpha = \frac{-b}{a}$$
$$\sum \alpha\beta = \frac{c}{a}$$
$$\alpha\beta\gamma = \frac{-d}{a}$$

From this we can find the coefficients given information about the roots.

Quartics

The general equation of a quartic can be written $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{3}{a} = 0$ with roots α, β, γ, δ. As before, it can also be expressed in the form

$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$. Therefore, we get that:

$$\sum \alpha = \frac{-b}{a}$$
$$\sum \alpha\beta = \frac{c}{a}$$
$$\sum \alpha\beta\gamma = \frac{-d}{a}$$
$$\alpha\beta\gamma\delta = \frac{c}{a}$$

Related Roots

If you are given equations for the roots in terms of the roots of a different polynomial you can substitute them in to find the new polynomial.

Example 1

$5x^3 - x^2 + 4x + 1 = 0$ has roots α, β, γ. Find a cubic with the roots $2\alpha + 3$, $2\beta + 3$, $2\gamma + 3$.

$$y = 2x + 3$$
$$\Rightarrow x = \frac{y-3}{2}$$

$$5(\frac{y-3}{2})^3 - (\frac{y-3}{2})^2 + 4(\frac{y-3}{2}) + 1 = 0$$
$$5(y-3)^3 - 2(y-3)^2 + 16(y-3) + 1 = 0$$
$$5(y^3 - 9y^2 + 27y - 27) - 2(y^2 - 6y + 9) + 16(y-3) + 1 = 0$$
$$5y^3 - 47y^2 + 163y - 193 = 0$$