Sequences and Series

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A sequence is an ordered list of terms. A series is the sum of a sequence.

Notation

We can write the terms of a sequence as $a_1, a_2, a_3, \ldots, a_r$
The sum of the terms $a_1 + a_2 + a_3 + \cdots + a_n$ can be written as $\sum_{r=1}^{n} a_r$

Summation Using Standard Results

There are several standard results that can be used to more easily evaluate series.

$$\sum_{1}^{n} r = \frac{(n+1)n}{2}$$

$$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{1}^{n} r^3 = \frac{(n+1)^2 n^2}{4}$$

Example 1

Find $\sum_{r=1}^{n}(2r+1)(3r+4)$

$$\begin{aligned}\sum_{r=1}^{n}(2r+1)(3r+4) &= \sum_{r=1}^{n} 6r^2+11r+4 \\ &= 6(\frac{n(n+1)(2n+1)}{6}) + 11(\frac{(n+1)n}{2}) + 4n \\ &= 2n^3 + \frac{17}{2}n^2 + \frac{21}{2}n \\ &= \frac{n}{2}(4n^2+17n+21)\end{aligned}$$

Method of Differences

This can be used to find series without using standard results.

Example 1

Find $\sum_{r=1}^{n} r(r+1)$ without quoting standard results.

$$\begin{aligned}&r(r+1)(r+2)-(r-1)(r)(r+1) \\ &= r(r+1)((r+2)-(r-1)) \\ &= 3r(r+1)\end{aligned}$$

$$\sum_{r=1}^{n} 3r(r+1) = \sum_{r=1}^{n} r(r+1)(r+2)-(r-1)(r)(r+1)$$

$$\begin{aligned}&= (1\times2\times3)-(0\times1\times2) \\ &+ (2\times3\times4)-(1\times2\times3) \\ &+(3\times4\times5)-(2\times3\times4) \\ &+\dots \\ &+ (n-2)(n-1)(n)-(n-3)(n-2)(n \\ &+ (n-1)(n)(n+1)-(n-2)(n-1)(n) \\ &+(n)(n+1)(n+2)-(n-1)(n)(n+1) \\ &=n(n+1)(n+2)-(0\times1\times2) \\ &=n(n+1)(n+2) \\ &\therefore \sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)\end{aligned}$$