Physics Beyond

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Kinematics

How to describe motion? (mathematically)

→ In maths we need a clear and precise definition.

→ Motion is change in position over time.

- ⇝ What is space?
- ⇝ What is time?

→ How to describe motion?

- ⇝ Need a reference point / origin.
- ⇝ Intuition about vectors.
  - ↪ Rule of displacement from one point to another along a straight line
  - ↪ Vector - from Latin "vehere" - to carry

To describe motion we need $f(t) \to x$ and $g(t) \to y$.

$$\begin{aligned} \vec{x} : \mathbb{R} &\to \text{Vector Space} \\ t &\mapsto \vec{x}(t) \end{aligned}$$

What is vector space?

→ Euclidean 3-dimensional space.

→ For example, vector space of a 3 tuple of reals can be written as $\mathbb{R}^3$

$\rightarrow$ We say V is a vector space if:

1. $\mathbb{R} \cdot V \rightarrow V$
2. Addition is commutative, associative, and has neutral element $\vec{O}$
3. $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v} \qquad \lambda, \mu \in \mathbb{R} \qquad \vec{v} \in V$
4. $(\lambda\mu)\vec{v} = \lambda(\mu\vec{v})$
5. $1 \cdot \vec{v} = \vec{v}$

$\rightarrow$ We need to introduce a coordinate system.

In 2 dimensions

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In n dimensions

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Where 1 is at index i

$\rightarrow$ We can describe using unitary vectors:

Two sets $A, B \qquad A \times B \quad \{(a, b) \mid a \in A, b \in B\}$

$A \times A = A^2$ Where $\times$ is the cartesian product

What is happening mathematically?

We have constructed a mapping $V \rightarrow \mathbb{R}^2$; it is $1 \rightarrow 1$
$\therefore$ For any vector, a pair of reals visa versa.

$$\vec{x} \mapsto (x_1, x_2) \qquad s.t \qquad \vec{x} = x_1e_1 + x_2e_2$$

1. Note that for a basis, e_1, e_2 this mapping is one to one and onto (bijection)
   $\vec{x}(f) = (ae_1, be_2) \leftarrow$ Linear combination of e_1 and e_2

2. This means that we have an inverse mapping.

$$\begin{aligned} \mathbb{R} &\to E \\ \begin{bmatrix} \lambda \\ \mu \end{bmatrix} &\mapsto \lambda e_1 + \mu e_2 \end{aligned}$$

   Coordinate systems translate $\mathbb{R}^2$ to vector E

3. This mapping depends on the chosen coordinate system.

   $\to$ Coordinate system: (origin, two basis vectors)
   - $\hookrightarrow$ $(\vec{o}, \vec{e}_1, \vec{e}_2)$
   - $\hookrightarrow$ $\vec{e}_1, \vec{e}_1$ are linearly independent.
   - $\hookrightarrow$ We assume $\vec{e}_1 \perp \vec{e}_1$ $\to$ Is orthogonal (meet at 90 deg / $\frac{\pi}{2}$)

We have an improvement of our description:

$$\vec{x} : \mathbb{R} \to \mathbb{R}$$

$$t \mapsto \vec{x}(t) \qquad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad x_1, x_2 : \mathbb{R} \to \mathbb{R}$$

Operators

$$\begin{aligned} + :\ & \mathbb{R}^n, \mathbb{R}^n \to \mathbb{R}^n \\ (\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}) &\mapsto \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \cdot :\ & \mathbb{R}, \mathbb{R}^n \to \mathbb{R}^n \\ (\lambda, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}) &\mapsto \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} \end{aligned}$$

$$\mathbb{R}^x = \{f : x \to \mathbb{R}\} \qquad \text{Where } x \text{ is any set and f is a function}$$

Maths foundations

Current foundation: Set theory. What does that mean?

→ Everything must be related back to sets.

→ Relatively modern idea (2nd half of 20th century)

Examples

1. Numbers:

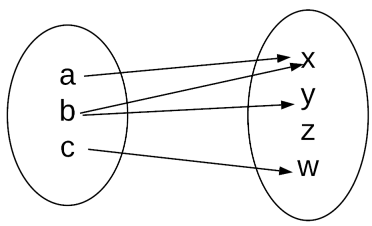
$$\begin{aligned}
&0 = \emptyset \text{ - empty set} \\
&1 = \{\emptyset\} \text{ - set of the empty set} \\
&2 = \{\emptyset, \{\emptyset\}\} \text{ - set of the empty set and a set of the empty set} \\
&\ \vdots
\end{aligned}$$

   This means that natural numbers can be modelled using set theory.

   → Model ideas = helps us to understand abstract ideas.

2. Ordered pair: (x, y) - order matters
   How can we model this using sets? (sets are unordered)
   $\{x, y\}$ doesn't work as unordered.
   $\{x, \{x, y\}\}$ could work for example.

3. Relation



   Relation $\mathbb{R} \subseteq A \times B$
   $(A \times B = \{(a, b) \mid a \in A, b \in B\}$
   Example 1:

$$\begin{aligned}
x \leqslant y &\in \mathbb{R} \\
\leqslant &\subseteq \mathbb{R} \times \mathbb{R} \\
(x, y) &\in \leqslant
\end{aligned}$$

Example 2 - Equivalence relation:
(Example of vectors)
<u>Vectors:</u>

(a) Idea: Rule of displacement of things along a straight line by a certain length (physics)
(b) Representation: Directed line segment

How to model a vector?
Consider the set L of all line segments in the plane

→ Subdivide L into disjointed subsets $[\vec{AB}]$ of line segments parallel, of same length and same orientation as $\vec{AB}$
$\vec{v} = [\vec{AB}]$ is a (model of a) vector.
→ Define a relation $\sim \subseteq L \times L$ ($\sim$ means equivalent to)

$$\begin{aligned} \vec{AB} \sim \vec{CD} :&\iff \text{They represent the same vector} \\ &\iff [\vec{AB}] = [\vec{CD}] \\ &\iff \vec{AB} \parallel \vec{CD},\ \left\|\vec{AB}\right\| = \left\|\vec{CD}\right\| \end{aligned}$$

In general, these three properties define what is called an equivalence relation.

4. Functions / maps / mapping
$\rho : A \to B \qquad x \mapsto f(x)$
Functions can be considered a type of relation. A relation is called a function if it is right-unique and left-total.

$$\begin{aligned} (x, y) \wedge (x, z) \Rightarrow & y = z \\ & vx \in A \\ & \exists y \in B \\ & (x, y) \in f \\ & y = f(x) \end{aligned}$$

Example:

$$\begin{aligned} &\rho : \mathbb{R} \to \mathbb{R} && x \mapsto x^2 \\ &\rho \subseteq \mathbb{R} \times \mathbb{R} && \rho = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \end{aligned}$$

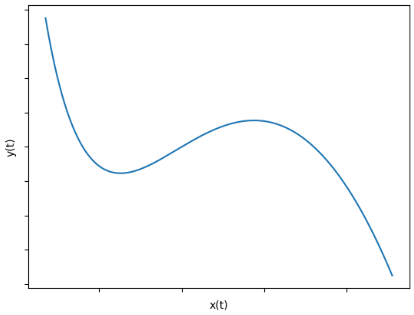
The relation is the graph of the function.

Motion

Two areas:

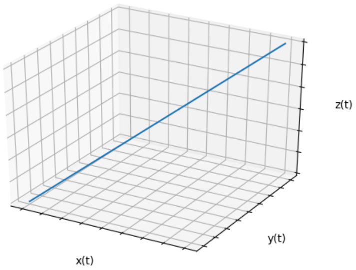
- Kinematics (mathematical modelling)
- Dynamics (what causes motion)

Def: Motion in the change in position in time.
position $\to$ space   time $\to$ time
$\rightsquigarrow$ Space + time too abstract $\to$ need to simplify
Space $\mapsto$ Euclidean geometry
Time $\mapsto$ Use $\mathbb{R}$ (Time is just a parameter)



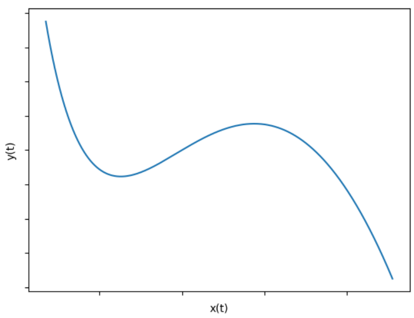
Say this is the graph of $p(t)$
$p : \mathbb{R} \to$ Space
$\rightsquigarrow$ Vectors + coordinates (reference frame)

$$
\begin{aligned}
x :& \mathbb{R} \to \mathbb{R}^3 \\
\vec{x} &= x_1 e_1 + x_2 e_2 + x_3 e_3 \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$



$\rightsquigarrow$ Algebra of geometric vectors translates directly into algebra of tuples.

$$\vec{x} + \vec{y} \longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$
$$\lambda \vec{x} \longleftrightarrow \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$$



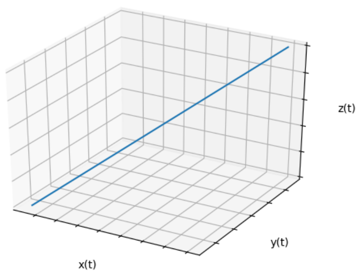
$$\begin{aligned} \vec{x} : &\mathbb{R} \to \mathbb{R}^3 \\ t &\mapsto \vec{x}(t) \end{aligned}$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Exaples

1. Motion along a line (with constant velocity)

$$\begin{aligned}\vec{x}(t) &= \vec{x_0} + t\vec{v} \\ \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} &= \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= \begin{bmatrix} x_{01} + tv_1 \\ x_{02} + tv_2 \\ x_{03} + tv_3 \end{bmatrix}\end{aligned}$$

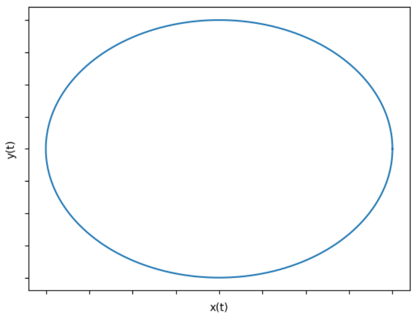


2. Circular motion
   $\vec{x}(t) = r \begin{bmatrix} \cos(\rho t) \\ \sin(\rho t) \end{bmatrix}$
   Where ρ some function and r is the radius.
   If it is uniform:

$$\vec{x}(t) = r \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}$$

$$\omega \in \mathbb{R}|\{0\} \qquad \omega = \frac{2\pi}{T}$$

   Where T is the time period.



3. Uniform spiral It is just like a circle, but with linear motion in the z-axis.

$$\vec{x}(f) = \begin{bmatrix} r\cos\omega t \\ r\sin\omega t \\ vt \end{bmatrix}$$