Physics Beyond

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Kinematics

How to describe motion? (mathematically)

→ In maths we need a clear and precise definition.

→ Motion is change in position over time.

⤳ What is space?

⤳ What is time?

→ How to describe motion?

⤳ Need a reference point / origin.

⤳ Intuition about vectors.

↪ Rule of displacement from one point to another along a straight line

↪ Vector - from Latin "vehere" - to carry

To describe motion we need $f(t) \to x$ and $g(t) \to y$.

$$\vec{x} : \mathbb{R} \to \text{Vector Space}$$
$$t \mapsto \vec{x}(t)$$

What is vector space?

→ Euclidean 3-dimensional space.

→ For example, vector space of a 3 tuple of reals can be written as $\mathbb{R}^3$

$\rightarrow$ We say V is a vector space if:

1. $\mathbb{R} \cdot V \rightarrow V$
2. Addition is commutative, associative, and has neutral element $\vec{O}$
3. $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$ $\qquad \lambda, \mu \in \mathbb{R} \qquad \vec{v} \in V$
4. $(\lambda\mu)\vec{v} = \lambda(\mu\vec{v})$
5. $1 \cdot \vec{v} = \vec{v}$

$\rightarrow$ We need to introduce a coordinate system.

In 2 dimensions

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In n dimensions

$$\vec{e_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Where 1 is at index i

$\rightarrow$ We can describe using unitary vectors:

Two sets $A, B \qquad A \times B \quad \{(a,b) \mid a \in A, b \in B\}$

$A \times A = A^2$ Where $\times$ is the cartesian product

What is happening mathematically?

We have constructed a mapping $V \rightarrow \mathbb{R}^2$; it is $1 \rightarrow 1$

$\therefore$ For any vector, a pair of reals visa versa.

$\vec{x} \mapsto (x_1, x_2) \qquad s.t \qquad \vec{x} = x_1 e_1 + x_2 e_2$

1. Note that for a basis, e_1, e_2 this mapping is one to one and onto (bijection)

$\vec{x}(f) = (ae_1, be_2) \leftarrow$ Linear combination of e_1 and e_2

2. This means that we have an inverse mapping.

$$\mathbb{R} \to E$$

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} \mapsto \lambda e_1 + \mu e_2$$

Coordinate systems translate $\mathbb{R}^2$ to vector E

3. This mapping depends on the chosen coordinate system.

$\to$ Coordinate system: (origin, two basis vectors)

$\hookrightarrow (\vec{o}, \vec{e_1}, \vec{e_2})$

$\hookrightarrow \vec{e_1}, \vec{e_1}$ are linearly independent.

$\hookrightarrow$ We assume $\vec{e_1} \perp \vec{e_1} \to$ Is orthogonal (meet at 90 deg / $\frac{\pi}{2}$)

We have an improvement of our description:

$$\vec{x} : \mathbb{R} \to \mathbb{R}$$

$$t \mapsto \vec{x}(t) \qquad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad x_1, x_2 : \mathbb{R} \to \mathbb{R}$$

Operators

$$+ : \mathbb{R}^n, \mathbb{R}^n \to \mathbb{R}^n$$

$$(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}) \mapsto \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$\cdot : \mathbb{R}, \mathbb{R}^n \to \mathbb{R}^n$$

$$(\lambda, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}) \mapsto \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$$

$$\mathbb{R}^x = \{f : x \to \mathbb{R}\} \qquad \text{Where } x \text{ is any set and f is a function}$$

Maths foundations

Current foundation: Set theory. What does that mean?

$\rightarrow$ Everything must be related back to sets.

$\rightarrow$ Relatively modern idea (2nd half of 20th century)

Examples

1. Numbers:

   $0 = \emptyset$ - empty set

   $1 = \{\emptyset\}$ - set of the empty set

   $2 = \{\emptyset, \{\emptyset\}\}$ - set of the empty set and a set of the empty set

   $\vdots$

   This means that natural numbers can be modelled using set theory.

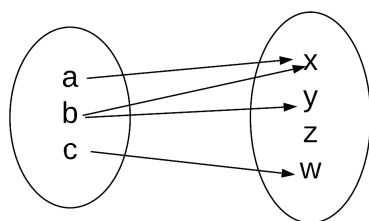 $\rightarrow$ Model ideas $=$ helps us to understand abstract ideas.

2. Ordered pair: (x, y) - order matters
   How can we model this using sets? (sets are unordered)
   $\{x, y\}$ doesn't work as unordered.
   $\{x, \{x, y\}\}$ could work for example.

3. Relation

   Relation $\mathbb{R} \subseteq A \times B$
   $(A \times B = \{(a, b) \mid a \in A, b \in B\}$
   Example 1:

   $x \leqslant y \in \mathbb{R}$

   $\leqslant \; \subseteq \mathbb{R} \times \mathbb{R}$

   $(x, y) \in \leqslant$

Example 2 - Equivalence relation:
(Example of vectors)
Vectors:

(a) Idea: Rule of displacement of things along a straight line by a certain length (physics)

(b) Representation: Directed line segment

How to model a vector?
Consider the set L of all line segments in the plane

$\rightarrow$ Subdivide L into disjointed subsets $[\vec{AB}]$ of line segments parallel, of same length and same orientation as $\vec{AB}$
$\vec{v} = [\vec{AB}]$ is a (model of a) vector.

$\rightarrow$ Define a relation $\sim \subseteq L \times L$ ($\sim$ means equivalent to)

$$\vec{AB} \sim \vec{CD} : \iff \text{ They represent the same vector}$$
$$\iff [\vec{AB}] = [\vec{CD}]$$
$$\iff \vec{AB} \parallel \vec{CD}, \ \left\|\vec{AB}\right\| = \left\|\vec{CD}\right\|$$

In general, these three properties define what is called an equivalence relation.

4. Functions / maps / mapping
$\rho : A \to B \qquad x \mapsto f(x)$
Functions can be considered a type of relation. A relation is called a function if it is right-unique and left-total.

$$(x, y) \wedge (x, z) \Rightarrow y = z$$
$$vx \in A$$
$$\exists y \in B$$
$$(x, y) \in f$$
$$y = f(x)$$

Example:

$\rho : \mathbb{R} \to \mathbb{R}$ $\qquad\qquad\qquad\qquad\qquad\qquad x \mapsto x^2$
$\rho \subseteq \mathbb{R} \times \mathbb{R}$ $\qquad\qquad\qquad\qquad \rho = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

The relation is the graph of the function.

Motion

Two areas:

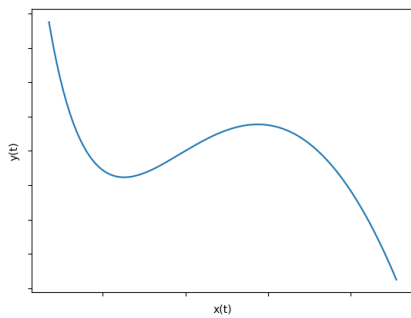- Kinematics (mathematical modelling)

- Dynamics (what causes motion)

<u>Def</u>: Motion in the change in position in time.
position $\to$ space   time $\to$ time
$\rightsquigarrow$ Space + time too abstract $\to$ need to simplify
Space $\mapsto$ Euclidean geometry
Time $\mapsto$ Use $\mathbb{R}$ (Time is just a parameter)
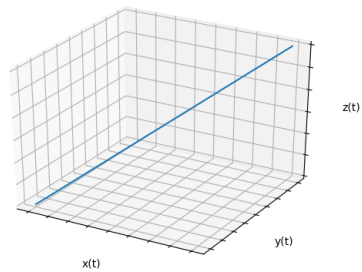

Say this is the graph of $p(t)$
$p : \mathbb{R} \to$ Space
$\rightsquigarrow$ Vectors + coordinates (reference frame)

$$x : \mathbb{R} \to \mathbb{R}^3$$
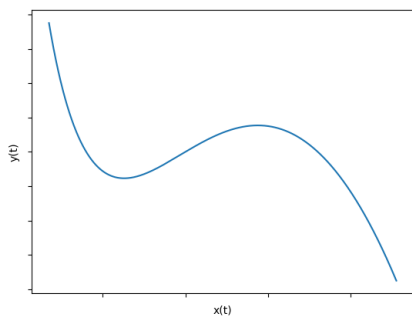$$\vec{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$$
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\rightsquigarrow$ Algebra of geometric vectors translates directly into algebra of tuples.

$$\vec{x} + \vec{y} \longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$\lambda\vec{x} \longleftrightarrow \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$$
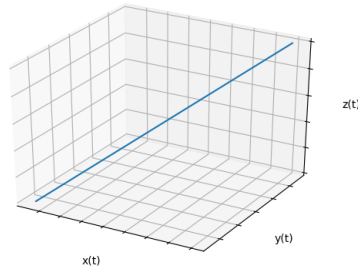

$$\vec{x} : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto \vec{x}(t)$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

7

Exaples

1. Motion along a line (with constant velocity)

$$\vec{x}(t) = \vec{x_0} + t\vec{v}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_{01} + tv_1 \\ x_{02} + tv_2 \\ x_{03} + tv_3 \end{bmatrix}$$

2. Circular motion

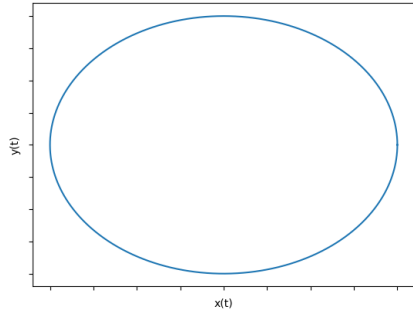$$\vec{x}(t) = r \begin{bmatrix} \cos(\rho t) \\ \sin(\rho t) \end{bmatrix}$$

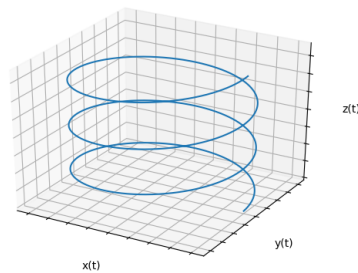Where ρ some function and r is the radius.
If it is uniform:

$$\vec{x}(t) = r \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}$$

$$\omega \in \mathbb{R} | \{0\} \qquad \omega = \frac{2\pi}{T}$$

Where T is the time period.

3. Uniform spiral It is just like a circle, but with linear motion in the z-axis.

$$\vec{x}(f) = \begin{bmatrix} r\cos\omega t \\ r\sin\omega t \\ vt \end{bmatrix}$$

Kinematics - reminder

⤳ Done with modelling motion mathematically?
↪ How to study the physics of motion?

⤳ What is the natural state of motion?
- Without physical interaction.

⤳ At rest? (Aristotle) ×

⤳ Uniform motion (Newton)
↪ Why?

↪ Changing reference frame to a moving one → object at rest relativity.

↪ But we cannot answer/ decide what the natural state of motion is in this stage.

Observation: State of motion changes due to physical interaction → Lets model physical interaction with forces. What is the change in motion due to force?
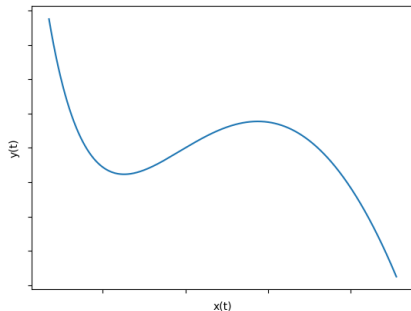
⇝ The shape of the trajectory is curved

⇝ We don't know how to model that!
↪ Back to maths

Kinematics 2

Modelling change in motion

Take two points on the above curve, $\vec{x}(t_0)$ and $\vec{x}(t)$. The change of displacement is $\Delta\vec{x}$. We know that velocity is change in displacement over change in time so we can find that using:

$$\vec{v} = \frac{\Delta\vec{x}}{\Delta t}$$
$$= \frac{\vec{x}(t) - \vec{t_0}}{t - t_0}$$

This gives us the average velocity between t_0 and t. We need to be able to find the velocity at every point in time t_0

$$\vec{v} : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto \text{velocity at t}$$

Problem: We need a second time $t = t_0 + h$ for $\vec{v}(t_0)$, but there is no natural choice for h. We notice for more precision we need to make h as small as possible!

$$\lim_{h \to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}$$
$$= \frac{d\vec{x}}{dt} \text{ Leibniz notation}$$
$$= \dot{\vec{x}}(t) \text{ Newton notation}$$

Differentiation on vectors is just element wise:

$$\frac{d}{dt} \begin{bmatrix} R\cos\omega t \\ R\sin\omega t \end{bmatrix} = \omega R \begin{bmatrix} -\sin\omega t \\ \cos\omega t \end{bmatrix}$$

Limits create paradoxes: Take $\lim_{h \to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}$. This works mathematically but causes a conceptual problem: How can something change at one point in time?

$\rightsquigarrow$ Solution: Use infinitesimals

$\rightarrow$ Numbers that are not provably not 0. Idea: If we zoom in on any curve enough, begins to look like a straight line: e.g. Earth looks flat.

How can we model this mathematically?

$\rightsquigarrow$ Around each point $\vec{x}(t_0)$ the curve is made up of infinitesimal line segments.

Geometrically: A tangent at $\vec{x}(t_0)$ touches the curve at $\vec{x}(t_0)$

$\hookrightarrow$ We take this to mean that it intersects with the curve at an infinitesimal line segment

$\hookrightarrow$ Infinitesimal line segment is the intersection of the tangent and the curve.

Problem: Do not have a rigorous definition of a tangent at a curve!

$\rightarrow$ But we know: Every line that intersects the circle in one points is a tangent $\rightsquigarrow$ Use that!
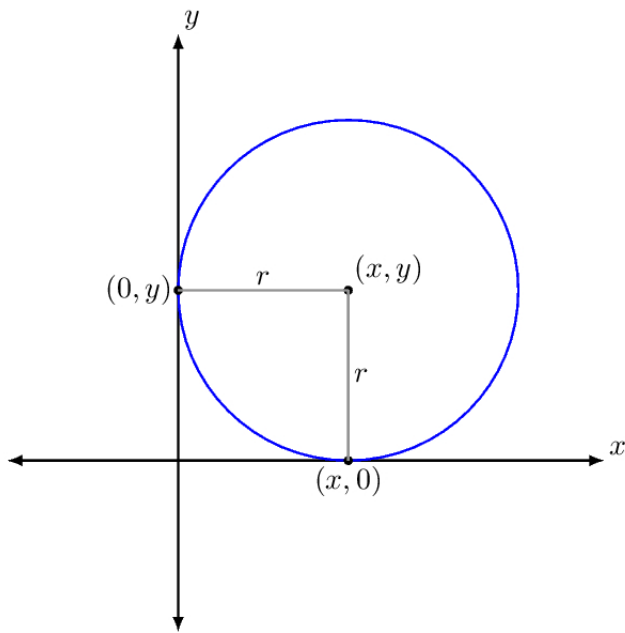
Take the radius, r, to be 1 and the center to be $(0, 1)$.

Eqation of circle:

$c : (x - 0)^2 + (y - 1)^2 = 1$

If we take our tangent, t to be the y axis:

$t : y = 0$

Sub into c:

$x^2 + 1 = 1$

$\Rightarrow x^2 = 0$

$\therefore c \cap t = \{(d, 0) \mid d^2 = 0\}$

⤳ Define first order infinitesimals as $D = \{d \in R \mid d^2 = 0\}$ where $R = \mathbb{R} \cup \{\text{infinitesimal numbers}\}$ Are such infinitesimals going to solve problems?

$$f(x) = x^2$$
$$\Rightarrow f(x+d) \qquad\qquad d \in D$$
$$= (x+d)^2$$
$$= x^2 + 2dx + d^2$$
$$= x^2 + 2dx$$
$$= f(x) + f'(x) \cdot d$$

$$f(x) = x^3$$
$$\Rightarrow f(x+d)$$
$$= (x+d)^3$$
$$= x^3 + 3x^2 d + 3xd^2 + d^3$$
$$= x^3 + 3x^2 d$$
$$= f(x) + f'(x) \cdot d$$