Physics Beyond

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Kinematics

How to describe motion? (mathematically)

- $\rightarrow$ In maths we need a clear and precise definition.
- $\rightarrow$ Motion is change in position over time.
  - $\rightsquigarrow$ What is space?
  - $\rightsquigarrow$ What is time?
- $\rightarrow$ How to describe motion?
  - $\rightsquigarrow$ Need a reference point / origin.
  - $\rightsquigarrow$ Intuition about vectors.
    - $\hookrightarrow$ Rule of displacement from one point to another along a straight line
    - $\hookrightarrow$ Vector - from Latin "vehere" - to carry

To describe motion we need $f(t) \to x$ and $g(t) \to y$.

$$\begin{aligned} \vec{x} : \mathbb{R} &\to \text{Vector Space} \\ t &\mapsto \vec{x}(t) \end{aligned}$$

What is vector space?

- $\rightarrow$ Euclidean 3-dimensional space.
- $\rightarrow$ For example, vector space of a 3 tuple of reals can be written as $\mathbb{R}^3$

$\rightarrow$ We say V is a vector space if:

1. $\mathbb{R} \cdot V \rightarrow V$
2. Addition is commutative, associative, and has neutral element $\vec{O}$
3. $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v} \qquad \lambda, \mu \in \mathbb{R} \qquad \vec{v} \in V$
4. $(\lambda\mu)\vec{v} = \lambda(\mu\vec{v})$
5. $1 \cdot \vec{v} = \vec{v}$

$\rightarrow$ We need to introduce a coordinate system.

In 2 dimensions

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In n dimensions

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Where 1 is at index i

$\rightarrow$ We can describe using unitary vectors:

Two sets $A, B \qquad A \times B \quad \{(a, b) \mid a \in A, b \in B\}$

$A \times A = A^2$ Where $\times$ is the cartesian product

What is happening mathematically?

We have constructed a mapping $V \rightarrow \mathbb{R}^2$; it is $1 \rightarrow 1$
$\therefore$ For any vector, a pair of reals visa versa.

$$\vec{x} \mapsto (x_1, x_2) \qquad s.t \qquad \vec{x} = x_1 e_1 + x_2 e_2$$

1. Note that for a basis, e_1, e_2 this mapping is one to one and onto (bijection)
   $\vec{x}(f) = (ae_1, be_2) \leftarrow$ Linear combination of e_1 and e_2

2. This means that we have an inverse mapping.

$$\begin{aligned} \mathbb{R} &\to E \\ \begin{bmatrix} \lambda \\ \mu \end{bmatrix} &\mapsto \lambda e_1 + \mu e_2 \end{aligned}$$

   Coordinate systems translate $\mathbb{R}^2$ to vector E

3. This mapping depends on the chosen coordinate system.

   $\to$ Coordinate system: (origin, two basis vectors)

   $\hookrightarrow$ $(\vec{o}, \vec{e}_1, \vec{e}_2)$

   $\hookrightarrow$ $\vec{e}_1, \vec{e}_1$ are linearly independent.

   $\hookrightarrow$ We assume $\vec{e}_1 \perp \vec{e}_1$ $\to$ Is orthogonal (meet at 90 deg / $\frac{\pi}{2}$)

We have an improvement of our description:

$$\begin{aligned} \vec{x} &: \mathbb{R} \to \mathbb{R} \\ t &\mapsto \vec{x}(t) \qquad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad x_1, x_2 : \mathbb{R} \to \mathbb{R} \end{aligned}$$

Operators

$$\begin{aligned} + : \ & \mathbb{R}^n, \mathbb{R}^n \to \mathbb{R}^n \\ (\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}) &\mapsto \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \cdot : \ & \mathbb{R}, \mathbb{R}^n \to \mathbb{R}^n \\ (\lambda, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}) &\mapsto \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} \end{aligned}$$

$$\mathbb{R}^x = \{f : x \to \mathbb{R}\} \qquad \text{Where } x \text{ is any set and f is a function}$$

Maths foundations

Current foundation: Set theory. What does that mean?

→ Everything must be related back to sets.

→ Relatively modern idea (2nd half of 20th century)

Examples

1. Numbers:

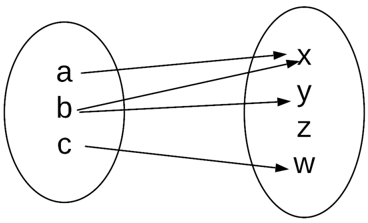
$$\begin{aligned} &0 = \emptyset \text{ - empty set} \\ &1 = \{\emptyset\} \text{ - set of the empty set} \\ &2 = \{\emptyset, \{\emptyset\}\} \text{ - set of the empty set and a set of the empty set} \\ &\ \vdots \end{aligned}$$

   This means that natural numbers can be modelled using set theory.

   → Model ideas = helps us to understand abstract ideas.

2. Ordered pair: (x, y) - order matters
   How can we model this using sets? (sets are unordered)
   $\{x, y\}$ doesn't work as unordered.
   $\{x, \{x, y\}\}$ could work for example.

3. Relation



   Relation $\mathbb{R} \subseteq A \times B$
   $(A \times B = \{(a, b) \mid a \in A, b \in B\}$
   Example 1:

$$\begin{aligned} &x \leqslant y \in \mathbb{R} \\ &\ \ \leqslant \subseteq \mathbb{R} \times \mathbb{R} \\ &\ (x, y) \in \leqslant \end{aligned}$$

Example 2 - Equivalence relation:
(Example of vectors)
<u>Vectors:</u>

(a) Idea: Rule of displacement of things along a straight line by a certain length (physics)

(b) Representation: Directed line segment

How to model a vector?
Consider the set L of all line segments in the plane

→ Subdivide L into disjointed subsets $[\vec{AB}]$ of line segments parallel, of same length and same orientation as $\vec{AB}$
$\vec{v} = [\vec{AB}]$ is a (model of a) vector.

→ Define a relation $\sim \subseteq L \times L$ ($\sim$ means equivalent to)

$$\begin{aligned} \vec{AB} \sim \vec{CD} :&\iff \text{They represent the same vector} \\ &\iff [\vec{AB}] = [\vec{CD}] \\ &\iff \vec{AB} \parallel \vec{CD},\ \left\|\vec{AB}\right\| = \left\|\vec{CD}\right\| \end{aligned}$$

In general, these three properties define what is called an equivalence relation.

4. Functions / maps / mapping
$\rho : A \to B \qquad x \mapsto f(x)$
Functions can be considered a type of relation. A relation is called a function if it is right-unique and left-total.

$$\begin{aligned} (x, y) \wedge (x, z) \Rightarrow & y = z \\ & vx \in A \\ & \exists y \in B \\ & (x, y) \in f \\ & y = f(x) \end{aligned}$$

Example:

$$\begin{aligned} &\rho : \mathbb{R} \to \mathbb{R} && x \mapsto x^2 \\ &\rho \subseteq \mathbb{R} \times \mathbb{R} && \rho = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \end{aligned}$$

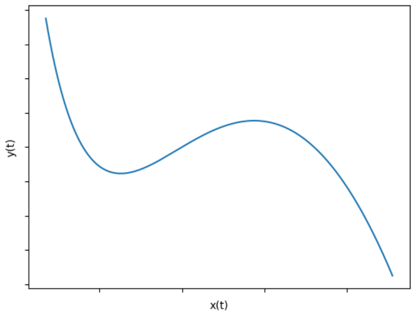
The relation is the graph of the function.

Motion

Two areas:

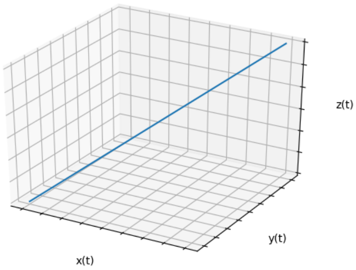
- Kinematics (mathematical modelling)
- Dynamics (what causes motion)

Def: Motion in the change in position in time.
position $\to$ space time $\to$ time
$\rightsquigarrow$ Space + time too abstract $\to$ need to simplify
Space $\mapsto$ Euclidean geometry
Time $\mapsto$ Use $\mathbb{R}$ (Time is just a parameter)



Say this is the graph of $p(t)$
$p : \mathbb{R} \to$ Space
$\rightsquigarrow$ Vectors + coordinates (reference frame)

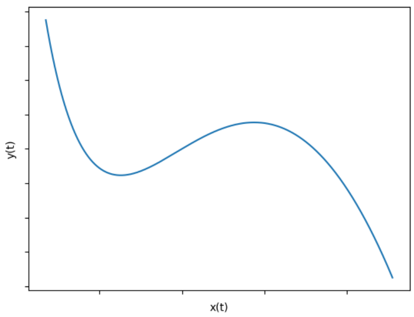
$$
\begin{aligned}
x :&\mathbb{R} \to \mathbb{R}^3 \\
\vec{x} &= x_1 e_1 + x_2 e_2 + x_3 e_3 \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$



$\rightsquigarrow$ Algebra of geometric vectors translates directly into algebra of tuples.

$$\vec{x} + \vec{y} \longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$\lambda \vec{x} \longleftrightarrow \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$$



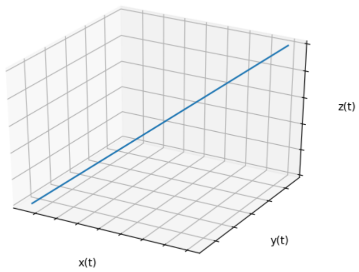
$$\begin{aligned} \vec{x} :& \mathbb{R} \to \mathbb{R}^3 \\ t &\mapsto \vec{x}(t) \end{aligned}$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Exaples

1. Motion along a line (with constant velocity)

$$\begin{aligned}\vec{x}(t) &= \vec{x_0} + t\vec{v} \\ \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} &= \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= \begin{bmatrix} x_{01} + tv_1 \\ x_{02} + tv_2 \\ x_{03} + tv_3 \end{bmatrix}\end{aligned}$$

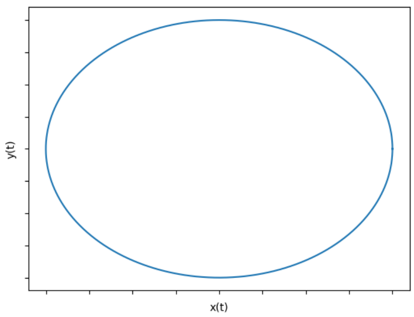


2. Circular motion
$\vec{x}(t) = r \begin{bmatrix} \cos(\rho t) \\ \sin(\rho t) \end{bmatrix}$
Where ρ some function and r is the radius.
If it is uniform:

$$\vec{x}(t) = r \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}$$

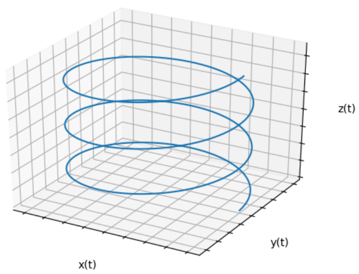
$$\omega \in \mathbb{R}|\{0\} \qquad \omega = \frac{2\pi}{T}$$

Where T is the time period.



3. Uniform spiral It is just like a circle, but with linear motion in the z-axis.

$$\vec{x}(f) = \begin{bmatrix} r\cos\omega t \\ r\sin\omega t \\ vt \end{bmatrix}$$



Kinematics - reminder

$\rightsquigarrow$ Done with modelling motion mathematically?
$\hookrightarrow$ How to study the physics of motion?

$\rightsquigarrow$ What is the natural state of motion?
- Without physical interaction.

$\rightsquigarrow$ At rest? (Aristotle) $\times$

$\rightsquigarrow$ Uniform motion (Newton)
$\hookrightarrow$ Why?

$\hookrightarrow$ Changing reference frame to a moving one $\rightarrow$ object at rest relativity.

$\hookrightarrow$ But we cannot answer/ decide what the natural state of motion is in this stage.

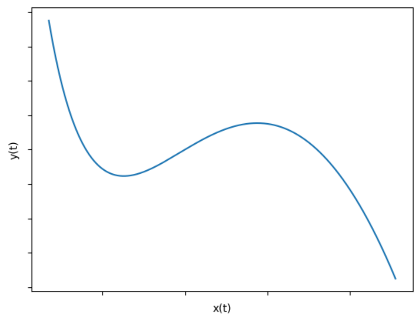
Observation: State of motion changes due to physical interaction $\rightarrow$ Lets model physical interaction with forces. What is the change in motion due to force?

$\rightsquigarrow$ The shape of the trajectory is curved

$\rightsquigarrow$ We don't know how to model that!
$\hookrightarrow$ Back to maths

Kinematics 2

Modelling change in motion



Take two points on the above curve, $\vec{x}(t_0)$ and $\vec{x}(t)$. The change of displacement is $\Delta\vec{x}$. We know that velocity is change in displacement over change in time so we can find that using:

$$\begin{aligned}\vec{v} &= \frac{\Delta\vec{x}}{\Delta t} \\ &= \frac{\vec{x}(t) - \vec{t_0}}{t - t_0}\end{aligned}$$

This gives us the average velocity between t_0 and t. We need to be able to find the velocity at every point in time t_0

$$\begin{aligned}\vec{v} : \mathbb{R} &\rightarrow \mathbb{R}^3 \\ t &\mapsto \text{velocity at t}\end{aligned}$$

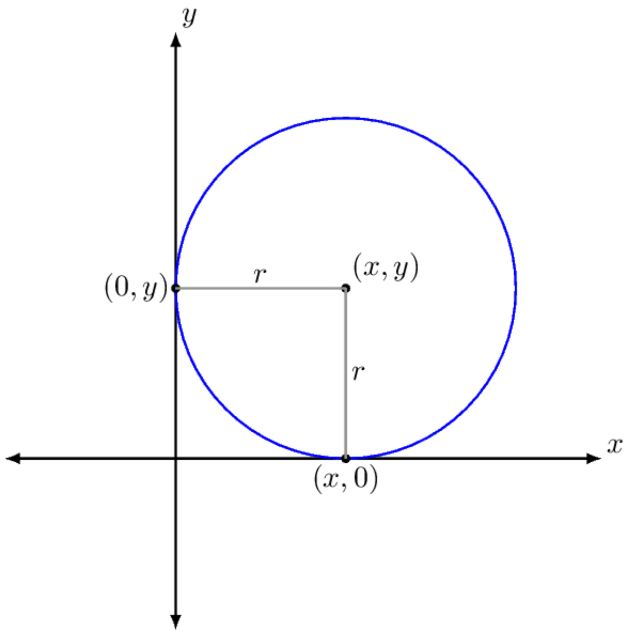
Problem: We need a second time $t = t_0 + h$ for $\vec{v}(t_0)$, but there is no natural choice for h. We notice for more precision we need to make h as small as possible!

$$\begin{aligned}
&\lim_{h\to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h} \\
=&\frac{d\vec{x}}{dt} \text{ Leibniz notation} \\
=&\dot{\vec{x}}(t) \text{ Newton notation}
\end{aligned}$$

Differentiation on vectors is just element wise:

$$\frac{d}{dt}\begin{bmatrix} R\cos\omega t \\ R\sin\omega t \end{bmatrix} = \omega R \begin{bmatrix} -\sin\omega t \\ \cos\omega t \end{bmatrix}$$

Limits create paradoxes: Take $\lim_{h\to 0} \frac{\vec{x}(t+h)-\vec{x}(t)}{h}$. This works mathematically but causes a conceptual problem: How can something change at one point in time?
$\rightsquigarrow$ Solution: Use infinitesimals
$\to$ Numbers that are not provably not 0. Idea: If we zoom in on any curve enough, begins to look like a straight line: e.g. Earth looks flat.
How can we model this mathematically?
$\rightsquigarrow$ Around each point $\vec{x}(t_0)$ the curve is made up of infinitesimal line segments.
Geometrically: A tangent at $\vec{x}(t_0)$ touches the curve at $\vec{x}(t_0)$
$\hookrightarrow$ We take this to mean that it intersects with the curve at an infinitesimal line segment
$\hookrightarrow$ Infinitesimal line segment is the intersection of the tangent and the curve.
Problem: Do not have a rigorous definition of a tangent at a curve!
$\to$ But we know: Every line that intersects the circle in one points is a tangent $\rightsquigarrow$ Use that!



Take the radius, r, to be 1 and the center to be $(0, 1)$.

Eqation of circle:
$c : (x - 0)^2 + (y - 1)^2 = 1$
If we take our tangent, t to be the y axis:
$t : y = 0$
Sub into c:
$x^2 + 1 = 1$
$\Rightarrow x^2 = 0$
$\therefore c \cap t = \{(d, 0) \mid d^2 = 0\}$

$\rightsquigarrow$ Define first order infinitesimals as $D = \{d \in R \mid d^2 = 0\}$ where $R = \mathbb{R} \cup \{\text{infinitesimal numbers}\}$ Are such infinitesimals going to solve problems?

$$\begin{aligned}
f(x) &= x^2 \\
&\Rightarrow f(x+d) \qquad\qquad d \in D \\
&\quad = (x+d)^2 \\
&\quad = x^2 + 2dx + d^2 \\
&\quad = x^2 + 2dx \\
&\quad = f(x) + f'(x) \cdot d
\end{aligned}$$

$$\begin{aligned}
f(x) &= x^3 \\
&\Rightarrow f(x+d) \\
&\quad = (x+d)^3 \\
&\quad = x^3 + 3x^2 d + 3xd^2 + d^3 \\
&\quad = x^3 + 3x^2 d \\
&\quad = f(x) + f'(x) \cdot d
\end{aligned}$$

Infinitesimals

The derivative is used to solve the tangent problem. We can see that infinitesimals solve this too.
Examples: sine and cosine
Take a unit circle:

$$\begin{aligned}
\sin x &= \sqrt{1 - \cos^2 x} \\
\Rightarrow \sin d &= \sqrt{1 - \cos^2 d}
\end{aligned}$$

Using and arc of angle α and arc length l

$$
\begin{aligned}
l &= 2\pi R \cdot \frac{\alpha}{2\pi} \\
&= R\alpha \\
\text{let } \alpha &= d \\
\Rightarrow l &= d
\end{aligned}
$$

$\therefore$ We are moving on a straight line and instead of an arc it is a right angled triangle

$$
\begin{aligned}
\Rightarrow \sin d &= d \\
\cos d &= \sqrt{1 - \sin^2(d)} \\
&= \sqrt{1 - d^2} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \sin(0 + d) = \sin(0) + \sin'(0) \cdot (d) \\
&\Rightarrow d = \sin'(0) \cdot d
\end{aligned}
$$

We would want this to be 1

$$
\begin{aligned}
\cos d &= \cos(0 + d) \\
&= \cos(0) + \cos'(0) \cdot d \\
&= 1 + \cos'(0) \cdot d \\
1 &= 1 + \cos'(0) \cdot d \\
\cos'(0) \cdot d &= 0 \\
cos'(0) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\sin(x + d) &= \sin(x) + \sin'(x) \cdot d \\
&= \sin(x) \cdot \cos(d) + \sin(d) \cdot \cos(x) \\
&= \sin(x) + d \cdot \cos(x) \\
\sin'(x) \cdot d &= \cos(x) \cdot d \\
\text{Would want } \sin'(x) &= \cos(x) \\
\cos(x + d) &=
\end{aligned}
$$

Limits

Definition

$$\lim_{x\to x_0} f(x) = c \qquad \forall\epsilon > 0 \exists\delta > 0$$
$$\forall x |x - x_0| < \delta \Rightarrow |f(x) - c| < \epsilon (\epsilon, \delta \in \mathbb{R})$$

For $v(t_0)$ this means :

$$\forall\epsilon > 0 \exists\delta > 0 \qquad \forall\delta \in \mathbb{R} \qquad |t - t_0| < \delta \Rightarrow |\frac{x(t) - x(t_0)}{t - t_0} - v(t_0)| < \epsilon$$
$$\cos'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$
$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Conclusion from examples: Infinitesimal solve the tangent problem easily and allow us to calculate the derivatives, provided that we can find a theory that allows us to cancel infinitesimals.

Theory of Infinitesimals

Need: A set of numbers, R with addition, multiplication, and $\mathbb{R} \subseteq R$

Why? So that we can do the same algebra as the reals (careful with division).

A set $D = \{d \in \mathbb{R} | d^2 = 0\}$
Satisfying the following axioms (Kock-Lavere Axiom)(K-l)
For all $g : 0 rightarrow R$
$a.d = b.d \qquad \forall d \in D$
$\Rightarrow a = b$ (Cancellation Lemma)
$\therefore$ There are unique $a, b \in R$ s.t. $g(d) = a + b.d \qquad \forall d \in D$

Explanation

1. All graphs of functions $D \to R$ Should be lines.
2. The equation of the line has the to be uniquely determined by g

Proposition

$a.d = b.d \qquad \forall d \in D$
$\Rightarrow a = b$

Proof

$$a.d = b.d \iff a.d - b.d = 0 \iff (a-b).d = 0$$
$$g : D \to R \qquad d \mapsto (a-b).d$$
$$\Rightarrow (K-L) g(d) = c + e.d$$
$$(but) g(d) = 0 + (a-b).d \Rightarrow c = 0, e = (a-b)$$

Since $a.d = b.d" we have g(d) = 0 \qquad \forall d \in D$ but by (K-L) $g(d) = c + e.d$ and $c = e = 0$ works. So $c = e = 0$ since c and e are unique.
$\to 0 = a - b \iff a = b$

Example

We have:

$$\sin d = d \tag{1}$$
$$\sin d = sin0 + sin0.d \tag{2}$$

$$(1 = 2) d = \sin'(0).d$$
$$1 = sin'(0)$$

Definition (Derivative)

Let $f : R \to R$, then according to (K-L), for each $x \in R$ there is a unique $f'(x) \in R$ s.t. $f(x+d) = f(x) + f'(x).d$

We call the map $f' : R \to R \qquad x \mapsto f'(x)$ the derivative of f.

Rules of Differentiation

$f : R \to R \qquad g : R \to R \qquad \lambda \in \mathbb{R}$

Linearity

$$
\begin{aligned}
&(f+\lambda g)' = f' + \lambda g' \\
&\quad \lambda g : R \to R \qquad x \mapsto \lambda g(x)
\end{aligned}
$$

Proof

$$
\begin{aligned}
(f+\lambda g)(x+d) &= (K-L)(f+\lambda g)(x) + (f+\lambda g)'(x).d \\
(f+\lambda g)(x+d) &= f(x+d) + \lambda g(x+d) \\
&= (K-L)f(x) + f'(x).d + \lambda(g(x) + g'(x).d) \\
&= (f(x) + \lambda g(x) + (f'(x) + \lambda g'(x)).d \\
&= (f+\lambda g)(x) + (f' + \lambda g')(x).d \\
&\Rightarrow (f+\lambda g)(x) + (f+\lambda g)'(x).d = (f+\lambda g)(x) + (f'+\lambda g')(x).d \\
&\Rightarrow (f+\lambda g)'(x).d = (f'+\lambda g')(x).d \\
&\Rightarrow (Calcellation Lemma)(f+\lambda g)'(x) = (f'+\lambda g')(x) \\
&\quad \text{If true for all x, funtions are equal} \\
&\Rightarrow (f+\lambda g)' = (f'+\lambda g')
\end{aligned}
$$

Product Rule

$$
f.g = R \to R \qquad x \mapsto f(x).g(x)
$$