

Physics Beyond

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December 3, 2019

Kinematics

How to describe motion? (mathematically)

- In maths we need a clear and precise definition.
- Motion is change in position over time.
 - ↪ What is space?
 - ↪ What is time?
- How to describe motion?
 - ↪ Need a reference point / origin.
 - ↪ Intuition about vectors.
 - ↪ Rule of displacement from one point to another along a straight line
 - ↪ Vector - from Latin "vehere" - to carry

To describe motion we need $f(t) \rightarrow x$ and $g(t) \rightarrow y$.

$$\begin{aligned}\vec{x} : \mathbb{R} &\rightarrow \text{Vector Space} \\ t &\mapsto \vec{x}(t)\end{aligned}$$

What is vector space?

- Euclidean 3-dimensional space.
- For example, vector space of a 3 tuple of reals can be written as \mathbb{R}^3

→ We say V is a vector space if:

1. $\mathbb{R} \cdot V \rightarrow V$
2. Addition is commutative, associative, and has neutral element $\vec{0}$
3. $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$ $\lambda, \mu \in \mathbb{R}$ $\vec{v} \in V$
4. $(\lambda\mu)\vec{v} = \lambda(\mu\vec{v})$
5. $1 \cdot \vec{v} = \vec{v}$

→ We need to introduce a coordinate system.

In 2 dimensions

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In n dimensions

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Where 1 is at index i

→ We can describe using unitary vectors:

Two sets A, B $A \times B = \{(a, b) \mid a \in A, b \in B\}$
 $A \times A = A^2$ Where \times is the cartesian product

What is happening mathematically?

We have constructed a mapping $V \rightarrow \mathbb{R}^2$; it is $1 \rightarrow 1$
 \therefore For any vector, a pair of reals visa versa.

$$\vec{x} \mapsto (x_1, x_2) \quad s.t \quad \vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$$

1. Note that for a basis, \vec{e}_1, \vec{e}_2 this mapping is one to one and onto (bijection)
 $\vec{x}(f) = (ae_1, be_2) \leftarrow$ Linear combination of \vec{e}_1 and \vec{e}_2

2. This means that we have an inverse mapping.

$$\begin{aligned} \mathbb{R} &\rightarrow E \\ \begin{bmatrix} \lambda \\ \mu \end{bmatrix} &\mapsto \lambda e_1 + \mu e_2 \end{aligned}$$

Coordinate systems translate \mathbb{R}^2 to vector E

3. This mapping depends on the chosen coordinate system.

→ Coordinate system: (origin, two basis vectors)

$$\hookrightarrow (\vec{o}, \vec{e}_1, \vec{e}_2)$$

$\hookrightarrow \vec{e}_1, \vec{e}_2$ are linearly independent.

\hookrightarrow We assume $\vec{e}_1 \perp \vec{e}_2 \rightarrow$ Is orthogonal (meet at 90 deg / $\frac{\pi}{2}$)

We have an improvement of our description:

$$\begin{aligned} \vec{x} : \mathbb{R} &\rightarrow \mathbb{R}^2 \\ t &\mapsto \vec{x}(t) \qquad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad x_1, x_2 : \mathbb{R} \rightarrow \mathbb{R} \end{aligned}$$

Operators

$$\begin{aligned} + : \mathbb{R}^n, \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) &\mapsto \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \cdot : \mathbb{R}, \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \left(\lambda, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) &\mapsto \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} \end{aligned}$$

$$\mathbb{R}^x = \{f : x \rightarrow \mathbb{R}\} \quad \text{Where } x \text{ is any set and } f \text{ is a function}$$

Maths foundations

Current foundation: Set theory. What does that mean?

→ Everything must be related back to sets.

→ Relatively modern idea (2nd half of 20th century)

Examples

1. Numbers:

$0 = \emptyset$ - empty set

$1 = \{\emptyset\}$ - set of the empty set

$2 = \{\emptyset, \{\emptyset\}\}$ - set of the empty set and a set of the empty set

\vdots

This means that natural numbers can be modelled using set theory.

→ Model ideas = helps us to understand abstract ideas.

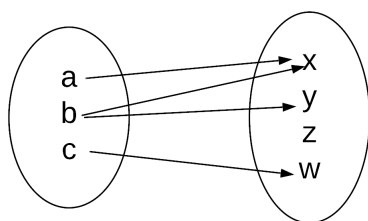
2. Ordered pair: (x, y) - order matters

How can we model this using sets? (sets are unordered)

$\{x, y\}$ doesn't work as unordered.

$\{x, \{x, y\}\}$ could work for example.

3. Relation



Relation $\mathbb{R} \subseteq A \times B$

$(A \times B = \{(a, b) \mid a \in A, b \in B\})$

Example 1:

$x \leq y \in \mathbb{R}$

$\leq \subseteq \mathbb{R} \times \mathbb{R}$

$(x, y) \in \leq$

Example 2 - Equivalence relation:

(Example of vectors)

Vectors:

(a) Idea: Rule of displacement of things along a straight line by a certain length (physics)

(b) Representation: Directed line segment

How to model a vector?

Consider the set L of all line segments in the plane

→ Subdivide L into disjointed subsets $[\vec{AB}]$ of line segments parallel, of same length and same orientation as \vec{AB}

$\vec{v} = [\vec{AB}]$ is a (model of a) vector.

→ Define a relation $\sim \subseteq L \times L$ (\sim means equivalent to)

$\vec{AB} \sim \vec{CD} : \iff$ They represent the same vector

$\iff [\vec{AB}] = [\vec{CD}]$

$\iff \vec{AB} \parallel \vec{CD}, \|\vec{AB}\| = \|\vec{CD}\|$

In general, these three properties define what is called an equivalence relation.

4. Functions / maps / mapping

$\rho : A \rightarrow B \quad x \mapsto f(x)$

Functions can be considered a type of relation. A relation is called a function if it is right-unique and left-total.

$(x, y) \wedge (x, z) \Rightarrow y = z$

$\forall x \in A$

$\exists y \in B$

$(x, y) \in f$

$y = f(x)$

Example:

$\rho : \mathbb{R} \rightarrow \mathbb{R}$

$\rho \subseteq \mathbb{R} \times \mathbb{R}$

$x \mapsto x^2$

$\rho = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

The relation is the graph of the function.

Motion

Two areas:

- Kinematics (mathematical modelling)
- Dynamics (what causes motion)

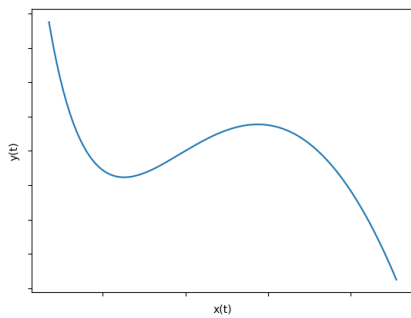
Def: Motion is the change in position in time.

position \rightarrow space time \rightarrow time

\leadsto Space + time too abstract \rightarrow need to simplify

Space \mapsto Euclidean geometry

Time \mapsto Use \mathbb{R} (Time is just a parameter)



Say this is the graph of $p(t)$

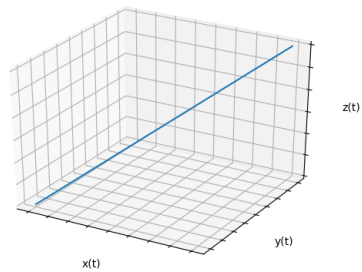
$p : \mathbb{R} \rightarrow \text{Space}$

\leadsto Vectors + coordinates (reference frame)

$$x : \mathbb{R} \rightarrow \mathbb{R}^3$$

$$\vec{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

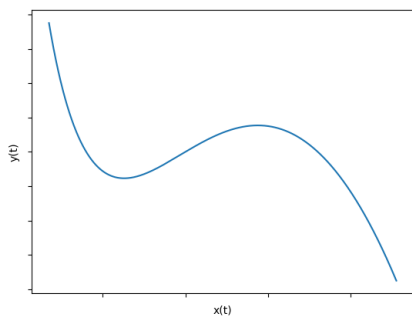
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



\rightsquigarrow Algebra of geometric vectors translates directly into algebra of tuples.

$$\vec{x} + \vec{y} \longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$\lambda \vec{x} \longleftrightarrow \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$$



$$\vec{x} : \mathbb{R} \rightarrow \mathbb{R}^3$$

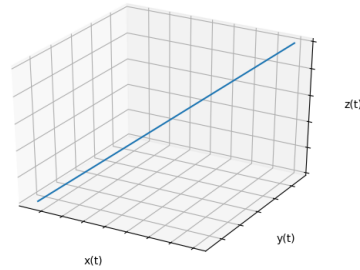
$$t \mapsto \vec{x}(t)$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Exaples

1. Motion along a line (with constant velocity)

$$\begin{aligned}\vec{x}(t) &= \vec{x}_0 + t\vec{v} \\ \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} &= \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= \begin{bmatrix} x_{01} + tv_1 \\ x_{02} + tv_2 \\ x_{03} + tv_3 \end{bmatrix}\end{aligned}$$



2. Circular motion

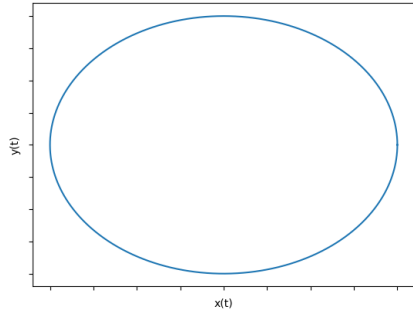
$$\vec{x}(t) = r \begin{bmatrix} \cos(\rho t) \\ \sin(\rho t) \end{bmatrix}$$

Where ρ some function and r is the radius.

If it is uniform:

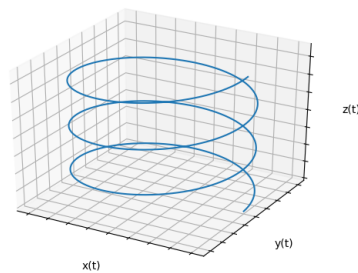
$$\begin{aligned}\vec{x}(t) &= r \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} \\ \omega &\in \mathbb{R} \setminus \{0\} \qquad \omega = \frac{2\pi}{T}\end{aligned}$$

Where T is the time period.



3. Uniform spiral It is just like a circle, but with linear motion in the z-axis.

$$\vec{x}(f) = \begin{bmatrix} r \cos \omega t \\ r \sin \omega t \\ vt \end{bmatrix}$$



Kinematics - reminder

- ↪ Done with modelling motion mathematically?
 - ↪ How to study the physics of motion?
- ↪ What is the natural state of motion?
 - Without physical interaction.
- ↪ At rest? (Aristotle) ×
- ↪ Uniform motion (Newton)
 - ↪ Why?