

Physics Beyond

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Kinematics

How to describe motion? (mathematically)

- In maths we need a clear and precise definition.
- Motion is change in position over time.
 - ↪ What is space?
 - ↪ What is time?
- How to describe motion?
 - ↪ Need a reference point / origin.
 - ↪ Intuition about vectors.
 - ↪ Rule of displacement from one point to another along a straight line
 - ↪ Vector - from Latin "vehere" - to carry

To describe motion we need $f(t) \rightarrow x$ and $g(t) \rightarrow y$.

$$\begin{aligned}\vec{x} : \mathbb{R} &\rightarrow \text{Vector Space} \\ t &\mapsto \vec{x}(t)\end{aligned}$$

What is vector space?

- Euclidean 3-dimensional space.
- For example, vector space of a 3 tuple of reals can be written as \mathbb{R}^3

→ We say V is a vector space if:

1. $\mathbb{R} \cdot V \rightarrow V$
2. Addition is commutative, associative, and has neutral element $\vec{0}$
3. $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$ $\lambda, \mu \in \mathbb{R}$ $\vec{v} \in V$
4. $(\lambda\mu)\vec{v} = \lambda(\mu\vec{v})$
5. $1 \cdot \vec{v} = \vec{v}$

→ We need to introduce a coordinate system.

In 2 dimensions

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In n dimensions

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Where 1 is at index i

→ We can describe using unitary vectors:

Two sets A, B $A \times B = \{(a, b) \mid a \in A, b \in B\}$
 $A \times A = A^2$ Where \times is the cartesian product

What is happening mathematically?

We have constructed a mapping $V \rightarrow \mathbb{R}^2$; it is $1 \rightarrow 1$
 \therefore For any vector, a pair of reals visa versa.

$$\vec{x} \mapsto (x_1, x_2) \quad s.t \quad \vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$$

1. Note that for a basis, \vec{e}_1, \vec{e}_2 this mapping is one to one and onto (bijection)
 $\vec{x}(f) = (ae_1, be_2) \leftarrow$ Linear combination of \vec{e}_1 and \vec{e}_2

2. This means that we have an inverse mapping.

$$\begin{aligned} \mathbb{R} &\rightarrow E \\ \begin{bmatrix} \lambda \\ \mu \end{bmatrix} &\mapsto \lambda e_1 + \mu e_2 \end{aligned}$$

Coordinate systems translate \mathbb{R}^2 to vector E

3. This mapping depends on the chosen coordinate system.

→ Coordinate system: (origin, two basis vectors)

$$\hookrightarrow (\vec{o}, \vec{e}_1, \vec{e}_2)$$

↪ \vec{e}_1, \vec{e}_2 are linearly independent.

↪ We assume $\vec{e}_1 \perp \vec{e}_2 \rightarrow$ Is orthogonal (meet at 90 deg / $\frac{\pi}{2}$)

We have an improvement of our description:

$$\vec{x} : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$t \mapsto \vec{x}(t) \quad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad x_1, x_2 : \mathbb{R} \rightarrow \mathbb{R}$$

Operators

$$\begin{aligned} + : \mathbb{R}^n, \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) &\mapsto \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \cdot : \mathbb{R}, \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \left(\lambda, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) &\mapsto \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} \end{aligned}$$

$$\mathbb{R}^x = \{f : x \rightarrow \mathbb{R}\} \quad \text{Where } x \text{ is any set and } f \text{ is a function}$$

Maths foundations

Current foundation: Set theory. What does that mean?

→ Everything must be related back to sets.

→ Relatively modern idea (2nd half of 20th century)

Examples

1. Numbers:

$0 = \emptyset$ - empty set

$1 = \{\emptyset\}$ - set of the empty set

$2 = \{\emptyset, \{\emptyset\}\}$ - set of the empty set and a set of the empty set

\vdots

This means that natural numbers can be modelled using set theory.

→ Model ideas = helps us to understand abstract ideas.

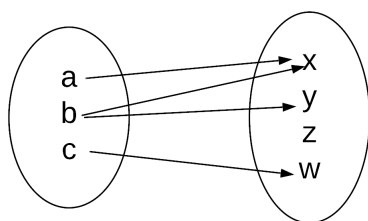
2. Ordered pair: (x, y) - order matters

How can we model this using sets? (sets are unordered)

$\{x, y\}$ doesn't work as unordered.

$\{x, \{x, y\}\}$ could work for example.

3. Relation



Relation $\mathbb{R} \subseteq A \times B$

$(A \times B = \{(a, b) \mid a \in A, b \in B\})$

Example 1:

$x \leq y \in \mathbb{R}$

$\leq \subseteq \mathbb{R} \times \mathbb{R}$

$(x, y) \in \leq$

Example 2 - Equivalence relation:

(Example of vectors)

Vectors:

- (a) Idea: Rule of displacement of things along a straight line by a certain length (physics)
- (b) Representation: Directed line segment

How to model a vector?

Consider the set L of all line segments in the plane

- Subdivide L into disjointed subsets $[\vec{AB}]$ of line segments parallel, of same length and same orientation as \vec{AB}
 $\vec{v} = [\vec{AB}]$ is a (model of a) vector.
- Define a relation $\sim \subseteq L \times L$ (\sim means equivalent to)

$$\begin{aligned} \vec{AB} \sim \vec{CD} : & \iff \text{They represent the same vector} \\ & \iff [\vec{AB}] = [\vec{CD}] \\ & \iff \vec{AB} \parallel \vec{CD}, \|\vec{AB}\| = \|\vec{CD}\| \end{aligned}$$

In general, these three properties define what is called an equivalence relation.

4. Functions / maps / mapping

$$\rho : A \rightarrow B \quad x \mapsto f(x)$$

Functions can be considered a type of relation. A relation is called a function if it is right-unique and left-total.

$$\begin{aligned} (x, y) \wedge (x, z) & \Rightarrow y = z \\ vx & \in A \\ \exists y & \in B \\ (x, y) & \in f \\ y & = f(x) \end{aligned}$$

Example:

$$\begin{aligned} \rho : \mathbb{R} & \rightarrow \mathbb{R} & x & \mapsto x^2 \\ \rho & \subseteq \mathbb{R} \times \mathbb{R} & \rho & = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \end{aligned}$$

The relation is the graph of the function.

Motion

Two areas:

- Kinematics (mathematical modelling)
- Dynamics (what causes motion)

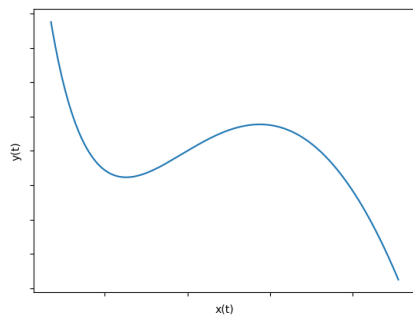
Def: Motion is the change in position in time.

position \rightarrow space time \rightarrow time

\leadsto Space + time too abstract \rightarrow need to simplify

Space \mapsto Euclidean geometry

Time \mapsto Use \mathbb{R} (Time is just a parameter)



Say this is the graph of $p(t)$

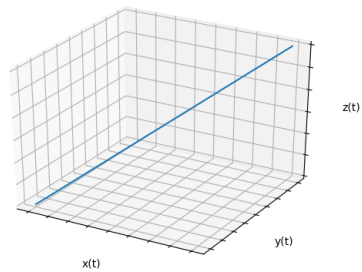
$p : \mathbb{R} \rightarrow \text{Space}$

\leadsto Vectors + coordinates (reference frame)

$$x : \mathbb{R} \rightarrow \mathbb{R}^3$$

$$\vec{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

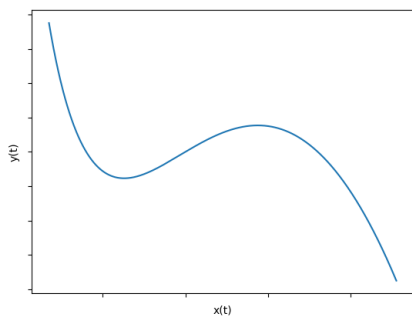
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



\rightsquigarrow Algebra of geometric vectors translates directly into algebra of tuples.

$$\vec{x} + \vec{y} \longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$\lambda \vec{x} \longleftrightarrow \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$$



$$\vec{x} : \mathbb{R} \rightarrow \mathbb{R}^3$$

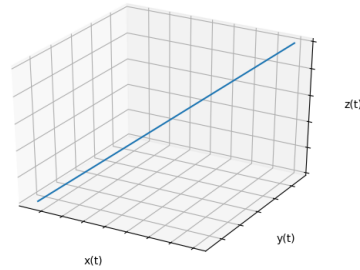
$$t \mapsto \vec{x}(t)$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Exaples

1. Motion along a line (with constant velocity)

$$\begin{aligned}\vec{x}(t) &= \vec{x}_0 + t\vec{v} \\ \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} &= \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= \begin{bmatrix} x_{01} + tv_1 \\ x_{02} + tv_2 \\ x_{03} + tv_3 \end{bmatrix}\end{aligned}$$



2. Circular motion

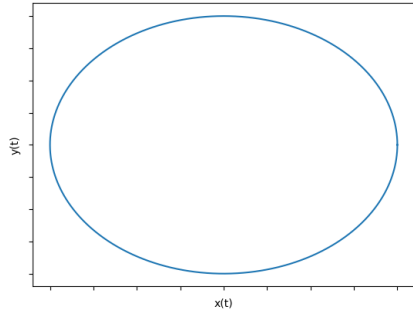
$$\vec{x}(t) = r \begin{bmatrix} \cos(\rho t) \\ \sin(\rho t) \end{bmatrix}$$

Where ρ some function and r is the radius.

If it is uniform:

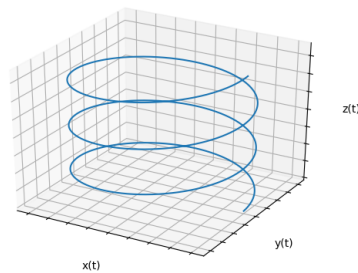
$$\begin{aligned}\vec{x}(t) &= r \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} \\ \omega &\in \mathbb{R} \setminus \{0\} \quad \omega = \frac{2\pi}{T}\end{aligned}$$

Where T is the time period.



3. Uniform spiral It is just like a circle, but with linear motion in the z-axis.

$$\vec{x}(f) = \begin{bmatrix} r \cos \omega t \\ r \sin \omega t \\ vt \end{bmatrix}$$



Kinematics - reminder

- ~> Done with modelling motion mathematically?
 - ↪ How to study the physics of motion?
- ~> What is the natural state of motion?
 - Without physical interaction.
- ~> At rest? (Aristotle) ×
- ~> Uniform motion (Newton)
 - ↪ Why?

↔ Changing reference frame to a moving one → object at rest relatively.

↔ But we cannot answer/ decide what the natural state of motion is in this stage.

Observation: State of motion changes due to physical interaction → Lets model physical interaction with forces. What is the change in motion due to force?

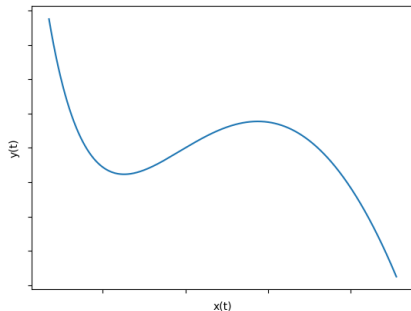
↪ The shape of the trajectory is curved

↪ We don't know how to model that!

↪ Back to maths

Kinematics 2

Modelling change in motion



Take two points on the above curve, $\vec{x}(t_0)$ and $\vec{x}(t)$. The change of displacement is $\Delta\vec{x}$. We know that velocity is change in displacement over change in time so we can find that using:

$$\begin{aligned}\vec{v} &= \frac{\Delta\vec{x}}{\Delta t} \\ &= \frac{\vec{x}(t) - \vec{x}(t_0)}{t - t_0}\end{aligned}$$

This gives us the average velocity between t_0 and t . We need to be able to find the velocity at every point in time t_0

$$\begin{aligned}\vec{v} &: \mathbb{R} \rightarrow \mathbb{R}^3 \\ t &\mapsto \text{velocity at } t\end{aligned}$$

Problem: We need a second time $t = t_0 + h$ for $\vec{v}(t_0)$, but there is no natural choice for h . We notice for more precision we need to make h as small as possible!

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h} \\ &= \frac{d\vec{x}}{dt} \text{ Leibniz notation} \\ &= \dot{\vec{x}}(t) \text{ Newton notation} \end{aligned}$$

Differentiation on vectors is just element wise:

$$\frac{d}{dt} \begin{bmatrix} R \cos \omega t \\ R \sin \omega t \end{bmatrix} = \omega R \begin{bmatrix} -\sin \omega t \\ \cos \omega t \end{bmatrix}$$

Limits create paradoxes: Take $\lim_{h \rightarrow 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}$. This works mathematically but causes a conceptual problem: How can something change at one point in time?

\rightsquigarrow Solution: Use infinitesimals

\rightarrow Numbers that are not provably not 0. Idea: If we zoom in on any curve enough, begins to look like a straight line: e.g. Earth looks flat.

How can we model this mathematically?

\rightsquigarrow Around each point $\vec{x}(t_0)$ the curve is made up of infinitesimal line segments.

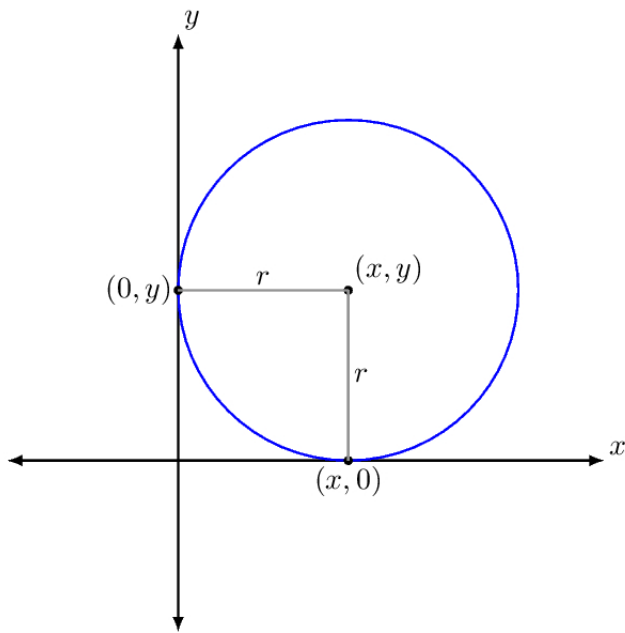
Geometrically: A tangent at $\vec{x}(t_0)$ touches the curve at $\vec{x}(t_0)$

\Leftrightarrow We take this to mean that it intersects with the curve at an infinitesimal line segment

\Leftrightarrow Infinitesimal line segment is the intersection of the tangent and the curve.

Problem: Do not have a rigorous definition of a tangent at a curve!

\rightarrow But we know: Every line that intersects the circle in one points is a tangent \rightsquigarrow Use that!



Take the radius, r , to be 1 and the center to be $(0, 1)$.

Equation of circle:

$$c : (x - 0)^2 + (y - 1)^2 = 1$$

If we take our tangent, t to be the y axis:

$$t : y = 0$$

Sub into c :

$$x^2 + 1 = 1$$

$$\Rightarrow x^2 = 0$$

$$\therefore c \cap t = \{(d, 0) \mid d^2 = 0\}$$

\rightsquigarrow Define first order infinitesimals as $D = \{d \in R \mid d^2 = 0\}$ where $R = \mathbb{R} \cup \{\text{infinitesimal numbers}\}$ Are such infinitesimals going to solve problems?

$$\begin{aligned}
 f(x) &= x^2 \\
 \Rightarrow f(x+d) & \qquad d \in D \\
 &= (x+d)^2 \\
 &= x^2 + 2dx + d^2 \\
 &= x^2 + 2dx \\
 &= f(x) + f'(x) \cdot d
 \end{aligned}$$

$$\begin{aligned}
 f(x) &= x^3 \\
 \Rightarrow f(x+d) & \\
 &= (x+d)^3 \\
 &= x^3 + 3x^2d + 3xd^2 + d^3 \\
 &= x^3 + 3x^2d \\
 &= f(x) + f'(x) \cdot d
 \end{aligned}$$

Infinitesimals

The derivative is used to solve the tangent problem. We can see that infinitesimals solve this too.

Examples: sine and cosine

Take a unit circle:

$$\begin{aligned}
 \sin x &= \sqrt{1 - \cos^2 x} \\
 \Rightarrow \sin d &= \sqrt{1 - \cos^2 d}
 \end{aligned}$$

Using and arc of angle α and arc length l

$$l = 2\pi R \cdot \frac{\alpha}{2\pi}$$

$$= R\alpha$$

$$\text{let } \alpha = d$$

$$\Rightarrow l = d$$

\therefore We are moving on a straight line and instead of an arc it is a right angled triangle

$$\Rightarrow \sin d = d$$

$$\cos d = \sqrt{1 - \sin^2(d)}$$

$$= \sqrt{1 - d^2}$$

$$= 1$$

$$\Rightarrow \sin(0 + d) = \sin(0) + \sin'(0) \cdot (d)$$

$$\Rightarrow d = \sin'(0) \cdot d$$

We would want this to be 1

$$\cos d = \cos(0 + d)$$

$$= \cos(0) + \cos'(0) \cdot d$$

$$= 1 + \cos'(0) \cdot d$$

$$1 = 1 + \cos'(0) \cdot d$$

$$\cos'(0) \cdot d = 0$$

$$\cos'(0) = 0$$

$$\sin(x + d) = \sin(x) + \sin'(x) \cdot d$$

$$= \sin(x) \cdot \cos(d) + \sin(d) \cdot \cos(x)$$

$$= \sin(x) + d \cdot \cos(x)$$

$$\sin'(x) \cdot d = \cos(x) \cdot d$$

Would want $\sin'(x) = \cos(x)$

$$\cos(x + d) =$$

Limits

$$\begin{aligned}\cos'(x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\ \sin'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}\end{aligned}$$

Conclusion from examples: Infinitesimal solve the tangent problem easily and allow us to calculate the derivatives, provided that we can find a theory that allows us to cancel infinitesimals.

Theory of Infinitesimals

Need: A set of numbers, R with addition, multiplication, and $\mathbb{R} \subseteq R$

Why? So that we can do the same algebra as the reals (careful with division).

A set $D = \{d \in \mathbb{R} | d^2 = 0\}$

Satisfying the following axioms (kock-lavere Axiom)(K-1)

For all $g : 0 \rightarrowtail R$

$$a.d = b.d \quad \forall d \in D$$

$\Rightarrow a = b$ (Cancellation Lemma)

\therefore There are unique $a, b \in R$ s.t. $g(d) = a + b.d \quad \forall d \in D$