RHS

$\alpha u + \beta u = \alpha(x, y) + \beta(x, y)$

$= (x, \alpha y) + (x, \beta y)$

$= (x + x, \alpha y + \beta y)$

$= (2x, \alpha y + \beta y)$

LHS $\neq$ RHS

$\therefore$ V is not a vector space over R.

Theorem: 1

statement

Let V be a vector space over a field F

then PT,

i. $\alpha \cdot 0 = 0 \ \forall \alpha \in F$

ii. $0v = 0 \ \forall v \in V$

iii. $(-\alpha)v = \alpha(-v) = -(\alpha v) \ \forall \alpha \in F$ and $v \in V$

iv. $\alpha v = 0 \Rightarrow \alpha = 0$ (or) $v = 0$

proof:

i. $\alpha \cdot 0 = \alpha(0 + 0)$

$\alpha 0 = \alpha 0 + \alpha 0$ [By cancellation law]

$0 = \alpha 0$

$\alpha 0 = 0$ Hence (i) proved.

ii. $0v = (0 + 0)v$

$0v = 0v + 0v$

$0 = 0v$

$0v = 0$ Hence (ii) proved

iii. To PT $(-\alpha)v = \alpha(-v) = -(\alpha v)$

$\alpha v + (-\alpha)v = (\alpha - \alpha)v$ [combining 1st & 3rd Term]

$= 0v$

$\alpha v + (-\alpha v) = 0$

$(-\alpha)v = -(\alpha v)$ — ①

[combining 2nd & 3rd Term]

$\alpha v + \alpha(-v) = \alpha(v - v)$

$= \alpha 0 = 0$

$\alpha(-v) = -(\alpha v)$ — ②

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From ① & ②.

$\alpha(-v) = (-\alpha)v = -(\alpha v)$ Hence (iii) proved.

iv. Let $\alpha v = 0$

If $\alpha = 0$ then there is nothing to prove.

If $\alpha \neq 0$ then α^{-1} exist.

$$\alpha v = 0$$
$$\alpha^{-1}(\alpha v) = \alpha^{-1}(0)$$
$$(\alpha^{-1}\alpha)v = 0$$
$$1 \cdot v = 0$$
$$v = 0$$

Hence (iv) proved.

$\alpha v = 0 \Rightarrow \alpha = 0$ (or) $v = 0$.

Theorem: 3

Statement

Let V be a vector space over a field F.

ST i. $\alpha(u-v) = \alpha u - \alpha v$

ii. $\alpha u = \alpha v$ and $\alpha \neq 0 \Rightarrow u = v$

iii. $\alpha u = \beta u$ and $u \neq 0 \Rightarrow \alpha = \beta$

proof:

i. $\alpha(u-v) = \alpha(u+(-v))$

$= \alpha u + \alpha(-v)$

$= \alpha u - \alpha v$

$\alpha(u-v) = \alpha u - \alpha v$ Hence (i) proved

ii. $\alpha \neq 0 \Rightarrow \alpha^{-1}$ exists

$$\alpha u = \alpha v$$
$$\alpha^{-1}(\alpha u) = \alpha^{-1}(\alpha v)$$
$$(\alpha^{-1}\alpha)u = (\alpha^{-1}\alpha)v$$
$$u = v$$

Hence (ii) proved.

iii. $\alpha u = \beta u$

$$\alpha u - \beta u = 0$$
$$(\alpha - \beta)u = 0 \quad [u \neq 0]$$
$$\alpha - \beta = 0$$
$$\alpha = \beta$$

Hence (iii) proved.

Subspaces

let V be a vector space over a field F. A non-empty subset W of V is called a subspace of V if W itself a vector space over F under the operation of V.

Theorem-3

statement :-

let V be a vector space over F. A non-empty subset W of V is a subspace of V iff W is closed with respective vector addition and scalar multiplication in V.

proof:

let W be a subspace of V.

Then by ~~addition~~ defn W itself is a vector space and hence W is closed with respective vector addition and scalar multiplication

Hence proved//

conversely

let W be a non-empty subset of V $\ni$: $u, v \in W \Rightarrow u+v \in W$

$u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$.

TO PT W is a subspace of V.

$\because$ W is non empty $\exists$ an element $u \in W$

$0u = 0 \in W$

$v \in W \Rightarrow -1 \cdot v = -v \in W$

Thus W contains 0 and the additive inverse of each element

Hence W is an additive subgroup of V

$u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$

since the elements of W are the elements of V the other axioms of a vector space are true in W.

Hence W is a subspace of V

Hence proved//

Theorem: 4

Statement :-

Let V be a vector space over a field R. A non-empty subset W of V is a subspace of V iff $u, v \in W$ and $\alpha, \beta \in F \Rightarrow \alpha u + \beta v \in W$.

proof:

Let W be a subspace of V

Let $u, v \in W$ and $\alpha, \beta \in F$

Since W is a subspace of V, W closed with respect to vector addition and scalar multiplication

$u, v \in W$ and $\alpha, \beta \in F \Rightarrow \alpha u, \beta v \in W$ [scalar multiplication]

$\alpha u + \beta v \in W$ [vector addition]

Hence proved.

conversely,

$u, v \in W$ and $\alpha, \beta \in F \Rightarrow \alpha u + \beta v \in W$.

$\Rightarrow$ Taking $\alpha = \beta = 1$ we have,

$u, v \in W \Rightarrow u + v \in W$

$\therefore W$ is closed with respect to vector addition

$\Rightarrow$ taking $\beta = 0$ we get,

$\alpha \in F, u \in W \Rightarrow \alpha u \in W$

$\therefore W$ is closed w.r.t scalar multiplication

Hence by the prev. theorem, W is a subspace.

Hence proved.

Note:

$\{0\}$ and V are subspaces of any vector space V. They are called Trivial subspace of V.

Examples

1. PT $W = \{(a, 0, 0) / a \in R\}$ is a subspace of R^3

sol:-

Let $u = (a, 0, 0)$ $v = (b, 0, 0)$ where $u, v \in W$ and $\alpha, \beta \in F$

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$$= \begin{pmatrix} \alpha a+\beta c & 0 \\ 0 & \alpha b+\beta d \end{pmatrix}$$

$\alpha u+\beta v \in W$

$\therefore$ W is a subspace of $M_2(R)$

Theorem. 5

Statement -

PT the union of two subspaces of a vector space need not be a subspace

proof

let $A = \{(a,0,0) / a \in R\}$ and $B = \{(0,b,0) / b \in R\}$

clearly, A & B are subspaces of R^3

To PT $A \cup B$ is not a subspaces of R^3.

let $u = (1,0,0)$ $v = (0,1,0) \in A \cup B$

Take $\alpha = \beta = 1$

Then $\alpha u+\beta v = 1(1,0,0) + 1(0,1,0)$

$= (1,1,0) \notin A \cup B$

$\therefore$ $A \cup B$ is not a subspace of R^3

Hence proved.

Theorem - 6

Statement :

PT the intersection of 2 subspaces of a vector space is a subspace

proof -

let A and B be 2 subspaces of a vector space over a field F

To PT $A \cap B$ is a subspace of V

clearly $0 \in A \cap B$ and hence $A \cap B$ is non-empty

let $u, v \in A \cap B$

Now let $u, v \in A$ and $u, v \in B$

Let $\alpha, \beta \in F$

since A is a subspace $\alpha u + \beta v \in A$ ..

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since B is a vector subspace $\alpha u+\beta v \in B$ 7

$\therefore \alpha u+\beta v \in A\cap B$

Hence by the theorem $A\cap B$ is a subspace of V.

Hence proved.

problems -

1 If A and B are subspaces of V P.T

$A+B = \{v \in V / v = a+b, a \in A, b \in B\}$ is a subspace of V.

further S.T A+B is the smallest subspace containing A and B i.e., if W is any subspace of V containing A and B then W contains A+B

sol:-

Let $v_1, v_2 \in A+B$ and $\alpha \in F$

Then $v_1 = a_1+b_1$ and $v_2 = a_2+b_2$ where $a_1, a_2 \in A$ & $b_1, b_2 \in B$

Now, $v_1+v_2 = (a_1+b_1) + (a_2+b_2)$

$= (a_1+a_2) + (b_1+b_2)$

$\in A+B$ $[\because a_1+a_2 \in A$ & $b_1+b_2 \in B]$

i.e., $v_1+v_2 \in A+B$

Also $\alpha v_1 = \alpha(a_1+b_1)$

$= \alpha a_1 + \alpha b_1 \in A+B$

$\therefore$ A+B is closed w.r.t addition & scalar multiplication

Hence by theorem 8, A+B is a subspace of V

To prove A+B is the smallest subspace containing A & B.

clearly, $A \subseteq A+B$ & $B \subseteq A+B$

Let W be any subspace of V containing A & B

we shall to P.T. $A+B \subseteq W$

Let $v \in A+B$

Then $v = a+b$ where $a \in A$ and $b \in B$

$A \subseteq W$, $a \in W$ and

$B \subseteq W$, $b \in W$

$\therefore a+b \in W$

i.e., $v \in W$

$\therefore$ $A+B \subseteq W$ so $A+B$ is the smallest subspace of V containing A & B

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Hence proved//

2. Let A and B be subspace of V.S Then $A \cap B = \{0\}$ iff every vector $v \in A+B$ can be uniquely expressed in the form $v = a+b$ where $a \in A$ & $b \in B$

Sol: Let $A \cap B = \{0\}$ Let $v \in A+B$.

To PT $v \in A+B$ is uniquely expressed in the form, $v = a+b$

Let $v = a_1+b_1 = a_2+b_2$ where $a_1, a_2 \in A$ $b_1, b_2 \in B$

Then $a_1 - a_2 = b_2 - b_1$ where $a_1 - a_2 \in A$ & $b_2 - b_1 \in B$

But $a_1 - a_2 \in A$ and $b_2 - b_1 \in A$ $[\because a_1 - a_2 = b_2 - b_1]$

$b_2 - b_1 \in B$ and $a_1 - a_2 \in B$ $[\because b_2 - b_1 = a_1 - a_2]$.

$\therefore$ $a_1 - a_2 \in A \cap B$ and $b_2 - b_1 \in A \cap B$

Since $A \cap B = \{0\}$; $a_1 - a_2 = 0$ & $b_2 - b_1 = 0$

$a_1 = a_2$ & $b_1 = b_2$

Hence the expression of v is uniquely expressed in the form $v = a+b$ where $a \in A$ & $b \in B$

Hence proved.

conversely.

suppose that any element in $A+B$ can be uniquely expressed in the form $a+b$ where $a \in A$ & $b \in B$.

To claim $A \cap B = \{0\}$, suppose $A \cap B \neq \{0\}$

Let $v \in A \cap B$ and $v \neq 0$

Then $0 = v - v = 0 + 0$

Thus 0 can be expressed in the form $a+b$ in 2 different ways which is a ~~c~~

Hence $A \cap B = \{0\}$. Hence proved//

Direct Sum

Defn:-

Let A and B be subspaces of vector space V. Then V is called the direct sum of A and B if (i) $A+B=V$, (ii) $A\cap B=\{0\}$

If V is the direct sum of A & B we write,

$V=A\oplus B$

Note:-

If $V=A\oplus B$ iff every element of V can be uniquely expressed in the form $a+b$ where $a\in A$ & $b\in B$.

Example

1. In $V_3(R)$ let $A=\{(a,b,0)/a,b\in R\}$ and $B=\{(0,0,c)/c\in R\}$

clearly A and B are subspaces of V and $A\cap B=\{0\}$

let $v=(a,b,c)\in V_3(R)$ Then $v=(a,b,0)+(0,0,c)$

Hence, $A+B=V_3(R)$

$\therefore V_3(R)=A\oplus B$

ii. In $M_2(R)$, let A be the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and B be the set of matrices of the form $\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$

clearly, A and B are subspaces of $M_2(R)$ and

$A\cap B=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\{0\}$

$A+B=\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$

$=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=M_2(R)$

$\therefore M_2(R)=A\oplus B$

Theorem-1

Statement :

let V be the vector space over F and 'W' a subspace of V. Let $V/W = \{W+v \mid v \in V\}$ then PT V/W is a vector space over F under the following operation

i. $(W+v_1) + (W+v_2) = W+v_1+v_2$

ii. $\alpha(W+v_1) = W+\alpha v_1$

proof:-

i. V/W is closed under vector addition.

ii. Associative :

$W+v_1, W+v_2, W+v_3 \in V/W$

$(W+v_1)+[(W+v_2)+(W+v_3)] = (W+v_1)+[W+v_2+v_3]$

$= W+v_1+v_2+v_3$

$= (W+v_1+v_2)+(W+v_3)$

$= [(W+v_1)+(W+v_2)]+(W+v_3)$

Addition is associative

iii. $(W+v_1)+(W+0) = W+v_1+0 = (W+0)+(W+v_1)$

$\therefore W+0$ is the identity element.

iv. $(W+v_1)+(W-v_1) = W+0$

$W-v_1$ is the inverse of $W+v_1$

v. $(W+v_1)+(W+v_2) = W+v_1+v_2$

$= (W+v_2)+v_1$

$= (W+v_2)+(W+v_1)$

$\therefore V/W$ is an abelian group under addition

vi. To PT $\alpha(u+v) = \alpha u+\alpha v$

$[\alpha((W+v_1)+(W+v_2))] = \alpha(W+v_1+v_2)$

$= W+\alpha(v_1+v_2)$

$= (W+\alpha v_1)+(W+\alpha v_2)$

$= \alpha(W+v_1)+\alpha(W+v_2)$

A linear transformation $T: V \to F$ is called a linear functional.

Example:

1. $T: V \to W$ defined by $T(v) = 0 \ \forall \ v \in V$ is a Trivial linear transformation.

2. $T: V \to V$ defined by $T(v) = v \ \forall \ v \in V$ is the identity Linear Transformation

3. Let V be vector space over a field F & W be a subspace of V. Then $T: V \to V/W$ defined by $T(v) = W + v$ is a linear Transformation.

sol:- Let $v_1, v_2 \in V$

i. To PT $T(v_1 + v_2) = T(v_1) + T(v_2)$

LHS:-

$$T(v_1 + v_2) = W + (v_1 + v_2)$$
$$= (W + v_1) + (W + v_2)$$
$$= T(v_1) + T(v_2) = RHS$$

ii. To PT $T(\alpha v_1) = \alpha T(v_1)$

LHS:

$$T(\alpha v_1) = W + \alpha v_1$$
$$= \alpha (W + v_1)$$
$$= \alpha T(v_1) = RHS$$

This is called natural homomorphism from $V \to V/W$ clearly T is onto and hence T is epimorphism

$T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by,

$T(a, b) = (8a - 3b, a + 4b)$ is a linear Transformation

proof:-

Let $u = (a, b)$ & $v = (c, d) \in \mathbb{R}^2$

i. $T(u + v) = T(u) + T(v)$

LHS:

$$T(u+v) = T((a, b) + (c, d))$$
$$= T[a + c, b + d]$$
$$= [8(a+c) - 3(b+d), (a+c) + 4(b+d)]$$
$$= 8a + 8c - 3b - 3d, a + c + 4b + 4d$$
$$= 8a - 3b + 8c - 3d, a + 4b + c + 4d$$

$= (2a-3b, a+4b) + (2c-3d, c+4d)$

$= T(a,b) + T(c,d)$

$= T(u) + T(v) = RHS$

ii. $T(\alpha u) = \alpha T(u)$

LHS:- $T(\alpha u) = T\alpha(a,b)$

$= T(\alpha a, \alpha b)$

$= 2\alpha a - 3\alpha b, \alpha a + 4\alpha b$

$= \alpha T(a,b)$

$= \alpha T(u) = RHS$

$\therefore T$ is linear Transformation

Example :

5. Let V be the set of all polynomials of degree $\leq n$ in $R[x]$ of including the zero polynomial. $T: V \to V$ defined by $T(f) = \frac{df}{dx}$ is a linear Transformation

sol:-

$T(f+g) = \frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} = T(f) + T(g)$

Also, $T(\alpha f) = \frac{d(\alpha f)}{dx} = \alpha \frac{df}{dx} = \alpha T(f)$.

6. Let V be the set of all polynomials of degree $\leq n$ in $R[x]$ of including the zero polynomial Then $T: V \to V_{n+1}(R)$ defined by, $T(a_0 + a_1 x + \cdots + a_n x^n) = (a_0, a_1, \ldots a_n)$ is a linear Transformation

sol:-

Let $f = a_0 + a_1 x + \cdots + a_n x^n$ and

$g = b_0 + b_1 x + \cdots + b_n x^n$

Then $f+g = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$

$\therefore T(f+g) [(a_0+b_0), (a_1+b_1), \ldots (a_n+b_n)]$

$= (a_0, a_1 \ldots a_n) + (b_0, b_1, \ldots b_n)$

$= T(f) + T(g)$

Also $T(\alpha f) = (\alpha a_0, \alpha a_1, \dots \alpha a_n)$

$= \alpha(a_0, a_1 \dots a_n)$

$= \alpha T(f)$

$\therefore T$ is Linear Transformation

clearly T is 1-1 & onto and Hence T is an isomorphism 14.

Theorem-8

Statement:

let $T: V \to W$ be a linear transformation then, $T(V) = \{T(v) / v \in V\}$ is a subspace of W.

proof:

Let w_1 & $w_2 \in T(V)$ and $\alpha \in F$

Then $\exists\ v_1, v_2 \in V \ni: T(v_1) = w_1$ & $T(v_2) = w_2$

Here $w_1 + w_2 = T(v_1) + T(v_2)$

$= T(v_1 + v_2)$ [$\because$ T is linear Transformation]

$\in T(V)$ ($\because v_1 + v_2 \in V$)

Similarly, $\alpha w_1 = \alpha \cdot T(v_1) = T(\alpha v_1)$

$\in T(V)$ [$\because$ T is Linear Transformation]

$\therefore T(V)$ is closed w.r.t vector addition & scalar multiplication

T(V) is a subspace of W

Hence proved//

Kernal of T [Def]

Let V & W be a vector space over a field F & $T: V \to W$ be a linear transformation Then the kernal of T is defined to be $\{v / v \in V$ & $T(v) = 0\}$ and is denoted by ker T

Thus $\ker T = \{v / v \in V$ & $T(v) = 0\}$

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Example

In eg-1 [previous] $Ker\,T = V$ 15

In eg-2 $Ker\,T = \{0\}$

In eg-5, $Ker\,T$ is the set of all constant polynomials

Note:-

$T: V \to W$ is a linear Transformation then T is a monomorphism iff $Ker\,T = \{0\}$

proof:

G.T $T: V \to W$ is a linear Transformation

let T be a monomorphism

ie, T is 1-1

TO PT $Ker\,T = \{0\}$

let $v \in Ker\,T$

Then by def, $T(v) = 0$

WKT, $T(v) = 0 = T(0) \Rightarrow v = 0$ [$\because T$ is 1-1]

$\therefore Ker\,T = \{0\}$

Hence proved

conversely,

let $Ker\,T = \{0\}$

TPT T is 1-1

let $T(v) = T(w)$

$T(v) - T(w) = 0$

$T(v-w) = 0$

$v - w = 0 \in Ker\,T = \{0\}$

$v - w = 0$

$v = w$

$\therefore T$ is 1-1

Hence proved//

Theorem-9

Fundamental Theorem of Homomorphism

Statement:

Let V and W be V.S over a field F and $T: V \to W$ be an epimorphism Then,

i. $\ker T = V_1$ is a subspace of V and

ii. $V/V_1 \cong W$ i.e., $\frac{V}{\ker T} \cong W$.

Proof:

i. G.T $T: V \to W$ be an epimorphism

To PT $\ker T = V_1$ is a subspace of V.

Let $V_1 = \ker T = \{v / v \in V \text{ } \& \text{ } T(v) = 0\}$

Now $T(0) = 0$; $0 \in \ker T = V_1$ $\qquad \because V_1$ is nonempty

Let $\alpha, \beta \in F$ & $v, w \in V_1$

since $v, w \in V_1 \Rightarrow T(v) = 0$; $T(w) = 0$

Now $T(\alpha v + \beta w) = T(\alpha v) + T(\beta w) = \alpha \cdot T(v) + \beta \cdot T(w)$

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0$

i.e., $T(\alpha v + \beta w) = 0$

$\alpha v + \beta w \in \ker T = V_1$ $\qquad \therefore V_1$ is a subspace of V.

ii. To PT $V/V_1 \cong W$

Define $\phi : V/V_1 \to W$ by $\phi(V_1 + v) = T(v)$ $\forall v \in V$

claim: ϕ is well defined,

Let $V_1 + v = V_1 + w$, $v, w \in V$

$v - w \in V_1 = \ker T$ $\qquad [\because Ha = Hb \Leftrightarrow ab^{-1} \in H]$

$T(v - w) = 0$

$T(v) - T(w) = 0$

$T(v) = T(w)$

$\phi(V_1 + v) = \phi(V_1 + w)$

$\therefore \phi$ is well defined.

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i. ϕ is 1-1 :

Let $\phi(V_1+v) = \phi(V_1+w)$

$T(v) = T(w)$

$T(v) - T(w) = 0$

$T(v-w) = 0$

$v-w \in \ker T = V_1$

$v-w \in V_1$ [$\because Ha = Hb \Leftrightarrow ab^{-1} \in H$]

$V_1+v = V_1+w$

$\therefore \phi$ is 1-1.

ii. ϕ is onto

let $w \in W$,

since T is onto $\exists$ an element $v \in V \ni : T(v) = w$

$\therefore \phi(V_1+v) = T(v) = w$

$\phi(V_1+v) = w$

$\therefore \phi$ is onto.

ϕ is homomorphism :-

Let V_1+v and $V_1+w \in V/V_1$ and $\alpha \in F$

i. $\phi((V_1+v)+(V_1+w)) = \phi[V_1+(v+w)]$

$= T(v+w)$

$= T(v) + T(w)$

$= \phi(V_1+v) + \phi(V_1+w)$.

ii. $\phi(\alpha(V_1+v)) = \phi(V_1+\alpha v)$

$= T(\alpha v)$

$= \alpha \cdot T(v)$

$= \alpha \cdot \phi(V_1+v)$

$\phi(\alpha(V_1+v)) = \alpha \cdot \phi(V_1+v)$

$\therefore \phi$ is homomorphism

ϕ is an isomorphism from V/V_1 to W.

Hence $V/V_1 \cong W$

Hence proved.

Theorem-10

statement:-

let V be a V.S over a field F. Let A and B be the subspaces of V Then PT $\frac{A+B}{A} \cong \frac{B}{A\cap B}$

proof:

Since A and B are subspaces of V, A+B is also a subspace of V containing A

Hence $\frac{A+B}{A}$ is a V.S over F.

An element of $\frac{A+B}{A}$ is of the form $A+(a+b)$, where $a \in A$ and $b \in B$

But $A+a=A$ $[\because a \in A]$

Hence an element of $\frac{A+B}{A}$ is of the form $A+b$.

consider a function $f: B \to \frac{A+B}{A}$ defined by

$f(b) = A+B$, $b \in B$

clearly f is onto.

let $A+b_1, A+b_2 \in \frac{A+B}{A}$, $b_1, b_2 \in B$

$f(b_1+b_2) = A+b_1+b_2$

$= (A+b_1)+(A+b_2)$

$= f(b_1)+f(b_2)$

$\Rightarrow f(\alpha b_1) = A+\alpha b_1$

$= \alpha(A+b_1)$

$= \alpha(f(b_1))$

$\therefore$ f is a linear Transformation

Also f is onto

Hence f is an epimorphism

let K be the kernal of f, then

$K = \{b / b \in B, f(b) = A\}$ [since $f(b) = A+b$]

let $b \in K$

$f(b) = A$

$A+b = A$

$b \in A$

since $b \in A$ and $b \in B$ we get $b \in A \cap B$

$\therefore K \subseteq A \cap B$.

$c \in A \cap B$

$c \in A$ and $c \in B$

$A + c = A$

$f(c) = A$

$c \in K$

$\therefore A \cap B \subseteq K$

$\therefore$ By the fundamental theorem of Homomorphism

$$\frac{B}{A \cap B} \cong \frac{A+B}{A}$$

Hence proved//

Theorem 11

Statement:

Let V and W be V.S over a field F. Let $L(V, W)$ represent the set of all linear transformation from V to W. Then $L(V, W)$ itself is V.S over F under addition & scalar multiplication defined by $(f+g)(v) = f(v) + g(v)$ and $(\alpha f)(v) = \alpha \cdot f(v)$.

proof:-

Let $f, g \in L(V, W)$ and $v_1, v_2 \in V$.

Then $(f+g)(v_1+v_2) = f(v_1+v_2) + g(v_1+v_2)$

$= f(v_1) + f(v_2) + g(v_1) + g(v_2)$

$= (f+g)v_1 + (f+g)v_2$

Also $(f+g)(\alpha v)$ $= f(\alpha v) + g(\alpha v)$

$= \alpha \cdot f(v) + \alpha \cdot g(v)$

$= \alpha[f(v) + g(v)]$

$(f+g)(\alpha v)$ $= \alpha (f+g) \cdot v$

Hence, $f+g \in L(V, W)$

Now $(\alpha f)(v_1+v_2) = (\alpha f)v_1 + (\alpha f)v_2$

$= \alpha \cdot f(v_1) + \alpha \cdot f(v_2)$

$= \alpha[f(v_1) + f(v_2)]$

$(\alpha f)(v_1+v_2) = \alpha \cdot f(v_1+v_2)$

Also, $(\alpha f)(\beta v) = \alpha[f(\beta v)]$

$= \alpha[\beta \cdot f(v)] = \beta[\alpha \cdot f(v)]$

Hence $\alpha f \in L(V, W)$

$\therefore L(V,W)$ is closed under addition and scalar multiplication.

Addition defined on $L(V,W)$ is obviously commutative & Associative

Then function $f : V \to W$ defined by $f(v) = 0$ & $v \in V$ is clearly a linear Transformation & is the additive identity of $L(V,W)$

Further $(-f) : V \to W$ defined by $(-f)v = -f(v)$ is the additive inverse of F.

Thus $L(V,W)$ is an abelian group under addition. The remaining axioms for a V.S can be easily verified

Hence $L(V,W)$ is a vector space.