$$= \frac{1}{27\,(x - 2/3 + 1)(x - 2/3 + 2)(x - 2/3 + 3)}$$

$$= \frac{1}{27}\left(x - 2/3\right)^{(-3)}$$

$$\Delta y = 1/27\,(-3)\left(x - 2/3\right)^{(-4)}$$

$$\Delta^2 y = 3/27 \times 4\,\left(x - 2/3\right)^{(-5)}$$

$$= \frac{12}{27\left[(x-2/3)+1\right]\left[(x-2/3)+2\right]\left[(x-2/3)+3\right]\left[(x-2/3)+4\right]\left[(x-2/3)+5\right]}$$

$$= \frac{12 \times 3}{27\,(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

$$\Delta^2 y = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

Hence showed.

14. Find the second differential of the polynomial $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ with $h = 2$

sol: First we shall express the given polynomial $f(x)$ in terms of factorial polynomial by synthetic division with $h = 2$

	1	-12	42	-30	9
0	↓	0	0	0	0
	1	-12	42	-30	$\boxed{9}$
2	↓	2	-20	44	
	1	-10	22	$\boxed{14}$	
4	↓	4	-24		
	1	-6	$\boxed{-2}$		
	↓	6			
	1	$\boxed{0}$			

$f(x) = x^{(4)} - 2x^{(2)} + 14x^{(1)} + 9$

$\Delta f(x) = 8x^{(3)} - 8x^{(1)} + 28$

$\Delta^2 f(x) = 48x^{(2)} - 16 \rightarrow 48\,x(x-2) - 16$

$\rightarrow 48x^2 - 96x - 16$

15. S.T $\Delta(5x^4+6x^3+x^2-x+7) = 20x^{(3)}+108x^{(2)}+108x^{(1)}+11$

sol

$y = 5x^4+6x^3+x^2-x+7$

Let $Ax^{(4)}+Bx^{(3)}+Cx^{(2)}+Dx^{(1)}+E$ be the factorial polynomial of y.

0	5	6	1	-1	7
	↓	0	0	0	0
1	5	6	1	-1	$\lfloor 7$
	↓	5	11	12	
2	5	11	12	$\lfloor 11$	
	↓	10	42		
3	5	21	$\lfloor 54$		
	↓	15			
	5	$\lfloor 36$			

$\therefore y = 5x^{(4)}+36x^{(3)}+54x^{(2)}+11x^{(1)}+7$

$\Delta y = 20x^{(3)}+108x^{(2)}+108x^{(1)}+11$

16. Find the fn whose first difference is $x^3+3x^2+6x+12$

sol

Given $\Delta y = x^3+3x^2+5x+12$

We express this in terms of factorial polynomial

0	1	3	5	12
	↓	0	0	0
1	1	3	5	$\lfloor 12$
	↓	1	4	
2	1	4	$\lfloor 9$	
	↓	2		
	1	$\lfloor 6$		

$\Delta y = x^{(3)}+6x^{(2)}+9x^{(1)}+12$

$\therefore y = \Delta^{-1}[x^{(3)}+6x^{(2)}+9x^{(1)}+12]$

$= \frac{x^{(4)}}{4}+2x^{(3)}+\frac{9x^{(2)}}{2}+12x^{(1)}+c$

$y = \frac{1}{4}[x(x-1)(x-2)(x-3)]+2[x(x-1)(x-2)]+\frac{9}{2}x(x-1)+12x+c$

17. If $y = \frac{1}{(3x+1)(3x+4)(3x+7)}$ S.T $\Delta^2 y = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$

sol

$y = \frac{1}{(3x+1)(3x+4)(3x+7)} \Rightarrow \frac{1}{27(x+1/3)(x+4/3)(x+7/3)}$

$= \frac{1}{27(x-2/3+1)(x-2/3+2)(x-2/3+3)} \Rightarrow y = \frac{1}{27}\left(x-\frac{2}{3}\right)^{-3}$

$\Delta y = \frac{1}{27}(-3)(x-2/3)^{-4} \Rightarrow \Delta^2 y = \frac{1}{27}3\times 4\,(x-2/3)^{-5}$

$$= \frac{18}{27\left((x-8/3)+1\right)\left((x-8/3)+2\right)\left((x-2/3)+3\right)\left((x-2/3)+4\right)\left((x-2/3)+5\right)}$$

$$= \frac{12 \times 36}{27\,(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

$$= \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

Other difference operators

In this section we introduce the shift operator E and averaging operator μ.

Def : The shift operator E is defined by,

$$E f(x) = f(x+h)$$

Hence $E^2 f(x) = E f(x+h) = f(x+2h)$

In general for any positive integer n

$$E^n f(x) = f(x+nh)$$

The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h)$$

For any real number n we have,

$$E^n f(x) = f(x+nh)$$

Note : $E^m E^n f(x) = E^{m+n} f(x)$

Defn :- The averaging operator μ is defined by,

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

There are several relations connecting the operators $\Delta, \nabla, \delta, E, \mu$ and the differentiation operator D. These results are presented in the following theorem

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Theorem 6.4 $\quad E = 1 + \Delta$

proof:-
$\Delta f(x) = f(x+h) - f(x)$
$= E\, f(x) - f(x)$
$= (E-1)\, f(x)$

Hence $\Delta = E - 1$

$\therefore E = \Delta + 1$

Theorem 6.5 $\quad \nabla = 1 - E^{-1}$

proof
$\nabla f(x) = f(x) - f(x-h)$
$= f(x) - E^{-1} f(x)$
$= (1 - E^{-1})\, f(x)$

Hence, $\nabla = 1 - E^{-1}$

Theorem 6.6 $\quad \delta = E^{1/2} - E^{-1/2}$

proof:-
$\delta f(x) = f(x + h/2) - f(x - h/2)$
$= E^{1/2} f(x) - E^{-1/2} f(x)$
$= (E^{1/2} - E^{-1/2})\, f(x)$

$\delta = E^{1/2} - E^{-1/2}$

Theorem: 6.7 $\quad \mu = \dfrac{E^{1/2} + E^{-1/2}}{2}$

proof:-
$\mu f(x) = \dfrac{f(x+h/2) + f(x-h/2)}{2} \rightarrow \dfrac{E^{1/2} f(x) + E^{-1/2} f(x)}{2}$

$\mu f(x) = \left(\dfrac{E^{1/2} + E^{-1/2}}{2}\right) f(x)$

$\mu = \dfrac{E^{1/2} + E^{-1/2}}{2}$

Theorem: 6.8 $\quad \delta = E^{1/2} \nabla$

proof:-
$\delta f(x) = (E^{1/2} - E^{-1/2})\, f(x)$
$= E^{1/2}(1 - E^{-1})\, f(x)$
$= E^{1/2} \nabla f(x)$ (using thm 6.5)

$\delta = E^{1/2} \nabla$

Theorem 6.9 $E = e^{hD}$

proof:

The Taylor's series expansion of $y = f(x)$ is gn by,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2 f''(x)}{2!} + \cdots + \frac{h^n f^{(n)}(x)}{n!} + \cdots$$

$$E f(x) = f(x) + h D[f(x)] + \frac{h^2}{2!} D^2[f(x)] + \cdots + \frac{h^n}{n!} D^n[f(x)] + \cdots$$

$$= \left[1 + hD + \frac{h^2}{2!} D^2 + \cdots + \frac{h^n}{n!} D^n + \cdots\right] f(x)$$

Hence, $E = 1 + hD + \frac{h^2 D^2}{2!} + \cdots + \frac{h^n D^n}{n!} + \cdots = e^{hD}$

Thus $E = e^{hD}$

Theorem 6.10

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]$$

proof:-

$E = e^{hD}$ [by theorem 6.9]

$\therefore hD = \log E = \log(1+\Delta)$

$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]$$

Note: Taking E as the fundamental operator we have expressed the other operator $\Delta, \nabla, \delta, \mu, D$ in terms of E as

1. $\Delta = E - 1$ 2. $\nabla = 1 - E^{-1}$ 3. $\delta = E^{1/2} - E^{-1/2}$

4. $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$ 5. $D = 1/h \log E$

problem

1. P.T $E\nabla = \nabla E = \Delta$

sol $E\nabla = E(1 - E^{-1}) = E - 1 = \Delta$

Also $\nabla E = (1 - E^{-1})E = E - 1 = \Delta$

2. P.T $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$

sol $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2})E^{1/2}$

$= E + 1$

$= (1 + \Delta) + 1$

$= 2 + \Delta$

3. P.T $\Delta\nabla = \Delta - \nabla = \delta^2$

Sol

$\Delta\nabla = (1-E^{-1})(E-1)$

$= E-1-1+E^{-1}$

$= E+E^{-1}-2$

$= (E^{1/2}-E^{-1/2})^2$

$\Delta\nabla = \delta^2$

Also, $\Delta - \nabla = (E-1)-(1-E^{-1})$

$= E-1-1+E^{-1}$

$= E+E^{-1}-2$

$= (E^{1/2}-E^{-1/2})^2$

$\Delta - \nabla = \delta^2$

4. P.T $E^{1/2} = \mu + 1/2\,\delta$

Sol WKT $\mu = 1/2(E^{1/2}+E^{-1/2})$ and $\delta = E^{1/2}-E^{-1/2}$

$\mu + 1/2\,\delta = \frac{1}{2}(E^{1/2}+E^{-1/2}) + \frac{1}{2}(E^{1/2}-E^{-1/2}) = E^{1/2}$

5. P.T $\mu\delta = \frac{\Delta}{2} + \frac{\Delta E^{-1}}{2}$

Sol

$\frac{\Delta}{2} + \frac{\Delta E^{-1}}{2} = \frac{\Delta}{2}(1+E^{-1})$

$= 1/2(E-1)(1+E^{-1})$ $(\because \Delta = E-1)$

$= 1/2(E+EE^{-1}-1-E^{-1})$

$= 1/2(E-E^{-1})$

$= \left(\frac{E^{1/2}+E^{-1/2}}{2}\right)(E^{1/2}-E^{-1/2})$

$= \mu\delta$ $\because$ [By defn of μ & δ]

6. P.T $1-e^{-hD} = \nabla$

Sol WKT $D = \frac{1}{h}\log E$

$\therefore hD = \log E$

$e^{hD} = E$

$\frac{1}{e^{hD}} = \frac{1}{E} \rightarrow e^{-hD} = E^{-1} \rightarrow 1-\nabla$

$\nabla = 1-e^{-hD}$

7. P.T $\Delta+\nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

Sol: WKT $\Delta = E-1$ and $\nabla = 1-E^{-1}$.

$$\frac{\Delta}{\nabla}-\frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1} \to \frac{(E-1)^2-(1-E^{-1})^2}{(1-E^{-1})(E-1)}$$

$$= \frac{(E-1-1+E^{-1})(E-1+1-E^{-1})}{(E-1-1+E^{-1})} \to E-1+1-E^{-1}$$

$$= \Delta+\nabla$$

8. P.T $\delta = \Delta E^{-1/2}$ and hence p.t $E = \left(\frac{\Delta}{\delta}\right)^2$

Sol: $\Delta E^{-1/2} f(x) = \Delta f(x-h/2)$

$$= f(x-h/2+h) - f(x-h/2)$$

$$= f(x+h/2) - f(x-h/2)$$

$$= \delta f(x)$$

$\Delta E^{-1/2} = \delta$

$E^{-1/2} = \delta/\Delta$

$E^{1/2} = \Delta/\delta$

$E = \left(\frac{\Delta}{\delta}\right)^2$

9. PT $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

Sol: WKT $E = e^{hD}$ (by thm 6 a)

$e^{hD} = 1+\Delta$

Taking logarithm on both sides we have $hD = \log(1+\Delta)$

Also $\nabla = 1-E^{-1}$

$E^{-1} = 1-\nabla$

$(e^{hD})^{-1} = 1-\nabla$

ie, $e^{-hD} = 1-\nabla$

Taking logarithm on both sides we have,

$-hD = \log(1-\nabla)$

$hD = -\log(1-\nabla)$

$\sinh(hD) = \frac{e^{hD}-e^{-hD}}{2}$ [by defn of hyperbolic fn]

$= \frac{E-E^{-1}}{2}$

$= \frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})$

$= \left(\frac{E^{1/2}+E^{-1/2}}{2}\right)(E^{1/2}-E^{-1/2})$

$= \mu\delta$

$\therefore hD = \sinh^{-1}(\mu\delta)$

10. P.T $\frac{1}{2}\delta^2 +$

Sol: $\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$

11. P $\nabla^r f($

Sol: WKT ∇

we prove

$\nabla f(x) =$

$\therefore$ The

Let us

$\therefore \nabla$

Now, ∇

for

$\therefore \Delta$

10. P.T $\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \Delta$

sol: $\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}\delta\left[\delta + 2\sqrt{1+\frac{\delta^2}{4}}\right]$

$= \frac{1}{2}\delta(\delta + \sqrt{4+\delta^2})$

$= \frac{1}{2}\delta[(E^{1/2}-E^{-1/2}) + \sqrt{4+(E^{1/2}-E^{-1/2})^2}]$

$= \frac{1}{2}\delta[(E^{1/2}-E^{-1/2}) + \sqrt{(E^{1/2}+E^{-1/2})^2}]$

$= \frac{1}{2}\delta[(E^{1/2}-E^{-1/2}) + (E^{1/2}+E^{-1/2})]$

$= \frac{1}{2}\delta[2E^{1/2}]$

$= \delta E^{1/2} \Rightarrow (E^{1/2}-E^{-1/2})E^{1/2}$

$= E-1$

$= \Delta$

11. P $\nabla^r f(x) = \Delta^r f(x-r)$ for any positive integer r.

sol: WKT $\nabla = 1-E^{-1}$ and $\Delta = E-1$

we prove the required result by induction on r when $r=1$

$\nabla f(x) = (1-E^{-1})f(x)$

$= f(x) - f(x-1)$

$= \Delta f(x-1)$

$\therefore$ The result is true for $r=1$

Let us assume that the result is true for $r=k$

$\therefore \nabla^k f(x) = \Delta^k f(x-k)$

Now, $\nabla^{k+1} f(x) = \nabla(\nabla^k f(x))$

$= \nabla(\Delta^k f(x-k))$

$= (1-E^{-1})\Delta^k f(x-k)$

$= \Delta^k f(x-k) - \Delta^k f(x-k-1)$

$= \Delta^k f(x-k) - \Delta^k f(x-k-1)$

$= \Delta^k[f(x-k) - f\, x-(k+1)]$

$= \Delta^k[\Delta f\, x-(k+1)]$

$= \Delta^{k+1} f(x-(k+1))$ Hence the result is true for $r=k+1$

$\therefore \Delta^r f(x) = \Delta^r f(x-r)$ for all natural numbers r.

12. Taking $h=1$ find $(\Delta+\nabla)^2 f(x)$ where $f(x) = x^2+x$

sol

$$(\Delta+\nabla)^2 f(x) = (E-1+1-E^{-1})^2 (x^2+x)$$

$$= (E-E^{-1})^2 (x^2+x)$$

$$= (E^2+E^{-2}-2)(x^2+x)$$

$$= [(x+2)^2+(x+2)] + [(x-2)^2+(x-2)] - 2(x^2+x)$$

$$= 8$$

13. P.T $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$

sol

$$y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1 = y_3 + (E-1)y_2 + (E-1)^2 y_1 + (E-1)^3 y_1$$

$$= y_3 + y_3 - y_2 + (E^2-2E+1)y_1 + (E^3-3E^2+3E-1)y_1$$

$$= 2y_3 - y_2 + y_3 - 2y_2 + y_1 + y_4 - 3y_3 + 3y_2 - y_1$$

$$= y_4$$

14. P.T $\Delta^2 y_2 = \nabla^2 y_4$

sol

$$\Delta^2 y_2 = (E-1)^2 y_2$$

$$= (E^2-2E+1) y_2$$

$$= y_4 - 2y_3 + y_2 \quad — ①$$

Also,

$$\nabla^2 y_4 = (1-E^{-1})^2 y_4$$

$$= (1-2E^{-1}+E^{-2}) y_4$$

$$= y_4 - 2y_3 + y_2 \quad — ②$$

From ① & ② we get

$$\Delta^2 y_2 = \nabla^2 y_4$$

15. given $u_0 = 2$, $u_1 = 11$, $u_2 = 90$, $u_3 = 200$, $u_4 = 100$, $u_5 = 8$

find $\Delta^5 u_0$ i. without constructing the difference table

ii. by constructing the difference table.

sol:-

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i. WKT $\nabla = 1 - E^{-1}$

$\therefore \nabla^5 u_5 = (1-E^{-1})^5 u_5$

$= u_5 - 5E^{-1}(u_5) + 10E^{-2}(u_5) - 10E^{-3}(u_5) + 5E^{-4}(u_5) - E^{-5}(u_5)$

$= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0$

$= 8 - 500 + 2000 - 800 + 55 - 2$

$\nabla^5 u_5 = +61$

ii. we have $\nabla^n u(x) = \Delta^n u(x-n)$

Hence $\nabla^5 u_5 = \Delta^5 u_0$

We construct the forward difference table.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$	$\Delta^5 u$
0	2	9	60	-9	-262	+61
1	11	69	51	-271	449	
2	80	120	-220	228		
3	200	-100	8			
4	100	-92				
5	8					

$\nabla^5 u_5 = \Delta^5 u_0 = +61$

16. If $u_0 = 1$ $u_1 = 5$ $u_2 = 8$ $u_3 = 3$ $u_4 = 7$ $u_5 = 0$ Find $\Delta^5 u_0$

sol: $\Delta^5 u_0 = (E-1)^5 u_0$

$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\ u_0$

$= E^5 u_0 - 5E^4 u_0 + 10E^3 u_0 - 10E^2 u_0 + 5E u_0 - u_0$

$= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0$

$= -35 + 30 - 80 + 25 - 1$

$\Delta^5 u_0 = -61$

17. Estimate the missing term in the following table

x	0	1	2	3	4
$u(x)$	1	3	9	-	81

Explain why the resulting value differs from 33

sol:

Let the missing term in $u(x)$ be a

consider $\Delta^4 u_0 = 0$ (since 4 values are given)

$\therefore (E-1)^4 u_0 = 0$

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$(E^4-4E^3+6E^2-4E+1)u_0=0$

$u_4-4u_3+6u_2-4u_1+u_0=0$

$81-4a+54-12+1=0$

$124-4a=0$

$a=31$

we understand from the data that $u(x)$ satisfies the relation $u(x)=3^x$ while estimating for $u(3)$ the basic assumption is that $u(x)$ is a polynomial of degree 3. But 3^x is not a polynomial but an exponential fn. Hence the assumption is voilated in this case and so we are not getting $u(3)=3^3=27$.

18. Gn an estimate of the population in 1971 from the following table

Year	1941	1951	1961	1971	1981	1991
population in lakhs	363	391	421	?	467	501

sol) Let the population in 1971 be a. let $u_0=363$ $u_1=391$ $u_2=421$ $u_3=a$ $u_4=467$ $u_5=501$

since five values are gn $\Delta^5 u_0=0$

$(E-1)^5u_0=0$

$(E^5-5E^4+10E^3-10E^2+5E-1)u_0=0$

$u_5-5u_4+10u_3-10u_2+5u_1-u_0=0$

$501-2335+10a-4210+1955-363=0$

$10a-4452=0$

$a=445.2$ lakhs

Hence the estimated population in 1971 is 445.2 lakhs

19. Given that $u_0+u_8=80$ $u_1+u_7=10$ $u_2+u_6=5$ $u_3+u_5=10$ Find u_4

sol! since 4 values are given $\Delta^4u(x)=0$ for all $n\geq 4$

In particular $\Delta^8u_0=0$ Hence $(E-1)^8u_0=0$

$u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$

$(u_0 + u_8) - 8(u_1 + u_7) + 28(u_2 + u_6) - 56(u_3 + u_5) + 70u_4 = 0$

$80 - 80 + 140 - 560 + 70u_4 = 0$

$70u_4 = 420$

$u_4 = 6$

20. Gn that $u_1 + u_2 + u_3 = 25$ $u_4 = 29$ $u_5 + u_6 = 113$ find the polynomial $u(x)$ and hence find u_{10}

sol) since three values are gn $u(x)$ is a polynomial of degree 2.

Let $u(x) = ax^2 + bx + c$

we note $u_1 = a + b + c$ $u_2 = 4a + 2b + c$ $u_3 = 9a + 3b + c$

Gn $u_1 + u_2 + u_3 = 25$

$14a + 6b + 3c = 25$ — ①

Now $u_4 = 29 \Rightarrow 16a + 4b + c = 29$ — ②

$u_5 + u_6 = 113 \Rightarrow 61a + 11b + 2c = 113$ — ③

solving ①, ② & ③ we get $a = 2$ $b = -1$ $c = 1$

$\therefore u(x) = 2x^2 - x + 1$

Now. $u_{10} = 100a + 10b + c = 200 - 10 + 1$

$u_{10} = 191$

21. If $u_1 = (12 - x)(4 + x)$ $u_2 = (5 - x)(4 - x)$ $u_3 = (x + 18)(x + 6)$ & $u_4 = 94$ obtain a value of x assuming second difference constant

sol) Since the second order differences are assumed to be constant the third order differences of the fn u will all be zero

$\Delta^3 u_1 = 0$ Hence $(E - 1)^3 u_1 = 0$

$(E^3 - 3E^2 + 3E - 1)u_1 = 0$

$u_4 - 3u_3 + 3u_2 - u_1 = 0$

$94 - 3(x + 18)(x + 6) + 3(5 - x)(4 - x) - (12 - x)(4 + x) = 0$

ie $x^2 - 107x - 218 = 0$

$(x - 109)(x + 2) = 0$

Hence, $x = 109$ $x = -2$

22. P.T $\left(\frac{\Delta^2}{E}\right) x^3 = 6x$

sol

$\left(\frac{\Delta^2}{E}\right) x^3 = \Delta^2 E^{-1} (x^3)$

$= \Delta^2 (x-1)^3$

$= \Delta^2 (x^3 - 3x^2 + 3x - 1)$

let $y = x^3 - 3x^2 + 3x - 1$

we now express y as factorial polynomial.

0	1	-3	3	-1
	↓	0	0	0
1	1	-3	3	$\lfloor -1$
	↓	1	-2	
2	1	-2	$\lfloor 1$	
	↓	2		
	1	$\lfloor 0$		

$y = x^{(3)} + x^{(1)} - 1$

$\Delta y = \Delta (x^{(3)} + x^{(1)} - 1)$

$= 3x^{(2)} + 1$

$\Delta^2 y = 3 \times 2 x^{(1)} + 0 = 6x$

Hence $\left(\frac{\Delta^2}{E}\right) x^3 = 6x$

23. P.T i. $\nabla^2 y_8 = y_8 - 2y_7 + y_6$ ii. $\delta^2 y_5 = y_6 - 2y_5 + y_4$

sol i. $\nabla^2 y_8 = \nabla (\nabla y_8)$

$= \nabla (y_8 - y_7)$

$= \nabla y_8 - \nabla y_7$

$= (y_8 - y_7) - (y_7 - y_6)$

$\nabla^2 y_8 = y_8 - 2y_7 + y_6$

ii.

$\delta^2 y_5 = \delta (\delta y_5)$

$= \delta \left(y_{\frac{11}{2}} - y_{\frac{9}{2}} \right)$

$= \delta y_{\frac{11}{2}} - \delta y_{\frac{9}{2}}$

$= (y_6 - y_5) - (y_5 - y_4)$

$\delta^2 y_5 = y_6 - 2y_5 + y_4$ //

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Q4 Explain the difference b/w $\left(\frac{\Delta^2}{E}\right) f(x)$ and $\frac{\Delta^2 f(x)}{E f(x)}$ &
find the value of these when $f(x) = x^2$

Sol

$$\left(\frac{\Delta^2}{E}\right) f(x) = \left((E-1)^2 E^{-1}\right) f(x)$$

$$= (E - 2 + E^{-1}) f(x)$$

$$= f(x+h) - 2 f(x) + f(x-h) \quad \text{— (1)}$$

Also, $$\frac{\Delta^2 f(x)}{E f(x)} = \frac{(E-1)^2 f(x)}{E f(x)}$$

$$= \frac{(E^2 - 2E + 1) f(x)}{f(x+h)}$$

$$= \frac{f(x+2h) - 2 f(x+h) + f(x)}{f(x+h)} \quad \text{— (2)}$$

From (1) & (2) we note that $\left(\frac{\Delta^2}{E}\right) f(x) \neq \frac{\Delta^2 f(x)}{E f(x)}$

Taking $f(x) = x^2$ in (1) & (2) we have.

$$\left(\frac{\Delta^2}{E}\right) f(x) = (x+h)^2 - 2x^2 + (x-h)^2 = 2h^2$$

$$\frac{\Delta^2 f(x)}{E f(x)} = \frac{(x+2h)^2 - 2(x+h)^2 + x^2}{(x+h)^2} = \frac{2h}{x+h}$$

Summation of series

The concept of finite differences can be applied to find the sum to n terms of given series

Let $S_n = v_1 + v_2 + \cdots + v_n = \sum_{i=1}^{n} v_i$

Let $v_i = \Delta u_i$ so that $u_i = \Delta^{-1} u_i$

$\therefore \; v_i = \Delta u_i = u_{i+1} - u_i$ (Taking $h=1$)

Thus, $v_1 = u_2 - u_1$

$v_2 = u_3 - u_2$

$v_n = u_{n+1} - u_n$

Here $S_n = v_1 + v_2 + \cdots + v_n = u_{n+1} - u_1 = \Delta^{-1} v_{n+1} - \Delta^{-1} v_1$

Theorem 6.11 Montmort's theorem

$$u_0 + u_1 x + u_2 x^2 + \cdots = \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2 u_0}{(1-x)^3} + \cdots$$

Proof :-

$$u_0 + u_1 x + u_2 x^2 + \cdots = u_0 + x u_1 + x^2 u_2 + \cdots$$

$$= u_0 + xEu_0 + x^2E^2u_0 + \cdots$$

$$= [1 + xE + x^2E^2 + \cdots] u_0$$

$$= (1-xE)^{-1} u_0$$

$$= \frac{1}{(1-xE)} u_0$$

$$= \frac{1}{(1-x(1+\Delta))} u_0$$

$$= \frac{1}{1-x-x\Delta} u_0$$

$$= \frac{1}{(1-x)\left(1-\frac{x\Delta}{1-x}\right)} u_0$$

$$= \frac{1}{(1-x)}\left(1-\frac{x\Delta}{1-x}\right)^{-1} u_0$$

$$= \frac{1}{1-x}\left[1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \cdots\right] u_0$$

$$= \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2 u_0}{(1-x)^3} + \cdots$$

problems

1. Sum the series to n terms of $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \cdots$

sol The n^{th} terms of the series is given by

$$V_n = n(n+1)(n+2) = (n+2)(n+1)n$$

$$= (n+2)^{(3)} \text{ (with } h=1)$$

$$\therefore S_n = \sum_{i=1}^{n} V_i = \Delta^{-1}V_{n+1} - \Delta^{-1}V_1$$

$$= \Delta^{-1}(n+3)^{3} - \Delta^{-1}V_1$$

$$S_n = \frac{(n+3)^{(4)}}{4} - 0$$

$$= \frac{1}{4}(n+3)(n+2)(n+1)n$$

$$S_n = \frac{1}{4}n(n+1)(n+2)(n+3).$$

2. Sum to n terms of the series $1\cdot3\cdot5 + 2\cdot4\cdot6 + \cdots$

Sol: The n^{th} term of the series is

$v_n = n(n+2)(n+4)$

$= n^3 + 6n^2 + 8n$

we express n^3+6n^2+8n as factorial polynomial with $h=1$.

0	1	6	8	0
	↓	0	0	0
1	1	6	8	$\lfloor 0$
	↓	1	7	
2	1	7	$\lfloor 15$	
	↓	2		
	1	$\lfloor 9$		

$\therefore \; v_n = n^{(3)} + 9n^{(2)} + 15n^{(1)}$

Now, $S_n = \sum_{i=1}^{n} v_i = \Delta^{-1} v_{n+1} - \Delta^{-1} v_1$

$= \Delta^{-1}\left[(n+1)^{(3)} + 9(n+1)^{(2)} + 15(n+1)^{(1)}\right] = 0$

$= \dfrac{(n+1)^{(4)}}{4} + \dfrac{9(n+1)^{(3)}}{3} + \dfrac{15(n+1)^{(2)}}{2}$

$= \dfrac{(n+1)n(n-1)(n-2)}{4} + 3(n+1)n(n-1) + \dfrac{15}{2}(n+1)n$

$= \dfrac{n(n+1)}{4}\left[n^2 - 3n + 2 + 12n - 12 + 30\right]$

$= \dfrac{n(n+1)(n^2+9n+20)}{4} \rightarrow \dfrac{n(n+1)(n+4)(n+5)}{4}$

3. Use the method of finite differences to prove $1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Sol:

The n^{th} term of the series is given by,

$v_n = n^2 = n(n-1) + n = n^{(2)} + n^{(1)}$

$\therefore \; S_n = \sum_{i=1}^{n} v_i = \Delta^{-1} v_{n+1} - \Delta^{-1} v_1$

$= \Delta^{-1}\left[(n+1)^{(2)} + (n+1)^{(1)}\right] - 0$

$= \dfrac{(n+1)^{(3)}}{3} - \dfrac{(n+1)^{(2)}}{2}$

$+ \frac{(n+1)n(n-1)}{3} + \frac{n(n-1)n}{2}$

$= n(n+1)\left(\frac{2n-2+3}{6}\right)$

$= \frac{n(n+1)(2n+1)}{6}$

4. Sum to n term of the series $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots$

sol :-

The n^{th} term of the series is given by,

$v_n = \frac{1}{n(n+1)(n+2)} = (n-1)^{(-3)}$

$S_n = \sum_{i=1}^{n} v_i = \Delta^{-1}_{v_{n+1}} - \Delta^{-1}_{v_1}$

Now,

$\Delta^{-1}_{v_{n+1}} = \Delta^{-1}_{n^{(-3)}} = \frac{-n^{(-2)}}{2}$

$= \frac{-1}{2(n+1)(n+2)}$

$\Delta^{-1}_{v_1} = \frac{-1}{4}$

$S_n = \frac{-1}{2}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{2}\right)$

$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$

$= \frac{n(n+3)}{4(n+1)(n+2)}$

5. Find the sum to infinity of the series $1\cdot2 + 2\cdot3x + 3\cdot4x^2 + \cdots$

sol

comparing the given series with $u_0 + u_1x + u_2x^2 + \cdots$ we get $u_0 = 2$ $u_1 = 6$ $u_2 = 12$ $u_3 = 20$

The difference table for these values is given below.

u	Δu	$\Delta^2 u$	$\Delta^3 u$
$u_0 = 2$			
	4		
$u_1 = 6$		2	
	6		0
$u_2 = 12$		2	
	8		
$u_3 = 20$			

By ment
$1\cdot2 + 2\cdot3x + 3$

By montmort's theorem.

$$1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \cdots = \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2 u_0}{(1-x)^3}$$

$$= \frac{2}{1-x} + \frac{4x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}$$

$$= \frac{2}{(1-x)^3}$$