Then, $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f} = \frac{ad+bc}{bd}+\frac{e}{f}$

$$= \frac{adf+bcf+ebd}{bdf}$$

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$$= \frac{a}{b}+\left[\frac{c}{d}+\frac{e}{f}\right]$$

* Existence of identity:

Let $\frac{0}{b} \in F$ where $b \neq 0$ is the identity element of F.

If $\frac{c}{d} \in F$, then $\frac{0}{b}+\frac{c}{d} = \frac{0+bc}{bd} = \frac{bc}{bd} = \frac{c}{d}$

$$\frac{0}{b}+\frac{c}{d} = \frac{c}{d}$$

* Existence of inverse:

Let $a/b \in F$

Then, $\exists$ an element $-a/b \in F$ $\ni$:

$$\left(\frac{a}{b}\right)+\left(-\frac{a}{b}\right) = \frac{ab-ab}{b^2} = \frac{0}{b^2} = 0 = \frac{0}{b}.$$

$\therefore -\frac{a}{b}$ is the inverse of a/b.

* Commutative:

Let, $\frac{a}{b}, \frac{c}{d} \in F$.

Then, $\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd} = \frac{cb+da}{db} = \frac{c}{d}+\frac{a}{b}$.

* Associative property:

Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \left(\frac{ac}{bd}\right)\cdot\frac{e}{f}$$

$$= \frac{(ac)\cdot e}{(bd)\cdot f} = \frac{a\cdot(ce)}{b\cdot(df)}$$

$$= \frac{a}{b}\cdot\left[\frac{c}{d}\cdot\frac{e}{f}\right].$$

element of finite order ... order of 1 is the characteristic of R. If it is of infinite order the characteristic (or) ring is zero.

proof:

Suppose, the order of one is n.

Then, n is the least +ve integer $\ni$: $n.1=0$.

ie) $1+1+1+ \dots 1$ (n times) $=0$.

Now, let $a \in R$.

Then, $na = a+a+a+ \dots a$ (n times)

$= 1.a+1.a+ \dots +1.a$ (n times)

$= a(1+1+1+ \dots +1)$

$na = a0 = 0$.

Thus, $na=0 \ \forall \ a \in R$.

Hence, the characteristic of a ring is n. If 1 is of infinite order.

Then, there is no +ve integer n, $\ni$: $n.1=0$.

Hence, the characteristic of ring is zero.

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Unit - 5

Subring:

Let R be a ring and S is the subset of R if $(S,+,\cdot)$ is itself a ring. Then S is called a subring of R.

Eg:

i) $(Z,+,\cdot)$ and $(2Z,+,\cdot)$ is a ring.

$\because 2Z \subseteq Z$, $(2Z,+,\cdot)$ is a subring of $(Z,+,\cdot)$

* f is homomorphism:

$$f(a+b) = \frac{a+b}{1}$$

$$= \frac{a}{1} + \frac{b}{1} = f(a) + f(b)$$

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$$f(ab) = \frac{ab}{1} = \frac{a}{1} \cdot \frac{b}{1} = f(a) \cdot f(b)$$

$\therefore$ f is homomorphism & f is an isomorphism.

$\therefore D \cong F$.

ie) D is embedded in F.

Hence the proof.

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ii) $(Q,+,\cdot)$ and $(Z,+,\cdot)$ are Ring.

$\because Z \subseteq Q$ $(Z,+,\cdot)$ is a Subring of $(Q,+,\cdot)$

iii) $(C,+,\cdot)$ and $(R,+,\cdot)$ are rings.

$\because R \subseteq C$, $(R,+,\cdot)$ is a subring of $(C,+,\cdot)$

Theorem:

A necessary and sufficient condition that a subset S of a Ring R is a Subring is that $a, b \in S$

$\Rightarrow a-b,\ ab \in S$.

(or)

A non-empty subset of ring R is a subring iff $a-b,\ ab \in S$.

proof:

Let S be a Subring of R.

$\therefore$ S is itself is a ring under $(+,\cdot)$

$\therefore$ S is a ring $a, b \in S \Rightarrow ab \in S$ and $b \in S \Rightarrow -b \in S$.

[$\because$ S is a group under '+'].

$\therefore a, -b \in S \Rightarrow a-b \in S$ [$\because$ by closure with respect to '+'].

$\therefore a, b \in S \Rightarrow a-b \in S,\ ab \in S$.

conversely,

Let $a, b \in S \Rightarrow a-b,\ ab \in S$.

$\therefore$ To P.T S is a subring of R.

G.T $a, b \in S \Rightarrow a-b,\ ab \in S$

$\therefore a, a \in S \Rightarrow a-a=0 \in S$.

$\therefore$ S is non-empty.

$0, b \in S \Rightarrow 0-b \in S \Rightarrow -b \in S$

$a, -b \in S \Rightarrow a-(-b) \in S$

$\Rightarrow a+b \in S$

$\therefore a, b \in S \Rightarrow a+b \in S$

Ring R is a

ubrings of R.

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of integer.

The construction of the quotient field in an integral domain is motivated at every step. By the, well known behaviours of the field of rational numbers.

W.K.T, every element in $\mathbb{Q}$ can be expressed as the quotient p/q where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Further, two fractions $2/3$ & $4/6$ represent the same rational numbers iff $ab = bc$

Also, $(a/b) + (c/d) = \frac{ad+bc}{bd}$

$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{ac}{bd}$.

Thus, the element of $\mathbb{Z}$ can be thought as fraction of the form $a/1$.

Theorem:

Any integral domain D can be embedded in a field F. And every element of F can be expressed as the quotient of two elements of D.

proof:

The construction of the field of the quotient F of an integral domain 'D' is carried out in the following steps,

i) Specify the elements of F.

ii) Define addition & multiplication in F.

iii) To prove F is a field under the operation.

iv) D can be embedded in F

* Step 1:

Let D be an integral domain.

Suppose, $S = \{a, b \,/\, a, b \in D, b \neq 0\}$.

consider, the equivalence class containing (a, b) and is defined by a/b.

(6)

Let, $F = \{ a/b \ / \ (a, b) \in S \}$.

* Step 2:

Let, $\frac{a}{b}, \frac{c}{d} \in F$

Define, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

& $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

clearly, $\frac{ad + bc}{bd}$ & $\frac{ac}{bd} \in F$.

we show that the operations '+' & '.' are well defined

Let $(a_1, b_1) \in \frac{a}{b}$ and $(c_1, d_1) \in \frac{c}{d}$.

$\because (a_1, b_1) \sim (a, b)$ and $(c_1, d_1) \sim (c, d)$

$a_1 b = b_1 a$ & $c_1 d = d_1 c$ — (1)

$\therefore a_1 b d d_1 = b_1 a d d_1$ and $c_1 d b b_1 = d_1 c b b_1$

$a_1 b d d_1 + c_1 d b b_1 = d_1 c b b_1 + b_1 a d d_1$

$[a_1 d_1 + b_1 c_1] bd = [ad + bc] b_1 d_1$

$$\frac{ad + bc}{bd} = \frac{a_1 d_1 + b_1 c_1}{b_1 d_1}$$

$$\therefore \frac{a}{b} + \frac{c}{d} = \frac{a_1}{b_1} + \frac{c_1}{d_1}$$

$\therefore$ '+' is well defined.

Also, From (1) $a_1 b c_1 d = b_1 a d_1 c$.

$\therefore (ac, bd) \sim (a_1 c_1, b_1 d_1)$

$$\therefore \frac{a}{b} \cdot \frac{c}{d} = \frac{a_1}{b_1} \cdot \frac{c_1}{d_1}.$$

'.' is well defined.

* Step 3:

To prove F is a field.

* Associative property

Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$

S is closed under addition.

The other two laws, also satisfied.

$\because$ They are true in R.

Hence, S is a subring of R.

Theorem:

The intersection of two subrings of a Ring R is a subring of R.

Proof:

Let R be Ring and A, B be any two subrings of R.

To p.T

$A \cap B$ is a subring of R.

Let, $a, b \in A \cap B$.

Then $a, b \in A$ and $a, b \in B$.

$\therefore$ A is a ring $a-b, ab \in A$ [by prev. thm]

$\therefore$ B is a ring $a-b, ab \in B$.

$\therefore a-b, ab \in A \cap B$

ie) $a, b \in A \cap B \Rightarrow a-b, ab \in A \cap B$.

$\therefore A \cap B$ is a subring of R.

Field of quotients of an integral domain:

The field in which every element on F can be expressed as a quotient of two elements on an integral domain is called the field of quotient of an integral domain.

Note:

Here, we construct a field F which contains the given integral domain 'D'.

For eg:

Z is contained in the field 'Q' and all the elements of Q can be expressed as Quotient's of integer.

The construction ...
domain is moti...
behaviours of ...
w.r.t, every ...
quotient p/q ...
Further ...
rational nu...
Also, (a, ...
$(\frac{a}{b}).($ ...
Thus, ...
of the form ...

Theorem:
Any i...
And every ...
of two el...

Proof:
The co...
an integral...
steps,
i) Speci...
ii) Define...
iii) To p...
iv) D c...

* Step 1 ...
Supp...

* Existence of multiplicative identity:

$\frac{1}{1}$ is the identity of F.

* Existence of multiplicative inverse:

If $\frac{a}{b}$ is a non-zero element of F then, $a \neq 0$.

$\frac{b}{a}$ is the multiplicative inverse of a/b.

element of F.

commutative with respect to '.'.

Let, $\frac{a}{b}, \frac{c}{d} \in E$

Then, $\frac{a}{c} \cdot \frac{b}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b}$

* Distributive law:

$$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b}\left(\frac{cf+ed}{df}\right)$$

$$= \frac{a(cf) + a(ed)}{b(df)}$$

$$= \frac{acf}{bdf} + \frac{aed}{bdf} = \frac{ac}{bd} + \frac{ae}{bf}.$$

$$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}.$$

$\therefore$ F is a field.

* Step 4:

To prove $D \cong F$.

$f : D \to F$ is defined by $f(a) = \frac{a}{1} \in D$.

* f is 1-1:

$f(a) = f(b) \Rightarrow \frac{a}{1} = \frac{b}{1} \Rightarrow a = b$.

$\therefore f$ is 1-1.

* f is onto:

Let $\frac{a}{1} = a \in F$

Then, $\exists$ an element, $a \in D \ni f(a) = a$.

$\therefore f$ is onto.

$\therefore f$ is bijection.

Define two elements (a,b) + $(c,d) \in S$ are defined to be equivalent iff ~~[illegible]~~ $ad = bc$ (i.e.)

$(a,b) \sim (c,d)$ iff $ad = bc$.

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claim,

The symbol equivalence is an equivalence relation in S.

* Reflexive:

Let $a, b \in S$

$\because$ D is commutative ring.

$\therefore ab = ba \;\forall\; a, b \in D$.

$\therefore (a,b) \sim (a,b)$

* Symmetric:

Let $(a,b) \sim (c,d)$

Then, $ad = bc$ (or) $bc = ad$

$\Rightarrow cd = ba$ [by commutative]

$\Rightarrow (c,d) \sim (a,b)$

$\therefore$ '$\sim$' is Symmetric.

* Transitive:

Let $(a,b) \sim (c,d) \sim (e,f)$

then,

$ad = bc$ and $cf = de$.

where, $a, b, c, d, e, f \in D$.

Then, $b \neq 0, d \neq 0, f \neq 0$.

$(ad)(cf) = (bc)(de) \Rightarrow a(dc)f = b(cd)e$ [by associative]

$(dc)(af) = (cd)(be)$

$(dc)(af) = (dc)(be)$

$af = be$ (by left cancellation law)

$\Rightarrow (a,b) \sim (e,f)$

$\Rightarrow$ '$\sim$' is transitive

$\therefore$ '$\sim$' is an equivalence relation in S.

[$\because$ A is dense in M iff A ~~a~~ intersects with every open ball].

Thus, A is dense in M.

Hence A is a countable dense subset of M.

$\therefore$ M is Separable. Hence proved.

3) P.T. any bdd Sequence in R has a Convergent sub seq.

Sol:

Let (x_n) be a bdd seq in R then $\exists$ a closed interval $[a,b]$ $\ni$: $x_n \in [a,b]$.

Thus (x_n) is a seq in the compact m.s $[a,b]$. [$\because$ By Heine Borel thm]

Since $[a,b]$ is compact it $\times$[has a convergent sub seq]$\times$ is sequentially compact.

By the defn of sequentially compact (x_n) has a convergent ~~se~~ subseq. Hence proved.

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