Thm - 3

Stmt: Let M be a me.s Let A be a connected subset of M. If B is subset of M $\ni$: $A \subseteq B \subseteq \bar{A}$ then B is connected.

Proof: Gn A is connected & B is subset of M $\ni$: $A \subseteq B \subseteq \bar{A}$

T.P. TB is connected.

suppose B is not connected

Then $B = B_1 \cup B_2$ where $B_1 \neq \phi$, $B_2 \neq \phi$ &

$B_1 \cap B_2 = \phi$, B_1 & B_2 are open sets since

B_1 & B_2 are open sets $\exists$ G_1 & G_2 $\ni$:

$B_1 = G_1 \cap B$, $B_2 = G_2 \cap B$.

$$B = B_1 \cup B_2$$

$$B = (G_1 \cap B) \cup (G_2 \cap B)$$

$$= (G_1 \cup G_2) \cap B.$$

$$B \subseteq G_1 \cup G_2$$

W.K.T $A \subseteq B$

$$\therefore A \subseteq G_1 \cup G_2$$

Then $A = (G_1 \cup G_2) \cap A$

$$= (G_1 \cap A) \cup (G_2 \cap A)$$

$\therefore$ Now $(G_1 \cap A)$ & $(G_2 \cap A)$ are open in A.

Also, $(G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A = (G_1 \cap G_2) \cap B$

$(G_1 \cap B) \cap (G_2 \cap B) = B_1 \cap B_2 = \phi$ $(\because A \subseteq B)$

$(G_1 \cap A) \cap (G_2 \cap A) = \phi$

Let us assume either $G_1 \cap A = \phi$ (or) $G_2 \cap A = \phi$

we assume $G_1 \cap A = \phi$

Then $A \subseteq G_1^c$

Since G_1 is open G_1^c is closed.

Also G_1^c is a closed set containing

A we know that $\bar{A}$ is the smallest closed set containing A.

$$\therefore \bar{A} \subseteq G_1^c$$

$$G_1 \cap \bar{A} \subseteq G_1 \cap G_1^c$$

$G_1 \cap \bar{A} \subseteq \phi \longrightarrow ①$

W.K.T $\phi \subseteq G_1 \cap \bar{A} \rightarrow ②$

from ①, ②

$G_1 \cap \bar{A} = \phi$

since $B \subseteq \bar{A}$ we have $G_1 \cap B = \phi$

$B_1 = \phi$ which is a $\Rightarrow\Leftarrow$ since $B_1 \neq \phi$

$\therefore$ B is connected.

Hence proved.

Thm - 4

Stmt:

Let A & B are connected subset of a m.s M & iff $A \cap B \neq \phi$ then P.T $A \cup B$ is connected.

Proof: Gn A & B are connected & $A \cap B \neq \phi$

T P T $A \cup B$ is connected.

Suppose $A \cup B$ is not connected then $\exists$ continuous onto fun

$f : A \cup B \rightarrow \{0, 1\}$. since $A \cap B \neq \phi$ we have $x_0 \in A \cap B$.

Then $x_0 \in A \cup B$

$\therefore f(x_0) = 0$ (or) 1

Case(i) Let $f(x_0) = 0$

Consider the restricted map

$f/_A : A \rightarrow \{0, 1\}$

since f is continuous.

Gn A is connected.

$\therefore f/_A$ is not onto

$\therefore f/_A(x) = 0 \ \forall \ x \in A$ (or) $f/_A(x) = 1 \ \forall \ x \in A$

Let $x_0 \in A$ & $f(x_0) = 0$

$\therefore f/A(x) = 0 \ \forall \ x \in A$.

$\therefore f(x) = 0 \ \forall \ x \in A$ —— ①

$f/_B : B \to \{0, 1\}$ is continuous.

Proceeding like above,

$f(x) = 0 \ \forall \ x \in B$ —— ②

from ① & ②

$\Rightarrow f(x) = 0 \ \forall \ x \in A \cup B$.

$\therefore f$ is not onto, which is a $\Rightarrow \Leftarrow$

$\therefore A \cup B$ is connected.

case (ii)

Let $f(x_0) = 1$

by similar argument as above we can get,

$f(x) = 1 \ \forall \ x \in A \cup B$.

$\therefore$ It is not onto which is a $\Rightarrow \Leftarrow$

$\therefore A \cup B$ is connected

Hence proved.

Thm 15

Stm

P.T A subspace of R is connected iff it is an interval.

Proof:

Let A be a connected subspace of R.

TPT A is an interval

Suppose A is not an interval. then $\exists$ $a, b, c \in R \ni a < b < c$ & $a, c \in A$ and $b \notin A$