Note:

Any polynomial is an entire function

(2)

12.1 Problem:

1\. An analytic function in a region with constant modulus is constant.

Sol:

Let $f(z) = u + iv$, be analytic in a domain D.

Given that,

$|f(z)|$ is constant.

$\sqrt{u^2+v^2} = c^2$ (say)

$\Rightarrow u^2+v^2 = c$ where c is a constant.

T.P.T. f is a constant.

Now, $u^2+v^2 = c$

Diff partially w. to. x & y. We get,

$2uu_x + 2vv_x = 0$

$2uu_y + 2vv_y = 0$

(i.e) $uu_x + vv_x = 0 \longrightarrow (1)$

$uu_y + vv_y = 0 \longrightarrow (2)$

Using C-R equation in (1) & (2)

$u_x = v_y$

$u_y = -v_x$

$uu_x - vu_y = 0$

$uu_y + vu_x = 0$.

$u_x + v_x = (x-y)(2x+4y) + (x^2+4xy+y^2)(1)$

$= (x-y)(2x+4y) + (x^2+4xy+y^2)$ $\to$ ①

$u_y + v_y = (x-y)(4x+2y) + (x^2+4xy+y^2)(-1)$

$= (x-y)(4x+2y) - (x^2+4xy+y^2)$ $\to$ ②

Since $f(z)$ is analytic, $u+v$ satisfy the C-R equation.

$u_x = v_y$ & $u_y = -v_x$

Using these equation in ① we get

$u_x - u_y = (x-y)(2x+4y) + (x^2+4xy+y^2)$

$u_x(z,0) - u_y(z,0) = (z-0)(2z+0) + (z^2+4z(0)+0)$

$= z(2z) + (z^2)$

$= 2z^2 + z^2$

$= 3z^2 \to$ ③

Using C-R equation in ② we get

$u_y + u_x = (x-y)(4x+2y) - (x^2+4xy+y^2)$

$u_y(z,0) + u_x(z,0) = (z-0)(4z+0) - (z^2+4z(0)+0)$

$= z(4z) - (z^2)$

$= 4z^2 - z^2$

$= 3z^2 \to$ ④

$= u(v_{xx} + v_{yy}) + v(u_{xx} + u_{yy}) + 2v_y u_x - 2u_x v_y$.

$= u(v_{xx} + v_{yy}) + v(u_{xx} + u_{yy})$

$= u(0) + v(0)$ (3)

$= 0$.

$\phi_{xx} + \phi_{yy} = 0$

Hence it is proved.

$\therefore$ The product uv is also a harmonic function.

8.1.13.

11. If $f(z)$ is analytic prove that,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\,|f'(z)|^2$$

~~proof~~

Let $f(z) = u + iv$

$\therefore |f(z)| = \sqrt{(u^2+v^2)}$

$|f(z)|^2 = u^2 + v^2$

Let $\phi = u^2 + v^2 = |f(z)|^2$

$$\frac{\partial\phi}{\partial x} = 2uu_x + 2vv_x$$

$$\frac{\partial^2\phi}{\partial x^2} = 2u_x u_x + 2uu_{xx} + 2vv_{xx} + 2v_x v_x$$

9 sol:

$\therefore u - v = e^{x}(\cos y - \sin y)$ (5)

10. Show that if u & v are conjugate harmonic functions the product uv is also a harmonic function.

sol: Given that, u & v are conjugate harmonic functions

$\therefore u_{xx} + u_{yy} = 0 \longrightarrow ①$

& $v_{xx} + v_{yy} = 0 \longrightarrow ②$

Let $\phi = uv$

T.P.T $\phi_{xx} + \phi_{yy} = 0$

$\phi_x = uv_x + u_x v$

$\phi_{xx} = uv_{xx} + v_x u_x + u_x v_x + v u_{xx}$

$\therefore \phi_{xx} = uv_{xx} + 2u_x v_x + v u_{xx} \longrightarrow ③$.

$\phi_y = uv_y + u_y v$

$\phi_{yy} = uv_{yy} + v_y u_y + u_y v_y + v u_{yy}$.

$\therefore \phi_{yy} = uv_{yy} + 2v_y u_y + v u_{yy} \longrightarrow ④$.

③ + ④

$$\phi_{xx} + \phi_{yy} = (uv_{xx} + 2u_x v_x + v u_{xx}) + (uv_{yy} + 2v_y u_y + v u_{yy})$$

$$= u(v_{xx} + v_{yy}) + v(u_{xx} + u_{yy}) + 2u_x v_x + 2v_y u_y$$

Let v be the harmonic conjugate of u.

$\therefore f(z) = u+iv$ is analytic

Let $\phi_1(x,y) = u_x$

$$= \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$$

$$\phi_1(z,0) = [\cos z - 2\sin z(0) + 2z + 4(0)]$$

$$= \cos z + 2z$$

$\phi_2(x,y) = u_y$

$$= \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$$

$$\phi_2(z,0) = \sin z(0) + 2\cos z(1) - (0) + 4z$$

$$= 2\cos z + 4z$$

$$\therefore f(z) = \int \phi_1(z,0) - i\phi_2(z,0)\, dz + c$$

$$= \int (\cos z + 2z) - i(2\cos z + 4z)\, dz + c$$

$$= \left[\int (\cos z + 2z)dz - i\int (2\cos z + 4z)\, dz\right] + c$$

$$= \left[\sin z + \frac{2z^2}{2}\right] - \left[i\left(2\sin z + \frac{4z^2}{2}\right)\right] + c$$

$$f(z) = \sin z + z^2 - 2i\sin z - 2iz^2 + c$$

$$\therefore f(z) = \sin z + z^2 - 2i\sin z - 2iz^2 + c$$

$= u + iv$

$\therefore v = 2y - 3x^2y + y^3$ (3)

Show that, $u(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic find an analytic function $f(z)$ interms of z.

Sol:

Given that,

$$u(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y + 2(-\sin x)\sinh y + 2x + 4y.$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$$

$$\frac{\partial^2 u}{\partial x^2} = (-\sin x)\cosh y - 2\cos x \sinh y + 2.$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y + 2\cos x \sinh y - 2$$

$\Rightarrow (-\sin x \cosh y - 2\cos x \sinh y + 2)$

$+ (\sin x \cosh y + 2\cos x \sinh y - 2) \Rightarrow 0.$

$\Rightarrow 0.$ $\therefore u$ is harmonic

$= 2(0) + 2(0) + 2u_x^2 + 2u_y^2 + 2v_x^2 + 2v_y^2$

$= 2[u_x^2 + u_y^2 + v_x^2 + v_y^2]$ (C-R eqn.)

$= 4[u_x^2 + u_y^2]$ $[\because u_x = v_y,\ u_y = -v_x]$

$= 4|f'(z)|^2$. (33)

Unit-2

Elementary Transformation:-

Definition:

A function $f: C \to C$ can be ...ered as a transformation from one

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \quad (\text{by } ①)$$

$$= 0 \qquad \left[\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right]$$

Now,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{-\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = 0 \qquad \left[\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right]$$

Definition:

Let $f = u + iv$ be an analytic function in a region D. Then v is said to be a conjugate harmonic function u.

Note:

Let f be an analytic function

$$\boxed{f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz + c}$$

where, $\phi_1(x,y) = v_y$ $f'(z) = u_x + iv_y$

& $\phi_2(x,y) = v_x$ $= v_y + iv_x$

$= \phi_1 + i\phi_2$

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Problem:

1. Prove that, $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.

sol:

Given that, $u = 2x - x^3 + 3xy^2$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \qquad \frac{\partial u}{\partial y} = 6xy$$

$$= \frac{\partial}{\partial x}(2 - 3x^2 + 3y^2) + \frac{\partial}{\partial y}(6xy)$$

$$= -6x + 6x$$

$$= 0.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \therefore u \text{ is harmonic}$$

$$\frac{f(z)-f(0)}{z} = \frac{(my^2)y^2}{(m^2y^4)+y^4} = \frac{my^4}{y^4(m^2+1)}$$

$$= \frac{m}{m^2+1} \quad (3)$$

The value of limit depends on m

$\therefore \lim_{z \to 0} \frac{f(z)-f(0)}{z}$ does not exist

$\therefore f$ is not differentiable at $z=0$.

19.12.19

Theorem 4:

If $f(z)$ is a differentiable function, the C-R equations can be put in the form

$$\frac{\partial f}{\partial \bar{z}} = 0$$

Proof:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} \longrightarrow (1)$$

Now,

$$z = x+iy \longrightarrow (2)$$

$$\therefore \bar{z} = x-iy \longrightarrow (3)$$

(2)+(3),

$$2x = z+\bar{z}$$

$$\therefore x = \frac{z+\bar{z}}{2}$$

(2)-(3)

$$z-\bar{z} = 2iy$$

$$y = \frac{z-\bar{z}}{2i}$$

$u_x(z,0) - u_y(z,0) = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$ (30)

$= \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z \rightarrow (3)$

Using C-R equation in (2) we get

$u_y + u_x = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$ [$\sin 0 = 0$, $\cos 0 = 1$]

$u_y(z,0) + u_x(z,0) = 0 \longrightarrow (4)$

Adding (3) & (4)

$2u_x(z,0) = -\operatorname{cosec}^2 z$

$u_x(z,0) = \frac{-1}{2}\operatorname{cosec}^2 z \longrightarrow (5)$

(4) - (3)

$2u_y(z,0) = \operatorname{cosec}^2 z$

$u_y(z,0) = \frac{1}{2}\operatorname{cosec}^2 z \longrightarrow (6)$

Now, $f(z) = u(z,0) + i v(z,0)$

$f'(z) = u_x(z,0) + i v_x(z,0)$

$= u_x(z,0) - i u_y(z,0)$

$= \frac{-1}{2}\operatorname{cosec}^2 z - \frac{i}{2}\operatorname{cosec}^2 z$

$= \frac{-1}{2}(1+i)\operatorname{cosec}^2 z$

Integrating

$f(z) = \frac{-1}{2}(1+i)\int \operatorname{cosec}^2 z \, dz$

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$= \frac{+1}{2}(1+i)(-\cot z) + c$

$\therefore f(z) = \left(\frac{1+i}{2}\right)\cot z + c.$

(57)

$f(z) = \int [\phi_1(z,0) + i\,\phi_2(z,0)]\,dz + c$

where $\phi_1 = u_x = \frac{-1}{2}\operatorname{cosec}^2 z.$

$\phi_2 = u_y = \frac{1}{2}\operatorname{cosec}^2 z.$

$= \int [\frac{-1}{2}\operatorname{cosec}^2 z - i\,\frac{1}{2}\operatorname{cosec}^2 z]\,dz$

$= \frac{-1}{2}(1+i)\int \operatorname{cosec}^2 z\,dz$

$= \frac{-1}{2}(1+i)(-\cot z) + c$

$\therefore f(z) = \left(\frac{1+i}{2}\right)\cot z + c$ //

Note:

i) $u+v = (x-y)(x^2+4xy+y^2)$

7) $f(z) = u+iv$ find the analytic function $f(z)$ in terms of z.

$u+iv = (x-y)(x^2+4xy+y^2)$

sol:

$u+v = (x-y)(x^2+4xy+y^2)$

Diff. parti. w.r.to. x & y

$$\frac{-y}{x^2+y^2} = \frac{-y}{x^2+y^2} = \phi'(x)$$

$\Rightarrow \phi'(x) = 0$

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$\Rightarrow \phi(x) = C_1$

② $\Rightarrow$ $v = \tan^{-1}\frac{y}{x} + c$

$\therefore f(z) = u + iv$

$= \log\sqrt{x^2+y^2} + i\left(\tan^{-1}\frac{y}{x}\right) + c$

4. Given $v = x^4 - 6x^2y^2 + y^4$ find $f(z)$ such that $f(z)$ is analytic

sol:

Given that $v = x^4 - 6x^2y^2 + y^4$

$\frac{\partial v}{\partial x} = 4x^3 - 12xy^2$ $\quad \frac{\partial v}{\partial y} = -12x^2y + 4y^3$

$\frac{\partial^2 v}{\partial x^2} = 12x^2 - 12y^2$ $\quad \frac{\partial^2 v}{\partial y^2} = -12x^2 + 12y^2$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\Rightarrow 12x^2 - 12y^2 - 12x^2 + 12y^2$

$\Rightarrow 0$

$\therefore v$ is harmonic function

Let u be the harmonic conjugate of v

7.1.18

H.W 5

6.

$= 2z(1+i)$

$f'(z) = (1+i)\,2z$

Harmonic function Definition:

Let $u(x,y)$ be a function of two variables x & y defined in a region D. $u(x,y)$ is said to be a harmonic function if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and this equation is called Laplace equation.

Theorem 5:

The real and imaginary parts of an the analytic function (or) harmonic function.

Proof:

Let $f(z) = u + iv$ be an analytic function. Therefore u & v satisfied the C-R equation. Hence $u_x = v_y$ & $u_y = -v_x$ $\longrightarrow (1)$

T.P.T u & v are harmonic function.

(ie) T.P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

$\therefore f(z) = u + iv$ is analytic

Let $\phi_1(x,y) = v_y$ (29)

$= -12x^2y + 4y^3$

and $\phi_2(x,y) = v_x$

$= 4x^3 - 12xy^2$

$\therefore \phi_1(z,0) = 0$

& $\phi_2(z,0) = 4z^3$

$\therefore f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz + c$

$= \int [0 + i\,4z^3]\, dz + c$

$= 4i \int z^3\, dz + c$

$= 4i \dfrac{z^4}{4} + c$

$= iz^4 + c$

$f(z) = iz^4 + c.$

7.1.18

H.W 5. If $f(z) = u + iv$ is an analytic function and $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ find $f(z)$.

6. find the analytic function $f(z) = u + iv$ if $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

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3. Show that $u = \log\sqrt{x^2+y^2}$ is harmonic and find its conjugates and also find the corresponding analytic function $f(z)$

Sol:

Given that $u = \log(\sqrt{x^2+y^2})$

$= \log(x^2+y^2)^{1/2}$

$= \frac{1}{2}\log(x^2+y^2)$

$\frac{\partial u}{\partial x} = \frac{1}{2} \times \frac{1}{x^2+y^2} \times 2x$ $\quad \frac{\partial u}{\partial y} = \frac{1}{2} \frac{1}{x^2+y^2} \times 2y$

$= \frac{x}{x^2+y^2}$ $\quad = \frac{y}{x^2+y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in D. Then v is a harmonic conjugate of u iff u is a harmonic conjugate of $-v$

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Milne-Thompson Method:

Let $u(x,y)$ be a given harmonic function. Let $f(z) = u + iv$ be an analytic function. Then, $\boxed{f'(z) = u_x + iv_x}$

$f'(z) = u_x + iv_x = u_x - iv_y$. Let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$.

W.K.T $x = \frac{z+\bar{z}}{2}$ & $y = \frac{z-\bar{z}}{2}$

hence,

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y)$$

$$= \phi_1\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right] - i\phi_2\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right]$$

put, $z = \bar{z}$

$\therefore f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

$$\Rightarrow \boxed{f'(z) = \int [\phi_1(z,0) - i\phi_2[z,0]]\,dz + c}$$

Note:

The analytic function $f(z)$ with a given harmonic function $v(x,y)$ as imaginary part. Then $f(z) =$

Sol: $u+iv = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Differentiate partially w.r.to x & y

$$\therefore u_x + iv_x = \frac{(\cosh 2y - \cos 2x)\cos 2x \cdot 2 - \sin 2x(\sin 2x \cdot 2)}{(\cosh 2y - \cos 2x)^2}$$

$$u_x + iv_x = \frac{2\cosh 2y \cdot \cos 2x - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} \to ①$$

$$u_y + iv_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y + iv_y = \frac{-2\sinh 2y \sin 2x}{(\cosh 2y - \cos 2x)^2} \to ②$$

Since $f(z)$ is analytic, $u+v$ satisfy the C-R equation.

$$u_x = v_y \ \& \ u_y = -v_x$$

Using these equation in ① we get,

$$u_x - iu_y = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$u_x(z,0) - iu_y(z,0) = \frac{2(1-\cos 2z)\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

(32)

$$\frac{\partial \phi}{\partial y} = 2uu_y + 2vv_y$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2uu_{yy} + 2u_y u_y + 2vv_{yy} + 2v_y v_y.$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (2u_x u_x + 2uu_{xx} + 2vv_{xx} + 2v_x v_x) + (2uu_{yy} + 2u_y u_y + 2vv_{yy} + 2v_y v_y).$$

$$= 2uu_{xx} + 2uu_{yy} + 2vv_{xx} + 2v_x^2 + 2u_y^2 + 2vv_{yy} + 2u_x^2 + 2v_y^2.$$

$$= 2uu_{xx} + 2u_x^2 + 2uu_{yy} + 2v_x^2 + 2vv_{xx} + 2vv_{yy} + 2v_y^2 + 2u_y^2$$

Given that $f(z)$ is analytic. Therefore u & v are harmonic function.

$\therefore u_{xx} + u_{yy} = 0.$

& $v_{xx} + v_{yy} = 0$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2u(0) + 2u_x^2 + 2u(0) + 2v_x^2 + 2v(0) + 2v(0)$$

$$= 2(uu_{xx} + uu_{yy}) + 2u_x^2 + 2(vv_{xx} + vv_{yy}) + 2v_x^2 + 2u_y^2$$

$\overline{f(z)} = u + i(-v)$

Since, $\overline{f(z)}$ is analytic, (10)

$$u_x = -v_y \;\&\; u_y = v_x.$$

Adding,

$$2u_x = 0 \;\&\; 2u_y = 0.$$

(ie) $u_x = 0, \quad u_y = 0.$

$$f'(z) = u_x + iv_x = 0.$$

$\Rightarrow f(z)$ is a constant.

3. If $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$ Prove that,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}} = 4 \frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right)$$

Sol:-

Let $z = x + iy.$

$\bar{z} = x - iy.$

$2x = z + \bar{z}; \quad 2iy = z - \bar{z}$

$x = \frac{z+\bar{z}}{2}; \quad y = \frac{z-\bar{z}}{2i}$

$\frac{\partial x}{\partial z} = \frac{1}{2}; \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}; \quad \frac{\partial y}{\partial z} = \frac{1}{2i};$

$\frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \frac{\partial y}{\partial z}.$$

Adding ③ & ④

$$2u_x(z,0) = 6z^2$$

$$u_x(z,0) = 3z^2 \longrightarrow ⑤$$

④ − ③

$$2u_y(z,0) = 0 \longrightarrow ⑥$$

$$u_y(z,0) = 0$$

Now, $f(z) = u(z,0) + iv(z,0)$

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz +$$

where $\phi_1 = u_x = 3z^2$

$\phi_2 = u_y = 0$

$$= \int [3z^2 - i(0)]\,dz$$

$$= \int [3z^2]\,dz$$

$$= \frac{\cancel{3}z^3}{\cancel{3}} + c$$

$$\therefore f(z) = z^3 + c$$

H.W.

8) Find the analytic function $f(z)$ if $u - v = e^x(\cos y - \sin y)$

9) Find a, so that $u = ax^2 - y^2 + xy$ is harmonic and find the analytic function $f(z)$

Given that $f(z) = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\theta/2\right)$

$= u+iv$

$u = \sqrt{r}\cos\theta/2 , \quad v = \sqrt{r}\sin\theta/2$

$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\theta/2 \qquad \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\theta/2$

$\frac{\partial u}{\partial \theta} = \sqrt{r}\,(-\sin\theta/2) \qquad \frac{\partial v}{\partial \theta} = \sqrt{r}\,(\cos\theta/2)\ 1/2$

$= -\sqrt{r}\sin\theta/2\,(1/2) \qquad = \frac{\sqrt{r}}{2}(\cos\theta/2)$

$\sqrt{r} = r^{1/2} = 2r^{1/2-1}$

$= 2r^{-1/2}$

$= 2\sqrt{r}$

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\frac{1}{2\sqrt{r}}\cos\theta/2 = \frac{1}{r}\frac{\sqrt{r}}{2}\cos\theta/2$

$\frac{1}{2\sqrt{r}} = \frac{\sqrt{r}}{\sqrt{r}\sqrt{r}\,2} \Rightarrow \frac{1}{2\sqrt{r}} = \frac{1}{2\sqrt{r}}$

$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta} \Rightarrow \frac{1}{2\sqrt{r}}\sin\theta/2 = \frac{-1}{r}\,\frac{-\sqrt{r}}{2}\sin\theta/2$

$\frac{1}{2\sqrt{r}} = \frac{\sqrt{r}}{\sqrt{r}\sqrt{r}\,2} \Rightarrow \frac{1}{2\sqrt{r}} = \frac{1}{2\sqrt{r}}$

Hence, the C-R equation is polar form are satisfied.

$\therefore$ It is differentiable

$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$ (by the theorem

Let v be the harmonic conjugate of u

$\therefore \phi(z) = u + iv$ is analytic.

Let $\phi_1(x, y) = u_x$ (6)

$= 2 - 3x^2 + 3y^2$

$\phi_1(z, 0) = 2 - 3z^2$

and $\phi_2(x, y) = u_y$

$= 6xy$

$\phi_2(z, 0) = 0$

$\phi(z) = \int \phi_1(x, y) + i\phi_2(x, y)\, dz + c$

$= \int \phi_1(z, 0) + i\phi_2(z, 0)\, dz + c$

$= \int [(2 - 3z^2) + 0]\, dz + c$

$= \int (2 - 3z^2)\, dz + c$

$= 2z - \frac{\cancel{3}z^3}{\cancel{3}} + c$

$\phi(z) = 2z - z^3 + c$

$= 2(x + iy) - (x + iy)^3 + c$

$= 2x + i2y - [x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3] + c$

$= 2x + 2iy - (x^3 + 3ix^2y + 3i^2xy^2 - iy^3) + c$

$= (2x + 2iy - x^3 - 3ix^2y + 3xy^2 + iy^3) + c$

$= (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3)$

$\boxed{2y}(a+1) = -2(bx+y)$

$$a+1 = \frac{-\cancel{2}(bx+y)}{\cancel{2}y}$$

(13)

$$a = \frac{-(bx+y)}{y} - 1$$

$$a = \frac{-(bx+y)-y}{y}$$

$\times\, a =$

$2ay - 2x = -(2bx + 2y)$

$ay - x + bx + y = 0$

$ay + y - x + bx = 0$

$y(a+1) - x(1-b) = 0$

$x(b-1) + y(a+1) = 0$

$b-1 = 0 \Rightarrow b = 1$

$a+1 = 0 \Rightarrow a = -1$

$f'(z) = u_x + iv_x$

$= (2x - 2y) + i(2x + 2y)$

$= 2[(x-y) + ix + iy]$

$= 2[(x+iy) + i(x+iy)]$

$= 2[z + iz)$

Analytic functions :-

(1)

Definition:

If f is analytic at a point "a" then, f is differentiable at "a".

A function "f" defined in a region "D" of the complex plane is said to be analytic at a point $a \in D$ if f is differentiable at every point of some neighbourwood (open ball) of "a".

Hence, f is analytic at "a" if $\exists$ $\varepsilon > 0$ such that f is differentiable at every point of the disc,

$$S(a, \varepsilon) = \{ z / |z-a| < \varepsilon \}.$$

If f is analytic at every point of a region D then, f is said to be analytic in D, a function which is analytic at every point of the complex plane is called an entire function (or) an integrable function

(iii) $uu_x - vv_y = 0 \longrightarrow (5)$

$vu_x + uu_y = 0 \longrightarrow (6)$

$(5) \times u \Rightarrow \quad u^2u_x - uv\cancel{v_y} = 0$

$(6) \times v \Rightarrow \quad v^2u_x + \cancel{v}uu_y = 0$

Adding $\quad (u^2+v^2)u_x = 0$

but, $u^2+v^2 = c$.

$\therefore cu_x = 0$ where $c \neq 0$

|||ly $u_x = 0$

$v_x = 0$

$\therefore f'(z) = u_x + iv_x = 0$

$\Rightarrow f(z)$ is constant.

2. If $f(z)$ & $\overline{f(z)}$ are analytic in a region D show that, $f(z)$ is a constant in D

Sol:

Let $f(z) = u+iv$.

$\therefore \overline{f(z)} = u - iv$

G.T. $f(z)$ is analytic in D.

Therefore, $u_x = v_y$

& $u_y = -v_x$.

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) - \frac{\partial}{\partial y}\left(\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{1}{2}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{\partial x}{\partial z}\right] + \left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{\partial y}{\partial z}\right]\right]$$

$$= \frac{1}{2}\left[\left[\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y}\right]\frac{1}{2} + \left(\frac{\partial^2}{\partial y \partial x} + i\frac{\partial^2}{\partial y^2}\right)\frac{1}{2i}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\left(i + \frac{1}{i}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\left(\frac{i^2+1}{i}\right)\right]$$

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$x = \frac{z+\bar{z}}{2}$; $y = \frac{z-\bar{z}}{2i}$

(4)

$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$; $\frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}$

Substitute these values in ①

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left[\frac{\partial f}{\partial x} + \frac{i}{i^2}\cdot\frac{\partial f}{\partial y}\right] \quad \times, \div \text{ by } (-i)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

$$\frac{\partial f}{\partial \bar{z}} = 0 \Longleftrightarrow \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

$$\Longleftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

i.e, $f_x = -i f_y$

Which is the Complex form of C-R equ.

∴ C-R equations can be put in the form

$\frac{\partial f}{\partial \bar{z}} = 0.$

Problem:

1. Show that $f(z) = \sqrt{r}\,(\cos \theta/2 + i\sin \theta/2)$ is differentiable for all where $r > 0$, $0 < \theta < 2\pi$ and find $f'(z)$.

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 0\right]$$

$$\therefore 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(2)

4. Test whether the following functions are analytic?

i) $z^3 + z$, ii) $e^x(\cos y + i\sin y)$,

iii) $e^x(\cos y - i\sin y)$. iv)

5. Determine the constant a & b (show) so that a function $\boxed{f(z) = x^2 + ay^2 + 2xy + i(bx^2)}$

$f(z) = (x^2 + ay^2 + 2xy) + i(bx^2 - y^2 + 2xy)$

Should be analytic and find $f'(z)$

Sol:

Given that,

$$f(z) = (x^2 + ay^2 + 2xy) + i(bx^2 - y^2 + 2xy)$$

$u = x^2 + ay^2 + 2xy$; $v = bx^2 - y^2 + 2xy$

$u_x = 2x + 2y$ $\qquad$ $v_x = 2bx + 2y$

$u_y = 2ay + 2x$ $\qquad$ $v_y = -2y + 2x$

$u_x = v_y$

$2x + 2y = -2y + 2x$

$u_y = -v_x$

$2ay + 2x = -2bx - 2y$

$$f'(z) = \frac{r}{z}\left[\frac{1}{2\sqrt{r}}\cos\theta/2 + i\,\frac{1}{2\sqrt{r}}\sin\theta/2\right]$$

$$= \frac{r}{z}\,\frac{1}{2\sqrt{r}}\left[\cos\theta/2 + i\sin\theta/2\right]$$

$$= \frac{\sqrt{r}\,\sqrt{r}}{z}\left[\frac{1}{2\sqrt{r}}\left(\cos\theta/2 + i\sin\theta/2\right)\right]$$

$$= \frac{1}{2z}\left[\sqrt{r}\left(\cos\theta/2 + i\sin\theta/2\right)\right]$$

$$= \frac{1}{2z}\left[r^{1/2}\,e^{i\theta/2}\right]$$

$$= \frac{1}{2z}\left[(re^{i\theta})^{1/2}\right] \quad \left[\because z = re^{i\theta}\right]$$

$$= \frac{1}{2z}\sqrt{z} \Rightarrow \frac{1}{2\sqrt{z}\sqrt{z}}\sqrt{z}$$

$$\boxed{f'(z) = 1/2\sqrt{z}}$$

2. Show that the function $f(z)$ is equal to,

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Satisfies C-R equations at the Origin, and f is not differentiable at $z=0$.

3. Show that $f(z) = \sin x \cosh y + i \cos x \sinh y$ is differentiable at every point

$$\Rightarrow \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

(2)

$$\Rightarrow \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$\Rightarrow 0$.

$\therefore$ u is hormonic function.

Let v be the hormonic conjugate of u

$\therefore$ $f(z) = u + iv$ is an analytic function

By C-R equation $u_x = v_y$, $u_y = -v_x$

$$u_x = v_y = \frac{x}{x^2+y^2} \quad (\text{by } ①)$$

Integrate w.r.to y

$$v = \int \frac{x}{x^2+y^2}\, dy = \int \frac{x}{x^2\left(1+\frac{y^2}{x^2}\right)}\, dy$$

$$= \tan^{-1}\frac{y}{x} + \phi(x) \longrightarrow ②$$

Now, $v_x = \dfrac{1}{1+\frac{y^2}{x^2}}\left(-\dfrac{y}{x^2}\right) + \phi'(x)$

$$= \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) + \phi'(x)$$

$v_x = -u_y$

where ... [illegible]

along the path $y = mx$

$\frac{f(x) - f(0)}{x} = \frac{[illegible]}{[illegible]}$

$= \frac{x}{x\sqrt{[illegible]}}$

[illegible]

The value of the limit depends on m

$\lim_{x \to 0} \frac{f(x) - f(0)}{x}$ does not exist

f is not differentiable at [illegible]

Show that $f(x) = \begin{cases} \frac{[illegible]}{x^2+y^2} & \text{if } [illegible] \\ 0 & \text{if } [illegible] \end{cases}$

is not differentiable at $(0,0)$

Sol:

$\frac{f(x) - f(0)}{x - 0} = \frac{[illegible]}{[illegible]}$

$= \frac{[illegible]}{[illegible]}$

$= [illegible]$

$= \frac{[illegible]}{[illegible]}$

along the path $x = my^2$

$\frac{h_2}{h}(\epsilon_2+i\epsilon_4)$

$= u_x - iu_y + \frac{h_1}{h}(\epsilon_1+i\epsilon_3) + \frac{h_2}{h}(\epsilon_2+i\epsilon_4)$

Since $\left|\frac{h_1}{h}\right| \le 1$, $\frac{h_1}{h}(\epsilon_1+i\epsilon_3) \to 0$ as $h_1 \to 0$

$|||^{ly}$ $\frac{h_2}{h}(\epsilon_2+i\epsilon_4) \to 0$ as $h_2 \to 0$.

$\therefore \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = u_x - iu_y$

Hence f is differentiable

Problems:

1. Show that the function $f(z) = \frac{z\,Re\,z}{|z|}$ (or) $f(z) = \begin{cases} \frac{z\,Re\,z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is not differentiable at $z=0$

Sol:

$\frac{f(z)-f(0)}{z-0} = \frac{\frac{\cancel{z}\,Re\,z}{|z|} - 0}{\cancel{z}}$

$= \frac{Re\,z}{|z|} = \frac{x}{\sqrt{x^2+y^2}}$