$\alpha u + \beta v = \alpha(a,0,0) + \beta(b,0,0)$

$= (\alpha a, 0, 0) + (\beta b, 0, 0)$

$= (\alpha a + \beta b, 0, 0)$

$\alpha u + \beta v \in W$

Hence W is a subspace of R^3

2. P.T $W = \{(a,b,0) / a,b \in R\}$ is a subspace of R^3

Sol:- Let $u = (a,b,0)$ $v = (c,d,0)$ where $u,v \in W$ and $\alpha, \beta \in F$

$\alpha u + \beta v = \alpha(a,b,0) + \beta(c,d,0)$

$= (\alpha a, \alpha b, 0) + (\beta c, \beta d, 0)$

$= (\alpha a + \beta c, \alpha b + \beta d, 0)$

$\alpha u + \beta v \in W$

$\therefore$ W is a subspace of R^3

3. In R^3 P.T $W = \{(ka, kb, kc) / k \in R\}$ is a subspace of R^3

Sol:- Let $u,v \in W$ and $\alpha, \beta \in R$ where,

$u = (k_1 a, k_1 b, k_1 c)$ & $v = (k_2 a, k_2 b, k_2 c)$

Now, $\alpha u + \beta v = \alpha(k_1 a, k_1 b, k_1 c) + \beta(k_2 a, k_2 b, k_2 c)$

$= [\alpha k_1 a, \alpha k_1 b, \alpha k_1 c] + [\beta k_2 a, \beta k_2 b, \beta k_2 c]$

$= (\alpha k_1 + \beta k_2) a, (\alpha k_1 + \beta k_2) b, (\alpha k_1 + \beta k_2) c$

ie., $\alpha u + \beta v \in W$.

By the theorem 4, W is a subspace of R^3.

4. $W = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} / a,b \in R \right\}$ is a subspace of $M_2(R)$

Sol:

Let $u = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $v = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ where $u,v \in W$ and $\alpha, \beta \in F$

$\alpha u + \beta v = \alpha \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \beta \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$

$= \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} + \begin{pmatrix} \beta c & 0 \\ 0 & \beta d \end{pmatrix}$

vii. TO PT $(\alpha+\beta)u = \alpha u+\beta u$

$$(\alpha+\beta)(w+v_1) = w+(\alpha+\beta)v_1$$
$$= w+\alpha v_1+\beta v_1$$
$$=(w+\alpha v_1)+(w+\beta v_1)$$
$$= \alpha(w+v_1)+\beta(w+v_1).$$

viii. TO PT $\alpha(\beta u) = (\alpha\beta)u$

$$\alpha(\beta(w+v_1)) = \alpha\ (w+\beta v_1)$$
$$= \alpha\beta\ (w+v_1)$$

ix. TO PT $1\cdot u = u$

$$1\cdot(w+v_1) = w+1\cdot v$$
$$= w+v.$$

Hence V/W is a vector space. Hence proved.

Note :-

The vector space V/W is called the quotient space of V by W.

Linear Transformation

Def:

Let V and W be vector spaces over a field F. A mapping $T: V \to W$ is called a homomorphism if

1. $T(u+v) = T(u)+T(v)$
ii. $T(\alpha u) = \alpha(T(u))$, where, $\alpha \in F$ and $u, v \in V$.

A homomorphism T of vector spaces is called a linear transformation.

Note:

i. If T is 1-1, then T is called monomorphism
ii. If T is onto, then T is called epimorphism
iii. If T is 1-1, and onto then T is called Isomorphism
iv. 2 v.s V and W are said to be isomorphic if $\exists$ an isomorphism $T: V \to W$ and $V \cong W$.