$$= \frac{1}{0.2}\left\{\frac{1.53}{2} - \frac{0.02}{12}\right\}$$ (9)

$$= \frac{1}{0.2}\{0.765 - 0.00167\}$$

$$= 5(0.765 - 0.00167)$$

$$= 5(0.76333)$$

$$= 3.81663$$

Ans:

$$\left(\frac{d\theta}{dt}\right)_{at\ t=0.6} = 3.82 \text{ radius/sec}$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{at\ t=0.6} = \frac{1}{0.04}(0.27 - 0)$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{at\ t=0.6} = 6.75 \text{ radius/sec}$$

Maxima and Minima

Problem

1) Find the maxima & minima value of y from the following table.

x	0	1	2	3	4	5
y	0	$1/4$	0	$9/4$	16	$225/4$

sol.

Here, $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$ & $h = 1$

w.k.t The Newton's forward interpolation formula is,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \quad \text{— (1)}$$

and, $$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{12}\Delta^4 y_0 + \cdots\right\} \quad \text{— (2)}$$

and,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0\right\} \quad \text{— (3)}$$

$$\frac{dy}{dx} = \frac{1}{10}(0.0198 + 0.0002 + 0.00003 - 0.00005 + 0.00006)$$

(9)

$$\frac{dy}{dx} = 0.002004.$$

$$\frac{d^2y}{dx^2} = \frac{1}{100}\left(-0.0004 - 0.0001 - \frac{11}{12}(0.0002)\right)$$

$$= \frac{1}{100}(-0.0004 - 0.0001 - 0.00018)$$

$$\frac{d^2y}{dx^2} = 0.0000068.$$

ii) Backward formula:

$$\frac{dy}{dx} = \frac{1}{10}\left(0.0183 - \frac{0.0004}{2} + 0 + \frac{0.0001}{4} + \frac{0.0003}{5}\right)$$

$$= \frac{1}{10}(0.0183 - 0.0002 + 0.000025 + 0.00006)$$

$$\frac{dy}{dx} = 0.001819.$$

$$\frac{d^2y}{dx^2} = \frac{1}{100}\left(-0.0004 + \frac{11}{12}(0.0001)\right)$$

$$\frac{d^2y}{dx^2} = -0.00000308$$

Ans:

i) Forward (at $x = 500$)

$$\frac{dy}{dx} = 0.002, \quad \frac{d^2y}{dx^2} = 0.0000068$$

ii) Backward (at $x = 550$)

$$\frac{dy}{dx} = 0.001819, \quad \frac{d^2y}{dx^2} = -0.00000308$$

5) The population in millions of a certain town is shown in the following table. Find the rate of growth of the population in 1961.

Year	1931	1941	1951	1961	1971
Population (y)	40.62	60.80	79.95	103.56	132.65

Problems

1) Find the value $\cos(1.74)$ from the following table

x	1.70	1.74	1.78	1.82	1.86
$\sin x$	0.9917	0.9857	0.9782	0.9691	0.9585

Sol:

Here $x_0 = 1.70$, $x_1 = 1.74$, $x_2 = 1.78$, $x_3 = 1.82$, $x_4 = 1.86$,

$h = 0.04$ and $x = 1.74$

Here $x = 1.74$ is nearer to beginning of the table we use

newton's forward interpolation formula,

$$\text{(i)}\quad \frac{dy}{dx} = \frac{1}{h}\left\{ \Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{(2p^3-9p^2+11p-3)}{12}\Delta^4 y_0 + \cdots \right\} \quad \text{--- (1)}$$

where, $p = \dfrac{x - x_0}{h}$

Here, $p = \dfrac{x - x_0}{h} = \dfrac{1.74 - 1.70}{0.04} = \dfrac{0.04}{0.04} = 1$

$\boxed{p = 1}$

The difference table is as follows:

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.70	0.9917				
		-0.0060			
1.74	0.9857		-0.0015		
		-0.0075		-0.0001	
1.78	0.9782		-0.0016		0.0002
		-0.0091		0.0001	
1.82	0.9691		-0.0015		
		-0.0106			
1.86	0.9585				

$\therefore$ (1) becomes,

$$\cos(1.74) = \frac{1}{0.04}\left\{ -0.0060 + \frac{2(1)-1}{2}(-0.0015) + \frac{3(1)^2 - 6(1) + 2}{6}(-0.0001) + \frac{2(1)^3 - 9(1)^2 + 11(1) - 3}{12}(0.0002) \right\}$$

where, $p = \frac{x - x_0}{h} = \frac{x-0}{1} = x. \Rightarrow \boxed{p = x}$ (13)

$$y(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{6}\Delta^3 y_0 + \frac{x(x-1)(x-2)(x-3)}{24}\Delta^4 y_0 \quad \text{--- (A)}$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2x-1)}{2}\Delta^2 y_0 + \frac{(3x^2-6x+2)}{6}\Delta^3 y_0 + \frac{2x^3-9x^2+11x-3}{12} + \right\} \text{--- (5)}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (x-1)\Delta^3 y_0 + \frac{6x^2-18x+11}{12}\Delta^4 y_0 + \cdots\right\} \text{--- (6)}$$

The tabular column as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0					
		0.25				
1	0.25		-0.5			
		-0.25		3	6	0
2	0		2.5			
		2.25		9	6	
3	2.25		11.5			
		13.74		15		
4	16		26.5			
		40.25				
5	56.25					

Now (5) becomes,

$$\frac{dy}{dx} = \frac{1}{1}\left\{0.25 + \frac{(2x-1)}{2}(-0.5) + \frac{3x^2-6x+2}{6}(3) + \frac{2x^3-9x^2-11x-3}{12}(6)\right\}$$

$$= 0.25 + \frac{(2x-1)}{2}(-0.5) + \frac{3x^2-6x+2}{2} + \frac{2x^3-9x^2+11x-3}{2} = 0.$$

$$= 0.25 - \frac{(2x-1)}{4} + \frac{3x^2-6x+2}{2} + \frac{2x^3-9x^2+11x-3}{2} = 0.$$

$$= \frac{1-2x+1+6x^2-12x+4+4x^3-18x^2+22x-6}{4} = 0.$$

$$\frac{4x^3-12x^2+8x}{4} = 0 \Rightarrow 4x^3-12x^2+8x = 0.$$

x	$y=\sqrt{x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
15	3.873				
		0.25			
17	4.123		-0.019		
		0.231		0.017	
19	4.354		-0.002		-0.031
		0.229		-0.014	
21	4.583		-0.016		
		0.213			
23	4.796				

$$\frac{dy}{dx} = \frac{1}{2}\left[0.25 + \frac{0.019}{2} + \frac{0.017}{3} + \frac{0.031}{4}\right]$$

$$= \frac{1}{2}(0.25 + 0.0095 + 0.0057 + 0.0075)$$

$$\frac{dy}{dx} = 0.13635 \Rightarrow 0.1364 \text{ (Approx)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}\left[-0.019 - 0.017 - \frac{11}{12}(0.031)\right]$$

$$= \frac{1}{4}(-0.019 - 0.017 - 0.0284)$$

$$\frac{d^2y}{dx^2} = -0.0161$$

Ans

$$\frac{dy}{dx} = 0.1364, \quad \frac{d^2y}{dx^2} = -0.0161.$$

3) $f'(50)$ and $f''(50)$ from the following table

x	50	51	52	53	54
$f(x)$	3.68	3.70	3.73	3.75	3.77

Sol.

Here $x=50$, is nearer to the beginning of the table we use newton's forward formula.

Here $x_0=50$, $x_1=51$, $x_2=52$, $x_3=53$, $x_4=54$.

Also, $x=50$ & $h=1$.

$$p = \frac{x-x_0}{h} = \frac{50-50}{1} = 0.$$

$$= \frac{1}{0.04}\left\{-0.0060+\frac{1}{2}(-0.0015)-\frac{1}{6}(-0.0001)+\frac{1}{12}(0.0002)\right\}$$

$$= \frac{1}{0.04}\left\{-0.0060-\frac{0.0015}{2}+\frac{0.0001}{6}+\frac{0.0002}{12}\right\}$$ (5)

$$= \frac{1}{0.04}\left\{-0.00600-0.00075+0.000167+0.00016\right\}$$

$$= \frac{1}{0.04}\left\{-0.00675+0.000334\right\}$$

$$= \frac{1}{0.04}(-0.006716)$$

$\cos(1.74) = -0.167915$

Ans:

$\cos(1.74) = -0.1679$ (Approx)

2) Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x=15$ where $y=\sqrt{x}$. Find the first & Second derivatives of $\sqrt{x}$ at $x=15$ from the following data

x	15	17	19	21	23
$\sqrt{x}$	3.873	4.123	4.354	4.583	4.796

sol:

Here $x=15$ is nearer to the beginning of the table we use Newton's forward formula.

Here, $x_0=15$, $x_1=17$, $x_2=19$, $x_3=21$, $x_4=23$

Also, $x=15$ & $h=2$

$$p = \frac{x-x_0}{h} = \frac{15-15}{2} = 0$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0-\frac{\Delta^2 y_0}{2}+\frac{\Delta^3 y_0}{3}-\frac{\Delta^4 y_0}{4}+\cdots\right\} \quad \text{--- (1)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0-\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0+\cdots\right\} \quad \text{--- (2)}$$

8) Find the minimum value of $f(x)$ which has the values.

(17)

x	0	2	4	6
$f(x)$	3	3	11	27

Sol.

Here, $x_0 = 0$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$ and $h = 6$

w.r.t Newton's forward interpolation formula is,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0 + \cdots \quad \text{—(1)}$$

and

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{12}\Delta^3 y_0 + \cdots\right\} \quad \text{—(2)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right\} \quad \text{—(3)}$$

where, $p = \frac{x - x_0}{h}$ Here, $p = \frac{x-0}{2} = x/2 \Rightarrow \boxed{p = x/2}$

$\therefore$ (1) becomes

$$y(x) = y_0 + \frac{x\Delta y_0}{2} + \frac{x/2\,(x/2 - 1)}{2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3}\Delta^3 y_0 + \frac{x(x-1)(x-2)(x-3)}{4}\Delta^4 y_0 + \cdots \quad \text{—(4)}$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2x/2 - 1)}{2}\Delta^2 y_0 + \left(\frac{3x^2+6x+2}{6}\right)\Delta^3 y_0 + \cdots\right\} \quad \text{—(5)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (x-1)\Delta^3 y_0 + \cdots\right\} \quad \text{—(6)}$$

The tabular column is as follows

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3			
		0		
2	3		8	
		8		0
4	11		8	
		16		
6	27			

5) W.K.T The Newton's backward interpolation formula is,

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots \quad (1)$$

where, $p = \frac{x - x_n}{h}$

(i.e) $y_x = y_n + p\nabla y_n + \frac{p^2+p}{2!}\nabla^2 y_n + \frac{(p^2+p)(p+2)}{6}\nabla^3 y_n + \frac{(p^2+p)(p^2+3p+2p+6)}{24}\nabla^4 y_n \dots$

$$y_x = y_n + p\nabla y_n + \frac{p^2+p}{2}\nabla^2 y_n + \frac{(p^3+3p^2+2p)}{6}\nabla^3 y_n + \frac{p^4+5p^3+6p^2+p^3+5p^2+6p}{24}\nabla^4 y_n + \dots$$

$$y_x = y_n + p\nabla y_n + \frac{p^2+p}{2}\nabla^2 y_n + \frac{p^3+3p^2+2p}{6}\nabla^3 y_n + \frac{p^4+6p^3+11p^2+6p}{24}\nabla^4 y_n + \dots$$

Now, $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$ —— (1)

$$\frac{dp}{dx} = \frac{d}{dx}\left\{\frac{x - x_n}{h}\right\}$$

$$= \frac{1}{h}\left\{\frac{d}{dx}(x) - \frac{d}{dx}(x_n)\right\}$$

$$\frac{dp}{dx} = \frac{1}{h}$$

$$\frac{dy}{dp} = \frac{d}{dp}(y_n) + \frac{d}{dp}(p)\nabla y_n + \frac{1}{2}\left\{\frac{d}{dp}(p^2) + \frac{d}{dp}(p)\right\}\nabla^2 y_n +$$

$$+ \frac{1}{6}\left\{\frac{d}{dp}(p^3) + 3\frac{d}{dp}(p^2) + 2\frac{d}{dp}(p)\right\}\nabla^3 y_n$$

$$+ \frac{1}{24}\left\{\frac{d}{dp}(p^4) + 6\frac{d}{dp}(p^3) + 11\frac{d}{dp}(p^2) + 6\frac{d}{dp}(p)\right\}\nabla^4 y_n + \dots$$

$$= \nabla y_n + \frac{1}{2}\{2p+1\}\nabla^2 y_n + \frac{1}{6}\{3p^2+6p+2\}\nabla^3 y_n + \frac{1}{24}\{4p^3+18p^2+22p+6\}\nabla^4 y_n + \dots$$

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{(3p^2+6p+2)}{6}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\nabla^4 y_n + \dots$$

(1) becomes,

$$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{(2p+1)}{2}\nabla^2 y_n + \frac{(3p^2+6p+2)}{6}\nabla^3 y_n + \frac{2(2p^3+9p^2+11p+3)}{24}\nabla^4 y_n + \dots\right.$$

$$\frac{dy}{dp} = \frac{d}{dp}\left\{ y_0 + p\left(\frac{\nabla y_0 + \nabla y_{-1}}{2}\right) + \frac{p^2}{2!}\nabla^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\nabla^3 y_{-1} + \nabla^3 y_{-2}}{2}\right) \right. \quad (3)$$

$$\left. + \frac{p^2(p^2-1)}{4!}\nabla^4 y_{-2} + \cdots \right\}$$

$$= \frac{d}{dp}(y_0) + \frac{d}{dp}(p)\left(\frac{\nabla y_0 + \nabla y_{-1}}{2}\right) + \frac{1}{2}\frac{d}{dp}(p^2)\nabla^2 y_{-1} + \frac{1}{6}\left\{\frac{d}{dp}(p^3) - \frac{d}{dp}(p)\right\}$$

$$\frac{\nabla^3 y_{-1} + \nabla^3 y_{-2}}{2} + \frac{1}{24}\left\{\frac{d}{dp}(p^2)\right\}\nabla^4 y_{-2} + \cdots$$

$$= \frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{1}{2}\left\{2p(\Delta^2 y_{-1})\right\} + \frac{1}{6}\left\{3p^2 - 1\right\}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}$$

$$+ \frac{1}{24}\left\{4p^3 - 2p\right\}\Delta^4 y_{-2} + \cdots$$

$$= \frac{\Delta y_0 + \Delta y_{-1}}{2} + p(\Delta^2 y_{-1}) + (3p^2-1)\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12}\right] + (2p^3 - p)\frac{\Delta^4 y_{-2}}{12} + \cdots$$

(1) becomes,

$$* \quad \frac{dy}{dx} = \frac{1}{h}\left\{ \frac{\Delta y_0 + \Delta y_{-1}}{2} + p(\Delta^2 y_{-1}) + (3p^2-1)\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12}\right] \right.$$

$$\left. + (2p^3 - p)\left[\frac{\Delta^4 y_{-2}}{12}\right] + \cdots \right\}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{1}{h^2}\left\{ \frac{d}{dp}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{d}{dp}(p)\,(\Delta^2 y_{-1}) + \left\{3\frac{d}{dp}(p^2) - \frac{d}{dp}(1)\right\}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12}\right) \right.$$

$$\left. + \left\{2\frac{d}{dp}(p^3) - \frac{d}{dp}(p)\right\}\left(\frac{\Delta^4 y_{-2}}{12}\right) + \cdots \right.$$

$$= \frac{1}{h^2}\left\{ \Delta^2 y_{-1} + 6p\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12}\right) + (6p^2-1)\frac{\Delta^4 y_{-2}}{12} + \cdots \right\}$$

$$* \quad \frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{ \Delta^2 y_{-1} + \frac{p}{2}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + (6p^2-1)\frac{\Delta^4 y_{-2}}{12} + \cdots \right\}$$

At $x = x_0$, $p = 0$

$$* \quad \therefore \frac{dy}{dx} \text{ at } x = x_0 \Rightarrow = \frac{1}{h}\left\{ \frac{\Delta y_0 + \Delta y_{-1}}{2} - \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12}\right) + \cdots \right\}$$

$$* \quad \left(\frac{d^2y}{dx^2}\right) \text{ at } x = x_0 \Rightarrow = \frac{1}{h}\left\{ \Delta^2 y_{-1} - \frac{\Delta^4 y_{-2}}{12} + \cdots \right\}$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right\} \quad \text{--- (1)}$$

(9)

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right\} \quad \text{--- (2)}$$

The diff table is as follows.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	3.68				
		0.02			
			0.01		
51	3.70			-0.02	
		0.03			0.03
52	3.73		-0.01		
		0.02		0.01	
53	3.75		0		
		0.02			
54	3.77				

$$\Rightarrow \frac{dy}{dx} = 0.02 - \frac{0.01}{2} - \frac{0.02}{3} - \frac{0.03}{4}$$

$$= 0.02 - 0.005 - 0.0067 - 0.0075$$

$$\frac{dy}{dx} = 0.0008$$

$$\Rightarrow \frac{d^2y}{dx^2} = 0.01 + 0.02 + \frac{11}{12}(0.03)$$

$$= 0.0575$$

Ans: $\frac{dy}{dx} = 0.0008$; $\frac{d^2y}{dx^2} = 0.0575$

4) From the following data obtain the 1^{st} & 2^{nd} derivatives of $y = \log_e x$ i) at $x = 500$, ii) $x = 550$.

x	500	510	520	530	540	550
$y = \log_e x$	6.2146	6.2344	6.2538	6.2729	6.2916	6.3099.

Also, calculate the actual value of the derivatives at this point.

Sol:

i) Here $x = 500$ is nearer to the beginning of the table.

To find $\frac{dy}{dx}$ (velocity).

Sol:

Here $x = 1961$ is nearer to the beginning of the table we use Newton's backward formula.

Here, $x_0 = 1931$, $x_1 = 1941$, $x_2 = 1951$, $x_3 = 1961$, $x_4 = x_n = 1971$

$$p = \frac{x - x_n}{h} = \frac{-(1971 - 1961)}{10} = \frac{-10}{10} = -1$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{6}\nabla^3 y_n - \frac{1}{12}\nabla^4 y_n + \cdots\right\}$$

The diff table is as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
					-4.47
			4.46		
1951	79.95	23.61		1.02	
			5.48		
1961	103.56	29.09			
1971	132.65				

$$\frac{dy}{dx} = \frac{1}{10}\left(29.09 - \frac{5.48}{2} - \frac{1.02}{6} + \frac{4.47}{12}\right)$$

$$= \frac{1}{10}(29.09 - 2.74 - 0.17 + 0.3725)$$

$$\frac{dy}{dx} = 2.65525$$

Ans:

$$\frac{dy}{dx} = 2.65525.$$

b) A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of time t (seconds).

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

Calculate the angular velocity and the

$4x(x^2-3x+2)=0$

$x(x^2-3x+2)=0$ (14)

$x=0$ (or) $x^2-3x+2=0$

$x=0,1,2$

i) When, $x=0$, $\frac{d^2y}{dx^2}=\{-0.5-(3)+\frac{11}{12}(6)\}$

$=(-0.5-3+\frac{11}{2}) = \frac{-1-6+11}{2} = 4/2$

$=2>0$.

ii) when $x=1$, $\frac{d^2y}{dx^2}=(-0.5-\frac{1}{12}(6)) = -0.5-\frac{1}{2}$

$=\frac{-1-1}{2}=\frac{-2}{2}=-1<0$

iii) when $x=2$, $\frac{d^2y}{dx^2}=(-0.5+3-\frac{1}{12}(6)) = (-0.5+3-1/2)$

$=\frac{-1+6-1}{2}=4/2=2>0$.

$\therefore y(x)$ attains min, when $x=0,2$ & $y(x)$ attains max, when $x=-1$ the max value is

$=0+2(0.25)+\frac{2}{2}(-0.5)=0.5-0.5=0$.

$\therefore y(0)=0$, $y(2)=0$, $y(1)=0.25$

2) From the following table find the maximum value of $f(x)$.

x	0	1	2	3	4	5
$f(x)$	58	43	40	45	52	60

Sol:

Here $x_0=0$, $x_1=1$, $x_2=2$, $x_3=3$, $x_4=4$ & $h=1$, $x_5=5$

W.K.T Newton's forward interpolation formula is,

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{6}\Delta^3 y_0+\frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0 \quad \to (1)$$

We use newton's forward formula

Here, $x_0 = 500$, $x_1 = 510$, $x_2 = 520$, $x_3 = 530$, $x_4 = 540$, $x_5 = 550$

Also, $x = 500$, $h = 10$ (8)

$$p = \frac{x - x_0}{h} = \frac{500 - 500}{10} = 0.$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{ \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} \right\}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \right\}.$$

ii) Here $x = 550$ is nearer to the beginning of the table we use

Newton's backward formula.

$$x_n = 550$$

$$p = \frac{x - x_n}{h} = \frac{550 - 550}{10} = 0$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{ \nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n \right\}.$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n \right\}.$$

x	$y = \log_e x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
500	6.2146					
		0.0198				
510	6.2344		-0.0004			
		0.0194		0.001		
520	6.2538		-0.0003		-0.0002	
		0.0191		-0.0001		0.0003
530	6.2729		-0.0004		0.0001	
		0.0187		0		
540	6.2916		-0.0004			
		0.0183				
550	6.3099					

i) Forward formula:

$$\frac{dy}{dx} = \frac{1}{10}\left(0.0198 + 0.0002 + \frac{0.0001}{3} - \frac{0.0002}{4} + \frac{0.0003}{5} \right)$$

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$$\therefore I = \int_{x_0}^{x_0+nh} f(x)\,dx \quad \text{—} ①.$$

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W.K.T $p = \frac{x-x_0}{h} \Rightarrow x = x_0+ph$.

$\therefore dx = d(x_0+ph) = d(x_0)+h\,dp = h\,dp$.

① when $x = x_0 \Rightarrow x_0 = x_0+ph$

$\Rightarrow \boxed{p=0}$ (forward)

② when $x = x_0+nh \Rightarrow x_0+nh = x_0+ph$

$\Rightarrow \boxed{p=n}$ W.K.T Newton's interpolation formula is,

$$y(x) = y(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

ie) $$f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p^2-p}{2}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{6}\Delta^3 y_0 + \cdots$$

$\therefore$ ① becomes, $I = \int_0^n f(x_0+ph)\,h\,dp$.

$$= h\int_0^n \left\{ y_0 + p\Delta y_0 + \frac{p^2-p}{2}\Delta^2 y_0 + \cdots \right\} dp$$

$$= h\left\{ y_0\int_0^n dp + \left(\int_0^n p\,dp\right)\Delta y_0 + \left(\int_0^n p^2 dp - \int_0^n p\,dp\right)\Delta^2 y_0 \right.$$

$$\left. + \left(\int_0^n p^3 dp - 3\int_0^n p^2 dp + 2\int_0^n p\,dp\right)\frac{\Delta^3 y_0}{6} + \cdots \right\}$$

$$= h\left\{ y_0[p]_0^n + \left[\frac{p^2}{2}\right]_0^n \Delta y_0 + \left(\left[\frac{p^3}{3}\right]_0^n - \left[\frac{p^2}{2}\right]_0^n\right)\frac{\Delta^2 y_0}{2} \right.$$

$$\left. + \left(\left[\frac{p^4}{4}\right]_0^n - 3\left[\frac{p^3}{3}\right]_0^n + 2\left[\frac{p^2}{2}\right]_0^n\right)\frac{\Delta^3 y_0}{6} + \cdots \right\}$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)\,dx = h\left\{ ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \cdots \right\} \text{—} ②$$

This formula is known as Newton's cote's quadrature formula.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{d}{dp}\left\{\frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\nabla^4 y_n + \right.\right.$$

$$= \frac{1}{h^2}\left\{\frac{d}{dp}(\nabla y_n) + \frac{1}{2}\left\{2\frac{d}{dp}(p) + \frac{d}{dp}(1)\right\}\nabla^2 y_n + \frac{1}{6}\left\{3\frac{d}{dp}(p^2) + 6\frac{d}{dp}(p) + \frac{d}{dp}(2)\right\}\nabla^3 y_n + \frac{1}{24}\left\{4\frac{d}{dp}(p^3) + 18\frac{d}{dp}(p^2) + 22\frac{d}{dp}(p) + \frac{d}{dp}(6)\right\}\nabla^4 y_n \ldots\right.$$

$$= \frac{1}{h^2}\left[0 + \frac{1}{2}(2)\nabla^2 y_n + \frac{1}{6}\{6p+6\}\nabla^3 y_n + \frac{1}{24}\{12p^2+36p+22\}\nabla^4 y_n \ldots\right]$$

(ii) * $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{1}{12}\{6p^2+18p+11\}\nabla^4 y_n + \ldots\right\}$

At $x = x_n$

$p = \frac{x-x_n}{h} \Rightarrow \boxed{p=0}$

(i) * $\frac{dy}{dx}$ at $x = x_n \Rightarrow = \frac{1}{h}\left\{\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right\}$

(iii) * $\frac{d^2y}{dx^2}$ at $x = x_n \Rightarrow = \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right\}$

3) Stirling's Formula:

W.K.T Stirling's formula is,

$$y_x = y_0 + p\left(\frac{\nabla y_0 + \nabla y_{-1}}{2}\right) + \frac{p^2}{2!}\nabla^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\nabla^3 y_{-1} + \nabla^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\nabla^4 y_{-2} + \ldots$$

where, $p = \frac{x-x_0}{h}$

Now, $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$ — ①

$$\frac{dp}{dx} = \frac{d}{dx}(p) = \frac{1}{h}\frac{d}{dx}(x-x_0) = \frac{1}{h}(1)$$

$$\frac{dp}{dx} = \frac{1}{h}.$$

acceleration of the rod when $t=0.6$ Seconds.

Sol:

Here $t=0.6$ is in the middle of the table we use Stirling's formula to find angular velocity and acceleration.

(a) To find $\left(\frac{d\theta}{dt}\right)$ at $t=0.6$ and $\left(\frac{d^2\theta}{dt^2}\right)$ at $t=0.6$

Now, $p = \frac{t-t_0}{h}$

choose $t_0 = 0.6$ and here $h=0.2$

$p = \frac{0.6-0.6}{0} = 0 \Rightarrow \boxed{p=0}$

W.K.T

$$\left(\frac{d\theta}{dt}\right)_{at\ t=t_0} = \frac{1}{h}\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12} + \cdots\right\} \quad \text{— (1)}$$

$$\text{and}\left(\frac{d^2\theta}{dt^2}\right)_{at\ t=t_0} = \frac{1}{h^2}\left\{\Delta^2 y_{-1} - \frac{\Delta^4 y_{-2}}{12} + \cdots\right\} \quad \text{— (2)}$$

The tabular column as follows:

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
0	0					
		0.12				
			0.25			
0.2	0.12					
				0.01		
		0.37				
					0	
			0.26			0
0.4	0.49					
				0.01		
		0.63				
					0	
			0.27			
0.6	1.12					
				0.01		
		0.9				
			0.28			
0.8	2.02					
		1.18				
1.0	3.20					

Here $\Delta\theta_0 = 0.9$, $\Delta\theta_{-1} = 0.63$, $\Delta^3\theta_{-1} = 0.01$, $\Delta^3\theta_{-2} = 0.01$,

$\Delta^2\theta_{-1} = 0.27$, $\Delta^4\theta_{-2} = 0$

$\therefore$ (1) becomes,

$$\left(\frac{d\theta}{dt}\right)_{at\ t=0.6} = \frac{1}{h}\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12} + \cdots\right\}$$

$$= \frac{1}{0.2}\left\{\frac{0.9+0.63}{2} - \frac{0.01+0.01}{12}\right\}$$

and,

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{24}\Delta^4 y_0 + \right\} \quad \text{— (2)}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right\} \quad \text{— (3)}$$

where, $p = \frac{x - x_0}{h}$, Here, $p = \frac{x-0}{1} = x \Rightarrow \boxed{p = x}$

$\therefore$ (1) becomes,

$$y(x) = y_0 + x\,\Delta y_0 + \frac{x(x-1)}{2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3}\Delta^3 y_0 + \cdots \quad \text{— (4)}$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2x-1)}{2}\Delta^2 y_0 + \frac{3x^2-6x+2}{6}\Delta^3 y_0 + \cdots\right\} \quad \text{— (5)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (x-1)\Delta^3 y_0 + \cdots\right\} \quad \text{— (6)}$$

The tabular column is as follows.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	58					
		−15				
			12			
1	43			−4		
		−3			−2	
2	40		8			7
		5		−6	5	
3	45		2			
		7		1		
4	52		1			
		8				
5	60					

(5) becomes,

$$\frac{dy}{dx} = 0 \Rightarrow 1\left\{-15 + \frac{(2x-1)}{2}(12) + \frac{3x^2-6x+2}{6}(-4)\right\} = 0.$$

$$-15 + (2x+1)6 - \frac{(3x^2-6x+2)}{6}(4) = 0.$$

$$-15 + 12x - 6 - \frac{12x^2 - 24x + 8}{6} = 0.$$

$$\frac{-90 + 72x - 36 - 2x^2 + 24x - 8 - 2x^2 + 9x^2 - 11x + 3}{6} = 0.$$

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$$\frac{-12x^2+96x-134}{6}=0$$

$$\frac{2(-6x^2+48x-67)}{6}=0$$

$-6x^2+48x-67=0 \Rightarrow 6x^2-48x+67=0$

$$x=\frac{48\pm\sqrt{(48)^2-4(6)(67)}}{2(6)}=\frac{48\pm\sqrt{2304-1608}}{12}$$

$$=\frac{48\pm\sqrt{696}}{12}=\frac{48\pm 26.4}{12}$$

$$=\frac{48+26.4}{12},\ \frac{48-26.4}{12}$$

$$=\frac{74.4}{12},\ \frac{21.6}{12}$$

$x=6.2,\ 1.8$

i) when, $x=1.8$,

$$\frac{d^2y}{dx^2}=1\left(12+(1.8-1)(-4)\right)=12+0.8(-4)=12-3.2=8.8>0.$$

ii) when, $x=6.2$,

$$\frac{d^2y}{dx^2}=1\left(12+(6.2-1)(-4)\right)=12+5.2(-4)=12-20.8=-8.8<0$$

$y(x)$ attains min value when $x=1.8$ & $y(x)$ attains max value when $x=6.2$.

The min value is $=58+(1.8)(-1.5)+\frac{(1.8)(1.8-1)}{2}(12)$

$$+\frac{1.8(1.8-1)(1.8-2)}{6}(-4)$$

$$=58-27+(1.8)(0.8)6+\frac{1.8(0.8)(-0.2)(-4)}{6}$$

$$=58-27+8.64+\frac{1.152}{6}=58-27+8.64+0.192$$

$=39.832$.

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∴ (5) becomes,

$$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{h}\left\{\Delta y_0 + \frac{2(x/2)-1}{2}\Delta^2 y_0 + \right\} = 0$$

$$\frac{1}{2}\left\{0 + \frac{x-1}{2}(8)\right\} = 0$$

$$\frac{1}{2}(x-1)4 = 0$$

$$4x - 4 = 0 \Rightarrow 4x = 4 \Rightarrow \boxed{x=1}$$

When $x = 1$,

$$\frac{d^2y}{dx^2} = \frac{1}{4}\left\{8 + (x-1)(0)\right\}$$

$$= \frac{1}{4}\left\{8 + (1/2 - 1)0\right\}$$

$$= 1/4\,(8) = 2 > 0.$$

∴ $y(x)$ attain min when $x = 1$,

The min value is $= 3 + \left\{1/2\,(0)\right\} + \frac{1/2\,(1/2 - 1)}{2}(8) + 0$

$$= 3 + 1/2\,(-1/2)4$$

$$= 3 - 1$$

$$y(1) = 2.$$

Unit - 5

Numerical Integration

Newton's Cote's Quadrature Formula

Let the function be $y = f(x)$, let $y_0 = f(x_0)$, $y_1 = f(x_1) \ldots y_n = f(x_n)$, let $I = \int_a^b f(x)\,dx$

Divide the interval $[a, b]$ into the subintervals such that $a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h \ldots x_n = x_0 + nh$ where h is the length of the interval.

Simpson's $\frac{1}{3}^{rd}$ Rule

(5)

Put, $n=2$ in (2).

The values of x are $x_0, x_1, x_2 \ldots$

Here all the differences of 3^{rd} order and higher order will become zero.

$$\therefore \int_{x_0}^{x_0+2h} f(x)\,dx = h\left\{2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2}\right\}$$

$$= h\left\{2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right\}$$

$$= h\left\{2y_0 + 2(y_1 - y_0) + \tfrac{1}{3}(\Delta y_1 - \Delta y_0)\right\}$$

$$= h\left\{2y_0 + 2y_1 - 2y_0 + \tfrac{1}{3}(y_2 - y_1 - y_1 + y_0)\right\}$$

$$= \frac{h}{3}\left\{6y_1 + y_2 - y_1 - y_1 + y_0\right\}$$

$$= \frac{h}{3}\left\{y_2 + 4y_1 + y_0\right\}$$

$$= \frac{h}{3}\left\{y_0 + 4y_1 + y_2\right\}$$

Ily,
$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left\{y_2 + 4y_3 + y_4\right\}$$

$$\int_{x_0+4h}^{x_0+6h} f(x)\,dx = \frac{h}{3}\left\{y_4 + 4y_5 + y_6\right\}$$

$\vdots$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left\{y_{n-2} + 4y_{n-1} + y_n\right\}$$

Adding all these equations,

$$\int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+4h} f(x)\,dx + \cdots + \int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\{y_0 + 4y_1 + y_2 + y_2 + 4y_3$$
$$+ y_4 + y_4 + 4y_5 + y_6 + \cdots + y_{n-1}$$
$$+ 4y_{n-1} + y_n\}$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left\{(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\right\}$$

This formula is known as Simpson's $\frac{1}{3}^{rd}$ rule formula.

Simpson's $3/8^{th}$ Rule:

put, $n=3$ in (2)

· The values of x are x_0, x_1, x_2, x_3

· The 4^{th} order differences and higher order differences will becomes zero

$$\int_{x_0}^{x_0+3h} f(x)\,dx = h\left\{3y_0 + \tfrac{9}{2}\Delta y_0 + \left(\tfrac{27}{3} - \tfrac{9}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6}\right\}$$

$$= h\left\{3y_0 + \tfrac{9}{2}(y_1 - y_0) + \tfrac{9}{4}(y_2 - 2y_1 + y_0) + \tfrac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right\}$$

$$= h\left\{\frac{24y_0 + 36y_1 - 36y_0 + 18y_2 - 36y_1 + 18y_0 + 3y_3 - 9y_2 + 9y_1 - 3y_0}{8}\right\}$$

i.e.) $$\int_{x_0}^{x_0+3h} f(x)\,dx = \frac{h}{8}\{3y_0 + 9y_1 + 9y_2 + 3y_3\}$$

$$= \frac{3h}{8}\{y_0 + 3y_1 + 3y_2 + y_3\}$$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$
$= (y_2 - y_1) - (y_1 - y_0)$
$= y_2 - y_1 - y_1 + y_0$
$= y_2 - 2y_1 + y_0$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$= (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)$
$= \{(y_3 - y_2) - (y_2 - y_1)\}$
$- \{(y_2 - y_1) - (y_1 - y_0)\}$
$= y_3 - 3y_2 + 3y_1 - y_0$

lly) $$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\{y_3 + 3y_4 + 3y_5 + y_6\}$$

$$\int_{x_0+6h}^{x_0+9h} f(x)\,dx = \frac{3h}{8}\{y_6 + 3y_7 + 3y_8 + y_9\}$$

⋮

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\{y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\}$$

Adding all these equalities,

$$\int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \int_{x_0+6h}^{x_0+9h} f(x)\,dx + \dots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{3h}{8}\left\{(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + (y_6 + 3y_7 + 3y_8 + y_9) + \dots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\right\}$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\{(y_0+y_n)+3(y_1+y_4+\cdots+y_{n-2})+3(y_2+y_5+y_8+\cdots+y_{n-1})+2(y_3+y_6+\cdots+y_{n-3})\}$$ (23)

This formula is known as simpson's $3/8^{th}$ rule formula.

P.T the error in Trapezoidal rule is of order h^2.

Error in Trapezoidal Rule

Let the function be $y=f(x)$.

Expand $f(x)$ by Taylor's series about the pt x_0.

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\cdots \quad (1)$$

$$\therefore \int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} f(x_0)\,dx + \int_{x_0}^{x_1}(x-x_0)f'(x_0)\,dx + \int_{x_0}^{x_1}\frac{(x-x_0)^2}{2!}f''(x_0) + \int_{x_0}^{x_1}\frac{(x-x_0)^3}{3!}f'''(x_0)+\cdots$$

$$= f(x_0)\int_{x_0}^{x_1} dx + f'(x_0)\int_{x_0}^{x_1}(x-x_0)\,d(x-x_0) + \frac{f''(x_0)}{2}\int_{x_0}^{x_1}(x-x_0)^2\,d(x-x_0) + \frac{f'''(x_0)}{6}\int_{x_0}^{x_1}(x-x_0)^3\,d(x-x_0)$$

$[\because d(x-x_0)=dx]$.

$$= f(x_0)[x]_{x_0}^{x_1} + f'(x_0)\left[\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_1} + \left[\frac{(x-x_0)^3}{3}\right]_{x_0}^{2}\frac{f''(x_0)}{2} + \left[\frac{(x-x_0)^4}{4}\right]_{x_0}^{x_1}\frac{f'''(x_0)}{6}+\cdots$$

ie)
$$\int_{x_0}^{x_1} f(x)\,dx = (x_1-x_0)f(x_0)+\frac{(x_1-x_0)^2}{2}f'(x_0)+\frac{(x_1-x_0)^3}{6}f''(x_0)+\frac{(x-x_0)^4}{24}f'''(x_0)+\cdots$$

$$\int_{x_0}^{x_1} f(x)\,dx = hy_0+\frac{h^2}{2}y_0'+\frac{h^3}{6}y_0''+\frac{h^4}{24}y_0'''+\cdots$$

Trapezoidal Rule

put, $n=1$ in (2)

Derive Trapezoidal rule.

The values of x are x_0, x_1.

Here all the differences except Δy_0 will be zero.

$$x_0+h \Rightarrow \int_{x_0}^{x_0+h} f(x)\,dx = h\left\{y_0 + \tfrac{1}{2}\Delta y_0\right\}$$

$$= h\left\{y_0 + \tfrac{1}{2}(y_1-y_0)\right\}$$

$$= h\left\{\frac{2y_0+y_1-y_0}{2}\right\}$$

$$= \tfrac{h}{2}(y_0+y_1)$$

$|||^{ly}$, x_0+2h

$$\Rightarrow \int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1+y_2)$$

$$\int_{x_0+2h}^{x_0+3h} f(x)\,dx = \frac{h}{2}(y_2+y_3)$$

$\vdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1}+y_n)$$

Adding all these eqn's,

$$\int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\{y_0+y_1+y_1+y_2+y_2+y_3 + \cdots + y_{n-1}+y_n\}$$

$$\text{i.e.)} \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left\{(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right\}$$

Here $y_0 = 0.2$, $y_1 = 0.1429$, $y_2 = 0.1111$, $y_3 = 0.0909$,

$y_4 = 0.0769$, $y_5 = 0.0667$, $y_6 = 0.0588$, $y_7 = 0.0526$,

$y_8 = 0.0476$, $y_9 = 0.0435$, $y_{10} = 0.04$

w.r.T trapezoidal rule is,

$$\int_a^b f(x)\,dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\}$$

$$\int_0^5 \frac{1}{4x+5}\,dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\}$$

$$= \frac{0.5}{2}\{(0.2+0.04)+2(0.1429+0.1111+0.0909+0.0769+0.0667+0.0588+0.0526+0.0476+0.0435)\}$$

$$= 1/4\{0.24+2(0.691)\}$$

$$= 1/4(0.24+1.382) \Rightarrow 1/4(1.622)$$

$$= 0.4055$$

Ans:

$$\int_0^5 \frac{1}{4x+5}\,dx = 0.4055$$

3) Evaluate $\int_0^{\pi/2} \sin x\,dx$ by simpson's $1/3^{rd}$ rule dividing the range into 6 equal parts ($n=6$).

Sol.

G.T $n=6$.

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} \Rightarrow \pi/12$$

$$\boxed{h = \pi/12}$$

Let, $y(x) = \sin x$.

$\therefore$ The tabular column is as follows:

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$y=\sin x$	0	0.2588	0.5002	0.7071	0.8660	0.965629	1

$$= \frac{0.2}{2}\{(1+0.5)+2(0.9615)+0.8621+0.7353+0.6098)\}$$

$$= \frac{1}{10}\{1.5+6.3374\}$$ (26)

$$= \frac{1}{10}(7.8374)$$

$$= 0.7837$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.7837 \quad \text{---} (1)$$

Now, $\int_0^1 \frac{1}{1+x^2}dx = [\tan^{-1}x]_0^1$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \tan^{-1}(\tan \pi/4) - \tan^{-1}(\tan 0)$$

$$= \pi/4 - 0 = \pi/4 \quad \text{---} (2)$$

From (1) & (2),

$$\Rightarrow \pi/4 = 0.7837$$

$$\pi = 0.7837 \times 4$$

Ans:

$$\pi = 3.1348.$$

2) Evaluate $\int_0^5 \frac{dx}{4x+5}$ by using trapezoidal rule with 11 coordinates

Sol:

G.T there are 11 co-ordinates.

Number of intervals, $n = 10$.

Now, $h = \frac{b-a}{n} = \frac{5-0}{10} = 5/10 = 0.5$

$$\boxed{h = 0.5}$$

Let, $y(x) = \frac{1}{4x+5}$

The tabular column is as follows.

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = 1/(4x+5)$	0.2	0.1429	0.1111	0.0909	0.077	0.0667	0.0588	0.0526	0.0476	0.0435	0.04

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$$|E| = \left| -\frac{h^3}{12}\{y_0'' + y_1'' + y_2'' + \dots + y_{n-1}''\}\right| \le \frac{h^3}{12}\{|y_0''| + |y_1''| + \dots + |y_{n-1}''|\}$$

Let, $M = \text{Max}\{y_0'', y_1'', \dots y_{n-1}''\}$

$\left[x_n = x_0 + nh \right.$

$nh = x_n - x_0$

$h = \frac{x_n - x_0}{n}$

$h = \frac{b-a}{n}$

$\left. \Rightarrow n = \frac{b-a}{h} \right]$

$\therefore |E| \le n \frac{h^3}{12} M$

25

$\therefore |E| \le \left(\frac{b-a}{h}\right)\frac{h^3}{12} M$

(i.e) $|E| \le (b-a)\frac{h^2}{12} M$.

Note:

The Error in Trapezoidal rule is of order h^2.

Problems:

1) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = 0.2$.

Hence determine the value of π?

(or)

Evaluate Trapezoidal rule by taking 5 intervals $\left[h = \frac{1-0}{5} = \frac{1}{5} = 0.2\right]$

Sol:

Here $y(x) = \frac{1}{1+x^2}$ and $x_0 = 0$.

The tabular column as follows:

x	0	0.2	0.4	0.6	0.8	1.0
y	1	0.9615	0.8621	0.7353	0.6098	0.5

Here, $y_0 = 1$, $y_1 = 0.9615$, $y_2 = 0.8621$, $y_3 = 0.7353$,

$y_4 = 0.6098$, $y_5 = 0.5$

W.K.T the trapezoidal rule is,

$$\int_a^b f(x)\,dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$$

$$\int_0^7 \left(\frac{1}{4x+5}\right)dx = \frac{h}{2}\{(y_0+y_2) + 2(y_1 + y_2 + y_3$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\{(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\}$$

Here, $y_0=0$, $y_1=0.2588$, $y_2=0.5$, $y_3=0.7071$, $y_4=0.8660$,

$y_5=0.9656$, $y_6=1$.

(29)

W.K.T Simpson's $1/3^{rd}$ rule is,

$$\int_a^b f(x)\,dx = \frac{h}{3}\{(y_0+y_n)+2(y_2+y_4+\dots+y_{n-2})+4(y_1+y_3+\dots+y_{n-1})\}$$

$$\int_0^{\pi/2} \sin x\,dx = \frac{h}{3}\{(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\}$$

$$= \frac{\pi}{12\times 3}\{(0+1)+2(0.5+0.8660)+4(0.2588+0.7071+0.9659)\}$$

$$= \frac{180}{36}\{1+2.732+7.7272\}$$

$$= 5\{11.4592\}$$

$$= 57.296 \qquad = \frac{22}{7\times 36}(1+2.732+7.7272)$$

$$= \frac{22}{252}(11.4592)$$

$$\int_0^{\pi/2} \sin x\,dx = 57.296 = \frac{252.1024}{252} = 1.0004 \text{ (Approx)}.$$

4) Evaluate $\int_0^1 \frac{dx}{1+x}$ using i) Trapezoidal rule, ii) Simpson's $\frac{1}{3}^{rd}$ rule iii) Simpson's $3/8^{th}$ rule, iv) Find the error in each method by comparing with actual integration upto 4 places of decimals Take $h=1/6$ for all cases.

Sol:

$$h = \frac{b-a}{n} = \frac{1-0}{6} = 1/6.$$

let $y(x) = \frac{1}{1+x}$. G.T $h = 1/6$.

The tabular column is as follows:

x	0	$1/6$	$1/3$	$1/2$	$2/3$	$5/6$	1
$y=\frac{1}{1+x}$	1	$6/7$	$3/4$	$2/3$	$3/5$	$6/4$	0.5
		0.8571	0.75	0.6667	0.6	0.5455	

$$\int_{x_0}^{x_1} y\,dx = hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0'' + \frac{h^4}{24}y_0''' + \cdots \quad \text{—— (2)}$$

$\because x_1 - x_0 = x_0 + h - x_0 = h$

$f(x) = y$

$f(x_0) = y_0$

$f'(x_0) = y_0'$

$f''(x_0) = y_0''$

Now the area of the trapezium in the interval $[x_0, x_1]$ is,

(24)

$$A_0 = \int_{x_0}^{x_1} y\,dx = \frac{h}{2}[y_0 + y_1] \quad \text{—— (3)}$$

put, $x = x_1$ in (1),

$$\Rightarrow f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) + \frac{(x_1 - x_0)^3}{3!} f'''(x_0) + \cdots$$

ie) $y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$

Substituting this value of y_1 in (3),

$$\Rightarrow A_0 = \frac{h}{2}\left\{y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots\right\}$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{4}y_0'' + \frac{h^4}{12}y_0''' + \cdots \quad \text{—— (4)}$$

(2) − (4)

$$\Rightarrow \int_{x_0}^{x_1} y\,dx - A_0 = \left\{hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0'' + \frac{h^4}{24}y_0''' + \cdots\right\}$$

$$- \left\{hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{4}y_0'' + \frac{h^4}{12}y_0''' + \cdots\right\}$$

$$= \frac{1}{6} - \frac{1}{4}h^3 y_0'' + \cdots$$

$$= \frac{-1}{12}h^3 y_0'' + \cdots$$

The Error in the interval $(x_0, x_1) \simeq -\frac{h^3}{12}y_0'' + \cdots$

The Error in the interval $(x_1, x_2) \simeq -\frac{h^3}{12}y_1'' + \cdots$

The Error in the interval $(x_2, x_3) \simeq -\frac{h^3}{12}y_2'' + \cdots$

The Error in the interval $(x_{n-1}, x_n) \simeq -\frac{h^3}{12}y_{n-1}'' + \cdots$

$\therefore$ The total Error E is given by, $[\because |a+b| \le |a| + |b|]$

$$E = -\frac{h^3}{12}\left\{y_0'' + y_1'' + y_2'' + \cdots + y_{n-1}''\right\}$$

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iv) $\int_0^1 \frac{dx}{1+x} = \int_0^1 \frac{d(1+x)}{1+x}$

$\log 2 = 0.3010$

$\log e = 0.4343$

$\log_e 2 = 0.6931$

(30)

$= [\log(1+x)]_0^1$

$= \log_e 2 - \log_e 1$

$= 0.6931$

Trepezoidal rule, Error = Exact value − Approx value

$= 0.6931 - 0.6949$

$= -0.0018$

Simpson's $1/3^{rd}$ rule Error $= 0.6931 - 0.6932$

$= -0.0001$.

Simpson's $3/8^{th}$ rule, Error $= 0.6931 - 0.6932$

$= -0.0001$.

W.t.T Weddley's rule is,

$$\int_a^b f(x)dx = \frac{3h}{10}\{(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5)$$
$$+ (2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11}) + \ldots$$
$$+ (2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n$$

Note:

For Weddles rule put, $n=6$ in newton's cote's quadrature formula.

Problems:

1) i) Evaluate $\int_0^6 \frac{dx}{1+x^2} \Rightarrow \tan^{-1}x$ taking $h=1$ using

ii) Trapezoidal rule, iii) Simpson's $1/3^{rd}$ rule,

iv) Simpson's $3/8^{th}$ rule, iv) Weddle's rule, v) Also check up by exact integration which rule gives the value closets

$$= \frac{4.0986}{3}$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3662.$$

iii) W.K.T the Simpson's $3/8^{th}$ rule is,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\{(y_0+y_6)+3(y_1+y_4)+3(y_2+y_5)+2y_3)\}$$

$$= 3/8\{(1+0.0270)+3(0.5+0.0588)+3(0.2+0.0385)$$
$$+2(0.1)\}$$

$$= 3/8\{1.0270+3(0.5588)+3(0.2358)+0.2\}$$

$$= 3/8(3.6189)$$

$$= 1.3571$$

iv) W.K.T the weddles rule is,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\{y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\}$$

$$= \frac{3}{10}\{1+5(0.5)+0.2+6(0.1)+0.0588+$$
$$5(0.03846)+(0.027)\}$$

$$= 0.3\{1+2.5+0.2+0.6+0.0588+0.1925+0.027\}$$

$$= 0.3(4.5783)$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3735$$

v) $$\int_0^6 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^6$$

$$= \tan^{-1}6 - \tan^{-1}0$$

$$= \tan^{-1}6 - \tan^{-1}(\tan 0)$$

$$= \tan^{-1}6$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.4056,,$$

Here, $y_0 = 1$, $y_1 = 0.8571$, $y_2 = 0.75$, $y_3 = 0.6667$, $y_4 = 0.6$;

$y_5 = 0.5455$; $y_6 = 0.5$

i) W.K.T Trapezoidal rule is,

$$\int_a^b f(x)dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\}$$

$$\therefore \int_0^1 \frac{dx}{1+x} = \frac{1/6}{2}\{(1+0.5)+2(1+0.8571) + 0.75+0.6667+ 0.6+0.5455)\}$$

$$= \frac{1}{12}\{1.5+2(3.4193)\} = 1/12\,(8.3386)$$

$$= 0.69488$$

$$= 0.6949$$

ii) W.K.T the Simpson's 1/3rd rule is,

$$\int_a^b f(x)dx = \frac{h}{3}\{(y_0+y_n)+2(y_2+y_4+\dots+y_{n-2}) + 4(y_1+y_3+\dots+y_{n-1})\}.$$

$$\int_0^1 \frac{dx}{1+x} = \frac{1/6}{3}\{(1.5)+2(0.75+0.6)+4((0.8571)+0.6667+0.5455)\}$$

$$= \frac{1}{18}\{1.5+2.7+8.2772\} = 1/18\,(12.4772)$$

$$= 0.69317 = 0.6932$$

iii) W.K.T the Simpson's 3/8th rule is,

$$\int_a^b f(x)dx = \frac{3h}{8}\{(y_0+y_6)+3(y_1+y_4)+3(y_2+y_5)+2y_3\}$$

$$\int_0^1 \frac{dx}{1+x} = \frac{3(1/6)}{8}\{(1.5)+3(0.8571+0.6)+3(0.75+0.5455) + 2(0.6667)\}$$

$$= 1/16\,\{1.5+4.3731+3.8865+1.3334\}$$

$$= 1/16\,\{11.0912\} = 0.6932.$$

to the actual value.

sol:

(31)

Let $y(x) = \frac{1}{1+x^2} \Rightarrow \tan^{-1} x$ G.T $h=1$

The tabular column is as follows

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.0385	0.0270

Here $y_0 = 1$, $y_1 = 0.5$, $y_2 = 0.2$, $y_3 = 0.1$, $y_4 = 0.0588$,

$y_5 = 0.0385$, $y_6 = 0.0270$

i) W.K.T the trapezoidal rule is,

$$\int_a^b f(x)dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\}$$

$$\therefore \int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}\{(1+0.0385)+2(0.5+0.2+0.1+0.0588+0.0385)\}$$

$$= \frac{1}{2}(1.0270+2(0.8773))$$

$$= 0.5(1.0270+1.7346)$$

$$= 0.5(2.8216)$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.4108.$$

ii) W.K.T the simpson's $1/3$rd rule is,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\{(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\}$$

$$= \frac{1}{3}\{(1+0.0270)+2(0.2+0.0588)+4(0.5+0.1+0.0385)\}$$

$$= 1/3\{1.0270+2(0.2588)+4(0.03851)\}$$

$$= 1/3(1.0270+0.5176+2.554)$$

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