$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y}{4} + \dots\right\} \quad (1)$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \dots\right\} \quad (2)$$

The difference table is as follows

x	$y=\sqrt{x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
15	3.873				
		0.25 Δy_0			
17	4.123		-0.019 $\Delta^2 y_0$		
		0.231		0.017 $\Delta^3 y_0$	
19	4.354		-0.002		-0.031 $\Delta^4 y_0$
		0.229		-0.014	
21	4.583		-0.016		
		0.213			
23	4.796				

$\Rightarrow$

$$\frac{dy}{dx} = \frac{1}{2}\left[0.25 + \frac{0.019}{2} + \frac{0.017}{3} + \frac{0.031}{4}\right]$$

$$= \frac{1}{2}(0.25 + 0.0095 + 0.0057 + 0.0075)$$

10

$$\frac{dy}{dx} = 0.13635 = 0.1364 \text{ (app)}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{1}{4}\left[-0.019 - 0.017 - \frac{11}{12}(0.031)\right]$$

$$= \frac{1}{4}[-0.019 - 0.017 - 0.0284]$$

$$\frac{d^2 y}{dx^2} = -0.0161$$

ANS:

$\frac{dy}{dx} = 0.1364$, $\frac{d^2 y}{}$

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$$= \frac{1}{0.04}\left\{-0.0060 - \frac{0.0015}{2} + \frac{0.0001}{6} + \frac{0.0002}{12}\right\}$$

$$= \frac{1}{0.04}\left\{-0.0060 - 0.00075 + 0.000167 + 0.00016\right\}$$

$$= \frac{1}{0.04}\left\{-0.00675 + 0.000334\right\}$$

$$= \frac{1}{0.04}\left\{-0.006716\right\}$$

$\cos(1.74) = -0.167915$

Ans:

$\cos(1.74) = -0.1679$ (approximation)

2) Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x=15$ where $y=\sqrt{x}$.

Find the first & Second derivatives of $\sqrt{x}$ at $x=15$ from the following data.

x	15	17	19	21	23
$\sqrt{x}$	3.873	4.123	4.354	4.583	4.796

Sol:

Here $x=15$, is nearer to the beginning of the table we use Newton's forward formula.

Here $x_0=15$, $x_1=17$, $x_2=19$, $x_3=21$, $x_4=23$.

Also, $x=15$ & $h=2$.

$$p = \frac{x-x_0}{h} = \frac{15-15}{2} = 0.$$

Now $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$ —— ①

$$\frac{dp}{dx} = \frac{d}{dx}\left\{\frac{x - x_n}{h}\right\}$$

$$= \frac{1}{h}\left\{\frac{d}{dx}(x) - \frac{d}{dx}(x_n)\right\}$$

$$\frac{dp}{dx} = \frac{1}{h}.$$

$$\frac{dy}{dp} = \frac{d}{dp}(y_n) + \frac{d}{dp}(P)\nabla y_n + \frac{1}{2}\left\{\frac{d}{dp}(P^2) + \frac{d}{dp}(P)\right\}\nabla^2 y$$
$$+ \frac{1}{6}\left\{\frac{d}{dp}(P^3) + 3\frac{d}{dp}(P^2) + 2\frac{d}{dp}(P)\right\}\nabla^3 y_n$$
$$+ \frac{1}{24}\left\{\frac{d}{dp}(P^4) + 6\frac{d}{dp}(P^3) + 11\frac{d}{dp}(P^2) + 6\frac{d}{dp}(P)\right\}\nabla^4 y_n + \dots$$

$$= \nabla y_n + \frac{1}{2}\left\{2P+1\right\}\nabla^2 y_n + \frac{1}{6}\left\{3P^2+6P+2\right\}\nabla^3 y_n + \frac{1}{24}\left\{4P^3+18P^2+22P+6\right\}\nabla^4 y_n + \dots$$

$$\frac{dy}{dp} = \nabla y_n + \frac{2P+1}{2}\nabla^2 y_n + \frac{(3P^2+6P+2)}{6}\nabla^3 y_n + \frac{4P^3+18P^2+22P+6}{24}\nabla^4 y_n + \dots$$

∴ ① becomes,

*
$$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{(2P+1)}{2}\nabla^2 y_n + \frac{(3P^2+6P+2)}{6}\nabla^3 y_n + \frac{4P^3+18P^2+22P+6}{24}\nabla^4 y_n + \dots\right\}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

table we use Newtons' forward interpolation formula,

(i.e.) $\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^2 y_0 + \frac{(2p^3-9p^2+11p-3)}{12}\Delta^4 y_0 + \cdots\right\}$ —①

where $p = \frac{x-x_0}{h}$.

Here, $p = \frac{x-x_0}{h} = \frac{1.74-1.70}{0.04} = \frac{0.04}{0.04} = 1.$

$\boxed{p=1}$

8

The difference table is as follows:

x	$y=\sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.70	0.9917				
		-0.0060 Δy_0			
1.74	0.9857		-0.0015 $\Delta^2 y_0$		
		-0.0075		-0.0001 $\Delta^3 y_0$	
1.78	0.9782		-0.0016		$+0.0002$ $\Delta^4 y_0$
		-0.0091		0.0001	
1.82	0.9691		-0.0015		
		-0.0106			
1.86	0.9585				

∴ ① becomes,

$\cos(1.74) = \frac{1}{0.04}\left\{-0.0060 + \frac{2(1)-1}{2}(-0.0015) + \frac{3(1)^2-6(1)+2}{6}(-0.0001) + \frac{2(1)^3-9(1)^2+11(1)-3}{12}(0.0002)\right\}$

$= \frac{1}{0.04}\left\{-0.0060 + \frac{1}{2}(-0.0015) - \frac{1}{6}(-0.0001) + \frac{1}{12}(0.0002)\right\}$

To find $\frac{dy}{dx}$ at $x = x_0$

Here $p = \frac{x - x_0}{h}$

$\Rightarrow \quad = \frac{x_0 - x_0}{h} = 0.$

3

$* \therefore \frac{dy}{dx}$ at $x = x_0 \Rightarrow \quad = \frac{1}{h}\left\{ \Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{-1}{4}\Delta^4 y_0 + \cdots \right\}$

$* \therefore \frac{d^2y}{dx^2}$ at $x = x_0 \Rightarrow \quad = \frac{1}{h^2}\left\{ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots \right\}$

sum J1(7M)

2) W.k.T. the Newton's backward interpolation formula is,

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \cdots$$

where $p = \frac{x - x_n}{h}$

ie) $y_x = y_n + p\nabla y_n + \frac{p^2+p}{2!}\Delta^2 y_n + \frac{(p^2+p)(p+2)}{6}\nabla^3 y_n + \frac{(p^2+p)(p^2+3p+2p+6)}{24}\nabla^4 y_n + \cdots$

$$y_x = y_n + p\nabla y_n + \frac{p^2+p}{2}\nabla^2 y_n + \frac{(p^3+3p^2+2p)}{6}\nabla^3 y_n + \frac{p^4+5p^3+6p^2+p^3+5p^2+6p}{24}\nabla^4 y_n + \cdots$$

$$y_x = y_n + p\nabla y_n + \frac{p^2+p}{2}\nabla^2 y_n + \frac{p^3+3p^2+2p}{6}\nabla^3 y_n + \frac{p^4+6p^3+11p^2+6p}{24}\nabla^4 y_n + \cdots$$

$\Rightarrow$

$$\frac{dy}{dx} = 0.02 - \frac{0.01}{2} - \frac{0.02}{3} - \frac{0.03}{4}$$

$$= 0.02 - 0.005 - 0.0067 - 0.0075$$

$$\frac{dy}{dx} = \cancel{-0.0442}. \quad = 0.0008$$

$$\Rightarrow \frac{d^2y}{dx^2} = 0.01 + 0.02 + \frac{11}{12}(0.03)$$

$$\frac{d^2y}{dx^2} = 0.0575$$

12

ANS:

$$\frac{dy}{dx} = \cancel{-0.0442}, \quad \frac{d^2y}{dx^2} = 0.0575$$
$$= 0.0008$$

01.08.2016

4) From the following data obtain the first and Second derivatives of $y = \log_e x$ i) at $x = 500$, ii) $x = 550$.

x	500	510	520	530	540	550
$y = \log_e x$	6.2146	6.2344	6.2358	6.2729	6.2916	6.3099.

Also, calculate the actual value of the derivatives at this point.

Sol:

i) Here $x = 500$ is nearer to the begin~~nig~~ing of the table we use Newton's forward formula.

Here $x_0 = 500$, $x_1 = 510$, $x_2 = 520$, $x_3 = 530$, $x_4 = 540$, ie) $\dfrac{dy}{dp} = \Delta y_0 + \frac{1}{2}\{2P-1\}\Delta^2 y_0 + \frac{1}{6}\{3P^2-6P+2\}\Delta^3 y_0 + \frac{1}{24}\{4P^3-18P^2+22P-6\}\Delta^4 y_0 + \cdots$

ie) $\dfrac{dy}{dp} = \Delta y_0 + \dfrac{(2P-1)}{2}\Delta^2 y_0 + \dfrac{(3P^2-6P+2)}{6}\Delta^3 y_0 + \dfrac{1}{12}\{2P^3-9P^2+11P-3\}\Delta^4 y_0 + \cdots$

2

$\therefore$ ① becomes,

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}.$$

$$* \quad \frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2P-1)}{2}\Delta^2 y_0 + \frac{(3P^2-6P+2)}{6}\Delta^3 y_0 + \frac{1}{12}\{2P^3-9P^2+11P-3\}\Delta^4 y_0 + \cdots\right\}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{d}{dp}\left\{\frac{1}{h}\left[\Delta y_0 + \frac{(2P-1)}{2}\Delta^2 y_0 + \frac{(3P^2-6P+2)}{6}\Delta^3 y_0 + \frac{1}{12}\{2P^3-9P^2+11P-3\}\Delta^4 y_0 + \cdots\right]\right\}\frac{1}{h}$$

$$= \frac{1}{h^2}\left\{\frac{d}{dp}(\Delta y_0) + \frac{1}{2}\left\{2\frac{d}{dp}(P) - \frac{d}{dp}(1)\right\}\Delta^2 y_0 + \frac{1}{6}\left\{3\frac{d}{dp}(P^2) - 6\frac{d}{dp}(P) + \frac{d}{dp}(2)\right\}\Delta^3 y_0 + \frac{1}{12}\left\{2\frac{d}{dp}(P^3) - 9\frac{d}{dp}(P^2) + 11\frac{d}{dp}(P) - \frac{d}{dp}(3)\right\}\Delta^4 y_0 + \cdots\right\}$$

$$= \frac{1}{h^2}\left\{\Delta^2 y_0 + \frac{1}{6}\{6P-6+0\}\Delta^3 y_0 + \frac{1}{12}\{6P^2+18P+11-0\}\Delta^4 y_0 + \cdots\right\}$$

$$* \quad \frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (P-1)\Delta^3 y_0 + \frac{1}{12}\{6P^2-18P+11\}\Delta^4 y_0 + \cdots\right\}$$

3) $f'(50)$ and $f''(50)$ from the following table

x	50	51	52	53	54
$f(x)$	3.68	3.70	3.73	3.75	3.77

Sol:

ANS: $\frac{d^2y}{dx^2} = 0.05751$ $\frac{dy}{dx} = 0.0008$

Here $x = 50$, is nearer to the begining of the table we use Newton's forward formula

Here $x_0 = 50$, $x_1 = 51$, $x_2 = 52$, $x_3 = 53$, $x_4 = 54$.

Also, $x = 50$ & $h = 1$

$$P = \frac{x - x_0}{h} = \frac{50 - 50}{1} = 0.$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right\} \quad \text{--- (1)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right\} \quad \text{--- (2)}$$

The diff table is as follows

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	3.68				
		0.02 Δy_0			
51	3.70		0.01 $\Delta^2 y_0$		
		0.03		−0.02 $\Delta^3 y_0$	
52	3.73		−0.01		0.03 $\Delta^4 y_0$
		0.02		0.01	
53	3.75		0		
		0.02			
54	3.77				

27.07.2016 UNIT-4

II
1M → Stirling's formula
7M → fwd, bwd

NUMERICAL DIFFERENTIATION

Study II (7M)

1) W.k.T. the Newtons forward interpolation formula is,

Find
i) 1st derivative
ii) 2nd derivative

$$y(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0 + \cdots$$

where $P = \frac{x - x_0}{h}$.

$$y(x) = y_0 + P\Delta y_0 + \frac{P^2 - P}{2}\Delta^2 y_0 + \frac{(P^3 - 3P^2 + 2P)}{6}\Delta^3 y_0 + \frac{(P^4 - 6P^3 + 11P^2 - 6P)}{24}\Delta^4 y_0 + \cdots$$

To find $\frac{dy}{dx}$:

Central interpolation
* forward ⇒ $0 < P < 1$
* backward ⇒ $-1 < P < 0$
* stirling ⇒ $-\frac{1}{4} < P < \frac{1}{4}$

Now $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$ ——— ①

Now $\frac{dp}{dx} = \frac{d}{dx}\left\{\frac{x - x_0}{h}\right\}$

$$= \frac{1}{h}\left\{\frac{d}{dx}(x) - \frac{d}{dx}(x_0)\right\}$$

$$= \frac{1}{h}[1] = \frac{1}{h}$$

$$\frac{dy}{dp} = \frac{d}{dp}(y_0) + \frac{d}{dp}(P)\Delta y_0 + \frac{1}{2}\left\{\frac{d}{dp}(P^2) - \frac{d}{dp}(P)\right\}\Delta^2 y_0 + \frac{1}{6}\left\{\frac{d}{dp}(P^3) - 3\frac{d}{dp}(P^2) + 2\frac{d}{dp}(P)\right\} + \frac{1}{24}\left\{\frac{d}{dp}(P^4) - 6\frac{d}{dp}(P^3) + 11\frac{d}{dp}(P^2) - 6\frac{d}{dp}(P)\right\}\Delta^4 y_0 + \cdots$$

$$= \frac{d}{dp}\left\{\frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\nabla^4 y_n + \cdots\right.\right.$$

$$= \frac{1}{h^2}\left\{\frac{d}{dp}(\nabla y_n) + \frac{1}{2}\left\{2\frac{d}{dp}(p) + \frac{d}{dp}(1)\right\}\nabla^2 y_n + \right.$$

$$\frac{1}{6}\left\{3\frac{d}{dp}(p^2) + 6\frac{d}{dp}(p) + \frac{d}{dp}(2)\right\}\nabla^3 y_n +$$

$$\frac{1}{24}\left\{4\frac{d}{dp}(p^3) + 18\frac{d}{dp}(p^2) + 22\frac{d}{dp}(p) + \frac{d}{dp}(6)\right\}\nabla^4 y_n + \cdots$$

$$= \frac{1}{h^2}\left[0 + \frac{1}{2}(2)\nabla^2 y_n + \frac{1}{6}\{6p+6\}\nabla^2 y_n + \frac{1}{24}\{12p^2+36p+22\}\nabla^4 y_n + \cdots\right.$$

* $$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{1}{12}\{6p^2+18p+11\}\nabla^4 y_n + \cdots\right\}$$

At $x = x_n$

$$p = \frac{x - x_n}{h} = 0 \Rightarrow \boxed{p=0}$$

* $\frac{dy}{dx}$ at $x = x_n \Rightarrow$ $$= \frac{1}{h}\left\{\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right\}$$

* $\frac{d^2y}{dx^2}$ at $x = x_n \Rightarrow$ $$= \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right\}$$

3) STIRILING'S FORMULA:

(change $\nabla \to \Delta$)

W.K.T. Strilings formula is, (-1, -2 are in suffix)

$$y_x = y_0 + p\left(\frac{\nabla y_0 + \nabla y_{-1}}{2}\right) + \frac{p^2}{2!}\nabla^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\nabla^3 y_{-1} + \nabla^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\nabla^4 y_{-2} + \cdots$$

where $p = \frac{x - x_0}{h}$

6

where $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$ —— (1)

$$\frac{dp}{dx} = \frac{d}{dx}(p) = \frac{1}{h}\frac{d}{dx}(x - x_0) = \frac{1}{h}(1)$$

$$\frac{dp}{dx} = \frac{1}{h}.$$

$$\frac{dy}{dp} = \frac{d}{dp}\left\{ y_0 + p\left(\frac{\nabla y_0 + \nabla y_{-1}}{2}\right) + \frac{p^2}{2!}\nabla^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\nabla^3 y_{-1} + \nabla^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\nabla^4 y_{-2} + \cdots \right\}$$

$$= \frac{d}{dp}(y_0) + \frac{d}{dp}(p)\left(\frac{\nabla y_0 + \nabla y_{-1}}{2}\right) + \frac{1}{2}\frac{d}{dp}(p^2)\nabla^2 y_{-1} + \frac{1}{6}\left\{\frac{d}{dp}(p^3) - \frac{d}{dp}(p)\right\}\frac{\nabla^3 y_{-1} + \nabla^3 y_{-2}}{2} + \frac{1}{24}\left\{\frac{d}{dp}(p^2)\right\}\nabla^4 y_{-2} + \cdots$$

$$= \frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{1}{2}\left\{2p(\Delta^2 y_{-1})\right\} + \frac{1}{6}\left\{3p^2 - 1\right\}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{24}\left\{4p^3 - 2p\right\}\Delta^4 y_{-2} + \cdots$$

$$= \frac{\Delta y_0 + \Delta y_{-1}}{2} + p(\Delta^2 y_{-1}) + (3p^2 - 1)\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12}\right] + (2p^3 - p)\frac{\Delta^4 y_{-2}}{12} + \cdots$$

① becomes,

$$* \quad \frac{dy}{dx} = \frac{1}{h}\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2} + p(\Delta^2 y_{-1}) + (3p^2 - 1)\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12}\right] + (2p^3 - p)\left[\frac{\Delta^4 y_{-2}}{12} + \cdots\right]\right.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

29.07.2016

$$= \frac{1}{h^2}\left\{\frac{d}{dp}\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right) + \frac{d}{dp}(p)(\Delta^2 y_{-1})\right.$$

$$+\left\{3\frac{d}{dp}(p^2) - \frac{d}{dp}(1)\right\}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{12}\right)$$

$$+ \left\{2\frac{d}{dp}(p^3) - \frac{d}{dp}(p)\right\}\left(\frac{\Delta^4 y_{-2}}{12}\right)+\cdots$$

$$= \frac{1}{h^2}\left\{\Delta^2 y_{-1} + 6p\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{\cancel{12}_2}\right) + (6p^2-1)\frac{\Delta^4 y_{-2}}{12}+\cdots\right.$$

I (1M)

$$* \frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_{-1} + \frac{p}{2}\left(\Delta^3 y_{-1}+\Delta^3 y_{-2}\right) + (6p^2-1)\frac{\Delta^4 y_{-2}}{12}+\cdots\right.$$

At $x=x_0$, $p=0$.

$\left[\therefore \left(\frac{dy}{dx}\right)$ at $x=x_0 = \frac{1}{h}\right]$

I (1M)

$$* \therefore \left(\frac{dy}{dx}\right)_{\text{at } x=x_0} = \frac{1}{h}\left\{\frac{\Delta y_0+\Delta y_{-1}}{2} - \left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{12}\right)+\cdots\right\}$$

I (1M)

$$* \left(\frac{d^2y}{dx^2}\right)_{\text{at } x=x_0} = \frac{1}{h^2}\left\{\Delta^2 y_{-1} - \frac{\Delta^4 y_{-2}}{12}+\cdots\right\}$$

PROBLEMS:

1) Find the value Cos (1.74) from the following table.

x	1.70	1.74	1.78	1.82	1.86
$\sin x$	0.9917	0.9857	0.9782	0.9691	0.9585

Sol:

Here $x_0=1.70$, $x_1=1.74$, $x_2=1.78$, $x_3=1.82$, $x_4=1.86$

$h=0.04$ and $x=1.74$.

Here $x=1.74$ is nearer to begining of the

Here $x_0 = 1931,\ x_1 = 1941,\ x_2 = 1951,\ x_3 = 1961,$

$x_4 = x_n = 1971$

$$p = \frac{x - x_n}{h} = \frac{-(1971 - 1961)}{10} = \frac{-10}{10} = -1$$

$$\left(\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{3}{2}\nabla^2 y_n + \frac{11}{6}\nabla^3 y_n + \frac{25}{11}\nabla^4 y_n\right\}\right.$$

$d^2]x$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{6}\nabla^3 y_n - \frac{1}{12}\nabla^4 y_n + \cdots\right\}$$

$$\frac{d^2y}{dx^2} =$$

The diff table is as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
1951	79.95		4.46		-4.47 $\nabla^4 y_n$
		23.61		1.02 $\nabla^3 y_n$	
1961	103.56		5.48 $\nabla^2 y_n$		
		29.09 ∇y_n			
1971	132.65				

$$\frac{dy}{dx} = \frac{1}{10}\left[29.09 - \frac{5.48}{2} - \frac{1.02}{6} + \frac{4.47}{12}\right]$$

Ans: $= \frac{1}{10}\left[29.09 - 2.74 - 0.17 + 0.3725\right]$

$$\frac{dy}{dx} = 2.65525$$

13

Also, $x = 500$, $h = 10$.

$$p = \frac{x - x_0}{h} = \frac{500-500}{10} = 0$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right\} \quad \text{--- (1)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right\} \quad \text{--- (2)}$$

The diff table is as follows.

Forward

x	$y = \log_e x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
500	6.2146					
		0.0198 (Δy_0)				
510	6.2344		−0.0184 ($\Delta^2 y_0$)			
		0.0014		0.0541 ($\Delta^3 y_0$)		
520	6.2358		0.0357		−0.1082 ($\Delta^4 y_0$)	
		0.0371		−0.0541		0.180
530	6.2729		−0.0184		0.0721	
		0.0187		0.018		
540	6.2916		−0.0004			
		0.0183				
550	6.3099					

Backward

4) From the following data obtain the 1st & 2nd derivatives of $y = \log_e x$ i) at $x = 500$, ii) $x = 550$.

x	500	510	520	530	540	550
$y = \log_e x$	6.2146	6.2344	6.2538	6.2729	6.2916	6.3099

Also, calculate the actual value of the derivatives at this pt.

Sol:

i) Here $x=500$ is nearer to the begining of the table. we use Newton's forward formula.

Here $x_0=500$, $x_1=510$, $x_2=520$, $x_3=530$, $x_4=540$,

$x_5=550$.

Also, $x=500$, $h=10$.

$$P=\frac{x-x_0}{h}=\frac{500-500}{10}=0.$$

$$\frac{dy}{dx}=\frac{1}{h}\left\{\Delta y_0-\frac{\Delta^2 y_0}{2}+\frac{\Delta^3 y_0}{3}-\frac{\Delta^4 y_0}{4}+\frac{\Delta^5 y_0}{5}\right\}$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\Delta^2 y_0-\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0\right\}$$

ii) Here $x=550$ is nearer to the begining of the table we use Newton's backward ~~forward~~. formula.

~~Here $x=550$,~~ $x_n=550$

$$P=\frac{x-x_n}{h}=\frac{550-550}{10}=0.$$

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\frac{1}{5}\nabla^5 y_n\right\}$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n\right\}$$

14

x	$y = \log_e x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
500	6.2146					
		0.0198 Δy_0				
510	6.2344		-0.0004 $\Delta^2 y_0$			
		0.0194		0.0001 $\Delta^3 y_0$		
520	6.2538		-0.0003		-0.0002 $\Delta^4 y_0$	
		0.0191		-0.0001		0.0003 $\Delta^5 y_0$ $\nabla^5 y_n$
530	6.2729		-0.0004		0.0001 $\nabla^4 y_n$	
		0.0187		0 $\nabla^3 y_n$		
540	6.2916		-0.0004 $\nabla^2 y_n$			
		0.0183 ∇y_n				
550	6.3099					

~~Ans:~~

i) Forward formula:

$$\frac{dy}{dx} = \frac{1}{10}\left[0.0198 + 0.0002 + \frac{0.0001}{3} - \frac{0.0002}{4} + \frac{0.0003}{5}\right]$$

$$= \frac{1}{10}\left[0.0198 + 0.0002 + 0.00003 - 0.00005 + 0.00006\right]$$

$$\frac{dy}{dx} = 0.002004$$

$$\cancel{\frac{dy}{dx}} =$$

5

$$\frac{d^2y}{dx^2} = \frac{1}{100}\left[-0.0004 - 0.0001 - \frac{11}{12}(0.0002)\right]$$

$$= \frac{1}{100}\left[-0.0004 - 0.0001 - 0.00018\right]$$

0.00045

$$\frac{d^2y}{dx^2} = 0.0000068.$$

~~4.5~~

ii) Backward:

$$\frac{dy}{dx} = \frac{1}{10}\left[0.0183 - \frac{0.0004}{2} + 0 + \frac{0.0001}{4} + \frac{0.0003}{5}\right]$$ 0.0009

21

where, $p = \frac{x - x_0}{h} = \frac{x-0}{1} = x$,

$$\boxed{P = x}$$

$$y(x) = y_0 + x\Delta y_0 + \frac{x(x-1)}{2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{6}\Delta^3 y_0 + \frac{x(x-1)(x-2)(x-3)}{24}\Delta^4 y_0 + \cdots \quad (4).$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2x-1)}{2}\Delta^2 y_0 + \frac{(3x^2-6x+2)}{6}\Delta^3 y_0 + \frac{2x^3-9x^2+11x-3}{12} + \cdots\right\} \quad (5)$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (x-1)\Delta^3 y_0 + \frac{6x^2-18x+11}{12}\Delta^4 y_0 + \cdots\right\} \quad (6)$$

The tabular column as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0 y_0					
		0.25 Δy_0				
1	0.25		−0.5 $\Delta^2 y_0$			
		−0.25		3 $\Delta^3 y_0$		
2	0		2.5		6 $\Delta^4 y_0$	
		2.25		9		0 $\Delta^5 y_0$
3	2.25		11.5		6	
		13.74		15		
4	16		26.5			
		40.25				
5	56.25					

Now (5) becomes,

$$\frac{dy}{dx} = \frac{1}{1}\left\{0.25 + \frac{(2x-1)}{2}(-0.5) + \frac{3x^2-6x+2}{6}(3) + \frac{2x^3-9x^2-11x-3}{12}(6)\right\} = 0.$$

03.08.2016

6) A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of time t (seconds).

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

Calculate the angular velocity and the acceleration of the rod when $t=0.6$ seconds

Sol:

Here $t=0.6$ is in the middle of the table we use ~~Stir~~ Striling's formula to find angular velocity and acceleration

ie) To find $\left(\frac{d\theta}{dt}\right)_{at\ t=0.6}$ and $\left(\frac{d^2\theta}{dt^2}\right)_{at\ t=0.6}$

Now, $p = \frac{t-t_0}{h}$.

Choose $t_0 = 0.6$ and here $h=0.2$.

18

$$p = \frac{0.6-0.6}{0}$$

$$\boxed{p=0}$$

W.K.T. $\left(\frac{d\theta}{dt}\right)_{at\ t=t_0} = \frac{1}{h}\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12} + \dots\right\}$ — (1)

and $\left(\frac{d^2\theta}{dt^2}\right)_{at\ t=t_0} = \frac{1}{h^2}\left\{\Delta^2 y_{-1} - \frac{\Delta^4 y_{-2}}{12} + \dots\right\}$ — (2)

$$= 0.25 + \frac{(2x-1)}{2}(-0.5) + \frac{3x^2-6x+2}{2} + \frac{2x^3-9x^2+11x-3}{2}$$

$$= 0.25 - \frac{(2x-1)}{4} + \frac{3x^2-6x+2}{2} + \frac{2x^3-9x^2+11x-3}{2} = 0$$

$$= \frac{1-2x+1+6x^2-12x+4+4x^3-18x^2+22x-6}{4} = 0.$$

$$\frac{4x^3-12x^2+8x}{4} = 0$$

22

$$4x^3-12x^2+8x = 0.$$

$$4x(x^2-3x+2) = 0.$$

$$x(x^2-3x+2) = 0.$$

$$x = 0 \text{ (or) } x^2-3x+2 = 0.$$

$$x = 0, 1, 2.$$

i) When $x=0$, $\frac{d^2y}{dx^2} = \{-0.5-(3)+\frac{11}{12}(6)\}$

$$= [-0.5-3+11/2] = \frac{-1-6+11}{2} = 4/2$$

$$= 2 > 0.$$

ii) When $x=1$, $\frac{d^2y}{dx^2} = [-0.5-\frac{1}{12}(6)] = -0.5-\frac{1}{2}.$

$$= \frac{-1-1}{2} = \frac{-2}{2} = -1 < 0.$$

iii) When $x=2$, $\frac{d^2y}{dx^2} = [-0.5+3-\frac{1}{12}(6)] = [-0.5+3-1/2]$

$$= \frac{-1+6-1}{2} = 4/2 = 2 > 0$$

∴ $y(x)$ attains min, when $x=0, 2$ & $y(x)$ attains max, when $x=1$ the max value is

$= 0 + 2(0.25) + \frac{2}{2}(-0.5) = 0.5 - 0.5 = 0.$

$\therefore y(0) = 0,\ y(2) = 0,\ y(1) = 0.25.$

× ——— × ——— ×

(Unit - 4 Continuation in new NA Note)

23

$40y_3 = 12240$

$y_3 = \frac{12240}{40}.$

$\boxed{y_3 = 306.}$

y_3 in ①

$\Rightarrow y_5 + 10(306) = 3450.$

$\boxed{y_5 = 390}$

[T12.

$$= \frac{1}{10}\left[0.0183 - 0.0002 + 0.000025 + 0.00000\ldots\right.$$

$$\frac{dy}{dx} = 0.001819.$$

$$\frac{d^2y}{dx^2} = \frac{1}{100}\left[-0.0004 + \frac{11}{12}(0.0001)\right]$$

$$\frac{d^2y}{dx^2} = -0.00000308$$

16

Ans:

i) Forward (at $x = 500$)

$$\frac{dy}{dx} = 0.002, \quad \frac{d^2y}{dx^2} = 0.0000068.$$

ii) Backward (at $x = 550$)

$$\frac{dy}{dx} = 0.001819, \quad \frac{d^2y}{dx^2} = -0.00000308.$$

5) The population in millions of a certain town is shown in the following table. Find the rate of growth of the population in 1961.

Year	1931	1941	1951	1961	1971
Population (y)	40.62	60.80	79.95	103.56	132.65

To find $\frac{dy}{dx}$ (velocity).

Sol:

Here $x = 1961$ is nearer to the begining of the table we use Newton's backward formula.

The tabular column as follows:

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
0	0					
		0.12				
0.2	0.12		0.25			
		0.37		0.01 $\Delta^3\theta_{-2}$		
0.4	0.49		0.26		0 $\Delta^4\theta_{-2}$	0
		0.63 $\Delta\theta_{-1}$		0.01 $\Delta^3\theta_{-2}$		
0.6	1.12		0.27 $\Delta^2\theta_{-1}$		0 $\Delta^4\theta_{-2}$	
		0.9 $\Delta\theta_0$		0.01 $\Delta^3\theta_{-1}$		
0.8	2.02		0.28 $\Delta^2\theta_0$			
		1.18				
1.0	3.20					

Here $\Delta\theta_0 = 0.9$, $\Delta\theta_{-1} = 0.63$, $\Delta^3\theta_{-1} = 0.01$,

$\Delta^3\theta_{-2} = 0.01$, $\Delta^2\theta_{-1} = 0.27$, $\Delta^4\theta_{-2} = 0$.

$\therefore$ ① becomes,

$$\left(\frac{d\theta}{dt}\right)_{at\ t=0.6} = \frac{1}{h}\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{12} + \cdots\right.$$

$$= \frac{1}{0.2}\left\{\frac{0.9+0.63}{2} - \frac{0.01+0.01}{12}\right\}$$

$$= \frac{1}{0.2}\left\{\frac{1.53}{2} - \frac{0.02}{12}\right\}$$

$$= \frac{1}{0.2}\left\{0.765 - 0.00167\right\}$$

$$= 5\left[0.765 - 0.00167\right]$$

$$= 5 \times 0.76333$$

$$= 3.81663.$$

ANS:

$$\left(\frac{d\theta}{dt}\right)_{at\ t=0.6} = 3.82 \text{ radians/sec.}$$

19

$$x = \frac{48 \pm \sqrt{(48)^2 - 4(6)(67)}}{2(6)} = \frac{48 \pm \sqrt{2304 - 1608}}{12}$$

$$= \frac{48 \pm \sqrt{696}}{12} = \frac{48 \pm 26.4}{12}$$

$$= \frac{48 + 26.4}{12}, \frac{48 - 26.4}{12}.$$

$$= \frac{74.4}{12}, \frac{21.6}{12}.$$

26

$$x = 6.2, 1.8.$$

i) When $x = 1.8$,

$$\frac{d^2y}{dx^2} = 1\left[12 + (1.8 - 1)(-4)\right] = 12 + 0.8(-4) = 12 - 3.2$$
$$= 8.8 > 0.$$

ii) When $x = 6.2$

$$\frac{d^2y}{dx^2} = 1\left[12 + (6.2 - 1)(-4)\right] = 12 + 5.2(-4) = 12 - 20$$
$$= -8.8 < 0.$$

$y(x)$ attains ~~maximum~~ min value when $x = 1.8$ & $y(x)$ attains ~~min~~ max value ~~at~~ when $x = 6.2$.

The min value is $= 58 + (1.8)(-1.5) + \frac{(1.8)(1.8-1)}{2}$

$$+ \frac{1.8(1.8-1)(1.8-2)}{6}(-4)$$

$$= 58 - 27 + (1.8)(0.8)6 + \frac{1.8(0.8)(-0.2)(-4)}{6}.$$

$$= 58 - 27 + 8.64 + \frac{1.152}{6}.$$

$$= 58 - 27 + 8.64 + 0.192$$

$$= 39.832.$$

The tabular column is as follows:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3			
2	3	0		
4	11	8	8	
6	27	16	8	0

28

$\therefore$ (5) becomes,

$$\frac{dy}{dx}=0 \Rightarrow \frac{1}{h}\left\{\Delta y_0 + \frac{2\left(\frac{x}{2}\right)-1}{2}\Delta^2 y_0 + \cdots\right\}=0.$$

$$\frac{1}{2}\left\{0+\frac{x-1}{2}(8)\right\}=0.$$

$$\frac{1}{2}(x-1)4=0.$$

$$4x-4=0 \Rightarrow 4x=4 \Rightarrow \boxed{x=1}$$

when $x=1$,

$$\frac{d^2y}{dx^2}=\frac{1}{4}\left\{8+(x-1)(0)\right\}$$

$$=\frac{1}{4}\left\{8+\left(\frac{1}{2}-1\right)0\right\}$$

$$=\frac{1}{4}(8)=2>0.$$

$\therefore y(x)$ attain max, when $x=1$,

The min value is $= 3+\left\{\frac{1}{2}(0)\right\}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}(8)+0$

$$=3+\frac{1}{2}\left(-\frac{1}{2}\right)4$$

$$=3-1$$

$$y(1)=2$$

2) From the following table find the maximum value of $f(x)$.

x	0	1	2	3	4	5
$f(x)$	58	43	40	45	52	60.

24

Sol:

Here $x_0=0, x_1=1, x_2=2, x_3=3, x_4=4$ & $h=1, x_5=5$

W.K.T. Newton's forward interpolation formula is,

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{6}\Delta^3 y_0+\frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0+\cdots \quad \text{—①}$$

and $$\frac{dy}{dx}=\frac{1}{h}\left\{\Delta y_0+\frac{2p-1}{2}\Delta^2 y_0+\frac{3p^2-6p+2}{6}\Delta^3 y_0+\frac{2p^3-9p^2+11p-3}{24}\Delta^4 y_0+\cdots\right] \quad \text{—②}$$

and

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\Delta^2 y_0+(p-1)\Delta^3 y_0+\frac{6p^2-18p+11}{12}\Delta^4 y_0+\cdots\right\} \quad \text{—③}$$

where $p=\frac{x-x_0}{h}$; Here $p=\frac{x-0}{1}=x \Rightarrow \boxed{p=x}$

$\therefore$ ① becomes,

$$y(x)=y_0+x\Delta y_0+\frac{x(x-1)}{2}\Delta^2 y_0+\frac{x(x-1)(x-2)}{3}\Delta^3 y_0+\cdots \quad \text{—④}$$

* W.K.T. Weddley's rule is,

$$\int_a^b f(x)\,dx = \frac{3h}{10}\{(y_0+5y_1+y_2+6y_3+y_4+5y_5) + (2y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}) + \cdots + (2y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3} + y_{n-2}+5y_{n-1}+y_n)\}$$

NOTE:

For weddles rule put $n=6$ in newton's cote's ~~defn~~ quadrature formula.

Pblms:

1) i) Evaluate $\int_0^6 \frac{dx}{1+x^2} \Rightarrow \tan^{-1}x$ taking $h=1$ using i) Trapezoidal rule, ii) Simpson's $\frac{1}{3}$rd rule, iii) Simpson's $\frac{3}{8}$th rule, iv) weddles rule, v) Also check up by direct integration which rule gives the value closets ~~dosets~~ to the actual value.

Sol:

Let $y(x) = \frac{1}{1+x^2} \Rightarrow \tan^{-1}x$ G.T. $h=1$.

The tablular column is as follows.

x	0	1	2	3	4	5	6
$y=\frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.0385	0.0270.

Here $y_0 = 1$, $y_1 = 0.5$, y ...

$$= \frac{1}{18}\{1.5 + 2.7 + 8.2772\} = \frac{1}{18}\{12.4772\}$$

$$= 0.69317 \approx 0.6932.$$

iii) W.K.T the Simpson's $\frac{3}{8}$th rule is,

$$\int_a^b f(x)dx = \frac{3h}{8}\{(y_0 + y_6) + (3(y_1 + y_4)) + 3(y_2 + y_5) + 2y_3\}$$

$$\int_0^1 \frac{dx}{1+x} = \frac{3(\frac{1}{6})}{8}\{(1.5) + 3(0.8571 + 0.6) + 3(0.75 + 0.5455) + 2(0.6667)\}$$

$$= \frac{1}{16}\{1.5 + 4.3731 + 3.8865 + 1.3334\}$$

$$= \frac{1}{16}\{11.0912\} = 0.69932.$$

iv) $$\int_0^1 \frac{dx}{1+x} = \int_0^1 \frac{d(1+x)}{1+x}$$

$\log 2 = 0.3010$

$\log e = 0.4343$

$\log_e 2 = 0.6931$

$$= [\log(1+x)]_0^1$$

$$= \log_e 2 - \log_e 1$$

$$= 0.6931.$$

15

Trapezoidal rule, Error = Exact value − Approx. val

$$= 0.6931 - 0.6949$$

$$= -0.0018.$$

Simpson's $\frac{1}{3}$rd rule Error $= 0.6931 - 0.6932$

$$= -0.0001$$

Simpson's $\frac{3}{8}$th rule, Error $= 0.6931 - 0.6932$

$$= -0.0001$$

Let $y(x) = \frac{1}{1+x}$. G.T. $h = 1/6$.

The tabular column is as follows.

x	0	$1/6$	$1/3$	$1/2$	$2/3$	$5/6$	1
$y = \frac{1}{1+x}$	1	$6/7$	$3/4$	$2/3$	$3/5$	$6/4$	0.5
		0.8571	0.75	0.6667	0.6	0.5435	

Here, $y_0 = 1$, $y_1 = 0.8571$, $y_2 = 0.75$, $y_3 = 0.6667$,

$y_4 = 0.6$; $y_5 = 0.5455$; $y_6 = 0.5$.

14

i) W.K.T. Trapezoidal rule is,

$$\int_a^b f(x)dx = \frac{h}{2}\left\{(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right\}$$

$$\therefore \int_0^1 \frac{dx}{1+x} = \frac{1/6}{2}\left\{(1+0.5) + 2(1+0.8571) + 0.75 + 0.6667 + 0.6 + 0.5455\right\}$$

$$= \frac{1}{12}\left\{1.5 + 2(3.4193)\right\}$$

$$= \frac{1}{12}(8.3386) = 0.69488 \simeq 0.6949.$$

ii) W.K.T. the Simpsons $\frac{1}{3}^{rd}$ rule is,

$$\int_a^b f(x)dx = \frac{h}{3}\left\{(y_0+y_n) + 2(y_2+y_4+\cdots+y_{n-2}) + 4(y_1+y_3+\cdots+y_{n-1})\right\}$$

$$\int_0^1 \frac{dx}{1+x} = \frac{1/6}{3}\left\{(1.5) + 2(0.75+0.6) + 4(0.8571 + 0.6667 + 0.5455)\right\}$$

Here, $y_0 = 0$, $y_1 = 0.2588$, $y_2 = 0.5$, $y_3 = 0.7071$, $y_4 = 0.8660$, $y_5 = 0.9656$, $y_6 = 1$.

W.K.T. Simpsons $\frac{1}{3}$rd rule is,

$$\int_a^b f(x)\,dx = \frac{h}{3}\left\{(y_0+y_n)+2(y_2+y_4+\cdots+y_{n-2}) + 4(y_1+y_3+\cdots+y_{n-1})\right\}$$

$$\int_0^{\pi/2} \sin x\,dx = \frac{h}{3}\left\{(y_0+y_1)+2(y_2+y_4)+4(y_1+y_3+y_5)\right\}$$

$$= \frac{\pi}{12\times 3}\left\{(0+1)+2(0.5+0.8660)+4(0.2588+0.7071+0.9656)\right\}$$

13

$$= \frac{180}{36}\left\{1+2.732+7.7272\right\}$$

$$= 5\left\{11.4592\right\}$$

$$= 57.296 \qquad = \frac{22}{7\times 36}\left\{1+2.732+7.7272\right\}$$

ANS:

$$= \frac{22}{252}\left\{11.4592\right\}$$

$$\int_0^{\pi/2} \sin x\,dx = 57.296 = \frac{252.1024}{252} = 1.0004 \text{ (approx)}.$$

4) Evaluate $\int_0^1 \frac{dx}{1+x}$ using i) Trapezoidal rule, ii) Simpsons 3rd rule, iii) Simpson's $\frac{3}{8}$th rule, iv) find the error in each method by comparing with actual integration up to 4 places of decimals. Take $h = \frac{1}{6}$ for all cases.

Sol: $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$.

3) Find the minimum value of $f(x)$ which has the values.

x	0	2	4	6.
$f(x)$	3	3	11	27.

27

Sol:

Here $x_0 = 0, x_1 = 2, x_2 = 4, x_3 = 6$ and $h = 6$.

W.K.T. Newton's fwd interpolation formula is,

$$y(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{6}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{24}\Delta^4 y_0 + \cdots \quad (1)$$

and

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2P-1)}{2}\Delta^2 y_0 + \frac{3P^2-6P+2}{12}\Delta^3 y_0 + \cdots\right\} \quad (2)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (P-1)\Delta^3 y_0 + \frac{6P^2-18P+11}{12}\Delta^4 y_0 + \cdots\right\} \quad (3)$$

where $p = \frac{x - x_0}{h}$ Here $P = \frac{x-0}{2} = x/2 \Rightarrow \boxed{P = x/2}$

$\therefore$ (1) becomes,

$$y(x) = y_0 + \frac{x\Delta y_0}{2} + \frac{x/2(x/2-1)}{2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3}\Delta^3 y_0 + \frac{x(x-1)(x-2)(x-3)}{4}\Delta^4 y_0 + \cdots \quad (4)$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2x/2-1)}{2}\Delta^2 y_0 + \left(\frac{3x^2+6x+2}{6}\right)\Delta^3 y_0 + \cdots\right\} \quad (5)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (x-1)\Delta^3 y_0 + \cdots\right\} \quad (6).$$

2) Evaluate $\int_0^5 \frac{dx}{4x+5}$ by using trapezoidal rule with 11 coordinates.

Sol:

G.T. there are 11 co-ordinates

Number of intervals $n = 10$.

Now $h = \frac{b-a}{n} = \frac{5-0}{10} = 5/10 = 1/2 = 0.5$

$\boxed{h = 0.5}$

Let $y(x) = \frac{1}{4x+5}$.

The tabular column is as follows.

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = \frac{1}{4x+5}$	0.2	0.1429	0.111	0.0909	0.077	0.0667	0.0588	0.0526	0.0476	0.0435	0.0[illegible]

Here $y_0 = 0.2$, $y_1 = 0.1429$, $y_2 = 0.1111$, $y_3 = 0.0909$,

$y_4 = 0.0769$, $y_5 = 0.0667$, $y_6 = 0.0588$, $y_7 = 0.0526$, $y_8 = 0.0476$

$y_9 = 0.0435$, $y_{10} = 0.04$.

W.k.T. trapezoidal rule is,

$$\int_a^b f(x)\,dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\}$$

$$\int_0^5 \frac{1}{4x+5}\,dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\}$$

$y_5 = 0.0385,\ y_6 = 0.0270.$

i) W.K.T. the trapezoidal rule is,

$$\int_a^b f(x)\,dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\}$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{1}{2}\{(1+0.0385)+2(0.5+0.2+0.1+0.0588+0.0385)\}$$

$$= \frac{1}{2}\{1.0270+2(0.8973)\}$$

$$= 0.5\{1.0270+1.7346\}$$

$$= 0.5(2.8216)$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.4108.$$

ii) W.K.T. the Simpsons $\frac{1}{3}^{rd}$ rule is,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\{(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\}$$

$$= \frac{1}{3}\{(1+0.0270)+2(0.2+0.0588)+4(0.5+0.1+0.0385)\}$$

$$= \frac{1}{3}\{1.0270+2(0.2588)+4(0.0385)\}$$

$$= \frac{1}{3}\{1.0270+0.5176+2.554\}$$

$$= \frac{4.0986}{3}$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3662.$$

17

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iii) W.K.T. the simpsons $\frac{3}{8}$th rule is,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\{(y_0+y_6)+3(y_1+y_4)+3(y_2+y_5)+2y_3\}$$

$$= \frac{3}{8}\{(1+0.0270)+3(0.5+0.0588)$$
$$+3(0.2+0.0385)+2(0.1)\}$$

$$= \frac{3}{8}\{1.0270+3(0.5588)+3(0.2358)+0.2\}$$

$$= \frac{3}{8}(3.6189)$$

$$= 1.3571.$$

iv) W.K.T. the weddles rules is,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\{y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\}$$

$$= \frac{3}{10}\{1+5(0.5)+0.2+6(0.1)+0.0588+$$
$$5(0.038346)+0.027\}$$

$$= 0.3\{1+2.5+0.2+0.6+0.0588+0.1925+0.027\}$$

$$= 0.3(4.5783)$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3735.$$

18

v) $$\int_0^6 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^6$$

$$= \tan^{-1}6 - \tan^{-1}0$$

$$= \tan^{-1}6 - \tan^{-1}(\tan 0)$$

$$= \tan^{-1}6$$

$= \frac{0.5}{2}\{(0.2+0.04)+2(0.1429+0.1111+0.0909$

$+0.0769+0.0667+0.0588+0.0526$

$+0.0476+0.0435)\}$

$= \frac{1}{4}\{0.24+2(0.691)\}$

$= \frac{1}{4}\{0.24+1.382\}$

$= \frac{1}{4}\{1.622\} = 0.4055$

ANS:

$$\int_0^5 \frac{1}{4x+5}\,dx = 0.4055.$$

3) Evaluate $\int_0^{\pi/2} \sin x\,dx$ by simpsons $\frac{1}{3}$rd rule dividing the range into 6 equal parts $(n=6)$.

Sol:

G.T. $n=6$.

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6}.$$

$$\boxed{h = \pi/12}$$

12.

Let $y(x) = \sin x$.

$\therefore$ The tabular column is as follows:

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$y=\sin x$	0	0.2588	0.5002	0.7071	0.8660	0.965629	1

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SIMPSON'S $\frac{3}{8}$ TH RULE

Put $n=3$ in (2)

$\therefore$ The values of x are x_0, x_1, x_2, x_3.

$\therefore$ The 4th order differences and higher order differences will becomes zero.

$$\int_{x_0}^{x_0+3h} f(x)\,dx = h\left\{3y_0 + \frac{9}{2}\Delta y_0 + \left(\frac{27}{3} - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6} + \cdots\right\}$$

$$= h\left\{3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right\}$$

$$= h\left\{\frac{24y_0 + 36y_1 - 36y_0 + 18y_2 - 36y_1 + 18y_0 + 3y_3 - 9y_2 + 9y_1 - 3y_0}{8}\right\}$$

$$= \frac{h\{24y_0 + 36y_1 - 36y_0 + 18y_2 - 36y_1 + 18y_0 + 3y_3 - 9y_2 + 9y_1 - 3y_0\}}{8}$$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$
$= (y_2 - y_1) - (y_1 - y_0)$
$= y_2 - y_1 - y_1 + y_0$
$= y_2 - 2y_1 + y_0$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$= (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)$
$= \{(y_3 - y_2) - (y_2 - y_1)\} - \{(y_2 - y_1) - (y_1 - y_0)\}$
$= y_3 - 3y_2 + 3y_1 - y_0$

i.e.) $$\int_{x_0}^{x_0+3h} f(x)\,dx = \frac{h}{8}\{3y_0 + 9y_1 + 9y_2 + 3y_3\}$$

$$= \frac{3h}{8}\{y_0 + 3y_1 + 3y_2 + y_3\}$$

Illy $$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\{y_3 + 3y_4 + 3y_5 + y_6\}$$

$$\int_{x_0+6h}^{x_0+9h} f(x)\,dx = \frac{3h}{8}\{y_6 + 3y_7 + 3y_8 + y_9\}$$

$\vdots$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\{y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\}$$

$$= h\left\{\frac{2y_0 + y_1 - y_0}{2}\right\}$$

$$= \frac{h}{2}(y_0 + y_1)$$

IIIly $x_0 + 2h$

$$\Rightarrow \int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_0+2h}^{x_0+3h} f(x)\,dx = \frac{h}{2}(y_2 + y_3).$$

$\vdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding all these eqns,

$$\int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\{y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots + y_{n-1} + y_n\}$$

* ie) $$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\}$$

SIMPSONS $\frac{1}{3}^{RD}$ RULE:

Put $n = 2$ in (2)

The values of x are x_0, x_1, x_2 —

Here all the differences of 3^{rd} order and higher order will become zero.

$$\therefore \int_{x_0}^{x_0+2h} f(x)\,dx = h\left\{2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2}\right\}$$

$$= h\left\{2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right\}$$

$$= h\left\{2y_0 + 2(y_1 - y_0) + \frac{1}{3}(\Delta y_1 - \Delta y_0)\right\}$$

$$= h\left\{2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - y_1 - y_1 + y_0)\right\}$$

$$= \frac{h}{3}\left\{6y_1 + y_2 - y_1 - y_1 + y_0\right\}$$

$$= \frac{h}{3}\left\{y_2 + 4y_1 + y_0\right\}$$

$$= \frac{h}{3}\left\{y_0 + 4y_1 + y_2\right\}$$

|||ly $$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left\{y_2 + 4y_3 + y_4\right\}$$

$$\int_{x_0+4h}^{x_0+6h} f(x)\,dx = \frac{h}{3}\left\{y_4 + 4y_5 + y_6\right\}$$

$\vdots$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left\{y_{n-2} + 4y_{n-1} + y_n\right\}$$

Adding all these eqns,

$$\int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+4h} f(x)\,dx + \cdots + \int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\{y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \cdots + y_{n-1} + 4y_{n-1} + y_n\}$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\}$$

This formula is known as Simpons $\frac{1}{3}$rd rule formula

4

ie) $y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$

Substituting this value of y_1 in ③

$$\Rightarrow A_0 = \frac{h}{2}\left\{ y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots \right\}$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{4}y_0'' + \frac{h^4}{12}y_0''' + \dots \quad \text{—— ④}$$

② - ④

$$\Rightarrow \int_{x_0}^{x_1} y\,dx - A_0 = \left\{ hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0'' + \frac{h^4}{24}y_0''' + \dots \right\}$$

$$- \left\{ hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{4}y_0'' + \frac{h^4}{12}y_0''' + \dots \right\}$$

$$= \frac{1}{6} - \frac{1}{4}h^3 y_0'' + \dots$$

$$= \frac{-1}{12}h^3 y_0'' + \dots$$

The Error in the interval $(x_0, x_1) \simeq -\frac{h^3}{12}y_0'' + \dots$

The error in the interval $(x_1, x_2) \simeq \frac{-h^3}{12}y_1'' + \dots$

The Error in the interval $(x_2, x_3) \simeq \frac{-h^3}{12}y_2'' + \dots$

The Error in the interval $(x_{n-1}, x_n) \simeq \frac{-h^3}{12}y_{n-1}'' + \dots$

$\therefore$ The total Error E is given by, $[\because |a+b| \le |a|+|b|]$

$$E = -\frac{h^3}{12}\left\{ y_0'' + y_1'' + y_2'' + \dots + y_{n-1}'' \right\}$$

$$|E| = \left| \frac{-h^3}{12}\left\{ y_0'' + y_1'' + y_2'' + \dots + y_{n-1}'' \right\} \right| \le \frac{h^3}{12}\left\{ |y_0''| + |y_1''| + \dots + |y_{n-1}''| \right\}$$

Let ~~$M = \max\{y_0'' + y_1'' + \dots$~~

$M = \max\{y_0'', y_1'', \dots, y_{n-1}''\}$

$\left[\because x_n = x_0 + nh \right.$

$nh = x_n - x_0$

$h = \frac{x_n - x_0}{n}$

$\left. h = \frac{b-a}{n} \right]$

$\left. \Rightarrow n = \frac{b-a}{h} \right]$

$\therefore |E| \le n \frac{h^3}{12} M$

$\therefore |E| \le \left(\frac{b-a}{h}\right) \frac{h^3}{12} M$

ie) $\boxed{|E| \le (b-a) \frac{h^2}{12} M}$

9

NOTE:

The Error in Trapezoidal rule is of order h^2.

PROBLEMS:

1) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = 0.2$ Hence determine the value of π?

(or)

Evaluate Trapezoidal rule by taking 5 interval

$\left[h = \frac{1-0}{5} = \frac{1}{5} = 0.2\right]$

Sol:

Here $y(x) = \frac{1}{1+x^2}$ and $x_0 = 0$.

The tabular column as follows:

x	0	0.2	0.4	0.6	0.8	1.0
y	1	0.9615	0.8621	0.7353	0.6098	0.5

Here $y_0 = 1$, $y_1 = 0.9615$, $y_2 = 0.8621$, $y_3 = 0.7353$, Adding all these equalities

$$\int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \int_{x_0+6h}^{x_0+9h} f(x)dx + \dots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)dx$$

$$= \frac{3h}{8}\{(y_0+3y_1+3y_2+y_3)+(y_3+3y_4+3y_5+y_6) + (y_6+3y_7+3y_8+y_9)+\dots+ (y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)\}$$

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\{(y_0+y_n)+3(y_1+y_4+\dots+y_{n-2}) +3(y_2+y_5+y_8+\dots+y_{n-1})+2(y_3+y_6+\dots+y_{n-3})\}$$

This formula is known as Simpson's $3/8$ th rule formula.

* P.T. the Error in Trapezoidal rule is of order h^2.

ERROR IN TRAPEZOIDEL RULE

Let the functn be $y=f(x)$.

Expand $f(x)$ by taylor's series about the pt

$$f(x) = f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\dots$$

$$\therefore \int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} f(x_0)dx + \int_{x_0}^{x_1} (x-x_0)f'(x_0)dx + \int_{x_0}^{x_1}\frac{(x-x_0)^2}{2!}f''(x_0) + \int_{x_0}^{x_1}\frac{(x-x_0)^3}{3!}f'''(x_0)+\dots$$

7

$$= f(x_0)\int_{x_0}^{x_1} dx + f'(x_0)\int_{x_0}^{x_1}(x-x_0)\,d(x-x_0) + \frac{f''(x_0)}{2}\int_{x_0}^{x_1}(x-x_0)^2\,d(x-x_0) + \frac{f'''(x_0)}{6}\int_{x_0}^{x_1}(x-x_0)^3\,d(x-x_0) + \cdots$$

$[\because d(x-x_0) = dx]$

$$= f(x_0)\,[x]_{x_0}^{x_1} + f'(x_0)\left[\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_1} + \left[\frac{(x-x_0)^3}{3}\right]_{x_0}^{x_1}\frac{f''(x_0)}{2} + \left[\frac{(x-x_0)^4}{4}\right]_{x_0}^{x_1}\frac{f'''(x_0)}{6} + \cdots$$

ie) $$\int_{x_0}^{x_1} f(x)\,dx = (x_1-x_0)f(x_0) + \frac{(x_1-x_0)^2}{2}f'(x_0) + \frac{(x_1-x_0)^3}{6}f''(x_0) + \frac{(x-x_0)^4}{24}f'''(0) + \cdots$$

$$\int_{x_0}^{x_1} f(x)\,dx = hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0'' + \frac{h^4}{24}y_0''' + \cdots$$

$$\int_{x_0}^{x_1} y\,dx = hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0'' + \frac{h^4}{24}y_0''' + \cdots \quad (2)$$

$\because x_1 - x_0 = x_0 + h - x_0 = h$

$f(x) = y$

$f(x_0) = y_0$

$f'(x_0) = y_0'$

$f''(x_0) = y_0''$

$\vdots$

Now the area of the trapezium in the interval $[x_0, x_1]$ is,

$$A_0 = \int_{x_0}^{x_1} y\,dx = \frac{h}{2}[y_0 + y_1] \quad (3)$$

Put $x = x_1$ in (1):

$$\Rightarrow f(x_1) = f(x_0) + (x_1-x_0)f'(x_0) + \frac{(x_1-x_0)^2}{2!}f''(x_0) + \frac{(x_1-x_0)^3}{3!}f'''(x_0) + \cdots$$

$y_4 = 0.6098$, $y_5 = 0.5$.

W.K.T, the trapezoidal rule is,

$$\int_a^b f(x)\,dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\}$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\{(y_0+y_5)+2(y_1+y_2+y_3+y_4)\}$$

$$= \frac{0.2}{2}\{(1+0.5)+2(0.9615+0.8621+0.7353+0.6098)\}$$

$$= \frac{1}{10}\{1.5+6.3374\}$$

$$= \frac{1}{10}\{7.8374\}$$

$$= 0.7837$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.7837 \quad \text{--- (1)}$$

10

Now $\int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1}x]_0^1$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \tan^{-1}(\tan \pi/4) - \tan^{-1}(\tan 0)$$

$$= \pi/4 - 0 = \pi/4 \quad \text{--- (2)}$$

From (1) & (2)

$$\Rightarrow \pi/4 = 0.7837$$

ANS: $\pi = 0.7837 \times 4$

$\pi = 3.1348$.

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UNIT-5

NUMERICAL INTEGRATION

NEWTONS COTE'S QUADRATURE FORMULA

Let the function be $y=f(x)$, let $y_0=f(x_0)$, $y_1=f(x_1), \dots y_n=f(x_n)$, let $I=\int_a^b f(x)\,dx$.

Divide the interval $[a,b]$ into the subintervals such that $a=x_0$, $x_1=x_0+h$, $x_2=x_0+2h, \dots, x_n=x_0+nh$ where h is the length of the interval.

$$\therefore I=\int_{x_0}^{x_0+nh} f(x)\,dx \quad \text{--- (1)}.$$

W.K.T. $p=\dfrac{x-x_0}{h} \Rightarrow x=x_0+ph$.

$\therefore dx=d(x_0+ph)=d(x_0)+h\,dp=h\,dp$

(1) where $x=x_0 \Rightarrow x_0=x_0+ph$

$\Rightarrow \boxed{P=0}$ (forward) (2) where $x=x_0+nh \Rightarrow x_0+nh=x_0+ph$

$\Rightarrow \boxed{P=n}$ W.K.T. Newtons interpolation formula is,

$$y(x)=y(x_0+ph)=y_0+P\Delta y_0+\frac{P(P-1)}{2!}\Delta^2 y_0+\frac{P(P-1)(P-2)}{3}\Delta^3 y_0+\dots$$

ie) $f(x_0+ph)=y_0+P\Delta y_0+\dfrac{P^2-P}{2}\Delta^2 y_0+\dfrac{P^3-3P^2+2P}{6}\Delta^3 y_0+\dots$

(1) Becomes, $I=\int_0^n f(x_0+ph)\,h\,dp$

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$$= h\int_0^n \left\{ y_0 + p\Delta y_0 + \frac{p^2-p}{2}\Delta^2 y_0 + \dots \right\} dp$$

$$= h\left\{ y_0\int_0^n dp + \left(\int_0^n p\,dp\right)\Delta y_0 + \left(\int_0^n p^2 dp - \int_0^n p\,dp\right)\frac{\Delta^2 y_0}{2} + \left(\int_0^n p^3 dp - 3\int_0^n p^2 dp + 2\int_0^n p\,dp\right)\frac{\Delta^3 y_0}{6} + \dots \right\}$$

$$= h\left\{ y_0[p]_0^n + \left[\frac{p^2}{2}\right]_0^n \Delta y_0 + \left(\left[\frac{p^3}{3}\right]_0^n - \left[\frac{p^2}{2}\right]_0^n\right)\frac{\Delta^2 y_0}{2} + \left(\left[\frac{p^4}{4}\right]_0^n - 3\left[\frac{p^3}{3}\right]_0^n + 2\left[\frac{p^2}{2}\right]_0^n\right)\frac{\Delta^3 y_0}{6} + \dots \right\}$$

$$* \therefore \int_{x_0}^{x_0+nh} f(x)\,dx = h\left\{ ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \dots \right\} \quad \text{--- (2)}$$

This formula is known as Newton's Cote's quadrature formula.

v.v.Imp (P.B)

TRAPEZOIDAL RULE.

Put $n = 1$ in (2).

Derive Trapezoidal rule.

The values of x are x_0, x_1

Here all the differences except Δy_0 will be zero.

$$x_0 + h \Rightarrow \int_{x_0}^{x_0+h} f(x)\,dx = h\left\{ y_0 + \tfrac{1}{2}\Delta y_0 \right\}$$

$$= h\left\{ y_0 + \tfrac{1}{2}(y_1 - y_0) \right\}$$