$$= T_4\left[T_3\left(1/cz+d\right)\right]$$

$$= T_4\left[\left(b-ad/c\right)\frac{1}{cz+d}\right]$$

$$= \frac{b-ad/c}{cz+d} + \frac{a}{c} = T(z).$$

$\therefore$ ~~Anky~~ Any Bilinear Transformation can be expressed as a product of Translation, rotation, magnification (or) Inversion.

Cross Ratio (Defn)

Let z_1, z_2, z_3, z_4 be four distinct pts in the extent complex plane. The cross Ratio of these points denoted by (z_1, z_2, z_3, z_4) is defined by,

$$(z_1, z_2, z_3, z_4) = \begin{cases} \dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} & \text{is none of } z_1, z_2, z_3, z_4 \text{ is } \infty \\ (z_1-z_3)/(z_1-z_4) & \text{is } z_2 = \infty \\ (z_2-z_4)/(z_1-z_4) & \text{is } z_3 = \infty \\ (z_1-z_3)/(z_2-z_3) & \text{is } z_4 = \infty \\ (z_2-z_4)/(z_2-z_3) & \text{is } z_1 = \infty \end{cases}$$

Note: Four distinct points z_1, z_2, z_3, z_4 are collinear (or) concyclic iff (z_1, z_2, z_3, z_4) is real

Theorem:1

Any bilinear transformation preserves cross ratio.

proof:-

Let $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ be the bilinear transformation

Let z_1, z_2, z_3, z_4 be four distinct points. Let their images Under this transformation be w_1, w_2, w_3, w_4 respectively.

we assume that z_i & w_i are different from ∞

TPT:

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$$

$$w_i = \frac{az_i + b}{cz_i + d} \quad , i = 1, 2, 3 \ldots$$

$$w_1 - w_3 = \frac{az_1 + b}{cz_1 + d} - \frac{az_3 + b}{cz_3 + d}$$

$$= \frac{(az_1 + b)(cz_3 + d) - (az_3 + b)(cz_1 + d)}{(cz_1 + d)(cz_3 + d)}$$

$$= \frac{(az_1cz_3 + bcz_3 + daz_1 + db) - (cz_1az_3 + bcz_1 + daz_3 + bd)}{(cz_1 + d)(cz_3 + d)}$$

$$= \frac{az_1cz_3 + bcz_3 + daz_1 + db - cz_1az_3 - bcz_1 - daz_3 - bd}{(cz_1 + d)(cz_3 + d)}$$

$$= \frac{bcz_3 + daz_1 - bcz_1 - daz_3}{(cz_1 + d)(cz_3 + d)}$$

common

$$= \frac{ad(z_1 - z_3) + bc(z_3 - z_1)}{(cz_1 + d)(cz_3 + d)} = \frac{ad(z_1 - z_3) - bc(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)}$$

$$= \frac{(ad - bc)(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)} = k_1(z_1 - z_3)$$

where $k_1 = \dfrac{ad - bc}{(cz_1 + d)(cz_3 + d)}$

$$w_2 - w_4 = \frac{az_2 + b}{cz_2 + b} - \frac{az_4 + b}{cz_4 + d}$$

$$= \frac{(az_2 + b)(cz_4 + d) - (az_4 + b)(cz_2 + d)}{(cz_2 + d)(cz_4 + d)}$$

$$= \frac{(az_2cz_4 + bcz_4 + az_2d + bd) - az_4cz_2 + az_4d + bcz_2 + bd}{(cz_2 + d)(cz_4 + d)}$$

$$= \frac{az_2cz_4 + bcz_4 + az_2d + bd - az_4cz_2 - az_4d - bcz_2 - bd}{(cz_2 + d)(cz_4 + d)}$$

$$= \frac{bcz_4 + adz_2 - bcz_2 - daz_4}{(cz_2 + d)(cz_4 + d)}$$

$$= \frac{bc(z_4 - z_2) + ad(z_2 - z_4)}{(cz_2 + d)(cz_4 + d)}$$

$$w_2 - w_3 = \frac{az_2+b}{cz_2+d} - \frac{az_3+b}{cz_3+d}$$

$$= \frac{(az_2+b)(cz_3+d) - (az_3+b)(cz_2+d)}{(cz_2+d)(cz_3+d)}$$

$$= \frac{(az_2cz_3 + daz_2 + bcz_3 + bd) - (az_3cz_2 + daz_3 + bcz_2 + bd)}{(cz_2+d)(cz_3+d)}$$

$$= \frac{az_2cz_3 + daz_2 + bcz_3 + bd - az_3cz_2 - daz_3 - bcz_2 - bd}{(cz_2+d)(cz_3+d)}$$

$$= \frac{bcz_3 + daz_2 - bcz_2 - daz_3}{(cz_2+d)(cz_3+d)}$$

$$= \frac{ad(z_2-z_3) + bc(z_3-z_2)}{(cz_2+d)(cz_3+d)}$$

$$= \frac{ad(z_2-z_3) - bc(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

$$w_2 - w_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)} = K_4(z_2-z_3)$$

where $K_4 = \frac{ad-bc}{(cz_2+d)(cz_3+d)}$

$$= \frac{ad(z_2-z_4) - bc(z_2-z_4)}{(cz_2+d)(cz_4+d)}$$

$$= \frac{(ad-bc)(z_2-z_4)}{(cz_2+d)(cz_4+d)} = K_2(z_2-z_4)$$

$$(w_1-w_3)(w_2-w_4) = K_1K_2(z_1-z_3)(z_2-z_4)$$

$$= K(z_1-z_3)(z_2-z_4) \quad \text{— ①}$$

Similarly,

$$w_1 - w_4 = \frac{az_1+b}{cz_1+d} - \frac{az_4+b}{cz_4+d}$$

$$= \frac{(az_1+b)(cz_4+d) - (az_4+b)(cz_1+d)}{(cz_1+d)(cz_4+d)}$$

$$= \frac{az_1cz_4 + az_1d + bcz_4 + bd - az_4cz_1 - az_4d - bcz_1 - bd}{(cz_1+d)(cz_4+d)}$$

$$= \frac{bcz_4 + adz_1 - bcz_1 - adz_4}{(cz_1+d)(cz_4+d)}$$

$$= \frac{bc(z_4-z_1) + ad(z_1-z_4)}{(cz_1+d)(cz_4+d)}$$

$$= \frac{ad(z_1-z_4) - bc(z_1-z_4)}{(cz_1+d)(cz_4+d)} = \frac{(ad-bc)(z_1-z_4)}{(cz_1+d)(cz_4+d)} = K_3(z_1-z_4)$$

where $K_3 = \frac{ad-bc}{(cz_1+d)(cz_4+d)}$

$$(w_1-w_4)(w_2-w_3) = K_3K_4(z_2-z_3)(z_1-z_4)$$

$$(w_1-w_4)(w_2-w_3) = K(z_1-z_4)(z_2-z_3) \quad \text{— ②}$$

where $K = K_3K_4$

$$\frac{①}{②} \Rightarrow \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{K(z_1-z_3)(z_2-z_4)}{K(z_1-z_4)(z_2-z_3)}$$

$$\therefore (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

Any bilinear transformation preserves cross ratio. this

Note. proof is similar when z_i (or) w_i is ∞.

The bilinear transformation which the three points z_1, z_2, z_3 to three points w_1, w_2, w_3 is given by $(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3)$

problem

1. Find the bilinear transformation which the pts $z = (2, i, -2)$ onto $w = (1, i, -1)$ respectively

sol:-

$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

$(w, 1, i, -1) = (z, 2, i, -2)$

$$\frac{(w-i)(1+1)}{(w+1)(1-i)} = \frac{(z-i)(2+2)}{(z+2)(2-i)}$$

$$\frac{2(w-i)}{(w+1)(1-i)} = \frac{4(z-i)}{(z+2)(2-i)}$$

$$\frac{w-i}{w-wi+1-i} = \frac{2(z-i)}{2z-zi+4-2i}$$

$(w-i)(2z-zi+4-2i) = 2(z-i)(w-wi+1-i)$

$2zw+4w-2iw-zwi-2zi-4i-2-z = 2zw-2zwi+2z-2zi-2iw-2w-2i-2$

$4w-2iw-zwi-4i-2-z+2zwi-2z+2iw+2w+2i+2=0$

$6w+iwz-2i-3z=0$

$iwz+6w=3z+2i$

$w(iz+6)=3z+2i$

$$w=\frac{3z+2i}{iz+6}$$

H.W

1. $z=-i, 0, i$ and $w=-1, i, 1$

sol:

$(w_1, w_2, w_3, w_4) = z_1, z_2, z_3, z_4$

$(w, -1, i, 1) = (z, -i, 0, i)$

$$\frac{(w-i)(-1-1)}{(w-1)(-1-i)} = \frac{(z-0)(-i-i)}{(z-i)(-i-0)}$$

$$\left[\because \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}\right]$$

$$\frac{-2(w-i)}{(w-1)(-1-i)} = \frac{-2i(z)}{(z-i)(-i)}$$

$$\frac{-2(w-i)}{-w-wi+1+i} = \frac{-2iz}{-zi-1}$$

$(-w+i)(-zi-1) = -2iz(-w-wi+1+i)$

$-zwi+w+z-i = izw-zw-iz+z$

$w-i+zw+iz=0$

$w(1+z)-i(1-z)=0$

$i(1-z)=w(1+z)$

$$w = \frac{i(1-z)}{(1+z)}$$

3. $z = 1, i, 0$ and $w = 0, 2, -i$ $\therefore (w_1 - w_3)$

sol

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

$$(w, 0, 2, -i) = (z, 1, -i, 0)$$

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$$\frac{(w-2)(0+i)}{(w+i)(-2)} = \frac{(z+i)(0+1)}{(z-0)(1+i)}$$

$$\frac{i(w-2)}{-2w-2i} = \frac{z+i}{z(1+i)}$$

$$\frac{wi-2i}{-2w-2i} = \frac{z+i}{z+zi}$$

$$(wi-2i)(z+zi) = (z+i)(-2w-2i)$$

$$zwi - wz - 2(z+2z) = (-2wz - 2iz - 2iw + 2)$$

$$zwi + wz + 2z - 2 + 2iw = 0$$

$$w(zi + z + 2i) = 2 - 2z$$

$$w = \frac{2(1-z)}{zi + z + 2i}$$

4. $z = (-1, 1, \infty)$ and $w = (-i, -1, i)$

sol

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

$$\left[\because \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}\right]$$

$$w, -i, -1, i = z, -1, 1, \infty$$

$$\frac{(w+1)(-i-i)}{(w-i)(-i+1)} = \frac{(z-1)(-1-\infty)}{(z-\infty)(-1-1)}$$

ie., $$\frac{z-1}{-1-1} = \frac{(w+1)(-i-i)}{(w-i)(-i+1)}$$

$$\frac{z-1}{-2} = \frac{-2wi-2i}{-wi+w-1-i}$$

$$(z-1)(-wi+w-1-i) = -2(-2wi-2i)$$

$$-wiz + wz - z - iz + wi - w + 1 + i = 4wi + 4i$$

$$-wiz + wz - z - iz + wi - w + 1 + i - 4wi - 4i = 0$$

$$-wiz + wz - z - iz - 3wi - w + 1 - 3i = 0$$

$$w(-iz + z - 3i - 1) = z + iz - 1 + 3i$$

$$w = \frac{z + zi - 1 + 3i}{-iz + z - 3i - 1}$$

5.

sol

$(w, w_1, w_2, w_3) = (z_1, z_2, z_3, z_4)$

$(w, i, -1, -i) = (z, 0, 1, \infty)$

$$\frac{(w+1)(i+i)}{(w+i)(i+1)} = \frac{(z-1)(0-\infty)}{(z-\infty)(0-1)}$$

$$\left[\because \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} \quad \frac{(z_1-z_3)(z_2-z_4)}{w(z_1-z_4)(z_2-z_3)}\right]$$

$$\frac{2i(w+1)}{(i+1)(w+i)} = \frac{z-1}{-1}$$

$$\frac{-2iw-2i}{iw-1+w+i} = z-1$$

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$-2iw-2i = (z-1)(iw+w-1+i)$

$-2iw-2i = ziw+zw-z+zi-iw-w+1-i$

$-2iw-ziw-zw+z-zi+iw+w-1-i=0$

$w(-2i-zi-z+i+1) = -z+zi+1+i$

$$w = \frac{-z(1-i)+i+1}{-i+zi-z+1}$$

$$= \frac{-z(1-i)+(i+1)}{-z(i+1)-(i-1)} \quad \frac{z(1-i)-(i+1)}{z(i+1)-(-i+1)}$$

$$w = \frac{z(1-i)-(i+1)}{z(i+1)-(1-i)}.$$