UNIT-3

INNER PRODUCT SPACE

Defn:

Let V be a v.s. over F. An inner pdt of V is a functn which assigns to each ordered pair of vectors u, v in V a scalar in F denoted by $\langle u, v \rangle$, satisfying the following condition

Inner pdt of

i) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

ii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

iii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ where $\overline{\langle v, u \rangle}$ is the complex conjugate of $\langle u, v \rangle$.

iv) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$.

A v.s. with an inner pdt defined on it is called an inner pdt. space.

An inner pdt space is called an Euclidean Space or unitary Space according as F is the field of real numbers or complex numbers.

NOTE:

i) If F is the field of real numbers then condition (iii) takes the form $\langle u,v\rangle = \langle v,u\rangle$

ii) $\langle u,\alpha v\rangle = \bar{\alpha}\langle u,v\rangle$

For $\langle u,\alpha v\rangle = \overline{\langle \alpha v,u\rangle}$

$= \overline{\alpha\langle v,u\rangle}$

$= \bar{\alpha}\,\overline{\langle v,u\rangle}$

$= \bar{\alpha}\langle u,v\rangle$

iii) $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$

For, $\langle u,v+w\rangle = \overline{\langle v+w,u\rangle}$

$= \overline{\langle v,u\rangle + \langle w,u\rangle}$

$= \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle}$

$= \langle u,v\rangle + \langle u,w\rangle$

iv) $\langle u,0\rangle = \langle 0,u\rangle = 0$

For, $\langle u,0\rangle = \langle u,0.0\rangle$

$= \bar{0}\langle u,0\rangle$

$= 0\langle u,0\rangle$

$= 0.$

III^{ly} $\langle 0,v\rangle = 0$

eg:

1) P.T $V_n(R)$ is a real inner pdt space (IPS) with inner pdt defined by,

$$\langle x, y \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \quad \text{where}$$

$x = (x_1, x_2, \ldots, x_n)$ & $y = (y_1, y_2, \ldots, y_n)$.

This is called std inner pdt of $V_n(R)$.

Proof:

Let $x, y, z \in V_n(R)$ and $\alpha \in R$

i) To prove $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

L.H.S:

$$\langle x+y, z \rangle = \langle [(x_1+y_1), (x_2+y_2) \ldots (x_n+y_n)], [(z_1, z_2, \ldots, z_n)] \rangle$$

$$= (x_1+y_1)z_1 + (x_2+y_2)z_2 + \cdots + (x_n+y_n)z_n$$

$$= (x_1z_1 + x_2z_2 + \cdots + x_nz_n) + (y_1z_1 + \cdots + y_nz_n)$$

$$= \langle x, z \rangle + \langle y, z \rangle$$

$$= R.H.S.$$

03.03.2017

ii) To prove $\langle \alpha \cdot x, y \rangle = \alpha \langle x, y \rangle$

L.H.S:

$$\langle \alpha x, y \rangle = \langle (\alpha x_1, \alpha x_2, \ldots, \alpha x_n), (y_1, y_2, \ldots, y_n) \rangle$$

$$= \alpha x_1y_1 + \alpha x_2y_2 + \cdots + \alpha x_ny_n.$$

$$= \alpha(x_1y_1 + x_2y_2 + \cdots + x_ny_n)$$

$$= \alpha \langle x, y \rangle$$

$$= R.H.S.$$

iii) To prove $\langle x, y\rangle = \langle y, x\rangle$

L.H.S:

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$
$$= y_1x_1 + y_2x_2 + \cdots + y_nx_n$$
$$= \langle y, x\rangle$$
$$= R.H.S.$$

4

iv) To prove $\langle x, x\rangle \geq 0$ & $\langle x, x\rangle = 0$ iff $x = 0$.

Now $\langle x, x\rangle = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$ & $\langle x, x\rangle = 0$ iff $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$.

$$\Rightarrow x_1 = x_2 = \cdots = x_n = 0.$$
$$\therefore \langle x, x\rangle = 0 \text{ iff } x = 0.$$

$\therefore V_n(R)$ is a real IPS.

2) P.T. $V_n(C)$ is a complex IPS with inner pdt defined by $\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$ where $x = (x_1, x_2, \ldots, x_n)$ & $y = (y_1, y_2, \ldots, y_n)$

Proof:

Let $x, y, z \in V_n(C)$ & $\alpha \in C$.

i) To prove $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

L.H.S: $\langle x+y, z\rangle = \langle ((x_1+y_1) + (x_2+y_2) + \cdots + (x_n+y_n)), (z_1, z_2, \ldots, z_n)\rangle$

$$= (x_1+y_1)\bar{z}_1 + (x_2+y_2)\bar{z}_2 + \cdots + (x_n+y_n)\bar{z}_n$$
$$= (x_1\bar{z}_1 + y_1\bar{z}_1) + (x_2\bar{z}_2 + y_2\bar{z}_2) + \cdots + (x_n\bar{z}_n + y_n\bar{z}_n)$$
$$= (x_1\bar{z}_1 + x_2\bar{z}_2 + \cdots + x_n\bar{z}_n) + (y_1\bar{z}_1 + y_2\bar{z}_2 + \cdots + y_n\bar{z}_n)$$
$$= \langle x, z\rangle + \langle y, z\rangle = R.H.S.$$

ii) To prove $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$.

L.H.S:

$\langle \alpha x, y\rangle = \langle (\alpha x_1, \alpha x_2, \dots, \alpha x_n), (y_1, y_2, \dots, y_n)\rangle$

$= (\alpha x_1 \bar{y}_1 + \alpha x_2 \bar{y}_2 + \dots + \alpha x_n \bar{y}_n)$

$= \alpha (x_1 \bar{y}_1 + x_2 \bar{y}_2 + \dots + x_n \bar{y}_n)$

$= \alpha \langle x, y\rangle$

$=$ R.H.S.

5

iii) To prove $\langle x, y\rangle = \overline{\langle y, x\rangle}$

R.H.S:

$\overline{\langle y, x\rangle} = \overline{y_1 \bar{x}_1 + y_2 \bar{x}_2 + \dots + y_n \bar{x}_n}$

$= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \dots + \bar{y}_n x_n$

$= x_1 \bar{y}_1 + x_2 \bar{y}_2 + \dots + x_n \bar{y}_n$

$= \langle x, y\rangle =$ R.H.S.

iv) To prove $\langle x, x\rangle \geq 0$ & $\langle x, x\rangle = 0$ iff $x = 0$.

$\langle x, x\rangle = x_1 \bar{x}_1 + x_2 \bar{x}_2 + \dots + x_n \bar{x}_n$

$= |x_1| + |x_2| + \dots + |x_n| \geq 0$ $\quad [\because z\bar{z} = |z|]$

$\langle x, x\rangle = 0 \Rightarrow |x_1| + |x_2| + \dots + |x_n| = 0$.

$\langle x, x\rangle = 0$ iff $x = 0$.

$\therefore V_n(C)$ is a Complex IPS.

3) Let V be the set of all continuous real valued functn defined on a $[0,1]$ P.T. V is a real IPS with inner pdt defined by $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$.

Proof:

Let $f, g, h \in V$ & $\alpha \in R$.

i) TPT $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$

L.H.S: $\langle f+g, h\rangle = \int_0^1 [(f(t)+g(t))\cdot h(t)]\,dt$.

$= \int_0^1 f(t)\cdot h(t)\,dt + \int_0^1 g(t)\cdot h(t)\,dt$

$= \langle f, h\rangle + \langle g, h\rangle$

$=$ R.H.S.

ii) TPT $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$

L.H.S: $\langle \alpha f, g\rangle = \int_0^1 \alpha f(t)\, g(t)\,dt$.

$= \alpha\int_0^1 f(t)\, g(t)\,dt$.

$= \alpha\langle f, g\rangle =$ R.H.S.

iii) TPT $\langle f, g\rangle = \langle g, f\rangle$.

R.H.S: $\langle g, f\rangle = \int_0^1 g(t)\cdot f(t)\,dt$

$= \int_0^1 f(t)\cdot g(t)\,dt$

$= \langle f, g\rangle =$ L.H.S.

iv) TPT $\langle f, f\rangle \geq 0$.

$\langle f, f\rangle = \int_0^1 f(t)\cdot f(t)\,dt$

$= \int_0^1 [f(t)]^2\,dt \geq 0$ & $\langle f, f\rangle = 0$ iff $f = 0$.

$\langle f, f\rangle = 0 \Rightarrow f = 0$.

$\therefore$ V is Real IPS.

Scanned with CamScanner

4) S.T. $V_2(R)$ is an IPS with inner pdt defined by $\langle x,y\rangle = x_1y_1 + x_2y_1 \mp x_1y_2 + 4x_2y_2$ where $x=(x_1,x_2)$ & $y=(y_1,y_2)$

Sol:

Let $x,y,z \in V_2(R)$ & $\alpha \in R$ where

$x=(x_1,x_2)$, $y=(y_1,y_2)$, $z=(z_1,z_2)$.

i) T.P $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$

L.H.S:

$$\langle x+y,z\rangle = \langle (x_1,x_2)+(y_1,y_2), z\rangle$$

$$= \langle (x_1+y_1, x_2+y_2),(z_1,z_2)\rangle$$

$$= [(x_1+y_1)z_1 + (x_2+y_2)z_1 \mp (x_1+y_1)z_2 + 4(x_2+y_2)z_2]$$

$$= x_1z_1 + y_1z_1 + x_2z_1 + y_2z_1 \mp x_1z_2 \mp y_1z_2 + 4x_2z_2 + 4y_2z_2$$

$$= (x_1z_1 + x_2z_1 \mp x_1z_2 + 4x_2z_2) + (y_1z_1 + y_2z_1 \mp y_1z_2 + 4y_2z_2)$$

$$= \langle x,z\rangle + \langle y,z\rangle$$

$= R.H.S$ //.

ii) TPT $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$

L.H.S: $\langle \alpha x,y\rangle = \langle(\alpha x_1,\alpha x_2),(y_1,y_2)\rangle$

$$= \alpha x_1y_1 + \alpha x_2y_1 \mp \alpha x_1y_2 + 4\alpha x_2y_2$$

$$= \alpha(x_1y_1 + x_2y_1 \mp x_1y_2 + 4x_2y_2)$$

$$= \alpha\langle x,y\rangle$$

$= R.H.S$ //.

iii) TPT $\langle x, y\rangle = \langle y, x\rangle$

L.H.S:

$\langle x, y\rangle = x_1 y_1 + x_2 y_1 \mp x_1 y_2 + 4x_2 y_2$

$= y_1 x_1 + y_1 x_2 \mp y_2 x_1 + 4y_2 x_2$ 8

$= \langle y, x\rangle$

$= R.H.S$ //

iv) TPT $\langle x, x\rangle \geq 0$.

L.H.S: $\langle x, x\rangle = x_1 x_1 + x_2 x_1 \mp x_1 x_2 + 4x_2 x_2$

$= (x_1)^2 + \cancel{x_2 x_1 \mp x_1 x_2} + 4(x_2)^2$

$= \cancel{(x_1)^2 + 2x_1 x_2 + 4(x_2)^2}$

$= (x_1)^2 + 4(x_2)^2 \geq 0$.

$= \cancel{(x_1 + x_2)^2 + 3x_2^2 \geq 0}$. [$\because$ add $3x_2^2$]

& $\langle x, x\rangle = 0$ iff $x = 0$.

$\langle x, x\rangle \Rightarrow \cancel{[(x_1 + x_2)^2 + 3x_2^2 = 0]}$. $(x_1)^2 + 4(x_2)^2 = 0$.

$\Rightarrow \cancel{(x_1 + x_2)^2 = 0 \ \& \ 3x_2^2 = 0}$. $x_1^2 = 0$ & $4(x_2)^2 = 0$

$\Rightarrow x_1 = 0, x_2 = 0 \Rightarrow x = 0$.

$\therefore V_2(R)$ is a IPS.

5) S.T. In an IPS i) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

ii) $\langle u, \alpha v + \beta w\rangle = \bar{\alpha}\langle u, v\rangle + \bar{\beta}\langle u, w\rangle$

iii) $\langle \alpha u + \beta v, \gamma w + \delta z\rangle = \alpha\bar{\gamma}\langle u, w\rangle + \alpha\bar{\delta}\langle u, z\rangle$
$+ \beta\bar{\gamma}\langle v, w\rangle + \beta\bar{\delta}\langle v, z\rangle$.

where $\alpha, \beta, \gamma, \delta \in F$ & $u, v, w, z \in V$.

07.03.2017

Sol:

i) TP $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

L.H.S:

$<\alpha u+\beta v, w> = <\alpha u, w> + <\beta v, w>$ [$\because$ By defn (ii) in IPS]

$= \alpha <u,w> + \beta <v,w>$

$=$ R.H.S.//.

(ii) L.H.S:

$<u, \alpha v+\beta w> = <u,\alpha v> + <u,\beta w>$

$= \overline{<\alpha v,u>} + \overline{<\beta w,u>}$

$= \bar{\alpha}\, \overline{<v,u>} + \bar{\beta}\, \overline{<w,u>}$

$= \bar{\alpha} <u,v> + \bar{\beta} <u,w>$.

$=$ R.H.S.

iii) ~~L.H.S~~

Let $p = \gamma w + \delta z$

L.H.S: $<\alpha u+\beta v, \gamma w+\delta z> = <\alpha u+\beta v, p>$

$= <\alpha u, p> + <\beta v, p>$

$= \alpha <u,p> + \beta <v,p>$.

$= \alpha <u, \gamma w+\delta z> + \beta <v, \gamma w+\delta z>$

$= \alpha[\bar{\gamma}<u,w> + \bar{\delta}<u,z>] + \beta[\bar{\gamma}<v,w> + \bar{\delta}<v,z>]$

$= \alpha\bar{\gamma}<u,w> + \alpha\bar{\delta}<u,z> + \beta\bar{\gamma}<v,w> + \beta\bar{\delta}<v,z>$

$=$ R.H.S. //.

Defn:

Let V be an IPS and let $x \in V$.

The norm or length of x denoted by $\|x\|$ (norm x) and is defined by,

v.v.imp 1M

$$\|x\| = \sqrt{<x,x>}$$

v.v.imp 1M x is called a unit vector if $\|x\| = 1$

08.03.2017

Pblms :-

1) Let V be the v.s of polynomial with inner pdt gn by $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$.

Let $f(t) = t+2$, $g(t) = t^2-2t-3$. Find i) $\langle f,g\rangle$, ii) $\|f\|^2$

Sol:

i) $$\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt.$$ 10

$$= \int_0^1 (t+2)(t^2-2t-3)\,dt$$

$$= \int_0^1 (t^3-2t^2-3t+2t^2-4t-6)\,dt.$$

$$= \int_0^1 (t^3-7t-6)\,dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$$

$$= \left(\frac{1}{4} - \frac{7}{2} - 6\right) - (0-0-0)$$

$$= \frac{1}{4} - \frac{14}{4} - \frac{6}{1}$$

$$= \frac{1-14-24}{4} = \frac{1-38}{4} = \frac{-37}{4}$$

ii) $\|f\|^2$

$$\|f\|^2 = \langle f,f\rangle$$

$$= \int_0^1 f(t)\cdot f(t)\,dt.$$

$$= \int_0^1 (t+2)(t+2)\,dt.$$

$$= \int_0^1 (t^2+2t+2t+4)\,dt.$$

$$= \int_0^1 (t^2+4t+4)\,dt.$$

$$= \left[\frac{t^3}{3} + \frac{\overset{2}{\cancel{4}}t^2}{\cancel{2}} + 4t\right]_0^1$$

$$= \frac{1}{3} + 2 + 4$$

$$= \frac{1+6+12}{3}$$

$$= \frac{19}{3}$$

$$\Rightarrow \|f\|^2 = 19/3$$

$$\|f\| = \sqrt{19/3}$$

NOTE:

Find the norm of the following vectors.

1) $(1,1,1) = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.
2) $(1,2,3) = \sqrt{1+4+9} = \sqrt{14}$.
3) $(3,-4,0) = \sqrt{9+16} = \sqrt{25} = 5$

THEOREM-1:

Statement

P.T. the norm defined in an IPS V has the following properties.

i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.
ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$
iii) $|\langle x,y\rangle| \leq \|x\| \, \|y\|$ (Schward's inequality). (ask it alone)
iv) $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality)

Proof:

i) $\|x\| = \sqrt{\langle x,x\rangle}$

W.K.T. $\langle x,x\rangle \geq 0$.

$$\therefore \sqrt{\langle x,x\rangle} \geq 0.$$

$$\Rightarrow \|x\| \geq 0.$$

TPT $\|x\| = 0 \Leftrightarrow x = 0$.

$$\|x\| = 0 \Leftrightarrow \sqrt{\langle x,x\rangle} = 0.$$

$$\langle x,x\rangle = 0 \Rightarrow x = 0.$$

Hence $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.

ii) To P.T. $\|\alpha x\| = |\alpha| \|x\|$.

$$\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle$$

$$= \alpha \langle x, \alpha x \rangle$$

$$= \alpha \bar{\alpha} \langle x, x \rangle.$$

$$\|\alpha x\|^2 = |\alpha^2| \cdot \langle x, x \rangle.$$

12

Taking square root.

$$\Rightarrow \|\alpha x\| = |\alpha| \sqrt{\langle x, x \rangle}.$$

$$\|\alpha x\| = |\alpha| \|x\|.$$

iii) To P.T. $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$.

The inequality is true when $x = 0$ (or) $y = 0$.

Hence let $x \neq 0$ and $y \neq 0$.

Consider $z = y - \frac{\langle y, x \rangle}{\|x\|^2} \cdot x$.

W.K.T. $\langle z, z \rangle \geq 0$.

$$\Rightarrow \langle y - \frac{\langle y, x \rangle}{\|x\|^2} \cdot x, y - \frac{\langle y, x \rangle}{\|x\|^2} \cdot x \rangle \geq 0.$$

$$\Rightarrow \langle 1 \cdot y + \left(\frac{-\langle y, x \rangle}{\|x\|^2}\right) \cdot x, 1 \cdot y + \left(\frac{-\langle y, x \rangle}{\|x\|^2}\right) \cdot x \rangle \geq 0 \quad \text{--- (1)}$$

W.K.T.

$$\langle \alpha u + \beta v, \gamma w + \delta z \rangle = \alpha \bar{\gamma} \langle u, w \rangle + \alpha \bar{\delta} \langle u, z \rangle + \beta \bar{\gamma} \langle v, w \rangle + \beta \bar{\delta} \langle v, z \rangle$$

(Theorem)

Here $\alpha = \gamma = 1 \Rightarrow \bar{\alpha} = \bar{\gamma} = 1$.

$$\beta = \delta = \frac{-\langle y, x \rangle}{\|x\|^2} \Rightarrow \bar{\beta} = \bar{\delta} = \frac{+\overline{\langle y, x \rangle}}{\|x\|^2}.$$

$u = w = y$ and $v = z = x$.

$$① \Rightarrow 1 \cdot 1 \langle y, y\rangle + 1 \cdot \left[\overline{-\frac{\langle y, x\rangle}{\|x\|^2}}\right] \langle y, x\rangle + \left(-\frac{\langle y, x\rangle}{\|x\|^2}\right) \cdot 1 \langle x, y\rangle + \left(-\frac{\langle y, x\rangle}{\|x\|^2}\right)\left(\overline{-\frac{\langle y, x\rangle}{\|x\|^2}} \langle x, x\rangle\right) \geq 0.$$

$$\Rightarrow \|y\|^2 - \frac{\overline{\langle y, x\rangle}}{\|x\|^2} \langle y, x\rangle - \frac{\langle y, x\rangle}{\|x\|^2} \langle x, y\rangle + \frac{\langle y, x\rangle}{\|x\|^2} \cdot \frac{\overline{\langle y, x\rangle}}{\|x\|^2} \|x\|^2 \geq 0.$$

$$\Rightarrow \|y\|^2 - \frac{\langle x, y\rangle}{\|x\|^2} \overline{\langle x, y\rangle} - \frac{\langle y, x\rangle}{\|x\|^2} \langle x, y\rangle + \frac{\langle y, x\rangle}{\|x\|^2} \langle x, y\rangle \geq 0.$$

$$\Rightarrow \|y\|^2 - \frac{\langle x, y\rangle}{\|x\|^2} \overline{\langle x, y\rangle} \geq 0$$

$[\because z\bar{z} = |z|^2]$

$$\Rightarrow \|y\|^2 \geq \frac{|\langle (x, y)\rangle|^2}{\|x\|^2}$$

13

$$\|x\|^2 \|y\|^2 \geq |\langle x, y\rangle|^2.$$

$$|\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2$$

Taking square root,

$$\Rightarrow |\langle x, y\rangle| \leq \|x\| \, \|y\|.$$

iv) To p.T: $\|x+y\| \leq \|x\| + \|y\|$

Now, $\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle 1 \cdot x + 1 \cdot y, 1 \cdot x + 1 \cdot y\rangle$ —— ②

W.K.T, $\langle \alpha u + \beta v, \gamma w + \delta z\rangle = \alpha\bar{\gamma} \langle u, w\rangle + \alpha\bar{\delta} \langle u, z\rangle + \beta\bar{\gamma} \langle v, w\rangle + \beta\bar{\delta} \langle v, z\rangle$

Here, $\alpha = \beta = \gamma = \delta = 1$ then $\bar{\alpha} = \bar{\beta} = \bar{\gamma} = \bar{\delta} = 1$.

$u = w = x$; $v = z = y$

$$② \Rightarrow \|x+y\|^2 = 1 \cdot 1 \langle x, x\rangle + 1 \cdot 1 \langle x, y\rangle + 1 \cdot 1 \langle y, x\rangle + 1 \cdot 1 \langle y, y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2.$$

① + ② ⇒

$$\|x+y\|^2 + \|x-y\|^2 = \|x\|^2 + \|y\|^2 + \|x\|^2 + \|y\|^2$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right].$$

Hence proved

2) S.T. in a real IPS, if $\langle x, y\rangle = 0$ then p.T.
$\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Sol:

From the prev thm,

$$\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2.$$

$$= \|x\|^2 + 0 + \|y\|^2 \qquad [\because \langle x, y\rangle = 0]$$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

15

ORTHOGONALITY

Let V be an IPS and let $x, y \in V$.
x is said to be orthogonal to y, if $\langle x, y\rangle = 0$.

V.V.X IMP. Q

NOTE:

i) x is orthogonal to $y \Rightarrow \langle x, y\rangle = 0$

$\Rightarrow \overline{\langle x, y\rangle} = \overline{0}$

$\Rightarrow \langle y, x\rangle = 0.$

$\Rightarrow y$ is orthogonal to x.

Thus x & y are orthogonal iff $\langle x, y\rangle = 0$.

ii) x is orthogonal to $y \Rightarrow \alpha x$ is orthogonal to y.

V.V.IMP

iii) 0 is orthogonal to every vector in V and 0 is the only vector with this property.

iii) x_1, x_2 are orthogonal to $y \Rightarrow x_1 + x_2$ is orthogonal to y

$$= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2 \qquad [\because z+\bar{z} = 2\,\mathrm{Re}(z)$$

$$\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \qquad \therefore \mathrm{Re}(z) \le |z|]$$

$$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$\le (\|x\| + \|y\|)^2.$$

$[z = x+iy$, $|z| = \sqrt{x^2+y^2}$, $\mathrm{Re}(z) = x$, $x \le \sqrt{x^2+y^2}$, $\mathrm{Re}(z) \le |z|]$

ie) $\|x+y\|^2 \le (\|x\| + \|y\|)^2$.

Taking square root on both sides, we get,

$$\|x+y\| \le \|x\| + \|y\|.$$

Hence proved.

14

Pblms:

1) S.T. In any IPS V, $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Sol:

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \langle 1\cdot x + 1\cdot y, 1\cdot x + 1\cdot y\rangle$$

W.K.T,

$$\langle \alpha u + \beta v, \gamma w + \delta z\rangle = \alpha\bar{\gamma}\langle u, w\rangle + \alpha\bar{\delta}\langle u, z\rangle + \beta\bar{\gamma}\langle v, w\rangle + \beta\bar{\delta}\langle v, z\rangle.$$

Hence, $\alpha = \beta = \gamma = \delta = 1$

$\bar{\alpha} = \bar{\beta} = \bar{\gamma} = \bar{\delta} = 1$.

$u = w = x$; $v = z = y$.

$$\|x+y\|^2 = 1\cdot 1\langle x, x\rangle + 1\cdot 1\langle x, y\rangle + 1\cdot 1\langle y, x\rangle + 1\cdot 1\langle y, y\rangle.$$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2.$$

$$\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2 \quad \text{——— (1)}$$

$$\text{IIIly}, \quad \|x-y\|^2 = \|x\|^2 - 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2 \quad \text{——— (2)}.$$

Scanned with CamScanner

10.03.2017

Defn:

*Let V be an IPS a set S of vectors in V is said to be orthogonal set if any 2 distinct vectors in S are orthogonal.

*S is said to be ortho normal set if S is orthogonal and $\|x\| = 1 \ \forall x \in S$.

eg: 16

The std basis $E_1 = \{e_1, e_2, \ldots, e_n\}$ in R^n or C^n is an orthogonal set with respect to the std inner pdt.

THEOREM-2:

Statement

Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of non-zero vectors in an IPS, then S is L.I.

Proof:

G.T. $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of non-zero vectors in an IPS V.

To P.T. S is L.I

Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$.

Then $\langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_1 \rangle = \langle 0, v_1 \rangle = 0$.

$\langle \alpha_1 v_1, v_1 \rangle + \langle \alpha_2 v_2, v_1 \rangle + \cdots + \langle \alpha_n v_n, v_1 \rangle = 0$.

$\alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \cdot 0 + \cdots + \alpha_n \cdot 0 = 0$ [$\because S$ is orthogonal]

$\alpha_1 \langle v_1, v_1 \rangle = 0 \Rightarrow \alpha_1 = 0$. [$\because v_1$ is non-zero].

$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$.

Hence S is L.I.

Hence proved.

THEOREM-3:

Statement

Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of non-zero vectors in V. $v \in V$ and $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

Then P.T. $\alpha_K = \frac{\langle v, v_K \rangle}{\|v_K\|^2}$.

Proof:

G.T. $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$.

$\langle v, v_K \rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_K \rangle$

$= \langle \alpha_1 v_1, v_K \rangle + \langle \alpha_2 v_2, v_K \rangle + \cdots + \langle \alpha_n v_n, v_K \rangle$

$= \alpha_1 \langle v_1, v_K \rangle + \alpha_2 \langle v_2, v_K \rangle + \cdots + \alpha_K \langle v_K, v_K \rangle + \cdots + \alpha_n \langle v_n, v_K \rangle$

$= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 + \cdots + \alpha_K \langle v_K, v_K \rangle + \cdots + \alpha_n \cdot 0$

[$\because$ S is orthogonal]

$\langle v, v_K \rangle = \alpha_K \langle v_K, v_K \rangle$.

$\langle v, v_K \rangle = \alpha_K \cdot \|v_K\|^2$.

17

$\alpha_K = \frac{\langle v, v_K \rangle}{\|v_K\|^2}$.

Hence proved.

13.3.17

compulsory

THEOREM-4

Statement

Every finite dimensional IPS has an ~~orthogonal~~ orthonormal basis.

Proof:

Let V be a finite dimensional IPS.

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. from this basis we shall construct an orthonormal basis $\{w_1, w_2, \ldots, w_n\}$ by means of a construction known as Gram-Schmidt orgonilization process.

(may be asked in this type name)

First we take $w_1 = v_1$.

Let $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$.

We claim that, $w_2 \neq 0$.

Suppose if $w_2 = 0$, then $\frac{v_2}{\|v_1\|}$ is a scalar multiple of w_1 and hence of v_1 which is a $\Rightarrow\Leftarrow$.

Since v_1, v_2 are L.I.

Hence $w_2 \neq 0$.

Also $\langle w_2, w_1 \rangle = \langle v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} \cdot w_1, w_1 \rangle$

$= \langle v_2 - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} \cdot v_1, v_1 \rangle$ 18

$[\because w_1 = v_1]$

$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} \langle v_1, v_1 \rangle$

$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} \|v_1\|^2$.

$= \langle v_2, v_1 \rangle - \langle v_2, v_1 \rangle$

$\langle w_2, w_1 \rangle = 0$.

13.03.2017

Now, Suppose that we have constructed non-zero orthogonal vectors $w_1, w_2, \ldots, w_K$ then we put,

$$w_{K+1} = v_{K+1} - \sum_{j=1}^{K} \frac{\langle v_{K+1}, w_j\rangle}{\|w_j\|^2} \cdot w_j$$

We claim that $w_{K+1} \neq 0$.

Suppose if $w_{K+1} = 0$ then v_{K+1} is a linear combination of $w_1, w_2, \ldots, w_K$ and hence it is a L.C of $v_1, v_2, \ldots, v_K$ which is a $\Rightarrow\Leftarrow$

Since $v_1, v_2, \ldots, v_{K+1}$ are L.I, hence $w_{K+1} \neq 0$.

$\langle w_{K+1}, w_i \rangle = \langle v_{K+1} - \sum_{j=1}^{K} \frac{\langle v_{K+1}, w_j\rangle}{\|w_j\|^2} w_j, w_i \rangle$

$= \langle v_{K+1}, w_i \rangle - \sum_{j=1}^{K} \frac{\langle v_{K+1}, w_j\rangle}{\|w_j\|^2} \langle w_j, w_i \rangle$

$$= \langle v_{k+1}, w_i \rangle - \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} . \langle w_i, w_i \rangle$$

(distinct vectors are zero if & since it is orthogonal)

(So that we only write equal vectors)

$$= \langle v_{k+1}, w_i \rangle - \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} \cancel{\|w_i\|^2}$$

$$= \langle v_{k+1}, w_i \rangle - \langle v_{k+1}, w_i \rangle$$

$\langle w_{k+1}, w_i \rangle = 0$.

Thus continuing in this way we obtain a non-zero orthogonal set $\{w_1, w_2, \dots, w_n\}$ (by prev thm-2).

$\therefore$ This set is L.I and hence it is a basis

To obtain an ~~orthogonal~~ normal basis ~~use~~ we replace each w_i by, $\frac{w_i}{\|w_i\|}$ Hence proved.

PROBLEMS:

1) Apply Gram-schmidt process to construct an orthonormal basis for $V_3(R)$ with the std inner pdt for the basis $\{v_1, v_2, v_3\}$ where $v_1 = (1,0,1)$. $v_2 = (1,3,1)$, $v_3 = (3,2,1)$.

Sol:

Let $w_1 = v_1 = (1,0,1)$.

Then $\|w_1\|^2 = \langle w_1, w_1 \rangle$.

$$= \langle (1,0,1), (1,0,1) \rangle$$

$$= 1^2 + 0^2 + 1^2 = 2.$$

$$\|w_1\| = \sqrt{2}.$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$\langle v_2, w_1 \rangle = \langle (1,3,1), (1,0,1) \rangle$$

$$= (1 \times 1) + (3 \times 0) + (1 \times 1)$$

$$= 1 + 1 = 2.$$

Scanned with CamScanner

$w_2 = (1,3,1) - \frac{2}{2} \cdot (1,0,1)$

$= (1,3,1) - (1,0,1)$

$\therefore w_2 = (0,3,0)$

$\|w_2\|^2 = \langle w_2, w_2 \rangle = 0 + 3^2 + 0 = 9$

$\|w_2\| = 3.$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} \cdot w_2$$

$\langle v_3, w_1 \rangle = \langle (3,2,1), (1,0,1) \rangle.$

$= 3 + 0 + 1 = 4.$

$\langle v_3, w_2 \rangle = \langle (3,2,1), (0,3,0) \rangle = 0 + 6 + 0 = 6.$

$w_3 = (3,2,1) - \frac{4}{2}(1,0,1) - 6/9 (0,3,0).$

$= (3,2,1) - 2(1,0,1) - \frac{2}{3}(0,3,0).$

$= (3,2,1) - (2,0,2) - (0, 6/3, 0)$

$= (3,2,1) - (2,0,2) - (0,2,0).$

$= (3-2-0), (2-0-2), (1-2-0).$

$w_3 = (1,0,-1)$

$\|w_3\|^2 = \langle w_3, w_3 \rangle$

$= 1^2 + 1^2 = 2.$

$\|w_3\|^2 = 2 \Rightarrow \|w_3\| = \sqrt{2}.$

The orthogonal basis is $\{(1,0,1), (0,3,0), (1,0,-1)\}$.

ANS:

Hence the orthonormal basis is,

$$\left\{ \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right), (0,1,0), \left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right) \right\}$$

$\left[\because \text{orthonormal} \Rightarrow \frac{w_i}{\|w_i\|}\right]$

14.03.2017

2) $v_1 = (1,-1,0)$, $v_2 = (2,-1,-2)$, $v_3 = (1,-1,-2)$.

Sol:

Let $w_1 = v_1 = (1,-1,0)$.

21

Then $\|w_1\|^2 = \langle w_1, w_1 \rangle$

$= \langle (1,-1,0), (1,-1,0) \rangle$

$= 1^2 + 1^2 + 0^2 = 2.$

$\|w_1\|^2 = 2.$

$\|w_1\| = \sqrt{2}.$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1$$

$\langle v_2, w_1 \rangle = \langle (2,-1,-2), (1,-1,0) \rangle$

$= 2 \times 1 + (-1) \times (-1) + (-2) \times 0$

$= 2 + 1 - 0$

$= 3.$

$w_2 = (2,-1,-2) - \frac{3}{2}(1,-1,0)$

$= (2,-1,-2) - (3/2, -3/2, 0).$

$w_2 = (1/2, 1/2, -2)$

$\|w_2\|^2 = \cancel{1/2} \langle w_2, w_2 \rangle$

$= (1/2)^2 + (1/2)^2 + (-2)^2$

$= 1/4 + 1/4 + 4.$

$= 1/2 + 4.$

$\|w_2\|^2 = 9/2.$

$\|w_2\| = 3/\sqrt{2}.$

Scanned with CamScanner

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} \cdot w_2 .$$

22

$$\langle v_3, w_1\rangle = \langle (1,-1,-2), (1,-1,0)\rangle$$

$$= 1\times 1 + (-1)\times(-1) + (-2)\times 0 .$$

$$= 1+1+0 = 2$$

$$\langle v_3, w_2\rangle = \langle (1,-1,-2), (1/2, 1/2, -2)\rangle .$$

$$= 1\times 1/2 + (-1)\times 1/2 + (-2)\times(-2)$$

$$= 1/2 - 1/2 + 4 = 4 .$$

$$w_3 = (1,-1,-2) - \frac{2}{2}(1,-1,0) - \frac{4}{(3/\sqrt{2})^2}(1/2, 1/2, -2)$$

$$= (1,-1,-2) - (1,-1,0) - \frac{4\times 2}{9}(1/2, 1/2, -2) .$$

$$= (1,-1,-2) - (1,-1,0) - 8/9(1/2, 1/2, -2) .$$

$$= (1,-1,-2) - (1,-1,0) - (4/9, 4/9, -16/9)$$

$$= (1-1-4/9, -1+1-4/9, -2-0+16/9)$$

$$w_3 = (-4/9, -4/9, -2/9)$$

$$\|w_3\|^2 = (-4/9)^2 + (-4/9)^2 + (-2/9)^2$$

$$= 16/81 + 16/81 + 4/81$$

$$\|w_3\|^2 = 36/81$$

$$\|w_3\| = 6/9 = 2/3 .$$

The orthogonal basis is $\{(1,-1,0), (\frac{1}{2}, \frac{1}{2}, -2), (-\frac{4}{9}, -\frac{4}{9}, -\frac{2}{9})\}$.

Ans:

$(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{-2\sqrt{2}}{3})$

The orthonormal basis is $\{(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \cancel{(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{-2\sqrt{2}}{3})}, (-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})\}$.

15.03.2017

3) Let V be the set of all polynomial of degree ≤ 2 with zero polynomial. V is a real IPS with inner pdt defined by $\langle f, g\rangle = \int_{-1}^{1} f(x) g(x)\,dx$. Starting with the basis $\{1, x, x^2\}$ obtain an orthonormal basis for V.

Sol:

Let $v_1 = 1$, $v_2 = x$, $v_3 = x^2$.

Let $\omega_1 = v_1$.

Then $\|\omega_1\|^2 = \langle \omega_1, \omega_1 \rangle$

$= \int_{-1}^{1} 1.1\,dx = \int_{-1}^{1} dx = [x]_{-1}^{1} = 1-(-1) = 1+1.$

$\|\omega_1\|^2 = 2.$

$\|\omega_1\| = \sqrt{2}.$

$$\omega_2 = v_2 - \frac{\langle v_2, \omega_1 \rangle}{\|\omega_1\|^2} \cdot \omega_1$$

$\langle v_2, \omega_1 \rangle = \langle x, 1 \rangle$

$= \int_{-1}^{1} x.1\,dx = \int_{-1}^{1} x\,dx$

$= \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.$

$\omega_2 = x - \frac{0}{2} \cdot 1 = x$

$\|\omega_2\|^2 = \langle \omega_2, \omega_2 \rangle = \langle x, x \rangle.$

$$= \int_{-1}^{1} x \cdot x\,dx = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = 2/3.$$

$$\|w_2\| = \sqrt{2/3}.$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} \cdot w_2.$$

$$\langle v_3, w_1\rangle = \langle x^2, 1\rangle = \int_{-1}^{1} x^2 \cdot 1 \cdot dx$$

24

$$= \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{3} + \frac{1}{3} = 2/3.$$

$$\langle v_3, w_2\rangle = \langle x^2, x\rangle = \int_{-1}^{1} x^2 \cdot x\,dx$$

$$= \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4} - 1/4 = 0.$$

$$w_3 = x^2 - \frac{2/3}{2} \cdot 1 - \frac{0}{2/3} \cdot x.$$

$$= x^2 - 2/6 = x^2 - 1/3$$

$$\|w_3\|^2 = \langle w_3, w_3\rangle.$$

$$= \langle x^2 - 1/3, x^2 - 1/3\rangle.$$

$$= \int_{-1}^{1} (x^2 - 1/3)(x^2 - 1/3)\,dx.$$

$$= \int_{-1}^{1} \left(x^4 - x^2/3 - x^2/3 + 1/9\right) dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2x^2}{3} + 1/9\right) dx$$

$$= \left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right]_{-1}^{1}$$

$$= \left[\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right] - \left[\frac{-1}{5} + \frac{2}{9} - 1/9\right]$$

$$= \left[\frac{1}{5} - \frac{1}{9}\right] - \left[-1/5 + 1/9\right] = \frac{1}{5} - \frac{1}{9} + \frac{1}{5} - 1/9$$

$$= \frac{2}{5} - \frac{2}{9}$$

$$= \frac{18-10}{45} = 8/45$$

$$\|w_3\|^2 = 8/45.$$

$$\|w_3\| = \sqrt{8/45} = \frac{2\sqrt{2}}{3\sqrt{5}}.$$

25

orthogonal basis $(1, x, x^2 - 1/3)$

orthonormal is $\left[\frac{1}{\sqrt{2}}, \frac{x}{\sqrt{2/3}}, \frac{x^2 - 1/3}{2\sqrt{2}/3\sqrt{5}}\right]$

ANS:

i.e., $\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}x}{\sqrt{2}}, \frac{3\sqrt{5}\,(x^2 - 1/3)}{2\sqrt{2}}\right)$

16.03.2017

4) Find the vector of unit length which is orthogonal to $(1, 3, 4)$ in $V_3(R)$ with std inner pdt

Sol:

Let $x = \{x_1, x_2, x_3\}$ be any vector orthogonal to $(1, 3, 4)$.

Then $x_1 + 3x_2 + 4x_3 = 0$.

Any soln of this eqn gives a vector orthogonal to $(1,3,4)$. for eg, $x = (1, 1, -1)$ is orthogonal to $(1, 3, 4)$.

$$\|x\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}.$$

ANS: Hence a unit vector orthogonal to $(1, 3, 4)$ is $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$.

ORTHOGONAL COMPLEMENT

Defn:

* Let V be an IPS. Let S be a subset of V. The orthogonal complement of S denoted by $\underline{S^\perp}$ ($\to$ S complement) is the set of all vectors in V which are orthogonal to every vector of S.

ie) $S^\perp = \{x / x \in V \ \& \ \langle x, u\rangle = 0 \ \forall u \in S\}$.

eg:

$V^\perp = \{\bar{0}\}$ and $\{\bar{0}\}^\perp = V$. 26

$\{0\} \to$ It denotes set

$0 \to$ It denotes elt

Since 0 is the only vector ~~and {0}~~ which is orthogonal to every vector.

NOTE: *

i) Let $S = \{(x,0,0) / x \in R\} \subseteq V_3(R)$ with a std inner pdt. Then $S^\perp = \{(0,y,z) / y, z \in R\}$. ie) the orthogonal complement of the x-axis is the $\underline{yz\text{-plane}}$.

for y-axis — xz-plane $(x,0,z)$

z " $\to$ xy "

THEOREM-5:

Statement

If S is any subset of V then $\underline{S^\perp}$ is a $\underline{\text{subspace}}$ of V.

Pf:

Clearly $0 \in S^\perp$.

$\therefore S^\perp \neq \phi$.

Let $x, y \in S^\perp$ and $\alpha, \beta \in F$

$\langle x, u\rangle = \langle y, u\rangle = 0$.

$\langle \alpha x + \beta y, u\rangle = \langle \alpha x, u\rangle + \langle \beta y, u\rangle$

$= \alpha\langle x, u\rangle + \beta\langle y, u\rangle$

$= \alpha(0) + \beta . 0 = 0$.

Scanned with CamScanner

$\therefore \langle \alpha x+\beta y, u\rangle = 0$.

$\therefore \alpha x+\beta y \in S^{\perp}$

Hence $S^{\perp}$ is a Subspace of V.

THEOREM-6:

Statement

Let V be a finite dimensional IPS. Let W be a subspace of V. Then P.T. V is the direct sum of W and $W^{\perp}$ ie) $V = W \oplus W^{\perp}$

Pf:

To P.T. $V = W \oplus W^{\perp}$

ie) to prove $W \cap W^{\perp} = \{0\}$ and $W + W^{\perp} = V$.

Let $v \in W \cap W^{\perp}$.

Then $v \in W$ and $v \in W^{\perp}$

$v \in W^{\perp} \Rightarrow v$ is orthogonal to every elt in W.

In particular v is orthogonal to itself.

ie) $\langle v, v\rangle = 0$.

Hence $v = 0$.

Hence $W \cap W^{\perp} = \{0\}$. — ①

ii) Let $\{v_1, v_2, \dots, v_r\}$ be an orthonormal basis for W.

Let $v \in V$.

Consider, $v_0 = v - \langle v, v_1\rangle v_1 - \langle v, v_2\rangle v_2 - \dots - \langle v, v_r\rangle v_r$

$\langle v_0, v_i\rangle = \langle v, v_i\rangle - \langle v, v_1\rangle\langle v_1, v_i\rangle - \langle v, v_2\rangle\langle v_2, v_i\rangle \dots$

$- \langle v, v_i\rangle\langle v_i, v_i\rangle \dots - \langle v, v_r\rangle\langle v_r, v_i\rangle$

$\langle v_0, v_i\rangle = \langle v, v_i\rangle - \langle v, v_i\rangle\langle v_i, v_i\rangle$ [$\because \langle v_i, v_j\rangle = 0 \ \forall i \neq j$]

$= \langle v, v_i\rangle - \langle v, v_i\rangle$ [$\because \langle v_i, v_i\rangle = 1$]

$= 0$

$\therefore v_0$ is orthogonal to each of $v_1, v_2, \dots, v_r$ and hence orthogonal to every vector in W.

Hence $v_0 \in W^{\perp}$ and

$$v = [\langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + \dots + \langle v, v_r\rangle v_r + v_0]$$

$$\Rightarrow v \in W + W^{\perp}$$

$$V = W + W^{\perp} \text{ --- } (2)$$

28

From (1) & (2)

$V = W \oplus W^{\perp}$. Hence proved.

Corollary

$$\dim V = \dim W + \dim W^{\perp}$$

Pf:

$\dim V = \dim (W + W^{\perp})$ [$\because$ If $V = A \oplus B$ then $\dim V = \dim A + \dim B$] (Refer thm-20 in Unit-1)

$= \dim W + \dim W^{\perp}$.

THEOREM-7:

Statement

Let V be any finite dimensional IPS. Let W be a subspace of V. then $(W^{\perp})^{\perp} = W$.

Pf:

Let $w \in W$ then for any $u \in W^{\perp}$

$\langle w, u\rangle = 0$.

Hence, $w \in (W^{\perp})^{\perp}$

Then $W \subseteq (W^{\perp})^{\perp}$

By prev thm, $V = W \oplus W^{\perp} \Rightarrow \dim V = \dim W + \dim W^{\perp}$ --- (1)

Also, $V = W^{\perp} \oplus (W^{\perp})^{\perp} \Rightarrow \dim V = \dim W^{\perp} + \dim (W^{\perp})^{\perp}$ --- (2)

Equate (1) & (2),

Hence $\dim W = \dim (W^{\perp})^{\perp}$

From (1) & (2); $W = (W^{\perp})^{\perp}$. Hence proved.

Pblms:

1) Let V be an IPS and S_1 & S_2 be subset of V. then $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$.

Sol:

Let $u \in S_2^{\perp}$

Then $\langle u, v \rangle = 0 \ \forall \ v \in S_2$.

but $S_1 \subseteq S_2$.

Hence $\langle u, v \rangle = 0 \ \forall \ v \in S_1$

Hence $u \in S_1^{\perp}$

Thus $S_2^{\perp} \subseteq S_1^{\perp}$.

29

2) Let W_1 and W_2 be subspaces of a finite dimensional IPS. Then i) $(W_1+W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$;
ii) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$. V.V.X.I.M.

Sol:

i) W.K.T. $W_1 \subseteq W_1 + W_2$.

$(W_1+W_2)^{\perp} \subseteq W_1^{\perp}$ (By the abv thm).

$|||^{ly}$ $(W_1+W_2)^{\perp} \subseteq W_2^{\perp}$.

Hence $(W_1+W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$ —— (1)

$\Rightarrow$ TP $W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1+W_2)^{\perp}$.

Now, let $w \in W_1^{\perp} \cap W_2^{\perp}$.

Then $w \in W_1^{\perp}$ and $w \in W_2^{\perp}$.

$\therefore \langle w, u \rangle = 0 \ \forall \ u \in W_1$ and W_2.

Now, let $v \in W_1 + W_2$ then $v = v_1 + v_2$. where $v_1 \in W_1$ & $v_2 \in W_2$

$\therefore \langle w, v \rangle = \langle w, v_1 + v_2 \rangle$

$= \langle w, v_1 \rangle + \langle w, v_2 \rangle = 0 + 0 = 0$ $[\because v_1 \in W_1$ & $v_2 \in W_2]$

Hence $w \in (W_1+W_2)^{\perp}$

$\therefore W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1+W_2)^{\perp}$ —— (2).

From (1) & (2) we get,

$(W_1+W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

ii) Proof is similar that of (i).

H.W.:

5) $v_1 = (1,0,1),\ v_2 = (1,0,-1),\ v_3 = (0,3,4)$.

Sol:

Let $w_1 = v_1 = (1,0,1)$

Then $\|w_1\|^2 = \langle (1,0,1),(1,0,1) \rangle$

$= 1^2+0^2+1^2 = 2.$

$\|w_1\|^2 = 2.$

$\|w_1\| = \sqrt{2}.$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1.$$

$\langle v_2, w_1 \rangle = \langle (1,0,-1),(1,0,1) \rangle$

$= 1\times1 + 0 + (-1)\times 1 = 1-1 = 0.$

$w_2 = v_2 = (1,0,-1).$

$\|w_2\|^2 = 1^2+0^2+(-1)^2 = 1+1 = 2$

$\|w_2\| = \sqrt{2}.$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} \cdot w_2.$$

$\langle v_3, w_1 \rangle = \langle (0,3,4),(1,0,1) \rangle.$

$= 0+0+4 = 4.$

$\langle v_3, w_2 \rangle = \langle (0,3,4),(1,0,-1) \rangle$

$= 0+0-4 = -4.$

$\cancel{w_3 = (0,3,4) - \frac{(0,3,4)}{2}}$

$w_3 = (0,3,4) - \frac{4}{2}\cdot(1,0,1) + \frac{4}{2}(1,0,-1)$

$= (0,3,4) - 2(1,0,1) + 2(1,0,-1).$

$= (0,3,4) + (-2,0,-2) + (2,0,-2)$

$w_3 = (0,3,0).$

$\|w_3\|^2 = 0^2+3^2+0^2 = 3^2 = 9.$

31

$\|w_3\| = 3.$

The orthogonal basis is $\{(1,0,1),(1,0,-1),(0,3,0)\}$.

ANS:

The orthonormal basis is $\{(1/\sqrt{2},0,1/\sqrt{2}),(1/\sqrt{2},0,-1/\sqrt{2}),(0,1,0)\}$

× —— × —— ×

18.03.2017

UNIT-4 (Seminar portion)

(15)

THEORY OF MATRICES

ALGEBRA OF MATRICES

Defn:

An $m\times n$ matrix A is an array of mn numbers a_{ij} where $1\le i\le m$, $1\le j\le n$ arranged in m rows and n columns as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We shall denote the matrix by the symbol (a_{ij}). If $m=n$ then A is called a square matrix of order n.

Defn:

eg: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Two matrices $A=(a_{ij})$ and $B=(b_{ij})$ are said to be equal if A and B have the same number of rows and columns and the corresponding entries in the two matrices are same.

ADDITION OF MATRICES

We have already defined the addition of two $m\times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$ by