GANGA GUIDE

MATHEMATICS

10

Based on the New Textbook 2019

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FOREWORD

This **GANGA GUIDE** for **Mathematics** for **Class X** has been prepared written in the hope of sharing the excitement experienced in the study of Mathematics strictly in accordance with the text book published by our Government of TamilNadu in 2019.

We have come forward to facilitate learning maths in a more meaningful and pleasant manner.

The most salient features of this comprehensive guide.

- **The Concept of the content given in the text book is printed at the beginning of every topic as "KEY POINTS".**
- **All the textual example problems are given for all the 8 Units.**
- The exercise problems have been solved with finest solutions illustrated so charmingly by the author of this Guide.
- The Guide contains newly-designed problems (15 to 20% will be asked in the Public Exam) in the form of Objective type, Short and Long answer types.

Every effort has been taken to learn mathematics tension-free. The Ganga Guide provides exam-oriented learning inputs that will go a long way in assisting the students to come out successful in the public exam.

With the **Ganga Guide** for **Mathematics X** we wish the students and the teachers a very happy, pleasant, fruitful academic year.

-Publisher

10 MATHEMATICS

CONTENTS

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CHAPTER 1 RELATIONS AND FUNCTIONS

I. ORDERED PAIR AND CARTESIAN PRODUCT :

Key Points

- ✓ If A and B are two non-empty sets, then the set of all ordered pairs (a, b) such that $a \in A$, $b \in B$ is called the Cartesian Product of A and B, and is denoted by $A \times B$. Thus, $A \times B = \{(a, b) \mid a \in A, b \in B\}$.
- ✓ $A \times B$ is the set of all possible ordered pairs between the elements of A and B such that the first coordinate is an element of A and the second coordinate is an element of B.
- ✓ $B \times A$ is the set of all possible ordered pairs between the elements of A and B such that the first coordinate is an element of B and the second coordinate is an element of A.
- ✓ If $a = b$ then $(a, b) = (b, a)$.
- ✓ The "cartesian product" is also referred as "cross product".
- ✓ In general $A \times B \neq B \times A$, but $n(A \times B) = n(B \times A)$.
- ✓ $A \times B = \phi$ if and only if $A = \phi$ or $B = \phi$.
- ✓ If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$.
- ✓ For any three sets A, B, C we have

  (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Example 1.1

If $A = \{1, 3, 5\}$ and $B = \{2, 3\}$ then (i) find $A \times B$ and $B \times A$. (ii) Is $A \times B = B \times A$? If not why ? (iii) Show that $n(A \times B) = n(B \times A) = n(A) \times n(B)$.

Solution :

Given that $A = \{1, 3, 5\}$ and $B = \{2, 3\}$

(i) $A \times B = \{1, 3, 5\} \times \{2, 3\}$

$= \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$ ...(1)

$B \times A = \{2, 3\} \times \{1, 3, 5\}$

$= \{(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$ . ..(2)

(ii) From (1) and (2) we conclude that $A \times B \neq B \times A$ as $(1, 2) \neq (2, 1)$ and $(1, 3) \neq) (3, 1)$, etc.

(iii) $n(A) = 3$; $n(B) = 2$.

From (1) and (2) we observe that,

$n(A \times B) = n(B \times A) = 6$;

we see that, $n(A) \times n(B) = 3 \times 2 = 6$ and

$n(B) \times n(A) = 2 \times 3 = 6$

Hence, $n(A \times B) = n(B \times A) = n(A) \times n(B) = 6.$

Thus, $n(A \times B) = n(B \times A) = n(A) \times n(B).$

Example 1.2

If $A \times B = \{(3, 2), (3, 4), (5, 2), (5, 4)\}$ then find A and B.

Solution :

$A \times B = \{(3, 2), (3, 4), (5, 2), (5, 4)\}$

We have

A = {set of all first coordinates of elements of $A \times B$}. Therefore $A = \{3, 5\}$

B = {set of all second coordinates of elements of $A \times B$}. Therefore $B = \{2, 4\}$

Thus $A = \{3, 5\}$ and $B = \{2, 4\}$.

Example 1.3

Let $A = \{x \in N \mid 1 < x < 4\}$, $B = \{x \in W \mid 0 \leq x < 2\}$ and $C = \{x \in N \mid x < 3\}$.

Then verify that (i) $A\times(B\cup C) = (A\times B) \cup (A\times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution :

$A = \{x \in N \mid 1 < x < 4\} = \{2, 3\}$,

$B = \{x \in W \mid 0 \leq x < 2\} = \{0, 1\}$.

$C = \{x \in N \mid x < 3\} = \{1, 2\}$

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$B \cup C = \{0, 1\} \cup \{1, 2\} = \{0, 1, 2\}$

$A \times (B \cup C) = \{2, 3\} \times \{0, 1, 2\}$

$= \{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$ ... (1)

$A \times B = \{2, 3\} \times \{0, 1\}$

$= \{(2, 0), (2, 1), (3, 0), (3, 1)\}$

$A \times C = \{2, 3\} \times \{1, 2\}$

$= \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$(A \times B) \cup (A \times C) = \{(2, 0), (2, 1), (3, 0), (3, 1)\} \cup \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$= \{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$ ... (2)

From (1) and (2),

$A \times (B \cup C) = (A \times B) \cup (A \times C)$ is verified.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$B \cap C = \{0, 1\} \cap \{1, 2\} = \{1\}$

$A \times (B \cap C) = \{2, 3\} \times \{1\}$

$= \{(2, 1), (3, 1)\}$ ... (3)

$A \times B = \{2, 3\} \times \{0, 1\}$

$= \{(2, 0), (2, 1), (3, 0), (3, 1)\}$

$A \times C = \{2, 3\} \times \{1, 2\}$

$= \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$(A \times B) \cap (A \times C)$

$= \{(2, 0), (2, 1), (3, 0), (3, 1)\} \cap \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$= \{(2, 1), (3, 1)\}$ ... (4)

From (3) and (4),

$A\times(B \cap C) = (A \times B) \cap (A \times C)$ is verified.

EXERCISE 1.1

1. Find $A \times B$, $A \times A$ and $B \times A$

(i) $A = \{2, -2, 3\}$ and $B = \{1, -4\}$

(ii) $A = B = \{p, q\}$ (iii) $A = \{m, n\}$; $B = \phi$

Solution:

(i) Given $A = \{2, -2, 3\}$, $B = \{1, -4\}$.

$A \times B = \{(2, 1), (2, -4), (-2, 1), (-2, -4), (3, 1), (3, -4)\}$

$A \times A = \{(2, 2), (2, -2), (2, 3), (-2, 2), (-2, -2), (-2, 3), (3, 2), (3, -2), (3, 3)\}$

$B \times A = \{(1, 2), (1, -2), (1, 3), (-4, 2), (-4, -2), (-4, 3)\}$

(ii) Given $A = B = \{p, q\}$

$A \times B = \{(p, p), (p, q), (q, p), (q, q)\}$

$A \times A = \{(p, p), (p, q), (q, p), (q, q)\}$

$B \times A = \{(p, p), (p, q), (q, p), (q, q)\}$

(iii) $A = \{m, n\}, B = \phi$

If $A = \phi$ (or) $B = \phi$, then $A \times B = \phi$.

and $B \times A = \phi$

$A \times B = \phi$ and $B \times A = \phi$

$A \times A = \{(m, m), (m, n), (n, m), (n, n)\}$

2. Let A = {1, 2, 3} and B = {*x* | *x* is a prime number less than 10}. Find A × B and B × A.

Solution :

Given $A = \{1, 2, 3\}$, $B = \{x \mid x$ is a prime number less than $10\}$.

$B = \{2, 3, 5, 7\}$

$A \times B = \{(1, 2), (1, 3), (1, 5), (1, 7), (2, 2), (2, 3), (2, 5), (2, 7), (3, 2), (3, 3), (3, 5), (3, 7)\}$

$B \times A = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (5, 1), (5, 2), (5, 3), (7, 1), (7, 2), (7, 3)\}$

3. If B × A = {(– 2, 3), (– 2, 4), (0, 3), (0, 4), (3, 3), (3, 4)} find A and B.

Solution :

Given $B \times A = \{(-2, 3), (-2, 4), (0, 3), (0, 4), (3, 3), (3, 4)\}$

$\therefore B = \{-2, 0, 3\}, A = \{3, 4\}$

4. If A = {5, 6}, B = {4, 5, 6}, C = {5, 6, 7}. Show that A × A = (B × B) ∩ (C × C).

Solution :

Given $A = \{5, 6\}$, $B = \{4, 5, 6\}$, $C = \{5, 6, 7\}$

LHS : $A \times A = \{5, 6\} \times \{5, 6\}$

$= \{(5, 5), (5, 6), (6, 5), (6, 6)\}$ ...(1)

RHS : $B \times B = \{4, 5, 6\} \times \{4, 5, 6\}$

$= \{(4, 4), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$

$C \times C = \{5, 6, 7\} \times \{5, 6, 7\}$

$= \{(5, 5), (5, 6), (5, 7), (6, 5), (6, 6), (6, 7), (7, 5), (7, 6), (7, 7)\}$

$\therefore (B \times B) \cap (C \times C) = \{(5, 5), (5, 6), (6, 5), (6, 6)\}$ ...(2)

$\therefore$ From (1) and (2).

LHS = RHS

5. Given A = {1, 2, 3}, B = {2, 3, 5}, C = {3, 4} and D = {1, 3, 5}, check if (A ∩ C) × (B ∩ D) = (A × B) ∩ (C × D) is true ?

Solution :

Given $A = \{1, 2, 3\}$, $B = \{2, 3, 5\}$,

$C = \{3, 4\}$, $D = \{1, 3, 5\}$

$A \cap C = \{3\}$, $B \cap D = \{3, 5\}$

$\therefore (A \cap C) \times (B \cap D) = \{(3, 3), (3, 5)\}$ ... (1)

$A \times B = \{(1, 2), (1, 3), (1, 5), (2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 5)\}$

$C \times D = \{(3, 1), (3, 3), (3, 5), (4, 1), (4, 3), (4, 5)\}$

$\therefore (A \times B) \cap (C \times D) = \{(3, 3), (3, 5)\}$ ...(2)

$\therefore$ From (1) and (2)

LHS = RHS.

6. Let A = {$x \in W \mid x < 2$}, B = {$x \in N \mid 1 < x \leq 4$} and C = {3, 5}. Verify that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Solution :

Given $A = \{x \in W \mid x < 2\} \Rightarrow A = \{0, 1\}$

$B = \{x \in N \mid 1 < x \leq 4\}$

$\Rightarrow B = \{2, 3, 4\}$

$C = \{3, 5\}$

(i) To verify :

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

$B \cup C = \{2, 3, 4, 5\}$

$\therefore A \times (B \cup C) = \{(0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5)\}$ ...(1)

$A \times B = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$

$A \times C = \{(0, 3), (0, 5), (1, 3), (1, 5)\}$

$\therefore (A \times B) \cup (A \times C) = \{(0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5)\}$ ...(2)

$\therefore$ From (1) and (2) LHS = RHS.

(ii) To verify : $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$B \cap C = \{3\}$

$\therefore A \times (B \cap C) = \{(0, 3), (1, 3)\}$ ...(1)

$A \times B = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$

$A \times C = \{(0, 3), (0, 5), (1, 3), (1, 5)\}$

$\therefore (A \times B) \cap (A \times C) = \{(0, 3), (1, 3)\}$ ...(2)

$\therefore$ From (1) and (2), LHS = RHS.

(iii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

$A \cup B = \{0, 1, 2, 3, 4\}$

$\therefore (A \cup B) \times C = \{(0, 3), (0, 5), (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5)\}$ ...(1)

$A \times C = \{(0, 3), (0, 5), (1, 3), (1, 5)\}$

$B \times C = \{(2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5)\}$

$\therefore (A \times C) \cup (B \times C) = \{(0, 3), (0, 5), (1, 3), (1, 5), (2, 3), (2, 5) (3, 3), (3, 5), (4, 3), (4, 5)\}$ ...(2)

$\therefore$ From (1) and (2) LHS = RHS.

7. Let A = The set of all natural numbers less than 8, B = The set of all prime numbers less than 8, C = The set of even prime number. Verify that

(i) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(ii) $A \times (B - C) = (A \times B) - (A \times C)$

Solution :

Given $A = \{1, 2, 3, 4, 5, 6, 7\}$

$B = \{1, 3, 5, 7\}$

$C = \{2\}$

(i) To verify : $(A \cap B) \times C = (A \times C) \cap (B \times C)$

$A \cap B = \{1, 3, 5, 7\}$

$\therefore (A \cap B) \times C = \{(1, 2), (3, 2), (5, 2), (7, 2)\}$ ... (1)

$A \times C = \{(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (7, 2)\}$

$B \times C = \{(1, 2), (3, 2), (5, 2), (7, 2)\}$ ... (2)

$\therefore$ From (1) and (2), LHS = RHS.

(ii) To verify : $A \times (B - C) = (A \times B) - (A \times C)$

$B - C = \{1, 3, 5, 7\}$

$\therefore A \times (B - C) = \{(1, 1), (1, 3), (1, 5), (1, 7),$
$(2, 1), (2, 3), (2, 5), (2, 7),$
$(3, 1), (3, 3), (3, 5), (3, 7),$
$(4, 1), (4, 3), (4, 5), (4, 7),$
$(5, 1), (5, 3), (5, 5), (5, 7),$
$(6, 1), (6, 3), (6, 5), (6, 7),$
$(7, 1), (7, 3), (7, 5), (7, 7),\}$

$A \times B = \{(1, 1), (1, 3), (1, 5), (1, 7),$
$(2, 1), (2, 3), (2, 5), (2, 7),$
$(3, 1), (3, 3), (3, 5), (3, 7),$
$(4, 1), (4, 3), (4, 5), (4, 7),$
$(5, 1), (5, 3), (5, 5), (5, 7),$
$(6, 1), (6, 3), (6, 5), (6, 7),$
$(7, 1), (7, 3), (7, 5), (7, 7),\}$

$A \times C = \{(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (7, 2)\}$

$\therefore (A \times B) - (A \times C)$
$= \{(1, 1), (1, 3), (1, 5), (1, 7),$
$(2, 1), (2, 3), (2, 5), (2, 7),$
$(3, 1), (3, 3), (3, 5), (3, 7),$
$(4, 1), (4, 3), (4, 5), (4, 7),$
$(5, 1), (5, 3), (5, 5), (5, 7),$
$(6, 1), (6, 3), (6, 5), (6, 7),$
$(7, 1), (7, 3), (7, 5), (7, 7),\}$ ...(2)

$\therefore$ From (1) and (2), LHS = RHS.

II. RELATIONS :

Key Points

✓ Let A and B be any two non-empty sets. A 'relation' R from A to B is a subset of $A \times B$ satisfying some specified conditions. If $x \in A$ is related to $y \in B$ through R, then we write it as $x\text{R}y$. $x\text{R}y$ if and only if $(x, y) \in \text{R}$.

✓ A relation may be represented algebraically eithe rby the roster method or by the set builder method.

✓ An arrow diagram is a visual representation of a relation.

✓ A relation which contains no element is called a "Null relation".

✓ If $n(A) = p$, $n(B) = q$, then the total number of relations that exist between A and B is 2^{pq}.

Example 1.4

Let A = {3, 4, 7, 8} and B = {1, 7, 10}. Which of the following sets are relations from A to B ?

(i) $R_1 = \{(3, 7), (4, 7), (7, 10), (8, 1)\}$

(ii) $R_2 = \{(3, 1), (4, 12)\}$

(iii) $R_3 = \{(3, 7), (4, 10), (7, 7), (7, 8), (8, 11), (8, 7), (8, 10)\}$

Solution :

$A \times B = \{(3, 1), (3, 7), (3, 10), (4, 1), (4, 7), (4,10), (7,1), (7,7), (7,10), (8, 1), (8, 7), (8, 10)\}$

(i) We note that $R_1 \subseteq A \times B$. Thus, R_1 is a relation from A to B.

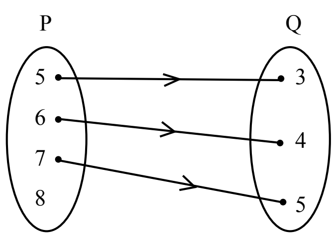
(ii) Here, $(4, 12) \in R_2$, but $(4, 12) \notin A \times B$. So, R_2 is not a relation from A to B.

(iii) Here, $(7, 8) \in R_3$, but $(7, 8) \notin A \times B$. So R_3 is not a relation from A to B.

Example 1.5

The arrow diagram shows a relationship between the sets P and Q. Write the relation in (i) Set builder form (ii) Roster form (iii) What is the domain and range of R.

Solution :



(i) Set builder form of

$R = \{(x, y) \mid y = x - 2, x \in P, y \in Q\}$

(ii) Roster form $R = \{(5, 3), (6, 4), (7, 5)\}$

(iii) Domain of $R = \{5, 6, 7\}$ and range of

$R = \{3, 4, 5\}$

EXERCISE 1.2

1. Let A = {1, 2, 3, 7} and B = {3, 0, – 1.7), which of the following are relation from A to B ?

(i) $R_1 = \{(2, 1), (7, 1)\}$
(ii) $R_2 = \{(-1, 1)\}$
(iii) $R_3 = \{(2, -1), (7, 7), (1, 3)\}$
(iv) $R_4 = \{(7, -1), (0, 3), (3, 3), (0, 7)\}$

Solution :

Given $A = \{1, 2, 3, 7\}$, $B = \{3, 0, -1, 7\}$

$\therefore A \times B = \{(1, 3), (1, 0), (1, -1), (1, 7), (2, 3), (2, 0), (2, -1), (2, 7), (3, 3), (3, 0), (3, -1), (3, 7), (7, 3), (7, 0), (7, -1), (7, 7)\}$

i) $R_1 = \{(2, 1), (7, 1)\}$

$(2, 1) \in R_1$ but $(2, 1) \notin A \times B$

$\therefore R_1$ is not a relation from A to B.

ii) $R_2 = \{(-1, 1)\}$

$(-1, 1) \in R_2$ but $(-1, 1) \notin A \times B$

$\therefore R_2$ is not a relation from A to B.

iii) $R_3 = \{(2, -1), (7, 7), (1, 3)\}$

We note that $R_3 \subseteq A \times B$

$\therefore R_3$ is a relation.

iv) $R_4 = \{(7, -1), (0, 3), (3, 3), (0, 7)\}$

$(0, 3), (0, 7) \in R_4$ but not in $A \times B$.

$\therefore R_4$ is not a relation.

2. Let A = {1, 2, 3, 4 ....., 45} and R be the relation defined as "is square of" on A. Write R as a subset of A × A. Also, find the domain and range of R.

Solution :

Given $A = \{1, 2, 3, 4, ... 45\}$

R : "is square of"

$R = \{1, 4, 9, 16, 25, 36\}$

Clearly R is a subset of A.

$\therefore$ Domain $= \{1, 2, 3, 4, 5, 6\}$

$\therefore$ Range $= \{1, 4, 9, 16, 25, 36\}$

3. A Relation R is given by the set {(*x, y*) / *y* = *x* + 3, *x* ∈ {0, 1, 2, 3, 4, 5}}. Determine its domain and range.

Solution :

Given R = $\{(x, y) / y = x + 3,$

$x \in \{0, 1, 2, 3, 4, 5\}\}$

$x = 0 \Rightarrow y = 3$

$x = 1 \Rightarrow y = 4$

$x = 2 \Rightarrow y = 5$

$x = 3 \Rightarrow y = 6$

$x = 4 \Rightarrow y = 7$

$x = 5 \Rightarrow y = 8$

$\therefore$ R = {(0, 3), (1, 4), (2, 5), (3, 6), (4, 7), (5, 8)}

$\therefore$ Domain : {0, 1, 2, 3,4, 5}

Range : {3, 4, 5, 6, 7, 8}

4. Represent each of the given relations by (a) an arrow diagram, (b) a graph and (c) a set in roster form, wherever possible.

(i) {(*x, y*) | *x* = 2*y*, *x* ∈ {2, 3, 4, 5}, *y* ∈ {1, 2, 3, 4}}

(ii) {(*x, y*) | *y* = *x* + 3, *x, y* are natural numbers < 10}

Solution :

i) $\{(x, y) \mid x = 2y, x \in \{2, 3, 4, 5\}, y \in \{1, 2, 3, 4\}\}$

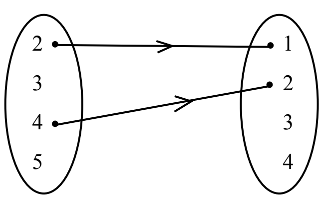
a) Arrow diagram :

$y = 1 \Rightarrow x = 2$

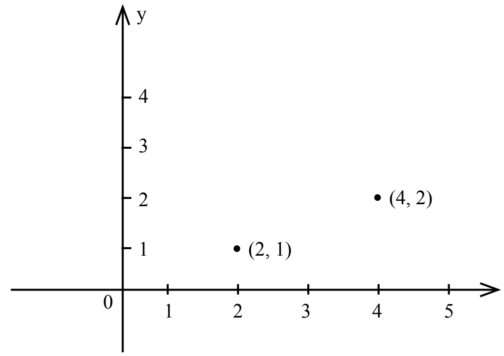
$y = 2 \Rightarrow x = 4$

$y = 3 \Rightarrow x = 6$

$y = 4 \Rightarrow x = 8$



b) Graph :

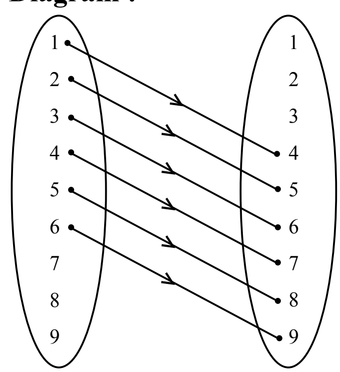


c) a set in roster : z {(2, 1), (4, 2)}

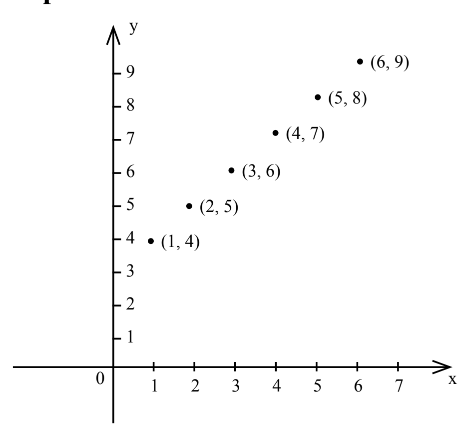
ii) $\{(x, y) \mid y = x + 3,$

x, y are natural numbers < 10}

a) Arrow Diagram :



b) Graph :



c) a set in roster :

= {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}

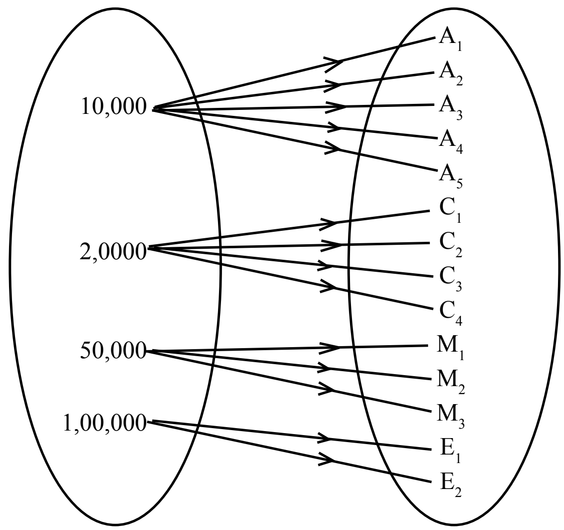
5. A company has four categories of employees given by Assistants (A), Clerks (C), Managers (M) and an Executive Officer (E). The company provide ₹10,000, ₹25,000, ₹50,000 and ₹1,00,000 as salaries to the people who work in the categories A, C, M and E respectively. If A_1, A_2, A_3, A_4 and A_5 were Assistants ; C_1, C_2, C_3, C_4 were Clerks ; M_1, M_2, M_3 were managers and E_1, E_2 were Executive officers and if the relation R is defined by xRy, where x is the salary given to person y, express the relation R through an ordered pair and an arrow diagram.

Solution :

a) Ordered Pair :

{(10000, A_1), (10000, A_2), (10000, A_3), (10000, A_4), (10000, A_5), (25000, C_1), (25000, C_2), (25000, C_3), (25000, C_4), (50000, M_1), (50000, M_2), (50000, M_3), (100000, E_1), (100000, E_{2s}).

b) Arrow Diagram :



III. FUNCTIONS :

Key Points

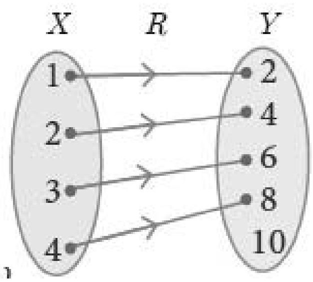
- ✓ A relation f between two non-empty sets X and Y is called a function from X to Y if, for each $x \in$ X there exists only one $y \in$ Y such that $(x, y) \in f$. That is, $f = \{(x, y) \mid$ for all $x \in$ X, $y \in$ Y$\}$.
- ✓ If f : X $\to$ Y is a function then the set X is called the domain of the function f and the set Y is called its co-domain.
- ✓ If $f(a) = b$, then b is called 'image' of a under f and a is called a 'pre-image' of b.
- ✓ The set of all images of the elements of X under f is called the 'range' of f.
- ✓ f : X $\to$ Y is a function only if
  (i) every element in the domain of f has an image.
  (ii) the image is unique.
- ✓ If A and B are finite sets such that n(A) = p, n(B) = q then the total number of functions that exist between A and B is q^p.
- ✓ The range of a function is a subset of its co-domain.

Example 1.6

Let X = {1, 2, 3, 4} and Y = {2, 4, 6, 8, 10} and R = {(1, 2), (2, 4), (3, 6), (4, 8)}. Show that R is a function and find its domain, co-domain and range ?

Solution :

Pictorial representation of R is given in Figure. From the diagram, we see that for each $x \in$ X, there exists only one $y \in$Y. Thus all elements in X have only one image in Y. Therefore R is a function. Domain X = {1, 2, 3, 4}; Co-domain Y = {2, 3, 6, 8, 10}; Range of f = {2, 4, 6, 8}.



Example 1.7

A relation 'f' is defined by $f(x) = x^2 - 2$ where, $x \in \{-2, -1, 0, 3\}$

(i) List the elements of f (ii) If f a function ?

Solution :

$f(x) = x^2 - 2$ where $x \in \{-2, -1, 0, 3\}$

(i) $f(-2) = (-2)^2 - 2 = 2$;

$f(-1) = (-1)^2 - 2 = -1$

$f(0) = (0)^2 - 2 = -2$; $f(3) = (3)^2 - 2 = 7$

Therefore, f = {(–2, 2), (–1, –1), (0, –2), (3, 7)}

(ii) We note that each element in the domain of f has a unique image. Therefore f is a function.

Example 1. 8

If X = {– 5, 1, 3, 4} and Y = {a, b, c}, then which of the following relations are functions from X to Y ?

(i) $R_1 = \{(-5, a), (1, a), (3, b)\}$

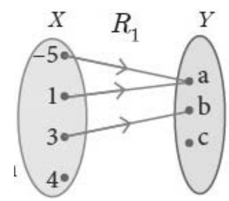
(ii) $R_2 = \{(-5, b), (1, b), (3, a), (4, c)\}$

(iii) $R_3 = \{(-5, a), (1, a), (3, b), (4, c), (1, b)\}$

Solution :

(i) $R_1 = \{(-5, a), (1, a), (3, b)\}$

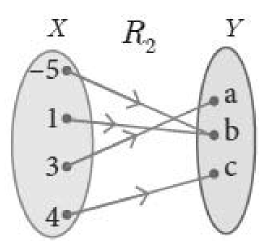
We may represent the relation R_1 in an arrow diagram.



(ii) $R_2 = \{(-5, b), (1, b), (3, a), (4, c)\}$

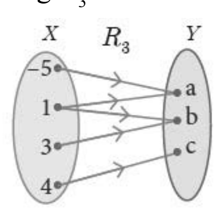
Arrow diagram of R_2 is shown in Figure.

R_2 is a function as each element of X has an unique image in Y.



(iii) $R_3 = \{(-5, a), (1, a), (3, b), (4, c), (1, b)\}$

Representing R_3 in an arrow diagram.



R_3 is not a function as $1 \in X$ has two images $a \in Y$ and $b \in Y$.

Note that the image of an element should always be unique.

Example 1.9

Given $f(x) = 2x - x^2$, find (i) f(1) (ii) f(x+1) (iii) f(x) + f(1)

Solution :

(i) Replacing x with 1, we get

$f(1) = 2(1) - (1)^2 = 2 - 1 = 1$

(ii) Replacing x with $x + 1$, we get

$$f(x+1) = 2(x+1) - (x+1)^2$$
$$= 2x + 2 - (x^2 + 2x + 1) = -x^2 + 1$$

(iii) $f(x) + f(1) = (2x - x^2) + 1 = -x^2 + 2x + 1$

[Note that $f(x) + f(1) \neq f(x+1)$. In general $f(a+b)$ is not equal to $f(a) + f(b)$].

EXERCISE 1.3

1. Let $f = \{(x, y) \mid x, y \in N \text{ and } y = 2x\}$ be a relation on N. Find the domain, co-domain and range. Is this relation a function ?

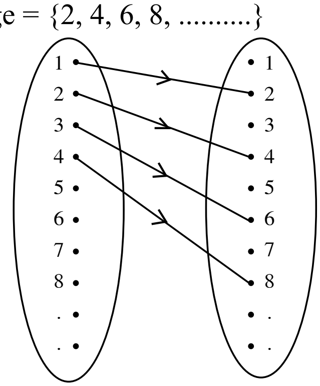
Solution :

Given $f = \{(x, y) \mid x, y \in N \text{ and } y = 2x\}$

Domain = {1, 2, 3, 4, ...}

Co-domain = {1, 2, 3, 4, ...}

Range = {2, 4, 6, 8, ...}



Yes, f is a function.

Since all the elements of domain will be mapped into a unique element in co-domain.

2. Let X = {3, 4, 6, 8}. Determine whether the relation $R = \{(x, f(x)) \mid x \in X, f(x) = x^2 + 1\}$ is a function from X to N ?

Solution :

Given X = {3, 4, 6, 8}

$R = \{(x, f(x)) \mid x \in X, f(x) = x^2 + 1\}$

$x = 3 \Rightarrow f(x) = f(3) = 9 + 1 = 10$

$x = 4 \Rightarrow f(x) = f(4) = 16 + 1 = 17$

$x = 6 \Rightarrow f(x) = f(6) = 36 + 1 = 37$

$x = 8 \Rightarrow f(x) = f(8) = 64 + 1 = 65$

R = {(3, 10), (4, 17), (6, 37), (8, 65)}

$\therefore$ The relation $R : X \to N$ is a function.

3. Given the function $f : x \to x^2 - 5x + 6$, evaluate

(i) f(– 1) (ii) f(2a) (iii) f(2) (iv) f(x – 1)

Solution :

Given $f : x \to x^2 - 5x + 6$

$\Rightarrow f(x) = x^2 - 5x + 6$

(i) $f(-1) = (-1)^2 - 5(-1) + 6$
$= 1 + 5 + 6$
$= 12$

(ii) $f(2a) = (2a)^2 - 5(2a) + 6$
$= 4a^2 - 10a + 6$

(iii) $f(2) = 2^2 - 5(2) + 6$
$= 4 - 10 + 6$
$= 0$

(iv) $f(x-1) = (x-1)^2 - 5(x-1) + 6$
$= x^2 - 2x + 1 - 5x + 5 + 6$
$= x^2 - 7x + 12$

4. A graph representing the function f(x) is given in figure it is clear that f(9) = 2.

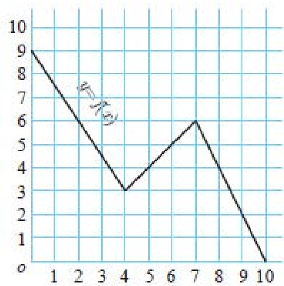
(i) Find the following values of the function

(a) f(0) (b) f(7) (c) f(2) (d) f(10)

(ii) For what value of x is f(x) = 1?

(iii) Describe the following (i) Domain (ii) Range.

(iv) What is the image of 6 under f?



Solution :

(i) a) f(0) = 9 b) f(7) = 6

c) f(2) = 6 d) f(10) = 0

(ii) When x = 9.5, f(x) = 1.

(iii) a) Domain : $\{x / 0 \le x \le 10, x \in R\}$

b) Range : $\{x / 0 \le x \le 9, x \in R\}$

(iv) Image of 6 = f(6) = 5.

---

5. Let f(x) = 2x + 5. If $x \neq 0$ then find $\dfrac{f(x+2)-f(2)}{x}$.

Solution :

Given

$$f(x) = 2x + 5$$
$$f(x+2) = 2(x+2) + 5 = 2x + 9$$
$$f(2) = 2(2) + 5 = 9$$
$$\therefore \frac{f(x+2)-f(x)}{x} = \frac{2x+9-9}{x} = 2$$

6. A function f is defined by f(x) = 2x – 3

(i) find $\dfrac{f(0)+f(1)}{2}$

(ii) find x such that f(x) = 0

(iii) find x such that f(x) = x

(iv) find x such that f(x) = f(1 – x).

Solution :

Given f(x) = 2x – 3

$$i)\quad \frac{f(0)+f(1)}{2} = \frac{(-3)+(-1)}{2} = \frac{-4}{2} = -2$$

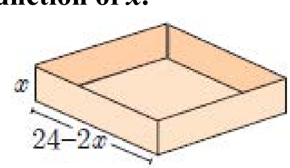
(ii) $f(x) = 0 \Rightarrow 2x - 3 = 0 \Rightarrow 2x = 3 \Rightarrow x = \dfrac{3}{2}$

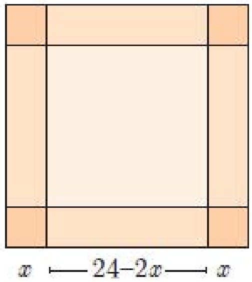
(iii) $f(x) = x \Rightarrow 2x - 3 = x \Rightarrow 2x - x = 3 \Rightarrow x = 3$

(iv) $f(x) = 1 - x \Rightarrow 2x - 3 = 1 - x \Rightarrow 2x + x = 1 + 3 \Rightarrow 3x = 4 \Rightarrow x = \dfrac{4}{3}$

---

7. An open box is to be made from a square piece of material, 24 cm on a side, by cutting equal squares from the corners and turning up the sides as shown in the figure. Express the volume V of the box as a function of x.





Solution :

From the given data, it is clear that length l = breadth, $b = 24 - 2x$ cm, height = x cm.

$\therefore$ Volume of the box, V $= lbh$

$$= (24 - 2x)^2 x$$
$$= (576 + 4x^2 - 96x)\, x$$
$$= 4x^3 - 96x^2 + 576x$$

$\therefore$ Volume is expressed as a function of x.

8. A function f is defined by $f(x) = 3 - 2x$. Find x such that $f(x^2) = (f(x))^2$.

Solution :

Given $\quad f(x) = 3 - 2x$ and

$$f(x^2) = (f(x))^2$$
$$\Rightarrow \quad 3 - 2x^2 = (3 - 2x)^2$$
$$\Rightarrow \quad 3 - 2x^2 = 9 + 4x^2 - 12x$$
$$\Rightarrow \quad 6x^2 - 12x + 6 = 0$$
$$\Rightarrow \quad x^2 - 2x + 1 = 0$$
$$\Rightarrow \quad (x - 1)^2 = 0$$
$$\Rightarrow \quad x = 1 \text{ (twice)}$$

9. A plane is flying at a speed of 500km per hour. Express the distance *d* travelled by the plane as function of time *t* in hours.

Solution :

Given, Speed of the plane = 500 km / h

time = t hrs

distance = d km

$\therefore$ Distance = Time × Speed

$\therefore$ d = 500t

10. The data in the adjacent table depicts the length of a woman's forehand and her corresponding height. Based on this data, a student finds a relationship between the height (y) and the forehand length (x) as $y = ax + b$, where a, b are constants.

Length 'x' of forehand (in cm)	Height 'y' (in inches)
45.5	65.5
35	56
45	65
50	69.5
55	74

(i) Check if this relation is a function.

(ii) Find a and b.

(iii) Find the height of a woman whose forehand length is 40 cm.

(iv) Find the length of forehand of a woman if her height is 53.3 inches.

Solution :

i) From the table, it is clear that each element of domain is having a unique image in co-domain and hence it is a function.

ii) $y = ax + b$

when $x = 55, y = 75 \Rightarrow \quad 75 = 55a + b$

when $x = 42, y = 62 \Rightarrow \quad 62 = 42a + b$

Subtracting, $\quad 13 = 13a$

$$a = 1$$

$\therefore \quad 75 = 55(1) + b$

$\therefore \quad b = 75 - 55 = 20$

$\therefore \quad a = 1, \quad b = 20$

iii) When $x = 48$; $y = ?$

$$\therefore\ y = ax + b$$
$$\Rightarrow y = x + 20$$
$$\text{when } x = 48,\ y = 48 + 20$$
$$\therefore\ y = 68$$

$\therefore$ Height of a woman = 68 inches.

iv) When $y = 60.54$ inches, $x = ?$

$$\therefore\ y = x + 20$$
$$\Rightarrow 60.54 = x + 20$$
$$\therefore\ x = 40.54 \text{ cm}$$

IV. REPRESENTATION OF FUNCTIONS AND TYPE OF FUNCTIONS :

Key Points

- ✓ A function may be represented by

  (a) a set of ordered pairs (b) a table form (c) an arrow diagram (d) a graphical form
- ✓ Every function can be represented by a curve in a graph. But not every curve drawn in a graph will represent a function.
- ✓ A curve drawn in a graph represents a function, if every vertical line intersects the curve in at most one point.
- ✓ Any equation represented in a graph is usually called a 'curve'.
- ✓ A function $f : A \to B$ is called one - one function if distinct elements of A have distinct images in B.
- ✓ If for all a_1, a_2 A, $f(a_1) = f(a_2)$ implies $a_1 = a_2$, then f is called **one-one function.**
- ✓ A function $f : A \to B$ is called **many-one function** if two or more elements of A have same image in B.
- ✓ A function $f : A \to B$ is called many-one if f it is not one-one.
- ✓ A function $f : A \to B$ is said to be **onto function** if the range of f is equal to the co-domain of f.
- ✓ That is a every element in the co-domain B has a pre-image in the domain A.
- ✓ An onto function is also called a **surjection.**
- ✓ If $f : A \to B$ is an onto function then, the range of f = B. That is f (A) = B.
- ✓ A function $f : A \to B$ is called an **into function** if there exists atleast one element in B which is not the image of any element of A.
- ✓ The range of f is a proper subset of the co-domain of f.
- ✓ A function $f : A \to B$ is called 'into' if it is not 'onto'.
- ✓ If a function $f : A \to B$ is both **one-one and onto**, then f is called a **bijection** from A to B.

- ✓ A function represented in a graph is one-one, if every horizontal line intersects the curve in at most one point.
- ✓ A function $f : A \to B$ is called a **constant function** if the range of f contains only one element. That is $f(x) = c$, for all $x \in A$ and for some fixed $c \in B$.
- ✓ Let A be a non-empty set. Then the function $f : A \to A$ defined by $f(x) = x$ for all $x \in A$ is called an **identity function** on A and is denoted by I_A.
- ✓ A function $f : A \to B$ is called a **real valued function** if the range of f is a subset of the set of all real numbers R. That is, $f(A) \subseteq R$.

Example 1.10

Using vertical line test, determine which of the following curves (Fig.1.18(a), 1.18(b), 1.18(c), 1.18(d)) represent a function ?

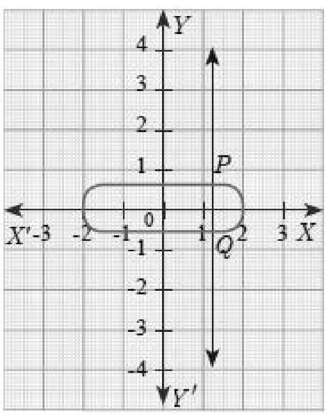


Fig. (1.18(a)

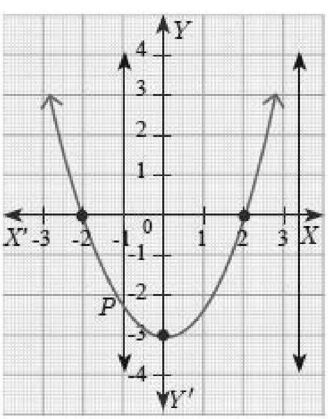


Fig. 1.18(b)

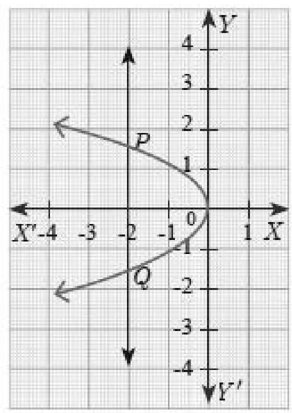
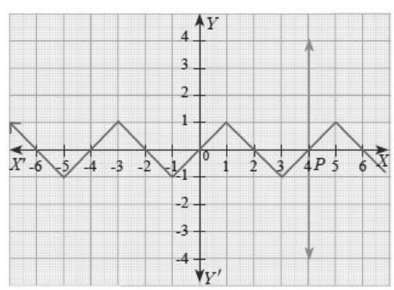


Fig. 1.18(c)

Solution :

The curves Fig.1.18(a) and Fig.1.18(c) do not represent a function as the vertical lines meet the curves in two points P and Q.



The curves in Fig.1.18(b) and Fig.1.18(d) represent a function as the vertical lines meet the curve in at most one point.

Example 1.11

Let A = {1, 2, 3, 4} and B = {2, 5, 8, 11, 14} be two sets. Let $f : A \rightarrow B$ be a function given by $f(x) = 3x - 1$. Represent this function

(i) by arrow diagram

(ii) in a table form

(iii) as a set of ordered pairs

(iv) in a graphical form

Solution :

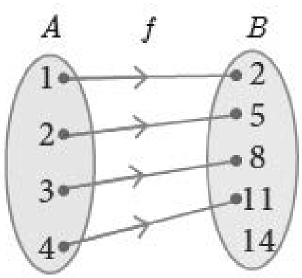
A = {1, 2, 3, 4} ; B = {2, 5, 8, 11, 14} ; $f(x) = 3x - 1$

$f(1) = 3(1) - 1 = 3 - 1 = 2$; $f(2) = 3(2) - 1 = 6 - 1 = 5$

$f(3) = 3(3) - 1 = 9 - 1 = 8$; $f(4) = 4(3) - 1 = 12 - 1 = 11$

(i) Arrow diagram

Let us represent the function $f : A \rightarrow B$ by an arrow diagram



(ii) Table form

The given function f can be represented in a tabular form as given below

x	1	2	3	4
$f(x)$	2	5	8	11

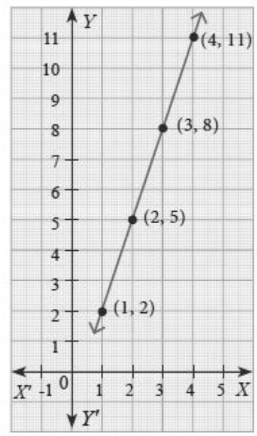
(iii) Set of ordered pairs

The function f can be represented as a set of ordered pair as

f = {(1, 2), (2, 5), (3, 8), (4, 11)}

(iv) Graphical form

In the adjacent xy-plane the points (1, 2), (2, 5), (3, 8), (4, 11) are plotted



Example 1.12

Using horizontal line test (Fig. 1.35(a), 1.35(b), 1.35(c)), determine which of the following functions are one-one.

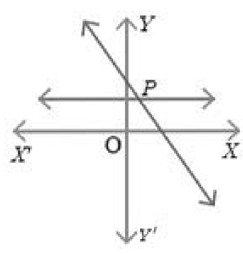


Fig. 1.35 (a)

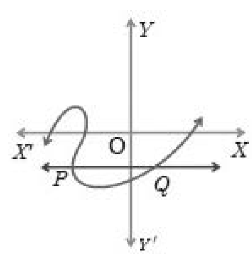


Fig. 1.35(b)

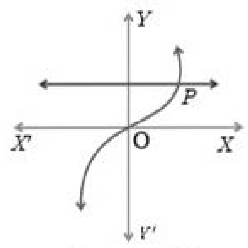


Fig. 1.35(c)

Solution :

The curves in Fig.1.35(a) and Fig. 1.35(c) represent a one-one function as the horizontal lines meet the curves in only one point P.

The curve Fig.1.35(b) does not represent a one-one function, since, the horizontal line meet the curve in two points P and Q.

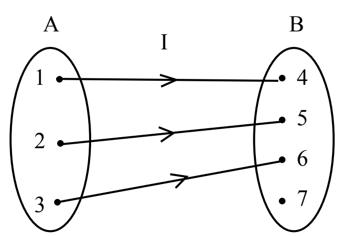
Example 1.13

Let A = {1, 2, 3}, B = {4, 5, 6, 7| and f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one but not onto function.

Solution:

A = {1, 2, 3}, B = {4, 5, 6, 7} ; f = {(1, 4), (2, 5), (3, 6)}

Then f is a function from A to B and for different elements in A, there are different images in B. Hence f is one-one function. Note that the element 7 in the co-domain does not have any pre-image in the doman. Hence f is not onto.



Example 1.14

If A = {–2, –1, 0, 1, 2} and $f : A \to B$ is an onto function defined by $f(x) = x^2 + x + 1$ then find B.

Solution :

Given A= {–2, –1, 0, 1, 2} and $f(x) = x^2 + x + 1$

$f(-2) = (-2)^2 + (-2) + 1 = 3$;

$f(-1) = (-1)^2 + (-1) + 1 = 1$

$f(0) = 0^2 + 0 + 1 = 1$;

$f(1) = 1^2 + 1 + 1 = 3$

$f(2) = 2^2 + 2 + 1 = 7$

Since, f is an onto function, range of $f = B =$ co-domain of f.

Therefore, B = {1, 3, 7}.

Example 1.15

Let f be a function $f : N \to N$ be defined by $f(x) = 3x + 2, x \in N$

(i) Find the images of 1, 2, 3 (ii) Find the pre-images of 29, 53

(iii) Identify the type of function

Solution : $f(x) = 3x + 2, x \in N$

(i) If $x = 1$, $f(1) = 3(1) + 2 = 5$

If $x = 2$, $f(2) = 3(2) + 2 = 8$

If $x = 3$, $f(3) = 3(3) + 2 = 11$

The images of 1, 2, 3 are 5, 8, 11 respectively.

(ii) If x is the pre-image of 29, then $f(x) = 29$.

Hence $3x + 2 = 29$

$3x = 27 \Rightarrow x = 9$.

Similarly, if x is the pre-image of 53, then $f(x) = 53$. Hence $3x + 2 = 53$

$3x = 51 \Rightarrow x = 17$.

Thus the pre-images of 29 and 53 are 9 and 17 respectively.

(iii) Since different elements of N have different images in the co-domain, te function f is one-one function.

The co-domain of f is N.

But the range of $f = \{5, 8, 11, 14, 17, ...\}$ is a proper subset of N. Therefore f is not an onto function. That is, f is an into function.

Thus f is one-one and into function.

Example 1.16

Forensic scientists can determine the height (in cms) of a person based on the length of their thigh bone. They usually do so using the function $h(b) = 2.47b + 54.10$ where b is the length of the thigh bone.

(i) Check if the function h is one-one

(ii) Also find the height of a person if the length of his thigh bone is 50 cms.

(iii) Find the length of the thigh bone if the height of a person is 147.96 cms.

Solution :

(i) To check if h is one-one, we assume that $h(b_1) = h(b_2)$.

Then we get, $2.47b_1 + 54.10 = 2.47b_2 + 54.10$

$$2.47b_1 = 2.47b_2 \Rightarrow b_1 = b_2$$

Thus, $h(b_1) = h(b_2) \Rightarrow b_1 = b_2$. So, the function h is one-one.

(ii) If the length of the thigh bone $b = 50$, then the height is

$$h(50) = (2.47 \times 50) + 54.10 = 177.6 \text{ cms.}$$

(iii) If the height of a person is 147.96 cms, then $h(b) = 147.96$ and so the length of the thigh bone is given by $2.47b + 54.10 = 147.96$.

$$b = \frac{93.86}{2.47} = 38$$

Therefore, the length of the thigh bone is 38 cms.

Example 1.17

Let f be a function from R to R defined by $f(x) = 3x - 5$. Find the values of a and b given that $(a, 4)$ and $(1, b)$ belong to f.

Solution :

$f(x) = 3x - 5$ can be written as

$f = \{(x, 3x - 5) \mid x \in R\}$

$(a, 4)$ means the image of a is 4.

That is, $f(a) = 4$

$3a - 5 = 4 \Rightarrow a = 3$

$(1, b)$ means the image of 1 is b.

That is, $f(1) = b \Rightarrow b = -2$

$3(1) - 5 = b \Rightarrow b = -2$

Example 1.18

The distance S (in kms) travelled by a particle in time 't' hours is given by $S(t) = \frac{t^2 + t}{2}$. Find the distance travelled by the particle after

(i) three and half hours.

(ii) eight hours and fifteen minutes.

Solution :

The distance travelled by the particle in time t hours is given by $S(t) = \frac{t^2 + t}{2}$.

(i) $t = 3.5$ hours. Therefore,

$$S(3.5) = \frac{(3.5)^2 + 3.5}{2} = \frac{15.75}{2} = 7.875$$

The distance travelled in 3.5 hours is 7.875 kms.

(ii) $t = 8.25$ hours. Therefore, $S(8.25) =$

$$\frac{(8.25)^2 + 8.25}{2} = \frac{76.3125}{2} = 38.15625$$

The distance travelled in 8.25 hours is 38.16 kms, approximately.

Example 1.19

If the function $f : R \to R$ defined by

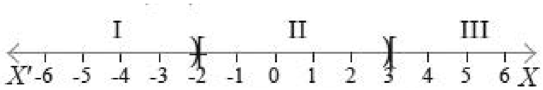
$$f(x) = \begin{cases} 2x+7, x<-2 \\ x^2-2, -2 \le x < 3, \\ 3x-2,\ x \ge 3 \end{cases}$$ then find the values of

(i) f(4) (ii) f(– 2) (iii) f(4) + 2f(1)

(iv) $\dfrac{f(1)-3f(4)}{f(-3)}$

Solution :

The function f is defined by three values in intervals I, II, III as shown below



$f(x) = 2x + 7$ $\quad f(x) = x^2 - 2$ $\quad f(x) = 3x - 2$

For a given value of $x = a$, find out the interval at which the point a is located, there after find $f(a)$ using the particular value defined in that interval.

(i) First, we see that,

$x = 4$ lie in the third interval.

Therefore, $f(x) = 3x - 2$;

$f(4) = 3(4) - 2 = 10$

(ii) $x = -2$ lies in the second interval.

Therefore, $f(x) = x^2 - 2$;

$f(-2) = (-2)^2 - 2 = 2$

(iii) From (i), $f(4) = 10$.

To find $f(1)$, first we see that $x = 1$ lies in the second interval.

Therefore, $f(x) = x^2 - 2$;

$f(1) = (1)^2 - 2 = -1$

So, $f(4) + 2f(1) = 10 + 2(-1) = 8$

(iv) We know that $f(1) = -1$ and $f(4) = 10$.

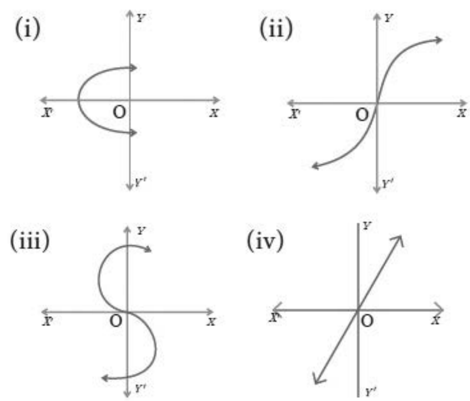
For finding $f(-3)$, we see that $x = -3$, lies in the first interval.

Therefore, $f(x) = 2x + 7$; thus, $f(-3) = 2(-3) + 7 = 1$

Hence, $\dfrac{f(1)-3f(4)}{f(-3)} = \dfrac{-1-3(10)}{1} = -31$

EXERCISE 1.4

1. Determine whether the graph given below represent functions. Give reason for your answers concerning each graph.



Solution :

(i) The curve do not represent a function since it meets y-axis at 2 points.

(ii) The curve represents a function as it meets x-axis or y-axis at only one point.

(iii) The curve do not represent a function since it meets y-axis at 2 points.

(iv) The line represents a function as it meets axes at origin.

2. Let $f : A \to B$ be a function define by $f(x) = \dfrac{x}{2} - 1$, where A = {2, 4, 6, 10, 12}, B = {0, 1, 2, 4, 5, 9}. Represent f by

(i) set of ordered pairs ; (ii) a table ; (iii) an arrow diagram ; (iv) a graph

Solution :

Given $f(x) = \frac{x}{2} - 1$

$x = 2 \Rightarrow f(2) = 1 - 1 = 0$

$x = 4 \Rightarrow f(4) = 2 - 1 = 1$

$x = 6 \Rightarrow f(6) = 3 - 1 = 2$

$x = 10 \Rightarrow f(10) = 5 - 1 = 4$

$x = 12 \Rightarrow f(12) = 6 - 1 = 5$

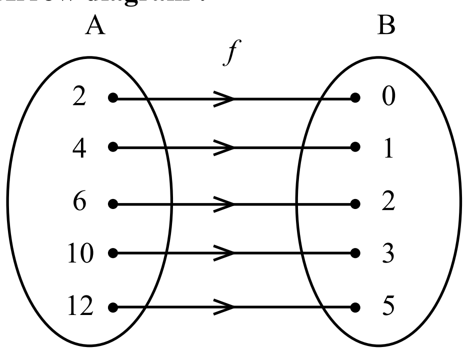
(i) Set of order pairs :

$f = \{(2, 0), (4, 1), (6, 2), (10, 4), (12, 5)\}$

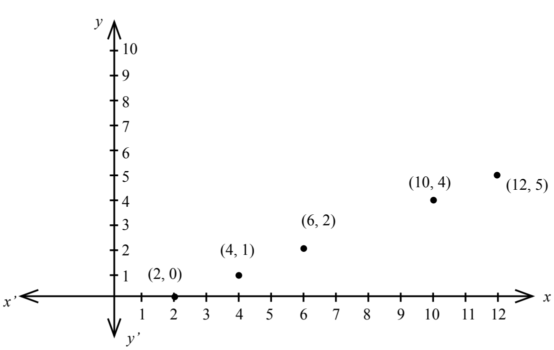
(ii) Table :

x	2	4	6	10	12
f(x)	0	1	2	4	5

(iii) Arrow diagram :



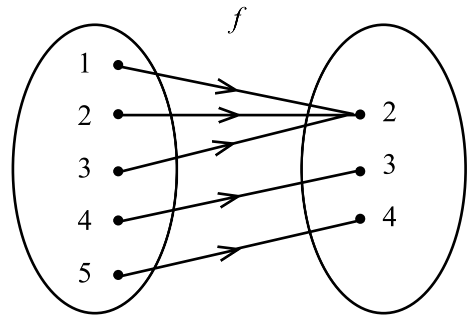
(iv) Graph



3. **Represent the function $f = \{(1, 2), (2, 2), (3, 2), (4, 3), (5, 4)\}$ through**
(i) an arrow diagram (ii) a table form (iii) a graph

Solution :

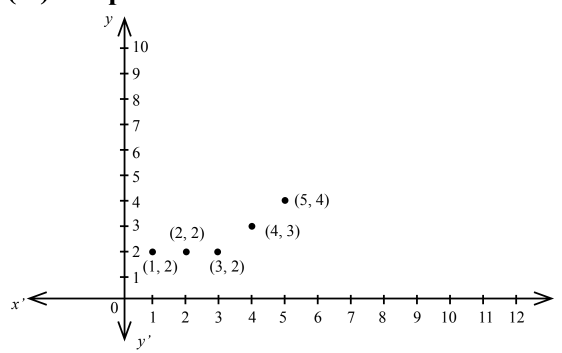
(i) Arrow Diagram :



(ii) Table Form :

x	1	2	3	4	5
f(x)	2	2	2	3	4

(iii) Graph :



4. **Show that the function $f : \mathrm{N} \to \mathrm{N}$ defined $f(x) = 2x - 1$ is one-one but not onto.**

Solution :

Given $f : \mathrm{N} \to \mathrm{N}$ defined by $f(x) = 2x - 1$.

$x = 1 \Rightarrow f(1) = 2 - 1 = 1$

$x = 2 \Rightarrow f(2) = 4 - 1 = 3$

$x = 3 \Rightarrow f(3) = 6 - 1 = 5$

$x = 4 \Rightarrow f(4) = 8 - 1 = 7$ ....

It is clear that f is a function from $N \to N$ and for different elements in domain, there are different images in co-domain.

$\therefore f$ is one to one function.

But co-domain is N and Range = {1,3,5, 7, ...}

$\therefore$ Range $\neq$ Co-domain.

$\therefore f$ is not on-to.

5. Show that the function $f : N \to N$ defined by $f(m) = m^2 + m + 3$ is one-one function.

Solution :

Given $f : N \to N$ defined by $f(m) = m^2 + m + 3$

$f(m) = m^2 + m + 3$

$m = 1 \Rightarrow f(1) = 1 + 1 + 3 = 5$

$m = 2 \Rightarrow f(2) = 4 + 2 + 3 = 9$

$m = 3 \Rightarrow f(3) = 9 + 3 + 3 = 15$

$m = 4 \Rightarrow f(4) = 16 + 4 + 3 = 23$ ....

For different elements of domain, there are different images in co-domain.

$\therefore f$ is one-one function.

6. Let A = {1, 2, 3, 4} and B = N. Let $f : A \to B$ be defined by $f(x) = x^3$ then,

(i) find the range of f

(ii) identify the type of function

Solution :

Given A = {1, 2, 3, 4}, B = N

$f(x) = x^3$

$x = 1 \Rightarrow f(1) = 1$

$x = 2 \Rightarrow f(2) = 8$

$x = 3 \Rightarrow f(3) = 27$

$x = 4 \Rightarrow f(4) = 64$

(i) Range of f = {1, 8, 27, 64}

(ii) f is one-one (diff. elements have diff. images) and

f is into (Range $\neq$ co-domain)

7. In each of the following cases state whether the function is bijective or not. Justify your answer.

(i) $f : R \to R$ defined by $f(x) = 2x + 1$

(ii) $f : R \to R$ defined by $f(x) = 3 - 4x^2$

Solution :

(i) $f : R \to R$ defined by $f(x) = 2x + 1$

Let $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 + 1 = 2x_2 + 1$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore f$ is 1 – 1 function.

$y = 2x + 1$

$\therefore 2x = y - 1$

$\Rightarrow x = \frac{y-1}{2}$

$\therefore f(x) = 2\left(\frac{y-1}{2}\right) + 1$

$= y$

$\therefore f$ is onto.

$\therefore f$ is one-one and onto

$\Rightarrow f$ is bijective.

(ii) $f : R \to R$ defined by $f(x) = 3 - 4x^2$.

Let $f(x_1) = f(x_2)$

$3 - 4x_1^2 = 3 - 4x_2^2$

$x_1^2 = x_2^2$

$x_1 = x_2$ (or) $x_1 = -x_2$

f is not 1 – 1.

(Example : when $x = -1$, $f(x) = f(-1) = -1$

when $x = 1$, $f(x) = f(1) = -1$

$\therefore$ Two different elements in domain have same images in co-domain.

Also, any even number in the co-domain is not image of any element x in the domain.

$\therefore f$ is not onto

$\therefore f$ is not bijective.

8. Let A = {–1, 1} and B = {0, 2}. If the function f : A → B defined by $f(x) = ax + b$ is an onto function ? Find a and b.

Solution :

Given A = {–1, 1}, B = {2, 2}

$f(x) = ax + b$ is on to function.

$\therefore f(-1) = 0 \Rightarrow -a + b = 0$ — (1)

$f(1) = 2 \Rightarrow a + b = 2$ — (2)

Solving (1) and (2)

$$2b = 2$$
$$b = 1$$
$$\Rightarrow a = 1$$

$\therefore a = 1, b = 1$

9. If the function f is defined by $f(x) = \begin{cases} x+2 & if\ x>1 \\ 2 & if\ -1\le x\le 1 \\ x-1 & if\ -3<x<-1 \end{cases}$; find the values of
(i) $f(3)$ (ii) $f(0)$ (iii) $f(-1, 5)$ (iv) $f(2) + f(-2)$

Solution :

$$\text{Given } f(x) = \begin{cases} x+2 & if\ x>1 \\ 2 & if\ -1\le x\le 1 \\ x-1 & if\ -3<x<-1 \end{cases}$$

(i) $f(3) = 3 + 2$ $(\because 3 \in (1 < x < \infty))$

$= 5$

(ii) $f(0) = 2$ $(\because 0 \in (-1 \le x \le 1))$

(iii) $f(-1.5) = -1.5 - 1$ $(\because -1.5 \in (-3 < x < -1))$

$= -2.5$

(iv) $f(2) + f(-2)$ $(\because 2 \in (1 < x < \infty)$

$= (2 + 2) + (-2 - 1)$ $(\because -2 \in (-3 < x < -1)$

$= 4 - 3$

$= 1$

10. A function f : [– 5, 9] → R is defined as follows :

$$f(x) = \begin{cases} 6x+1 & if\ -5\le x<2 \\ 5x^2-1 & if\ \ 2\le x<6 \\ 3x-4 & if\ \ 6\le x<6 \end{cases}$$

Find (i) $f(-3) + f(2)$ (ii) $f(7) - f(1)$
(iii) $2f(4) + f(8)$ (iv) $\dfrac{2f(-2) - f(6)}{f(4) + f(-2)}$

Solution :

$$\text{Given } f(x) = \begin{cases} 6x+1 & if\ -5\le x<2 \\ 5x^2-1 & if\ \ 2\le x<6 \\ 3x-4 & if\ \ 6\le x<9 \end{cases}$$

(i) $f(-3) + f(2)$ $(\because -3 \in (-5 \le x < 2))$

$= [6(-3) + 1] + [5(4) - 1]$ and

$= -17 + 19$ $2 \in (2 \le x < 6)$

$= 2$

(ii) $f(7) - f(1)$ $(\because 7 \in (6 \le x \le 9))$

$= [3(7) - 4] - [6(1) + 1]$ and

$= 17 - 7$ $1 \in (-5 \le x < 2)$

$= 10$

(iii) $2f(4) + f(8)$ $(\because 4 \in (2 \le x < 6)$

$= 2[5(16) - 1] + [3(8) - 4]$ and

$= 2[79] + 20$ $8 \in (6 \le x \le 9)$

$= 158 + 20$

$= 178$

(iv)

$$\frac{2f(-2)-f(6)}{f(4)+f(-2)} \qquad (\because -2 \in (-5 \le x < 2) \quad 6 \in (6 \le x \le 9) \quad -2 \in (-5 \le x < 2))$$

$$= \frac{2(-11)-14}{79+(-11)}$$

$$= \frac{-22-14}{79-11}$$

$$= \frac{-36}{68}$$

$$= \frac{-9}{17}$$

11. The distance S an object travels under the influence of gravity in time *t* seconds is given by S(t) = $\frac{1}{2}gt^2 + at + b$ where, (*g* is the acceleration due to gravity), *a, b* are constants. Check if the function S(t) is one-one.

Solution :

Given S(t) = $\frac{1}{2}gt^2 + at + b$

To check s(t) is one-one, we consider,

$s(t_1) = s(t_2)$

$$\Rightarrow \frac{1}{2}gt_1^2 + at_1 + \cancel{b} = \frac{1}{2}gt_2^2 + at_2 + \cancel{b}$$

$$\Rightarrow \frac{1}{2}g(t_1^2 - t_2^2) + a(t_1 - t_2) = 0$$

$$\Rightarrow (t_1 - t_2)\left[\frac{1}{2}g(t_1 + t_2) + a\right] = 0$$

$$\Rightarrow t_1 - t_2 = 0 \qquad (\because \tfrac{1}{2}g(t_1+t_2) + a \neq 0$$

$$\Rightarrow t_1 = t_2$$

$\therefore$ s(t) is one-one.

12. The function 't' which maps temperature in Celsius (C) into temperature in Fahrenheit (F) is defined by t(C) = F where F = $\frac{9}{5}$C + 32, Find,

(i) t(0) (ii) t(28) (iii) t(– 10)

(iv) the value of C when t(C) = 212

(v) the temperature when the Celsius value is equal to the Farenheit value.

Solution :

Given t(C) = F = $\frac{9C}{5} + 32$

$$(i)\quad t(0) = \frac{9(0)}{5} + 32 = 32^o F$$

$$(ii)\quad t(28) = \frac{9(28)}{5} + 32 = 50.4 + 32 = 82.4^o F$$

$$(iii)\quad t(-10) = \frac{9(-10)}{5} + 32 = -18 + 32 = 14^o F$$

(iv) When $t(c) = 212$,

$$212 = \frac{9C}{5} + 32$$

$$\Rightarrow \frac{9C}{5} = 180$$

$$\Rightarrow C = 180 \times \frac{5}{9}$$

$$\Rightarrow C = 100^o C$$

(v) When Celsius value = Farenheit value,

$$C = \frac{9C}{5} + 32$$

$$\Rightarrow \quad 5C = 9C + 160$$

$$\Rightarrow -4C = 160$$

$$\Rightarrow \quad C = -40^o$$

V. COMPOSITION OF FUNCTIONS :

Key Points

- ✓ Let $f : A \to B$ and $g : B \to C$ be two functions. Then the **composition of f and g** denoted by $g \circ f$ is defined as the function $g \circ f(x) = g(f(x))$ for all $x \in A$.
- ✓ Generally, $f \circ g \neq g \circ f$ for any two functions f and g. So, composition of functions is not commutative.
- ✓ Composition of three fuctions is always associative. That is $f \circ (g \circ h) = (f \circ g) \circ h$.
- ✓ A function $f : R \to R$ defined $f(x) = mx + c$, $m \neq 0$ is called a **linear function**. Geometrically this represents a striaght line in the graph.
- ✓ $f : R \to [0, \infty)$ defined $f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$ is called a **modulus (or) Absolute value function.**
- ✓ Modulus function is not a linear function but it is composed of two linear functions x and $-x$.
- ✓ A function $f : R \to R$ defined by $f(x) = ax^2 + bx + c$, $(a \neq 0)$ is called a **quadratic function.**
- ✓ A function $f : R \to R$ defined by $f(x) = ax^3 + bx^2 + cx + d$, $(a \neq 0)$ is called a **cubic function.**
- ✓ A function $f : R \to \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called a **reciprocal function.**
- ✓ A function $f : R \to R$ defined by $f(x) = c$, for all $x \in R$ is called a **constant function.**

Example 1.20

Find $f \circ g$ and $g \circ f$ when $f(x) = 2x + 1$ and $g(x) = x^2 - 2$.

Solution :

$f(x) = 2x + 1$, $g(x) = x^2 - 2$

$f \circ g(x) = f(g(x)) = f(x^2 - 2)$

$= 2(x^2 - 2) + 1 = 2x^2 - 3$

$g \circ f(x) = g(f(x)) = g(2x + 1)$

$= (2x + 1)^2 - 2 = 4x^2 + 4x - 1$

Thus $f \circ g = 2x^2 - 3$, $g \circ f = 4x^2 + 4x - 1$.

From the above, we see that $f \circ g \neq g \circ f$.

Example 1.21

Represent the function $f(x) = \sqrt{2x^2 - 5x + 3}$ as a composition of two functions.

Solution :

We set

$f_2(x) = 2x^2 - 5x + 3$ and $f_1(x) = \sqrt{x}$

Then,

$f(x) = \sqrt{2x^2 - 5x + 3} = \sqrt{f_2(x)}$

$= f_1[f_2(x)]. = f_1 f_2(x)$

Example 1.22

If $f(x) = 3x - 2$, $g(x) = 2x + k$ and if $f \circ g = g \circ f$, then find the value of k.

Solution :

$f(x) = 3x - 2$, $g(x) = 2x + k$

$f \circ g(x) = f(g(x))$

$= f(2x + k) = 3(2x + k) - 2 = 6x + 3k - 2$

Thus, $f \circ g(x) = 6x + 3k - 2.$

$g \circ f(x) = g\,(3x - 2) = 2\,(3x - 2) + k$

Thus, $g \circ f(x) = 6x - 4 + k.$

Given that $f \circ g = g \circ f$

Therefore, $6x + 3k - 2 = 6x - 4 + k$

$6x - 6x + 3k - k = -4 + 2 \Rightarrow k = -1$

Example 1.23

Find k if $f \circ f(k) = 5$ where $f(k) = 2k - 1$.

Solution :

$f \circ f(k) \quad = f(f(k))$

$= 2\,(2k - 1) - 1 = 4k - 3.$

Thus, $f \circ f(k) \quad = 4k - 3$

But, it is given that $f \circ f(k) = 5$

Therefore $4k - 3 = 5 \Rightarrow k = 2.$

Example 1.24

$f(x) = 2x + 3$, $g(x) = 1 - 2x$ and $h(x) = 3x$. Prove that $f \circ (g \circ h) = (f \circ g) \circ h$.

Solution :

$f(x) = 2x + 3$, $g(x) = 1 - 2x$, $h(x) = 3x$

Now, $(f \circ g)(x) = f(g(x)) = f(1 - 2x)$

$= 2\,(1 - 2x) + 3 = 5 - 4x$

Then, $(f \circ g) \circ h\,(x) = (f \circ g)(h(x)) = (f \circ g)\,(3x)$

$= 5 - 4(3x) = 5 - 12x$ ...(1)

$(g \circ h)\,(x) = g(h(x)) = g(3x) = 1 - 2(3x) = 1 - 6x$

So, $f \circ (g \circ h)\,(x) = f(1 - 6x) = 2\,(1 - 6x) + 3$

$= 5 - 12x$ ...(2)

From (1) and (2), we get $f \circ (g \circ h) = (f \circ g) \circ h$.

Example 1.25

Find x if $gff(x) = fgg(x)$, given $f(x) = 3x + 1$ and $g(x) = x + 3$.

Solution :

$gff(x) = g\,[f\,\{f(x)\}]$ (This mean "g of f of f of x")

$= g\,[f(3x + 1)] = g\,[3(3x+1)+1] = g(9x + 4)$

$g(9x + 4) = [(9x + 4) + 3] = 9x + 7$

$fgg(x) = f[g\{g(x)\}]$ (This means "f of g of x")

$= f\,[g(x + 3)] = f\,[(x + 3) + 3] = f(x + 6)$

$f(x + 6) = [3\,(x + 6) + 1] = 3x + 19$

These two quantities being equal, we get $9x + 7 = 3x + 19$. Solving this equation we obtain $x = 2$.

EXERCISE 1.5

1. Using the functions f and g given below, find $f \circ g$ and $g \circ f$. Check whether $f \circ g = g \circ f$.

(i) $f(x) = x - 6,\ g(x) = x^2$

(ii) $f(x) = \dfrac{2}{x},\ g(x) = 2x^2 - 1$

(iii) $f(x) = \dfrac{x+6}{3},\ g(x) = 3 - x$

(iv) $f(x) = 3 + x,\ g(x) = x - 4$

(v) $f(x) = 4x^2 - 1,\ g(x) = 1 + x$

Solution :

(i) $f(x) = x - 6,\ g(x) = x^2$

$(f \circ g)\,(x) = f(g(x))$

$= f(x^2)$

$= x^2 - 6$

$(g \circ f)\,(x) = g\,(f(x))$

$= g\,(x - 6)$

$= (x - 6)^2$

$= x^2 - 12x + 36$

$\therefore f \circ g \neq g \circ f$

(ii) $f(x) = \frac{2}{x}, g(x) = 2x^2 - 1$

$$(f \circ g)(x) = f(g(x))$$
$$= f(2x^2 - 1)$$
$$= \frac{2}{2x^2 - 1}$$
$$(g \circ f)(x) = g(f(x))$$
$$= g\left(\frac{2}{x}\right)$$
$$= 2\left(\frac{2}{x}\right)^2 - 1$$
$$= \frac{8}{x^2} - 1$$
$$\therefore f \circ g \neq g \circ f$$

(iii) $f(x) = \frac{x+6}{3}, g(x) = 3 - x$

$$(f \circ g)(x) = f(g(x))$$
$$= f(3 - x)$$
$$= \frac{(3-x)+6}{3}$$
$$= \frac{9-x}{3}$$
$$(g \circ f)(x) = g(f(x))$$
$$= g\left(\frac{x+6}{3}\right)$$
$$= 3 - \frac{x+6}{3}$$
$$= \frac{9-x-3}{3}$$
$$= \frac{6-x}{3}$$
$$\therefore f \circ g \neq g \circ f$$

(iv) $f(x) = 3 + x, g(x) = x = 4$

$$(f \circ g)(x) = f(g(x))$$
$$= f(x - 4)$$
$$= 3 + (x - 4)$$
$$= x - 1$$
$$(g \circ f)(x) = g(f(x))$$
$$= g(3 + x)$$
$$= 3 + x - 4$$
$$= x - 1$$
$$\therefore f \circ g = g \circ f$$

(v) $f(x) = 4x^2 - 1, g(x) = 1 + x$

$$(f \circ g)(x) = f(g(x))$$
$$= f(1 + x)$$
$$= 4(1 + x^2) - 1$$
$$= 4(1 + x^2 + 2x) - 1$$
$$= 4x^2 + 8x + 3$$
$$(g \circ f)(x) = g(f(x))$$
$$= g(4x^2 - 1)$$
$$= 1 + 4x^2 - 1$$
$$= 4x^2$$
$$\therefore f \circ g \neq g \circ f$$

---

2. Find the value of k, such that $f \circ g = g \circ f$.

(i) $f(x) = 3x + 2$, $g(x) = 6x - k$

(ii) $f(x) = 2x - k$, $g(x) = 4x + 5$

Solution :

(i) $f(x) = 3x + 2$; $g(x) = 6x - k$

$$\text{Given } f \circ g = g \circ f$$
$$\Rightarrow (f \circ g)(x) = (g \circ f)(x)$$
$$\Rightarrow f(g(x)) = g(f(x))$$
$$\Rightarrow f(6x - k) = g(3x + 2)$$

$\Rightarrow 3(6x - k) + 2 = 6(3x + 2) - k$

$\Rightarrow 18x - 3k + 2 = 18x + 12 - k$

$\Rightarrow -3k + 2 = 12 - k$

$\Rightarrow -2k = 10$

$\Rightarrow \; k = \dfrac{-10}{2} = -5$

(ii) $f(x) = 2x - k$; $g(x) = 4x + 5$

$$
\begin{aligned}
\text{Given } f \circ g &= g \circ f \\
\Rightarrow (f \circ g)(x) &= (g \circ f)(x) \\
\Rightarrow f(g(x)) &= g(f(x)) \\
\Rightarrow f(4x + 5) &= g(2x - k) \\
\Rightarrow 2(4x + 5) - k &= 4(2x - k) + 5 \\
\Rightarrow 8x + 10 - k &= 8x - 4k + 5 \\
\Rightarrow 10 - k &= -4k + 5 \\
\Rightarrow -k + 4k &= 5 - 10 \\
\Rightarrow 3k &= -5
\end{aligned}
$$

$\Rightarrow$ —

3. If $f(x) = 2x - 1$, $g(x) = \dfrac{x+1}{2}$, show that $f \circ g = g \circ f = x$.

Solution :

Given $f(x) = 2x - 1$, $g(x) = \dfrac{x+1}{2}$

$$
\begin{aligned}
\therefore f \circ g &= (f \circ g)(x) \\
&= f(g(x)) \\
&= f\left(\frac{x+1}{2}\right) \\
&= 2\left(\frac{x+1}{2}\right) - 1 \\
&= x + 1 - 1 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
g \circ f &= (g \circ f)(x) \\
&= g(f(x)) \\
&= g(2x - 1) \\
&= \frac{2x - 1 + 1}{2} \\
&= x
\end{aligned}
$$

$$\therefore f \circ g = g \circ f = x$$

4. (i) If $f(x) = x^2 - 1$, $g(x) = x - 2$ find a, if $g \circ f(a) = 1$.

(ii) Find k, if $f(k) = 2k - 1$ and $f \circ f(k) = 5$.

Solution :

(i) $f(x) = x^2 - 1$, $g(x) = x - 2$

$$
\begin{aligned}
\text{Given } g \circ f(a) &= 1 \\
\Rightarrow g(f(a)) &= 1 \\
\Rightarrow g(a^2 - 1) &= 1 \\
\Rightarrow a^2 - 1 - 2 &= 1 \\
\Rightarrow a^2 - 3 &= 1 \\
\Rightarrow a^2 &= 4 \\
\therefore a &= \pm 2
\end{aligned}
$$

(ii) $f(k) = 2k - 1$

$$
\begin{aligned}
\Rightarrow f \circ f(k) &= 5 \\
\Rightarrow f(f(k)) &= 5 \\
\Rightarrow f(2k - 1) &= 5 \\
\Rightarrow 2(2k - 1) - 1 &= 5 \\
\Rightarrow 4k - 2 &= 6 \\
\Rightarrow 4k &= 8 \\
\therefore k &= 2
\end{aligned}
$$

5. Let A, B, C $\subseteq$ N and a function $f : A \to B$ be defined by $f(x) = 2x + 1$ and $g : B \to C$ be defined by $g(x) = x^2$. Find the range of $f \circ g$ and $g \circ f$.

Solution :

$f : A \to B,\ g : B \to C$ where $A, B, C \subseteq N$.

$f(x) = 2x + 1,\ g(x) = x^2$

Range of f o g :

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 1$$

$\therefore$ Range of f o g = $\{y / y = 2x^2 + 1, x \in N\}$.

Range of g o f :

$$(g \circ f)(x) = g(f(x)) = g(2x+1) = (2x+1)^2$$

$\therefore$ Range of g o f = $\{y/y = (2x+1)^2, x \in N\}$.

6. Let $f(x) = x^2 - 1$. Find (i) f o f (ii) f o f o f

Solution :

Given $f(x) = x^2 - 1$

a) $f \circ f = ?$

$$(f \circ f)(x) = f(f(x)) = f(x^2 - 1) = (x^2-1)^2 - 1 = x^4 - 2x^2 + 1 - 1 = x^4 - 2x^2$$

b) $f \circ f \circ f = ?$

$$(f \circ f \circ f)(x) = f \circ f(f(x))$$
$$= f \circ f(x^2 - 1)$$
$$= (x^2-1)^4 - 2(x^2-1)^2$$
$$= (x^2-1)^2[(x^2-1)^2 - 2]$$
$$= (x^4 - 2x^2 + 1)(x^4 - 2x^2 - 1)$$
$$= (x^4 - 2x^2)^2 - 1$$

$((a+b)(a-b) = a^2 - b^2)$

7. If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = x^5$ and $g(x) = x^4$ then check if f, g are one-one and f o g is one-one ?

Solution :

Given $f(x) = x^5$, $g(x) = x^4$

Let A be the domain.

B be the co-domain.

For every element $\in$ A, there is a unique image in B. Since f is an odd function

$\therefore f$ is $1 - 1$.

But $g(x)$ is an even function.

$\therefore$ Two elements of domain will have the since image in co-domain.

$\therefore g$ is not $1 - 1$.

8. Consider the functions $f(x)$, $g(x)$, $h(x)$ as given below. Show that $(f \circ g) \circ h = f \circ (g \circ h)$ in each case.

(i) $f(x) = x - 1$, $g(x) = 3x + 1$ and $h(x) = x^2$

(ii) $f(x) = x^2$, $g(x) = 2x$ and $h(x) = x + 4$

(iii) $f(x) = x - 4$, $g(x) = x^2$ and $h(x) = 3x - 5$

Solution :

(i) Given $f(x) = x - 1$, $g(x) = 3x + 1$, $h(x) = x^2$

To verify : $(f \circ g) \circ h = f \circ (g \circ h)$

$$(f \circ g)(x) = f(g(x)) = f(3x+1) = 3x + 1 - 1 = 3x$$

$$\therefore ((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = (f \circ g)(x^2) = 3x^2 \quad \text{--- (1)}$$

$(g \circ h)(x) = g(h(x))$
$= g(x^2)$
$= 3x^2 + 1$

$\therefore (f \circ (g \circ h))(x) = f((g \circ h)(x))$
$= f(3x^2 + 1)$
$= 3x^2 + 1 - 1$
$= 3x^2$ — (2)

$\therefore$ From (1) and (2), $(f \circ g) \circ h = f \circ (g \circ h)$

(ii) Given $f(x) = x^2$, $g(x) = 2x$, $h(x) = x + 4$

$(f \circ g)(x) = f(g(x))$
$= f(2x)$
$= (2x)^2 = 4x^2$

$\therefore ((f \circ g) \circ h)(x) = (f \circ g)(h(x))$
$= (f \circ g)(x + 4)$
$= 4(x + 4)^2$ — (1)

$(g \circ h)(x) = g(h(x))$
$= g(x + 4)$
$= 2(x + 4)$
$= 2x + 8$

$\therefore (f \circ (g \circ h))(x) = f(2x + 8)$
$= (2x + 8)^2$
$= (2(x + 4))^2$
$= 4(x + 4)^2$ — (2)

$\therefore$ From (1) and (2),

$(f \circ g) \circ h = f \circ (g \circ h)$.

(iii) Given $f(x) = x - 4$, $g(x) = x^2$, $h(x) = 3x - 5$

$(f \circ g)(x) = f(g(x))$
$= f(x^2)$
$= x^2 - 4$

$\therefore ((f \circ g) \circ h)(x) = (f \circ g)(h(x))$
$= (f \circ g)(3x - 5)$
$= (3x - 5)^2 - 4$ — (1)

$(g \circ h)(x) = g(h(x))$
$= g(3x - 5)$
$= (3x - 5)^2$

$\therefore (f \circ (g \circ h))(x) = f(g \circ h)(x))$
$= f(3x - 5)^2$
$= (3x - 5)^2 - 4$
— (2)

$\therefore$ From (1) and (2),

$(f \circ g) \circ h = f \circ (g \circ h)$.

9. Let $f = \{(-1, 3), (0, -1), (2, -9)\}$ be a linear function from Z into Z. Find $f(x)$.

Solution :

Given $f = \{(-1, 3), (0, -1), (2, -9)\}$ is a linear function from Z into Z.

Let $y = ax + b$

When $x = -1, y = 3 \Rightarrow$

$3 = -a + b$ — (1)

When $x = 0, \quad y = -1 \Rightarrow -1 = 0 + b$

$\therefore b = -1$

$\therefore (1) \Rightarrow 3 = -a - 1$

$\Rightarrow a = -4$

$\therefore a = -4, b = -1$

$\therefore y = -4x - 1$ is the required linear function.

10. In electrical circuit theory, a circuit C(t) is called a linear circuit if it satisfies the superposition principle given by $C(at_1 + bt_2) = aC(t_1) + bC(t_2)$, where a, b are constants. Show that the circuit C(t) = 3t is linear.

Solution :

Given C(t) = 3t

To Prove : C(t) is linear.

$C(at_1) = 3at_1$, $C(bt_2) = 3bt^2$

Adding,

$$C(at_1) + C(bt_2) = 3at_1 + 3bt_2 = 3(at_1 + bt_2)$$

which is the principle of super position.

$\therefore$ C(t) = 3t is a linear function.

EXERCISE 1.6

Multiple choice questions :

1. **If n (A × B) = 6 and A = {1, 3} then n(B) is**

(1) 1 (2) 2 (3) 3 (4) 6

Hint : **Ans : (3)**

n(A) = 2, n (A × B) = 6 ⇒ n(B)

= n (A × B) / n(A)

= 6/2 = 3

2. **A = {a, b, p}, B = {2, 3}, C = {p, q, r, s} then n[(A ∪ C) × B] is**

(1) 8 (2) 20 (3) 12 (4) 16

Hint : **Ans : (3)**

A ∪ C = {a, b, p, q, r, s}, B = {2, 3}

n [(A ∪ C) × B] = 6 × 2 =12

3. **If A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8} then state which of the following statement is true ?**

(1) (A × C) ⊂ (B × D)

(2) (B × D) ⊂ (A × C)

(3) (A × B) ⊂ (A × D)

(4) (D × A) ⊂ (B × A)

Hint : **Ans : (1)**

It is clearly (A × C) ⊂ (B × D).

4. **If there are 1024 relations from a set A = {1, 2, 3, 4, 5} to a set B, then the number of elements in B is**

(1) 3 (2) 2 (3) 4 (4) 8

Hint : **Ans : (2)**

n(A) = 5 = p

$\therefore$ No. of relations from A to B = 1024

$\Rightarrow \quad 2^{5q} = 1024$

$\Rightarrow \quad (32)^q = (32)^2$

$\Rightarrow \quad q = 2$

$\therefore$ n(B) = 2

5. **The range of the relation R = {(x, x^2) | x is a prime number less than 13} is**

(1) {2, 3, 5, 7}

(2) {2, 3, 5, 7, 11}

(3) {4, 9, 25, 49, 121}

(4) {1, 4, 9, 25, 49, 121}

Hint : **Ans : (3)**

Prime no's less than 13 = {2, 3, 5, 7, 11}

$\therefore$ Range of R = {4, 9, 25, 49, 121},

($\because$ R = {(x, x^2)})

6. **If the ordered pairs (a + 2, 4) and (5, 2a + b) are equal then (a, b) is**

(1) (2, – 2) (2) (5, 1)

(3) (2, 3) (4) (3, – 2)

Hint : **Ans : (4)**

$$\begin{array}{l|l} a+2=5, & 2a+b=4 \\ \Rightarrow a=3 & 6+b=4 \\ & \Rightarrow b=-2 \end{array}$$

7. Let n(A) = *m* and n(B) = *n* then the total number of non-empty relations thatn can be defined from A to B is

(1) m^n (2) n^m (3) $2^{mn} - 1$ (4) 2^{mn}

Hint : **Ans : (4)**

Total no. of non-empty relations from

A to B = $2^{n(A)\,.\,n(B)} = 2^{mn}$

8. If {(*a*, 8), (6, *b*)} represents an identity function, then the value of *a* and *b* are respectively

(1) (8, 6) (2) (8, 8) (3) (6, 8) (4) (6, 6)

Hint : **Ans : (1)**

$(a, 8), (6, b) \Rightarrow$ identity function

$\therefore a = 8, b = 6$

9. Let A = {1, 2, 3, 4} and B = {4, 8, 9, 10}. A function *f* : A → B given by *f* = {(1, 4), (2, 8), (3, 9), (4, 10)} is a

(1) Many-one function
(2) Identity function
(3) One-to-one function
(4) Into function

Hint : **Ans : (3)**

Diff. elements of A have diff. images in B.

10. If $f(x) = 2x^2$ and $g(x) = \frac{1}{3x}$, then $f \circ g$ is

(1) $\frac{3}{2x^2}$ (2) $\frac{2}{3x^2}$ (3) $\frac{2}{9x^2}$ (4) $\frac{1}{6x^2}$

Hint : **Ans : (3)**

$$(f \circ g)(x) = f(g(x))$$
$$= f\left(\frac{1}{2x}\right)$$
$$= 2\left(\frac{1}{3x}\right)^2$$
$$= \frac{2}{9x^2}$$

11. If *f* : A → B is a bijective function and if n(B) = 7, then n(A) is equal to

(1) 7 (2) 49 (3) 1 (4) 14

Hint : **Ans : (1)**

$f : A \to B$ is bijective (1 – 1 and onto) and n(B) = 7 $\therefore$ n(A) = 7

12. Let *f* and *g* be two functions given by
$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 7)\}$
$g = \{(0, 2), (1, 0), (2, 4), (-4, 2), (7, 0)\}$
then the range of *f* o *g* is

(1) {0, 2, 3, 4, 5} (2) {– 4, 1, 0, 2, 7}
(3) {1, 2, 3, 4, 5} (4) {0, 1, 2}

Hint : **Ans : (4)**

$(f \circ g)(0) = f(g(0)) = f(2) = 0$

$(f \circ g)(1) = f(g(1)) = f(0) = 1$

$(f \circ g)(2) = f(g(2)) = f(4) = 2$

$(f \circ g)(-4) = f(g(-4)) = f(2) = 0$

$(f \circ g)(7) = f(g(7)) = f(0) = 1$

$\therefore$ Range = {0, 1, 2}

13. Let $f(x) = \sqrt{1+x^2}$ then

(1) $f(xy) = f(x)\,.\,f(y)$
(2) $f(xy) \geq f(x)\,.\,f(y)$
(3) $f(xy) \leq f(x)\,.\,f(y)$
(4) None of these

Hint : **Ans : (3)**

$$f(x) = \sqrt{1+x^2}$$
$$\therefore f(y) = \sqrt{1+y^2}$$
$$\therefore f(xy) = \sqrt{1+x^2y^2}$$
$$f(x).f(y) = \sqrt{(1+x^2)(1+y^2)}$$
$$= \sqrt{1+x^2+y^2+x^2+y^2}$$
$$\geq \sqrt{1+x^2y^2}$$
$$\geq f(xy)$$
$$\therefore f(xy) \leq f(x).f(y)$$

14. If $g = \{(1,1), (2,3), (3,5), (4,7)\}$ is a function given by $g(x) = ax + \beta$ then the values of α and β are

(1) $(-1, 2)$ (2) $(2, -1)$

(3) $(-1, -2)$ (4) $(1, 2)$

Hint : **Ans : (3)**

$g(x) = \alpha x + \beta$

$\Rightarrow 1 = \alpha + \beta,\ 3 = 2\alpha + \beta,\ 5 = 3\alpha + \beta$

on Substracting, $\alpha = 2 \Rightarrow \beta = -1$

15. $f(x) = (x+1)^3 - (x-1)^3$ represents a function which is

(1) linear (2) cubic

(3) reciprocal (d) quadratic

Hint : **Ans : (4)**

$f(x) = (x+1)^3 - (x-1)^3$

$= (x^3 + 3x^2 + 3x + 1) - (x^3 - 3x^2 + 3x - 1)$

$= 6x^2 + 2$, a quadratic function.

UNIT EXERCISE - 1

1. If the ordered pairs $(x^2 - 3x, y^2 + 4y)$ and $(-2, 5)$ are equal, then find x and y.

Solution :

Given $(x^2 - 3x, y^2 + 4y) = (-2, 5)$

$\therefore x^2 - 3x = -2$	$y^2 + 4y = 5$
$\Rightarrow x^2 - 3x + 2 = 0$	$y^2 + 4y - 5 = 0$
$\Rightarrow (x-2)(x-1) = 0$	$(y+5)(y-1) = 0$
$\therefore x = 2, 1$	$y = -5, 1$

2. The Cartesian product A × A has 9 elements among which (–1, 0) and (0, 1) are found. Find the set A and the remaining elements of A × A.

Solution :

Given $n(A \times A) = 9$ and $(-1, 0), (0, 1) \in A \times A$

$\therefore$ A = {–1, 0, 1} and the remaining elements of

$A \times A = \{(-1,-1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)\}$

3. Given that $f(x) = \begin{cases} \sqrt{x-1} & x \geq 1 \\ 4 & x < 1 \end{cases}$. Find

(i) $f(0)$ (ii) $f(3)$ (iii) $f(a+1)$ in terms of a. (Given that $a \geq 0$)

Solution :

Given $f(x) = \begin{cases} \sqrt{x-1} & x \geq 1 \\ 4 & x < 1 \end{cases}$

$i)\quad f(0) = 4$

$ii)\quad f(3) = \sqrt{3-1} = \sqrt{2}$

$iii)\quad f(a+1) = \sqrt{a+1-1} \qquad (\because a \geq 0$

$= \sqrt{a} \qquad \Rightarrow a + 1 \geq 1$

4. Let A = {9, 10, 11, 12, 13, 14, 15, 16, 17} and let f : A → N be defined by f(n) = the highest prime factor of $n \in$ A. Write f as a set of ordered pairs and find the range of f.

Solution :

Given A = {9, 10, 11, 12, 13, 14, 15, 16, 17}

$f : A \to N$ defined by $f(n)$ = highest prime factor of $n \in A$.

$f(9) = 3$ ($\because$ 3 is the highest prime factor of 9)

$f(10) = 5$ ($\because 10 = 5 \times 2$)

$f(11) = 11$ ($\because$ 11 is a prime no.)

$f(12) = 3$ ($\because 12 = 3 \times 2 \times 2$)

$f(13) = 13$ ($\because$ 13 is a prime no.)

$f(14) = 7$ ($\because 14 = 7 \times 2$)

$f(15) = 5$ ($\because 15 = 5 \times 3$)

$f(16) = 2$ ($\because 16 = 2 \times 2 \times 2 \times 2$)

$f(17) = 17$ ($\because$ 17 is a prime number)

f = {(9, 3), (10, 5), (11, 11), (12, 3), (13, 13), (14, 7), (15, 5), (16, 2), (17, 17)}

Range of f = {2, 3, 5, 7, 11, 13, 17}

5. Find the domain of the function

$$f(x) = \sqrt{1+\sqrt{1-\sqrt{1-x^2}}}.$$

Solution :

Given $f(x) = \sqrt{1+\sqrt{1-\sqrt{1-x^2}}}$

If $x > 1$ and $x < -1$, $f(x)$ leads to unreal

$\therefore$ The domain of $f(x) = \{-1, 0, 1\}$

6. If $f(x) = x^2$, $g(x) = 3x$ and $h(x) = x - 2$. Prove that $(f \circ g) \circ h = f \circ (g \circ h)$.

Solution :

Given $f(x) = x^2$, $g(x) = 3x$, $h(x) = x - 2$

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(3x) \\
&= (3x)^2 = 9x^2 \\
((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \\
&= (f \circ g)(x-2) \\
&= 9(x-2)^2 \quad \text{--- (1)} \\
(g \circ h)(x) &= g(h(x)) \\
&= g(x-2) \\
&= 3(x-2) = 3x - 6 \\
\therefore (f \circ (g \circ h))(x) &= f(3x-6) \\
&= (3x-6)^2 \\
&= (3(x-2))^2 \\
&= 9(x-2)^2 \quad \text{--- (2)}
\end{aligned}$$

$\therefore$ From (1) and (2),

$(f \circ g) \circ h = f \circ (g \circ h)$.

7. Let A = {1, 2} and B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}. Verify whether A × C is a subset of B × D =?

Solution :

Given A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6}, D = {5, 6, 7, 8}

A × C = {(1, 5), (1, 6), (2, 5), (2, 6)}

B × D = {**(1, 5), (1, 6)**, (1, 7), (1, 8), **(2, 5), (2, 6)**, (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8).

Clearly A × C is a subset of B × D.

8. If $f(x) = \frac{x-1}{x+1}$, $x \neq 1$ show that $f(f(x)) = -\frac{1}{x}$, provided $x \neq 0$.

Solution :

Given $f(x) = \dfrac{x-1}{x+1}$

$$f(f(x)) = f\left(\frac{x-1}{x+1}\right)$$

$$= \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{\frac{x-1-x-1}{x+1}}{\frac{x-1+x+1}{x+1}}$$

$$= \frac{-2}{2x} = \frac{-1}{x}$$

9. The functions f and g are defined by $f(x) = 6x + 8$; $g(x) = \frac{x-2}{3}$

(i) Calculate the value of $gg\left(\frac{1}{2}\right)$

(ii) Write an expression for $gf(x)$ in its simplest form.

Solution :

Given $f(x) = 6x + 8$, $g(x) = \dfrac{x-2}{3}$

i) $$gg\left(\frac{1}{2}\right) = g\left(g\left(\frac{1}{2}\right)\right)$$

$$= g\left(\frac{\frac{1}{2}-2}{3}\right)$$

$$= g\left(\frac{-3}{6}\right)$$

$$= g\left(\frac{-1}{2}\right)$$

$$= \frac{-\frac{1}{2}-2}{3} = \frac{-5}{6}$$

ii) $gf(x) = g(f(x))$

$= g(6x+8)$

$= \dfrac{6x+8-2}{3}$

$= \dfrac{6x+6}{3}$

$= 2x+2$

$= 2(x+1)$

10. Write the domain of the following real functions

(i) $f(x) = \dfrac{2x+1}{x-9}$ **(ii)** $p(x) = \dfrac{-5}{4x^2+1}$

(iii) $g(x) = \sqrt{x-2}$ **(iv)** $h(x) = x+6$

Solution :

i) $f(x) = \dfrac{2x+1}{x-9}$

If $x = 9, f(x) \to \infty$

The domain is R – {9}

ii) $f(x) = \dfrac{-5}{4x^2+1}$

If $x \to$ c, $4x^2 + 1$ does not tend to ∞.

$\therefore$ The domain is R.

iii) $g(x) = \sqrt{x-2}$

The function exists only if $x \geq 2$

$\therefore$ The domain is $[2, \infty)$.

iv) $h(x) = x + 6$

$h(x)$ exists for all real numbers.

$\therefore$ The domain is R.

PROBLEMS FOR PRACTICE

1. If A × B = {$(a, x), (a, y), (b, x), (b, y)$}, then find A and B.

(Ans: A = {*a, b*}, B = {*x, y*})

2. If A = {1, 2, 3}, B = {3, 4}, C = {4, 5, 6}, find i) A×(B ∩ C) (ii) (A × B) ∩ (A × C).

(Ans : {(1, 4), (2, 4), (3, 4)}

3. Fnd the Cartesian product of 3 sets.

A = {1, 2}, B = {3, 4}, C = {$x : x \in$ N and $4 \leq x \leq 6$}.

{Ans : {(1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 4), (1, 4, 5), (1, 4, 6), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (2, 4, 5), (2, 4, 6)}

4. If A = {*a, d*}, B = {*b, c, e*}, C = {*b, c, f*}, then verify A × (B ∪ C) = (A × B) ∪ (A × C).

5. If n(P) = 3, n(Q) = 4, (r, 4), (g, 1), (w, 3), (r, 9) are in P × Q, find P and Q and also the remaining elements of P × Q.

(Ans : (r, 1), (r, 3), (g, 3), (g, 4), (g, 9), (w, 1), (w, 4), (w, 9))

6. Find x and y if $(x + 3, 5) = (6, 2x + y)$.

(Ans : 3, – 1)

7. If A = {1, 2, 3, 4, 5, 6}, R = {$(a, b) \mid b$ is exactly divisible by a}. Find the range of R.

(Ans : {1, 2, 3, 4, 5, 6}

8. If the ordered pairs $(x, -1), (5, y)$ belong to the set {$(a, b) \mid b = 2a - 3$}, find x and y.

(Ans : 1, 7)

9. Express R={$(x, y) \mid x^2 + y^2 = 25, x . y \in$ N}.

(Ans : (0, 5), (3, 4), (4, 3), (5, 0)}

10. If n(A) = 3 and B = {2, 3, 4, 6, 7, 8}, find the number of relations from A to B.

(Ans : 2^{18})

11. Find the domain of the following functions:

(i) $f(x)=\dfrac{1}{\sqrt{x-5}}$ (ii) $g(x)=\dfrac{3}{2-x^2}$

(Ans : i) $(5, \infty)$ ii) $R - \{-\sqrt{2}, \sqrt{2}\}$

12. If A and B are 2 sets having 3 elements in common. If n(A) = 5, n(B) = 4, find n(A × B) and n[(A × B) ∩ (B ×A)].

(Ans : 20, 9)

13. Which of the following relations are functions and write them in ordered pairs also.

i) $\{(x, y) : y=x^2, x\in\{1, 2\}, y\in\{1,2,4, 6, 8\}\}$

ii) $\{(x, y) : y > 2x+1, x\{1, 2\}, y\in\{2, 4, 6\}\}$.

(Ans : (i) a function (ii) not a function)

14. If $f : R \to R$ is defined by $f(x) = 2x^3 - 5$, show that f is bijective.

15. Prove that the function $f: N \to N$ defined by $f(x) = x^2 + x + 1$ is one-one but not onto.

16. If $f: R \to R$ is defined by $f(x) = x^2 - 3x + 2$, find $f \circ f$.

(Ans : $x^4 - 6x^3 + 10x^2 - 3x$)

17. Let $p(x) = \begin{cases} 4x-5, & x\geq 2 \\ 1-x, & x<2 \end{cases}$,

$q(x) = \begin{cases} 3x+7 & x\leq 3 \\ 7 & x>3 \end{cases}$

are functions from $R \to R$, find

(i) (p o q) (3) (ii) (q o p) (10).

(Ans : 59, 7)

18. If $f: R\to R$ be defined by $f(x)=(3-x^3)^{1/3}$, find $(f \circ f)(x)$.

(Ans : x)

19. If $f(x)=\sqrt{x-2}$, find $(f \circ f \circ f)(38)$.

(Ans : 0)

20. A function $f: [-3, 7) \to R$ is defined by

$$f(x)=\begin{cases} 4x^2-1, & -3\leq x<2 \\ 3x-2, & 2\leq x\leq 4 \\ 2x-3, & 4<x<7 \end{cases}$$

Find (i) $\dfrac{f(3)+f(-1)}{2f(6)-f(1)}$ (ii) $f(-2)-f(4)$

(Ans : $\dfrac{2}{3}$, 5)

21. The following table represents a function from A = {5, 6, 8, 10} to B = {19, 15, 9, 11} where $f(x) = 2x - 1$. Find a and b.

x	5	6	8	10
f(x)	a	11	6	19

(Ans : a = 9, b= 15)

22. A function $f : [1, 6) \to R$ is defined as follows :

$$f(x)=\begin{cases} 1+x, & 1\leq x<2 \\ 2x-1, & 2\leq x<4 \\ 3x^2-10, & 4\leq x<6 \end{cases}$$

Find (i) $f(2)-f(4)$ (ii) $2f(5)-3f(1)$

(Ans : – 35, 124)

23. If $f(x) = lx + 5$, $g(x) = 2x + 3$, find l such that $f \circ g = g \circ f$.

(Ans : l = 8)

24. Given $f(x)=x+\dfrac{1}{2}$, $g(x)=\dfrac{1}{2}x+3$, $h(x)=x^2$, verify that $(f \circ g) \circ h = f \circ (g \circ h)$.

OBJECTIVE TYPE QUESTIONS

1. $X = \{8^n - 7n - 1 / n \in N\}$, $Y = \{49N - 49 / n \in N\}$, then

   (a) $X \subset Y$ (b) $Y \subset X$

   (c) $X = Y$ (d) $X \cap Y = \phi$.

   Ans : (c)

2. If $A = \{a, b\}$, $B = \{c, d\}$, $C = \{d, e\}$, then $\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$ is equal to

   (a) $A \cap (B \cup C)$ (b) $A \cup (B \cap C)$

   (c) $A \times (B \cup C)$ (d) $A \times (B \cap C)$

   Ans : (c)

3. If $n(A) = 4$, $n(B) = 3$, $n(A \times B \times C) = 24$, then $n(C) =$

   (a) 288 (b) 1 (c) 2 (d) 17

   Ans : (c)

4. If $n(A) = 4$, $n(B) = 5$, $n(A \cap B) = 3$, then $n((A \times B) \cap (B \times A))$ is

   (a) 8 (b) 9 (c) 10 (d) 11

   Ans : (b)

5. $f : R \to R$ given by $f(x) = x + \sqrt{x^2}$ is

   (a) 1 – 1 (b) onto

   (c) bijective (d) none of these

   Ans : (d)

6 Which of the following functions from z to itself are bijections ?

   (a) $f(x) = x^3$ (b) $f(x) = x + 2$

   (c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + x$

   Ans : (b)

7. If $f(x) = \dfrac{1}{x^2}$, $g(x) = 1 + x$, then $g \circ f$ is

   (a) $(1 + x)^2$ (b) $\dfrac{x^2}{x^2+1}$

   (c) $\dfrac{1}{(1+x)^2}$ (d) $\dfrac{x^2+1}{x^2}$

   Ans : (d)

8. $f : X \to Y$ where $X = \{-1, -2, -3\}$, $Y = \{3, 4, 5\}$ is given by $f(x) = x + 6$, $x \in X$, then f is

   (a) onto (b) many to one

   (c) constant function (d) bijective

   Ans : (d)

9. The range of $f : Z \to Z$ given by $f(x) = x^2$ is

   (a) Q (b) Z (c) W (d) N

   Ans : (c)

10. The value of a in order that $\{(4, 3), (7, a)\}$ may represent a constant function is

    (a) 7 (b) 4

    (c) 3 (d) none of these

    Ans : (c)

11. The co-domain of $f : A \to R$ where $A = \{2, 3, 5\}$, $f(x) = 2x - 1$ can be taken as

    (a) $\{3, 5, 7\}$ (b) $\{3, 5, 9, 11\}$

    (c) $\{5, 9\}$ (d) $\{3, 5\}$

    Ans : (b)

CHAPTER 2 NUMBERS AND SEQUENCES

I. EUCLID'S DIVISION LEMMA AND ALGORITHM

Key Points

- ✓ Let a and b ($a > b$) be any two positive integers. Then, there exist unqiue integers q and r such that $a = bq + r$, $0 \le r < b$. (Euclid's Division Lemma)

  The remainder is always less than the divisor.

  If $r = 0$ then $a = bq$ so b divides a.

  Similarly, if b divides a then $a = bq$.
- ✓ If $a$and b are any two integers then there exist unique integers q and $r$such that $a = bq + r$, where $0 \le r < |b|$.
- ✓ If a and b are positive integers such that $a = bq + r$, then every common divisor of a and b is a common divisor of b and r and vice-versa. (Euclid's Division Algorithm)
- ✓ If a, b are two positive integers with $a > b$ then G.C.D of (a, b) = GCD of $(a - b, b)$.
- ✓ Two positive integers are said to be relatively prime or co prime if their. Highest Common Factor is 1.

Example 2.2

Find the quotient and remainder when a is divided by b in the following cases (i) $a = -12, b = 5$ (ii) $a = 17, b = -3$ (iii) $a = -19, b = -4$.

Solution :

(i) $a = -12, b = 5$

By Euclid's division lemma

$$a = bq + r, \text{ where } 0 \le r < |b|$$

$$-12 = 5 \times (-3) + 3 \quad 0 \le r < |5|$$

Therefore, Quotient $q = -3$, Remainder r = 3.

(ii) $a = 17 \qquad b = -3$

By Eculid's division lemma

$$a = bq + r, \text{ where } 0 \le r < |b|$$

$$17 = (-3) \times (-5) + 2, \quad 0 \le r < |-3|$$

Therefore, Quotient $q = -5$, Remainder r = 2.

(iii) $a = -19 \qquad b = -4$

By Eculid's division lemma

$$a = bq + r, \text{ where } 0 \le r < |b|$$

$$-19 = (-4) \times (5) + 1, \quad 0 \le r < |-4|$$

Therefore, Quotient $q = 5$, Remainder r = 1.

Example 2.3

Show that the square of an odd integer is of the form $4q + 1$, for some integer q.

Solution :

Let x be any odd integer. Since any odd integer is one more than an even integer, we have $x = 2k + 1$, for some integer k.

$$\begin{aligned} x^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4k\,(k + 1) + 1 \\ &= 4q + 1, \end{aligned}$$

where $q = k\,(k + 1)$ is some integer.

Example 2.4

If the Highest Common Factor of 210 and 55 is expressible in the form $55x - 325$, find x.

Solution :

Using Euclid's Division Algorithm, let us find the HCF of given numbers

$$\begin{aligned} 210 &= 55 \times 3 + 45 \\ 55 &= 45 \times 1 + 10 \\ 45 &= 10 \times 4 + 5 \\ 10 &= 5 \times 2 + 0 \end{aligned}$$

The remainder is zero.

So, the last divisor 5 is the Highest Common Factor (HCF) of 210 and 55.

Since, HCF is expressible in the form $55x - 325 = 5$

gives $55x = 330$

Hence $x = 6$

Example 2.5

Find the greatest number that will divide 445 and 572 leaving remainders 4 and 5 respectively.

Solution :

Since the remainders are 4, 5 respectively the required number is the HCF of the number $445 - 4 = 441$, $572 - 5 = 567$.

Hence, we will determine the HCF of 441 and 567. Using Euclid's Division Algorithm, we have

$$\begin{aligned} 567 &= 441 \times 1 + 126 \\ 441 &= 125 \times 3 + 63 \\ 126 &= 63 \times 2 + 0 \end{aligned}$$

Therefore HCF of 441,567 = 63 and so the required number is 63.

Example 2.6

Find the HCF of 396, 504, 636.

Solution :

To find HCF of 396 and 504

Using Euclid's division algorithm we get

$504 = 396 \times 1 + 108$

The remainder is $108 \neq 0$

Again applying Euclid's division algorithm

$396 = 108 \times 3 + 72$

The remainder is $72 \neq 0$,

Again applying Euclid's division algorithm

$108 = 72 \times 1 + 36$

The remainder is $36 \neq 0$,

Again applying Euclid's division algorithm

$72 = 36 \times 2 + 0$

Here the remainder is zero. Therefore HCF of 396, 504 = 36.

To find the HCF of 636 and 36.

Using Euclid's division algorithm we get

$636 = 36 \times 17 + 24$

The remainder is $24 \neq 0$

Again applying Euclid's division algorithm $36 = 24 \times 1 + 12$

The remainder is $12 \neq 0$

Again applying Euclid's division algorithm $24 = 12 \times 2 + 0$

Here the remainder is zero. Therefore HCF of 636, 36 = 12

Therefore Highest Common Factor of 396, 504 and 636 is 12.

EXERCISE 2.1

1. Find all positive integers, when divided by 3 leaves remainder 2.

Solution:

To find all positive intergers which when divided by 3 leaves remainder 2.

i.e. $a \equiv 2 \pmod 3$

$\Rightarrow$ $a - 2$ is divisible by 3

$\Rightarrow$ $a = 3K + 2$

$\therefore$ a takes the following values

a = 2, 5, 8, 11, 14, ... when K = 0, 1, 2, 3, 4, ...

2. A man has 532 flower pots. He wants to arrange them in rows such that each row contains 21 flower pots. Find the number of completed rows and how many flower pots are left over

Solution :

No. of flower pots = 532

All pots to be arranged in rows

& each row to contain 21 flower pots.

$\therefore\ 532 = 21q + r$

$$\begin{array}{r|r} & 25 \\ \hline 21 & 532 \\ & 42 \\ \hline & 112 \\ & 105 \\ \hline & 7 \end{array}$$

$\Rightarrow 532 = 21 \times 25 + 7$

$\therefore$ Number of completed rows = 25

Number of flower pots left out = 7

3. Prove that the product of two consecutive positive integers is divisible by 2.

Solution :

Let the 2 consecutive positive integers be

$x, x + 1$

$\therefore$ Product of 2 integers $= x(x + 1)$

$= x^2 + x$

Case (i)

If x is an even number

Let $x = 2k$

$\therefore\ x^2 + x = (2k)^2 + 2k$

$= 2k(2k + 1)$, divisible by 2

Case (ii)

If x is an odd number,

Let $x = 2k + 1$

$x^2 + x = (2k + 1)^2 + (2k + 1)$

$= 4k^2 + 4k + 1 + 2k + 1$

$= 4k^2 + 6k + 2$

$= 2(2k^2 + 3k + 2)$ divisible by 2

$\therefore$ Product of 2 consecutive positive integers is divisible by 2.

4. When the positive integers *a, b* and *c* are divided by 13, the respective remainders are 9,7 and 10. Show that *a + b + c* is divisible by 13.

Solution :

When a is divided by 13, remainder is 9

i.e., $a = 13q + 9$ .... (1)

When b is divided by 13, remainder is 7

i.e., $b = 13q + 7$ .... (2)

When c is divided by 13, remainder is 11

i.e., $c = 13q + 11$ .... (3)

Adding (1), (2) & (3)

$a + b + c = 39q + 26$

$= 13\ (2q + 2)$

= divisible by 13

Hence proved.

---

5. Prove that square of any integer leaves the remainder either 0 or 1 when divided by 4.

Solution :

Case (i)

Let x be the even number

Let $x = 2n$

$x^2 = (2n)^2$

$= 4n^2$, which leaves remainder 0 when divided by 4.

Case (ii)

Let x be the odd number

$\therefore$ Let $x = 2n + 1$

$x^2 = (2n + 1)^2$

$= 4n^2 + 4n + 1$

$= 4\ (n^2 + n) + 1$ leaves remainder 1 when divided by 4.

Hence proved.

---

6. Use Euclid's Division Algorithm to find the Highest Common Factor (HCF) of

(i) 340 and 412 (ii) 867 and 255

(iii)10224 and 9648 (iv) 84, 90 and 120

Solution :

i) HCF of 340, 412 by Euclid's algorithm.

First we should divide 412 by 340.

$412 = 340 \times 1 + 72$

$340 = 72 \times 4 + 52$

$72 = 52 \times 1 + 20$

$52 = 20 \times 2 + 12$

$20 = 12 \times 1 + 8$

$12 = 8 \times 1 + 4$

$8 = (4) \times 2 + 0$

$\therefore$ The last divisor "4" is the HCF.

ii) HCF of 867 and 255.

$867 = 255 \times 3 + 102$

$255 = 102 \times 2 + 51$

$102 = (51) \times 2 + 0$

$\therefore$ The last divisor "51" is the HCF.

iii) HCF of 10224 and 9648

$10224 = 9648 \times 1 + 576$

$9648 = 576 \times 16 + 432$

$576 = 432 \times 1 + 144$

$432 = (144) \times 3 + 0$

$\therefore$ The last divisior "144" is the HCF.

iv) HCF of 84, 90, and 120

First we find HCF of 84 & 90

$90 = 84 \times 1 + 6$

$84 = (6) \times 14 + 0$

$\therefore$ HCF of 84, 90 is 6

Next, we find HCF of 120 and 6.

$120 = 6 \times 20 + 0$

$\therefore$ HCF of 84, 90 and 120 = 6

7. Find the largest number which divides 1230 and 1926 leaving remainder 12 in each case.

Solution :

To find the largest number which dividies 1230 and 1926 leaving remainder 12

i.e., HCF of 1230 - 12 and 1926 - 12

i.e., HCF of 1218 and 1914

First we divide 1914 by 1218

$1914 = 1218 \times 1 + 696$

$1218 = 696 \times 1 + 522$

$696 = 522 \times 1 + 174$

$522 = (174) \times 3 + 0$

$\therefore$ HCF = 174

$\therefore$ The required largest number = 174.

8. If d is the Highest Common Factor of 32 and 60, find x and y satisfying d = $32x + 60y$.

Solution :

Given d is the HCF of 32 and 60

$\therefore \quad 60 = 32 \times 1 + 28$ .... (1)

$32 = 28 \times 1 + 4$ .... (2)

$28 = (4) \times 7 + 0$

$\therefore d = 4$

$= 32 - (28 \times 1)$ [From (2)]

$= 32 \times 1\ (60 - 32)$ [From (1)]

$= 32 - (1 \times 60 - 1 \times 32)$

$= 32 - 1 \times 60 + 1 \times 32$

$= 32\ (1 + 1) + 60\ (-1)$

$d = 32\ (2) + 60\ (-1)$

$\Rightarrow 32x + 60y = 32\ (2) + 60\ (-1)$

$\therefore$ x = 2, y = −1

9. A positive integer when divided by 88 gives the remainder 61. What will be the remainder when the same number is divided by 11?

Solution :

The standard form is $a = bq + r$

$\Rightarrow a = 88q + 61$

$a = (11 \times 8q) + (55 + 6)$

$a = 11\ [8q + 5] + 6$

$\therefore$ When the same positive integer is divided by 11 the remainder is 6.

10. Prove that two consecutive positive integers are always coprime.

Solution :

Let x, $x + 1$ be two consecutive integers.

Let G.C.D. of $(x, x + 1)$ be 'n'

Then 'n' divides $x + 1 - x$

i.e., $n = 1$

G.C.D. of $(x, x + 1) = 1$

x & $x + 1$ are Co-prime.

II. FUNDAMENTAL THEOREM OF ARITHMETIC :

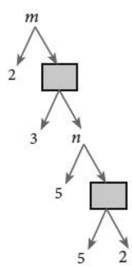
Key Points

- ✓ Every natural number except 1 can be factorized as a product of primes and this factorization is unique except for the order in which the prime factors are written.
- ✓ If a prime number p divides ab then either p divides a or p divides b. That is p divides at least one of them.
- ✓ If a composite number n divides ab, then n neither divide a nor b. For example, 6 divides 4×3 but 6 neither divide 4 nor 3.

Example 2.7

In the given factor tree, find the numbers m and n.

Solution :



Value of the first box from bottom = $5 \times 2 = 10$

Value of $n = 5 \times 10 = 50$

Value of the second box from bottom

$= 3 \times 50 = 150$

Value of $m = 2 \times 150 = 300$

Thus, the required numbers are $m = 300$, $n = 50$.

Example 2.8

Can the number 6^n, n being a natural number end with the digit 5 ? Give reason for your answer.

Solution :

Since $6^n = (2 \times 3)^n = 2^n \times 3^n$,

2 is a factor of 6^n. So, 6^n is always even.

But any number whose last digit is 5 is always odd.

Hence, 6^n cannot end with the digit 5.

Example 2.9

Is $7 \times 5 \times 3 \times 2 + 3$ a composite number ? Justify your answer.

Solution :

Yes, the given number is a composite number, because

$7 \times 5 \times 3 \times 2 + 3 = 3\times(7\times5\times2+1) = 3 \times 71$

Since the given number can be factorized in terms of two primes, it is a composite number.

Example 2.10

'a' and 'b' are two positive integers such that $a^b \times b^a = 800$. Find 'a' and 'b'.

Solution :

The number 800 can be factorized as

$800 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^5 \times 5^2$

Hence, $a^b \times b^a = 2^5 \times 5^2$

This implies that $a = 2$ and $b = 5$ (or) $a = 5$ and $b = 2$.

EXERCISE 2.2

1. For what values of natural number n, 4^n can end with the digit 6?

Solution :

$4^n = (2 \times 2)^n$

$= 2^n \times 2^n$

Since 2 is a factor of 4, 4^n is always even.

$\therefore$ If 4^n is to be end with 6, n should be even.

$\therefore$ For the even powers of 'n', 4^n will be ended with even no.

2. If m, n are natural numbers, for what values of m, does $2^n \times 5^m$ ends in 5?

Solution :

Let $x = 2^n \times 5^m$

Since m and n are natural, numbers and 2^n is even.

for any value of m, $2^n \times 5^m$ will not be ended in 5.

$\therefore$ No value of m will make x true.

3. Find the HCF of 252525 and 363636.

Solution :

Given numbers are 252525, 363636

By using prime factorization

5	252525
5	50505
3	10101
7	3367
	481

2	363636
2	181818
3	90909
3	30303
3	10101
7	3367
	481

$\therefore 252525 = 5 \times 5 \times \underline{3} \times \underline{7} \times \underline{481}$

$363636 = 2 \times 2 \times \underline{3} \times 3 \times 3 \times \underline{7} \times \underline{481}$

$\therefore$ HCF $= 3 \times 7 \times 481$

$= 10,101$

4. If $13824 = 2^a \times 3^b$ then find a and b.

Solution :

Given $2^a \times 3^b = 13824$

$\Rightarrow 2^a \times 3^b = 2^9 \times 3^2$

$\therefore a = 9, b = 2$

2	13824
2	6912
2	3456
2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	3

5. If $p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4} = 113400$ where p_1, p_2, p_3, p_4 are primes in ascending order and x_1, x_2, x_3, x_4 are integers, find the value of p_1, p_2, p_3, p_4 and x_1, x_2, x_3, x_4.

Solution :

$p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4} = 113400$

2	113400
2	56700
2	28350
3	14175
3	4725
3	1575
3	525
5	175
5	35
	7

$\therefore$ $113400 = 2^3 \times 3^4 \times 5^2 \times 7^1$

$\therefore$ $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$ and

$x_1 = 3, x_2 = 4, x_3 = 2, x_4 = 1$

6. Find the LCM and HCF of 408 and 170 by applying the fundamental theorem of arithmetic.

Solution :

Given no's are 408, 170

$$\begin{array}{r|l} 2 & 408 \\ \hline 2 & 204 \\ \hline 2 & 102 \\ \hline 3 & 51 \\ \hline & 17 \end{array} \qquad \begin{array}{r|l} 2 & 170 \\ \hline 5 & 85 \\ \hline & 17 \end{array}$$

$\therefore$ $408 = 2^3 \times 3 \times 17$

$170 = 2 \times 5 \times 17$

$\therefore$ H.C.F $= 2 \times 17 = 34$

L.C.M $= 2^3 \times 17 \times 5 \times 3 = 2040$

7. Find the greatest number consisting of 6 digits which is exactly divisible by 24, 15, 36?

Solution :

First, we find the L.C.M of 24, 15, 36

$$\begin{array}{r|lll} 3 & 24, & 15, & 36 \\ \hline 2 & 8, & 5, & 12 \\ \hline 2 & 4, & 5, & 6 \\ \hline & 2, & 5, & 3 \end{array}$$

L.C.M $= 5 \times 3^2 \times 2^3$

$= 5 \times 9 \times 8$

$= 360$

The greatest 6 digit no. is 999999

$$\begin{array}{r|l|l} 360 & 999999 & 277 \\ & 720 & \\ \hline & 2799 & \\ & 2520 & \\ \hline & \;\;2799 & \\ & \;\;2520 & \\ \hline & \;\;\;279 & \\ \hline \end{array}$$

$\therefore$ Required greatest number

$= 999999 - 279$ (remainder)

$= 999720$

8. What is the smallest number that when divided by three numbers such as 35, 56 and 91 leaves remainder 7 in each case?

Solution :

The required number is the LCM of (35, 56, 91) + remainder 7

$35 = 7 \times 5$

$56 = 7 \times 2 \times 2 \times 2$

$91 = 7 \times 13$

$\therefore$ L.C.M $= 7 \times 5 \times 13 \times 8$

$= 3640$

$\therefore$ The required number is $3640 + 7$

$= 3647$

9. Find the least number that is divisible by the first ten natural numbers.

Solution :

The required number is the LCM of

(1, 2, 3, ... 10)

$2 = \underline{2} \times 1$

$4 = \underline{2} \times 2$

$6 = 3 \times \underline{2}$

$8 = 2 \times 2 \times \underline{2}$

$9 = 3 \times 3$

$10 = 5 \times \underline{2}$ and 1, 3, 5, 7

L.C.M $= 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7$

$= 2520$

III. MODULAR ARITHMETIC :

Key Points

- ✓ Two integers a and b are congruence modulo n if they differ by an integer multiple of n. That $b - a = kn$ for some integer k. This can also be written as $a \equiv b \pmod{n}$.
- ✓ Here the number n is called modulus. In other words, $a \equiv b \pmod{n}$ means $a - b$ is divisible by n.
- ✓ The equation $n = mq + r$ through Euclid's Division lemma can also be written as $n \equiv r$ (mod m).
- ✓ Two integers a and b are congruent modulo m, written as $a \equiv b \pmod{m}$, if they leave the same remainder when divided by m.
- ✓ a, b, c and d are integers and m is a positive integer such that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then

  (i) $(a + c) \equiv (b + d) \pmod{m}$ (ii) $(a - c) \equiv (b - d) \pmod{m}$ (iii) $(a \times c) \equiv (b \times d) \pmod{m}$
- ✓ If $a \equiv b \pmod{m}$ then

  (i) $ac \equiv bc \pmod{m}$ (ii) $a \pm c \equiv b \pm c \pmod{m}$ for any integer c

Example 2.11

Find the remainders when 70004 and 778 is divided by 7.

Solution :

Since 70000 is divisible by 7

$$70000 \equiv 0 \pmod{7}$$
$$70000 + 4 \equiv 0 + 4 \pmod{7}$$
$$70004 \equiv 4 \pmod{7}$$

Therefore, the remainder when 70004 is divided by 7 is 4.

Since 777 is divisible by 7

$$777 \equiv 0 \pmod{7}$$
$$777 + 1 \equiv 0 + 1 \pmod{7}$$
$$778 \equiv 1 \pmod{7}$$

Therefore, the remainder when 778 is divided by 7 is 1.

Example 2.12

Determine the value of d such that $15 \equiv 3 \pmod{d}$.

Solution :

$15 \equiv 3 \pmod{d}$ means $15 - 3 = kd$, for some integer k.

$12 = kd$. gives d divides 12.

The divisors of 12 are 1, 2, 3, 4, 6, 12. But d should be larger than 3 and so the possible values for d are 4, 6, 12.

Example 2.13

Find the least positive value of x such that

(i) $67 + x \equiv 1 \pmod{4}$ (ii) $98 \equiv (x + 4) \pmod{5}$

Solution :

(i) $67 + x \equiv 1 \pmod{4}$

$67 + x - 1 = 4n$, for some integer n

$66 + x = 4n$

$66 + x$ is a multiple of 4.

Therefore, the least positive value of x must be 2, since 68 is the nearest multiple of 4 more than 66.

(ii) $98 \equiv (x + 4) \pmod 5$

$98 - (x + 4) = 5n$, for some integer n.

$94 - x = 5n$

$94 - x$ is a multiple of 5.

Therefore, the least positive value of x must be 4

Since $94 - 4 = 90$ is the nearest multiple of 5 less than 94.

Example 2.14

Solve $8x \equiv 1 \pmod{11}$

Solution :

$8x \equiv 1 \pmod{11}$ can be written as $8x - 1 = 11k$, for some integer k.

$$x = \frac{11k+1}{8}$$

When we put $k = 5, 13, 21, 29, \ldots$ then $11k + 1$ is divisible by 8.

$$x = \frac{11\times 5+1}{8} = 7$$

$$x = \frac{11\times 13+1}{8} = 18$$

Therefore, the solutions are 7, 18, 29, 40, ...

Example 2.15

Compute x, such that $10^4 \equiv x \pmod{19}$

Solution :

$10^2 = 100 \equiv 5 \pmod{19}$

$10^4 = (10^2)^2 \equiv 5^2 \pmod{19}$

$10^4 = 25 \pmod{19}$

$10^4 = 6 \pmod{19}$ (since $25 = 6 \pmod{19}$)

Therefore, $x = 6$.

Example 2.16

Find the number of integer solutions of $3x \equiv 1 \pmod{15}$.

Solution : $3x \equiv 1 \pmod{15}$ can be written as

$3x - 1 = 15k$ for some integer k

$$3x = 15k + 1$$

$$x = \frac{15k+1}{3}$$

$$x = 5k + \frac{1}{3}$$

Since $5k$ is an integer, $5k + \frac{1}{3}$ cannot be an integer.

So there is no integer solution.

Example 2.17

A man starts his journey from Chennai to Delhi by train. He starts at 22.30 hours on Wednesday. If it takes 32 hours of travelling time and assuming that the train is not late, when will he reach Delhi ?

Solution :

Starting time 22.30, Travelling time 32 hours. Here we use modulo 24.

The reaching time is

$22.30 + 32 \pmod{24} \equiv 54.30 \pmod{24}$

$\equiv 6.30 \pmod{24}$

(Since $32 = (1 \times 24) + 8$

Thursday Friday)

Example 2.18

Kala and Vani are friends, Kala says, "Today is my birthday" and she asks Vani, "When will you celebrate your birthday?" Vani replies, "Today is Monday and I celebrated my birthday 75 days ago". Find the day when Vani celebrated her birthday.

Solution :

Let us associate the numbers 0, 1, 2, 3, 4, 5, 6 to represent the weekdays from Sunday to Saturday respecitvely.

Vani says today is Monday. So the number for Monday is 1. Since Vani's birthday was 75 days ago, we have to subtract 75 from 1 and take the modulo 7, since a week contain 7 days.

$-74 \pmod 7 \equiv -4 \pmod 7 \equiv 7-4 \pmod 7 \equiv 3 \pmod 7$

(Since, $-74-3=-77$ is divisible by 7)

Thus, $1-75 \equiv 3 \pmod 7$

The day for the number 3 is Wednesday.

Therefore, Vani's birthday must be on Wednesday.

EXERCISE 2.3

1. Find the least positive value of x such that

(i) $71 \equiv x \pmod 8$ (ii) $78 + x \equiv 3 \pmod 5$

(iii) $89 \equiv (x+3) \pmod 4$

(iv) $96 \equiv \frac{x}{7} \pmod 5$ (v) $5x \equiv 4 \pmod 6$

Solution :

i) $71 \equiv x \pmod 8$

$71 - x = 8n$

$\therefore$ $71 - x$ is a multiple of 8

$\therefore$ The least +ve value of x is 7

($\because$ $71 - 7 = 64$ is the nearest multiple of 8) less than 71.

ii) $78 + x = 3 \pmod 5$

$\Rightarrow 78 + x - 3 = 5n$

$\Rightarrow 75 + x$ is a multiple of 5

$\therefore$ The least +ve value of x is 5

($\because$ $75 + 5 = 80$, is the nearest multiple of 5 above 75)

iii) $89 \equiv (x + 3) \pmod 4$

$\Rightarrow 89 - x - 3 = 4n$

$\Rightarrow 86 - x$ is a multiple of 4

$\therefore$ The least +ve value is 2

($\because$ $86 - 2 = 84$ is the nearest multiple of 4 less than 86)

iv) $96 \equiv \frac{x}{7} \pmod 5$

$\Rightarrow 96 - \frac{x}{7} = 5n$

$\Rightarrow 96 - \frac{x}{7}$ is a multiple of 5

$\therefore$ The least +ve value of x is 7

($\because$ $96 - 1 = 95$ is a multiple of 5)

v) $5x \equiv 4 \pmod 6$

$\Rightarrow 5x - 4 = 6n$

$\therefore$ $5x - 4$ is a multiple of 6

$\therefore$ The least +ve value of x is 2

($\because$ $5(2) - 4 = 6$ is a multiple of 6)

2. If x is congruent to 13 modulo 17 then $7x - 3$ is congruent to which number modulo 17?

Solution :

Given $x \equiv 13 \pmod{17}$

$\Rightarrow$ $x - 13$ is a multiple of 17

$\therefore$ The least +ve value of x is 30

$\therefore$ $7x - 3 \equiv y \pmod{17}$

$\Rightarrow$ $7(30) - 3 \equiv y \pmod{17}$

$\Rightarrow$ $207 \equiv y \pmod{17}$

$\therefore y = 3$ ($\because$ $207 - 3 = 204$ is divisible by 17)

3. Solve $5x \equiv 4 \pmod 6$

Solution :

Given $5x \equiv 4 \pmod 6$

$\Rightarrow$ $5x - 4$ is divisible by 6

$\therefore$ $x = 2, 8, 14, \ldots\ldots\ldots$ (by assumption)

4. Solve $3x - 2 \equiv 0$ (mod 11)

Solution :

Given $3x - 2 \equiv 0 \pmod{11}$

$\therefore$ $3x - 2$ is divisible by 11

$\therefore$ The possible values of x are 8, 19, 30, ...

5. What is the time 100 hours after 7 a.m.?

Solution :

Formula :

$t + n = f \pmod{24}$

$t \to$ current time

$n \to$ no. of hrs.

$f \to$ future time

$100 + 7 = f \pmod{24}$

$\Rightarrow$ $107 - f$ is divisible by 24

$\therefore$ $f = 11$ so that $107 - 11 = 96$ is divisible by 24.

$\therefore$ The time is 11 A.M.

6. What is the time 15 hours before 11 p.m.?

Solution :

Formula :

$t - n \equiv p \pmod{24}$	$t \to$ Current time
$\Rightarrow 11 - 15 = -4 \equiv -1 \times 24 + 20$	$n \to$ no of hrs
$\equiv 20 \pmod{24}$	$p \to$ past time

$\therefore$ The time 15 hours in the past was 8 p.m.

7. Today is Tuesday. My uncle will come after 45 days. In which day my uncle will be coming?

Solution :

Today is Tuesday

Day after 45 days = ?

When we divide 45 by 7, remainder is 3.

$\therefore$ The 3rd day from Tuesday is <u>Friday</u>

8. Prove that $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n.

Solution :

When n = 1, 2n + 6 × 9n

= 2 + (6 × 9) = 56, divisible by 7.

9. Find the remainder when 2^{81} is divided by 17.

Solution :

To find the remainder when 2^{81} is divided by 17.

$$2^4 = 16 = -1 \pmod{17}$$

$$\Rightarrow 2^{80} = (2^4)^{20} = (-1)^{20} = 1$$

$$\therefore 2^{81} = 2^{80} \times 2^1 = 1 \times 2 = 2$$

10. The duration of flight travel from Chennai to London through British Airlines is approximately 11 hours. The airplane begins its journey on Sunday at 23:30 hours. If the time at Chennai is four and half hours ahead to that of London's time, then find the time at London, when will the flight lands at London Airport.

Solution :

Formula :

$t + n \equiv f \pmod{24}$	$t \to$ present time
$\Rightarrow 23.30 + 11 = f \pmod{24}$	$n \to$ no of hours
$\Rightarrow 34.30 = f \pmod{24}$	$f \to$ future time
$\therefore 34.30 - f$ is divisible by 24	
$\Rightarrow$ $f = 10.30$ (a.m)	

But the time difference between London & Chennai is 4.30 hrs.

$\therefore$ Flight reaches London Airpot at

= 10.30 hrs − 4.30 hrs

= 6 AM next day i.e. 6 AM on Monday

IV. SEQUENCES :

Key Points

- ✓ A real valued sequence is a function defined on the set of natural numbers and taking real values.
- ✓ A sequence can be considered as a function defined on the set of natural numbers.
- ✓ Though all the sequences are functions, not all the functions are sequences.

Example 2.19

Find the next three terms of the sequences.

(i) $\frac{1}{2}, \frac{1}{6}, \frac{1}{14}, ...$ (ii) 5, 2 – 1, – 4, ...

(iii) 1, 0.1, 0.01, ...

Solution :

(i) $\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \ldots$

+ 4 + 4 + 4

In the above sequence the numerators are same and the denominator is increased by 4.

So the next three terms are

$$= \frac{1}{14 \quad 4} = \frac{1}{18}$$

$$= \frac{1}{18 \quad 4} = \frac{1}{22}$$

$$= \frac{1}{22 \quad 4} = \frac{1}{26}$$

(ii) 5, 2, –1, –4, ...

– 3 – 3 – 3

Here each term is decreased by 3. So the next three terms are – 7, – 10, – 13.

(iii) 1, 0.1, 0.01, ...

+ 10 + 10

Here each term is divided by 10. Hence, the next three terms are

$$a_4 = \frac{0.01}{10} = 0.001$$

$$a_5 = \frac{0.001}{10} = 0.0001$$

$$a_6 = \frac{0.0001}{10} = 0.00001$$

Example 2.20

Find the general term for the following sequences

(i) 3, 6, 9, ... (ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...$

(iii) 5, – 25, 125. ....

Solution :

(i) 3, 6, 9, ...

Here the terms are multiples of 3. So the general term is

$$a_n = 3n,$$

(ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...$

$$a_1 = \frac{1}{2};\ a_2 = \frac{2}{3};\ a_3 = \frac{3}{4}$$

We see that the numerator of n^{th} terms is n, and the denominator is one more than the numerator. Hence, $a_n = \frac{n}{n+1}$, $n \in \mathrm{N}$

(iii) 5, –25, 125, ...

The terms of the sequence have + and – sign alternatively and also they are in powers of 5.

So the general terms $a_n = (-1)^{n+1} 5^n$, $n \in \mathrm{N}$

Example 2.21

The general term of a sequence is defined as

$$a_n = \begin{cases} n(n+3); n \in N \text{ is odd} \\ n^2+1 \quad ; n \in N \text{ is even} \end{cases}$$

Find the eleventh and eighteenth terms.

Solution :

To find a_{11}, since 11 is odd, we put

$n = 11$ in $a_n = n\ (n + 3)$

Thus, the eleventh term

$a_{11} = 11\ (11 + 3) = 154,$

To find a_{18}, since 18 is even we put

$n = 18$ in $a_n = n^2 + 1$

Thus, the eighteenth term

$a_{18} = 18^2 + 1 = 325.$

Example 2.22

Find the first five terms of the following sequence.

$$a_1 = 1,\ a_2 = 1,\ a_n = \frac{a_{n-1}}{a_{n-2}+3};\ n \geq 3,\ n \in N$$

Solution :

The first two terms of this sequence are given by $a_1 = 1$. $a_2 = 1$. The third term a_3 depends on the first and second terms.

$$a_3 = \frac{a_{3-1}}{a_{3-2}+3} = \frac{a_2}{a_1+3} = \frac{1}{1+3} = \frac{1}{4}$$

Similarly the fourth term a_4 depends upon a_2 and a_3.

$$a_4 = \frac{a_{4-1}}{a_{4-2}+3} = \frac{a_3}{a_2+3} = \frac{\frac{1}{4}}{1+3} = \frac{\frac{1}{4}}{4} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

In the same way, the fifth term a_5 can be calculated as

$$a_5 = \frac{a_{5-1}}{a_{5-2}+3} = \frac{a_4}{a_3+3} = \frac{\frac{1}{16}}{\frac{1}{4}+3} = \frac{1}{16} \times \frac{4}{13} = \frac{1}{52}$$

Therefore, the first five terms of the sequence are 1, 1, $\frac{1}{4}, \frac{1}{16}, \frac{1}{25}$

EXERCISE 2.4

1. Find the next three terms of the following sequence.

(i) 8, 24, 72, … **(ii) 5, 1,-3,…**

(iii) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16},$ ....

Solution :

i) Given sequence is 8, 24, 72,..........

Each number is multiplied by 3

$\therefore$ The next 3 terms in the sequence are

$72 \times 3 = 216$

$216 \times 3 = 648$

$648 \times 3 = 1944$

$\therefore$ 216, 648, 1944

ii) Given sequence is 5, 1, −3, ...

Each number is subtracted by 4

$-3 - 4 = -7$

$-7 - 4 = -11$

$-11 - 4 = -15$

$\therefore$ The next 3 terms in the sequence are $-7, -11, -15$

iii) Given sequence is $\frac{1}{4}, \frac{2}{9}, \frac{3}{16},$ ....

Each no. in Numerator is increased by 1 & all nos in denominator are consecutive square no's

$\therefore$ The next 3 terms are

$\frac{4}{25}, \frac{5}{36}, \frac{6}{49},$ ....

2. Find the first four terms of the sequences whose n^{th} terms are given by

(i) $a_n = n^3 - 2$ (ii) $a_n = (-1)^{n+1} n(n+1)$ (iii) $a_n = 2n^2 - 6$

Solution :

i) Given $a_n = n^3 - 2$

$n = 1 \Rightarrow a_1 = 1^3 - 2 = 1 - 2 = -1$

$n = 2 \Rightarrow a_2 = 2^3 - 2 = 8 - 2 = 6$

$n = 3 \Rightarrow a_3 = 3^3 - 2 = 27 - 2 = 25$

$n = 4 \Rightarrow a_4 = 4^3 - 2 = 64 - 2 = 62$

$\therefore$ The first 4 terms are $-1, 6, 25, 62$

ii) Given $a_n = (-1)^{n+1} n(n+1)$

$n = 1 \Rightarrow a_1 = (-1)^2 . 1(2) = 2$

$n = 2 \Rightarrow a_2 = (-1)^3 . 2(3) = -6$

$n = 3 \Rightarrow a_3 = (-1)^4 . 3(4) = 12$

$n = 4 \Rightarrow a_4 = (-1)^5 . 4(5) = -20$

$\therefore$ The first 4 terms are $2, -6, 12, -20$

iii) Given $a_n = 2n^2 - 6$

$n = 1 \Rightarrow a_1 = 2(1) - 6 = -4$

$n = 2 \Rightarrow a_2 = 2(4) - 6 = 2$

$n = 3 \Rightarrow a_3 = 2(9) - 6 = 12$

$n = 4 \Rightarrow a_4 = 2(16) - 6 = 26$

$\therefore$ The first 4 terms are $-4, 2, 12, 26$

3. Find the n^{th} term of the following sequences

(i) 2,5,10,17, ... (ii) $0, \frac{1}{2}, \frac{2}{3}$, ... (iii) 3,8,13,18, ...

Solution :

i) Given sequence is 2, 5, 10, 17, ...

$\Rightarrow 1^2 + 1, 2^2 + 1, 3^2 + 1, 4^2 + 1, \ldots\ldots\ldots\ldots$

$\therefore$ The n^{th} term of the sequence is

$$a_n = n^2 + 1$$

ii) Given sequence is $0, \frac{1}{2}, \frac{1}{3}$, ...

$$\Rightarrow \frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \ldots\ldots\ldots\ldots$$

$$\Rightarrow \frac{1-1}{1}, \frac{2-1}{2}, \frac{3-1}{3}, \ldots\ldots\ldots\ldots$$

$\therefore$ The n^{th} term of the sequence is

$$a_n = \frac{n-1}{n}$$

iii) Given sequence is 3, 8, 13, 18, ...

$\Rightarrow 5 - 2, 10 - 2, 15 - 2, 20 - 2, \ldots\ldots\ldots$

$\Rightarrow 5(1) - 2, 5(2) - 2, 5(3) - 2, 5(4) - 2$

$\therefore$ The n^{th} term of the sequence is

$$a_n = 5n - 2$$

4. Find the indicated terms of the sequences whose n^{th} terms are given by

(i) $a_n = \frac{5n}{n+2}$; a_6 and a_{13}

(ii) $a_n = -(n^2 - 4)$; a_4 and a_{11}

Solution :

i) Given $a_n = \frac{5n}{n+2}, \quad a_6, a_{13} = ?$

$$a_6 = \frac{5(6)}{6+2} = \frac{30}{8} = \frac{15}{4}$$

$$a_{13} = \frac{5(13)}{13+2} = \frac{65}{15} = \frac{13}{3}$$

ii) $a_n = -(n^2 - 4)$; $a_4, a_{11} = ?$

$a_4 = -(16 - 4) = -12$

$a_{11} = -(121 - 4) = -117$

5. Find a_8 and a_{15} whose n^{th} term is $a_n = \begin{cases} \frac{n^2-1}{n+3} & ; n \text{ is even}, n \in N \\ \frac{n^2}{2n+1} & ; n \text{ is odd}, \end{cases}$

Solution :

Given

$$a_n = \begin{cases} \dfrac{n^2-1}{n+3} & ; n \text{ is even}, n \in N \\ \dfrac{n^2}{2n+1} & ; n \text{ is odd}, \end{cases}$$

$$a_8 = \frac{8^2-1}{8+3} = \frac{63}{11}$$

$$a_{15} = \frac{15^2}{2(15)+1} = \frac{225}{31}$$

6. If $a_1 = 1$, $a_2 = 1$ and $a_n = 2a_{n-1} + a_{n-2}$, $n \geq 3$, $n \in$ N, then find the first six terms of the sequence.

Solution :

Given $a_1 = 1, a_2 = 1$

$a_n = 2\,a_{n-1} + a_{n-2}$, $n \geq 3$

$a_3 = 2a_2 + a_1$

$= 2\,(1) + 1$

$= 3$

$a_4 = 2a_3 + a_2$

$= 2\,(3) + 1$

$= 7$

$a_5 = 2a_4 + a_3$

$= 2\,(7) + 3$

$= 17$

$a_6 = 2a_5 + a_4$

$= 2\,(17) + 7$

$= 41$

$\therefore$ The first 6 terms are 1, 1, 3, 7, 17, 41

V. ARITHMETIC PROGRESSION :

Key Points

- ✓ An Arithmetic Progression is a sequence whose successive terms differ by a constant number.
- ✓ Let a and d be real numbers. Then the numbers of the form $a, a + d, a + 2d, a + 3d, a + 4d,...$ is said to form Arithmetic Progression denoted by A.P. The number 'a' is called the first term and 'd' is called the common difference.
- ✓ If there are finite numbers of terms in an A.P. then it is called Finite Arithmetic Progression. If there are infinitely many terms in an A.P. then it is called Infinite Arithmetic Progression.
- ✓ The n^{th} term denoted by t_n can be written as $t_n = a + (n - 1)d$.
- ✓ The common difference of an A.P. can be positive, negative or zero.
- ✓ An Arithmetic progression having a common difference of zero is called a constnat arithmetic progression.
- ✓ In a finite A.P. whose first term is a and last term l, then the number of terms in the A.P. is given by $l = a + (n - 1)d$ gives $n = \left(\dfrac{l-a}{d}\right) + 1$.
- ✓ If every term is added or subtracted by a constant, then the resulting sequence is also an A.P.

- ✓ In every term is multiplied or divided by a non-zero number, then the resulting sequence is also an A.P.
- ✓ If the sum of three consecutive terms of an A.P. is given, then they can be taken as $a-d$, a and $a+d$. Here the common difference is d.
- ✓ If the sum of four consecutive terms of an A.P. is given then, they can be taken as $a-3d$, $a-d$, $a+d$ and $a+3d$. Here common difference is $2d$.
- ✓ Three non-zero numbers a, b, c are in A.P. if and only if $\boxed{2b = a+c}$.

Example 2.23

Check whether the following sequences are in A.P. or not ?

(i) $x+2, 2x+3, 3x+4, ...$

(ii) 2, 4, 8, 16, ...

(iii) $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}, ...$

Solution :

To check that the given sequence is in A.P., it is enough to check if the differences between the consecutive terms are equal or not.

(i) $t_2 - t_1 = (2x+3) - (x+2) = x+1$

$t_3 - t_2 = (3x+4) - (2x+3) = x+1$

$t_2 - t_1 = t_3 - t_2$

Thus, the differnces between consecutive terms are equal.

Hence the sequence $x+2, 2x+3, 3x+4, ...$ is in A.P.

(ii) $t_2 - t_1 = 4 - 2 = 2$

$t_3 - t_2 = 8 - 4 = 4$

$t_2 - t_1 = t_3 - t_2$

Thus, the differences between consecutive terms are not equal. Hence the terms of the sequence 2, 4, 8, 16, ... are not in A.P.

(iii) $t_2 - t_1 = 5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$

$t_3 - t_2 = 7\sqrt{2} - 5\sqrt{2} = 2\sqrt{2}$

$t_4 - t_3 = 9\sqrt{2} - 7\sqrt{2} = 2\sqrt{2}$

Thus, the differences between consecutive terms are equal. Hence the terms of the sequence $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}, ...$ are in A.P.

Example 2.24

Write an A.P. whose first term is 20 and common difference is 8.

Solution :

First term $= a = 20$; common difference $= d = 8$

Arithmetic Progression is $a, a+d, a+2d, a+3d, ...$

In this case, we get 20, 20 + 8, 20 + 2(8), 20 + 3(8) ....

So, the required A.P. is 20, 28, 36, 44, ...

Example 2.25

Find the 15^{th}, 24^{th} and n^{th} term (general term) of an A.P. given by 3, 15, 27, 39, ...

Solution :

We have, first term $= a = 3$ and common difference $= d = 15 - 3 = 12$.

We know that n^{th} term (general term) of an A.P. with first term a and common difference d is given by $t_n = a + (n-1)d$

$t_{15} = a + (15-1)\,d = a + 14d$

$= 3 + 14(12) = 171$

$t_{24} = a + 23d = 3 + 23(12) = 279$

(Here $a = 3$ and $d = 12$)

The n^{th} (general term) term is given by

$t_n = a + (n-1)d$

Thus, $t_n = 3 + (n-1)\ 12$

$t_n = 12n - 9$

Example 2.26

Fiond the number of terms in the A.P. 3, 6, 9, 12, ... 111.

Solution :

First term $a = 3$; common difference $d = 6 - 3 = 3$; last term $l = 111$

We know that, $n = \left(\frac{l-a}{d}\right) + 1$

$$n = \left(\frac{111-3}{3}\right) + 1 = 37$$

Thus the A.P. contain 37 terms.

Example 2.27

Determine the general term of an A.P. whose 7th term is – 1 and 16th term is 17.

Solution :

Let the A.P. be $t_1, t_2, t_3, t_4, ...$

It is given that $t_7 = -1$ and $t_{16} = 17$

$a + (7-1)d = -1$ and $a + (16-1)d = 17$

$$a + 6d = -1 \quad ...(1)$$

$$a + 15d = 17 \quad ...(2)$$

Subtracting equation (1) from equation (2), we get 9d = 18 gives d = 2

Putting d = 2 in equation (1), we get $a + 12 = -1$ so $a = -13$

Hence, general term $t_n = a + (n-1)d$

$= -13 + (n-1) \times 2 = 2n - 15$

Example 2.28

If l^{th}, m^{th} and n^{th} terms of an A.P. are x, y, z respectively, then show that

(i) $x(m-n) + y(n-1) + z(l-m) = 0$

(ii) $(x-y)n + (y-z)l + (z-x)m = 0$

Solution :

(i) Let a be the first term and d be the common difference. It is given that

$t_l = x$, $t_m = y$, $t_n = z$

Using the general term formula

$$a + (l-1)\,d = x \quad ...(1)$$

$$a + (m-1)\,d = y \quad ...(2)$$

$$a + (n-1)\,d = z \quad ...(3)$$

We have, $x(m-n) + y(n-l) + s(l-m)$

$= a[(m-n) + (n-l) + (l-m)] + d[(m-n) + (l-1) + (n-l)(m-1) + (l-m)(n-1)]$

$= a[0] + d[lm - ln - m + n + mn - lm - n + l + ln - mn - l + m]$

$= a(0) + d(0) = 0$

(ii) On subtracting equation (2) from equation (1), equation (3) from equation (2) and equation (1) from equation (3), we get

$$x - y = (l-m)\,d$$

$$y - z = (m-n)\,d$$

$$z - x = (n-l)\,d$$

$(x-y)n + (y-z)l + (z-x)m = [(l-m)n + (m-n)l + (n-l)m]\,d$

$= [ln - mn + lm - nl + nm - lm]\,d = 0$

Example 2.29

In an A.P., sum of four consective terms is 28 and their sum of their squares is 276. Find the four numbers.

Solution :

Let us take the four terms in the form $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$.

Since sum of the four terms is 28,

$a - 3d + a - d + a + d + a + 3d = 28$

$4a = 28$ gives $a = 7$

Similarly, since sum of their squares is 276,

$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 276.$

$a^2 - 6ad + 9d^2 + a^2 - 2ad + d^2 + a^2 + 2ad + d^2 + a^2 + 6ad + 9d^2 = 276$

$4a^2 + 20d^2 = 276 \Rightarrow 4(7)^2 + 20d^2 = 276.$

$d^2 = 4$ gives $d = \pm 2$

If $d = 2$ then the four numbers are $7 - 3(2)$, $7 - 2$, $7 + 2$, $7 + 3(2)$

That is the four numbers are 1, 5, 9 and 13.

If $a = 7$, $d = -2$ then the four numbers are 13, 9, 5 and 1

Therefore, the four consecutive terms of the A.P. are 1, 5, 9 and 13.

Example 2.30

A mother divides ₹207 into three parts such that the amount are in A.P. and gives it to her three children. The product of the two least amounts that the children had ₹4623. Find the amount received by each child.

Solution :

Let the amount received by the three children be in the form of A.P. is given by

$a - d, a, a + d$. Since, sum of the amount is ₹207, we have

$(a - d) + a + (a + d) = 307$

$3a = 207$ gives $a = 69$

It is given that product of the two least amounts is 4623

$(a - d)\, a = 4623$

$(69 - d)\, 69 = 4623$

$d = 2$

Therefore, amount given by the mother to her three children are

₹(69 –2), ₹69, ₹(69 + 2). That is, ₹67, ₹69 and ₹71.

EXERCISE 2.5

1. Check whether the following sequences are in A.P.

(i) $a - 3, a - 5, a - 7, \ldots\ldots$

(ii) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots$

(iii) 9, 13, 17, 21, 25, ...

(iv) $\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots\ldots$

(v) $1, -1, 1, -1, 1, -1, \ldots\ldots$

Solution :

i) $a - 3, a - 5, a - 7, \ldots\ldots$

$t_2 - t_1 = (a - 5) - (a - 3)$

$= a - 5 - a + 3$

$= -2$

$t_3 - t_2 = (a - 7) - (a - 5)$

$= a - 7 - a + 5$

$= -2$

$\therefore t_2 - t_1 = t_3 - t_2$

$\therefore$ The given sequence is in A.P.

ii) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$

$t_2 - t_1 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$

$t_3 - t_2 = \frac{1}{4} - \frac{1}{3} = \frac{3-4}{12} = \frac{-1}{12}$

$\therefore t_2 - t_1 \neq t_3 - t_2$

$\therefore$ The given sequence is not in A.P.

iii) 9, 13, 17, 21, 25, ...

Each term of the sequence is increased by a constant number 4.

$\therefore$ The sequence is in A.P.

iv) $\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, ...$

$t_2 - t_1 = 0 - \left(\frac{-1}{3}\right) = \frac{1}{3}$

$t_3 - t_2 = \frac{1}{3} - 0 = \frac{1}{3}$

$t_4 - t_3 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$\therefore$ The sequence is in A.P.

v) 1, −1, 1, −1, 1, −1 ....

$t_2 - t_1 = -1 - 1 = -2$

$t_3 - t_2 = 1 - (-1) = 1 + 1 = 2$

$\therefore$ The sequence is not in A.P.

2. First term *a* and common difference *d* are given below. Find the corresponding A.P.

(i) $a = 5, d = 6$ (ii) $a = 7, d = -5$

(iii) $a = \frac{3}{4}, d = \frac{1}{2}$

Solution :

i) $a = 5$, d = 6

$\therefore$ The A.P. is $a, a + d, a + 2d, a + 3d$, ...

$= 5, 5 + 6, 5 + 2(6), 5 + 3(6), ...$

$= 5, 11, 17, 23, ...$

ii) $a = 7, d = -5$

$\therefore$ The A.P is

$= 7, 7 + (-5), 7 + 2(-5), 7 + 3(-5), ...$

$= 7, 2, -3, -8, ...$

iii) $a = \frac{3}{4}, d = \frac{1}{2}$

$\therefore$ The A.P is

$= \frac{3}{4}, \frac{3}{4} + \frac{1}{2}, \frac{3}{4} + 2\left(\frac{1}{2}\right), \frac{3}{4} + 3\left(\frac{1}{2}\right), ...$

$= \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, ...$

3. Find the first term and common difference of the Arithmetic Progressions whose n^{th} terms are given below

(i) $t_n = -3 + 2n$ (ii) $t_n = 4 - 7n$

Solution :

i) $t_n = -3 + 2n$

$n = 1 \Rightarrow t_1 = -3 + 2(1) = -1$

$n = 2 \Rightarrow t_2 = -3 + 2(2) = 1$

$n = 3 \Rightarrow t_3 = -3 + 2(3) = 3$ ....

$\therefore a = -1, \quad d = t_2 - t_1$

$= 1 - (-1)$

d = 2

ii) $t_n = 4 - 7n$

$n = 1 \Rightarrow t_1 = 4 - 7(1) = -3$

$n = 2 \Rightarrow t_2 = 4 - 7(2) = -10$

$n = 3 \Rightarrow t_3 = 4 - 7(3) = -17$

$\therefore a = -3, \quad d = t_2 - t_1$

$= -10 - (-3)$

$= -10 + 3$

d = −7

4. Find the 19th term of an A.P. −11, −15, −19,......

Solution :

Given A.P is −11, −15, −19, ...

$a = -11, \quad d = -15 - (-11)$

$= -15 + 11$

$= -4$

$\therefore$ The 19th term is

$t_{19} = a + (19 - 1)\,d$

$= a + 18\,d$

$= (-11) + 18(-4)$

$= -11 - 72$

$= -83$

5. Which term of an A.P. 16, 11, 6, 1,... is −54?

Solution :

Given A.P. is 16, 11, 6, 1, ... − 54

$a = 16, d = -5, t_n = -54$

$\Rightarrow \quad a + (n-1)\,d = -54$

$\Rightarrow \quad 16 + (n-1)(-5) = -54$

$\Rightarrow \quad 16 - 5n + 5 = -54$

$\Rightarrow \quad -5n + 21 = -54$

$\Rightarrow \quad -5n + 21 = -54$

$\Rightarrow \quad -5n = -54 - 21$

$\Rightarrow \quad -5n = -75$

$\therefore n = 15$

$\therefore$ 15th term of A.P. is − 54

6. Find the middle term(s) of an A.P. 9, 15, 21, 27,…,183.

Solution :

Given A.P is 9, 15, 21, 27, ... 183

$a = 9, \; d = 6, \; l = 183$

$n = \frac{l-a}{d} + 1$

$= \frac{183-9}{6} + 1$

$= \frac{174}{6} + 1$

$= 29 + 1$

$n = 30$

$\therefore$ Middle terms are $30/2, \; 30/2 + 1$

$= 15^{th}, \; 16^{th}$

$t_{15} = a + 14d$	$t_{16} = a + 15d$
$= 9 + 14(6)$	$= 9 + 15(6)$
$= 9 + 84$	$= 9 + 90$
$= 93$	$= 99$

$\therefore$ The 2 middle terms are 93, 99.

7. If nine times ninth term is equal to the fifteen times fifteenth term, show that six times twenty fourth term is zero.

Solution :

Given $9\,(t_9) = 15\,(t_{15})$

To Prove : $6\,(t_{24}) = 0$

$\Rightarrow \quad 9\,(t_9) = 15\,(t_{15})$

$\Rightarrow \quad 9\,(a + 8d) = 15\,(a + 14d)$

$\Rightarrow \quad 3\,(a + 8d) = 5\,(a + 14d)$

$\Rightarrow \quad 3a + 24d = 5a + 70d$

$\Rightarrow \quad 2a + 46d = 0$

$\Rightarrow \quad 2\,(a + 23d) = 0$

Multiplying 3 on both sides,

$\Rightarrow \quad 6\,(a + 23d) = 0$

$\Rightarrow \quad 6\,(t_{24}) = 0$

Hence proved.

8. If $3 + k$, $18 - k$, $5k + 1$ are in A.P. then find k.

Solution :

Given $3 + k$, $18 - k$, $5k + 1$ are in A.P.

$\Rightarrow (18 - k) - (3 + k) = (5k + 1) - (18 - k)$

$\Rightarrow \quad 15 - 2k = 6k - 17$

$\Rightarrow \quad -8k = -32$

$\Rightarrow \quad k = 4$

9. Find x, y and z, given that the numbers x, 10, y, 24, z are in A.P.

Solution :

Given that x, 10, y, 24, z are in A.P.

$\therefore$ y is the arithmetic mean of 10 & 24

$$\Rightarrow y = \frac{10+24}{2} = \frac{34}{2} = 17$$

$\therefore$ x, 10, y, 24, z are in A.P.

Clearly $d = 7$

$\therefore x = 10 - 7 = 3 \quad \& \; z = 24 + 7 = 31$

$\therefore x = 3, y = 17, z = 31$

10. In a theatre, there are 20 seats in the front row and 30 rows were allotted. Each successive row contains two additional seats than its front row. How many seats are there in the last row?

Solution :

By the data given,

$a = 20, d = 2, n = 30$

$t_{30} = a + 29d$

$= 20 + 29(2)$

$= 20 + 58$

$= 78$

$\therefore$ The no. of seats in 30^{th} row = 78

11. The sum of three consecutive terms that are in A.P. is 27 and their product is 288. Find the three terms.

Solution :

Let the 3 consecutive terms in an A.P. be

$a - d, a, a + d$

i) Sum of 3 terms = 27

$\Rightarrow \quad a - d + a + a + d = 27$

$\Rightarrow \quad 3a = 27$

$a = 9$

ii) Product of 3 terms = 288

$\Rightarrow \quad (a - d) \,.\, a \,.\, (a + d) = 288$

$\Rightarrow \quad a^2 (a^2 - d^2) = 288$

$\Rightarrow \quad 9 (81 - d^2) = 288$

$\Rightarrow \quad 81 - d^2 = 32$

$\Rightarrow \quad d^2 = 49$

$\Rightarrow \quad d = \pm 7$

a = 9, d = 7 $\Rightarrow$ the 3 terms are 2, 9, 16

a = 9, d = −7 $\Rightarrow$ the 2 terms are 16, 9, 2

12. The ratio of 6^{th} and 8^{th} term of an A.P. is 7 : 9. Find the ratio of 9^{th} term to 13^{th} term.

Solution :

Given $\dfrac{t_6}{t_8} = \dfrac{7}{9}$

$\Rightarrow \dfrac{a+5d}{a+7d} = \dfrac{7}{9}$

$\Rightarrow 9a + 45d = 7a + 49d$

$\Rightarrow \quad 2a = 4d$

$\Rightarrow \quad a = 2d \quad$ .....(1)

$\therefore \quad \dfrac{t_9}{t_{13}} = \dfrac{a+8d}{a+12d}$

$= \dfrac{2d+8d}{2d+12d}$ (from (1))

$= \dfrac{10d}{14d}$

$= \dfrac{5}{7}$

$\therefore t_9 : t_{13} = 5 : 7$

13. In a winter season let us take the temperature of Ooty from Monday to Friday to be in A.P. The sum of temperatures from Monday to Wednesday is 0° C and the sum of the temperatures from Wednesday to Friday is 18° C. Find the temperature on each of the five days.

Solution :

Let the temperature from Monday to Friday respectively be

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d$$

i) Given $a + (a+d) + (a+2d) = 0$

$$3a + 3d = 0$$
$$a + d = 0$$
$$a = -d$$

ii) Given $(a+2d) + (a+3d) + (a+4d) = 18$

$\Rightarrow \quad 3a + 9d = 18$

$\Rightarrow \quad -3d + 9d = 18$

$\Rightarrow \quad 6d = 18$

$\Rightarrow \quad d = 3$

$\therefore \quad a = -3$

The temperature of each of the 5 days

-3^0 C, 0^0 C, 3^0 C, 6^0 C, 9^0 C

14. Priya earned ₹ 15,000 in the first year. Thereafter her salary increased by ₹ 1500 per year. Her expenses are ₹ 13,000 during the first year and the expenses increases by ₹ 900 per year. How long will it take for her to save ₹ 20,000.

Solution :

	1st year	**2nd year**
Salary :	₹15,000	₹16,500
Expense :	₹13,000	₹13,900
Savings :	₹2,000	₹2,600

$\therefore$ the yearly savings are

₹2,000, ₹2,600, ₹3,200, ... form an A.P with $a = 2{,}000$, $d = 600$, $t_n = 20{,}000$

$a + (n-1)\,d = 20{,}000$

$\Rightarrow \quad 2{,}000 + (n-1)\,600 = 20{,}000$

$\Rightarrow \quad 600\,n - 600 = 18{,}000$

$\Rightarrow \quad 600\,n = 18{,}600$

$\Rightarrow \quad n = \dfrac{186}{6}$

$n = 31$

After 31 years, her savings will be ₹20,000.

VI. ARITHMETIC SERIES :

Key Points

- ✓ The sum of terms of a sequence is called series.
- ✓ Let $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots$ be the sequence of real numbers. Then the real numbers $a_1 + a_2 + a_3 + \ldots.$ is defined as the series of real numbers.
- ✓ If a series has finite number of terms then it is called a Finite series. If a series has infinite number of terms then it is called Infinite series.
- ✓ Sum to n terms of an A.P. $S_n = \dfrac{n}{2}[2a + (n-1)d]$
- ✓ If the first term a, and the last term l (n^{th} term) are given then $S_n = \dfrac{n}{2}[a + l]$.

Example 2.31

Find the sum of first 15 terms of the A.P. $8, 7\frac{1}{4}, 6\frac{1}{2}, 5\frac{3}{4}, ...$

Solution :

Here the first term $a = 8$, common difference $d = 7\frac{1}{4} - 8 = -\frac{3}{4}$,

Sum of first n terms of an A.P.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{15} = \frac{15}{2}\left[2 \times 8 + (15-1)(-\frac{3}{4})\right]$$

$$S_{15} = \frac{15}{2}\left[16 - \frac{21}{2}\right] = \frac{165}{4}$$

Example 2.32

Find the sum of $0.40 + 0.43 + 0.46 + ... + 1$.

Solution :

Here the value of n is not given. But the last term is given. From this, we can find the value of n.

Given $a = 0.40$ and $l = 1$, we find $d = 0.43 - 0.40 = 0.03$.

$$\text{Therefore, } n = \left(\frac{l-a}{d}\right) + 1$$

$$= \left(\frac{1-0.40}{0.03}\right) + 1 = 21$$

Sum of first n terms of an A.P.

$$S_n = \frac{n}{2}[a + l]$$

Here, $n = 21$.

$$\text{Therefore,} \quad S_{21} = \frac{21}{2}[0.40 + 1] = 14.7$$

So, the sum of 21 terms of the given series is 14.7.

Example 2.33

How many terms of the series 1 + 5 + 9 + ... must be taken so that their sum is 190 ?

Solution :

Here we have to find the value of n, such that $S_n = 190$.

First term $a = 1$, common difference

$d = 5 - 1 = 4$.

Sum of first n terms of an A.P.

$$S_n = \frac{n}{2}[2a + (n-1)d] = 190$$

$$\frac{n}{2}[2 \times 1 + (n-1) \times 4] = 190$$

$$n[4n - 2] = 380$$

$$2n^2 - n - 190 = 0$$

$$(n-10)(2n+19) = 0$$

But $n = 10$ as $n = -\frac{19}{2}$ is impossible. Therefore, $n = 10$.

Example 2.34

The 13th term of an A.P. is 3 and the sum of first 13 terms is 234. Find the common difference and the sum of first 21 terms.

Solution :

Given the 13th term = 3 so,

$t_{13} = a + 12d = 3$ ...(1)

Sum of first 13 terms = 234 gives

$S_{13} = \frac{13}{2}[2a + 12d] = 234$

$2a + 12d = 36$ ...(2)

Solving (1) and (2) we get, $a = 33$, $d = \frac{-5}{2}$

Therefore, common difference is $\frac{-5}{2}$.

Sum of first 21 terms

$$S_{21}=\frac{21}{2}\left[2\times 33+(21-1)\times\left(-\frac{5}{2}\right)\right]$$

$$=\frac{21}{2}[66-50]=168$$

Example 2.35

In an AP. the sum of first n terms is $\frac{5n^2}{2}+\frac{3n}{2}$. Find the 17^{th} term.

Solution :

The 17^{th} term can be obtained by subtracting the sum of first 16 terms from the sum of first 17 terms.

$$S_{17}=\frac{5\times(17)^2}{2}+\frac{3\times 17}{2}=\frac{1445}{2}+\frac{51}{2}=748$$

$$S_{16}=\frac{5\times(16)^2}{2}+\frac{3\times 16}{2}=\frac{1280}{2}+\frac{48}{2}=664$$

Now, $t_{17}=S_{17}-S_{16}=748-664=84$

Example 2.36

Find the sum of all natural numbers between 300 and 600 which are divisible by 7.

Solution :

The natural numbres between 300 and 600 which are divisible by 7 are 301, 308, 315, ..., 595.

The sum of all natural numbers between 300 and 600 is 301 + 308 + 315 + ... + 595.

The terms of the above sereis are in A.P.

First term a = 301 ; common difference d = 7 ; Last term l = 595.

$$n=\left(\frac{l-a}{d}\right)+1=\left(\frac{595-301}{7}\right)+1=43$$

Since, $S_n=\frac{n}{2}[a+l]$,

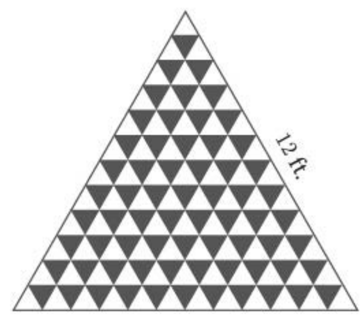
we have $S_{57}=\frac{43}{2}[301+595]=19264.$

Example 2.37

A mosaic is designed in the shape of an equilateral triangle, 12 ft on each side. Each tile in the mosaic is in the shape of an equilateral triangle of 12 inch side. The tiles are alternate in colour as shown in the figure. Find the number of tiles of each colour and total number of tiles in the mosaic.

Solution :

Since the mosaic is in th eshape of an equilateral triangle of 12ft, and the tile is in the shape of an equilateral triangle of 12 inch (1 ft), there will be 12 rows in the mosaic.



From the figure, it is clear that number of white tiles in each row are 1, 2, 3, 4, ... 12 which clearly forms an Arithmetic Progression.

Similarly the number of blue tiles in each row are 0, 1, 2, 3, ... 11 which is also an Arithmetic Progression.

Number of white tiles

$= 1 + 2 + 3 + ... + 12 = \frac{12}{2}[1 + 12] = 78$

Number of blue tiles

$= 0 + 1 + 2 + 3 + ...+ 11 = \frac{12}{2}[0 + 11] = 66$

The total number of tiles in the mosaic

= 78 + 66 = 144

Example 2.38

The houses of a street are numbered from 1 to 49. Senthil's house is numbered such that the sum of numbers of the houses prior to Senthil's house is equal to the sum of numbers of the houses following Senthil's house. Find Senthil's house number ?

Solution :

Let Senthil's house number be x.

It is given that $1 + 2 + 3 + ... + (x - 1)$

$= (x - 1) + (x + 2) + ... + 49$

$1 + 2 + 3 + ... + (x - 1)$

$= [1 + 2 + 3 + ... + 49] - [1 + 2 + 3 +.... + x]$

$$\frac{x-1}{2}[1+(x-1)] = \frac{49}{2}[1+49] - \frac{x}{2}[1+x]$$

$$\frac{x(x-1)}{2} = \frac{49\times 50}{2} - \frac{x(x+1)}{2}$$

$$x^2 - x = 2450 - x^2 - x \Rightarrow 2x^2 = 2450$$

$$x^2 = 1225 \text{ gives } x = 35$$

Therefore, Senthil's hosue number is 35.

Example 2.39

The sum of first, n, 2n and 3n terms of an A.P. are S_1, S_2 and S_3 respectively. Prove that $S_3 = 3(S_2 - S_1)$.

Solution :

If S_1, S_2 and S_3 are sum of first n, 2n and 3n terms of an A.P. respectively then

$$S_1 = \frac{n}{2}[2a+(n-1)d],\ S_2 = \frac{2n}{2}[2a+(2n-1)d],$$

$$S_3 = \frac{3n}{2}[2a+(3n-1)d]$$

Consider

$$S_2 - S_1 = \frac{2n}{2}[2a+(2n-1)d] - \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}[(4a+2(2n-1)d]-[2a+(n-1)d]]$$

$$S_2 - S_1 = \frac{n}{2}\times[2a+(3n-1)d]$$

$$3(S_2 - S_1) = \frac{3n}{2}[2a+(3n-1)d]$$

$$3(S_2 - S_1) = S_3$$

EXERCISE 2.6

1. **Find the sum of the following**

   (i) 3, 7, 11,… up to 40 terms.

   (ii) 102, 97, 92,… up to 27 terms.

   (iii) 6 + 13 + 20 + ... + 97

Solution :

i) Given A.P is 3, 7, 11, ... up to 40 terms

$a = 3,\ d = 4,\ n = 40$

$$S_n = n/2\,[2a + (n-1)\,d]$$

$$S_{40} = \frac{40}{2}[6+39(4)]$$

$$= 20[6+156]$$

$$= 20\times 162$$

$$= 3240$$

ii) Given A.P is 102, 97, 92, ... up to 27 terms

$a = 102,\ d = -5,\ n = 27$

$$S_n = n/2\,[2a + (n-1)\,d]$$

$$S_{27} = \frac{27}{2}[204+26(-5)]$$

$$= \frac{27}{2}[204-130]$$

$$= \frac{27}{2}\times 74$$

$$= 27\times 37$$

$$= 999$$

iii) Given 6 + 13 + 20 + ... + 97

$a = 6, d = 7, l = 97$

$$\therefore \ n = \frac{l-a}{d} + 1$$

$$= \frac{97-6}{7} + 1$$

$$= \frac{91}{7} + 1$$

$$= 13 + 1$$

$$= 14$$

$$\therefore \ S_n = \frac{n}{2}(a+l)$$

$$S_{14} = \frac{14}{2}(6+97)$$

$$= 7 \times 103$$

$$= 721$$

---

2. How many consecutive odd integers beginning with 5 will sum to 480?

Solution :

By the data given,

The series is 5 + 7 + 9 + ... n = 480

$\therefore \quad a = 5, d = 2, S_n = 480$

$$\Rightarrow \frac{n}{2}[2a + (n-1)d] = 480$$

$$\Rightarrow \frac{n}{2}[10 + (n-1)d] = 480$$

$$\Rightarrow \frac{n}{2}[5 + (n-1)] = 480$$

$$\Rightarrow n[n+4] = 480$$

$$\Rightarrow n^2 + 4n - 480 = 0$$

$$\Rightarrow (n+24)(n-20) = 0$$

$$\Rightarrow n = -24, \ n = 20$$

$$\therefore \ n = 20$$

---

3. Find the sum of first 28 terms of an A.P. whose n^{th} term is $4n - 3$.

Solution :

Given $\quad t_n = 4n - 3$

$n = 1 \Rightarrow t_1 = 4 - 3 = 1$

$n = 2 \Rightarrow t_2 = 8 - 3 = 5$

$n = 3 \Rightarrow t_3 = 12 - 3 = 9$

$\therefore \ a = 1, d = 5 - 1 = 4$

$$\therefore \ S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{28} = \frac{28}{2}[2 + 27(4)] = 14[2 + 108]$$

$$= 14 \times 110 = 1540$$

---

4. The sum of first n terms of a certain series is given as $2n^2 - 3n$. Show that the series is an A.P.

Solution :

Given $\quad S_n = 2n^2 - 3n$

$n = 1 \Rightarrow S_1 = 2 - 3 = -1$

$n = 2 \Rightarrow S_2 = 2(4) - 3(2) = 8 - 6 = 2$

$\therefore \ t_1 = a = -1, S_2 = 2$

$\Rightarrow \quad t_2 + t_1 = 2$

$\Rightarrow \quad t_2 - 1 = 2$

$\Rightarrow \quad t_2 = 3$

$\therefore \quad t_1 = -1, t_2 = 3, d = t_2 - t_1$

$= 4$

$\therefore \quad a = -1, d = 4$

$\therefore$ The series is $-1 + 3 + 7 +$ .... is an A.P.

---

5. The 104^{th} term and 4^{th} term of an A.P. are 125 and 0. Find the sum of first 35 terms.

Solution :

Given $t_{104} = 125, \quad t_4 = 0$

To find : S_{35}

$$\begin{aligned} a + 103\,d &= 125 \\ a + 3\,d &= 0 \\ \hline 100\,d &= 125 \end{aligned}$$

$$d = \frac{5}{4}$$

$$a + 3\left(\frac{5}{4}\right) = 0$$

$$\Rightarrow \quad a + \frac{15}{4} = 0$$

$$\Rightarrow \quad a = \frac{-15}{4}$$

$$\therefore S_n = \frac{n}{2}\,[2a + (n-1)d]$$

$$S_{35} = \frac{35}{2}\left[\frac{-15}{2} + (34)\left(\frac{5}{4}\right)\right]$$

$$= \frac{35}{2}\left[\frac{-15}{2} + \frac{85}{2}\right]$$

$$= \frac{35}{2} \times 35$$

$$= \frac{1225}{2}$$

$$= 612.5$$

---

6. Find the sum of all odd positive integers less than 450.

Solution :

To find the sum :

$1 + 3 + 5 + 7 + \ldots\ldots\ldots + 449$

$a = 1,\ d = 2,\ l = 449$

$$\therefore n = \frac{l-a}{d} + 1$$

$$= \frac{449-1}{2} + 1$$

$$= \frac{448}{2} + 1$$

$$= 224 + 1$$

$$= 225$$

$$\therefore S_n = \frac{n}{2}[a + l]$$

$$S_{225} = \frac{225}{2}[450]$$

$$= 225 \times 225$$

$$= 50{,}625$$

---

7. Find the sum of all natural numbers between 602 and 902 which are not divisible by 4.

Solution :

First we take the sum of the numbers from 603 to 901

$a = 603,\ d = 1,\ l = 901$

$$\therefore n = \frac{l-a}{d} + 1$$

$$= \frac{901-603}{1} + 1$$

$$= 298 + 1$$

$$= 299$$

$$\therefore S_n = \frac{n}{2}[a + l]$$

$$= \frac{299}{2} \times 1504$$

$$= 299 \times 752$$

$$= 224848$$

Next we take sum of all the no's between 602 & 902 which are divi. by 4

$a = 604, \quad d = 4, \quad l = 900$

$$\therefore n = \frac{l-a}{d} + 1$$

$$= \frac{900-604}{4} + 1$$

$$= \frac{296}{4} + 1$$

$$= 74 + 1$$

$$= 75$$

$$\begin{array}{r} 150 \\ 4\,\overline{)602} \\ 600 \\ \hline 2 \\ \hline \end{array}$$

$602 + (4 - 2)$

$= 602 + 2$

$= 604$

$$\therefore S_n = \frac{n}{2}[a + l]$$

$$S_{75} = \frac{75}{2} \times 1504$$

$$= 75 \times 752$$

$$= 56{,}400$$

$$\begin{array}{r} 225 \\ 4\,\overline{)902} \\ 900 \\ \hline 2 \\ \hline \end{array}$$

$= 902 - 2$

$= 900$

$\therefore$ Sum of no's which are not div. by 4

$$= 224848 - 56400 = 168448$$

8. Raghu wish to buy a laptop. He can buy it by paying ₹ 40,000 cash or by giving it in 10 installments as ₹ 4800 in the first month, ₹ 4750 in the second month, ₹ 4700 in the third month and so on. If he pays the money in this fashion, find

(i) total amount paid in 10 installments.

(ii) how much extra amount that he has to pay than the cost?

Solution :

Installment in 1st month = Rs. 4800

Installment in 2nd month = Rs. 4750

Installment in 3rd month = Rs. 4700

i.e., 4800, 4750, 4700, ... forms an A.P. with

$a = 4800,\ d = -50,\ n = 10$

i) Total amount paid in 10 installments

$$\therefore\ S_n = \frac{n}{2}\ [2a+(n-1)d]$$

$$\therefore\ S_{10} = 5\ [9600+9(-50)]$$

$$= 5\ [9600-450]$$

$$= 5\times 9150$$

$$= \text{Rs. } 45,750/-$$

ii) Amount he paid extra in installments

$= 45,750 - 40,000$

$=$ Rs. 5,750/-

9. A man repays a loan of ₹ 65,000 by paying ₹ 400 in the first month and then increasing the payment by ₹ 300 every month. How long will it take for him to clear the loan?

Solution :

Amounts of repayment in successive months

400 + 700 + 1000 + ... n months = ₹ 65,000

$a = 400,\ d = 300,\ S_n = 65,000$

$$\frac{n}{2}\ [2a+(n-1)\ d] = 65,000$$

$$\Rightarrow \frac{n}{2}\ [800+(n-1)\ 300] = 65,000$$

$$\Rightarrow n\ [400+(n-1)\ 150] = 65,000$$

$$\Rightarrow n\ [150n+250] = 65,000$$

$$\Rightarrow n\ [3n+5] = 1,300$$

$$\Rightarrow 3n^2+5n-1300 = 0$$

$$\begin{array}{c|c} 5 & -3900 \\ \hline -\frac{65}{3} & -\frac{60}{3} \end{array}$$

$$\frac{65}{3},\quad -20$$

$$\Rightarrow n = 20,\ -\frac{65}{3}$$

$$\boxed{\therefore\ n = 20}$$

10. A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two bricks less than the previous step.

(i) How many bricks are required for the top most step?

(ii) How many bricks are required to build the stair case?

Solution :

No. of bricks in bottom step = 100

No. of bricks in successive steps are

98, 96, 94, ...

$\therefore$ 100, 98, 96, 94, ... for 30 steps form an A.P. with

$a = 100,\ d = -2,\ n = 30$

i) No. of bricks used in the top most step

$$t_{30} = a + 29d$$

$$= 100 + 29\ (-2)$$

$$= 100 - 58$$

$$= 42$$

ii) Total no. of bricks used to build the stair case

$$S_{20} = \frac{30}{2}(100+42)$$
$$= 15 \times 142$$
$$= 2130$$

11. If S_1, S_2, S_3, ... S_m are the sums of n terms of m A.P.'s whose first terms are 1, 2, 3,...., m and whose common differences are 1, 3, 5, ..., $(2m - 1)$ respectively, then show that

$$S_1 + S_2 + S_3 + \dots\dots\dots + S_m = \frac{1}{2} mn\,(mn+1)$$

Solution :

1^{st} A.P. $\Rightarrow a = 1,\ d = 1$

$$\Rightarrow S_1 = \frac{n}{2}[2+(n-1)\,1]$$
$$= \frac{n}{2}[n+1]$$

2^{nd} A.P. $\Rightarrow a = 2,\ d = 3$

$$\Rightarrow S_2 = \frac{n}{2}[4+(n-1)\,3]$$
$$= \frac{n}{2}[3n+1]$$

m^{th} A.P. $\Rightarrow a = m,\ d = 2m-1$

$$\Rightarrow S_m = \frac{n}{2}[2m+(n-1)\,(2m-1)]$$
$$= \frac{n}{2}[2m+2mn - 2m - n+1]$$
$$= \frac{n}{2}[2mn - n+1]$$
$$= \frac{n}{2}[(2m-1)\,n+1]$$

$S_1 + S_2 + \dots + S_m$

$$= \frac{n}{2}(n+1) + \frac{n}{2}(3n+1) + \dots + \frac{n}{2}((2m-1)\,n+1)$$
$$= \frac{n}{2}[(n+3n+\dots+(2m-1)n) + (1+1+\dots\ m \text{ terms})]$$
$$= \frac{n}{2}[(n(1+3+5+\dots\dots+(2m-1) + m]$$
$$= \frac{n}{2}[(n(m^2) + m]$$
$$= \frac{n}{2}[m(mn+1)]$$
$$= \frac{1}{2}mn\,(mn+1)$$

= RHS

Hence Proved

12. Find the sum $\left[\frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \frac{5a-3b}{a+b} + \dots\dots \text{ to 12 terms}\right]$

Solution :

Given series is

$$\frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \frac{5a-3b}{a+b} + \dots\dots\dots \text{ to 12 terms}$$

1st term $= \dfrac{a-b}{a+b}$, Common diff $= \dfrac{2a-b}{a+b}$

$$S_{12} = \frac{12}{2}\left[2\left(\frac{a-b}{a+b}\right) + 11\left(\frac{2a-b}{a+b}\right)\right] \quad (\because S_n = \frac{n}{2}[2a+(n-1)d])$$
$$= 6\left[\frac{2a-2b+22a-11b}{a+b}\right]$$
$$= 6\left[\frac{24a-13b}{a+b}\right]$$
$$= \frac{6}{a+b}[24a-13b]$$

Hence proved.

Key Points

- ✓ A Geometric Progression is a sequence in which each term is obtained by multilplying a fixed non-zero number to the preceding term except the first term. The fixed number is called common ratio. The common ratio is usually denoted by *r*.
- ✓ Let a and $r \neq 0$ be real numbers. Then the numbers of the form $a, ar, ar^2, ... ar^{n-1}$ ... is called a Geometric Progression. The number '*a*' ios called the first term and number 'r' is called the common ratio.
- ✓ The general term or nth term of a G.P. is $t_n = ar^{n-1}$.
- ✓ When the product of three consecutive terms of a G.P. are given, we can take the three terms as $\frac{a}{r}, a, ar$.
- ✓ When the products of four consecutive terms are given for a G.P. then we can take the four terms as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
- ✓ When each term of a Geometric Progression is multiplied or divided by a non-zero constant then the resulting sequence is also a Geometric Progression.
- ✓ Three non-zero numbers a, b, c are in G.P. if and only if $b^2 = ac$.

Example 2.40

Which of the following sequences form a Geometric Progression ?

(i) 7, 14, 21, 28, ... (ii) $\frac{1}{2}$, 1, 2, 4, ...

(iii) 5, 25, 50, 75, ...

Solution :

To check if a given sequence form a G.P. we have to see if the ratio between successive terms are equal.

(i) 7, 14, 21, 28, ...

$$\frac{t_2}{t_1} = \frac{14}{7} = 2; \quad \frac{t_3}{t_2} = \frac{21}{14} = \frac{3}{2}; \quad \frac{t_4}{t_3} = \frac{28}{21} = \frac{4}{3}$$

Since the ratios between successive terms are not equal, the sequence 7, 14, 21, 28, ... is not a Geometric Progression.

(ii) $\frac{1}{2}$, 1, 2, 4, ...

$$\frac{t_2}{t_1} = \frac{1}{\frac{1}{2}} = 2; \quad \frac{t_3}{t_2} = \frac{2}{1} = 2; \quad \frac{t_4}{t_3} = \frac{4}{2} = 2$$

Here the ratios between successive terms are equal. Therefore the sequence $\frac{1}{2}$, 1, 2, 4, ... is a Geometric Progression with common ratio r = 2.

(iii) 5, 25, 50, 75, ...

$$\frac{t_2}{t_1} = \frac{25}{5} = 5; \quad \frac{t_3}{t_2} = \frac{50}{25} = 2; \quad \frac{t_4}{t_3} = \frac{75}{50} = \frac{3}{2}$$

Since the ratios between successive terms are not equal, the sequence 5, 25, 50, 75, ... is not a Geometric Progression.

Example 2.41

Find the geometric progression whose first term and common ratios are given by (i) $a = -7$, r = 6 (ii) $a = 256$, r = 0.5

Solution :

(i) The general form of Geometric progression is *a, ar, ar*^{3}, ...

$a = -7$, $ar = -7 \times 6 = -42$, $ar^2 = -7 \times 6^2 = -252$

Therefore the required Geometric Progression is $-7, -42, -252, ...$

(ii) The general form of Geometric progression is *a, ar, ar*^{3}, ...

$a = 256$, $ar = 256 \times 0.5 = 128$, $ar^3 = 256 \times (0.5)^2 = 64$

Example 2.42

Find the 8th term of the G.P. 9, 3, 1, ...

Solution :

To find the 8th term we have to use the nth term formula $t_n = ar^{n-1}$

First term $a = 9$, common ratio $r = \frac{t_2}{t_1} = \frac{3}{9} = \frac{1}{3}$

$$t_8 = 9 \times \left(\frac{1}{3}\right)^{8-1} = 9 \times \left(\frac{1}{3}\right)^7 = \frac{1}{243}$$

Therefore the 8th term of the G.P. is $\frac{1}{243}$.

Example 2.43

In a Geometric progression, the 4th term is $\frac{8}{9}$ and the 7th term is $\frac{64}{243}$. Find the Geometric Progression.

Solution :

4th term, $t_4 = \frac{8}{9}$ gives $ar^3 = \frac{8}{9}$ ...(1)

7th term, $t_7 = \frac{64}{243}$ gives $ar^6 = \frac{64}{243}$ ...(2)

Dividing (2) by (1) we get, $\frac{ar^6}{ar^3} = \frac{\frac{64}{243}}{\frac{8}{9}}$

$$r^3 = \frac{8}{27} \text{ gives } r = \frac{2}{3}$$

Substituting the value of r in (1), we get $a \times \left[\frac{2}{3}\right]^3 = \frac{8}{9} \Rightarrow a = 3$

Therefore the Geometric Progression is *a, ar, ar*^{3}, ... That is, 3, 2, $\frac{4}{3}$, ...

Example 2.44

The product of three consecutive terms of a Geometric Progression is 343 and their sum is $\frac{91}{3}$. Find the three terms.

Solution :

Since the product of 3 consecutive terms is given.

we can take them as $\frac{a}{r}$, *a, ar*.

Product of the terms = 343

$$\frac{a}{r} \times a \times ar = 343$$

$a^3 = 7^3$ gives $a = 7$

Sum of the terms = $\frac{91}{3}$

gives $7\left(\frac{1 + r + r^2}{r}\right) = \frac{91}{3}$

$3 + 3r + 3r^2 = 13r$ gives $3r^2 - 10r + 3 = 0$

$(3r - 1)(r - 3) = 0$ gives $r = 3$ or $r = \frac{1}{3}$

If $a = 7$, $r = 3$ then the three terms are $\frac{7}{3}$, 7, 21.

If $a = 7$, $r = \frac{1}{3}$ then the three terms are 21, 7, $\frac{7}{3}$.

Example 2.45

The present value of a machine is ₹40,000 and its value depreciates each year by 10%. Find the estimated value of the machine in the 6th year.

Solution :

The value of the machine at present is

₹40,000. Since it is depreciated at the rate of 10% after one year the value of the machine is 90% of the initial value.

That is the value of the machine at the end of the first year is $40,000 \times \frac{90}{100}$

After two years, the value of the machine is 90% of the value in the first year.

Value of the machine at the end of the 2nd year is $40,000 \times \left(\frac{90}{100}\right)^2$

Continuing this way, the value of the machine depreciates in the following way as

$$40000, 40000 \times \frac{90}{100}, 40000 \times \left(\frac{90}{100}\right)^2 \ldots$$

This sequence is in the form of G.P. with first term 40,000 and common ratio $\frac{90}{100}$. For finding the value of the machine at the end of 5th year (i.e. in 6th year), we need to find the sixth term of this G.P.

Thus, $n = 6,\ a = 40,000,\ r = \frac{90}{100}$.

Using $t_n = ar^{n-1}$, we have $t_6 = 40,000 \times \left(\frac{90}{100}\right)^{n-1}$

$$= 40000 \times \left(\frac{90}{100}\right)^5$$

$$t_6 = 40,000 \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10}$$

$$= 23619.6$$

Therefore the value of the machine in 6th year = ₹23619.60.

EXERCISE 2.7

1. Which of the following sequences are in G.P.?

(i) 3, 9, 27, 81,… (ii) 4, 44, 444, 4444,...

(iii) 0.5, 0.05, 0.005, …

iv) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots\ldots$ v) 1, −5, 25, −125 …

vi) 120, 60, 30, 18,… vii) $16, 4, 1, \frac{1}{4}, \ldots\ldots$

Solution :

i) Given sequence is 3, 9, 27, 81 ....

$$t_2/t_1 = 9/3 = 3$$

$$t_3/t_2 = 27/9 = 3$$

$$t_4/t_3 = 81/27 = 3$$

∴ The sequence is a G.P.

ii) Given sequence is 4, 44, 444, ...

$$t_2/t_1 = 44/4 = 11$$

$$t_3/t_2 = 444/44 = 111/11 \neq 11$$

$$\therefore t_2/t_1 \neq t_3/t_2$$

∴ The sequence is not a G.P.

iii) Given sequence is 0.5, 0.05, 0.005, ...

$$\frac{t_2}{t_1} = \frac{0.05}{0.5} = \frac{5}{50} = \frac{1}{10}$$

$$\frac{t_3}{t_2} = \frac{0.005}{0.05} = \frac{5}{50} = \frac{1}{10}$$

∴ The sequence is a G.P.

iv) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, ...$

$$\frac{t_2}{t_1} = \frac{1/6}{1/3} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{1/2}{1/6} = \frac{6}{12} = \frac{1}{2}$$

$\therefore$ The sequence is a G.P.

v) 1, − 5, 25, − 125, ...

$$\frac{t_2}{t_1} = -5$$

$$\frac{t_3}{t_2} = \frac{25}{-5} = -5$$

$$\frac{t_4}{t_3} = \frac{-125}{25} = -5$$

$\therefore$ The sequence is a G.P.

vi) 120, 60, 30, 18, ...

$$\frac{t_2}{t_1} = \frac{60}{120} = \frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{30}{60} = \frac{1}{2}$$

$$\frac{t_4}{t_3} = \frac{18}{30} \neq \frac{1}{2}$$

$\therefore$ The sequence is not a G.P.

vii) $16, 4, 1, 1/4, ...$

$$\frac{t_2}{t_1} = \frac{4}{16} = \frac{1}{4}$$

$$\frac{t_3}{t_2} = \frac{1}{4}$$

$$\frac{t_4}{t_3} = \frac{1}{4}$$

$\therefore$ The sequence is a G.P.

2. Write the first three terms of the G.P. whose first term and the common ratio are given below.

(i) $a = 6, r = 3$ (ii) $a = \sqrt{2}, r = \sqrt{2}$

(iii) $a = 1000, r = \frac{2}{5}$

Solution :

i) Given $a = 6, r = 3$

$\therefore$ The first 3 terms of the G.P. are

6, 18, 54, ...

ii) Given $a = \sqrt{2}, \; r = \sqrt{2}$

$\therefore$ The first 3 terms of the G.P. are

$\sqrt{2}, 2, 2\sqrt{2}, ...$

iii) $a = 1000, r = 2/5$

$\therefore$ The first 3 terms are

$$1000, \; 1000 \times 2/5, \; 1000 \times 2/5 \times 2/5$$

$= 1000, 400, 160, ...$

3. In a G.P. 729, 243, 81,… find t_7.

Solution :

Given G.P is 729, 243, 21, ...

$$a = 729, \; r = 8/243 = 1/3$$

$$\therefore \; t_n = a \,.\, r^{n-1}$$

$$\Rightarrow t_7 = a \,.\, r^6$$

$$= 729 \times \left(1/3\right)^6$$

$$= 3^6 \times \frac{1}{3^6} = 1$$

4. Find x so that $x + 6, x + 12$ and $x + 15$ are consecutive terms of a Geometric Progression.

Solution :

Given $x + 6, x + 12, x + 15$ are consecutive terms of a G.P.

$$\Rightarrow \quad \frac{x+12}{x+6} = \frac{x+15}{x+12}$$

$$\Rightarrow \quad (x+12)^2 = (x+15)(x+6)$$

$$\Rightarrow \quad x^2 + 24x + 144 = x^2 + 21x + 90$$

$$\Rightarrow \quad 3x = -54$$

$$x = -18$$

5. Find the number of terms in the following G.P.

i) 4, 8, 16, … , 8192 ?

ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\ldots, \frac{1}{2187}$?

Solution :

i) Given G.P is 4, 8, 16, ... 8192

$$\Rightarrow \quad a = 4, r = 2, t_n = 8192$$

$$\Rightarrow \quad a \,.\, r^{n-1} = 8192$$

$$\Rightarrow \quad 4 \times 2^{n-1} = 8192$$

$$\Rightarrow \quad 2^{n-1} = 2048$$

$$\Rightarrow \quad 2^{n-1} = 2^{11}$$

$$\Rightarrow \quad n - 1 = 11$$

$$\therefore \; n = 12$$

ii) Given G.P is $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\ldots, \frac{1}{2187}$

$$a = \frac{1}{3},\; r = \frac{1}{3},\; t_n = \frac{1}{2187}$$

$$\Rightarrow \quad a \,.\, r^{n-1} = \frac{1}{2187}$$

$$\Rightarrow \quad \frac{1}{3} . \left(\frac{1}{3}\right)^{n-1} = \frac{1}{2187}$$

$$\Rightarrow \quad \left(\frac{1}{3}\right)^{n-1} = \frac{3}{2187}$$

$$\Rightarrow \quad \left(\frac{1}{3}\right)^{n-1} = \frac{1}{729}$$

$$\Rightarrow \quad \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{6}$$

$$\therefore \; n - 1 = 6$$

6. In a G.P. the 9th term is 32805 and 6th term is 1215. Find the 12th term.

Solution :

Given $t_9 = 32805$, $t_6 = 1215$, $t_{12} = ?$

$$a \,.\, r^8 = 32805 \quad \ldots\ldots\ldots\ldots (1)$$

$$a \,.\, r^5 = 1215 \quad \ldots\ldots\ldots\ldots (2)$$

$$(1) \div (2) \Rightarrow r^3 = \frac{32805}{1215}$$

$$\Rightarrow \quad r^3 = 27$$

$$\Rightarrow \quad r = 3$$

Sub. $r = 3$ in (2)

$$a \times 3^5 = 1215$$

$$\Rightarrow \quad a \times 243 = 1215$$

$$\Rightarrow \quad a = \frac{1215}{243}$$

$$\Rightarrow \quad a = 5$$

$$\Rightarrow \quad \therefore \; t_{12} = a \,.\, r^{11}$$

$$= 5 \times 3^{11}$$

7. Find the 10th term of a G.P. whose 8th term is 768 and the common ratio is 2.

Solution :

Given $t_8 = 768$, $r = 2$

$$\Rightarrow \quad a \,.\, r^7 = 768$$

$$\Rightarrow \quad a \times 2^7 = 768$$

$$\Rightarrow \quad a \times 128 = 768$$

$$a = 6$$

$$\therefore \quad t_{10} = a \,.\, r^9$$

$$= 6 \times 2^9$$

$$= 6 \times 512$$

$$= 3072$$

8. If *a, b, c* are in A.P. then show that 3*a*, 3*b*, 3*c* are in G.P.

Solution :

Given a, b, c are in A.P.

$$\Rightarrow b = \frac{a+c}{2} \qquad ......(1)$$

To Prove : $3^a, 3^b, 3^c$ are in G.P.

i.e. TP : $(3^b)^2 = 3^a \cdot 3^c$

LHS : $(3^b)^2$

$= 3^{2b}$

$= 3^{a+c}$ (from (1))

$= 3^a \cdot 3^c$

$=$ RHS

$\therefore 3^a, 3^b, 3^c$ are in G.P.

9. In a G.P. the product of three consecutive terms is 27 and the sum of the product of two terms taken at a time is $\frac{57}{2}$. Find the three terms.

Solution :

Let the 3 consecutive terms of a G.P be $\frac{a}{r}, a, ar$

i) Product of 3 terms $= 27$

$$\Rightarrow \frac{a}{r} \times a \times ar = 27$$

$$\Rightarrow a^3 = 27$$

$$\therefore a = 3$$

ii) Sum of product of terms taken 2 at a time $= \frac{57}{2}$

$$i.e., \frac{a}{r} \times a + a \times ar + ar \times \frac{a}{r} = \frac{57}{2}$$

$$\Rightarrow a^2\left[\frac{1}{r} + r + 1\right] = \frac{57}{2}$$

$$\Rightarrow 9\left[\frac{1+r^2+r}{r}\right] = \frac{57}{2}$$

$$\Rightarrow \frac{1+r+r^2}{r} = \frac{\cancel{57}^{19}}{\cancel{9}_3 \times 2}$$

$$\Rightarrow \frac{1+r+r^2}{r} = \frac{19}{6}$$

$$\Rightarrow 6r^2 + 6r + 6 = 19r$$

$$\Rightarrow 6r^2 - 13r + 6 = 0$$

$$\Rightarrow r = 3/2,\ 2/3$$

$\therefore a = 3,\ r = 3/2 \Rightarrow$ 3 terms are $2, 3, 9/2$

&

$\therefore a = 3,\ r = 3/2 \Rightarrow$ 3 terms are $9/2, 3, 2$

10. A man joined a company as Assistant Manager. The company gave him a starting salary of ₹ 60,000 and agreed to increase his salary 5% annually. What will be his salary after 5 years?

Solution :

Given, initial salary = Rs. 60,000

Annual increment = 5%

Salary increment at the end of 1 year =

$$60,0\cancel{00} \times \frac{5}{1\cancel{00}} = 3000$$

$\therefore$ Continuing this way, we want to find the total salary after 5 years.

$$A = P\left(1 + r/100\right)^n$$

$$= 60,000\left(1 + 5/100\right)^n$$

$$= 60,000 \times \left(\frac{105}{100}\right)^5$$

$$= 60,000 \times (1.05)^5$$

$$= \text{Rs. } 76,600$$

log 60,000 = 4.7782
5 log (1.05) = 0.1060
5.8842
Antilog 76,600

11. Sivamani is attending an interview for a job and the company gave two offers to him.

Offer A: ₹ 20,000 to start with followed by a guaranteed annual increase of 6% for the first 5 years.

Offer B: ₹ 22,000 to start with followed by a guaranteed annual increase of 3% for the first 5 years.

What is his salary in the 4^{th} year with respect to the offers A and B?

Solution :

Offer A :

P = ₹ 20,000 $r = 6\%$

$n = 3$ (in the 4^{th} year)

$$A = P\left(1+\frac{r}{100}\right)^3$$

log 20,000 = 4.3010
3 log (1.06) = 0.0759
4.3769
Antilog 23820

$$= 20,000\left(1+\frac{6}{100}\right)^3$$
$$= 20,000\left(\frac{106}{100}\right)^3$$
$$= 20,000\,(1.06)^3$$
$$= 23,820$$

Offer B :

P = ₹ 22,000 $r = 3\%$

$n = 3$ (in the 4^{th} year)

$$A = P\left(1+\frac{r}{100}\right)^3$$

log 22,000 = 4.3424
3 log (1.03) = 0.0384
4.3808

$= 22,000 \times (1.03)^3$

= ₹ 24040 Antilog 24040

12. If a, b, c are three consecutive terms of an A.P. and x, y, z are three consecutive terms of a G.P. then prove that $x^{b-c} \times y^{c-a} \times z^{a-b} = 1$.

Solution :

Given a, b, c are consecutive terms of A.P.

$\Rightarrow a, a + d, a + 2d, ...$

x, y, z are consecutive terms of G.P.

$\Rightarrow x, xr, xr^2, ...$

T.P : $x^{b-c} \times y^{c-z} \times z^{a-b} = 1$

LHS : $x^{b-c} \times y^{c-a} \times z^{a-b} = x^{-d} \times (xr)^{2d} \times (xr^2)^{-d}$

$= x^0 \times r^{2d} \times r^{-2d} = x^0 \times r^0 = 1$

= RHS. Hence proved.

VIII. GEOMETRIC SERIES :

Key Points

- ✓ A series whose terms are in Geometric progression is called Geometric series.
- ✓ The sum to n terms is $S_n = \dfrac{a(r^n - 1)}{r - 1}$, r ≠ 1.
- ✓ If r = 1, then $S_n = a + a + a + ... + a = na$.
- ✓ The sum of infinite terms of a G.P. is given by $a + ar + ar^2 + ar^3 + ... = \dfrac{a}{1-r}$, –1 < r < 1.

Example 2.46

Find the sum of 8 terms of the G.P.

1, –3, 9, –27, ...

Solution :

Here the first term $a = 1$,

common ratio $r = \dfrac{-3}{1} = -3 < 1$, Here $n = 8$.

Sum to n terms of a G.P. is

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ if } r \neq 1$$

Hence, $$S_n = \frac{1((-3)^8 - 1)}{(-3) - 1} = \frac{6561 - 1}{-4} = -1640$$

Example 2.47

Find the first term of G.P. in which $S_6 = 4095$ and r = 4.

Solution :

Common ratio = 4 > 1, Sum of first 6 terms $S_6 = 4095$

Hence, $$S_n = \frac{a(r^n - 1)}{r - 1} = 4095$$

Since, r = 4, $$\frac{a(4^6 - 1)}{4 - 1} = 4095 \text{ gives}$$

$$a \times \frac{4095}{3} = 4095$$

First term $a = 3$.

Example 2.48

How many terms of the series 1 + 4 + 16 + ... make the sum 1365 ?

Solution :

Let n be the number of terms to be added to get the sum 1365

$$a = 1,\ r = \frac{4}{1} = 4 > 1$$

$$S_n = 1365 \text{ gives } \frac{a(r^n - 1)}{r - 1} = 1365$$

$$\frac{1(4^n - 1)}{4 - 1} = 1365 \text{ so, } (4^n - 1) = 4095$$

$$4^n = 4096 \text{ then } 4^n = 4^6$$

$$n = 6$$

Example 2.49

Find the sum 3 + 1 + – + ... ∞

Solution :

Here $a = 3,\ r = \dfrac{t_2}{t_1} = \dfrac{1}{3}$

$$\text{Sum of infinite terms} = \frac{a}{1-r} = \frac{3}{1 - \frac{1}{3}} = \frac{9}{2}$$

Example 2.50

Find the rational form of the number 0.6666 ...

Solution :

We can express the number 0.6666 ... as follows

0.6666... = 0.6 + 0.06+ 0.006 + 0.0006 + ...

We now see that numbers 0.6, 0.06, 0.006 ... forms an G.P. whose first term a = 0.6 and common ratio r = $\frac{0.06}{0.6}$ = 0.1. Also –1 < r = 0.1 < 1

Using the infinite G.P. formula, we have

$$0.6666... = 0.6 + 0.06 + 0.006 + 0.0006 + ...$$

$$= \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{2}{3}$$

Thus the rational number equivalent of 0.6666 ... is $\frac{2}{3}$

Example 2.51

Find the sum to n terms of the series 5 + 55 + 555 + ...

Solution :

The series is neither Arithmetic nor Geometric series. So it can be split into two series and then find the sum.

5 + 55 + 555 + ... + n terms = 5 [1 + 11 + 111 + ... + n terms]

$$= \frac{5}{9}[9 + 99 + 999 + ... + n \text{ terms}]$$

$$= \frac{5}{9}[(10-1) + (100-1) + (1000-1) + ... + n \text{ terms}]$$

$$= \frac{5}{9}[(10 + 100 + 1000 + ... + n \text{ terms}) - n]$$

$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{(10-1)} - n\right] = \frac{50(10^n - 1)}{81} - \frac{5n}{9}$$

Example 2.52

Find the least positive integer n such that 1 + 6 + 6^2 + ... + 6^n > 5000.

Solution :

We have to find the least number of terms for which the sum must be greater than 5000.

That is, to find the least value of n, such that S_n > 5000

We have $$S_n = \frac{a(r^n - 1)}{r-1} = \frac{1(6^n - 1)}{6-1} = \frac{6^n - 1}{5}$$

$$S_n > 5000 \text{ gives } \frac{6^n - 1}{5} > 5000$$

$$6^n - 1 > 25000 \text{ gives } 6^n > 25001$$

Since, $6^5 = 7776$ and $6^6 = 46656$

The least positive value of n is 6 such that 1 + 6 + 62 + ... + 6^n > 5000.

Example 2.53

A person saved money every year, half as much as he could in the previous year. If he had totally saved ₹ 7875 in 6 years then how much did he save in the first year ?

Solution :

Total amount saved in 6 years is S_6 = 7875

Since he saved half as much money as every year he saved in the previous year.

We have $r = \frac{1}{2} < 1$

$$\frac{a(1-r^n)}{1-r} = \frac{a\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = 7875$$

$$\frac{a\left(1-\frac{1}{64}\right)}{\frac{1}{2}} = 7875 \text{ gives } a \times \frac{63}{32} = 7875$$

$$a = \frac{7875 \times 32}{63} \text{so, } a = 4000$$

The amount saved in the first year is ₹4000.

EXERCISE 2.8

1. Find the sum of first *n* terms of the G.P.
(i) $5, -3, \frac{9}{5}, -\frac{27}{25}, ...$ (ii) 256, 64, 16, ...

Solution :

i) Given G.P is $5, -3, \frac{9}{5}, \frac{-27}{25}, ...$

$a = 5,\ r = {}^{-3}\!/_{5} < 1$

$\therefore\ S_n = a\,.\frac{1-r^n}{1-r}$

$= (5)\times\left(\frac{1-\left(-\frac{3}{5}\right)^n}{1-\left(-\frac{3}{5}\right)}\right)$

$= (5)\times\left(\frac{1-\left(-\frac{3}{5}\right)^n}{\frac{8}{5}}\right)$

$= \frac{25}{8}\left(1-\left(-\frac{3}{5}\right)^n\right)$

ii) Given G.P is 256, 64, 16, ...

$a = 256,\ r = \frac{1}{4} < 1$

$\therefore\ S_n = a\,.\frac{1-r^n}{1-r}$

$= 256\times\frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}}$

$= 256\times\frac{1-\left(\frac{1}{4}\right)^n}{\frac{3}{4}}$

$= \frac{1024}{3}\left(1-\left(\frac{1}{4}\right)^n\right)$

2. Find the sum of first six terms of the G.P. 5, 15, 45, …

Solution :

Given G.P is 5, 15, 45, ...

$a = 5, r = 3 > 1$

$S_n = a\,.\frac{r^n-1}{r-1}$

$\therefore\ S_6 = 5\,.\frac{3^6-1}{3-1}$

$= \frac{5}{2}\times 728$

$= 5\times 364$

$= 1820$

3. Find the first term of the G.P. whose common ratio 5 and whose sum to first 6 terms is 46872.

Solution :

Given $r = 5, S_6 = 46872$

$S_n = a\,.\frac{r^n-1}{r-1}$

$\Rightarrow\ a\times\frac{5^6-1}{4} = 46872$

$\Rightarrow\ a\,(5^6-1) = 46872\times 4$

$\Rightarrow\ a\,(15624) = 46872\times 4$

$\therefore\ a = \frac{46872\times 4}{15624}$

$= 3\times 4$

$a = 12$

4. Find the sum to infinity of
(i) 9 + 3 + 1 + ...
(ii) $21 + 14 + \frac{28}{3} + ...$

Solution :

i) 9 + 3 + 1 + ... is a geometric series

with $a = 9,\ r = \frac{1}{3} < 1$

$S_\infty = \frac{a}{1-r} = \frac{9}{1-\frac{1}{3}}$

$= \frac{9}{\frac{2}{3}}$

$= \frac{27}{2}$

ii) $21+14+\frac{28}{3}, \ldots\ldots\ldots$ is geo. series

with a = 21, $r = \frac{14}{21} = \frac{2}{3} < 1$

$$\therefore S_\infty = \frac{a}{1-r} = \frac{21}{1-\frac{2}{3}}$$

$$= \frac{21}{1-\frac{2}{3}}$$

$$= \frac{21}{\frac{1}{3}}$$

$$= 63$$

---

5. If the first term of an infinite G.P. is 8 and its sum to infinity is $\frac{32}{3}$ then find the common ratio.

Solution :

$$\text{Given } a = 8,\ S_\infty = \frac{32}{3},\ r = ?$$

$$\Rightarrow \frac{a}{1-r} = \frac{32}{3}$$

$$\Rightarrow \frac{\overset{1}{\cancel{8}}}{1-r} = \frac{\overset{4}{\cancel{32}}}{3}$$

$$\Rightarrow 3 = 4-4r$$

$$\Rightarrow 4r = 1$$

$$\therefore r = \frac{1}{4}$$

---

6. Find the sum to n terms of the series
(i) 0.4 + 0.44 + 0.444 + ... to *n* terms
(ii) 3 + 33 + 333 + ... to *n* terms

Solution :

i) 0.4 + 0.44 + 0.444 + ... to *n* terms

$$= \frac{4}{10}+\frac{44}{100}+\frac{444}{1000}+\ldots\ldots\text{ to } n \text{ terms}$$

$$= 4\left[\frac{1}{10}+\frac{11}{100}+\frac{111}{1000}+\ldots\ldots n \text{ terms}\right]$$

$$= \frac{4}{9}\left[\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+\ldots\ldots n \text{ terms}\right]$$

$$= \frac{4}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\ldots\ldots n \text{ terms}\right]$$

$$= \frac{4}{9}\left[(1+1+1+\ldots\ldots n \text{ terms}) - \left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots\ldots n \text{ terms}\right)\right]$$

$$= \frac{4}{9}\left[n-\frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$$

$$= \frac{4}{9}\left[n-\frac{1}{9}\left(1-\left(\frac{1}{10}\right)^n\right)\right]$$

ii) 3 + 33 + 333 + ... upto *n* terms

$$= 3\,(1+11+111+\ldots\ldots+n \text{ terms})$$

$$= \frac{3}{9}(9+99+999+\ldots\ldots+n \text{ terms})$$

$$= \frac{3}{9}[(10-1)+(100-1)+(1000-1)+\ldots\ldots n \text{ terms}]$$

$$= \frac{3}{9}[(10+100+1000+\ldots\ldots n \text{ terms}] - [(1+1+1+\ldots\ldots n \text{ terms}]$$

$$= \frac{3}{9}\left[10.\left(\frac{10^n-1}{n}\right)-n\right]$$

$$= \frac{30}{81}(10^n-1)-\frac{3n}{9}$$

$$= \frac{10}{27}(10^n-1)-\frac{n}{3}$$

---

7. Find the sum of the Geometric series 3 + 6 + 12 + ... + 1536.

Solution :

Given 3 + 6 + 12 + ... + 1536 is a geometric series

$$a = 3,\ r = 2,\ t_n = 1536$$

$\Rightarrow \quad a \cdot r^{n-1} = 1536$

$\Rightarrow \quad 3 \cdot 2^{n-1} = 1536$

$$\Rightarrow \quad 2^{n-1} = \frac{1536}{3}$$

$$\Rightarrow \quad 2^{n-1} = 512 = 2^9$$

$$\therefore n-1 = 9$$

$$n = 10$$

$$\therefore \quad S_n = a \cdot \frac{r^n - 1}{r-1}$$

$$S_{10} = 3 \times \frac{2^{10}-1}{2-1}$$

$$= 3\,(1023)$$

$$= 3069$$

8. Kumar writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they continue the process similarly. Assuming that the process is unaltered and it costs ₹ 2 to mail one letter, find the amount spent on postage when 8th set of letters is mailed.

Solution :

By the data given,

The number of mails delivered are

4, 4 × 4, 4 × 4 × 4, ...

i.e., 4, 16, 64, ... 8th set of letters.

Each mail costs ₹ 2

∴ The total cost is

(4 × 2) + (16 × 2) + (64 × 2) + ... 8th set

= 8 + 32 + 28 + ... 8th set (which forms)

form a geometric series with $a = 8$, $r = 4$, $n = 8$

$$\therefore S_n = a \cdot \frac{r^n - 1}{r-1}$$

$$S_8 = 8 \cdot \frac{4^8 - 1}{3}$$

$$= 8 \times \frac{65535}{3}$$

$$= 8 \times 21845$$

$$= ₹\,174760$$

9. Find the rational form of the number $0.\overline{123}$.

Solution :

Let $\quad x = 0.\overline{123}$

$x = 0.123123123 \quad .....(1)$

$\Rightarrow 1000\,x = 123.123123 \;.......$

$\Rightarrow 1000\,x = 123.\overline{123} \qquad .....(2)$

$\therefore (2) - (1) \Rightarrow 999x = 123$

$$\Rightarrow x = \frac{123}{999}$$

$$\therefore x = \frac{41}{333}$$

10. If $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...$ *n* terms then prove that

$$(x - y)\, S_n = \left| \frac{x^2\,(x^n - 1)}{x - 1} - \frac{y^2\,(y^n - 1)}{y - 1} \right|$$

Solution :

Given

$S_n = (x + y)\,(x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ... + n$ terms

$\Rightarrow \quad (x - y)\, S_n = (x - y)\,(x + y) + (x - y)\,(x^2 + xy + y^2) + (x - y)\,(x^3 + x^2y + xy^2 + y^3) + ...\, n$ terms

$\Rightarrow \quad (x - y)\, S_n = (x^2 - y^2) + (x^3 + y^3) + (x^4 - y^4) + ...\, n$ terms

$(x^2 + x^3 + x^4 + \text{........} \; n \text{ terms})$

$- (y^2 + y^3 + y^4 + \text{........} \; n \text{ terms}),$

both of them are geometric series

$a = x^2, r = x \;\&\; a = y^2, r = y$

$$\therefore \; (x-y)\,S_n = \frac{x^2(x^n-1)}{x-1} - \frac{y^2(y^n-1)}{y-1}$$

$$\left(\because S_n = a\,.\frac{r^n-1}{r-1}\right)$$

Hence proved.

IX. SPECIAL SERIES :

Key Points

- ✓ The sum of first n natural numbers $1 + 2 + 3 + \text{.....} + n = \dfrac{n(n+1)}{2}$
- ✓ The sum of squares of first n natural numbers

  $$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
- ✓ The sum of cubes of first n natural numbers $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$
- ✓ The sum of first n odd natural numbers $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

Example 2.54

Find the value of (i) $1 + 2 + 3 + \text{....} + 50$ (ii) $16 + 17 + 18 + \text{....} + 75$

Solution :

(i) $1 + 2 + 3 + \ldots + 50$

Using, $1 + 2 + 3 + \text{....} + n = \dfrac{n(n+1)}{2}$

$$1 + 2 + 3 + \text{....} + 50 = \frac{50 \times (50+1)}{2} = 1275$$

(ii) $16 + 17 + 18 + \ldots + 75$

$= (1+2+3 + \ldots + 75) - (1 + 2 + 3 + \ldots + 15)$

$$= \frac{75(75+1)}{2} - \frac{15(15+1)}{2}$$

$= 2850 - 120 = 2730$

Example 2.55

Find the sum of (i) $1 + 3 + 5 + \text{....} +$ to 40 terms (ii) $2 + 4 + 6 + \text{....} + 80$ (iii) $1 + 3 + 5 + \text{....} + 55$

Solution :

(i) $1 + 3 + 5 + \text{.....} + 40 \text{ terms} = 40^2 = 1600$

(ii) $2 + 4 + 6 + \text{.....} + 80$

$= 2\,(1 + 2 + 3 + \text{....} + 40)$

$$= 2 \times \frac{40 \times (40+1)}{2} = 1640$$

(iii) $1 + 3 + 5 + \text{....} + 55$

Here the number of terms is not given. Now we have to find the number of terms using the formula, $n = \dfrac{(l-a)}{d} + 1$ gives $n = \dfrac{(55-1)}{2} + 1 = 28$.

Therefore, $1 + 3 + 5 + \text{....} + 55$

$= (28)^2 = 784.$

Example 2.56

Find the sum of (i) $1^2 + 2^2 + ... + 19^2$

(ii) $5^2 + 10^2 + 15^2 + ... + 105^2$

(iii) $15^2 + 16^2 + 17^2 + ... 28^2$

Solution :

(i) $1^2+2^2+.....+19^2 = \frac{19\times(19+1)\ (2\times19+1)}{6}$

$= \frac{19\times20\times39}{6} = 2470$

(ii) $5^2+10^2+15^2+....+105^2$

$= 5^2(1^2+2^2+3^2+....+21^2)$

$= 25\times\frac{25\times(21+1)\ (2\times21+1)}{6}$

$= \frac{25\times21\times22\times43}{6} = 82775$

(iii) $15^2+16^2+17^2+....28^2$

$= (1^2+2^2+3^2+....+28^2)$

$-(1^2+2^2+3^2+.....+14^2)$

$= \frac{28\times29\times57}{6} - \frac{14\times15\times29}{6}$

$= 7714 - 1015 = 6699$

Example 2.57

Find the sum of (i) $1^3 + 2^3 + 3^3 + ... + 16^3$
(ii) $9^3 + 10^3 + ... + 21^3$

Solution :

(i) $1^3+2^3+3^3+....+16^3 = \left[\frac{16\times(16+1)}{2}\right]^2$

$= (136)^2 = 18496$

(ii) $9^3+10^3+....+21^3 = (1^3+2^3+3^3+.....+21^3)$

$-(1^3+2^3+3^3+....+8^3)$

$= \left[\frac{21\times(21+1)}{2}\right]^2 = \left[\frac{8\times(8+1)}{2}\right]^2$

$= (231)^2 - (36)^2 = 52065$

Example 2.58

If $1 + 2 + 3 + ... + n = 666$ then find n.

Solution :

Since, $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$,

we have $\frac{n(n+1)}{2} = 666$

$n^2 + n - 1332 = 0$ gives $(n + 37)(n - 36) = 0$

So, $n = -37$ or $n = 36$

But $n \neq -37$ (Since n is a natural number) ;
Hence $n = 36$.

EXERCISE 2.9

1. Find the sum of the following series

i) $1 + 2 + 3 + ... + 60$

ii) $3 + 6 + 9 + ... + 96$

iii) $51 + 52 + 53 + ... + 92$

iv) $1 + 4 + 9 + 16 + ... + 225$

v) $6^2 + 7^2 + 8^2 + ... + 21^2$

vi) $10^3 + 11^3 + 12^3 + ... + 20^3$

vii) $1 + 3 + 5 + ... + 71$

Solution :

i) $1 + 2 + 3 + ... + 60$

$\sum_{k=1}^{n} K = \frac{n(n+1)}{2}$

$= \frac{60\times61}{2}$

$= 30\times61$

$= 1830$

ii) $3 + 6 + 9 + ... + 96$

$= 3\ (1 + 2 + 3 + ... + 32)$

$= 3\left(\frac{32\times33}{2}\right)$

$= 3\times16\times33$

$= 1584$

iii) $51 + 52 + 53 + ... + 92$

$= (1 + 2 + 3 + ... + 92) - (1 + 2 + 3 + ... + 50)$

$= \frac{92\times93}{2} - \frac{50\times51}{2}$

$= 46\times93 - 25\times51$

$= 4278 - 1275$

$= 3003$

iv) $1 + 4 + 9 + 16 + ... + 225$

$= 1^2 + 2^2 + 3^2 + ... + 15^2$

$$\sum_{k=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$$

$= \frac{15\times16\times31}{6}$

$= 1240$

v) $6^2 + 7^2 + 8^2 + ... + 21^2$

$= (1^2 + 2^2 + 3^2 + ... + 21^2) - (1^2 + 2^2 + ... + 5^2)$

$= \frac{21\times22\times43}{6} - \frac{5\times6\times11}{6}$

$= 3311 - 55$

$= 3256$

vi) $10^3 + 11^3 + 12^3 + ... + 20^3$

$= (1^3 + 2^3 + ... + 20^3) - (1^3 + 2^3 + ... + 9^3)$

$$\sum_{k=1}^{n} K^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$= \left(\frac{20\times21}{2}\right)^2 - \left(\frac{9\times10}{2}\right)^2$

$= (210)^2 - (45)^2$

$= 44100 - 2025$

$= 42075$

vii) $1 + 3 + 5 + ... + 71$

$a = 1, d = 2, l = 71$

$\therefore n = \frac{l-a}{d} + 1$

$= \frac{71-1}{2} + 1$

$= 36$

$\therefore 1 + 3 + 5 + ... + 71 = (36)^2$

$(\because 1 + 3 + 5 + ... + n \text{ terms} = n^2)$

$= 1296$

2. If $1 + 2 + 3 + ... + k = 325$, then find $1^3 + 2^3 + 3^3 + ... + k^3$

Solution :

Given $1 + 2 + 3 + ... + k = 325$

$\Rightarrow \frac{k(k+1)}{2} = 325$

$\therefore 1^3 + 2^3 + 3^3 + ... + k^3$

$= \left(\frac{k(k+1)}{2}\right)^2$

$= (325)^2$

$= 105625$

3. If $1^3 + 2^3 + 3^3 + ... + k^3 = 44100$, then find $1 + 2 + 3 + ... + k$.

Solution :

Given $1^3 + 2^3 + 3^3 + ... + k^3 = 44100$

$\Rightarrow \left(\frac{k(k+1)}{2}\right)^2 = 44100$

$\Rightarrow \frac{k(k+1)}{2} = 210$

$\Rightarrow 1 + 2 + 3 + ... + k = 210$

4. How many terms of the series $1^3 + 2^3 + 3^3 + \ldots\ldots$ should be taken to get the sum 14400?

Solution :

Given $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + k^3 = 14400$

$$\Rightarrow \left(\frac{k(k+1)}{2}\right)^2 = 14400$$

$$\Rightarrow \frac{k(k+1)}{2} = 120$$

$$\Rightarrow k^2 + k - 240 = 0$$

$$\Rightarrow (k+16)(k-15) = 0$$

$$\therefore k = -16,\ k = 15$$

But $k \neq -16$

$$\therefore\ k = 15$$

5. The sum of the squares of the first *n* natural numbers is 285, while the sum of their cubes is 2025. Find the value of *n*.

Solution :

Give sum of the squares of first 'n' natural numbers = 285

i.e., $\dfrac{n(n+1)(2n+1)}{6} = 285$ ....(1)

and Sum of their cubes = 2025

i.e., $\left(\dfrac{n(n+1)}{2}\right)^2 = 2025$

$\Rightarrow\ n\left(\dfrac{n+1}{2}\right) = 45$ ....(2)

Sub (2) in (1)

(1) $\Rightarrow \dfrac{n(n+1)}{2} \times \dfrac{2n+1}{3} = 285$

$$\Rightarrow 45 \times \frac{2n+1}{3} = 285$$

$$\Rightarrow 2n+1 = \frac{285}{15} = 19$$

$$\Rightarrow 2n = 18$$

$$\therefore\ n = 9$$

6. Rekha has 15 square colour papers of sizes 10 cm, 11 cm, 12 cm,…, 24 cm. How much area can be decorated with these colour papers?

Solution :

Given sides of 15 square Colour papers are

10 cm, 11 cm, 12 cm, ... 24 cm

$\therefore$ its area $= 10^2 + 11^2 + 12^2 + \ldots\ldots\ldots + 24^2$

$= (1^2 + 2^2 + 3^2 + \ldots\ldots + 24^2) - (1^2 + 2^2 + \ldots\ldots + 9^2)$

$$= \frac{24 \times 25 \times 49}{6} - \frac{9 \times 10 \times 19}{6}$$

$$= 4900 - 285$$

$$= 4615 \text{ cm}^2$$

7. Find the sum of the series to $(2^3 - 1^3) + (4^3 - 3^3) + (6^3 - 5^3) + \ldots\ldots$ to

(i) *n* terms (ii) 8 terms

Solution :

To find the sum of the series :

i) $(2^3 - 1^3) + (4^3 - 3^3) + (6^3 - 5^3) + \ldots\ldots\ldots$ n terms

$= (2^3 + 4^3 + 6^3 + \ldots\ldots\ldots\ n \text{ terms})$

$- (1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ \text{n terms})$

$$\sum_1^n (2n)^3 - \sum_1^n (2n-1)^3$$

$$= \sum_1^n [(2n)^3 - (2n-1)^3]$$

$$(\because a^3 - b^3 = (a-b)(a^2 + ab + b^2))$$

$$= \sum_1^n [(2n - 2n + 1)(4n^2 + 2n(2n-1) + (2n-1)^2]$$

$$= \sum_1^n [4n^2 + 4n^2 - 2n + 4n^2 - 4n + 1]$$

$$= \sum_1^n [12n^2 - 6n + 1]$$

$$= 12\sum n^2 - 6\sum n + \sum 1$$

$$= \overset{2}{\cancel{12}}\left[\frac{n(n+1)(2n+1)}{\cancel{6}}\right] - \overset{3}{\cancel{6}}\left[\frac{n(n+1)}{\cancel{2}}\right] + n$$

$$= n(n+1)[4n+2-3]+n$$

$$= (n^2+n)(4n-1)+n$$

$$= 4n^3+4n^2-n^2-n+n$$

$$= 4n^3+3n^2$$

ii) When n = 8,

$S_8 = 4(8^3) + 3(8^2)$

$= 4(512) + 3(64)$

$= 2048 + 192$

$= 2240$

EXERCISE 2.10

Multiple choice questions :

1. Euclid's division lemma states that for positive intgers *a* and *b*, there exist unique integers *q* and *r* such that *a* = *bq* + *r*, where *r* must satisfy.

(1) $1 < r < b$ (2) $0 < r < b$
(3) $0 \leq r < b$ (4) $0 < r \leq b$

Hint : **Ans : (3)**

By definition of Euclid's lemma $0 \leq r < b$

2. Using Euclid's division lemma, if the cube of any positive integer is divided by 9 then the possible remainders are

(1) 0, 1, 8 (2) 1, 4, 8
(3) 0, 1, 3 (4) 1, 3, 5

Hint : **Ans : (1)**

$x^3 \equiv y \pmod 9$
when $x = 3$, $y = 0$ (27 is div. by 9)
when $x = 4$, $y = 1$ (63 is div. by 9)
when $x = 5$, $y = 8$ (117 is div. by 9)
$\therefore$ The remainders are 0, 1, 8, ...

3. If the HCF of 65 and 117 is expressible in the form of 65m – 117, then the value of *m* is

(1) 4 (2) 2 (3) 1 (4) 3

Hint : **Ans : (2)**

HCF of 65, 117 is 13

65m – 117 = 13

$\Rightarrow$ 65m = 130

$\Rightarrow$ m = 2

4. The sum of the exponents of the prime factors in the prime factorization of 1729 is

(1) 1 (2) 2 (3) 3 (4) 4

Hint : **Ans : (3)**

$1729 = 7 \times 13 \times 19$

$= 7^1 \times 13^1 \times 19^1$

$\therefore$ Sum of the exponents = 1 + 1 + 1 = 3

5. The least number that is divisible by all the numbers from 1 to 10 (both inclusive) is

(1) 2025 (2) 5220 (3) 5025 (d) 2520

Hint : **Ans : (4)**

Refer 9th sum in Ex. 2.2

6. $7^{4k} \equiv$ _______ (mod 100)

(1) 1 (2) 2 (3) 3 (4) 4

Hint : **Ans : (1)**

If k = 1, 7^4 leaves remainder 1 modulo 100.

7. Given $F_1 = 1$, $F_2 = 3$ and $F_n = F_{n-1} + F_{n-2}$ then F_5 is

(1) 3 (2) 5 (3) 8 (4) 11

Hint : **Ans : (4)**

$F_3 = F_2 + F_1 = 4$
$F_4 = F_3 + F_2 = 7$
$F_5 = F_4 + F_3 = 4 + 7 = 11$

8. The first term of an arithmetic progression is unity and the common difference is 4. Which of the following will be a term of this A.P.

(1) 4551 (2) 10091
(3) 7881 (4) 13531

Hint : Ans : (3)

$a = 1, d = 4$

$\therefore$ The A.P is 1, 5, 9, 13, ... leaves remainder 1 when divided by 4.

$\therefore$ 7881 leaves remainder 1 when divided by 4.

9. If 6 times 6th term of an A.P. is equal to 7 times the 7th term, then the 13th term of the A.P. is

(1) 0 (2) 6 (3) 7 (4) 13

Hint : Ans : (1)

$6(t_6) = 7\,(t_7) \Rightarrow 6\,(a + 5d) = 7\,(a + 6d)$

$\Rightarrow 6a + 30d = 7a + 42d$

$\Rightarrow a + 12d = 0$

$\Rightarrow t_{13} = 0$

10. An A.P. consists of 31 terms. If its 16th term is *m*, then the sum of all the terms of this A.P. is

(1) 16m (2) 62m (3) 31m (d) $\frac{31}{2}$m

Hint : Ans : (3)

$n = 31,\ a + 15d = m\quad S_{31} = \frac{31}{2}\,[2a + 30d]$

$= 31\,(a + 15d)$

$= 31m$

11. In an A.P., the first term is 1 and the common difference is 4. How many terms of the A.P. must be taken for their sum to be equal to 120?

(1) 6 (2) 7 (3) 8 (4) 9

Hint : Ans : (3)

$a = 1,\ d = 4,\ S_n = 120$

$\Rightarrow \frac{n}{2}(2a+(n-1)d)=120$

$\Rightarrow \frac{n}{2}(2+(n-1)4)=120$

$\Rightarrow n(1+2n-2)=120$

$\Rightarrow n(2n-1)=120$

$\Rightarrow 2n^2-n-120=0$

$\Rightarrow n=8$

-1	-240
$\frac{-16}{2}$	$\frac{15}{2}$

$-8,\ \frac{15}{2}$

12. If A = 2^{65} and B = $2^{64} + 2^{63} + 2^{62} + ... + 2^0$ which of the following is true ?

(1) B is 2^{64} more than A

(2) A and B are equal

(3) B is larger than A by 1

(4) A is larger than B by 1

Hint : Ans : (4)

2^4 is greater than $2^0 + 2^1 + 2^2 + 2^3$ by 1

2^5 is greater than $2^0 + 2^1 + 2^2 + 2^3 + 2^4$ by 1

$\therefore$ 2^{65} is greater than $2^0 + 2^1 + ... + 2^{64}$ by 1

$\therefore$ A is larger than B by 1.

13. The next term of the sequence $\frac{3}{16}, \frac{1}{8}, \frac{1}{12}, \frac{1}{18}, ...$ is

(1) $\frac{1}{24}$ (2) $\frac{1}{27}$ (3) $\frac{2}{3}$ (4) $\frac{1}{81}$

Hint : Ans : (2)

$$r = \frac{1/8}{3/16} = \frac{1}{8}\times\frac{16}{3} = \frac{2}{3}$$

$$\therefore \text{Next term of the sequence} = \frac{1}{18}\times\frac{2}{3} = \frac{1}{27}$$

14. If the sequence t_1, t_2, t_3, ... are in A.P. then the sequence t_6, t_{12}, t_{18}, ... is

(1) a Geometric Progression

(2) an Arithmetic Progression

(3) neither an Arithmetic Progression nor a Geometric Progression

(4) a constant sequence

Hint : **Ans : (3)**

Obivously they should be in A.P.

15. The vlaue of $(1^3 + 2^3 + 3^3 + ... + 15^3) - (1 + 2 + 3 + ... + 15)$ is

(1) 14400 (2) 14200

(3) 14280 (4) 14520

Hint : **Ans : (3)**

$$\left(\frac{15\times16}{2}\right)^2 - \frac{15\times16}{2}$$

$$= 14400 - 120$$

$$= 14280$$

UNIT EXERCISE - 2

1. Prove that $n^2 - n$ divisible by 2 for every positive interger n.

Solution :

Any positive integer is of the form $2q$ (or) $2q + 1$ for some integer q.

i) $n^2 - n = (2q)^2 - 2q$

$= 2q\,(2q - 1)$

which is divisible by 2.

ii) $n^2 - n = (2q + 1)^2 - (2q + 1)$

$= (2q + 1)\,(2q + 1 - 1)$

$= 2q\,(2q + 1)$, which is divisible by 2.

Hence proved.

2. A milk man has 175 litres of cow's milk and 105 litres of buffalow's milk. He wishes to sell the milk by filling the two types of milk in cans of equal capacity. Claculate the following (i) Capacity of a can (ii) Number of cans of cow's milk (iii) Number of cans of buffalow's milk.

Solution :

Cow's milk = 175 lrs.

Buffalow's milk = 105 lrs.

Since he wish to sell the milk by filling the 2 types of milk in cans of equal capacity,

i) Capacity of a can = HCF of 175 and 105

= 35 litres

ii) Number of cans of Cow's milk = $\frac{175}{35} = 5$

iii) Number of cans of buffalow's milk

$= \frac{105}{35} = 3$

3. When the positive integers a, b and c are divided by 13 the respective remainders are 9, 7 and 10. Find the remainder when a + 2b + 3c is divided by 13.

Solution :

Let $a = 13q + 9$

$b = 13q + 7 \Rightarrow 2b = 26q + 14$

$c = 13q + 10 \Rightarrow 3c = 39q + 30$

$a + 2b + c = (13q + 9) + (26q + 14) + (39q + 30)$

$= 78q + 53$

$= 13\,(6q) + 13(4) + 1$

$\therefore$ When $a + 2b + 3c$ is divided by 13, the remainder is 1.

4. Show that 107 is of the form 4q + 3 for any integer *q*.

Solution :

When 107 is divided by 4,

$107 = 4(26) + 3$

This is of the form

$107 = 4q + 3$ for $q = 26$.

5. If $(m + 1)^{th}$ term of an A.P. is twice the $(n + 1)^{th}$ term, then prove that $(3m + 1)^{th}$ term is twice the $(m + n + 1)^{th}$ term.

Solution :

Given $t_{m+1} = 2(t_{n+1})$

$a + (m + 1 - 1)\,d = 2\,(a + (n + 1 - 1)\,d)$

$a + md = 2\,(a + nd)$

$a + md = 2a + 2nd$ — (1)

To Prove : $t_{3m+1} = 2\,(t_{m+n+1})$

LHS : t_{3m+1}

$= a + (3m + 1 - 1)d$

$= a + 3md$

$= (a + md) + 2md$

$= 2a + 2nd + 2md$ (from (1))

$= 2\,[a + (m + n)d]$

$= 2\,[t_{m+n+1}]$

= RHS

Hence proved.

6. Find the 12^{th} term from the last term of the A.P – 2, – 4, – 6, ... –100.

Solution :

Given A.P is – 2, – 4, – 6, ... – 100

To find : t_{12} from the last term

$a = -100,\ d = 2$

$t_{12} = a + 11d$

$= -100 + 11(2)$

$= -100 + 22$

$= -78$

7. Two A.P.'s have the same common difference. The first term of one A.P. is 2 and that of the other is 7. Show that the difference between their 10^{th} terms is the same as the difference between their 21^{st} terms, which is the same as the difference between any two corresponding terms.

Solution :

1^{st} A.P	2^{nd} A.P.
$a = 2,\ d = d$	$a = 7,\ d = d$
$t_{10} = a + 9d$	$T_{10} = a + 9d$
$= 2 + 9d$	$= 7 + 9d$
$t_{21} = a + 20d$	$T_{21} = a + 20d$
$= 2 + 20d$	$= 7 + 20d$

$\therefore T_{10} - t_{10} = 5$ and $T_{21} - t_{21} = 5 = T_n - t_n = 5$

8. A man saved ₹16500 in ten years. In each year after the first he saved ₹100 more than he did in the preceding year. How much did he save in the first year ?

Solution :

Given S_n = ₹ 16500, d = ₹ 100, n = 10 in A.P.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\Rightarrow \frac{10}{2}[2a + 9(100)] = 16500$$

$$\Rightarrow 2a + 900 = \frac{16500}{5}$$

$$\Rightarrow 2a + 900 = 3300$$

$$\Rightarrow 2a = 2400$$

$$\therefore a = 1200$$

$\therefore$ He saved Rs. 1200 in 1^{st} year.

9. Find the G.P. in which the 2nd term is $\sqrt{6}$ and the 6th term is $9\sqrt{6}$.

Solution :

Given $t_2 = \sqrt{6},\ t_6 = 9\sqrt{6}$ in G.P.

$$\Rightarrow a.r = \sqrt{6},\quad a.r^5 = 9\sqrt{6}$$

$$\therefore r^4 = 9 \qquad \text{(when divide)}$$

$$\Rightarrow r = \sqrt{3}$$

$$\therefore a \times \sqrt{3} = \sqrt{6}$$

$$\therefore a = \sqrt{2}$$

$\therefore$ The G.P is

$\sqrt{2}, \sqrt{6}, \sqrt{18}, ...$

(or)

$\sqrt{2}, \sqrt{6}, 3\sqrt{2}, ...$

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10. The value of a motor cycle depreciates at the rate of 15% per year. What will be the value of the motor cycle 3 year hence, which is now purchased for ₹ 45,000 ?

Solution :

P = ₹ 45000, n = 3, r = 15% (depreciation)

$$A = P\left(1 - \frac{r}{100}\right)^n$$

$$= 45,000\left(1 - \frac{15}{100}\right)^3$$

$$= 45,000 \times \frac{85}{100} \times \frac{85}{100} \times \frac{85}{100}$$

$$= 27,635.625$$

$$= 27636$$

PROBLEMS FOR PRACTICE

1. Express the number $0.\overline{3178}$ in the form of $\frac{a}{b}$.

   (Ans: $\frac{3178}{999}$)

2. A class of 20 boys and 15 girls is divided into n groups so that each group has x boys and y girls. Find x, y and n.

   (Ans : n = 7, x = 4, y = 3)

3. Show that 7^n cannot end with digit zero for any natural number.

4. Find the HCF of the following numbers by using Euclid's division algorithm.

   i) 867, 255 ii) 1656, 4025

   iii) 180, 252, 324 iv) 92690, 7378

   v) 134791, 6341, 6339

   (Ans : i) 51 ii) 23 iii) 36 iv) 31 v) 1)

5. Use Euclid's lemma, show that the square of any positive integer is either of the form 3m (or) 3m+1 for same integer m.

6. Find the largest number which divides 70 and 125 leaving remainder 5 and 8 respectively.

   (Ans : 13)

7. Find the largest positive integer that will divide 398, 436 and 542 that leaves remainders 7, 11, 15 respectively.

   (Ans : 17)

8. If HCF of 144 and 180 is expressed in the form 13m – 3, find m.

   (Ans : 3)

9. If d is the HCF of 56 and 72, find x and y satisfying $d = 56x + 72y$. Also show that x and y is not unique.

(Ans : 4 and –3, –68, 53)

10. Find the largest number of four digits exactly divisible by 12, 15, 18 and 27.

(Ans : 9720)

11. Write the first 6 terms of the sequence whose n^{th} term is

i) $a_n = \begin{cases} n & , \text{ if } n = 1,2,3 \\ a_{n-1} + a_{n-2} + a_{n-3}, & \text{if } n > 3 \end{cases}$

ii) $a = \dfrac{3n-2}{3^{n-1}}$

(Ans : i) 1, 2, 3, 5, 8, 13 ii) $1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \ldots$**)**

12. Find the indicated terms in each of the following :

i) $a_n = (-1)^n \cdot 2^{n+3}(n+1)$; a_5, a_8

ii) $a_n = (-1)^n, (1 - n + n^2)$; a_2, a_9

(Ans : i) – 1536, 18432 ii) 3, –73)

13. How many temrs are there in the A.P.

$-1, \frac{-5}{6}, \frac{-2}{3}, \ldots\ldots, \frac{10}{3}$?

(Ans : 27)

14. Find the 40^{th} term of an A.P whose 5^{th} term is 41 and 11^{th} term is 71.

(Ans : 216)

15. If 7^{th} term of an A.P is $\frac{1}{9}$ and 9th term is $\frac{1}{7}$.

(Ans : 1)

16. Find the middle term of the A.P 213, 205, 197, ...37

(Ans :125)

17. The first term of an A.P is 5, the last term is 45. Sum of all its terms is 400. Find the number of terms and the common difference of A.P.

(Ans : n = 16, d = $\frac{8}{3}$**)**

18. The 24^{th} term of an A.P is twice its 10^{th} term. Show that its 72^{nd} term is 4 times its 15^{th} term.

19. Find the sum of all two digit odd positive numbers.

(Ans : 2475)

20. Which term of the sequence

$20, 19\frac{1}{4}, 18\frac{1}{2}$ .... is the first negative term ?

(Ans : 28)

21. Find the 18^{th} term of the A.P from right end 3, 7, 11, ... 407.

(Ans : 339)

22. How many consecutive integers beginning with 10 must be taken for their sum to be 2035 ?

(Ans : 55)

23. Sum of 3 numbers in an A.P is 54 and their product is 5670. Find the 3 numbers.

(Ans : 15, 18, 21)

24. Find the sum of all natural numbers between 201 and 399 that are divisible by 5.

25. Find : $\left(4 - \frac{1}{n}\right) + \left(7 - \frac{2}{n}\right) + \left(10 - \frac{3}{n}\right) + \ldots\ldots$ up to n terms.

(Ans : $\frac{3n^2 + 4n - 1}{2}$**)**

26. The sum of first 'n' terms of an A.P is $5n - n^2$. Find the n^{th} term of the A.P.

(Ans : – 2 (n – 3))

27. If 7 times the 7^{th} term of an A.P is equal to 11 times its 11^{th} term, show that its 18^{th} term is 0.

28. Find the sum of 22 terms of the A.P $x + y, x - y, x - 3y$, ...

(Ans : 22 (*x* – 20*y*))

29. If the ratio between the sums of *n* terms of AP's is (7n + 1) : (4n + 27), find the ratio of their 11th terms.

(Ans : 148 : 111)

30. A man gets the initial salary of 5200 p.m. and receive an automatic increase of 320 in the very next month and each month hereafter. Find i) his salary in 10th month ii) total earings during the 1st year.

(Ans : 8080, 83520)

31. Sum of 3 terms of a G.P is $\frac{39}{10}$ and their product is 1. Find the G.P.

(Ans : $\frac{5}{2}, 1, \frac{2}{5}$**)**

32. If 4^{th} and 7^{th} terms of a G.P are 54 and 1458 respectively, find G.P.

(Ans : 2, 6, 18, 54, ...)

33. In the series 18, – 12, 8, ..., which term is $\frac{512}{729}$?

(Ans : 9^{th} term)

34. Find the sum : 0.2 + 0.92 + 0.992 + ... to *n*terms.

35. Find the sum of the series :

$9^{1/3}\ 9^{1/9}\ 9^{1/27}$ .... ∞. **(Ans : 3)**

36. A farmer buys a used tractor for Rs. 12000. He pays Rs. 6000 and agress to pay the balance in annual instalment of Rs. 500 plus 12% interest on u paid amount. How much will the tractor cost him ?

(Ans : Rs. 16,680)

37. The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the series.

(Ans : $5 + \frac{10}{3} + \frac{20}{9} + ...$**)**

38. Evaluate : $1^2 + 3^2 + 5^2 + ... 29^2$

(Ans : 4495)

39. Evaluate : $8^3 + 9^3 + ... + 17^3$

40. The sum of the squares of first 'n' natural numbers is 285, while the sum of their cubes is 2025. Find 'n'.

(Ans : 9)

OBJECTIVE TYPE QUESTIONS

1. When the denominator number of $\frac{257}{500}$ writes in the form of $2^m \times 5^n$, then m + n is

(a) 6 (b) 5

(c) 23 (d) none of these

Ans : (b)

2. The common difference of the A.P $\frac{1}{2q}, \frac{1-2q}{2q}, \frac{1-4q}{2q}$, ...

(a) 0 (b) 1 (c) – 1 (d) q

Ans : (c)

3. The first 3 terms of A.P are 3y – 1, 3y + 5 and 5y + 1, then *y* is

(a) 5 (b) 1 (c) – 5 (d) 4

Ans : (a)

4. For an A.P, $S_n = n^2 - n + 1$, the 2^{nd} term is

   (a) 2 (b) 3 (c) 4 (d) – 2

   Ans : (b)

5. The next term of an A.P :

   – 12, – 9, – 6, – 3, ... is

   (a) 3 (b) 6

   (c) 0 (d) none of these

   Ans : (c)

6 The sum of 6 terms of the A.P 1, 3, 5, 7, ... is

   (a) 25 (b) 49 (c) 36 (d) 30

   Ans : (c)

7. Which term of the series – 3, – 1, 5, ... is 53 ?

   (a) 12 (b) 13 (c) 14 (d) 15

   Ans : (d)

8. The common ratio of the G.P $\frac{-5}{2}, \frac{25}{4}, \frac{-125}{8}$, ... is

   (a) 15 (b) $\frac{35}{4}$ (c) $\frac{-5}{2}$ (d) $\frac{5}{2}$

   Ans : (c)

9. If the 3^{rd} term of G.P is 4, then the product of its first 5 terms is

   (a) 4^3 (b) 4^5 (c) 4^4 (d) 4^2

   Ans : (b)

10. If *a, b, c* are in A.P., *a, b, d*in G.P, then *a, a – b, d – c* will be in

    (a) A.P (b) G.P

    (c) A.P and G.P (d) none of these

    Ans : (b)

11. The 3^{rd} term of a G.P is the square of first term. If the 2^{nd} term is 8, then the 6^{th} term is

    (a) 120 (b) 124 (c) 128 (d) 132

    Ans : (d)

12. If $a_1 = a_2 = 2$, $a_n = a_{n-1} - 1$, then a_5 is

    (a) 1 (b) – 1 (c) 0 (d) – 2

    Ans : (b)

13. The 9^{th} term of the series 27 + 9 + $5\frac{2}{5}$ + ... is

    (a) $1\frac{10}{17}$ (b) $\frac{10}{17}$ (c) $\frac{16}{27}$ (d) $\frac{17}{27}$

    (Ans : (a))

14. The n^{th} term of the series 3.8 + 6.11 + 9.14 + 12.17 + ... will be

    (a) 3n (n + 5) (b) n (n + 5)

    (c) n (3n + 5) (d) 3n (3n + 5)

    Ans : (d)

15. If the n^{th} term of a G.P 5, $\frac{-5}{2}$, $\frac{5}{4}$, ... is $\frac{5}{1024}$, then n is

    (a) 11 (b) 10 (c) 9 (d) 4

    Ans : (a)

16. If 1 + 2 + 3 + ... + n = K, then $1^3 + 2^3 + 3^3 + ... + n^3$ is

    (a) K^3 (b) K^2

    (c) $\frac{K(K+1)}{2}$ (d) $(K + 1)^3$

    Ans : (b)

CHAPTER 3 ALGEBRA

I. SIMULTANEOUS LINEAR EQUATIONS IN THREE VARIABLES :

Key Points

- ✓ Any first degree equation containing two variables x and y is called a linear equation in two variables. The general form of linear equation in two variables x and y is $ax + by + c = 0$, where atleast one of a, b is non-zero and a, b, c are real numbers.
- ✓ The general form of a linear equation in three variables x, y and z is $ax + by + cz + d = 0$ where a, b, c, d are real numbers, and atleast one of a, b, c is non-zero.
- ✓ A linear equation in two variables of the form $ax + by + c = 0$, represents a straight line.
- ✓ A linear equation in three variables of the form $ax + by + cz + d = 0$, represents a plane.

Example 3.1

The father's age is six times his son's age. Six years hence the age of father will be four times his son's age. Find the present ages (in years) of the son and father.

Solution :

Let the present age of father be x years and the present age of son be y years

Given, $x = 6y$ — (1)

$x + 6 = 4(y + 6)$ — (2)

Substituting (1) in (2), $6y + 6 = 4(y + 6)$

$6y + 6 = 4y + 24$ gives, $y = 9$

Therefore, son's age = 9 years and father's age = 54 years.

Example 3.2

Solve $2x - 3y = 6$, $x + y = 1$

Solution :

$2x - 3y = 6$ — (1)

$x + y = 1$ — (2)

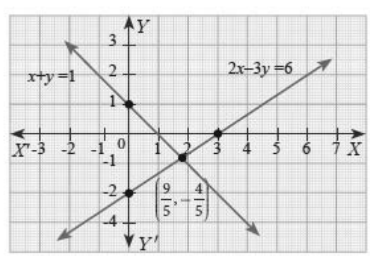
(1) × 1 gives, $2x - 3y = 6$

(2) × 2 gives, $2x + 2y = 2$

$-5y = 4$ gives, $y = \frac{-4}{5}$

Substituting $y = \frac{-4}{5}$ in (2), $x - \frac{4}{5} = 1$ we get,

$x = \frac{9}{5}$



Therefore $x = \frac{9}{5}, y = \frac{-4}{5}$

Example 3.3

Solve the following system of linear equations in three variables

$3x - 2y + z = 2$, $2x + 3y - z = 5$, $x + y + z = 6$.

Solution :

$3x - 2y + z = 2$ — (1)

$2x + 3y - z = 5$ — (2)

$x + y + z = 6$ — (3)

Adding (1) and (2), $3x - 2y + z = 2$

$2x + 3y - z = 5$ (+)

$5x + y = 7$ — (4)

Adding (2) and (3), $2x + 3y - z = 5$

$x + y + z = 6$ (+)

$3x + 4y = 11$ — (5)

$4 \times (4) - (5)$ $20x + 4y = 28$

$3x + 4y = 11$ (–)

$17x = 17$ gives, $x = 1$

Substituting $x = 1$ in (4), $5 + y = 7$ gives, $y = 2$

Substituting $x = 1$, $y = 2$ in (3), $1 + 2 + z = 6$ we get, $z = 3$

Therefore, $x = 1$, $y = 2$, $z = 3$

Example 3.4

In an interschool atheletic meet, with 24 individual events, securing a total of 56 points, a first place secures 5 points, a second place secures 3 points, and a third place secures 1 point. Having as many third place finishers as first and second place finishers, find how many athletes finished in each place.

Solution :

Let the number of I, II and III place finishers be x , y and z respectively.

Total number of events = 24;

Total number of points = 56.

Hence, the linear equations in three variables are

$x + y + z = 24$ — (1)

$5x + 3y + z = 56$ — (2)

$x + y = z$ — (3)

Substituting (3) in (1) we get, $z + z = 24$ gives, $z = 12$

Therefore, (3) equation will be, $x + y = 12$

(2) is $5x + 3y = 44$

$3 \times (3)$ is $3x + 3y = 36$ (–)

$2x = 8$ we get, $x = 4$

Substituting $x = 4$, $z = 12$ in (3) we get,

$y = 12 - 4 = 8$

Therefore, Number of first place finishers is 4

Number of second place finishers is 8

Number of third place finishers is 12.

Example 3.5

Solve $x + 2y - z = 5$; $x - y + z = -2$;

$-5x - 4y + z = -11$

Solution :

Let, $x + 2y - z = 5$ — (1)

$x - y + z = -2$ — (2)

$-5x - 4y + z = -11$ — (3)

Adding (1) and (2) we get,

$x + 2y - z = 5$

$x - y + z = -2$ (+)

$2x + y = 3$ — (4)

Subtracting (2) and (3),

$$x - y + z = -2$$
$$-5x - 4y + z = -11 \quad (-)$$
$$6x + 3y = 9 \quad —(5)$$

Dividing by 3 $2x + y = 3$

Substracting (4) and (5),

$$2x + y = 3$$
$$2x + y = 3$$
$$0 = 0$$

Here we arrive at an identity $0 = 0$.

Hence the system has an infinite number of solutions.

Example 3.6

Solve $3x + y - 3z = 1$; $-2x - y + 2z = 1$; $-x - y + z = 2$.

Solution :

Let $3x + y - 3z = 1$ — (1)

$-2x - y + 2z = 1$ — (2)

$-x - y + z = 2$ — (3)

Adding (1) and (2), $3x + y - 3z = 1$

$$-2x - y + 2z = 1 \quad (+)$$
$$x - z = 2 \quad —(4)$$

Adding (1) and (3), $3x + y - 3z = 1$

$$-x - y + z = 2 \quad (+)$$
$$2x - 2z = 3 \quad —(5)$$

Now, (5) – 2 × (4) we get,

$$2x - 2z = 3$$
$$2x - 2z = 4 \quad (-)$$
$$0 = -1$$

Here we arrive at a contradiction as $0 \neq -1$.

This means that the system is inconsistent and has no solution.

Example 3.7

Solve $\frac{x}{2} - 1 = \frac{y}{6} + 1 = \frac{z}{7} + 2$; $\frac{y}{3} + \frac{z}{2} = 13$

Solution :

Considering, $\frac{x}{2} - 1 = \frac{y}{6} + 1$

$$\frac{x}{2} - \frac{y}{6} = 1 + 1 \text{ gives,}$$

$$\frac{6x - 2y}{12} = 2 \text{ we get, } 3x - y = 12 \quad —(1)$$

Considering, $\frac{x}{2} - 1 = \frac{z}{7} + 2$

$$\frac{x}{2} - \frac{z}{7} = 1 + 2 \text{ gives,}$$

$$\frac{7x - 2z}{14} = 3 \text{ we get, } 7x - 2z = 42 \quad —(2)$$

Also, from $\frac{y}{3} + \frac{z}{2} = 13$ gives,

$$\frac{2y + 3z}{6} = 13 \text{ we get, } 2y + 3z = 78 \quad —(3)$$

Eliminating z from (2) and (3)

(2) × 3 gives, $21x - 6z = 126$

(3) × 2 gives, $4y + 6z = 156$ (+)

$$21x + 4y = 282$$

(1) × 4 gives, $12x - 4y = 48$ (+)

$$33x = 330 \quad \text{so, } x = 10$$

Substituting $x = 10$ in (1), $30 - y = 12$ we get, $y = 18$

Substituting $x = 10$ in (2), $70 - 2z = 42$ then, $z = 14$

Therefore, $x = 10$, $y = 18$, $z = 14$.

Example 3.8

Solve :

$$\frac{1}{2x}+\frac{1}{4y}-\frac{1}{3z}=\frac{1}{4}\ ;\ \frac{1}{x}=\frac{1}{3y};\ \frac{1}{x}-\frac{1}{5y}+\frac{4}{z}=2\frac{2}{15}$$

Solution :

Let $\frac{1}{x}=p,\frac{1}{y}=q,\frac{1}{z}=r$

The given equations are written as

$$\frac{p}{2}+\frac{q}{4}-\frac{r}{3}=\frac{1}{4}$$

$$p=\frac{q}{3}$$

$$p-\frac{q}{5}+4r=2\frac{2}{15}=\frac{32}{15}$$

By simplifying we get,

$6p + 3q - 4r = 3$ — (1)

$3p = q$ — (2)

$15p - 3q + 60r = 32$ — (3)

Substituting (2) in (1) and (3) we get,

$15p - 4r = 3$ — (4)

$6p + 60r = 32$ reduces to

$3p + 30r = 16$ — (5)

Solving (4) and (5),

$15p - 4r = 3$

$15p + 150r = 80$ (–)

$-154r = -77$ we get, $r = \frac{1}{2}$

Substituting $r = \frac{1}{2}$ in (4) we get, $15p - 2 = 3$ gives, $p = \frac{1}{3}$

From (2), $q = 3p$ we get $q = 1$

Therefore, $x=\frac{1}{p}=3,\ y=\frac{1}{q}=1,\ z=\frac{1}{r}=2$.

That is, $x = 3$, $y = 1$, $z = 2$.

Example 3.9

The sum of thrice the first number, second number and twice the third number is 5. If thrice the second number is subtracted from the sum of first number and thrice the third we get 2. If the third number is subtracted from the sum of twice the first, thrice the second, we get 1. Find the numbers.

Solution :

Let the three numbers be x, y, z

From the given data we get the following equations,

$3x + y + 2z = 5$ — (1)

$x + 3z - 3y = 2$ — (2)

$2x + 3y - z = 1$ — (3)

(1) × 1 gives, $3x + y + 2z = 5$

(2) × 3 gives, $3x - 9y + 9z = 6$ (–)

$10y - 7z = -1$ — (4)

(1) × gives, $6x + 2y + 4z = 10$

(3) × 3 gives, $6x + 9y - 3z = 3$ (–)

$-7y + 7z = 7$

Adding (4) and (5), $10y - 7z = -1$

$-7y + 7z = 7$

$3y = 6$ gives $y = 2$

Substituting $y = 2$ in (5), $-14 + 7z = 7$ gives,

$z = 3$

Substituting $y = 2$ and $z = 3$ in (1),

$3x + 2 + 6 = 5$ we get $x = -1$

Therefore, $x = -1$, $y = 2$, $z = 3$.

EXERCISE 3.1

1. Solve the following system of linear equations in three variables

(i) $x+y+z=5$; $2x-y+z=9$; $x-2y+3z=16$

(ii) $\frac{1}{x}-\frac{2}{y}+4=0$; $\frac{1}{y}-\frac{1}{z}+1=0$; $\frac{2}{z}+\frac{3}{x}=14$

(iii) $x+20=\frac{3y}{2}+10=2z+5=110-(y+z)$

Solution:

i) Given

$x+y+z=5$ — (1)

$2x-y+z=9$ — (2)

$x-2y+3z=16$ — (3)

$(1)-(3) \Rightarrow 3y-2z=-11$ — (4)

$(2) \Rightarrow 2x-y+z=9$

$(1)\times 2 \Rightarrow 2x+2y+2z=10$ (–)

Subtracting $-3y-z=-1$ — (5)

Solving (4) & (5)

$3y-2z=-11$

$-3y-z=-1$

Adding $-3z=-12$

$z=4$

Sub $z=4$ in (5)

$-3y-4=-1$

$\Rightarrow -3y=3$

$\Rightarrow y=-1$

sub $y=-1$, $z=4$ in (1)

$\Rightarrow x-1+4=5$

$\Rightarrow x=2$

$\therefore$ Solution set :

$x=2, y=-1, z=4$

(ii) $\frac{1}{x}-\frac{2}{y}+4=0$

$\frac{1}{y}-\frac{1}{z}+1=0$

$\frac{2}{z}+\frac{3}{x}=14$

Let $1/x=a$, $1/y=b$, $1/z=c$

$\therefore$ $a-2b=-4$ — (1)

$b-c=-1$ — (2)

$2c+3a=14$ — (3)

Solving (1) & (2)

$(1)\times 1 \Rightarrow a-2b=-4$

$(2)\times 2 \Rightarrow 2b-2c=-2$

Adding $a-2c=-6$ — (4)

Solving (3) & (4)

$3a+2c=14$

$a-2c=-6$

Adding $4a=8$

$a=2$

$\therefore$ (1) $\Rightarrow -2b=-6$

$b=3$

(2) $\Rightarrow 3-c=-1$

$\Rightarrow c=4$

$\therefore a=2, b=3, c=4$

$\Rightarrow x=1/2,\ y=1/3,\ z=1/4$

$\therefore$ Solution set :

$\left\{1/2,\ 1/3,\ 1/4\right\}$

(iii) $x+20=\frac{3y}{2}+10=2z+5=110-(y+z)$

$x+20=\frac{3y}{2}+10$

$\Rightarrow \quad 2x+40=3y+20$

$\Rightarrow \quad \boxed{2x-3y=-20}$ .... (1)

$\frac{3y}{2}+10=2z+5$

$\Rightarrow \quad 3y+20=4z+10$

$\Rightarrow \quad \boxed{3y-4z=-10}$ .... (2)

$2z+5=110-y-z$

$\Rightarrow \quad \boxed{3z+y=105}$ .... (3)

Solving (1) & (2)

$2x-3y=-20$

$3y-4z=-10$

Adding $2x-4z=-30$ .... (4)

Solving (2) & (3)

(2) $\Rightarrow \quad 3y-4z=-10$

(3) × 3 $\Rightarrow \quad 9z+3y=315$

Subtracting, $-13z=-325$

$z=25$

Sub $z=25$ in (4)

$2x-100=-30$

$\Rightarrow \quad 2x=70$

$\Rightarrow \quad x=35$

Sub $x=35$ in (1)

$70-3y=-20$

$3y=90$

$\therefore y=30$

$\therefore$ Solution : $x=35, y=30, z=35$

2. Discuss the nature of solutions of the following system of equations

i) $x+2y-z=6$; $-3x-2y+5z=-12$; $x-2z=3$

ii) $2y+z=3(-x+1)$; $-x+3y-z=-4$; $3x+2y+z=-\frac{1}{2}$

iii) $\frac{y+z}{4}=\frac{z+x}{3}=\frac{x+y}{2}$; $x+y+z=27$

Solution :

Given $\quad x+2y-z=6$ — (1)

$-3x-2y+5z=-12$ — (2)

$x-2z=3$ — (3)

Adding (1) & (2) $\Rightarrow \quad -2x+4z=-6$

÷ by (−2) $\Rightarrow \quad x-2z=3$ ..... (4)

Subtracting (3) & (4)

$x-2z=3$

$x-2z=3$

$0=0$

$\therefore$ The system of equation has infinite number of solutions.

ii) Given

$2y+z=3(-x+1) \Rightarrow$

$3x+2y+z=3$ .... (1)

$-x+3y-z=-4$ .... (2)

$3x+2y+z=-\frac{1}{2}$ .... (3)

Solving (1) & (2)

(1) + (2) $\Rightarrow 2x+5y=-1$ .... (4)

Solving (2) + (3)

(2) + (3) $\Rightarrow 2x+5y=-9/2$ .... (5)

Solving (4) & (5)

$(4) - (5) \Rightarrow \quad 2x + 5y = -1$

$\quad 2x + 5y = -9/2$

$0 = 9/2 - 1$

$0 = 7/2$

This is a contradiction

Since $0 \neq 7/2$

$\therefore$ The system has no solution.

iii) Given $\frac{y+z}{4} = \frac{z+x}{3} = \frac{x+y}{2}; x+y+z = 27$

$\Rightarrow \frac{y+z}{4} = \frac{z+x}{3}$ & $\frac{z+x}{3} = \frac{x+y}{2}$

$\Rightarrow 3y + 3z = 4z + 4x$ & $2z + 2x = 3x + 3y$

$\Rightarrow 4x - 3y + z = 0$ & $x + 3y - 2z = 0$

$\therefore 4x - 3y + z = 0$ .... (1)

$x + 3y - 2z = 0$ .... (2)

$x + y + z = 27$ .... (3)

From (1) & (2)

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ ie $\frac{4}{1} \neq \frac{-3}{3} \neq \frac{-1}{2}$

From (2) & (3), $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ ie $1 \neq 3 \neq -2$

$\therefore$ The system of equations has a unique solutions.

3. Vani, her father and her grand father have an average age of 53. One-half of her grand father's age plus one-third of her father's age plus one fourth of Vani's age is 65. Four years ago if Vani's grand-father was four times as old as Vani then how old are they all now ?

Solution :

Let the present age of Vani, her father, grand father be x, y, z respectively.

By data given,

$\frac{x+y+z}{3} = 53 \Rightarrow x+y+z = 159$ .... (1)

$\frac{1}{2}z + \frac{1}{3}y + \frac{1}{4}x = 65$

$\Rightarrow \frac{6z+4y+3x}{12} = 65 \Rightarrow$

$3x + 4y + 6z = 780$ .... (2)

$(z-4) = 4(x-4) \Rightarrow 4x - z = 12$ .... (3)

Solving (1) & (2)

$(1) \times (4) \Rightarrow 4x + 4y + 4z = 636$

$(2) \Rightarrow x + 4y + 6z = 780$

Subtracting $3x - 2z = -144$ .... (4)

Solving (3) & (4)

$(3) \times (2) \Rightarrow 8x - 2z = 24$

$(4) \Rightarrow 3x - 2z = -144$

Subtracting $7x = 168$

$x = \frac{168}{7} = 24$

Sub x = 24 in (3)

$96 - z = 12$

$z = 84$

$\therefore (1) \Rightarrow 24 + y + 84 = 159$

$\Rightarrow y = 51$

$\therefore$ Vani' present age = 24 years

Father's present age = 51 years

Grand father's age = 84 years

4. **The sum of the digits of a three-digit number is 11. If the digits are reversed, the new number is 46 more than five times the former number. If the hundreds digit plus twice the tens digit is equal to the units digit, then find the original three digit number ?**

Solution :

Let x be the 100's digit

y be the 10's digit

z be the unit's digit of a 3 digit no.

$\therefore$ The required number is $100x + 10y + z$ and the reversed number is $100z + 10y + x$

Given $\boxed{x+y+z=11}$ .... (1)

$\therefore$ By the data given,

$100z + 10y + x = 5(100x + 10y + z) + 46$

$\Rightarrow \boxed{499x + 40y - 95z = -46}$ .... (2)

and

also $x + 2y = z$

$\Rightarrow \boxed{x + 2y - z = 0}$ .... (3)

Adding (1) & (3), $2x + 3y = 11$ .... (4)

Solving (1) & (2)

$(1) \times 95 \Rightarrow \quad 95x + 95y + 95z = 1045$

$(2) \times 1 \Rightarrow \quad 499x + 40y - 95z = -46$

Subtracting, $\quad 94x + 135y = 999$

$\div$ by 27 $\Rightarrow \quad 22x + 5y = 37$

.... (5)

Solving (4) & (5)

$(4) \times 11 \Rightarrow \quad 22x + 33y = 121$

$(5) \Rightarrow \quad 22x + 5y = 37$

Subtracting, $\quad 28y = 84$

$y = 3$

Sub $y = 3$ in (4)

$2x + 9 = 11$

$2x = 2$

$x = 1$

Sub in (1)

$x + y + z = 11$

$\Rightarrow 1 + 3 + z = 11$

$\therefore z = 4$

$\therefore$ The 3 digit no is

$100x + 10y + z$

$= 100(1) + 10(3) + 7$

$= 137$ and reversed digit number 731.

5. **There are 12 pieces of five, ten and twenty rupee currencies whose total value is ₹ 105. When first 2 sorts are interchanged in their numbers its value will be increased by ₹ 20. Find the number of currencies in each sort.**

Solution :

Let x be the number of 5 rupee currencies

Let y be the number of 10 rupee currencies

Let z be the number of 20 rupee currencies

By the data given,

$x + y + z = 12$ .... (1)

$5x + 10y + 20z = 105$ .... (2)

$10x + 5y + 20z = 125$ .... (3)

Solving (1) & (2)

$$(1) \times 5 \Rightarrow 5x + 5y + 5z = 60$$
$$(2) \Rightarrow 5x + 10y + 20z = 105$$

Subtracting, $-5y - 15z = -45$

$\Rightarrow y + 3z = 9$ .... (4)

Solving (2) & (3)

$$(2) \times 2 \Rightarrow 10x + 20y + 40z = 210$$
$$(3) \Rightarrow 10x + 5y + 20z = 125$$

Subtracting, $15y + 20z = 85$

$\Rightarrow 3y + 4z = 17$ ..... (5)

$$(4) \times 3 \Rightarrow 3y + 9z = 27$$
$$(5) \Rightarrow 3y + 4z = 17$$
$$5z = 10$$

Sub z = 2 in (4)

$y + 6 = 9 \Rightarrow y = 3$

$\therefore$ (1) $\Rightarrow x + 3 + 2 = 12 \Rightarrow x = 7$

$\therefore$ No. of 5 rupee notes = 7

No. of 10 rupee notes = 3

No. of 20 rupee notes = 2

II. GCD AND LCM OF POLYNOMIALS :

Key Points

- ✓ **Step 1 :** First, divide $f(x)$ by $g(x)$ to obtain $f(x) = g(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. Then, deg $[r(x)] <$ deg $[g(x)]$.
- ✓ **Step 2 :** If the remainder $r(x)$ is non-zero, divide $g(x)$ by $r(x)$ to obtain $g(x) = r(x)\,q(x) + r_1(x)$ where $r_1(x)$ is the new remainder. Then deg$[r_1(x)] <$ deg$[r(x)]$. If the remainder $r_1(x)$ is zero, then $r(x)$ is the required GCD.
- ✓ **Step 3 :** If $r_1(x)$ is non-zero, then continue th eprocess until we get zero as remainder. The divisor at this stage will be the required GCD.
- ✓ If the $f(x)$ and $g(x)$ are two polynomials of same degree then the polynomial carrying the highest coefficient will be the dividend.
- ✓ The Least Common Multiple of two or more algebraic expressions is the expression of lowest degree (or power) such that the expressions exactly divide it.

Example 3.10

Find the GCD of the polynomials $x^3 + x^2 - x + 2$ and $2x^3 - 5x^2 + 5x - 3$.

Solution :

Let $f(x) = 2x^3 - 5x^2 + 5x - 3$ and $g(x) = x^3 + x^2 - x + 2$

$$\begin{array}{r|l} & \quad 2 \\ \hline x^3 + x^2 - x + 2 & 2x^3 - 5x^2 + 5x - 3 \\ & 2x^3 + 2x^2 - 2x + 4 \quad (-) \\ \hline & \quad -7x^2 + 7x - 7 \end{array}$$

$-7\,(x^2 - x + 1)$

$-7\,(x^2 - x + 1) = 0$, note that -7 is not a divisor of g(x)

Now dividing $g(x) = x^3 + x^2 - x + 2$ by the new remainder $x^2 - x + 1$ (leaving the constant factor), we get

$$\begin{array}{r|ll}
 & x+2 & \\ \hline
x^2-x+1 & x^3+x^2-x+2 & \\
 & x^3-x^2+x & (-) \\ \hline
 & 2x^2-2x+2 & \\
 & 2x^2-2x+2 & (-) \\ \hline
 & 0 &
\end{array}$$

Here, we get zero remainder

Therefore, GCD $(2x^3 - 5x^2 + 5x - 3, x^3 + x^2 - x + 2) = x^2 - x + 1$

Example 3.11

Find the GCD of $6x^3 - 30x^2 + 60x - 48$ and $3x^3 - 12x^2 + 21x - 18$

Solution :

Let $f(x) = 6x^3 - 30x^2 + 60x - 48 = 6(x^3 - 5x^2 + 10x - 8)$ and

$g(x) = 3x^3 - 12x^2 + 21x - 18 = 3(x^3 - 4x^2 + 7x - 6)$

Now, we shall find the GCD of $x^3 - 5x^2 + 10x - 8$ and $x^3 - 4x^2 + 7x - 6$

$$\begin{array}{r|ll}
 & 1 & \\ \hline
x^3-5x^2+10x-8 & x^3-4x^2+7x-6 & \\
 & x^3-5x^2+10x-8 & (-) \\ \hline
 & x^2-3x+2 &
\end{array}$$

$$\begin{array}{r|ll}
 & x-2 & \\ \hline
x^2-3x+2 & x^3-5x^2+10x-8 & \\
 & x^3-3x^2+2x & (-) \\ \hline
 & -2x^2+8x-8 & \\
 & -2x^2+6x-4 & (-) \\ \hline
 & 2x-4 & \\
 & = 2(x-2) &
\end{array}$$

$$\begin{array}{r|ll}
 & x-1 & \\ \hline
x-2 & x^2-3x+2 & \\
 & x^2-2x & (-) \\ \hline
 & -x+2 & \\
 & -x+2 & (-) \\ \hline
 & 0 &
\end{array}$$

Here, we get zero as remainder.

GCD of leading coefficients 3 and 6 is 3.

Thus, GCD $[(6x^3 - 30x^2 + 60x - 48, 3x^3 - 12x^2 + 21x - 18)] = 3(x - 2)$

Example 3.12

Find the LCM of the following

i) $8x^4 y^2, 48x^2y^4$ ii) $5x - 10, 5x^2 - 20$

iii) $x^4 - 1, x^2 - 2x + 1$ iv) $x^3-27, (x - 3)^2, x^2 - 9$

Solution :

i) $8x^4 y^2, 48x^2y^4$

First let us find the LCM of the numerical coefficients.

That is, LCM (8, 48)

$= 2 \times 2 \times 2 \times 6 = 48$

$$\begin{array}{r|l}
2 & 8, 48 \\ \hline
2 & 4, 24 \\ \hline
2 & 2, 12 \\ \hline
 & 1, 6
\end{array}$$

Then find the LCM of the terms involving variables.

That is, LCM $(x^4y^2, x^2y^4) = x^4y^4$

Finally find the LCM of the given expression.

We condclude that the LCM of the given expression is the product of the LCM of the numerical coefficient and the LCM of the terms with variables.

Therefore, LCM $(8x^4 y^2, 48x^2y^4) = 48x^4 y^4$

ii) $5x - 10, 5x^2 - 20$

$5x - 10 = 5(x - 2)$

$5x^2 - 20 = 5(x^2 - 4) = 5(x + 2)(x - 2)$

Therefore, LCM $[(5x - 10), (5x^2 - 20)]$

$= 5(x + 2)(x - 2)$

iii) $x^4 - 1, x^2 - 2x + 1$

$x^4 - 1 = (x^2)^2 - 1 = (x^2 + 1)(x^2 - 1)$

$= (x^2 + 1)(x + 1)(x - 1)$

$x^2 - 2x + 1 = (x - 1)^2$

Therefore, LCM $[(x^4 - 1), (x^2 - 2x + 1)]$

$= (x^2 + 1)(x + 1)(x - 1)^2$

iv) $x^3 - 27, (x - 3)^2, x^2 - 9$

$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$;

$(x - 3)^2 = (x - 3)^2$; $(x^2 - 9)$

$= (x + 3)(x - 3)$

Therefore, LCM $[(x^3 - 27), (x - 3)^2, (x^2 - 9)] = (x - 3)^2 (x + 3)(x^2 + 3x + 9)$

EXERCISE 3.2

1. Find the GCD of the given polynomials

i) $x^4 + 3x^3 - x - 3, x^3 + x^2 - 5x + 3$

ii) $x^4 - 1, x^3 - 11x^2 + x - 11$

iii) $3x^4 + 6x^3 - 12x^2 - 24x, 4x^4 + 14x^3 + 8x^2 - 8x$

iv) $3x^3 + 3x^2 + 3x + 3, 6x^3 + 12x^3 + 6x + 12$

Solution :

i) Let $f(x) = x^4 + 3x^3 - x - 3$

$g(x) = x^3 + x^2 - 5x + 3$

To find the GCD of f(x), g(x)

Divide f(x) by g(x)

$$
\begin{array}{r|l}
 & x + 2 \\
\hline
x^3 + x^2 - 5x + 3 & x^4 + 3x^3 + 0x^2 - x - 3 \\
 & x^4 + x^3 - 5x^2 + 3x \\
\hline
 & 2x^3 + 5x^2 - 4x - 3 \\
 & 2x^3 + 2x^2 - 10x + 6 \\
\hline
 & 3x^2 + 6x - 9
\end{array}
$$

$= 3(x^2 + 2x - 3) \neq 0$

Now , divide g(x) by $x^2 + 2x - 3$ (excluding 3)

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
x^2 + 2x - 3 & x^3 + x^2 - 5x + 3 \\
 & x^3 + 2x^2 - 3x \\
\hline
 & - x^2 - 2x + 3 \\
 & - x^2 - 2x + 3 \\
\hline
 & 0
\end{array}
$$

∴ Remainder becomes 0.

∴ The corresponding quotient is the HCF

∴ HCF $= x^2 + 2x - 3$

ii) Let $f(x) = x^4 - 1$

$g(x) = x^3 - 11x^2 + x - 11$

$$
\begin{array}{r|l}
 & x + 11 \\
\hline
x^3 - 11x^2 + x - 11 & x^4 - 1 \\
 & x^4 - 11x^3 + x^2 - 11x \\
\hline
 & 11x^3 - x^2 + 11x - 1 \\
 & 11x^3 - 121x^2 + 11x - 121 \\
\hline
 & 120x^2 + 120
\end{array}
$$

$= 120(x^2 + 1) \neq 0$

Now, divide g(x) by $x^2 + 1$

$$\begin{array}{r|l} & x - 11 \\ \hline x^2 + 1 & x^3 - 11x^2 + x - 11 \\ & x^3 + \ \ 0x^2 + x \\ \hline & \quad -11x^2 - 11 \\ & \quad -11x^2 - 11 \\ \hline & \qquad 0 \end{array}$$

$\therefore$ HCF $= x^2 + 1$

iii) Let $f(x) = 4x^4 + 14x^3 + 8x^2 - 8x$

$= 2x\,(2x^3 + 7x^2 + 4x - 4)$

$g(x) = 3x^4 + 6x^3 - 12x^2 - 24x$

$= 3x\,(1x^3 + 2x^2 - 4x - 8)$

$$\begin{array}{r|l} & 2 \\ \hline x^3 + 2x^2 - 4x - 8 & 2x^3 + 7x^2 + 4x - \ 4 \\ & 2x^3 + 4x^2 - 8x - 16 \\ \hline & \quad 3x^2 + 12x + 12 \end{array}$$

$= 3\,(x^2 + 4x + 4) \neq 0$

$$\begin{array}{r|l} & x - 2 \\ \hline x^2 + 4x + 4 & x^3 + 2x^2 - 4x - 8 \\ & x^3 + 4x^2 + 4x \\ \hline & \quad -2x^2 - 8x - 8 \\ & \quad -2x^2 - 8x - 8 \\ \hline & \qquad 0 \end{array}$$

$\therefore$ G.C.D $= x\,(x^2 + 4x + 4)$

iv) Let $f(x) = 6x^3 + 12x^2 + 6x + 12$

$= 6\,(x^3 + 2x^2 + x + 2)$

$g(x) = 3x^3 + 3x^2 + 3x + 3$

$= 3\,(x^3 + x^2 + x + 1)$

GCD of 6, 3 = 3

$$\begin{array}{r|l} & 1 \\ \hline x^3 + x^2 + x + 1 & x^3 + 2x^2 + x + 2 \\ & x^3 + \ \ x^2 + x + 1 \\ \hline & \quad x^2 + 1 \end{array}$$

$= x^2 + 1 \neq 0$

$$\begin{array}{r|l} & x + 1 \\ \hline x^2 + 1 & x^3 + x^2 + x + 1 \\ & x^3 + 0 \ + x + 0 \\ \hline & \quad x^2 + 1 \\ & \quad x^2 + 1 \\ \hline & \qquad 0 \end{array}$$

$\therefore$ G.C.D $= 3\,(x^2 + 1)$

2. Find the LCM of the given expressions.

(i) $4x^2y$, $8x^3y^2$ ii) $-9a^3b^2$, $12a^2b^2c$,

iii) $16m$, $-12m^2n^2$, $8n^2$

iv) $p^2 - 3p + 2$, $p^2 - 4$

v) $2x^2 - 5x - 3$, $4x^2 - 36$

vi) $(2x^2 - 3xy)^2$, $(4x - 6y)^2$, $8x^3 - 27y^3$

Solution :

i) $4x^2y = \underline{2} \times \underline{2} \times x^2 \times y$

$8x^2y^2 = \underline{2} \times \underline{2} \times 2 \times x^3 \times y^2$

$\therefore$ LCM $= 2 \times 2 \times 2 \times x^3 \times y^2$

$= 8x^3y^2$

ii) $-9a^3b^2 = -\underline{3} \times 3 \times a^3 \times b^2$

$12a^2b^2c = \underline{3} \times 2 \times 2 \times a^2 \times b^2 \times c$

$\therefore$ LCM $= -3 \times 3 \times 2 \times 2 \times a^3 \times b^2 \times c$

$= -36\,a^3b^2c$

iii) $16m = \underline{2} \times \underline{2} \times (2) \times 2 \times m$

$-12m^2n^2 = -\underline{3} \times \underline{2} \times \underline{2} \times m^2 \times n^2$

$8n^2 = \underline{2} \times \underline{2} \times (2) \times n^2$

$\therefore$ LCM $= 2 \times 2 \times 2 \times 2 \times -3 \times m^2 \times n^2$

$= -48\,m^2n^2$

iv) $p^2 - 3p + 2 = (\underline{p-2})(p-1)$

$p^2 - 4 = (\underline{p-2})(p+2)$

$\therefore$ LCM $= (p-2)(p-1)(p+2)$

v) $2x^2 - 5x - 3 = (\underline{x-3})(2x+1)$

$4x^2 - 36 = (2x)^2 - (6)^2$

-5	-6
-6	$+1$
2	2

$-3, \quad \frac{1}{2}$

$= (2x+6)(2x-6)$

$= 2(x+3)\,2(x-3)$

$= 4(x+3)(x-3)$

$\therefore$ LCM $= 4(x-3)(x+3)(2x+1)$

vi) $(2x^2 - 3xy)^2 = (x(2x-3y))^2$

$= x^2 (2x-3y)^2$

$(4x - 6y)^3 = (2(2x-3y))^3$

$= 8(2x-3y)^3$

$8x^3 - 27y^3 = (2x)^3 - (3y)^3$

$= (2x-3y)(4x^2 + 6xy + 9y^2)$

$\therefore$ LCM $= (2x-3y)^3\, 8x^2 (4x^2 + 6xy + 9y^2)$

$= 8x^2 (2x-3y)^3 (4x^2 + 6xy + 9y^2)$

III. RELATIONSHIP BETWEEN LCM AND GCD:

Key Points

✓ The product of two polynomials is the product of their LCM and GCD. That is, $f(x) \times g(x) =$ LCM$[f(x), g(x)] \times$ GCD $[f(x), g(x)]$.

EXERCISE 3.3

1. Find the LCM and GCD for the following and verify that f(x) × g(x) = LCM × GCD i) $21x^2y, 35xy^2$ ii) $(x^3 - 1)(x + 1), (x^3 + 1)$ iii) $(x^2y + xy^2), (x^2 + xy)$

Solution :

i) Given $f(x) = 21x^2y = \underline{7} \times 3 \times x^2 \times y$

$g(x) = 35xy^2 = \underline{7} \times 5 \times x \times y^2$

$\therefore$ GCD $= 7 \times x \times y = 7xy$

$\therefore$ LCM $= 7 \times x^2 \times y^2 \times 15 = 105x^2y^2$

$\therefore f(x) \times g(x) = 21x^2y \times 35xy^2$

$= 735\, x^3y^3$

$\therefore$ LCM × GCD $= 105x^2y^2 \times 7xy$

$= 735\, x^3y^3$

$\therefore f(x) \times g(x) =$ LCM × GCD

ii) Given $f(x) = (x^3 - 1)(x + 1)$

$= (x-1)(x^2+x+1)\,\underline{(x+1)}$

$g(x) = x^3 + 1$

$= \underline{(x+1)}\,(x^2 - x + 1)$

$\therefore$ GCD $= x + 1$

$\therefore$ LCM $= (x+1)(x-1)(x^2+x+1)(x^2-x+1)$

$= (x^3+1)(x^3-1)$

$= x^6 - 1$

$\therefore f(x) \times g(x) = (x-1)(x^2+x+1)(x+1)(x+1)(x^2-x+1)$

$= (x^3-1)(x+1)(x^3+1)$

$= (x+1)(x^6-1)$

$\therefore$ LCM × GCD $= (x^6-1)(x+1)$

$= 735\, x^3y^3$

$\therefore f(x) \times g(x) =$ LCM × GCD

iii) $f(x) = x^2y + xy^2$

$= xy(x + y)$

$g(x) = x^2 + xy$

$= x(x + y)$

$\therefore$ GCD $= x(x + y)$

$\therefore$ LCM $= x(x + y)\, y$

$= (x^3 + 1)(x^3 - 1)$

$= x^6 - 1$

$\therefore f(x) \times g(x) = xy(x + y) \times x(x + y)$

$= x^2y(x + y)^2$

$\therefore$ LCM $\times$ GCD $= xy(x + y)\, x\, (x + y)$

$= x^2y(x + y)^2$

$\therefore f(x) \times g(x) =$ LCM $\times$ GCD

2. Find the LCM of each pair of the following polynomials

i) $a^2 + 4a - 12$, $a^2 - 5a + 6$ whose GCD is $a - 2$

ii) $x^4 - 27a^3x$, $(x - 3a)^2$ whose GCD is $(x - 3a)$

Solution :

i) Let $f(x) = a^2 + 4a - 12$

$= (a + 6)(a - 2)$

$g(x) = a^2 - 5a + 6$

$= (a - 3)(a - 2)$

GCD $= a - 2$

$$\therefore \text{LCM} = \frac{f(x) \times g(x)}{GCD}$$

$$= \frac{(a+6)(a-2) \times (a-3)(a-2)}{a-2}$$

$= (a + 6)(a - 3)(a - 2)$

ii) Let $f(x) = x^4 - 27a^3x$

$= x(x^3 - 27a^3)$

$= x(x^3 - (3a)^3)$

$= x[(x - 3a)(x^2 + 3ax + 9a^2)]$

$g(x) = (x - 3a)^2$

$= (x - 3a)(x - 3a)$

GCD $= x - 3a$

$$\therefore \text{LCM} = \frac{f(x) \times g(x)}{GCD}$$

$$= \frac{x(x-3a)(x^2+3ax+9a^2)(x-3a)^2}{x-3a}$$

$= x(x^2 + 3ax + 9a^2)(x - 3a)^2$

3. Find the GCD of each pair of the following polynomials

i) $12(x^4 - x^3)$, $8(x^4 - 3x^3 + 2x^2)$ whose LCM is $24x^3(x - 1)(x - 2)$

ii) $(x^3 + y^3)$, $(x^4 + x^2y^2 + y^4)$ whose LCM is $(x^3 + y^3)(x^2 + xy + y^2)$

Solution :

i) $f(x) = 12(x^4 - x^3)$

$= 12x^3(x - 1)$

$g(x) = 8(x^4 - 3x^3 + 2x^2)$

$= 8 \times x^2(x^2 - 3x + 2)$

$= 8 \times x^2(x - 2)(x - 1)$

LCM $= 24x^3(x - 1)(x - 2)$

$$\therefore \text{GCD} = \frac{f(x) \times g(x)}{LCM}$$

$$= \frac{12x^3(x-1)\, 8x^2(x-2)(x-1)}{24x^3(x-1)(x-2)}$$

$= 4x^2(x - 1)$

ii) $f(x) = (x^3 + y^3) = (x + y)(x^2 - xy + y^2)$

$g(x) = x^4 + x^2y^2 + y^4 = (x^2 - xy + y^2)(x^2 + xy + y^2)$

LCM $= (x^3 + y^3)(x^2 + xy + y^2)$

$= (x + y)(x^2 - xy + y^2)(x^2 + xy + y^2)$

$$\therefore \text{GCD} = \frac{f(x) \times g(x)}{LCM}$$

$$= \frac{(x+y)(x^2-xy+y^2) \times (x^2-xy+y^2)(x^2+xy+y^2)}{(x+y)(x^2-xy+y^2)(x^2+xy+y^2)}$$

$= x^2 - xy + y^2$

4. Given the LCM and GCD of the two polynomials p(x) and q(x) find the unknown polynomial in the following table

S. No.	LCM	GCD
i)	$a^3 - 10a^2 + 11a + 70$	$a - 7$
ii)	$(x^2 + y^2)(x^4 + x^2y^2 + y^4)$	$(x^2 - y^2)$

p(x)	q(x)
$a^2 - 12a + 35$	
	$(x^4 - y^4)(x^2 + y^2 - xy)$

Solution :

i) Given LCM = $a^3 - 10a^2 + 11a + 70$,

GCD = $a - 7$

$p(x) = a^2 - 12a + 35$, $q(x) = ?$

$$q(x) = \frac{\text{LCM} \times \text{GCD}}{p(x)} = \frac{(a^3 - 10a^2 + 11a + 70)(a-7)}{(a-7)(a-5)}$$

$$= \frac{(a-5)(a^5 - 5a - 14)(a-7)}{(a-7)(a-5)}$$

$$= a^2 - 5a - 14$$

$$q(x) = (a-7)(a+2)$$

ii) Given LCM = $(x^2 + y^2)(x^4 + x^2y^2 + y^4)$

GCD = $(x^2 - y^2)$

$q(x) = (x^4 - y^4)(x^2 + y^2 - xy)$ $p(x) = ?$

$$p(x) = \frac{\text{LCM} \times \text{GCD}}{q(x)}$$

$$= \frac{\cancel{(x^2+y^2)}(x^4 + x^2y^2 + y^4)\cancel{(x^2-y^2)}}{\cancel{(x^2+y^2)}\cancel{(x^2-y^2)}(x^2 + y^2 - xy)}$$

$$= \frac{(x^2 + y^2 - xy)(x^2 - xy + y^2)}{x^2 + y^2 - xy}$$

$$= x^2 + xy + y^2$$

IV. RATIONAL EXPRESSIONS AND EXCLUDED VALUES:

Key Points

- ✓ An expression is called a rational expression if it can be written in the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.
- ✓ A rational expression is the ratio of two polynomials.
- ✓ A rational expression $\frac{p(x)}{q(x)}$ is said to be in its lowest form if GCD $(p(x), q(x)) = 1$.
- ✓ A value that makes a rational expression (in its lowest form) undefined is called an excluded value.

Example 3.13

Reduce the rational expressions to its lowest form

$(i)\ \frac{x-3}{x^2-9}$ $\quad (ii)\ \frac{x^2-16}{x^2+8x+16}$

Solution :

$$(i)\ \frac{x-3}{x^2-9} = \frac{x-3}{(x+3)(x-3)} = \frac{1}{x+3}$$

$$(ii)\ \frac{x^2-16}{x^2+8x+16} = \frac{(x+4)(x-4)}{(x+4)^2} = \frac{x-4}{x+4}$$

Example 3.14

Find the excluded values of the following expressions (if any).

$i)\ \frac{x+10}{8x}$ $\quad (ii)\ \frac{7p+2}{8p^2+13p+5}$ $\quad (iii)\ \frac{x}{x^2+1}$

Solution :

$i)\ \frac{x+10}{8x}$

The expression $\frac{x+10}{8x}$ is undefined when $8x = 0$ or $x = 0$. Hence the excluded value is 0.

(ii) $\dfrac{7p+2}{8p^2+13p+5}$

The expression $\dfrac{7p+2}{8p^2+13p+5}$ is undefined when

$8p^2 + 13p + 5 = 0$ that is

$(8p + 5)(p + 1) = 0$

$p = \dfrac{-5}{8},\ p = -1.$

The excluded values are $\dfrac{-5}{8}$ and -1.

(iii) $\dfrac{x}{x^2+1}$

Here $x^2 \geq 0$ for all x.

Therefore, $x^2 + 1 \geq 0 + 1 = 1$. Hence, $x^2 + 1 \neq 0$ for any x.

Therefore, there can be no real excluded values for the given rational expression $\dfrac{x}{x^2+1}$

EXERCISE 3.4

1. Reduce each of the following rational expressions to its lowest form.

i) $\dfrac{x^2-1}{x^2+x}$ ii) $\dfrac{x^2-11x+18}{x^2-4x+4}$

iii) $\dfrac{9x^2+81x}{x^3+8x^2-9x}$ iv) $\dfrac{p^2-3p-40}{2p^3-24p^2+64p}$

Solution :

i) $\dfrac{x^2-1}{x^2+x}$

$= \dfrac{(x+1)(x-1)}{x(x+1)}$

$= \dfrac{x-1}{x}$

ii) $\dfrac{x^2-11x+18}{x^2-4x+4}$

$= \dfrac{(x-9)(x-2)}{(x-2)(x-2)}$

$= \dfrac{x-9}{x-2}$

iii) $\dfrac{9x^2+81x}{x^3+8x^2-9x}$

$= \dfrac{9x(x+9)}{x(x^2+8x-9)}$

$= \dfrac{9(x+9)}{(x+9)(x-1)}$

$= \dfrac{9}{x-1}$

iv) $\dfrac{p^2-3p-40}{2p^3-24p^2+64p}$

$= \dfrac{p^2-3p-40}{2p(p^2-12p+32)}$

$= \dfrac{(p-8)(p+5)}{2p(p-8)(p+4)}$

$= \dfrac{p+5}{2p(p-4)}$

2. Find the excluded values, if any of the following expressions.

i) $\dfrac{y}{y^2-25}$ ii) $\dfrac{t}{t^2-5t+6}$

iii) $\dfrac{x^2+6x+8}{x^2+x-2}$ iv) $\dfrac{x^3-27}{x^3+x^2-6x}$

Solution :

i) $\dfrac{y}{y^2-25} = \dfrac{y}{(y+5)(y-5)}$

The expression is undefined if $y = -5$, $y = 5$

$\therefore$ The excluded values are $5, -5$

ii) $\dfrac{t}{t^2-5t+6}$

$= \dfrac{t}{(t-3)(t-2)}$

The expression is undefined if t = 3, t = 2

$\therefore$ The excluded values are 3, 2

iii) $\dfrac{x^2+6x+8}{x^2+x-2}$

$= \dfrac{(x+4)(x+2)}{(x+2)(x-1)}$

$= \dfrac{x+4}{x-1}$

This expression is not defined if x = 1

$\therefore$ The excluded values is 1

iv) $\dfrac{x^3-27}{x^3+x^2-6x}$

$= \dfrac{x^3-3^3}{x(x^2+x-6)}$

$= \dfrac{(x-3)(x^2+3x+9)}{x(x+3)(x-2)}$

This expression is not defined for
x = 0, x = − 3, x = 2

$\therefore$ The excluded values are 0, − 3, 2

V. MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS:

Key Points

✓ If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are two rational expressions wehre $q(x) \neq 0$, $s(x) \neq 0$, their product is

$$\frac{p(x)}{q(x)} \times \frac{r(x)}{s(x)} = \frac{p(x)\times r(x)}{q(x)\times s(x)}$$

✓ If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are two rational expressions wehre $q(x), s(x) \neq 0$, then,

$$\frac{p(x)}{q(x)} \div \frac{r(x)}{s(x)} = \frac{p(x)}{q(x)} \times \frac{s(x)}{r(x)} = \frac{p(x)\times s(x)}{q(x)\times r(x)}$$

Example 3.15

i) Multiply $\dfrac{x^3}{9y^2}$ by $\dfrac{27y}{x^5}$ ii) Multiply $\dfrac{x^4b^2}{x-1}$ by $\dfrac{x^2-1}{a^4b^3}$

Solution :

i) $\dfrac{x^3}{9y^2} \times \dfrac{27y}{x^5} = \dfrac{3}{x^2y}$

ii) $\dfrac{x^4b^2}{x-1} \times \dfrac{x^2-1}{a^4b^3} = \dfrac{x^4\times b^2}{x-1} \times \dfrac{(x+1)(x-1)}{a^4\times b^3} = \dfrac{x^4(x+1)}{a^4b}$

Example 3.16

Find

i) $\dfrac{14x^4}{y} \div \dfrac{7x}{3y^4}$ ii) $\dfrac{x^2-16}{x+4} \div \dfrac{x-4}{x+4}$

iii) $\dfrac{16x^2-2x-3}{3x^2-2x-1} \div \dfrac{8x^2+11x+3}{3x^2-11x-4}$

Solution :

i) $\dfrac{14x^4}{y} \div \dfrac{7x}{3y^4} = \dfrac{14x^4}{y} \times \dfrac{3y^4}{7x} = 6x^3y^3$

ii) $\dfrac{x^2-16}{x+4} \div \dfrac{x-4}{x+4} = \dfrac{(x+4)(x-4)}{(x+4)} \times \left(\dfrac{x+4}{x-4}\right) = x+4$

iii) $\dfrac{16x^2-2x-3}{3x^2-2x-1} \div \dfrac{8x^2+11x+3}{3x^2-11x-4}$

$$= \frac{16x^2-2x-3}{3x^2-2x-1} \times \frac{3x^2-11x-4}{8x^2+11x+3}$$

$$= \frac{(8x+3)(2x-1)}{(3x+1)(x-1)} \times \frac{(3x+1)(x-4)}{(8x+3)(x+1)}$$

$$= \frac{(2x-1)(x-4)}{(x-1)(x+1)} = \frac{2x^2-9x+4}{x^2-1}$$

EXERCISE 3.5

1. Simplify

i) $\dfrac{4x^2y}{2z^2} \times \dfrac{6xz^3}{20y^4}$ ii) $\dfrac{p^2-10p+21}{p-7} \times \dfrac{p^2+p-12}{(p-3)^2}$

iii) $\dfrac{5t^3}{4t-8} \times \dfrac{6t-12}{10t}$

Solution :

i) $\dfrac{4x^2y}{2z^2} \times \dfrac{6xz^3}{20y^4}$

$$= \frac{\cancel{4}^{2}\, x^2\, \cancel{y}}{\cancel{2}\, \cancel{z^2}} \times \frac{\cancel{6}^{3}\, x z^{\cancel{3}^{1}}}{\cancel{20}_{\cancel{10}_{5}}\, y^{\cancel{4}^{3}}}$$

$$= \frac{3}{5}\frac{x^3z}{y^3}$$

ii) $\dfrac{p^2-10p+21}{p-7} \times \dfrac{p^2+p-12}{(p-3)^2}$

$$= \frac{\cancel{(p-7)}\ \cancel{(p-3)}}{\cancel{p-7}} \times \frac{(p+4)\ \cancel{(p-3)}}{\cancel{(p-3)^2}}$$

$$= p+4$$

iii) $\dfrac{5t^3}{4t-8} \times \dfrac{6t-12}{10t}$

$$= \frac{5t^3}{4(t-2)} \times \frac{6(t-2)}{10t}$$

$$= \frac{3t^2}{4}$$

2. Simplify

i) $\dfrac{x+4}{3x+4y} \times \dfrac{9x^2-16y^2}{2x^2+3x-20}$

ii) $\dfrac{x^3-y^3}{3x^2+9xy+6y^2} \times \dfrac{x^2+2xy+y^2}{x^2-y^2}$

Solution :

i) $\dfrac{x+4}{3x+4y} \times \dfrac{9x^2-16y^2}{2x^2+3x-20}$

$$= \frac{x+4}{3x+4y} \times \frac{(3x+4y)(3x-4y)}{(x+4)(2x-5)}$$

$$= \frac{3x-4y}{2x-5}$$

3	-40
8	-5
2	2
4,	$-5/2$

$$ii)\ \frac{x^3-y^3}{3x^2+9xy+6y^2}\times\frac{x^2+2xy+y^2}{x^2-y^2}$$

$$=\frac{\cancel{(x-y)}\,(x^2+xy+y^2)}{3(x^2+3xy+2y^2)}\times\frac{\cancel{(x+y)}\ \cancel{(x+y)}}{\cancel{(x+y)}\ \cancel{(x-y)}}$$

$$=\frac{(x^2+xy+y^2)\ \cancel{(x+y)}}{3(x+2y)\ \cancel{(x+y)}}$$

$$=\frac{x^2+xy+y^2}{3(x+2y)}$$

3. Simplify

$$i)\ \frac{2a^2+5a+3}{2a^2+7a+6}\div\frac{a^2+6a+5}{-5a^2-35a-50}$$

$$ii)\ \frac{b^2+3b-28}{b^2+4b+4}\div\frac{b^2-49}{b^2-5b-14}$$

$$iii)\ \frac{x+2}{4y}\div\frac{x^2-x-6}{12y^2}$$

$$iv)\ \frac{12t^2-22t+8}{3t}\div\frac{3t^2+2t-8}{2t^2+4t}$$

Solution :

$$i)\ \frac{2a^2+5a+3}{2a^2+7a+6}\div\frac{a^2+6a+5}{-5a^2-35a-50}$$

$$=\frac{(2a+3)\,(a+1)}{(2a+3)\,(a+2)}\times\frac{-5(a^2+7a+10)}{(a+5)\,(a+1)}$$

$$=\frac{\cancel{(2a+3)}\ \cancel{(a+1)}}{\cancel{(2a+3)}\ \cancel{(a+2)}}\times\frac{-5\,\cancel{(a+5)}\ \cancel{(a+2)}}{\cancel{(a+5)}\ \cancel{(a+1)}}$$

$$=-5$$

$$ii)\ \frac{b^2+3b-28}{b^2+4b+4}\div\frac{b^2-49}{b^2-5b-14}$$

$$=\frac{(b+7)\,(b-4)}{(b+2)\,(b+2)}\times\frac{(b-7)\,(b+2)}{(b+7)\,(b-7)}$$

$$=\frac{b-4}{b+2}$$

$$iii)\ \frac{x+2}{4y}\div\frac{x^2-x-6}{12y^2}$$

$$=\frac{x+2}{4y}\times\frac{12y^2}{(x-3)\,(x+2)}$$

$$=\frac{3y}{x-3}$$

$$iv)\ \frac{12t^2-22t+8}{3t}\div\frac{3t^2+2t-8}{2t^2+4t}$$

$$=\frac{2(6t^2-11t+4)}{3t}\times\frac{2t(t+2)}{3t^2+2t-8}$$

$$=\frac{2(3t-4)\,(2t-1)}{3t}\times\frac{2t\,(t+2)}{(3t-4)\,(t+3)}$$

$$=\frac{4(2t-1)}{3}$$

4. If x = $\frac{a^2+3a-4}{3a^2-3}$ and y = $\frac{a^2+2a-8}{2a^2-2a-4}$ find the value of $x^2\,y^{-2}$.

Solution :

Given

$$x=\frac{a^2+3a-4}{3a^2-3}\ ,\quad y=\frac{a^2+2a-8}{2a^2-2a-4}$$

$$=\frac{(a+4)\,(a-1)}{3(a+1)\,(a-1)}\quad =\frac{(a+4)\,(a-2)}{2(a-2)\,(a+1)}$$

$$=\frac{a+4}{3(a+1)}\quad =\frac{a+4}{2(a+1)}$$

$$\therefore x^2y^{-2}=\frac{x^2}{y^2}$$

$$=\frac{(a+4)^2}{9(a+1)^2}\times\frac{4(a+1)^2}{(a+4)^2}$$

$$=\frac{4}{9}$$

5. If a polynomial p(x) = x² − 5x − 14 is divided by another polynomial q(x) we get $\frac{x-7}{x+2}$ find q(x).

Solution :

Given

$$\frac{p(x)}{q(x)} = \frac{x-7}{x+2}$$

$$\Rightarrow \frac{x^2-5x-14}{q(x)} = \frac{x-7}{x+2}$$

$$\Rightarrow \frac{(x-7)(x+2)}{q(x)} = \frac{x-7}{x+2}$$

$$\Rightarrow q(x) = (x+2)^2$$

$\Rightarrow q(x) = x^2+4x+4$ is another polynomial

VI. ADDITION AND SUBTRACTION OF RATIONAL EXPRESSION:

Example 3.17

Find $\frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$

Solution :

$$= \frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$$

$$= \frac{(x^2+20x+36)-(x^2+12x+4)}{x^2-3x-28}$$

$$= \frac{8x+32}{x^2-3x-28} = \frac{8(x+4)}{(x-7)(x+4)} = \frac{8}{x-7}$$

Example 3.18

Simplify $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$

Solution :

$$= \frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$$

$$= \frac{1}{(x-2)(x-3)} + \frac{1}{(x-2)(x-1)} - \frac{1}{(x-5)(x-3)}$$

$$= \frac{(x-1)(x-5)+(x-3)(x-5)-(x-1)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x^2-6x+5)+(x^2-8x+15)-(x^2-3x+2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{x^2-11x+8}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x-9)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{x-9}{(x-1)(x-3)(x-5)}$$

EXERCISE 3.6

1. Simplify

i) $\frac{x(x+1)}{x-2} + \frac{x(1-x)}{x-2}$ *ii)* $\frac{x+2}{x+3} + \frac{x-1}{x-2}$

iii) $\frac{x^3}{x-y} + \frac{y^3}{y-x}$

Solution :

i) $\frac{x(x+1)}{x-2} + \frac{x(1-x)}{x-2}$

$$= \frac{x^2+x+x-x^2}{x-2}$$

$$= \frac{2x}{x-2}$$

ii) $\frac{x+2}{x+3} + \frac{x-1}{x-2}$

$$= \frac{(x-2)(x+2)+(x+3)(x-1)}{(x+3)(x-2)}$$

$$= \frac{(x^2-4)+(x^2+2x-3)}{(x+3)(x-2)}$$

$$= \frac{2x^2+2x-7}{(x+3)(x-2)}$$

$$iii)\ \frac{x^3}{x-y}+\frac{y^3}{y-x}$$

$$=\frac{x^3}{x-y}-\frac{y^3}{x-y}$$

$$=\frac{x^3-y^3}{x-y}$$

$$=\frac{(x-y)(x^2+xy+y^2)}{x-y}$$

$$=x^2+xy+y^2$$

2. Simplify

$$i)\ \frac{(2x+1)(x-2)}{x-4}-\frac{(2x^2-5x+2)}{x-4}$$

$$ii)\ \frac{4x}{x^2-1}-\frac{x+1}{x-1}$$

Solution :

$$i)\ \frac{(2x+1)(x-2)}{x-4}-\frac{(2x^2-5x+2)}{x-4}$$

$$=\frac{2x^2-3x+2-2x^2+5x-2}{x-4}$$

$$=\frac{2x-4}{x-4}=\frac{2(x-2)}{x-4}$$

$$ii)\ \frac{4x}{x^2-1}-\frac{x+1}{x-1}$$

$$=\frac{4x}{(x+1)(x-1)}-\frac{x+1}{x-1}$$

$$=\frac{4x-(x+1)^2}{(x+1)(x-1)}$$

$$=\frac{4x-(x^2+2x+1)}{(x+1)(x-1)}$$

$$=\frac{-x^2+2x-1}{(x+1)(x-1)}$$

$$=\frac{-(x^2-2x+1)}{(x+1)(x-1)}$$

$$=\frac{-(x-1)(x-1)}{(x+1)(x-1)}$$

$$=\frac{1-x}{1+x}$$

3. Subtract $\frac{1}{x^2+2}$ from $\frac{2x^3+x^2+3}{(x^2+2)^2}$

Solution :

$$=\frac{2x^3+x^2+3}{(x^2+2)^2}-\frac{1}{x^2+2}$$

$$=\frac{(2x^3+x^2+3)-(x^2+2)}{(x^2+2)^2}$$

$$=\frac{2x^3+x^2+3-x^2-2}{(x^2+2)^2}$$

$$=\frac{2x^3+1}{(x^2+2)^2}$$

4. Which rational expression should be subtracted from $\frac{x^2+6x+8}{x^3+8}$ to get $\frac{3}{x^2-2x+4}$.

Solution :

$$\frac{x^2+6x+8}{x^3+8}-\frac{3}{x^2-2x+4}$$

$$=\frac{(x+4)\cancel{(x+2)}}{\cancel{(x+2)}(x^2-2x+4)}-\frac{3}{x^2-2x+4}$$

$$=\frac{x+4-3}{x^2-2x+4}=\frac{x+1}{x^2-2x+4}$$

5. If $A = \dfrac{2x+1}{2x-1}, B = \dfrac{2x-1}{2x+1}$ find

$$\frac{1}{A-B} - \frac{2B}{A^2-B^2}$$

Solution :

$$= \frac{1}{A-B} - \frac{2B}{A^2-B^2}$$

$$= \frac{1}{A-B} - \frac{2B}{(A+B)(A-B)}$$

$$= \frac{A+B-2B}{(A+B)(A-B)}$$

$$= \frac{A-B}{(A+B)(A-B)}$$

$$= \frac{1}{A+B}$$

$$= \frac{1}{\frac{2x+1}{2x-1} + \frac{2x-1}{2x+1}}$$

$$= \frac{1}{\frac{(2x+1)^2 + (2x-1)^2}{4x^2-1}}$$

$$= \frac{4x^2-1}{2[4x^2+1]} \qquad \left(\because (a+b)^2 + (a-b)^2 = 2(a^2+b^2)\right)$$

6. If $A = \dfrac{x}{x+1}, B = \dfrac{1}{x+1}$. prove that

$$\frac{(A+B)^2 + (A-B)^2}{A \div B} = \frac{2(x^2+1)}{x(x+1)^2}$$

Solution :

Given

$$A = \frac{x}{x+1}, B = \frac{1}{x+1}$$

$$= \frac{(A+B)^2 + (A-B)^2}{A/B}$$

$$= 2(A^2+B^2) \times \frac{B}{A}$$

$$= 2\left[\frac{x^2}{(x+1)^2} + \frac{1}{(x+1)^2}\right] \times \frac{1/x+1}{x/x+1}$$

$$= \frac{2(1+x^2)}{(x+1)^2} \times \frac{1}{x}$$

$$= \frac{2(x^2+1)}{x(x+1)^2}$$

= RHS

Hence Proved

7. Pari needs 4 hours to complete a work. His friend Yuvan needs 6 hours to complete the same work. How long will it take to complete if they work together?

Solution :

Time taken by Pari to Complete a work = 4 hrs.

$\therefore$ In 1 hour, he completes $1/4$ part of work

Time taken by Yuvan to complete the same work= 6 hrs.

$\therefore$ In 1 hour,he completes $1/6$ part of work

$\therefore$ Total work completion by both in 1 hr

$$= 1/4 + 1/6$$

$$= \frac{3+2}{12}$$

$$= \frac{5}{12} \text{ part of work}$$

$\therefore$ Time taken by both together to complete the work = $12/5$ hrs

$$= 2\,2/5 \text{ hrs}$$

= 2 hrs 24 min.

8. Iniya bought 50 kg of fruits consisting of apples and bananas. She paid twice as much per kg for the apple as she did for the banana. If Iniya bought ₹ 1800 worth of apples and ₹ 600 worth bananas, then how many kgs of each fruit did she buy?

Solution :

Let x be the weight and p be the price of an apple.

Let y be the weight and q be the price of a banana

$\therefore$ By data given, $x + y = 50$ — (1)

$px = 1800$ — (2)

$qy = 600$ — (3)

Also, $p = 2q$.

$\therefore (2) \Rightarrow 2qx = 1800$

$\Rightarrow q = \dfrac{900}{x}$

$(3) \Rightarrow \dfrac{900}{x} y = 600$

$\Rightarrow y = \dfrac{2x}{3}$

$(1) \Rightarrow x + \dfrac{2x}{3} = 50 \Rightarrow 5x = 150$

$\Rightarrow x = 30$

$y = 50 - 30 = 20.$

$\therefore$ She bought 30 kg of apple &20 kg of banana.

VII. SQUARE ROOT OF POLYNOMIALS BY FACTORISATION :

Key Points

- ✓ The square root of a given positive real number is another number which when multiplied with itself is the given number
- ✓ $|q(x)| = \sqrt{p(x)}$ where $|q(x)|$ is the absolute value of $q(x)$.

Example 3.19

Find the square root of the following expressions

(i) $256(x-a)^8(x-b)^4(x-c)(x-d)^{20}$ ii) $\dfrac{144\,a^8\,b^{12}\,c^{16}}{81\,f^{12}\,g^4\,h^{14}}$

Solution :

$$i)\ \sqrt{256\,(x-a)^8\,(x-b)^4\,(x-c)^{16}\,(x-d)^{20}} = 16\left|\,(x-a)^4\,(x-b)^2\,(x-c)^8\,(x-d)^{10}\,\right|$$

$$ii)\ \sqrt{\frac{144\,a^8\,b^{12}\,c^{16}}{81\,f^{12}\,g^4\,h^{14}}} = \frac{4}{3}\left|\frac{a^4\,b^6\,c^8}{f^6\,g^2\,h^7}\right|$$

Example 3.20

Find the square root of the following expressions

(i) $16x^2 + 9y^2 - 24xy + 24x - 18y + 9$ (ii) $(6x^2 + x - 1)(3x^2 + 2x - 1)(2x^2 + 3x + 1)$

$$iii)\ \left[\sqrt{15}x^2 + \left(\sqrt{3}+\sqrt{10}\right)x + \sqrt{2}\right]\left[\sqrt{5}x^2 + (2\sqrt{5}+1)\,x + 2\right]\left[\sqrt{3}x^2 + (\sqrt{2}+2\sqrt{3})x + 2\sqrt{2}\right]$$

Solution :

i) $\sqrt{16x^2+9y^2-24xy+24x-18y+9}$

$$=\sqrt{(4x)^2+(-3y)^2+(3)^2+2(4x)(-3y)+2(-3y)(3)+2(4x)(3)}$$

$$=\sqrt{(4x-3y+3)^2}=|4x-3y+3|$$

ii) $\sqrt{(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)}$

$$=\sqrt{(3x-1)(2x+1)(3x-1)(x+1)(2x+1)(x+1)}=|(3x-1)(2x+1)(x+1)|$$

iii) First let us factorize the polynomials

$$\begin{aligned}\sqrt{15}x^2+\left(\sqrt{3}+\sqrt{10}\right)x+\sqrt{2} &= \sqrt{15}x^2+\sqrt{3}x+\sqrt{10}x+\sqrt{2}\\ &= \sqrt{3}x\left(\sqrt{5}x+1\right)+\sqrt{2}\left(\sqrt{5}x+1\right)\\ &= \left(\sqrt{5}x+1\right)\times\left(\sqrt{3}x+\sqrt{2}\right)\\ \sqrt{5}x^2+\left(2\sqrt{5}+1\right)x+2 &= \sqrt{5}x^2+2\sqrt{5}x+x+2\\ &= \sqrt{5}x(x+2)+1(x+2)=\left(\sqrt{5}x+1\right)(x+2)\\ \sqrt{3}x^2+\left(\sqrt{2}+2\sqrt{3}\right)x+2\sqrt{2} &= \sqrt{3}x^2+\sqrt{2}x+2\sqrt{3}x+2\sqrt{2}\\ &= x\left(\sqrt{3}x+\sqrt{2}\right)+2\left(\sqrt{3}x+\sqrt{2}\right)=(x+2)\left(\sqrt{3}x+\sqrt{2}\right)\end{aligned}$$

Therefore,

$$\sqrt{\left[\sqrt{15}x^2+\left(\sqrt{3}+\sqrt{10}\right)x+\sqrt{2}\right]\left[\sqrt{5}x^2+\left(2\sqrt{5}+1\right)x+2\right]\left[\sqrt{3}x^2+\left(\sqrt{2}+2\sqrt{3}\right)x+2\sqrt{2}\right]}$$

$$=\sqrt{\left(\sqrt{5}x+1\right)\left(\sqrt{3}x+\sqrt{2}\right)\left(\sqrt{5}x+1\right)(x+2)\left(\sqrt{3}x+\sqrt{2}\right)(x+2)}=\left|\left(\sqrt{5}x+1\right)\left(\sqrt{3}x+\sqrt{2}\right)(x+2)\right|$$

EXERCISE 3.7

1. Find the square root of the following rational expressions.

i) $\dfrac{400\,x^4y^{12}z^{16}}{100\,x^8y^4z^4}$ *ii)* $\dfrac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}$

iii) $\dfrac{121(a+b)^2\,(x+y)^8\,(b-c)^8}{81(b-c)^4\,(a-b)^{12}\,(b-c)^4}$

Solution :

i) $\sqrt{\dfrac{400\,x^4y^{12}z^{16}}{100\,x^8y^4z^4}}$

$$=\frac{20}{10}\left|\frac{x^2y^6z^8}{x^4y^2z^2}\right|$$

$$=2\left|\frac{y^4z^6}{x^2}\right|$$

$$ii)\ \sqrt{\frac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}}$$

$$=\sqrt{\frac{\left(\sqrt{7}x+\sqrt{2}\right)^2}{\left(x-\frac{1}{4}\right)^2}}$$

$$=\left|\frac{\sqrt{7}x+\sqrt{2}}{x-\frac{1}{4}}\right|$$

$$=4\left|\frac{\sqrt{7}x+\sqrt{2}}{4x-1}\right|$$

$$iii)\ \sqrt{\frac{121(a+b)^8\,(x+y)^8\,(b-c)^8}{81\,(b-c)^4\,(a-b)^{12}\,(b-c)^4}}$$

$$=\frac{11}{9}\left|\frac{(a+b)^4\,(x+y)^4\,(b-c)^4}{(b-c)^2\,(a-b)^6\,(b-c)^2}\right|$$

$$=\frac{11}{9}\left|\frac{(a+b)^4\,(x+y)^4}{(a-b)^6}\right|$$

2. Find the square root of the following

i) $4x^2 + 20x + 25$

ii) $9x^2 - 24xy + 30xz - 40yz + 25z^2 + 16y^2$

iii) $1+\frac{1}{x^6}+\frac{2}{x^3}$

iv) $(4x^2-9x+2)(7x^2-13x-2)(28x^2-3x-1)$

v) $\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)$

Solution :

$$i)\ \sqrt{4x^2+20x+25}$$

$$=\sqrt{(2x+5)^2}$$

$$=|2x+5|$$

$$ii)\ \sqrt{9x^2-24xy+30xz-40yz+25z^2+16y^2}$$

$$=\sqrt{(3x-4y+5z)^2}$$

\

$$=|3x-4y+5z|$$

$$iii)\ \sqrt{1+\frac{1}{x^6}+\frac{2}{x^3}}$$

$$=\sqrt{\left(1+\frac{1}{x^3}\right)^2}$$

$$=\left|1+\frac{1}{x^3}\right|$$

$$iv)\ \sqrt{(4x^2-9x+2)(7x^2-13x-2)(28x^2-3x-1)}$$

$$=\sqrt{(4x-1)(x-2)(7x+1)(x-2).(7x+1)(4x-1)}$$

(by factorisation)

$$=\sqrt{(7x+1)^2\,(4x-1)^2\,(x-2)^2}$$

$$=|(7x+1)(4x-1)(x-2)|$$

$$v)\ \sqrt{(2x^2+\tfrac{17}{6}x+1),(\tfrac{3}{2}x^2+4x+2),(\tfrac{4}{3}x^2+\tfrac{11}{3}x+2)}$$

$$=\sqrt{\frac{(12x^2+17x+6)}{6}.\frac{(3x^2+8x+4)}{2}.\frac{(4x^2+11x+6)}{3}}$$

$$=\sqrt{\frac{(4x+3)(3x+2).(3x+2)(x+2).(4x+3)(x+2)}{36}}$$

$$=\tfrac{1}{6}\sqrt{(4x+3)^2.(3x+2)^2.(x+2)^2}$$

$$=\tfrac{1}{6}\,|(4x+3)(3x+2)(x+2)|$$

VIII. SQUARE ROOT OF POLYNOMIALS BY DIVISION METHOD:

Key Points

- ✓ The long division method in finding the square root of a polynomial is useful when the degree of the polynomial is higher.
- ✓ Before proceeding to find the square root of a polynomial, one has to ensure that the degrees of the variables are in descending or ascending order.

Example 3.21

Find the square root of $64x^4 - 16x^3 + 17x^2 - 2x + 1$

Solution :

$$\begin{array}{r|l r}
 & 8x^2 - x + 1 & \\
\hline
8x^2 & 64x^4 - 16x^3 + 17x^2 - 2x + 1 & \\
 & 64x^4 & (-) \\
\hline
16x^2 - x & -16x^3 + 17x^2 & \\
 & -16x^3 + x^2 & (-) \\
\hline
16x^2 - 2x + 1 & 16x^2 - 2x + 1 & \\
 & 16x^2 - 2x + 1 & (-) \\
\hline
 & 0 & \\
\hline
\end{array}$$

Therefore

$$\sqrt{64x^4 - 16x^3 + 17x^2 - 2x + 1} = |8x^2 - x + 1|$$

Example 3.22

Find the square root of the expression $\frac{4x^2}{y^2} + \frac{20x}{y} + 13 - \frac{30y}{x} + \frac{9y^2}{x^2}$

Solution :

$$\begin{array}{r|l r}
 & \frac{2x}{y} + 5 - \frac{3y}{x} & \\
\hline
\frac{2x}{y} & \frac{4x^2}{y^2} + \frac{20x}{y} + 13 - \frac{30y}{x} + \frac{9y^2}{x^2} & \\
 & \frac{4x^2}{y^2} & (-) \\
\hline
\frac{4x}{y} + 5 & \frac{20x}{y} + 13 & \\
 & \frac{20x}{y} + 25 & (-) \\
\hline
\frac{4x}{y} + 10 - \frac{3y}{x} & -12 - \frac{30y}{x} + \frac{9y^2}{x^2} & \\
 & -12 - \frac{30y}{x} + \frac{9y^2}{x^2} & (-) \\
\hline
 & 0 & \\
\hline
\end{array}$$

Hence,

$$\sqrt{\frac{4x^2}{y} + \frac{20x}{y} + 13 - \frac{30y}{x} + \frac{9y^2}{x^2}} = \left|\frac{2x}{y} + 5 - \frac{3y}{x}\right|$$

Example 3.23

If $9x^4 + 12x^3 + 28x^2 + ax + b$ is a perfect square, find the values of a and b.

Solution :

$$
\begin{array}{r|ll}
 & 3x^2 + 2x + 4 & \\
\hline
3x^2 & 9x^4 + 12x^3 + 28x^2 + ax + b & \\
 & 9x^4 & (-) \\
\hline
6x^2 + 2x & 12x^3 + 28x^2 & \\
 & 12x^3 + 4x^2 & (-) \\
\hline
6x^2 + 4x + 4 & 24x^2 + ax + b & \\
 & 24x^2 + 16x + 16 & (-) \\
\hline
 & 0 &
\end{array}
$$

Because the given polynomial is a perfect square $a - 16 = 0$, $b - 16 = 0$

Therefore $a = 16$, $b = 16$.

EXERCISE 3.8

1. Find the square root of the following polynomials by division method

(i) $x^4 - 12x^3 + 42x^2 - 36x + 9$

(ii) $37x^2 - 28x^3 + 4x^4 + 42x + 9$

(iii) $16x^4 + 8x^2 + 1$

(iv) $121x^4 - 198x^3 - 183x^2 + 216x + 144$

Solution :

(i) $x^4 - 12x^3 + 42x^2 - 36x + 9$

$$
\begin{array}{r|l}
 & x^2 - 6x + 3 \\
\hline
x^2 & x^4 - 12x^3 + 42x^2 - 36x + 9 \\
 & x^4 \\
\hline
2x^2 - 6x & -12x^3 + 42x^2 \\
 & -12x^3 + 36x^2 \\
\hline
2x^2 - 12x + 3 & 6x^2 - 36x + 9 \\
 & 6x^2 - 36x + 9 \\
\hline
 & 0
\end{array}
$$

$\therefore \sqrt{x^4 - 12x^3 + 42x^2 - 36x + 9} = \left| x^2 - 6x + 3 \right|$

(ii) $37x^2 - 28x^3 + 4x^4 + 42x + 9$

$$
\begin{array}{r|l}
 & 2x^2 - 7x - 3 \\
\hline
2x^2 & 4x^4 - 28x^3 + 37x^2 + 42x + 9 \\
 & 4x^4 \\
\hline
4x^2 - 7x & -28x^3 + 37x^2 \\
 & -28x^3 + 49x^2 \\
\hline
4x^2 - 14x - 3 & -12x^2 + 42x + 9 \\
 & -12x^2 + 42x + 9 \\
\hline
 & 0
\end{array}
$$

$\therefore \sqrt{4x^4 - 28x^3 + 37x^2 + 42x + 9} = \left| 2x^2 - 7x - 3 \right|$

(iii) $16x^4 + 8x^2 + 1$

$$
\begin{array}{r|l}
 & 4x^2 + 1 \\
\hline
4x^2 & 16x^4 + 0x^3 + 8x^2 + 0x + 1 \\
 & 16x^4 \\
\hline
8x^2 + 1 & 8x^2 + 1 \\
 & 8x^2 + 1 \\
\hline
 & 0
\end{array}
$$

$\therefore \sqrt{16x^4 + 8x^2 + 1} = \left| 4x^2 + 1 \right|$

(iv) $121x^4 - 198x^3 - 183x^2 + 216x + 144$

$$
\begin{array}{r|l}
 & 11x^2 - 9x - 12 \\
\hline
11x^2 & 121x^4 - 198x^3 - 183x^2 + 216x + 144 \\
 & 121x^4 \\
\hline
22x^2 - 9x & -198x^3 - 183x^2 \\
 & -198x^3 + 81x^2 \\
\hline
22x^2 - 18x - 12 & -264x^2 + 216x + 144 \\
 & -264x^2 + 216x + 144 \\
\hline
 & 0
\end{array}
$$

$\therefore \sqrt{121x^4 - 198x^3 - 183x^2 + 216x + 144}$

$= \left| 11x^2 - 9x - 12 \right|$

2. Find the square root of the expression $\frac{x^2}{y^2} - \frac{10x}{y} + 27 - \frac{10y}{x} + \frac{y^2}{x^2}$

Solution :

$\frac{x}{y} - 5 + \frac{y}{x}$

$\frac{x}{y}$ | $\frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2}$

$\frac{x^2}{y^2}$

$\frac{2x}{y} - 5$ | $-10\frac{x}{y} + 27$

$-10\frac{x}{y} + 25$

$\frac{2x}{y} - 10 + \frac{y}{x}$ | $2 - 10\frac{y}{x} + \frac{y^2}{x^2}$

$2 - 10\frac{y}{x} + \frac{y^2}{x^2}$

0

$\therefore \sqrt{\frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2}} = \left| \frac{x}{y} - 5 + \frac{y}{x} \right|$

3. Find the values of a and b if the following polynomials are perfect squares

(i) $4x^4 - 12x^3 + 37x^2 + bx + a$ (ii) $ax^4 + bx^3 + 361x^2 + 220x + 100$

Solution :

$2x^2 - 3x + 7$

$2x^2$ | $4x^4 - 12x^3 + 37x^2 + bx + a$

$4x^4$

$4x^2 - 3x$ | $-12x^3 + 37x^2$

$-12x^3 + 9x^2$

$4x^2 - 6x + 7$ | $28x^2 + bx + a$

$28x^2 - 42x + 49$

(+) (–)

0

(The given polynomial is a perfect square)

$\therefore$ $a - 49 = 0$, $b + 42 = 0$

$a = 49$ $b = -42$

(ii) $ax^4 + bx^3 + 361x^2 + 220x + 100$

Solution :

$10 + 11x + 12x^2$

10 | $100 + 220x + 361x^2 + 6x^3 + ax^4$

100

$20 + 11x$ | $220x + 361x^2$

$220x + 121x^2$

$20 + 22x + 12x^2$ | $240x^2 + bx^3 + ax^4$

$240x^2 + 264x^3 + 144x^4$

(–) (–)

0

(The given polynomial is a perfect square)

$\therefore$ $a - 144 = 0$, $b - 264 = 0$

$a = 144$ $b = 264$

4. Find the values of m and n if the following expressions are perfect squares

i) $\frac{1}{x^4} - \frac{6}{x^3} + \frac{13}{x^2} + \frac{m}{x} + n$

ii) $x^4 - 8x^3 + mx^2 + nx + 16$

Solution :

$\frac{1}{x^2} - \frac{3}{x} + 2$

$\frac{1}{x^2}$ | $\frac{1}{x^4} - \frac{6}{x^3} + \frac{13}{x^2} + \frac{m}{x} + n$

$\frac{1}{x^4}$

$\frac{2}{x^2}-\frac{3}{x}$	$\frac{-6}{x^3}+\frac{13}{x^2}$ $\frac{-6}{x^3}+\frac{a}{x^2}$
$\frac{2}{x^2}-\frac{6}{x}+2$	$\frac{4}{x^2}+\frac{m}{x}+n$ $\frac{4}{x^2}-\frac{12}{x}+4$ (+) (−)
	0

(The given polynomial is a perfect square)

$\therefore$ m + 12 = 0, n – 4 = 0

m = – 12 n = 4

ii) $x^4 - 8x^3 + mx^2 + nx + 16$

	$x^2 - 4x + 4$
x^2	$x^4 - 8x^3 + mx^2 + nx + 16$ x^4
$2x^2 - 4x$	$-8x^3 + mx^2$ $-8x^3 + 16x^2$
$2x^2 - 8x + 4$	$(m-16)x^2 + nx + 16$ $8x^2 - 32x + 16$ (–) (+) (–)
	0

(The given polynomial is a perfect square)

$\therefore$ m – 16 – 8 = 0, n + 32 = 0

m = 24 n = – 32

IX. QUADRATIC EQUATIONS, ZEROS AND ROOTS :

Key Points

- ✓ A quadratic expression is an expression of degree n in variable x is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots. + a_{n-1}x + a_n$ where $a_0 \neq 0$ and $a_1, a_2, \ldots..a_n$ are real numbers, $a_0, a_1, a_2, \ldots\ldots a_n$ are called coefficients of the expression.
- ✓ An expression of degree 2 is called a Quadratic Expression which is expressed as $p(x) = ax^2 + bx + c$, $a \neq 0$ and a, b, c are real numbers.
- ✓ Let $p(x)$ be a polynomial. $x = a$ is called zero of $p(x)$ if $p(a) = 0$.
- ✓ Let $ax^2 + bx + c = 0$, $(a \neq 0)$ be a quadratic equation. The values of x such that the expression $ax^2 + bx + c$ becomes zero are called roots of the quadratic equation $ax^2 + bx + c = 0$.
- ✓ If α and β are roots of a quadratic equation $ax^2 + bx + c = 0$ then

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

- ✓ $\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = \frac{-b}{a}$
- ✓ $\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) \times \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{c}{a}$
- ✓ x^2 - (sum of roots) x + product of roots = 0 is the general form of the quadratic equation when the roots are given.

Example 3.24

Find the zeroes of the quadratic expression $x^2 + 8x + 12$.

Solution :

Let $p(x) = x^2 + 8x + 12 = (x + 2)(x + 6)$

$p(-2) = 4 - 16 + 12 = 0$

$p(-6) = 36 - 48 + 12 = 0$

Therefore -2 and -6 are zeros of $p(x)=x^2+8x+12$

Example 3.25

Write down the quadratic equation in general form for which sum and product of the roots are given below.

$i)\ 9, 14 \qquad ii)\ -\frac{7}{2}, \frac{5}{2} \qquad iii)\ -\frac{3}{5}, -\frac{1}{2}$

Solution :

(i) General form of the quadratic equation when the roots are given is

x^2 – (sum of the roots) x + product of the roots $= 0$

$x^2 - 9x + 14 = 0$

$ii)\ x^2 - \left(-\frac{7}{2}\right)x, \frac{5}{2} = 0$ gives $2x^2 + 7x + 5 = 0$

$iii)\ x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{1}{2}\right) = 0 \Rightarrow \frac{10x^2 + 6x - 5}{10} = 0$

Therefore $10x^2 + 6x - 5 = 0$

Example 3.26

Find the sum and product of the roots for each of the following quadratic equations :

(i) $x^2 + 8x - 65 = 0$ (ii) $2x^2 + 5x + 7 = 0$ (iii) $kx^2 - k^2x - 2k^3 = 0$

Solution :

Let α and β be the roots of the given quadratic equation

(i) $x^2 + 8x - 65 = 0$

$a = 1, b = 8, c = -65$

$$\alpha + \beta = -\frac{b}{a} = -8 \text{ and } \alpha\beta = \frac{c}{a} = -65$$

$$\alpha + \beta = -8 \ ; \ \alpha\beta = -65$$

(ii) $2x^2 + 5x + 7 = 0$

$a = 2, b = 5, c = 7$

$$\alpha + \beta = -\frac{b}{a} = \frac{-5}{2} \text{ and } \alpha\beta = \frac{c}{a} = \frac{7}{2}$$

$$\alpha + \beta = -\frac{5}{2} \ ; \ \alpha\beta = \frac{7}{2}$$

(iii) $kx^2 - k^2x - 2k^3 = 0$

$a = k, b = -k^2, c = -2k^3$

$$\alpha + \beta = -\frac{b}{a} = \frac{-(-k^2)}{k}$$

$$\text{and } \alpha\beta = \frac{c}{a} = \frac{-2k^3}{k} = -2k^2$$

EXERCISE 3.9

1. Determine the quadratic equations, whose sum and product of roots are

$i)\ -9, 20 \qquad ii)\ \frac{5}{3}, 4$

$iii)\ \frac{-3}{2}, -1 \qquad iv)\ -(2-a)^2, (a+5)^2$

Solution :

i) Given

Sum of the roots, SOR = -9

Product of the roots, POR = 20

The required quadratic equation is

x^2 – (SOR) x + (POR) $= 0$

$\Rightarrow \quad x^2 - (-9)x + 20 = 0$

$\Rightarrow \quad x^2 + 9x + 20 = 0$

ii) Given SOR = $\frac{5}{3}$, POR = 4

$\therefore$ The required quadratic equation is

$$x^2 - \frac{5}{3}x + 4 = 0$$

$$\Rightarrow \quad 3x^2 - 5x + 12 = 0$$

iii) Given SOR = $\frac{-3}{2}$, POR = –1

$\therefore$ The required equation is

$$x^2 - \left(\frac{-3}{2}\right)x + (-1) = 0$$

$$\Rightarrow \quad 2x^2 + 3x - 2 = 0$$

iv) Given SOR = $-(2-a)^2$, POR = $(a+5)^2$

$\therefore$ The required equation is

$$x^2 - (-(2-a)^2)x + (a+5)^2 = 0$$

$$\Rightarrow \quad x^2 + (2-a)^2x + (a+5)^2 = 0$$

2. Find the sum and product of the roots for each of the following quadratic equations

(i) $x^2 + 3x - 28 = 0$ (ii) $x^2 + 3x = 0$

iii) $3 + \frac{1}{a} = \frac{10}{a^2}$ (iv) $3y^2 - y - 4 = 0$

Solution :

i) $x^2 + 3x - 28 = 0$

Given equation is $x^2 + 3x - 28 = 0$

$a = 1, b = 3, c = -28$

$\therefore$ Sum of the roots = $\alpha + \beta = \frac{-}{\ } = \frac{-}{\ } = -$

Product of the roots = $\alpha\beta = \frac{c}{a} = \frac{-28}{1} = -28$

ii) $x^2 + 3x = 0$

$a = 1, b = 3, c = 0$

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-3}{1} = -3$$

$$\alpha\beta = \frac{c}{a} = \frac{0}{1} = 0$$

iii) Given $3 + \frac{1}{a} = \frac{10}{a^2}$

$$\Rightarrow \quad \frac{3a+1}{a} = \frac{10}{a^2}$$

$$\Rightarrow \quad 3a + 1 = \frac{10}{a}$$

$$\Rightarrow \quad 3a^2 + a - 10 = 0$$

A = 3, B = 1, C = –10

$$\therefore \alpha + \beta = \frac{-B}{A} = \frac{-1}{3}$$

$$\alpha\beta = \frac{C}{A} = \frac{-10}{3}$$

(iv) $3y^2 - y - 4 = 0$

$a = 3, b = -1, c = -4$

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{1}{3}$$

$$\alpha\beta = \frac{c}{a} = \frac{-4}{3}$$

X. SOLVING QUADRATIC EQUATIONS BY FACTORISATION :

Example 3.27

Solve $2x^2 - 2\sqrt{6}x + 3 = 0$

Solution :

$$2x^2 - 2\sqrt{6}x + 3 = 2x^2 - \sqrt{6}x - \sqrt{6}x + 3$$

(by spliting the middle term)

$$= \sqrt{2}x\left(\sqrt{2}x - \sqrt{3}\right) - \sqrt{3}\left(\sqrt{2}x - \sqrt{3}\right)$$

$$= \left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x - \sqrt{3}\right)$$

Now, equating the factors to zero we get,

$$\left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x - \sqrt{3}\right) = 0$$

$$\sqrt{2}x - \sqrt{3} = 0 \quad \text{or} \quad \sqrt{2}x - \sqrt{3} = 0$$

$$\sqrt{2}x = \sqrt{3} \quad \text{or} \quad \sqrt{2}x = \sqrt{3}$$

Therefore the solution is x = $\dfrac{\sqrt{3}}{\sqrt{2}}$

Example 3.28

Solve $2m^2 + 19m + 30 = 0$

Solution :

$2m^2 + 19m + 30 = 2m^2 + 4m + 15m + 30$

$= 2m(m + 2) + 15(m + 2)$

$= (m + 2)(2m + 15)$

Now, equating the factors to zero we get,

$(m + 2)(2m + 15) = 0$

$m + 2 = 0$ gives, $m = -2$ or $2m + 15 = 0$

we get, $m = \dfrac{-15}{2}$

Therefore the roots are -2, $\dfrac{-15}{2}$

Some equations which are not quadratic can be solved by reducing them to quadratic equations by suitable substitutions. Such examples are illustrated below.

Example 3.29

Solve $x^4 - 13x^2 + 42 = 0$

Solution :

Let $x^2 = a$. Then, $(x^2)^2 - 13x^2 + 42 = a^2 - 13a + 42 = (a - 7)(a - 6)$

Given, $(a - 7)(a - 6) = 0$ we get, $a = 7$ or 6.

Since $a = x^2$, $x^2 = 7$ then, $x = \pm\sqrt{7}$ or $x^2 = 6$ we get, $x = \pm\sqrt{6}$

Therefore the roots are $x = \pm\sqrt{7}$, $\pm\sqrt{6}$

Example 3.30

Solve $\dfrac{x}{x-1} + \dfrac{x-1}{x} = 2\dfrac{1}{2}$

Solution :

Let $y = \dfrac{x}{x-1}$ then $\dfrac{1}{y} = \dfrac{x-1}{x}$

Therefore, $\dfrac{x}{x-1} + \dfrac{x-1}{x} = 2\dfrac{1}{2}$ becomes $y + \dfrac{1}{y} = \dfrac{5}{2}$

$$2y^2 - 5y + 2 = 0 \text{ then, } y = \frac{1}{2},\ 2$$

$\dfrac{x}{x-1} = \dfrac{1}{2}$ we get, $2x = x - 1$ implies $x = -1$

$\dfrac{x}{x-1} = 2$ we get, $x = 2x - 2$ implies $x = 2$

Therefore the roots are $x = -1, 2$

EXERCISE 3.10

1. Solve the following quadratic equations by factorization method

i) $4x^2 - 7x - 2 = 0$

ii) $3(p^2 - 6) = p(p + 5)$

iii) $\sqrt{a(a-7)} = 3\sqrt{2}$

iv) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

v) $2x^2 - x + \frac{1}{8} = 0$

Solution :

i) Given $4x^2 - 7x - 2 = 0$

$$\begin{array}{c|c} -7 & -8 \\ \hline -8 & +1 \end{array}$$

$\Rightarrow \quad 4x^2 - 8x + x - 2 = 0$

$\Rightarrow \quad 4x(x - 2) + 1(x - 2) = 0$

$\Rightarrow \quad (4x + 1)(x - 2) = 0$

$\Rightarrow \quad 4x = -1$ (or) $x - 2 = 0$

$x = -1/4$ (or) $x = 2$

Roots are $\left\{-1/4, 2\right\}$

---

ii) Given $3(p^2 - 6) = p(p + 5)$

$$\begin{array}{c|c} -5 & -36 \\ \hline -9 & +4 \\ \hline 2 & 2 \end{array}$$

$\Rightarrow 3p^2 - 18 = p^2 + 5p$

$\Rightarrow 2p^2 - 5p - 18 = 0$

$\Rightarrow \quad p = 9/2, -2$

$\frac{-9}{2}, \ 2$

---

iii) Given $\sqrt{a(a-7)} = 3\sqrt{2}$

Squaring on both sides

$a^2 - 7a = 18$

$a^2 - 7a - 18 = 0$

$(a - 9)(a + 2) = 0$

$a = 9$ (or) -2

Roots are $9, -2$

---

iv) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

$$\begin{array}{c|c} 7 & 10 \\ \hline 5 & 2 \end{array}$$

$\Rightarrow \quad \sqrt{2}x^2 + 2x + 5x + 5\sqrt{2} = 0$

$\Rightarrow \quad \sqrt{2}x(x + \sqrt{2}) + 5(x + \sqrt{2}) = 0$

$\Rightarrow \quad (\sqrt{2}x + 5) + (x + \sqrt{2}) = 0$

$\therefore \quad \sqrt{2}x + 5 = 0$ (or) $x + \sqrt{2} = 0$

$\Rightarrow \quad x = -5/\sqrt{2}$ (or) $x = -\sqrt{2}$

$\therefore$ Roots are $-5/\sqrt{2}, -\sqrt{2}$

---

v) $2x^2 - x + \frac{1}{8} = 0$

$\Rightarrow \quad 16x^2 - 8x + 1 = 0$

$\Rightarrow \quad 16x^2 - 4x - 4x + 1 = 0$

$\Rightarrow \quad 4x(4x - 1) - 1(4x - 1) = 0$

$\Rightarrow \quad (4x - 1)(4x - 1) = 0$

$\therefore 4x - 1 = 0, 4x - 1 = 0$

$x = 1/4, 1/4$

$\therefore$ Roots are $1/4, 1/4$

---

2. The number of volleyball games that must be scheduled in a league with n teams is given by $G(n) = \frac{n^2 - n}{2}$ where each team plays with every other team exactly once. A league schedules 15 games. How many teams are in the league?

Solution :

By data given, $G(n) = \frac{n^2 - n}{2} = 15$

$\Rightarrow \quad n^2 - n = 30$

$\Rightarrow \quad n^2 - n - 30 = 0$

$\Rightarrow \quad (n - 6)(n + 5) = 0$

$\Rightarrow \quad n = 6, -5$

$\therefore$ Number of terms in the league = 6

XI. SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARES AND BY FORMULA:

Key Points

- ✓ **Step 1** Write the quadratic equation in general form $ax^2 + bx + c = 0$.
- ✓ **Step 2** Divide both sides of the equation by the coefficient of x^2 if it is not 1.
- ✓ **Step 3** Shift the constant term to the right hand side.
- ✓ **Step 4** Add the square of one-half of the coefficient of x to both sides.
- ✓ **Step 5** Write the left hand side as a square and simplify the right hand side.
- ✓ **Step 6** Take the square root on both sides and solve for x.
- ✓ The formula for finding roots of a quadratic equation $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 3.31

Solve $x^2 - 3x - 2 = 0$

Solution :

$x^2 - 3x - 2 = 0$

$x^2 - 3x = 2$ (Shifting the Constant to RHS)

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = 2 + \left(\frac{3}{2}\right)^2$$

$\left(\text{Add } \left|\frac{1}{2}(\text{co-efficient of } x)\right|^2 \text{ to both sides}\right)$

$$\left(x - \frac{3}{2}\right)^2 = \frac{17}{4}$$

(writing the LHS as complete square)

$$x - \frac{3}{2} = \pm\frac{\sqrt{17}}{2}$$

(Taking the square root on both sides)

$$x = \frac{3}{2} + \frac{\sqrt{17}}{2} \text{ or } x = \frac{3}{2} - \frac{\sqrt{17}}{2}$$

Therefore, $x = \dfrac{3+\sqrt{17}}{2}, \dfrac{3-\sqrt{17}}{2}$

Example 3.32

Solve $2x^2 - x - 1 = 0$

Solution :

$2x^2 - x - 1 = 0$

$$x^2 - \frac{x}{2} - \frac{1}{2} = 0$$

$(\div 2 \text{ make co-efficient of } x^2 \text{ as } 1)$

$$x^2 - \frac{x}{2} = \frac{1}{2}$$

$$x^2 - \frac{x}{2} + \left(\frac{1}{4}\right)^2 = \frac{1}{2} + \left(\frac{1}{4}\right)^2$$

$$\left(x - \frac{1}{4}\right)^2 = \frac{9}{16} = \left(\frac{3}{4}\right)^2$$

$$x - \frac{1}{4} = \pm\frac{3}{4} \Rightarrow x = 1, -\frac{1}{2}$$

Example 3.33

Solve $x^2 + 2x - 2 = 0$ by formula method

Solution :

Compare $x^2 + 2x - 2 = 0$ with the standard form $ax^2 + bx + c = 0$

a = 1, b = 2, c = − 2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

substituting the values of a, b and c in the formula we get,

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}$$

Therefore, $x = -1 + \sqrt{3},\ -1 - \sqrt{3}$

Example 3.34

Solve $2x^2 - 3x - 3 = 0$ by formula method

Solution :

Compare $2x^2 - 3x - 3 = 0$ with the standard form $ax^2 + bx + c = 0$

a = 2, b = −3, c = − 3

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

substituting the values of a, b and c in the formula we get,

$$= \frac{-(-3) \pm \sqrt{(-3)\ \ -4(2)(-3)}}{2(2)} = \frac{3 \pm \sqrt{33}}{4}$$

Therefore, $x = \dfrac{3 + \sqrt{33}}{4},\ x = \dfrac{3 - \sqrt{33}}{4}$

Example 3.35

Solve $3p^2 + 2\sqrt{5}\,p - 5 = 0$ by formula method

Solution :

Compare $3p^2 + 2\sqrt{5}\,p - 5 = 0$ with the standard form $ax^2 + bx + c = 0$

a = 3, b = $2\sqrt{5}$, c = − 5

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

substituting the values of a, b and c in the formula we get,

$$p = -2\sqrt{5} \pm \sqrt{\frac{\left(2\sqrt{5}\right)^2 - 4(3)(-5)}{2(3)}}$$

$$= \frac{-2\sqrt{5} \pm \sqrt{80}}{6}$$

$$= \frac{-\sqrt{5} \pm 2\sqrt{5}}{3}$$

Therefore, $x = \dfrac{\sqrt{5}}{3},\ -\sqrt{5}$

Example 3.36

Solve $pqx^2 = (p + q)^2 x + (p + q)^2 = 0$ by formula method

Solution :

Compare the coefficients of the given equation with the standard form $ax^2 + bx + c = 0$

a = pq, b = − $(p + q)^2$, c = $(p + q)^2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

substituting the values of a, b and c in the formula we get,

$$x = \frac{\left[-(p+q)^2\right] \pm \sqrt{\left[-(p+q)^2\right]^2 - 4(pq)(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^4 - 4(pq)(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2\left[(p+q)^2 - 4pq\right]}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2(p^2 + q^2 + 2pq - 4pq)}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2(p-q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm (p+q)(p-q)}{2pq}$$

Therefore, $x = \frac{p+q}{2pq} \times pq, \ \frac{p+q}{2pq} \times 2q$

we get, $x = \frac{p+q}{q}, \ \frac{p+q}{p}$

EXERCISE 3.11

1. Solve the following quadratic equations by completing the square method

i) $9x^2 - 12x + 4 = 0$ ii) $\frac{5x+7}{x-1} = 3x+2$

Solution :

i) Given equation is

$$9x^2 - 12x + 4 = 0$$

$$\Rightarrow 9x^2 - 12x = -4$$

$$\Rightarrow x^2 - \tfrac{12}{9}x = -\tfrac{4}{9} \qquad (\div \text{ by } 9)$$

$$\Rightarrow x^2 - \tfrac{4}{3}x = -\tfrac{4}{9}$$

$$\Rightarrow x^2 - \tfrac{4}{3}x + \left(\tfrac{2}{3}\right)^2 = -\tfrac{4}{9} + \left(\tfrac{2}{3}\right)^2$$

$$\Rightarrow \left(x - \tfrac{2}{3}\right)^2 = -\tfrac{4}{9} + \tfrac{4}{9}$$

$$\Rightarrow \left(x - \tfrac{2}{3}\right)^2 = 0$$

$$\Rightarrow \left(x - \tfrac{2}{3}\right)\left(x - \tfrac{2}{3}\right) = 0$$

$$\Rightarrow x = \tfrac{2}{3}, \tfrac{2}{3}$$

$$\therefore \text{Solution set} = \left\{\tfrac{2}{3}, \tfrac{2}{3}\right\}$$

ii) Given equation is

$$\frac{5x+7}{x-1} = 3x+2$$

$$\Rightarrow 5x + 7 = (3x+2)(x-1)$$

$$\Rightarrow 5x + 7 = 3x^2 - x - 2$$

$$\Rightarrow 3x^2 - 6x - 9 = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow x^2 - 2x = 3$$

$$\Rightarrow x^2 - 2x + 1 = 3 + 1$$

$$\Rightarrow (x-1)^2 = 4$$

$$\Rightarrow (x-1) = \pm 2$$

$$\Rightarrow x - 1 = 2, \ x - 1 = -2$$

$$\Rightarrow x = 3, \quad x = -1$$

Solution set = $\{3, -1\}$

2. Solve the following quadratic equations by formula method

i) $2x^2 - 5x + 2 = 0$

ii) $\sqrt{2}f^2 - 6f + 3\sqrt{2} = 0$

iii) $3y^2 - 20y - 23 = 0$

iv) $36y^2 - 12ay + (a^2 - b^2) = 0$

Solution :

i) Given equation is

$$2x^2 - 5x + 2 = 0$$

$$a = 2, b = -5, c = 2$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{25-16}}{4}$$

$$= \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm 3}{4}$$

$$= \frac{5+3}{4}, \frac{5-3}{4}$$

$$= 2, \ \frac{1}{2}$$

ii) Given equation is

$$\sqrt{2}f^2 - 6f + 3\sqrt{2} = 0$$

$$a = \sqrt{2},\ b = -6,\ c = 3\sqrt{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4(\sqrt{2})(3\sqrt{2})}}{2\sqrt{2}}$$

$$= \frac{6 \pm 2\sqrt{36 - 24}}{2\sqrt{2}}$$

$$= \frac{6 \pm 2\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{6 + 2\sqrt{3}}{2\sqrt{2}}, \frac{6 - 2\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{3 + \sqrt{3}}{\sqrt{2}}, \frac{3 - \sqrt{3}}{\sqrt{2}}$$

iii) Given equation is

$3y^2 - 20y - 23 = 0$

$a = 3,\ b = -20,\ c = -23$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{20 \pm \sqrt{400 - 4(3)(-23)}}{6}$$

$$= \frac{20 \pm 2\sqrt{400 + 276}}{6}$$

$$= \frac{20 \pm \sqrt{676}}{6}$$

$$= \frac{20 \pm 26}{6}$$

$$= \frac{20 + 26}{6}, \frac{20 - 26}{6}$$

$$= \frac{46}{6}, \frac{-6}{6}$$

$$= \frac{23}{3}, -1$$

iv) Given equation is

$36y^2 - 12ay + (a^2 - b^2) = 0$

$A = 36,\ B = -12a,\ C = a^2 - b^2$

$$\therefore\ y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{12a \pm \sqrt{144a^2 - 4(36)(a^2 - b^2)}}{2(36)}$$

$$= \frac{12a \pm \sqrt{144a^2 - 144(a^2 - b^2)}}{72}$$

$$= \frac{12a \pm \sqrt{144b^2}}{72}$$

$$= \frac{12a \pm 12b}{72}$$

$$= \frac{a \pm b}{6}$$

$$= \frac{a + b}{6}, \frac{a - b}{6}$$

3. A ball rolls down a slope and travels a distance $dt = t^2 - 0.75t$ feet in t seconds. Find the time when the distance travelled by the ball is 11.25 feet.

Solution :

By data given,

$d = t^2 - 0.75t$ where $d = 11.25$ ft

$\Rightarrow\quad t^2 - 0.75\,t = 11.25$

$\Rightarrow\quad t^2 - 0.75\,t - 11.25 = 0$

$\Rightarrow\quad (t - 3.75)(t + 3) = 0$

$\Rightarrow\quad t - 3.75 = 0,\quad t + 3 = 0$

$\therefore\ t = 3.75\qquad t = -3$

But $t \neq -3$

$\therefore\ t = 3.75$ sec

XII. SOLVING PROBLEMS INVOLVING QUDRATIC EQUATIONS :

Example 3.37

The product of Kumaran's age (in years) two years ago and his age four years from now is one more than twice his present age. What is his present age?

Solution :

Let the present age of Kumaran be x years.

Two years ago, his age $= (x-2)$ years.

Four years from now, his age $= (x+4)$ years.

Given, $(x-2)(x+4) = 1 + 2x$

$x^2 + 2x - 8 = 1 + 2x$ gives $(x-3)(x+3) = 0$ then, $x = \pm 3$

Therefore, $x = 3$ (Rejecting -3 as age cannot be negative)

Kumaran's present age is 3 years.

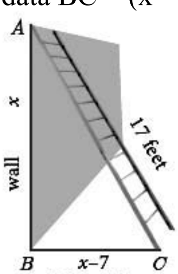
Example 3.38

A ladder 17 feet long is leaning against a wall. If the ladder, vertical wall and the floor from the bottom of the wall to the ladder form a right triangle, find the height of the wall where the top of the ladder meets if the distance between bottom of the wall to bottom of the ladder is 7 feet less than the height of the wall?

Solution :

Let the height of the wall AB = x feet

As per the given data BC = (x – 7) feet



In the right triangle ABC, AC = 17 ft,

BC = (x – 7) feet

By Pythagoras theorem, $AC^2 = AB^2 + BC^2$

$(17)^2 = x^2 + (x-7)^2$; $289 = x^2 + x^2 - 14x + 49$

$x^2 - 7x - 120 = 0$ hence, $(x-15)(x+8) = 0$ then, $x = 15$ (or) -8

Therefore, height of the wall AB = 15 ft (Rejecting -8 as height cannot be negative).

Example 3.39

A flock of swans contained x^2 members. As the clouds gathered, 10x went to a lake and one-eighth of the members flew away to a garden. The remaining three pairs played about in the water. How many swans were there in total?

Solution :

As given there are x^2 swans

As per the given data $x^2 - 10x - \frac{1}{8}x^2 = 6$ we get, $7x^2 - 80x - 48 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{80 \pm \sqrt{6400 - 4(7)(-48)}}{14}$$

$$= \frac{80 \pm 88}{14}$$

Therefore, $x = 12,\ -\frac{4}{7}$

Here $x = -\frac{4}{7}$ is not possible as the number of swans cannot be negative.

Hence, x = 12. Therefore total number of swans is $x^2 = 144$.

Example 3.40

A passenger train takes 1 hr more than an express train to travel a distance of 240 km from Chennai to Virudhachalam. The speed of passenger train is less than that of an express train by 20 km per hour. Find the average speed of both the trains.

Solution :

Let the average speed of passenger train be x km/hr.

Then the average speed of express train will be (x + 20) km/hr

Time taken by the passenger train to cover distance of 240 km = $\frac{240}{x}$ hr

Time taken by express train to cover distance of 240 km = $\frac{240}{x+20}$ hr

Given, $\frac{240}{x} = \frac{240}{x+20} + 1$

$240\left|\frac{1}{x} - \frac{1}{x+20}\right| = 1$ gives, $240\left|\frac{x+20-x}{x(x+20)}\right| = 1$

we get, $4800 = (x^2 + 20x)$

$x^2 + 2x - 4800 = 0$ gives, $(x + 80)(x - 60) = 0$ we get, $x = -80$ or 60.

Therefore x = 60 (Rejecting − 80 as speed cannot be negative)

Average speed of the passenger train is 60 km/hr

Average speed of the express train is 80 km/hr.

EXERCISE 3.12

1. If the difference between a number and its reciprocal is $\frac{24}{5}$, find the number.

Solution :

Let x be the required number

$\frac{1}{x}$ be its reciprocal

Given $x - \frac{1}{x} = \frac{24}{5}$

$\Rightarrow \quad \frac{x^2 - 1}{x} = \frac{24}{5}$

$\Rightarrow \quad 5x^2 - 5 = 24x$

$\Rightarrow \quad 5x^2 - 24x - 5 = 0$

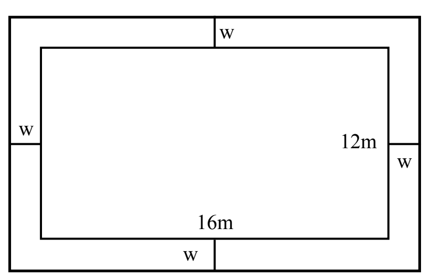
$\Rightarrow \quad x = 5, \ -1/5$

− 24	− 25
$\frac{-25}{5}$	$\frac{+1}{5}$
-5,	$\frac{1}{5}$

$\therefore$ The required numbers are $5, \ -1/5$

2. A garden measuring 12m by 16m is to have a pedestrian pathway that is '*w*' meters wide installed all the way around so that it increases the total area to 285 m². What is the width of the pathway?

Solution :



Given the dimensions of the garden

= 16m × 12m

Let 'w' be the equal width of the pedestrian pathway.

∴ By the data given,

56	− 372
$\frac{62}{4}$	$\frac{-6}{4}$
$\frac{31}{2}$	$\frac{-3}{2}$

$\Rightarrow \quad (16 + 2w)(12 + 2w) = 285$

$\Rightarrow \quad 4w^2 + 56w + 192 - 285 = 0$

$\Rightarrow \quad 4w^2 + 56w - 93 = 0$

$\Rightarrow \quad w = {}^{-31}\!/_2, \ {}^{3}\!/_2$

But w can't be -ve

$\therefore \ w = {}^{3}\!/_2$

= 1.5 m

∴ Width of the path away = 1.5 m

3. A bus covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hour more it would have taken 30 minutes less for the journey. Find the original speed of the bus.

Solution :

Let 'x' Km/hr be the original speed of the bus.

Distance covered at a uniform speed = 90 Km

$$\text{Time taken} = \frac{90}{x}$$

Had the speed been 15 Km/hr more,

$$\text{Time taken} = \frac{90}{x+15}$$

∴ By Data given

$$\frac{90}{x+15} - \frac{90}{x} = \frac{1}{2} \qquad \left(\because 30 \text{ min} = \frac{1}{2}\text{hr}\right)$$

$$\Rightarrow \quad 90\left(\frac{1}{x+15} - \frac{1}{x}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \frac{x - x - 15}{x(x+15)} = \frac{1}{180}$$

$$\Rightarrow \quad 180 \times (-15) = x^2 + 15x$$

$$\Rightarrow \quad x^2 + 15x - 2700 = 0$$

$$\Rightarrow \quad (x + 60)(x - 45) = 0$$

$$\therefore \quad x = -60, 45$$

∴ Original speed = 45 Km/hr

4. A girl is twice as old as her sister. Five years hence, the product of their ages (in years) will be 375. Find their present ages.

Solution :

Let the present ages of the girl and her sister be x, y

By data given,

i) x = 2y

ii) (x + 5) (y + 5) = 375

15	− 700
$\frac{35}{2}$	$\frac{-20}{2}$
$\frac{35}{2}$	−10

$\Rightarrow \quad (2y + 5)(y + 5) = 375$

$\Rightarrow \quad 2y^2 + 15y - 350 = 0$

$\Rightarrow \quad y = {}^{-35}\!/_2, 10$

y can't be -ve

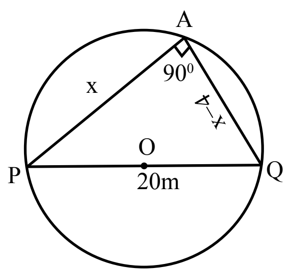
∴ y = 10

∴ x = 2y ⇒ x = 20

∴ Their present ages are 20, 10 years old.

5. A pole has to be erected at a point on the boundary of a circular ground of diameter 20 m in such a way that the difference of its distances from two diametrically opposite fixed gates P and Q on the boundary is 4 m. Is it possible to do so? If answer is yes at what distance from the two gates should the pole be erected?

Solution :



In the fig, PQ = 20 m = diameter of the circle with centre O.

Let A be the point of pole s.t

$AP - AQ = 4m \quad AP = x, AQ = x - 4$

Also, $PA^2 + QA^2 = PQ^2$

(Angle in a semicircle is 90^0)

$\Rightarrow \quad x^2 + (x-4)^2 = 20^2$

$\Rightarrow 2x^2 - 8x + 16 - 400 = 0$

$\Rightarrow \quad 2x^2 - 8x - 384 = 0$

$\Rightarrow \quad x^2 - 4x - 192 = 0$

$\Rightarrow \quad (x-16)(x+12) = 0$

$\therefore \quad x = 16$ m

$x = 16 \quad \Rightarrow \quad x - 4 = 12m$

$\therefore$ Pole should be erected at a distance of 16 m, 12m from the two gates.

6. From a group of $2x^2$ black bees, square root of half of the group went to a tree. Again eight-ninth of the bees went to the same tree. The remaining two got caught up in a fragrant lotus. How many bees were there in total ?

Solution :

Given number of black bees $= 2x^2$

By the data given,

$$2x^2 - x - \tfrac{8}{9}(2x^2) = 2$$

$$\Rightarrow \quad 2x^2\left(1 - \tfrac{8}{9}\right) - x = 2$$

$$\Rightarrow \quad 2x^2\left(\tfrac{1}{9}\right) - x = 2$$

$$\Rightarrow \quad 2x^2 - 9x = 18$$

$$\Rightarrow \quad 2x^2 - 9x - 18 = 0$$

$$\Rightarrow \quad x = 6, \ -\tfrac{3}{2}$$

But $x \neq -\tfrac{3}{2}$

$\therefore \ x = 6$

− 9	− 36
$\frac{-12}{2}$	$\frac{+3}{2}$
− 6	$\frac{3}{2}$

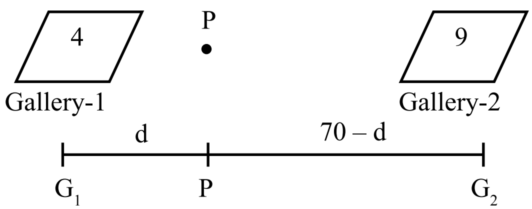
$\therefore$ Total number of bees $= 2x^2$

$= 2\,(36)$

$= 72$

7. Music is been played in two opposite galleries with certain group of people. In the first gallery a group of 4 singers were singing and in the second gallery 9 singers were singing. The two galleries are separated by the distance of 70 m. Where should a person stand for hearing the same intensity of the singers voice? (Hint: The ratio of the sound intensity is equal to the square of the ratio of their corresponding distances).

Solution :



Let the position of the person in between the 2 galleries be P, who stand at a distance of 'd' m.

$\therefore G_1 G_2 = 70$m, $G_1P =$ dm, $G_2P = (70 - d)$m

Since the ratio of sound intensity is equal to the square of the ratio of their corresponding sides,

$$\frac{4}{9} = \frac{d^2}{(70-d)^2}$$

$\Rightarrow$ $4(70 - d)^2 = 9d^2$

$\Rightarrow$ $4(4900 - 140d + d^2) = 9d^2$

$\Rightarrow$ $19600 - 560d + 4d^2 = 9d^2$

$\Rightarrow$ $5d^2 + 560d - 19600 = 0$

$\Rightarrow$ $d^2 + 112d - 3920 = 0$

$\Rightarrow$ $(d + 140)(d - 28) = 0$

$\Rightarrow$ $d = -140$ (or) 28

$\therefore d = 28$m

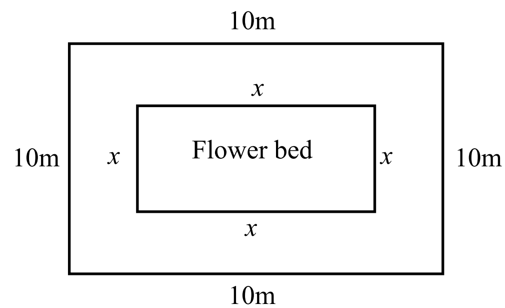
The person should stand 28m from gallery 1

(or)

42m from gallery-2 to hear the same intensity of the singers voice.

8. There is a square field whose side is 10 m. A square flower bed is prepared in its centre leaving a gravel path all round the flower bed. The total cost of laying the flower bed and gravelling the path at ₹ 3 and ₹ 4 per square metre respectively is ₹ 364. Find the width of the gravel path.

Solution :



Given, length of the square field = 10m

Let the side of the flower bed = x m

$\therefore$ Area of square field = 100 m^2 &

area of the flower bed = x^2 m^2

$\therefore$ Area of the gravel path = $(100 - x^2)$ m^2

Given cost of laying flower bed = ₹3 /m^2.

Cost of laying gravel path = ₹4 /m^2.

$\therefore$ By the problem,

$3x^2 + 4(100 - x^2) = 364$

$\Rightarrow$ $3x^2 + 400 - 4x^2 = 364$

$\Rightarrow$ $x^2 = 36$

$\Rightarrow$ $x = 6$

$\therefore$ Length of flower bed = 6m

$\therefore$ Width of the path $= \frac{10-6}{2}$

$= \frac{4}{2}$

$= 2m$

9. Two women together took 100 eggs to a market, one had more than the other. Both sold them for the same sum of money. The first then said to the second: "If I had your eggs, I would have earned ₹ 15", to which the second replied: "If I had your eggs, I would have earned ₹ $6\frac{2}{3}$ How many eggs did each had in the beginning?

Solution :

Let the number of eggs of woman 1 and 2 respectively be x, y, and their selling price be

$p, q, \quad \Rightarrow x + y = 100 \quad —(1)$

* If both of them sold the eggs for the equal sum of money, $px = qy$.
* By the data given in the problem,

$$py = 15, \quad qx = 6\frac{2}{3}$$

$$qx = \frac{20}{3}$$

$$p = \frac{15}{y}, \quad q = \frac{20}{3x}$$

Also,

$$px = qy \Rightarrow \frac{15}{y}x = \frac{20}{3x}y$$

$$\Rightarrow \frac{3x}{y} = \frac{4y}{3x}$$

$$\Rightarrow 9x^2 = 4y^2$$

$$\Rightarrow 9x^2 = 4(100 - x)^2 \qquad \text{(from (1))}$$

$$\Rightarrow 9x^2 = 4(x^2 - 200x + 10000)$$

$$\Rightarrow 5x^2 + 800x - 40000 = 0$$

$$\Rightarrow x^2 + 160x - 8000 = 0$$

$$\Rightarrow (x + 200)(x - 40) = 0$$

$$\Rightarrow x = 40$$

$$\therefore y = 60$$

$\therefore$ Woman 1 had 40 eggs and
Woman 2 had 60 eggs.

10. The hypotenuse of a right angled triangle is 25 cm and its perimeter 56 cm. Find the length of the smallest side.

Solution :

Given b = 25 cm, $a + b + c = 56$ cm

$$\Rightarrow \quad a = c = 56 - 25$$

$$\Rightarrow \quad a + c = 31$$

Let $a = x$, $c = 31 - x$

$\therefore$ In Δ ABC, $a^2 + c^2 = 25^2$

$$x^2 + (31 - x)^2 = 625$$

$$2x^2 - 62x + 336 = 0$$

$$x^2 - 31x + 168 = 0$$

$$(x - 24)(x - 7) = 0$$

$$x = 24, 7$$

$\therefore$ The sides of the Δ are 7cm, 24cm, 25cm

$\therefore$ Length of smallest side = 7 cm

XIII. NATURE OF ROOTS OF A QUADRATIC EQUATION :

Key Points

✓ The roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are found using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Values of Discriminant $\Delta = b^2 - 4ac$	**Nature of Roots**
$\Delta > 0$	Real and Unequal roots
$\Delta = 0$	Real and Equal roots
$\Delta < 0$	No Real root

Example 3.41

Determine the nature of roots for the following quadratic equations

i) $x^2 - x - 20 = 0$ ii) $9x^2 - 24x + 16 = 0$

iii) $2x^2 - 2x + 9 = 0$

Solution :

i) $x^2 - x - 20 = 0$

Here, $a = 1, b = -1, c = -20$

Now, $\Delta = b^2 - 4ac$

$\Delta = (-1)^2 - 4(1)(-20) = 81$

Here, $\Delta = 81 > 0$. So, the equation will have real and unequal roots

ii) $9x^2 - 24x + 16 = 0$

Here, $a = 9, b = -24, c = 16$

Now, $\Delta = b^2 - 4ac$

$\Delta = (-24)^2 - 4(9)(16) = 0$

Here, $\Delta = 0$. So, the equation will have real and unequal roots

iii) $2x^2 - 2x + 9 = 0$

Here, $a = 2, b = -2, c = 9$

Now, $\Delta = b^2 - 4ac$

$\Delta = (-2)^2 - 4(2)(8) = -68$

Here, $\Delta = -68 < 0$. So, the equation will have no real roots

Example 3.42

(i) Find the values of 'k', for which the quadratic equation $kx^2 - (8k + 4) + 81 = 0$ has real and equal roots ?

(ii) Find the values of 'k', such that quadratic equation $(k + 9)x^2 + (k + 1)x + 1 = 0$ has no real roots ?

Solution :

i) $kx^2 - (8k + 4) + 81 = 0$

Since the equation has real and equal roots, $\Delta = 0$.

That is, $b^2 - 4ac = 0$

Here, $a = k, b = -(8k + 4), c = 81$

That is, $[(8k + 4)]^2 - 4(k)(81) = 0$

$64k^2 + 64k + 16 - 324k = 0$

$64k^2 - 260k + 16 = 0$

dividing by 4 we get $16k^2 - 65k + 4 = 0$

$(16k - 1)(k - 4) = 0$ then, $k = \dfrac{1}{16}$ or $k = 4$

ii) $(k+9)x^2 + (k+1)x + 1 = 0$

Since the equation has no real roots, $\Delta < 0$

That is, $b^2 - 4ac < 0$

Here, $a = k+9$, $b = k+1$, $c = 1$

That is, $(k+1)^2 - 4(k+9)(1) < 0$

$$k^2 + 2k + 1 - 4k - 36 < 0$$

$$k^2 - 2k - 35 < 0$$

$$(k+5)(k-7) < 0$$

Therefore $-5 < k < 7$. {If $\alpha < \beta$ and if $(x-\alpha)(x-\beta) < 0$ then, $\alpha < x < \beta$).

Example 3.43

Prove that the equation $x^2(p^2+q^2) + 2x(pr+qs) + r^2 + s^2 = 0$ has no real roots. If $ps = qr$, then show that the roots are real and equal.

Solution :

The given quadratic equation is,

$$x^2(p^2+q^2) + 2x(pr+qs) + r^2 + s^2 = 0$$

Here, $a = p^2 + q^2$, $b = 2(pr+qs)$, $c = r^2 + s^2$

Now $\Delta = b^2 - 4ac$

$$= [2(pr+qs)]^2 - 4(p^2+q^2)(r^2+s^2)$$

$$= 4[p^2r^2 + 2pqrs + q^2s^2 - p^2r^2 - p^2s^2 - q^2r^2 - q^2s^2]$$

$$= 4[-p^2s^2 + 2pqrs - q^2r^2]$$

$$= -4[(ps-qr)^2] < 0 \quad \text{....... (1)}$$

Since, $\Delta = b^2 - 4ac < 0$, the roots are not equal.

If $ps = qr$ then $= -4[ps - qr]^2$

$$= -4[qr - qr]^2 = 0 \quad \text{(using (1))}$$

Thus $\Delta = 0$ if $ps = qr$ and so the roots will be real and equal.

EXERCISE 3.13

1. Determine the nature of the roots for the following quadratic equations

i) $15x^2 + 11x + 2 = 0$

ii) $x^2 - x + 1 = 0$

iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

iv) $9y^2 - 6\sqrt{2}\,y + 2 = 0$

v) $9a^2b^2x^2 - 24abcdx + 16c^2d^2 = 0$, $a \neq 0$ $b \neq 0$

Solution :

i) Given equation is $15x^2 + 11x + 2 = 0$

$$a = 15, b = 11, c = 2$$

$$\therefore \quad \Delta = b^2 - 4ac$$

$$= 121 - 4 \times 15 \times 2$$

$$= 121 - 120$$

$$= 1 > 0$$

$\therefore$ The equation will have real and unequal roots.

ii) Given equation is $x^2 - x - 1 = 0$

$$a = 1, b = -1, c = -1$$

$$\therefore \quad \Delta = b^2 - 4ac$$

$$= 1 - 4(1)(-1)$$

$$= 1 + 4$$

$$= 5 > 0$$

$\therefore$ The equation will have real and unequal roots.

iii) Given $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

$$a = \sqrt{2}, b = -3, c = 3\sqrt{2}$$

$$\therefore \quad \Delta = b^2 - 4ac$$

$$= 9 - 4(\sqrt{2})(3\sqrt{2})$$

$$= 9 - 24$$

$$= -15 < 0$$

$\therefore$ The roots are unreal.

iv) Given $9y^2 - 6\sqrt{2}\,y + 2 = 0$

$a = 9, b = -6\sqrt{2}, c = 2$

$\therefore \quad \Delta = b^2 - 4ac$

$= (-6\sqrt{2})^2 - 4(9)(2)$

$= 72 - 72$

$= 0$

$\therefore$ The roots are real & equal.

v) Given $9a^2b^2x^2 - 24abcdx + 16c^2d^2$

$= 0, a, b \neq 0$

$a = 9a^2b^2, b = -24abcd, c = 16c^2d^2$

$\therefore \quad \Delta = B^2 - 4AC$

$= 576\, a^2b^2c^2d^2 - 4(9a^2b^2)(16c^2d^2)$

$= 576\, a^2b^2c^2d^2 - 576\, a^2b^2c^2d^2$

$= 0$

$\therefore$ The roots are real & equal.

2. Find the value(s) of 'k' for which the roots of the following equations are real and equal.

i) $(5k - 6)x^2 + 2kx + 1 = 0$

ii) $kx^2 + (6k + 2)x + 16 = 0$

Solution :

i) Given equation is $(5k - 6)x^2 + 2kx + 1 = 0$ are real & equal

$a = 5k - 6, b = 2k, c = 1$

$\therefore \quad \Delta = b^2 - 4ac = 0$

$\Rightarrow \quad 4k^2 - 4(5k - 6)(1) = 0$

$\Rightarrow \quad 4k^2 - 20k + 24 = 0$

$\Rightarrow \quad k^2 - 5k + 6 = 0$

$\Rightarrow \quad (k - 3)(k - 2) = 0$

$\therefore k = 3, 2$

ii) Given the roots of $kx^2 + (6k + 2)x + 16 = 0$ are real & equal

$a = k, b = 6k + 2, c = 16$

$\therefore \quad \Delta = b^2 - 4ac = 0$

-10	9
$\frac{-9}{9}$	$\frac{-1}{9}$
-1	$\frac{-1}{9}$

$\Rightarrow \quad (6k + 2)^2 - 4(k)(16) = 0$

$\Rightarrow \quad 36k^2 + 24k + 4 - 64k = 0$

$\Rightarrow \quad 36k^2 - 40k + 4 = 0$

$\Rightarrow \quad 9k^2 - 10k + 1 = 0$

$\therefore k = 1, \frac{1}{9}$

3. If the roots of $(a - b)x^2 + (b - c)x + (c - a) = 0$ are real and equal, then prove that b, a, c are in arithmetic progression.

Solution :

i) Given equation is $(a - b)x^2 + (b - c)x + (c - a) = 0$ are real & equal

To prove : b, a, c are in A.P.

Here $A = a - b, B = b - c, C = c - a$

$\therefore \quad \Delta = B^2 - 4AC = 0$

$\Rightarrow \quad (b - c)^2 - 4(a - b)(c - a) = 0$

$\Rightarrow \quad (b^2 + c^2 - 2bc) - 4(ac - bc - a^2 + ab) = 0$

$\Rightarrow \quad b^2 + c^2 - 2bc - 4ac + 4bc + 4a^2 - 4ab = 0$

$\Rightarrow \quad 4a^2 + b^2 + c^2 - 4ab + 2bc - 4ac = 0$

$\Rightarrow \quad (2a - b - c)^2 = 0$

$\Rightarrow \quad 2a - b - c = 0$

$\Rightarrow \quad b + c = 2a$

$\Rightarrow \quad a = \frac{b + c}{2}$

$\therefore$ b, a, c are in A.P.

4. If a, b are real then show that the roots of the equation $(a-b)x^2-6(a+b)x-9(a-b) = 0$ are real and unequal.

Solution :

To prove the roots of $(a - b) x^2 - 6 (a + b) x - 9 (a - b) = 0$ are real & unequal

Here $A = a - b$, $B = -6 (a + b)$, $c = -9 (a - b)$

$\therefore \quad \Delta = B^2 - 4AC = 0$

$= 36 (a + b)^2 - 4 (a - b) (-9 (a - b)) = 0$

$= 36 (a + b)^2 + 36 (a - b)^2$

$= 36 [(a + b)^2 + (a - b)^2]$

$= 36 [2(a^2 + b^2)]$

$= 72 (a^2 + b^2) > 0,$ a & b are real

$\therefore$ The roots of the equation are real & unequal.

---

5. If the roots of the equation $(c^2 - ab) x^2 - 2 (a^2 - bc) x + b^2 - ac = 0$ are real and equal prove that either $a = 0$ (or) $a^3 + b^3 + c^3 = 3abc$.

Solution :

Given roots of the equation $(c^2 - ab) x^2 - 2 (a^2 - bc) x + b^2 - ac = 0$ are real & equal

To prove either $a = 0$ (or)

$a^3 + b^3 + c^3 = 3abc$

Here $A = c^2 - ab$, $B = -2 (a^2 - bc)$ $C = b^2 - ac$

Given $\Delta = B^2 - 4AC = 0$

$\Rightarrow \quad 4 (a^2 - bc)^2 - 4 (c^2 - ab) (b^2 - ac) = 0$

$\Rightarrow \quad (a^2 - bc)^2 - (c^2 - ab) (b^2 - ac) = 0$

$\Rightarrow \quad (a^4 + b^2c^2 - 2a^2bc) - (b^2c^2 - ab^3 - ac^3 + a^2bc) = 0$

$\Rightarrow \quad a^4 - 3a^2bc + ab^3 + ac^3 = 0$

$\Rightarrow \quad a (a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow \quad a = 0$ (or) $a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = 3abc$

Hence proved.

XIV. RELATION BEWEEN ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

Key Points

✓ Let α and β are the roots of the equation $ax^2 + bx + c = 0$ then.

$$\alpha+\beta = \frac{-b}{a} = \frac{-\text{ Co-efficient of } x}{\text{Co-efficient of } x^2}$$

$$\alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Co-efficient of } x^2}$$

Example 3.44

If the difference between the roots of the equation $x^2 - 13x + k = 0$ is 17 find k.

Solution :

$x^2 - 13x + k = 0$ here, $a = 1, b = -13, c = k$

Let α, β be the roots of the equation. Then

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-13)}{1} = 13 \quad \text{............ (1)}$$

Also $\alpha - \beta = 17$ .... (2)

(1) + (2) we get, $2\alpha = 30$ gives $\alpha = 15$

Therefore, $15 + \beta = 13$ (from (1)) gives $\beta = -2$

$$\text{But } \alpha\beta = \frac{c}{a} = \frac{k}{1} \text{ gives } 15 \times (-2) = k$$

we get, $k = -30$

Example 3.45

If α and β are the roots of $x^2 + 7x + 10 = 0$ find the values of

i) $(\alpha - \beta)$ ii) $\alpha^2 + \beta^2$ iii) $\alpha^3 - \beta^3$ iv) $\alpha^4 + \beta^4$

v) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ vi) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

Solution :

$x^2 + 7x + 10 = 0$ here, $a = 1, b = 7, c = 10$

If α and β be the roots of the equation then,

$$\alpha + \beta = \frac{-b}{a} = \frac{-7}{1} = -7 \,;\, \alpha\beta = \frac{c}{a} = \frac{10}{1} = 10$$

i) $$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= \sqrt{(-7)^2 - 4 \times 10} = \sqrt{9} = 3$$

ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-7)^2 - 2 \times 10 = 29$

iii) $\alpha^3 + \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta\,(\alpha - \beta)$

$= (3)^3 + 3(10)\,(3) = 117$

iv) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$= 29^2 - 2 \times (10)^2$

$= 641$ (since from (iii), $\alpha^2 + \beta^2 = 29$)

v) $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{49 - 20}{10} = \frac{29}{10}$$

vi) $$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{(-343) - 3\,(10 \times (-7))}{10}$$

$$= \frac{-343 + 210}{10} = \frac{-133}{10}$$

Example 3.46

If α, β are the roots of the equation $3x^2 + 7x - 2 = 0$, find the values of

i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ ii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

Solution :

$3x^2 + 7x - 2 = 0$ here, $a = 3, b = 7, c = -2$

since α, β are the roots of the equation

i) $$\alpha + \beta = \frac{-b}{a} = \frac{-7}{3} \,;\; \alpha\beta = \frac{c}{a} = \frac{-2}{3}$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{\left(\frac{-7}{3}\right)^2 - 2\left(\frac{-2}{3}\right)}{\frac{-2}{3}}$$

$$= \frac{-61}{6}$$

ii) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}=\dfrac{\alpha^3+\beta^3}{\alpha\beta}=\dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$$=\frac{\left(-\frac{7}{3}\right)^3-3\left(-\frac{2}{3}\right)\left(-\frac{7}{3}\right)}{-\frac{7}{3}}$$

$$=\frac{67}{9}$$

Example 3.47

If α, β are the roots of the equation $2x^2 - x - 1 = 0$, then form the equation whose roots are

i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ ii) $\alpha^2\beta, \beta^2\alpha$ iii) $2\alpha+\beta, 2\beta+\alpha$

Solution :

$2x^2 - x - 1 = 0$ here, $a = 2, b = -1, c = -1$

$$\alpha+\beta=\frac{-b}{a}=\frac{-(-1)}{2}=\frac{1}{2}\ ;\ \alpha\beta=\frac{c}{a}=-\frac{1}{2}$$

i) Given roots are $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

Sum of the roots $=\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\alpha+\beta}{\alpha\beta}=\dfrac{\frac{1}{2}}{-\frac{1}{2}}=-1$

Product of the roots $=\dfrac{1}{\alpha}\times\dfrac{1}{\beta}=\dfrac{1}{\alpha\beta}=\dfrac{1}{-\frac{1}{2}}=-2$

The required equation is x^2 – (Sum of the roots) x + (Product of the roots) = 0

$x^2 - (-1)x - 2 = 0$ gives $x^2 + x - 2 = 0$

ii) Given roots are $\alpha^2\beta, \beta^2\alpha$

Sum of the roots $\alpha^2\beta, \beta^2\alpha$

$= \alpha\beta(\alpha+\beta) = -\dfrac{1}{2}\left(\dfrac{1}{2}\right)=-\dfrac{1}{4}$

Product of the roots $(\alpha^2\beta), \times (\beta^2\alpha)$

$= \alpha^3\beta^3 = (\alpha\beta)^3 = \left(-\dfrac{1}{2}\right)^3=-\dfrac{1}{8}$

The required equation is $x^2 -$ (Sum of the roots) x + (Product of the roots) = 0

$x^2-\left(-\dfrac{1}{4}\right)x-\dfrac{1}{8}=0$ gives $8x^2+2x-1=0$

iii) $2\alpha+\beta, 2\beta+\alpha$

Sum of the roots $2\alpha+\beta+2\beta+\alpha$

$= 3(\alpha+\beta) = 3\left(\dfrac{1}{2}\right)=\dfrac{3}{2}$

Product of the roots $= (2\alpha + B)(2\beta+\alpha)$

$= 4\alpha\beta + 2\alpha^2 + 2\beta^2 = \alpha\beta$

$= 5\alpha\beta + 2(\alpha^2+\beta^2)$

$= 5\alpha\beta + 2[(\alpha+\beta)^2 - 2\alpha\beta]$

$=5\left(-\dfrac{1}{2}\right)+2\left[\dfrac{1}{4}-2\times-\dfrac{1}{2}\right]=0$

The required equation is $x^2 -$ (Sum of the roots)x + (Product of the roots) = 0

$x^2-\dfrac{3}{2}x+0=0$ gives $2x^2-3x=0$

EXERCISE 3.14

1. Write each of the following expression in terms of $\alpha + \beta$ and $\alpha\beta$.

i) $\frac{\alpha}{3\beta}+\frac{\beta}{3\alpha}$ ii) $\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}$

iii) $(3\alpha-1)(3\beta-1)$ iv) $\frac{\alpha+3}{\beta}+\frac{\beta+3}{\alpha}$

Solution :

i) $\frac{\alpha}{3\beta}+\frac{\beta}{3\alpha}$

$$=\frac{\alpha^2+\beta^2}{3\alpha\beta}$$

$$=\frac{\left(\alpha+\beta^2\right)-2\alpha\beta}{3\alpha\beta}$$

ii) $\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}$

$$=\frac{\beta+\alpha}{\alpha^2\beta^2}$$

$$=\frac{\alpha+\beta}{(\alpha\beta)^2}$$

iii) $(3\alpha-1)(3\beta-1)$

$$=9\alpha\beta-3\alpha-3\beta+1$$

$$=9\alpha\beta-3(\alpha+\beta)+1$$

iv) $\frac{\alpha+3}{\beta}+\frac{\beta+3}{\alpha}$

$$=\frac{\alpha^2+3\alpha+\beta^2+3\beta}{\alpha\beta}$$

$$=\frac{\alpha^2+\beta^2+3(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta+3(\alpha+\beta)}{\alpha\beta}$$

2. The roots of the equation $2x^2-7x+5=0$ are α and β. Without solving for the roots, find

i) $\frac{1}{\alpha}+\frac{1}{\beta}$ ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ iii) $\frac{\alpha+2}{\beta+2}+\frac{\beta+2}{\alpha+2}$

Solution :

Given α, β are the roots of $2x^2-7x+5=0$

$$a=2,\ b=-7,\ c=5$$

$$\alpha+\beta=-b/a=7/2$$

$$\alpha\beta=c/a=5/2$$

i) $\frac{1}{\alpha}+\frac{1}{\beta}$

$$=\frac{\alpha+\beta}{\alpha\beta}$$

$$=\frac{7/2}{5/2}$$

$$=7/5$$

ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$$=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{49/4-5}{5/2}=29/4\times 2/5=29/10$$

iii) $\dfrac{\alpha+2}{\beta+2}+\dfrac{\beta+2}{\alpha+2}$

$$=\frac{\alpha^2+4\alpha+4+\beta^2+4\beta+4}{(\alpha+2)(\beta+2)}$$

$$=\frac{(\alpha^2+\beta^2)+4(\alpha+\beta)+8}{\alpha\beta+2\alpha+2\beta+4}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta+4(\alpha+\beta)+8}{\alpha\beta+2(\alpha+\beta)+4}$$

$$=\frac{\frac{49}{4}-5+\overset{2}{\cancel{4}}\left(\frac{7}{\underset{1}{\cancel{2}}}\right)+8}{\frac{5}{2}+2\left(\frac{7}{2}\right)+4}$$

$$=\frac{\frac{49}{4}+3+14}{\frac{19}{2}+4}$$

$$=\frac{\frac{49+68}{4}}{\frac{27}{2}}$$

$$=\frac{\overset{13\ \cancel{29}}{\cancel{117}}}{\underset{2}{\cancel{4}}}\times\frac{\overset{1}{\cancel{2}}}{\underset{3\cancel{9}}{\cancel{27}}}$$

$$=\frac{13}{6}$$

3. The roots of the equation $x^2+6x-4=0$ are α, β. Find the quadratic equation whose roots are

i) α^2 and β^2 *ii)* $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$ *iii)* $\alpha^2\beta$ and $\beta^2\alpha$

Solution :

Given α, β are the roots of $x^2+6x-4=0$

$a = 1, b = 6, c = -4$

$$\alpha+\beta = {-b}/{a} = -6$$
$$\alpha\beta = {c}/{a} = -4$$

i) To find the equation whose roots are α^2, β^2

Sum $= \alpha^2+\beta^2$
$= (\alpha+\beta)^2-2\alpha\beta$
$= (-6)^2-2(-4)$
$= 36+8$
$= 44$

Product $= \alpha^2\beta^2$
$= (\alpha\beta)^2$
$= (-4)^2$
$= 16$

$\therefore$ The required equation is

x^2 −(Sum of the roots) x + Proof of the roots = 0

$\Rightarrow x^2-44x+16=0$

ii) To find the equation whose roots are

$\dfrac{2}{\alpha}, \dfrac{2}{\beta}$

Sum $= \dfrac{2}{\alpha}+\dfrac{2}{\beta}$

$= 2\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)$

$= 2\left(\dfrac{\alpha+\beta}{\alpha\beta}\right)=2\left(\dfrac{-6}{-4}\right)=3$

Product $= \dfrac{2}{\alpha}\cdot\dfrac{2}{\beta}$

$= \dfrac{4}{\alpha\beta}$

$= \dfrac{4}{-4}=-1$

$\therefore$ The required equation is $x^2-3x-1=0$

iii) To find the equation whose roots are $\alpha^2\beta, \beta^2\alpha$

Sum $= \alpha^2\beta + \beta^2\alpha$

$= \alpha\beta(\alpha+\beta)$

$= -4(-6)$

$= 24$

Product $= \alpha^2\beta, \alpha\beta^2$

$= (\alpha\beta)^3$

$= (-4)^3$

$= -64$

$\therefore$ The required equation is

$\Rightarrow \quad x^2 - 24x - 64 = 0$

4. If α, β are the roots of $7x^2 + ax + 2 = 0$ and if $\beta - \alpha = \frac{-13}{7}$ Find the values of a.

Solution :

Given α, β are the roots of $7x^2 + ax + 2 = 0$

$$\alpha+\beta = -a/7$$

$$\alpha\beta = 2/7$$

Also, $\beta - \alpha = -13/7$

$$\Rightarrow \alpha - \beta = 13/7$$

$$\Rightarrow (\alpha-\beta)^2 = \frac{169}{49}$$

$$\Rightarrow (\alpha+\beta)^2 - 4\alpha\beta = \frac{169}{49}$$

$$\Rightarrow \left(\frac{-a}{7}\right)^2 - 4\left(\frac{2}{7}\right) = \frac{169}{49}$$

$$\Rightarrow \frac{a^2}{49} - \frac{8}{7} = \frac{169}{49}$$

$$\Rightarrow \frac{a^2}{49} - \frac{56}{49} = \frac{169}{49}$$

$$\Rightarrow a^2 - 56 = 169$$

$$\Rightarrow a^2 = 225$$

$$\therefore a = 15, -15$$

5. If one root of the equation $2y^2 - ay + 64 = 0$ is twice the other then find the values of *a*.

Solution :

Let α, β be the roots of $2y^2 - ay + 64 = 0$

$$\alpha+\beta = a/2$$

$$\alpha\beta = 32$$

Given $\alpha = 2\beta$

$\therefore \alpha+\beta = a/2$

$\Rightarrow 3\beta = a/2$

$\Rightarrow \beta = a/6$

$\alpha\beta = 32$

$2\beta . \beta = 32$

$\Rightarrow \beta^2 = 16$

$\Rightarrow \beta = \pm 4$

$$\therefore a/6 = \pm 4$$

$$a = \pm 24$$

6. If one root of the equation $3x^2 + kx + 81 = 0$ (having real roots) is the square of the other then find k.

Solution :

Let α, β be the roots of $3x^2 + kx + 81 = 0$

$$\alpha+\beta = -k/3$$

$$\alpha\beta = 27$$

Given $\alpha = \beta^2$

$\therefore \alpha+\beta = -k/3$ .... (1)

$\alpha\beta = 27$

$\Rightarrow \beta^2 . \beta = 27$

$\Rightarrow \beta^3 = 27$

$\therefore \beta = 3$

$\therefore \alpha = 9$

$\therefore (1) \Rightarrow 9 + 3 = -k/3$

$\Rightarrow -k/3 = 12$

$\Rightarrow k = -36$

XV. QUADRATIC GRAPHS

Key Points

- ✓ A parabola represents a Quadratic function.
- ✓ A quadratic function has the form f $(x) = ax^2 + bx + c$, where a, b, c are constants, and $a \neq 0$.
- ✓ The coefficient a in the general equation is responsible for parabolas to open upward or downward and vary in "width" ("wider" or "skinnier"), but they all have the same basic " $\cup$ " shape.
- ✓ The greater the quadratic coefficient, the narrower is the parabola.
- ✓ The lesser the quadratic coefficient, the wider is the parabola.
- ✓ A parabola is symmetric with respect to a line called the axis of symmetry. The point of intersection of the parabola and the axis of symmetry is called the vertex of the parabola. The graph of any second degree polynomial gives a curve called "parabola".
- ✓ If the graph of the given quadratic equation intersect the X axis at two distinct points, then the given equation has two real and unequal roots.
- ✓ If the graph of the given quadratic equation touch the X axis at only one point, then the given equation has only one root which is same as saying two real and equal roots.
- ✓ If the graph of the given equation does not intersect the X axis at any point then the given equation has no real root.
- ✓ If the straight line intersects the parabola at two distinct points, then the x coordinates of those points will be the roots of the given quadratic equation.
- ✓ If the straight line just touch the parabola at only one point, then the x coordinate of the common point will be the single root of the quadratic equation.
- ✓ If the straight line doesn't intersect or touch the parabola then the quadratic equation will have no real roots.

Example 3.48

Discuss the nature of solutions of the following quadratic equations.

i) $x^2 + x - 12 = 0$ ii) $x^2 - 8x + 16 = 0$ iii) $x^2 + 2x + 5 = 0$

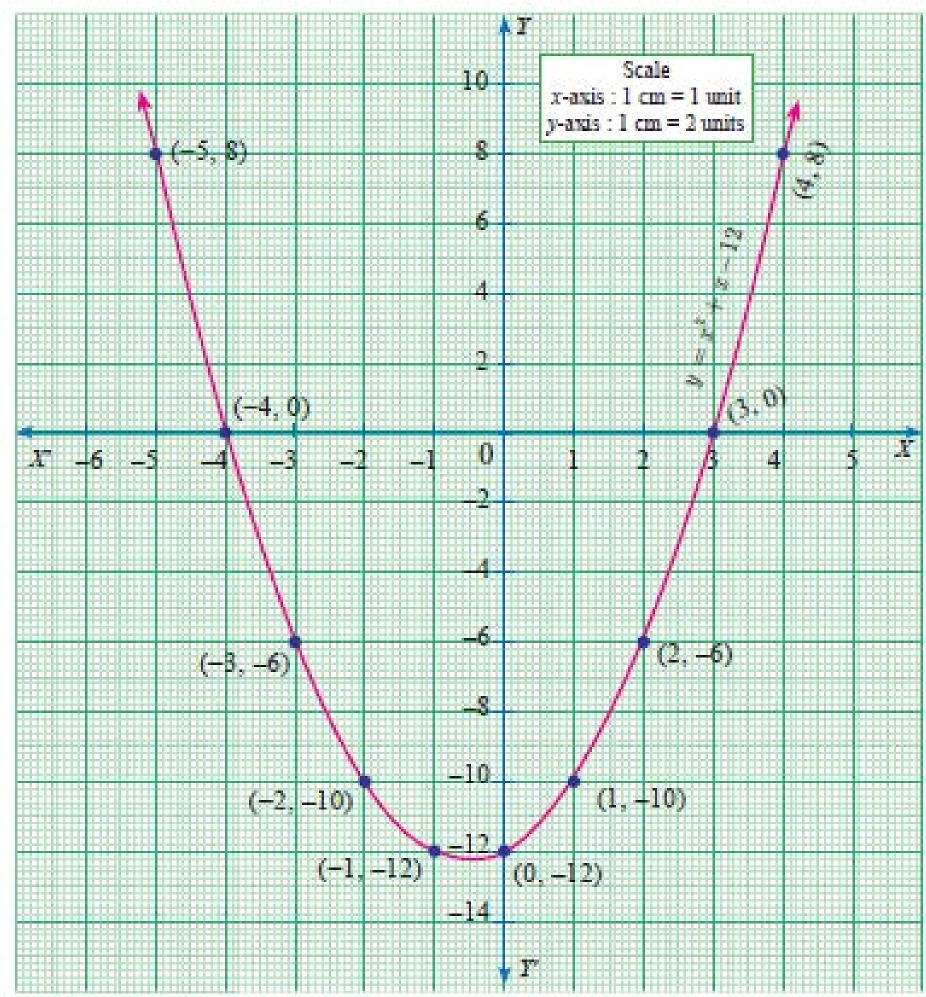
Solution :

i) $x^2 + x - 12 = 0$

Step 1 Prepare the table of values for the equation $y = x^2 + x - 12$

x	−5	−4	−3	−2	−1	0	1	2	3	4
y	8	0	−6	−10	−12	−12	−10	−6	0	8

Step 2 Plot the points for the above ordered pairs (x, y) on the graph using suitable scale.



Step 3 Draw the parabola and mark the co-ordinates of the parabola which intersect the X axis.

Step 4 The roots of the equation are the x coordinates of the intersecting points (–4, 0) and (3,0) of the parabola with the X axis which are −4 and 3 respectively.

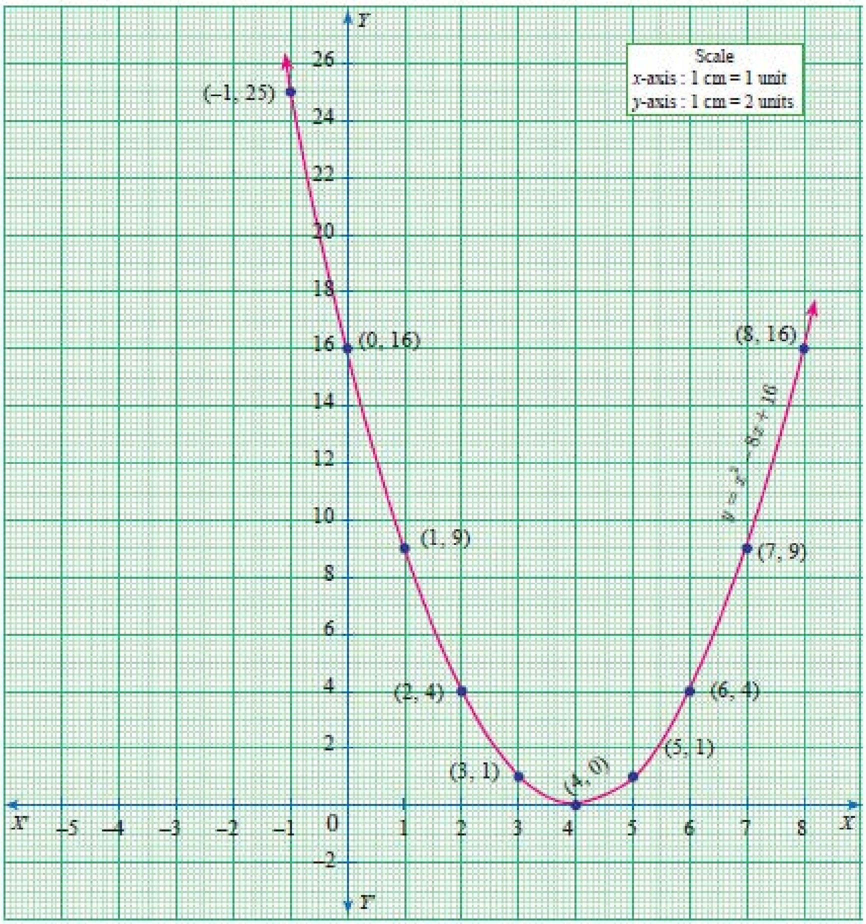
Since there are two points of intersection with the X axis, the quadratic equation $x^2 + x - 12$ has real and unequal roots.

ii) $x^2 - 8x + 16 = 0$

Step 1 Prepare the table of values for the equation $y = x^2 - 8x + 16$

x	−1	0	1	2	3	4	5	6	7	8
y	25	16	9	4	1	0	1	4	9	16

Step 2 Plot the points for the above ordered pairs (x, y) on the graph using suitable scale.



Step 3 Draw the parabola and mark the co-ordinates of the parabola which intersect with the X axis.

Step 4 The roots of the equation are the x coordinates of the intersecting points of the parabola with the X axis (4, 0) which is 4.

Since there is only one point of intersection with the X axis, the quadratic equation $x^2 - 8x + 16 = 0$ has real and equal roots.

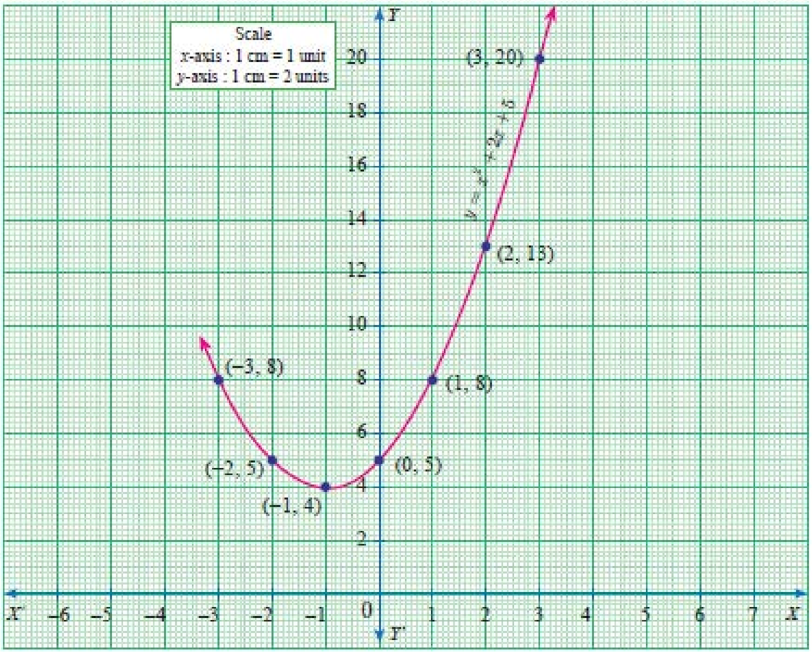
iii) $x^2 + 2x + 5 = 0$

Let $y = x^2 + 2x + 5$

Step 1 Prepare the table of values for the equation $y = x^2 + 2x + 5$

x	−3	−2	−1	0	1	2	3
y	8	5	4	5	8	13	20

Step 2 Plot the above ordered pairs (x, y) on the graph using suitable scale.



Step 3 Join the points by a free-hand smooth curve this smooth curve is the graph of

$y = x^2 + 2x + 5$

Step 4 The solutions of the given quadratic equation are the x coordinates of the intersecting points of the parabola the X axis.

Here the parabola doesn't intersect or touch the X axis.

So, we conclude that there is no real root for the given quadratic equation.

Example 3.49

Draw the graph of $y = 2x^2$ and hence solve $2x^2 - x - 6 = 0$

Solution :

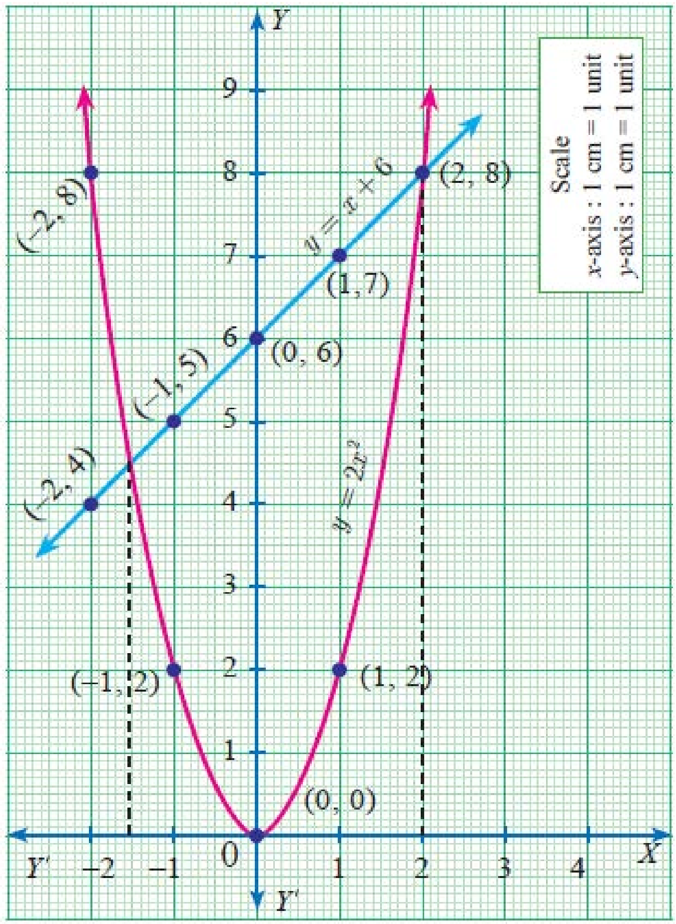
Step 1 Draw the graph of $y = 2x^2$ by preparing the table of values as below

x	−2	−1	0	1	2
y	8	2	0	2	8

Step 2 To solve $2x^2 - x - 6 = 0$, subtract $2x^2 - x - 6 = 0$ from $y = 2x^2$

$$\text{that is} \quad \begin{array}{l} y = 2x^2 \qquad (-) \\ 0 = 2x^2 - x - 6 \\ \hline y = x + 6 \\ \hline \end{array}$$

The equation $y = x + 6$ represents a straight line. Draw the graph of $y = x + 6$ by forming table of values as below



x	−2	−1	0	1	2
y	4	5	6	7	8

Step 3 Mark the points of intersection of the curve $y = 2x^2$ and the line $y = x + 6$. That is, (–1.5, 4.5) and (2,8)

Step 4 The x coordinates of the respective points forms the solution set {–1.5,2} for $2x^2 - x - 6 = 0$

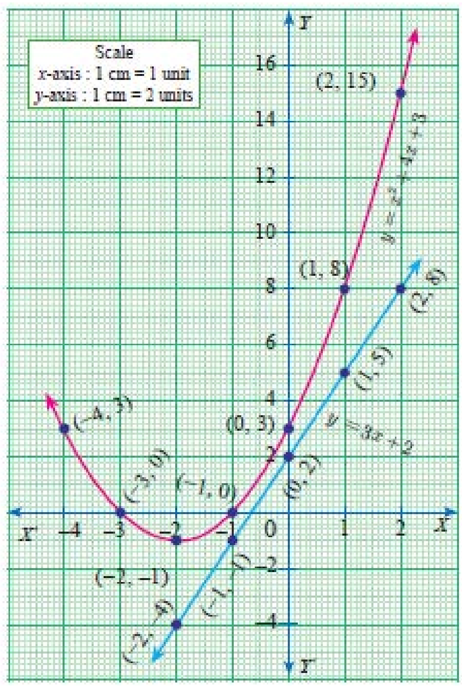
Example 3.50

Draw the graph of $y = x^2 + 4x + 3$ and hence find the roots of $x^2 + x + 1 = 0$

Solution :

Step 1 Draw the graph of $y = x^2 + 4x + 3$ by preparing the table of values as below

x	−4	−3	−2	−1	0	1	2
y	3	0	−1	0	3	8	15



Step 2 To solve $x^2 + x + 1 = 0$, subtract $x^2 + x + 1 = 0$ from $y = x^2 + 4x + 3$

$$\text{that is}\quad \begin{array}{r} y = x^2 + 4x + 3 \quad (-) \\ 0 = x^2 + x + 1 \\ \hline y = \qquad 3x + 2 \\ \hline \end{array}$$

The equation represent a straight line. Draw the graph of $y = 3x + 2$ by forming the table of values as below.

x	−2	−1	0	1	2
y	−4	−1	2	5	8

Step 3 Observe that the graph of $y = 3x + 2$ does not intersect or touch the graph of the parabola $y = x^2 + 4x + 3$

Thus $x^2 + x + 1 = 0$ has no real roots.

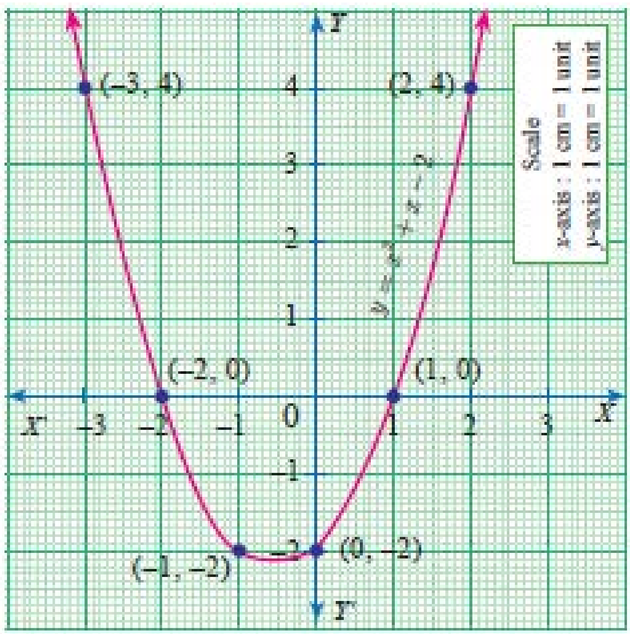
Example 3.51

Draw the graph of $y = x^2 + x - 2$ and hence solve $x^2 + x - 2 = 0$

Solution :

Step 1 Draw the graph of $y = x^2 + x - 2$ by preparing the table of values as below

x	−3	−2	−1	0	1	2
y	4	0	−2	−2	0	4



Step 2 To solve $x^2 + x - 2 = 0$, subtract $x^2 + x - 2 = 0$ from $y = x^2 + x - 2$

$$\text{that is} \quad \begin{array}{l} y = x^2 + x - 2 \quad (-) \\ 0 = x^2 + x + 2 \\ \hline y = \quad 0 \\ \hline \end{array}$$

The equation $y = 0$ represents the X axis.

Step 3 Mark the point of intersection of the curve $x^2 + x - 2$ with the X axis. That is (–2,0) and (1,0)

Step 4 The x coordinates of the respective points form the solution set $\{-2,1\}$ for $x^2 + x - 2 = 0$.

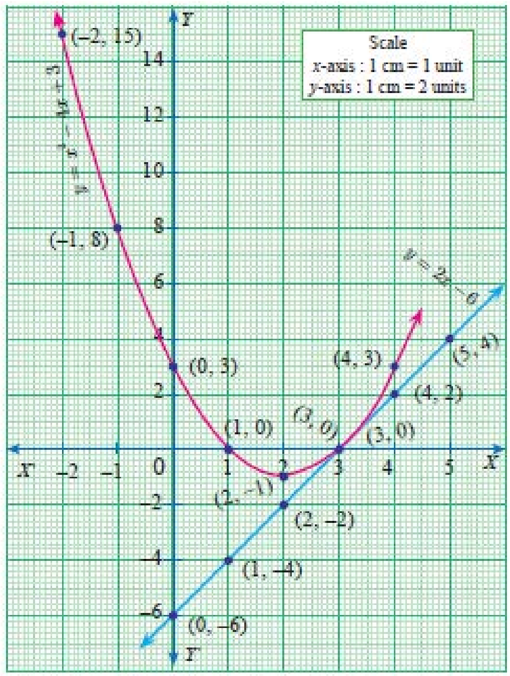
Example 3.52

Draw the graph of $y = x^2 - 4x + 3$ and use it to solve $x^2 - 6x + 9 = 0$

Solution :

Step 1 Draw the graph of $y = x^2 - 4x + 3$ by preparing the table of values as below

x	−2	−1	0	1	2	3	4
y	15	8	3	0	−1	0	3



Step 2 To solve $x^2 - 6x + 9 = 0$, subtract $x^2 - 6x + 9 = 0$ from $y = x^2 - 4x + 3$

$$\text{that is}\quad \begin{array}{r} y = x^2 - 4x + 3 \quad (-) \\ 0 = x^2 - 6x + 9 \\ \hline y = \quad 2x - 6 \\ \hline \end{array}$$

The equation $y = 2x - 6$ represents a straight line. Draw the graph of $y = 2x - 6$ forming the table of values as below.

x	0	1	2	3	4	5
y	−6	−4	−2	0	2	4

The line $y = 2x - 6$ intersect $y = x^2 - 4x + 3$ only at one point.

Step 3 Mark the point of intersection of the curve $y = x^2 - 4x + 3$ and $y = 2x - 6$ that is (3, 0)

Therefore, the x coordinate 3 is the only solution for the equation $x^2 - 6x + 9 = 0$.

EXERCISE 3.15

1. Graph the following quadratic equations and state their nature of solutions.

i) $x^2 - 9x + 20 = 0$ ii) $x^2 - 4x + 4 = 0$ iii) $x^2 + x + 7 = 0$

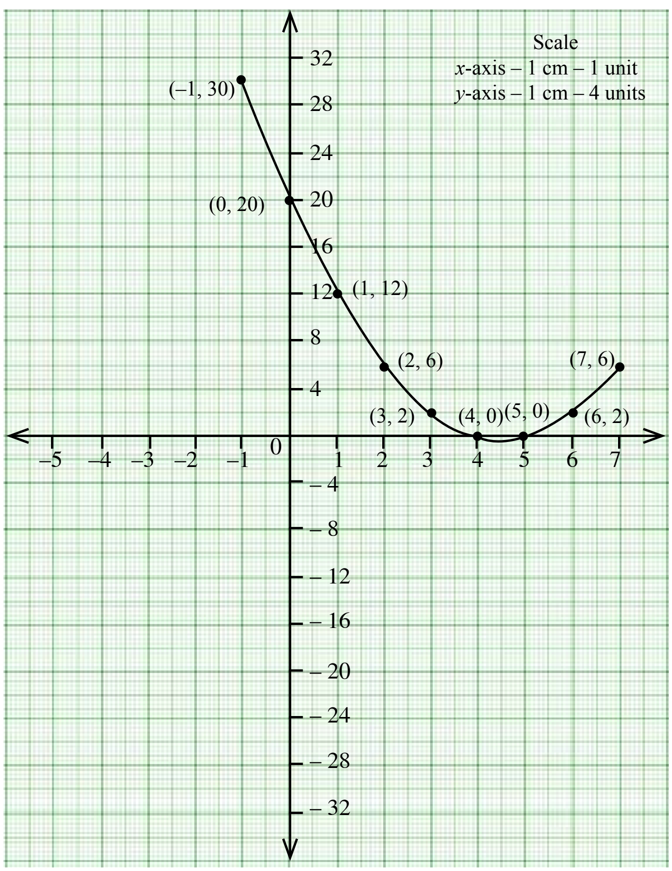
iv) $x^2 - 9 = 0$ v) $x^2 - 6x + 9 = 0$ vi) $(2x - 3)(x + 2) = 0$

Solution :

i) $x^2 - 9x + 20 = 0$

Let $y = x^2 - 9x + 20$

X :	– 2	– 1	0	1	2	3	4	5	6	7
x^2 :	4	1	0	1	4	9	16	25	36	49
$-9x$:	18	9	0	–9	–18	–27	–36	–45	–54	–63
20 :	20	20	20	20	20	20	20	20	20	20
$y = x^2 - 9x + 20$:	42	30	20	12	6	2	0	0	2	6



- Plot the points (–1, 30), (0, 20), (1, 12), (2, 6), (3, 2), (4, 0), (5, 0), (6, 2), (7, 6) on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = x^2 - 9x + 20$.
- Here, the curve meets x-axis at (4, 0), (5, 0).

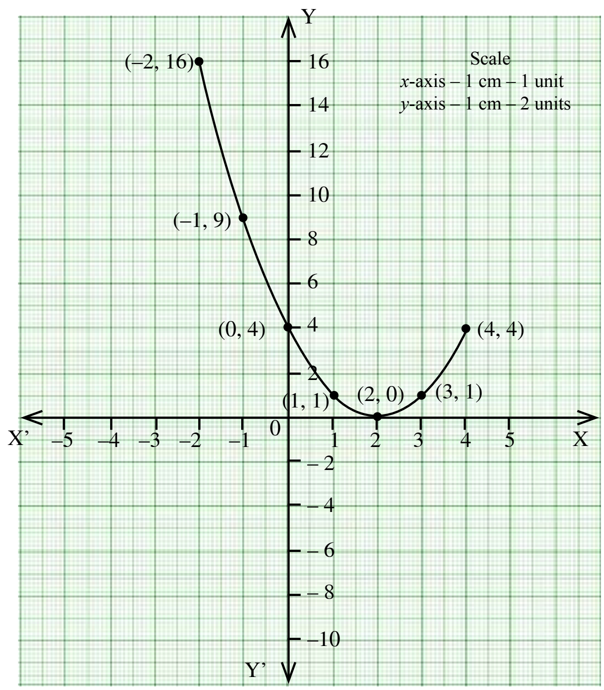
  ∴ The equation has real roots and x - coordinates of the points are $x = 4$, $x = 5$.

  ∴ Solution = {4, 5}

ii) $\mathbf{x^2 - 4x + 4 = 0}$

Let $y = x^2 - 4x + 4$

x :	– 2	– 1	0	1	2	3	4
x^2 :	4	1	0	1	4	9	16
$-4x$:	8	4	0	– 4	– 8	–12	–16
4 :	4	4	4	4	4	4	4
$y = x^2 - 4x + 4$:	16	9	4	1	0	1	4



- Plot the points (– 2, 16), (–1, 9), (0, 4), (1, 1), (2, 0), (3, 1), (4, 4) on the graph.
- Join all the points by a free-hand smoot curve. This curve is the graph of $y = x^2 - 4x + 4$.
- Here, the curve meets x-axis at (2, 0).

  ∴ The equation 2 equal roots.

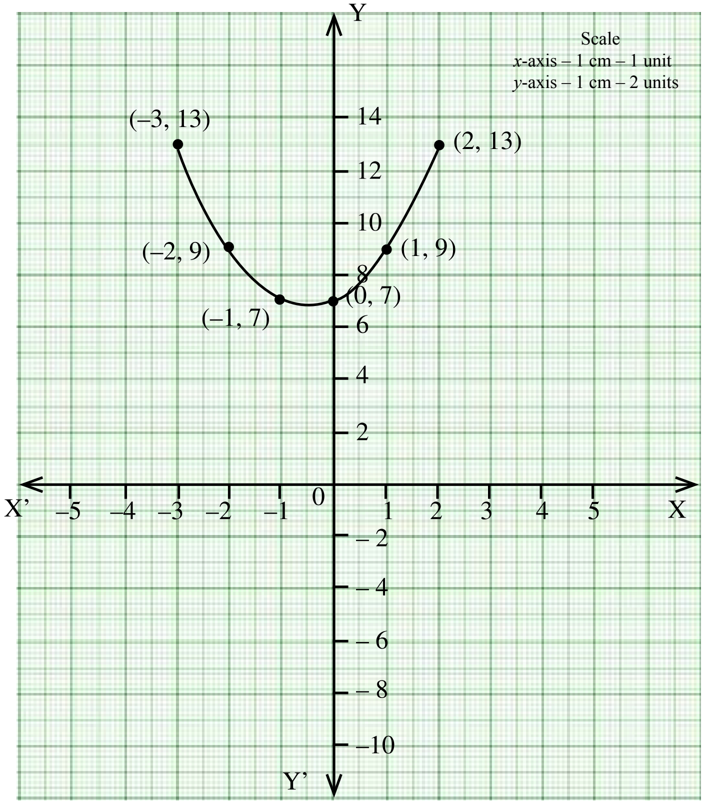
  ∴ The x - coordinates of the points is $x = 2$.

  ∴ Solution = {2, 2}

iii) $x^2 + x + 7 = 0$

Let y = $x^2 + x + 7$

x :	– 3	– 2	– 1	0	1	2	3
x^2 :	9	4	1	0	1	4	9
x :	–3	–2	–1	0	1	2	3
7 :	7	7	7	7	7	7	7
$y = x^2 + x + 7$:	13	9	7	7	9	13	19

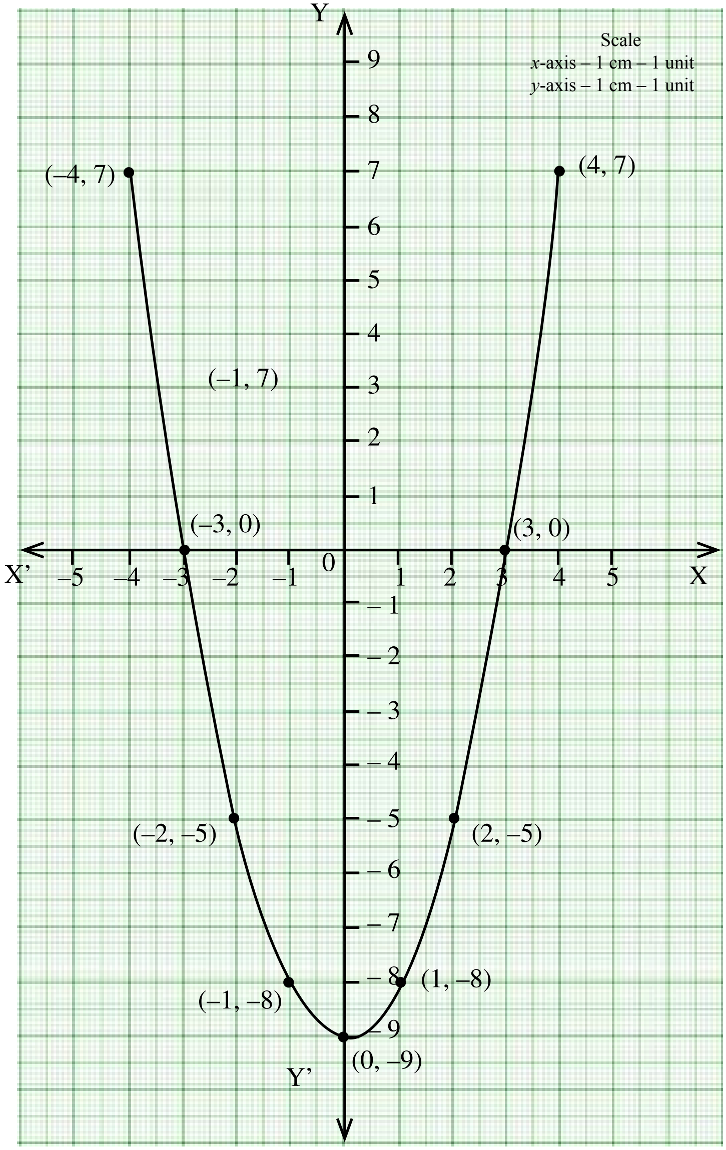


- Plot the points (–3, 13), (–2, 9), (–1, 7), (0, 7), (1, 9), (2, 13), (3, 19) on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = x^2 + x + 7$.
- Here the curve does not meet the x-axis and the curve has no real roots.

iv) $x^2 - 9 = 0$

Let y = $x^2 - 9$

x :	– 4	– 3	– 2	–1	0	1	2	3	4
x^2 : –9 :	16 –9	9 –9	4 –9	1 –9	0 –9	1 –9	4 –9	9 –9	16 –9
$y = x^2 - 9$:	7	0	–5	–8	–9	–8	–5	0	7



- Plot the points (–4, 7), (–3, 0), (–2, –5), (–1, –8), (0, –9) (1, –8), (2, –5), (3, 0), (4, 7) on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = x^2 - 9$.
- Here, the curve meets x-axis at 2 points (–3, 0), (3, 0)

  $\therefore$ The equation has real and unequal roots.

  $\therefore$ The x - coordinates are 3, –3 will be the solution.

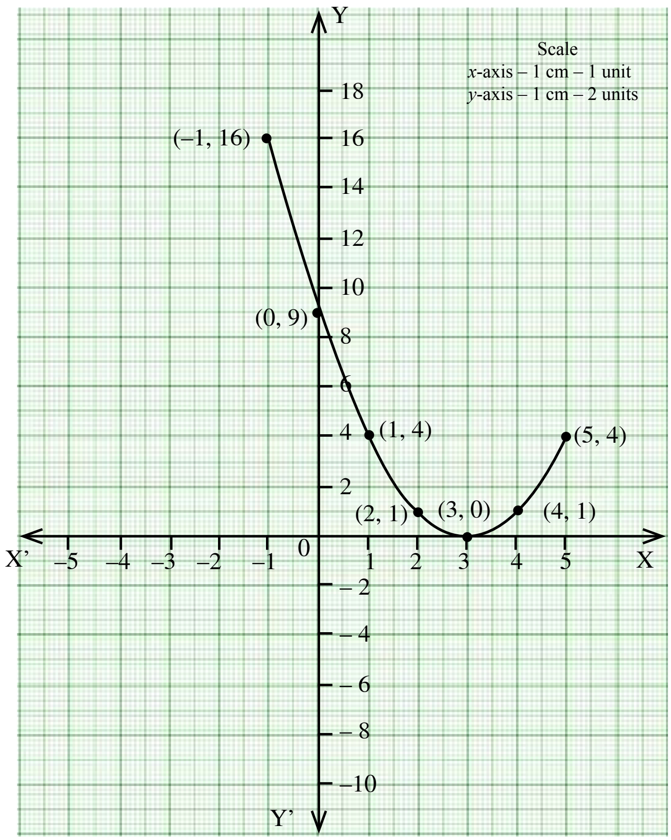
  $\therefore$ Solution = {–3, 3}

v) $\boldsymbol{x^2 - 6x + 9 = 0}$

Solution:

Let $y = x^2 - 6x + 9$

x :	-2	-1	0	1	2	3	4	5
x^2 :	4	1	0	1	4	9	16	25
$-6x$:	12	6	0	-6	-12	-18	-24	-30
9 :	9	9	9	9	9	9	9	9
$y = x^2 - 6x + 9$:	25	16	9	4	1	0	1	4



- Plot the points (–1, 16), (0, 9), (1, 4), (2, 1), (3, 0), (4, 1) (5, 4) on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = x^2 - 6x + 9$.
- Here, the curve meets x-axis at only one point (3, 0) and the equation has real and equal roots.

  $\therefore$ The x - coordinate 3 will be the solution.

  $\therefore$ Solution = {3, 3}

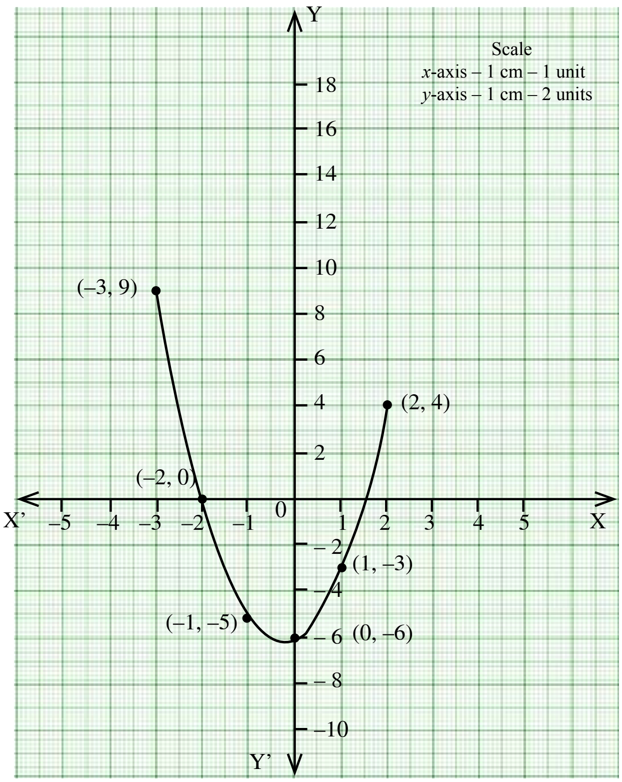
vi) $(2x - 3)(x + 2) = 0$

$\Rightarrow 2x^2 + x - 6 = 0$

Solution:

Let $y = 2x^2 + x - 6$

x :	– 3	– 2	–1	0	1	2
$2x^2$:	18	8	2	0	2	8
x :	–3	–2	–1	0	1	2
– 6 :	–6	–6	–6	– 6	– 6	– 6
$y = 2x^2 + x - 6$:	9	0	– 5	– 6	– 3	4



- Plot the points (–3, 9), (–2, 0), (–1, –5), (0, –6), (1, –3), (2, 4) on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = 2x^2 + x - 6$.
- Here, the curve meets x-axis at two points (–2, 0), (1.5, 0) and

$\therefore$ The equation has real and unequal roots.

$\therefore$ The x - coordinates are $x = -2, -1.5$ will be the solution.

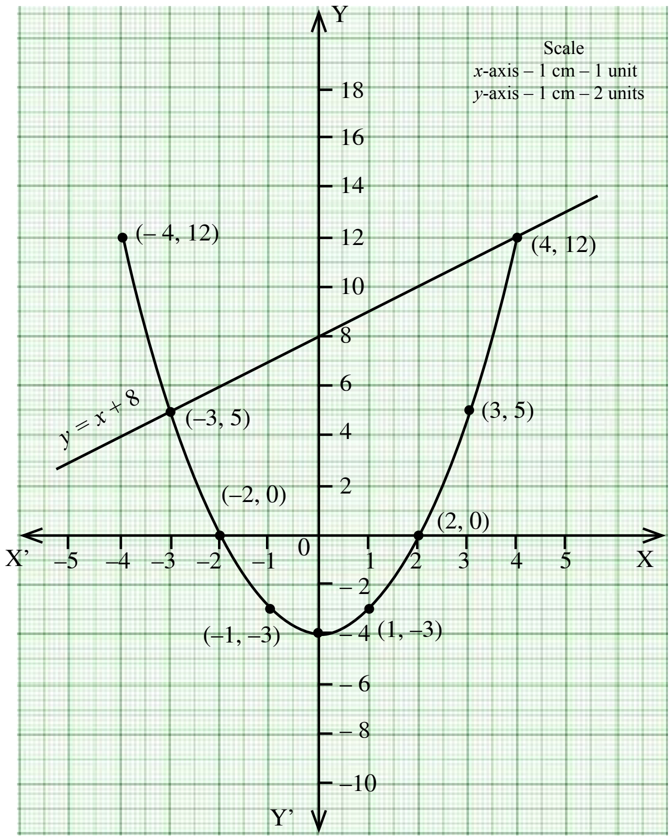
$\therefore$ Solution = {–2, 3/2}

2. Draw the graph of $y = x^2 - 4$ and hence solve $x^2 - x - 12 = 0$

Solution :

First, we draw the graph of $y = x^2 - 9$

x :	– 4	– 3	– 2	–1	0	1	2	3	4
x^2 :	16	9	4	1	0	1	4	9	16
–4 :	– 4	– 4	– 4	– 4	– 4	– 4	– 4	– 4	– 4
$y = x^2 - 4$:	12	5	0	– 3	– 4	– 3	0	5	12



- Plot the points (– 4, 12), (– 3, 5), (–2, 0), (–1, –3), (0, –4), (1, –3), (2, 0), (3, 5), (4, 12) on the graph.
- To solve $x^2 - x - 12 = 0$, subtract $x^2 - x - 12 = 0$ from $y = x^2 - 4$.

$$\text{from} \quad y = x^2 - 4$$

$$\begin{array}{l} y = x^2 + 0x - 4 \\ 0 = x^2 - x - 12 \\ \hline y = \quad\quad x + 8 \\ \hline \end{array}$$

- We draw the graph of $y = x + 8$.

x	– 4	– 3	– 2	– 1	0	1	2	3	4
y	4	5	6	7	8	9	10	11	12

- The line meets the curve at (–3, 5), (4, 12).

  $\therefore$ The x - coordinates $x = -3$, $x = 4$ will be the solution of $x^2 - x - 12 = 0$.

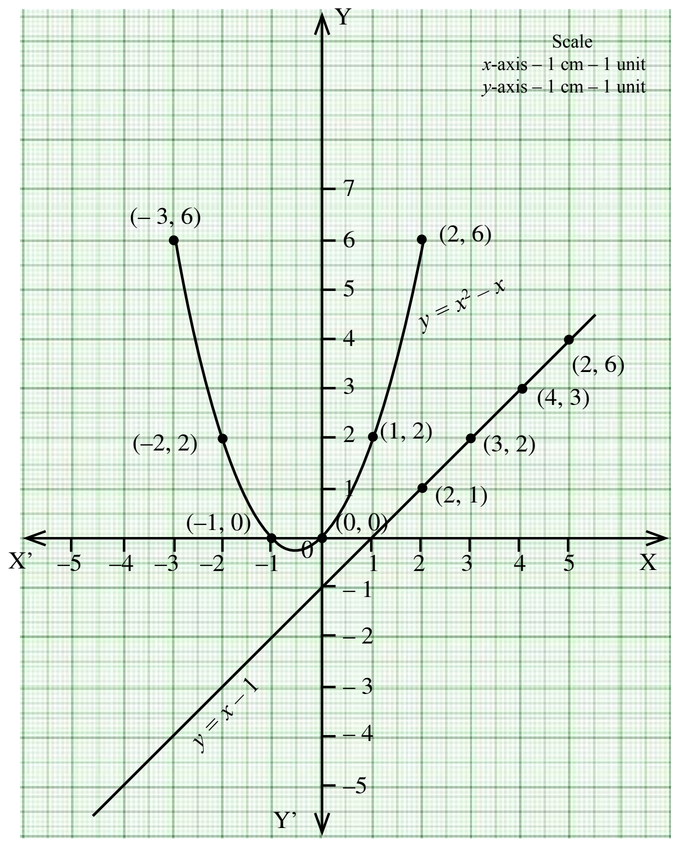
  $\therefore$ Solution = {–3, 4}

3. Draw the graph of $y = x^2 + x$ and hence solve $x^2 + 1 = 0$

Solution :

First, we draw the graph of $y = x^2 + x$.

x :	– 3	– 2	–1	0	1	2
x^2 :	9	4	1	0	1	4
x :	– 3	– 2	– 1	0	1	2
$y = x^2 + x$:	6	2	0	0	2	6



- Plot the points (–3, 6), (–2, 2), (–1, 0), (0, 0), (1, 2), (2, 6) on the graph.
- To solve $x^2 + 1 = 0$, subtract $x^2 + 1 = 0$ from $y = x^2 + x$.

$$\begin{array}{l} y = x^2 + \quad x \\ 0 = x^2 - 0x + 1 \\ \hline y = \qquad x - 1 \\ \hline \end{array}$$

- Draw the graph of $y = x - 1$.

x	– 4	– 3	– 2	– 1	0	1	2	3	4	5
y	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4

- The line $y = x - 1$ does not meet the curve $y = x^2 + x$ and the equation has no real roots.

4. Draw the graph of $y = x^2 + 3x + 2$ and use it to solve $x^2 + 2x + 1 = 0$

Solution :

First, we draw the graph of $y = x^2 + 3x + 2$.

x :	– 4	– 3	– 2	–1	0	1	2	3
x^2 :	16	9	4	1	0	1	4	9
$3x$:	– 12	– 9	– 6	– 3	0	3	6	9
2 :	2	2	2	2	2	2	2	2
$y = x^2 + 3x + 2$:	6	2	0	0	2	6	12	20

- Plot the points (–4, 6), (–3, 2), (–2, 0), (–1, 0), (0, 2), (1, 6), (2, 12), (3, 20) on the graph.
- Join all the points to draw a free-hand smooth curve.
- To solve $x^2 + 2x + 1 = 0$, subtract $x^2 + 2x + 1 = 0$ from $y = x^2 + 3x + 2$.

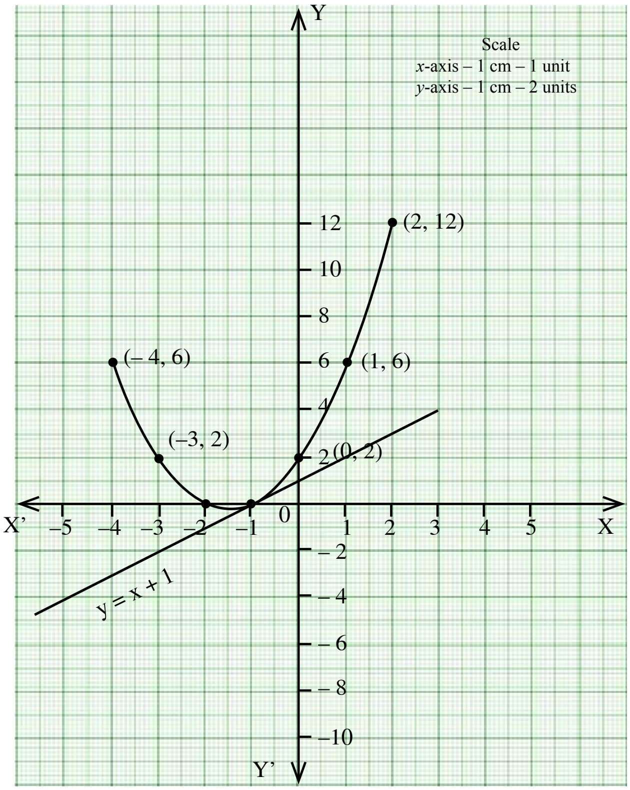
$$\begin{array}{l} y = x^2 + 3x + 2 \\ 0 = x^2 + 2x + 1 \\ \hline y = \qquad x + 1 \\ \hline \end{array}$$

- Draw the graph of $y = x + 1$.

x	– 4	– 3	– 2	– 1	0	1	2	3	4
y	– 3	– 2	– 1	0	1	2	3	4	5

- The line $y = x + 1$ meets the curve $y = x^2 + 3x + 2$ at (–1, 0) only and the equation $x^2+2x + 1 = 0$ has 2 equal roots.

$\therefore$ Solution = {–1, –1}



5. Draw the graph of $y = x^2 + 3x - 4$ and hence use it to solve $x^2 + 3x - 4 = 0$

Solution :

First, we draw the graph of $y = x^2 + 3x - 4$.

x :	– 5	– 4	– 3	–2	– 1	0	1	2	3
x^2 : $3x$: -4 :	25 –15 – 4	16 –12 –4	9 –9 –4	4 –6 –4	1 –3 – 4	0 0 – 4	1 3 – 4	4 6 – 4	9 9 – 4
$y = x^2 + 3x - 4$:	6	0	– 4	– 6	– 6	– 4	0	6	14

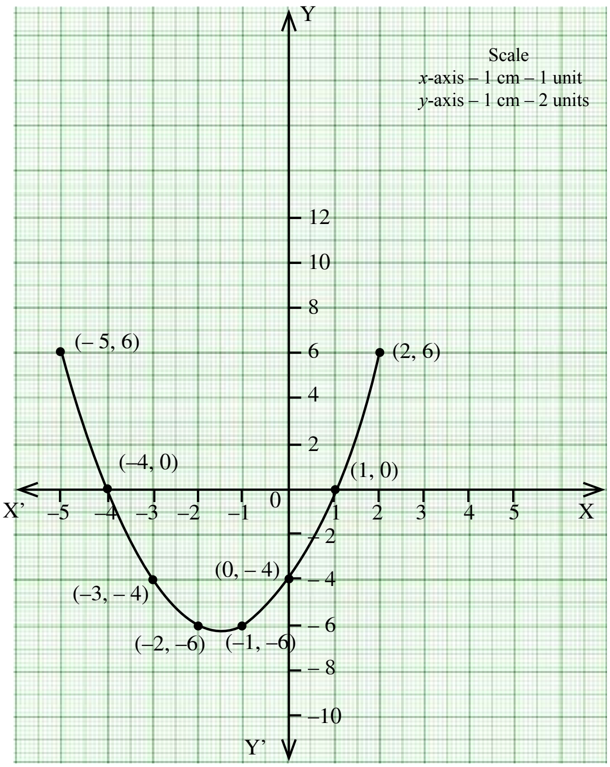
- Plot the points (– 4, 0), (–3, – 4), (– 2, – 6), (– 1, – 6), (0, – 4), (1, 0), (2, 6), (3, 14), (4, 24) on the graph.
- Join all the points to draw a free-hand smooth curve.
- To solve $x^2 + 3x - 4 = 0$, subtract $x^2 + 3x - 4 = 0$ from $y = x^2 + 3x - 4$.

$$\begin{array}{l} y = x^2 + 3x - 4 \\ 0 = x^2 + 3x - 4 \\ \hline y = \quad 0 \\ \hline \end{array}$$

which is the equation of x - axis.

- The curve meets x-axis at $(-4, 0)$, $(1, 0)$ and the x co-ordinates of the points $x = -4$, $x = 1$ will be the solution of $x^2 + 3x - 4 = 0$.

  $\therefore$ Solution = $\{-4, 1\}$

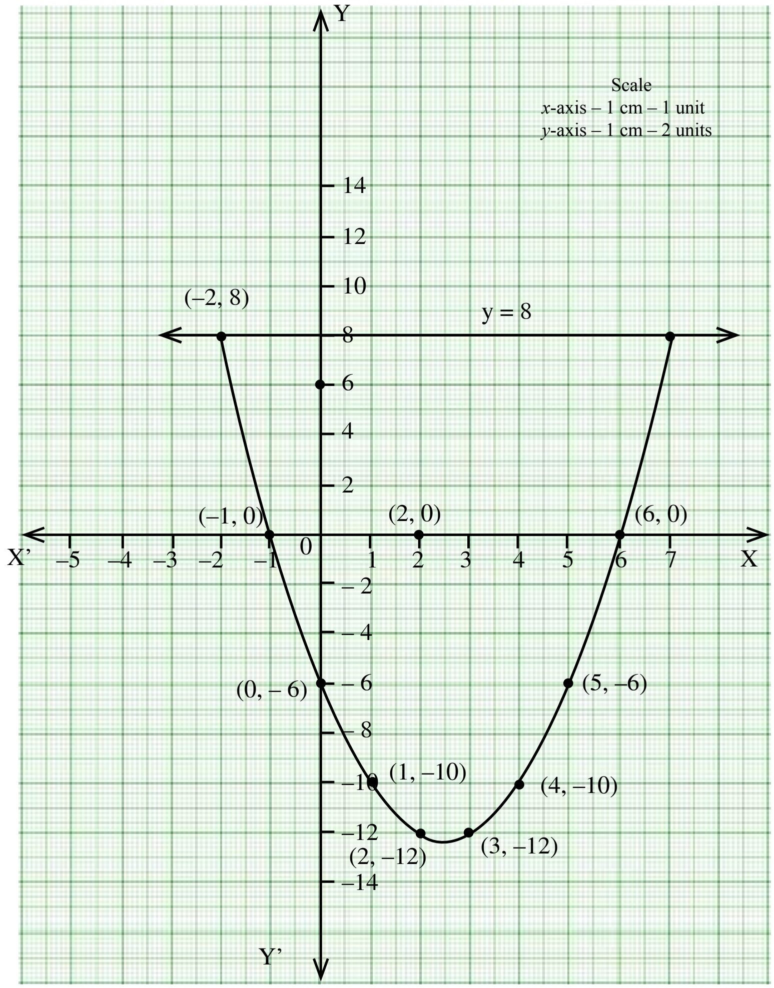


6. Draw the graph of $y = x^2 - 5x - 6$ and hence solve $x^2 - 5x - 14 = 0$

Solution :

First, we draw the graph of $y = x^2 - 5x + 6$.

x :	– 3	– 2	– 1	0	1	2	3	4	5	6	7
x^2 :	9	4	1	0	1	4	9	16	25	36	49
$-5x$:	15	10	5	0	– 5	–10	– 15	–20	–25	–30	–35
-6 :	– 6	– 6	– 6	– 6	– 6	– 6	– 6	– 6	– 6	– 6	– 6
$y = x^2 - 5x + 6$:	18	8	0	– 6	– 10	– 12	– 12	– 10	– 6	0	8



- Plot the points and join them by a hand-free smooth curve.
- To solve $x^2 - 5x - 14 = 0$, subtract $x^2 - 5x - 14 = 0$ from $y = x^2 - 5x - 6$.

$$\begin{array}{l} y = x^2 - 5x - \ \ 6 \\ 0 = x^2 - 5x - 14 \\ \hline y = \qquad 8 \\ \hline \end{array}$$

a line parallel to x-axis.

- The line $y = 8$ meets the curve $y = x^2 - 5x + 6$ at $(-2, 8)$, $(7, 8)$.

  The x co-ordinates of the points $x = -2$, $x = 7$ will be the solution of $x^2 - 5x - 14 = 0$.

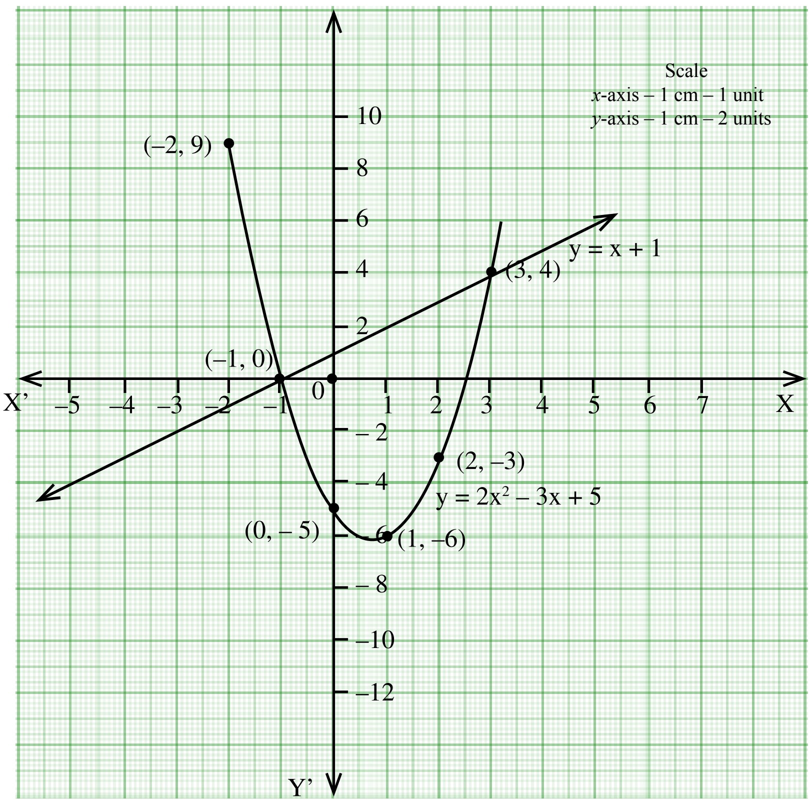
  $\therefore$ Solution = $\{-2, 7\}$

7. Draw the graph of $y = 2x^2 - 3x - 5$ and hence solve $2x^2 - 4x - 6 = 0$

Solution :

First, we draw the graph of $y = 2x^2 - 3x - 5$.

x :	– 2	– 1	0	1	2	3
$2x^2$:	8	2	0	2	8	18
$-3x$:	6	3	0	– 3	– 6	– 9
– 5 :	– 5	– 5	– 5	– 5	– 5	– 5
$y = 2x^2 - 3x - 5$:	9	0	– 5	– 6	– 3	4



- Plot the points on the graph and join all of them by a hand-free smooth curve.
- To solve $2x^2 - 4x - 6 = 0$, subtract it from $y = 2x^2 - 3x - 5$.

$$\begin{array}{l} y = 2x^2 - 3x - 5 \\ 0 = 2x^2 - 4x - 6 \\ \hline y = \qquad x + 1 \\ \hline \end{array}$$

- Draw the graph of $y = x + 1$.

x	-4	-3	-2	-1	0	1	2	3	4
y	-3	-2	-1	0	1	2	3	4	5

- The line $y = x + 1$ meets the curve $y = 2x^2 - 3x - 5$ at $(-1, 0)$, $(3, 4)$.

  The x co-ordinates of the points $x = -1$, $x = 3$ will be the solution set.

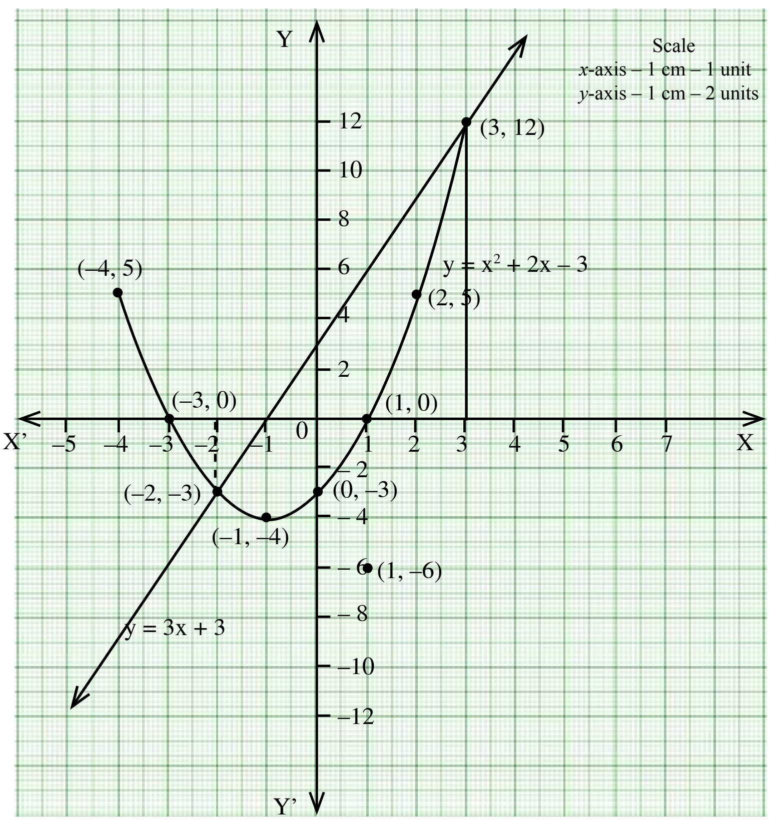
  $\therefore$ Solution = $\{-1, 3\}$

8. Draw the graph of $y = (x - 1)(x + 3)$ and hence solve $x^2 - x - 6 = 0$

Solution :

First, we draw the graph of $y = x^2 + 2x - 3$.

x :	-4	-3	-2	-1	0	1	2	3
x^2 :	16	9	4	1	0	1	4	9
$2x$:	-8	-6	-4	-2	0	2	4	6
-3 :	-3	-3	-3	-3	-3	-3	-3	-3
$y = x^2 + 2x - 3$:	5	0	-3	-4	-3	0	5	12



- Plot the points on the graph and join all of them by a hand-free smooth curve.
- To solve $x^2 - x - 6 = 0$, subtract it from $y = x^2 + 2x - 3$.

$$\begin{array}{l} y = x^2 + 2x - 3 \\ 0 = x^2 - x - 6 \\ \hline y = 3x + 3 \\ \hline \end{array}$$

- Draw the graph of $y = 3x + 3$.

x	-4	-3	-2	-1	0	1	2	3
y	-9	-6	-3	0	3	6	9	12

- The line meets the corve at $(-2, -3)$, $(3, 12)$.

  The x co-ordinates of the points $x = -2$, $x = 3$ which are the solution.

  $\therefore$ Solution = $\{-2, 3\}$

XVI. MATRICES

Key Points

✓ A matrix is a rectangular array of elements. The horizontal arrangements are called rows and vertical arrangements are called columns.

✓ If a matrix A has m number of rows and n number of columns, then the order of the matrix A is (Number of rows) × (Number of columns) that is, m × n.We read m × n as m cross n or m by n.

✓ General form of a matrix A with m rows and n columns (order m × n) can be written in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{13} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

where, a_{11}, a_{12} ..... denote entries of the matrix.

✓ a_{ij} is the element in the i^{th} row and j^{th} column and is referred as $(i, j)^{th}$ element.

✓ The total number of entries in the matrix $A = (a_{ij})_{m \times n}$ is mn.

✓ A matrix is said to be a **row matrix** if it has only one row and any number of columns. A row matrix is also called as a row vector.

- ✓ A matrix is said to be a **column matrix** if it has only one column and any number of rows. It is also called as a column vector.
- ✓ A matrix in which the number of rows is equal to the number of columns is called a **square matrix**. Thus a matrix $A = (a_{ij})_{m\times n}$ will be a square matrix if $m = n$.
- ✓ In a square matrix, the elements of the form $a_{11}, a_{22}, a_{33}, \ldots$ (i.e) a_{ii} are called leading diagonal elements.
- ✓ A square matrix, all of whose elements, except those in the leading diagonal are zero is called a **diagonal matrix.**
- ✓ A diagonal matrix in which all the leading diagonal elements are equal is called a **scalar matrix.**
- ✓ A square matrix in which elements in the leading diagonal are all "1" and rest are all zero is called an **identity matrix** (or) **unit matrix**.
- ✓ A matrix is said to be a **zero matrix** or null matrix if all its elements are zero.
- ✓ The matrix which is obtained by interchanging the elements in rows and columns of the given matrix A is called transpose of A and is denoted by A^T (read as A transpose).
- ✓ A square matrix in which all the entries above the leading diagonal are zero is called a lower **triangular matrix.**
- ✓ If all the entries below the leading diagonal are zero, then it is called an **upper triangular matrix.**
- ✓ Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B. That is, $a_{ij} = b_{ij}$ for all i, j.
- ✓ The negative of a matrix $A_{m\times n}$ denoted by $-A_{m\times n}$ is the matrix formed by replacing each element in the matrix $A_{m\times n}$ with its additive inverse.

Example 3.53

Consider the following information regarding the number of men and women workers in three factories I, II and III.

Factory	Men	Women
I	23	18
II	47	36
III	15	16

Represent the above information in the form of a matrix. What does the entry in the second row and first column represent?

Solution :

The information is represented in the form of a 3 × 2 matrix as follows

$$A = \begin{pmatrix} 23 & 18 \\ 47 & 36 \\ 15 & 16 \end{pmatrix}$$

The entry in the second row and first column represent that there are 47 men workers in factory II.

Example 3.54

If a matrix has 16 elements, what are the possible orders it can have?

Solution :

We know that a matrix of order m × n, has mn elements. Thus to find all possible orders of a matrix with 16 elements, we will find all ordered pairs of natural numbers whose product is 16.

Such ordered pairs are (1,16), (16,1), (4,4), (8,2), (2,8)

Hence possible orders are 1×16, 16×1, 4×4, 2×8, 8×2.

Example 3.55

Construct a 3 × 3 matrix whose elements are $a_{ij} = i^2 j^2$

Solution :

The general 3 × 3 matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad a_{ij} = i^2 j^2$$

$a_{11} = 1^2 \times 1^2 = 1 \times 1 = 1$;

$a_{12} = 1^2 \times 2^2 = 1 \times 4 = 4$;

$a_{13} = 1^2 \times 3^2 = 1 \times 9 = 9$;

$a_{21} = 2^2 \times 1^2 = 4 \times 1 = 4$;

$a_{22} = 2^2 \times 2^2 = 4 \times 4 = 16$;

$a_{23} = 2^2 \times 3^2 = 4 \times 9 = 36$;

$a_{31} = 3^2 \times 1^2 = 9 \times 1 = 9$;

$a_{32} = 3^2 \times 2^2 = 9 \times 4 = 36$;

$a_{33} = 3^2 \times 3^2 = 9 \times 9 = 81$;

Hence the required matrix is $A = \begin{pmatrix} 1 & 4 & 9 \\ 4 & 16 & 36 \\ 9 & 36 & 81 \end{pmatrix}$

Example 3.56

Find the value of a, b, c, d from the equation $\begin{pmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$

Solution :

The given matrices are equal. Thus all corresponding elements are equal.

Therefore, $a - b = 1$ .... (1)

$2a + c = 5$ .... (2)

$2a - b = 0$ .... (3)

$3c + d = 2$ .... (4)

(3) gives $2a - b = 0$

$2a = b$ .... (5)

Put $2a = b$ in equation (1),

$a - 2a = 1$ gives $a = -1$

Put $a = -1$ in equation (5),

$2(-1) = b$ gives $b = -2$

Put $a = -1$ in equation (2),

$2(-1) + c = 5$ gives $c = 7$

Put $c = 7$ in equation (4),

$3(7) + d = 2$ gives $d = -19$

Therefore, $a = -1, b = -2, c = 7, d = -19$

EXERCISE 3.16

1. In the matrix A = $\begin{pmatrix} 8 & 9 & 4 & 3 \\ -1 & \sqrt{7} & \frac{\sqrt{3}}{2} & 5 \\ 1 & 4 & 3 & 0 \\ 6 & 8 & -11 & 1 \end{pmatrix}$, write

(i) The number of elements

(ii) The order of the matrix

(iii) Write the elements a_{22}, a_{23}, a_{24}, a_{34}, a_{43}, a_{44}.

Solution :

i) A has 4 rows and 4 columns

Number of elements $= 4 \times 4$

$= 16$

ii) Order of the matrix $= 4 \times 4$

iii) $a_{22} = \sqrt{7}$, $a_{23} = \sqrt{3}/2$

$a_{24} = 5$ $a_{34} = 0$, $a_{43} = -11$, $a_{44} = 1$

2. If a matrix has 18 elements, what are the possible orders it can have? What if it has 6 elements?

Solution :

Given, a matrix has 18 elements

The possible orders of the matrix are

18×1, 1×18, 9×2, 2×9, 6×3, 3×6

If the matrix has 6 elements

The order are 1×6, 6×1, 3×2, 2×3

3. Construct a 3 × 3 matrix whose elements are given by

$i)\ a_{ij} = |i - 2j| \qquad ii)\ a_{ij} = \dfrac{(i+j)^3}{3}$

Solution :

i) Given $a_{ij} = |i - 2j|$, 3×3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$a_{11} = |1-2| = |-1| = 1$

$a_{12} = |1-4| = |-3| = 3$

$a_{13} = |1-6| = |-5| = 5$

$a_{21} = |2-2| = 0$

$a_{22} = |2-4| = |-2| = 2$

$a_{23} = |2-6| = |-4| = 4$

$a_{31} = |3-2| = |1| = 1$

$a_{32} = |3-4| = |-1| = 1$

$a_{33} = |3-6| = |-3| = 3$

$$\therefore\ A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix}$$

$ii)\ a_{ij} = \dfrac{(i+j)^3}{3}$

$a_{11} = 8/3$, $\quad a_{12} = 27/3 = 9$, $\quad a_{13} = 64/3$

$a_{21} = 27/3 = 9$, $a_{22} = 64/3$, $\quad a_{23} = 125/3$

$a_{31} = 64/3$, $\quad a_{32} = 125/3$, $\quad a_{33} = 216/3 = 72$

$$\therefore\ A = \begin{pmatrix} 8/3 & 9 & 64/3 \\ 9 & 64/3 & 125/3 \\ 64/3 & 125/3 & 72 \end{pmatrix}$$

4. **If** $\begin{pmatrix} 5 & 4 & 3 \\ 1 & -7 & 9 \\ 3 & 8 & 2 \end{pmatrix}$ **then find the transpose of A.**

Solution :

Given

$$A = \begin{pmatrix} 5 & 4 & 3 \\ 1 & -7 & 9 \\ 3 & 8 & 2 \end{pmatrix}$$

$$\therefore \; A^T = \begin{pmatrix} 5 & 1 & 3 \\ 4 & -7 & 8 \\ 3 & 9 & 2 \end{pmatrix}$$

---

5. **If A =** $\begin{pmatrix} \sqrt{7} & -3 \\ -\sqrt{5} & 2 \\ \sqrt{3} & -5 \end{pmatrix}$ **then find the transpose of-A.**

Solution :

Given

$$A = \begin{pmatrix} \sqrt{7} & -3 \\ -\sqrt{5} & 2 \\ \sqrt{3} & -5 \end{pmatrix}$$

$$-A = \begin{pmatrix} -\sqrt{7} & 3 \\ \sqrt{5} & -2 \\ -\sqrt{3} & 5 \end{pmatrix}$$

$$\therefore \text{Transpose of } -A = \begin{pmatrix} -\sqrt{7} & \sqrt{5} & -\sqrt{3} \\ 3 & -2 & 5 \end{pmatrix}$$

---

6. **If A =** $\begin{pmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & \frac{5}{2} \\ 8 & 3 & 1 \end{pmatrix}$ **then verify** $(A^T)^T = A$

Solution :

Given

$$A = \begin{pmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & 5/2 \\ 8 & 3 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 5 & -\sqrt{17} & 8 \\ 2 & 0.7 & 3 \\ 2 & 5/2 & 1 \end{pmatrix}$$

$$\left(A^T\right)^T = \begin{pmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & 5/2 \\ 8 & 3 & 1 \end{pmatrix} = A$$

---

7. **Find the values of x, y and z from the following equations**

$i)\ \begin{pmatrix} 12 & 3 \\ x & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} y & z \\ 3 & 5 \end{pmatrix}$ $\quad ii)\ \begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$

$iii)\ \begin{pmatrix} x+y+z \\ x+z \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

Solution :

i) Given

$$\begin{pmatrix} 12 & 3 \\ x & 3/2 \end{pmatrix} = \begin{pmatrix} y & z \\ 3 & 5 \end{pmatrix}$$

$$\Rightarrow \quad x = 3, \; y = 12, \; z = 3$$

ii) Given

$$\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$$

$\Rightarrow$ x + y = 6, xy = 8, 5 + z = 5

x = 2 (or) 4, $\Rightarrow$ z = 0

y = 4 (or) 2

iii) Given

$$\begin{pmatrix} x+y+z \\ x+z \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$$

$\Rightarrow$ x + y + z = 0	x + z = 5	y + z = 7
$\Rightarrow$ 5 + y = 9	$\Rightarrow$ x + 3 = 5	$\Rightarrow$ 4 + z = 7
$\Rightarrow$ y = 4	$\Rightarrow$ x = 2	$\Rightarrow$ z = 3

XVII. OPERATIONS ON MATRICES

Key Points

- ✓ Two matrices can be added or subtracted if they have the same order. To add or subtract two matrices, simply add or subtract the corresponding elements.
- ✓ We can multiply the elements of the given matrix A by a non-zero number k to obtain a new matrix kA whose elements are multiplied by k. The matrix kA is called scalar multiplication of A.
- ✓ The null matrix or zero matrix is the identity for matrix addition.
- ✓ If A be any given matrix then –A is the additive inverse of A.

Example 3.57

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix}$, find A + B

Solution :

$$A+B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+7 & 3+0 \\ 4+1 & 5+3 & 6+1 \\ 7+2 & 8+4 & 9+0 \end{pmatrix} = \begin{pmatrix} 2 & 9 & 3 \\ 5 & 8 & 7 \\ 9 & 12 & 9 \end{pmatrix}$$

Example 3.58

Two examinations were conducted for three groups of students namely group 1, group 2, group 3 and their data on average of marks for the subjects Tamil, English, Science and Mathematics are given below in the form of matrices A and B. Find the total marks of both the examinations for all the three groups.

$$A = \begin{array}{c} \\ \text{Group 1} \\ \text{Group 2} \\ \text{Group 3} \end{array} \begin{pmatrix} \text{Tamil} & \text{English} & \text{Science} & \text{Mathematics} \\ 22 & 15 & 14 & 23 \\ 50 & 62 & 21 & 30 \\ 53 & 80 & 32 & 40 \end{pmatrix}$$

$$B = \begin{array}{c} \\ \text{Group 1} \\ \text{Group 2} \\ \text{Group 3} \end{array} \begin{pmatrix} \text{Tamil} & \text{English} & \text{Science} & \text{Mathematics} \\ 20 & 38 & 15 & 40 \\ 18 & 12 & 17 & 80 \\ 81 & 47 & 52 & 18 \end{pmatrix}$$

Solution :

The total marks in both the examinations for all the three groups is the sum of the given matrices.

$$A+B = \begin{pmatrix} 22+20 & 15+38 & 14+15 & 23+40 \\ 50+18 & 62+12 & 21+17 & 30+80 \\ 53+81 & 80+47 & 32+52 & 40+18 \end{pmatrix} = \begin{pmatrix} 42 & 53 & 29 & 63 \\ 68 & 74 & 38 & 110 \\ 134 & 127 & 84 & 58 \end{pmatrix}$$

Example 3.59

If A = $\begin{pmatrix} 1 & 3 & -2 \\ 5 & -4 & 6 \\ -3 & 2 & 9 \end{pmatrix}$ B = $\begin{pmatrix} 1 & 8 \\ 3 & 4 \\ 9 & 6 \end{pmatrix}$ find A+B.

Solution :

It is not possible to add A and B because they have different orders.

Example 3.60

If A = $\begin{pmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{pmatrix}$ B = $\begin{pmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{pmatrix}$ then Find 2A + B.

Solution :

Since A and B have same order 3 × 3, 2A + B is defined.

$$\text{We have } 2A + \mathbf{B} = 2\begin{pmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{pmatrix} + \begin{pmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 16 & 12 \\ 2 & 6 & 18 \\ -8 & 6 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 27 & 9 \\ 1 & 8 & 22 \\ -1 & 11 & -2 \end{pmatrix}$$

Example 3.61

If $A = \begin{pmatrix} 5 & 4 & -2 \\ \frac{1}{2} & \frac{3}{4} & \sqrt{2} \\ 1 & 9 & 4 \end{pmatrix}$ B = $\begin{pmatrix} -7 & 4 & -3 \\ \frac{1}{4} & \frac{7}{2} & 3 \\ 5 & -6 & 9 \end{pmatrix}$ then Find 4A − 3B.

Solution :

Since A, B are of the same order 3 × 3, subtraction of 4A and 3B is defined.

$$4A-3B=4\begin{pmatrix} 5 & 4 & -2 \\ \frac{1}{2} & \frac{3}{4} & \sqrt{2} \\ 1 & 9 & 4 \end{pmatrix}-3\begin{pmatrix} -7 & 4 & -3 \\ \frac{1}{4} & \frac{7}{2} & 3 \\ 5 & -6 & 9 \end{pmatrix}$$

$$=\begin{pmatrix} 20 & 6 & -8 \\ 2 & 3 & 4\sqrt{2} \\ 4 & 36 & 16 \end{pmatrix}+\begin{pmatrix} 21 & -12 & 9 \\ -\frac{3}{4} & -\frac{21}{2} & -9 \\ -15 & 18 & -27 \end{pmatrix}$$

$$=\begin{pmatrix} 41 & 4 & 1 \\ \frac{5}{4} & -\frac{15}{2} & 4\sqrt{2}-9 \\ -11 & 54 & -11 \end{pmatrix}$$

Example 3.62

Find the value of a, b, c, d, x, y from the following matrix equation.

$$\begin{pmatrix} d & 8 \\ 3b & a \end{pmatrix}+\begin{pmatrix} 3 & a \\ -2 & -4 \end{pmatrix}=\begin{pmatrix} 2 & 2a \\ b & 4c \end{pmatrix}+\begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

Solution :

First, we add the two matrices on both left, right hand sides to get

$$\begin{pmatrix} d+3 & 8+a \\ 3b-2 & a-4 \end{pmatrix}=\begin{pmatrix} 2 & 2a+1 \\ b-5 & 4c \end{pmatrix}$$

Equating the corresponding elements of the two matrices, we have

d + 3 = 2 gives d = − 1

8 + a = 2a + 1 gives a = 7

3b − 2 = b − 5 gives b = $\frac{-3}{2}$

Substituting a = 7 in a − 4 = 4c gives c = $\frac{3}{4}$

Therefore, a = 7, b = $-\frac{3}{2}$ c = $\frac{3}{4}$, d = −1

Example 3.63

If

$$A=\begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix}, B=\begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix}, C=\begin{pmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{pmatrix}$$

compute the following :

(i) 3A + 2B − C ii) $\frac{1}{2}A-\frac{3}{2}B$

Solution :

i) $3A+2B-C$

$$=3\begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix}+2\begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix}-\begin{pmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 3 & 24 & 9 \\ 9 & 15 & 0 \\ 24 & 21 & 18 \end{pmatrix}+\begin{pmatrix} 16 & -12 & -8 \\ 4 & 22 & -6 \\ 0 & 2 & 10 \end{pmatrix}+\begin{pmatrix} -5 & -3 & 0 \\ 1 & 7 & -2 \\ -1 & -4 & -3 \end{pmatrix}$$

$$=\begin{pmatrix} 14 & 9 & 1 \\ 14 & 44 & -8 \\ 23 & 19 & 25 \end{pmatrix}$$

ii) $\frac{1}{2}A-\frac{3}{2}B$

$$=\frac{1}{2}(A-3B)$$

$$=\frac{1}{2}\left(\begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix}-3\begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix} + \begin{pmatrix} -24 & 18 & 12 \\ -6 & -33 & 9 \\ 0 & -3 & -15 \end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} -23 & 26 & 15 \\ -3 & -28 & 9 \\ 8 & 4 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{23}{2} & 13 & \frac{15}{2} \\ -\frac{3}{2} & -14 & \frac{9}{2} \\ 4 & 2 & -\frac{9}{2} \end{pmatrix}$$

EXERCISE 3.17

1. If $A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}, B = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix}$ **then verify that**

(i) A + B = B + A

ii) A + (−A) = (−A) + A = O

Solution :

Given $A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}, B = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix}$

i) To verify : A + B = B + A

A & B are of same order

$$\therefore\ A + B = \begin{pmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{pmatrix}$$

$$B + A = \begin{pmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{pmatrix}$$

$$\therefore\ A + B = B + A$$

ii) To verify : A + (− A) = (− A) + A = O

LHS : A + (− A)

$$= \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix} + \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

RHS : (− A) + A

$$= \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

∴ LHS = RHS

2. If $A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix}$

and $C = \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$ **then verify that**

A + (B + C) = (A + B) + C

Solution :

$$B + C = \begin{pmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{pmatrix}$$

$$A + (B + C) = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix} + \begin{pmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{pmatrix} \qquad \text{...........(1)}$$

$$A+B = \begin{pmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{pmatrix}$$

$$\therefore (A+B)+C = \begin{pmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{pmatrix} + \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{pmatrix} \qquad \text{...........(2)}$$

$\therefore$ From (1) & (2) LHS = RHS

3. Find X and Y if X + Y = $\begin{pmatrix} 7 & 0 \\ 3 & 5 \end{pmatrix}$ and X - Y = $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

Solution :

Given $X + Y = \begin{pmatrix} 7 & 0 \\ 3 & 5 \end{pmatrix}$ .... (1)

$X - Y = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ .... (2)

$$(1)+(2) \Rightarrow 2X = \begin{pmatrix} 10 & 0 \\ 3 & 9 \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} 5 & 0 \\ 3/2 & 9/2 \end{pmatrix}$$

$$(1)-(2) \Rightarrow 2Y = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}$$

$$\Rightarrow Y = \begin{pmatrix} 2 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

4. If $A = \begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}, B = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$ find the value of i) B − 5A ii) 3A − 9B

Solution :

Given

$$A = \begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}, B = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$$

$$i)\ B - 5A = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix} - \begin{pmatrix} 0 & 20 & 45 \\ 40 & 15 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -17 & -37 \\ -39 & -11 & -26 \end{pmatrix}$$

$$ii)\ 3A - 9B = \begin{pmatrix} 0 & 12 & 27 \\ 24 & 9 & 21 \end{pmatrix} - \begin{pmatrix} 63 & 27 & 72 \\ 9 & 36 & 81 \end{pmatrix}$$

$$= \begin{pmatrix} -63 & -65 & -45 \\ 15 & -27 & -60 \end{pmatrix}$$

5. Find the values of x, y, z if

$$i) \begin{pmatrix} x-3 & 3x-z \\ x+y+7 & x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}$$

ii) $(x \quad y-z \quad z+3) + (y \quad 4 \quad 3) = (4 \quad 8 \quad 16)$

Solution :

$$i)\ \text{Given} \begin{pmatrix} x-3 & 3x-z \\ x+y+7 & x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}$$

$\Rightarrow x - 3 = 1$	$3x - z = 0$	$x + y + 7 = 1$
$\therefore x = 4$	$12 - z = 0$	$\Rightarrow x + y = -6$
	$z = 12$	$\Rightarrow 4 + y = -6$
		$\Rightarrow y = -10$

ii) $(x \quad y-z \quad z+3) + (y \quad 4 \quad 3) = (4 \quad 8 \quad 16)$

$\Rightarrow x + y = 4$	$y - z + 4 = 8$	$z + 6 = 16$
$\Rightarrow x + 14 = 4$	$y - z = 4$	$z = 10$
$\Rightarrow x = -10$	$y - 10 = 4$	
	$\therefore y = 14$	

$\therefore x = -10, y = 14, z = 10$

6. Find x and y if $x\begin{pmatrix}4\\-3\end{pmatrix}+y\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$

Solution :

Given $x\begin{pmatrix}4\\-3\end{pmatrix}+y\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$

$\Rightarrow$ $4x - 2y = 4$ $-3x + 3y = 6$

$\Rightarrow$ $2x - y = 2$ .... (1)

$-x + y = 2$ .... (2)

(1) $\Rightarrow$ $2x - y = 2$

(2) $\Rightarrow$ $-x + y = 2$

Adding, $x = 4$

Sub $x = 4$ in (2)

$-4 + y = 2 \Rightarrow y = 6$

$\therefore$ $x = 4, y = 6$

7. Find the non-zero values of x satisfying the matrix equation

$$x\begin{pmatrix}2x & 2\\3 & x\end{pmatrix}+2\begin{pmatrix}8 & 5x\\4 & 4x\end{pmatrix}=2\begin{pmatrix}x^2+8 & 24\\10 & 6x\end{pmatrix}$$

Solution :

Given $x\begin{pmatrix}2x & 2\\3 & x\end{pmatrix}+2\begin{pmatrix}8 & 5x\\4 & 4x\end{pmatrix}=2\begin{pmatrix}x^2+8 & 24\\10 & 6x\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}2x^2 & 2x\\3x & x^2\end{pmatrix}+\begin{pmatrix}16 & 10x\\8 & 8x\end{pmatrix}=\begin{pmatrix}2x^2+16 & 48\\20 & 12x\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}2x^2+16 & 12x\\3x+8 & x^2+8x\end{pmatrix}=\begin{pmatrix}2x^2+16 & 48\\20 & 12x\end{pmatrix}$$

$\therefore$ $12x = 48 \Rightarrow x = 4$	$x^2 + 8x = 12x$
$3x + 8 = 20 \Rightarrow 3x = 12$	$\Rightarrow x^2 - 4x = 0$
$\Rightarrow x = 4$	$\Rightarrow x(x-4) = 0$
	$\Rightarrow x = 0, x = 4$

$\therefore$ $x = 4$

8. Solve for x, y $\begin{pmatrix}x^2\\y^2\end{pmatrix}+2\begin{pmatrix}-2x\\y\end{pmatrix}=\begin{pmatrix}5\\8\end{pmatrix}$

Solution :

Given $\begin{pmatrix}x^2\\y^2\end{pmatrix}+2\begin{pmatrix}-2x\\y\end{pmatrix}=\begin{pmatrix}5\\8\end{pmatrix}$

$\Rightarrow$ $x^2 - 4x = 5$	$y^2 - 2y = 8$
$\Rightarrow$ $x^2 - 4x - 5 = 0$	$\Rightarrow$ $y^2 - 2y - 8 = 0$
$\Rightarrow$ $(x-5)(x+1) = 0$	$\Rightarrow$ $(y-4)(y+2) = 0$
$\Rightarrow$ $x = 5, -1$	$\Rightarrow$ $\therefore$ $y = 4, y = -2$

XVIII. MULTIPLICATION OF MATRICES

Key Points

- ✓ To multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix.
- ✓ Matrices are multiplied by multiplying the elements in a row of the first matrix by the elements in a column of the second matrix, and adding the results.
- ✓ Matrix multiplication is not commutative in general.
- ✓ Matrix multiplication is distributive over matrix addition.
- ✓ Matrix multiplication is always associative.
- ✓ If A is a square matrix of order n´n and I is the unit matrix of same order then AI = IA = A.
- ✓ AB = 0 does not necessarily imply that A = 0 or B = 0 or both A,B = 0

Example 3.64

If $A=\begin{pmatrix}1&2&0\\3&1&5\end{pmatrix}, B=\begin{pmatrix}8&3&1\\2&4&1\\5&3&1\end{pmatrix}$ find AB.

Solution :

We observe that A is a 2 × 3 matrix and B is a 3 × 3 matrix, hence AB is defined and it will be of the order 2 × 3.

Given $A=\begin{pmatrix}1&2&0\\3&1&5\end{pmatrix}_{2\times3}, B=\begin{pmatrix}8&3&1\\2&4&1\\5&3&1\end{pmatrix}_{3\times3}$

$$AB=\begin{pmatrix}1&2&0\\3&1&5\end{pmatrix}\times\begin{pmatrix}8&3&1\\2&4&1\\5&3&1\end{pmatrix}$$

$$=\begin{pmatrix}8+4+0&3+8+0&1+2+0\\24+2+25&9+4+15&3+1+5\end{pmatrix}$$

$$=\begin{pmatrix}12&11&3\\51&28&9\end{pmatrix}$$

Example 3.65

If $A=\begin{pmatrix}2&1\\1&3\end{pmatrix}, B=\begin{pmatrix}2&0\\1&3\end{pmatrix}$ find AB and BA. Check if AB = BA.

Solution :

We observe that A is a 2 × 2 matrix and B is a 2 × 2 matrix, hence AB is defined and it will be of the order 2 × 2.

$$AB=\begin{pmatrix}2&1\\1&3\end{pmatrix}\times\begin{pmatrix}2&0\\1&3\end{pmatrix}=\begin{pmatrix}4+1&0+3\\2+3&0+9\end{pmatrix}=\begin{pmatrix}5&3\\5&9\end{pmatrix}$$

$$BA=\begin{pmatrix}2&0\\1&3\end{pmatrix}\times\begin{pmatrix}2&1\\1&3\end{pmatrix}=\begin{pmatrix}4+0&2+0\\2+3&1+9\end{pmatrix}=\begin{pmatrix}4&2\\5&10\end{pmatrix}$$

Therefore, AB ≠ BA

Example 3.66

If $A=\begin{pmatrix}2&-2\sqrt{2}\\\sqrt{2}&2\end{pmatrix}$ and $B=\begin{pmatrix}2&2\sqrt{2}\\-\sqrt{2}&2\end{pmatrix}$ Show that A and B satisfy commutative property with respect to matrix multiplication.

Solution :

We have to show that AB = BA

LHS : $AB=\begin{pmatrix}2&-2\sqrt{2}\\\sqrt{2}&2\end{pmatrix}\times\begin{pmatrix}2&2\sqrt{2}\\-\sqrt{2}&2\end{pmatrix}$

$$=\begin{pmatrix}4+4&4\sqrt{2}-4\sqrt{2}\\2\sqrt{2}-2\sqrt{2}&4+4\end{pmatrix}$$

$$=\begin{pmatrix}8&0\\0&8\end{pmatrix}$$

RHS : $BA=\begin{pmatrix}2&2\sqrt{2}\\-\sqrt{2}&2\end{pmatrix}\times\begin{pmatrix}2&-2\sqrt{2}\\\sqrt{2}&2\end{pmatrix}$

$$=\begin{pmatrix}4+4&-4\sqrt{2}+4\sqrt{2}\\-2\sqrt{2}+2\sqrt{2}&4+4\end{pmatrix}$$

$$=\begin{pmatrix}8&0\\0&8\end{pmatrix}$$

Hence LHS = RHS (ie) AB = BA

Example 3.67

Solve $\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}$

Solution :

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}_{2\times2}\times\begin{pmatrix}x\\y\end{pmatrix}_{2\times1}=\begin{pmatrix}4\\5\end{pmatrix}$$

By matrix multiplication $\begin{pmatrix}2x+y\\x+2y\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}$

Rewriting $2x + y = 4$ .... (1)

$x + 2y = 5$ .... (2)

(1) − 2 × (2) gives
$$\begin{array}{rl} 2x + y = & 4 \\ 2x + 4y = & 10 \quad (-) \\ \hline -3y = & -6 \end{array}$$
gives $y = 2$

Substituting $y = 2$ in (1), $2x + 2 = 4$ gives $x = 1$

Therefore, $x = 1, y = 2$.

Example 3.68

If $A = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

show that(AB)C = A(BC).

Solution :

LHS (AB) C

$$A = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}_{1\times3} \times \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}_{3\times2}$$

$$= (1-2+2 \quad -1-1+6) = (1 \quad 4)$$

$$(AB)\,C = (1 \quad 4)_{1\times2} \times \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= (1+8 \quad 2-4) = (9 \quad -2) \quad \text{......... (1)}$$

RHS A(BC)

$$BC = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}_{3\times2} \times \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}_{2\times2}$$

$$= \begin{pmatrix} 1-2 & 2+1 \\ 2+2 & 4-1 \\ 1+6 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}_{1\times3} \times \begin{pmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{pmatrix}_{3\times2}$$

$$A(BC) = (-1-4+14 \quad 3-3-2)$$

$$= (9 \quad -2) \quad \text{............ (2)}$$

From (1) and (2), (AB)C = A(BC).

Example 3.69

If $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ -4 & 2 \end{pmatrix}, C = \begin{pmatrix} -7 & 6 \\ 3 & 2 \end{pmatrix}$

verify that A (B + C) = AB + AC.

Solution :

LHS A (B + C)

$$B + C = \begin{pmatrix} 1 & 2 \\ -4 & 2 \end{pmatrix} + \begin{pmatrix} -7 & 6 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -6 & 8 \\ -1 & 4 \end{pmatrix}$$

$$A(B+C) = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} -6 & 8 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -6-1 & 8+4 \\ 6-3 & -8+12 \end{pmatrix} = \begin{pmatrix} -7 & 12 \\ 3 & 4 \end{pmatrix} \quad \text{... (1)}$$

RHS AB + AC

$$AB = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 1-4 & 2+2 \\ -1-12 & -2+6 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4 \\ -13 & 4 \end{pmatrix}$$

$$AC = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} -7 & 6 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -7+3 & 6+2 \\ 7+9 & -6+6 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 8 \\ 16 & 0 \end{pmatrix}$$

Therefore, $AB + AC = \begin{pmatrix} -3 & 4 \\ -13 & 4 \end{pmatrix} + \begin{pmatrix} -4 & 8 \\ 16 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -7 & 12 \\ 3 & 4 \end{pmatrix} \quad \text{...... (2)}$$

From (1) and (2), A (B + C) = AB + AC. Hence proved.

Example 3.70

If $A=\begin{pmatrix}1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}2 & -1\\ -1 & 4\\ 0 & 2\end{pmatrix}$ show that $(AB)^T = B^T A^T$.

Solution :

LHS $(AB)^T$

$$AB=\begin{pmatrix}1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}_{2\times3}\times\begin{pmatrix}2 & -1\\ -1 & 4\\ 0 & 2\end{pmatrix}_{3\times2}$$

$$=\begin{pmatrix}2-2+0 & -1+8+2\\ 4+1+0 & -2-4+2\end{pmatrix}=\begin{pmatrix}0 & 9\\ 5 & -4\end{pmatrix}$$

$$(AB)^T=\begin{pmatrix}0 & 9\\ 5 & -4\end{pmatrix}^T=\begin{pmatrix}0 & 5\\ 9 & -4\end{pmatrix} \quad ....(1)$$

RHS $(B^T A^T)$

$$B^T=\begin{pmatrix}2 & -1 & 0\\ -1 & 4 & 2\end{pmatrix},\ A^T=\begin{pmatrix}1 & 2\\ 2 & -1\\ 1 & 1\end{pmatrix}$$

$$B^T A^T=\begin{pmatrix}2 & -1 & 0\\ -1 & 4 & 2\end{pmatrix}_{2\times3}=\begin{pmatrix}1 & 2\\ 2 & -1\\ 1 & 1\end{pmatrix}_{3\times2}$$

$$=\begin{pmatrix}2-2+0 & 4+1+0\\ -1+8+2 & -2-4+2\end{pmatrix}$$

$$B^T A^T=\begin{pmatrix}0 & 5\\ 9 & -4\end{pmatrix} \quad ....(2)$$

From (1) and (2), $(AB)^T = B^T A^T$.

Hence proved.

EXERCISE 3.18

1. Find the order of the product matrix AB if

	(i)	(ii)	(iii)	(iv)	(v)
Orders of A	3 × 3	4 × 3	4 × 2	4 × 5	1 × 1
Orders of B	3 × 3	3 × 2	2 × 2	5 × 1	1 × 3

Solution :

i) A is of order $= 3 \times 3$

B is of order $= 3 \times 3$

$\therefore$ Order of AB $= 3 \times 3$

ii) $A \to 4 \times 3$, $B \to 3 \times 2$

$\therefore$ Order of AB $= 4 \times 2$

iii) $A \to 4 \times 2$, $B \to 2 \times 2$

$\therefore$ Order of AB $= 4 \times 2$

iv) $A \to 4 \times 5$, $B \to 5 \times 1$

$\therefore$ Order of AB $= 4 \times 1$

v) $A \to 1 \times 1$, $B \to 1 \times 3$

$\therefore$ Order of AB $= 1 \times 3$

2. If A is of order $p \times q$ and B is of order $q \times r$ what is the order of AB and BA?

Solution :

Given A is of order $p \times q$

B is of order $q \times r$

$\therefore$ Order of AB $= p \times r$

Order of BA is not defined

($\because$ No. of columns in B & no. of rows in A are not equal.)

3. A has 'a' rows and 'a + 3' columns. B has 'b' rows and '17–b' columns, and if both products AB and BA exist, find a, b?

Solution :

Given Order of A is $a \times (a + 3)$

Order of B is $b \times (17 - b)$

Product AB exist

$\Rightarrow$ $a + 3 = b$

(No. of columns in A = No. of rows in B)

$\Rightarrow$ $a - b = -3$........ (1)

Product BA exist

$\Rightarrow$ $17 - b = a$

(No. of columns in B = No. of rows in A)

$\Rightarrow$ $a + b = 17$ .... (2)

Solving (1) & (2)

$2a = 14$

$a = 7$

Sub $a = 7$ in (1)

$7 - b = -3$

$b = 10$

$\therefore$ $a = 7, b = 10$

4. If $A=\begin{pmatrix}2 & 5\\4 & 3\end{pmatrix}, B=\begin{pmatrix}1 & -3\\2 & 5\end{pmatrix}$ find AB, BA and check if AB = BA?

Solution :

Given $A=\begin{pmatrix}2 & 5\\4 & 3\end{pmatrix}, B=\begin{pmatrix}1 & -3\\2 & 5\end{pmatrix}$

$$AB=\begin{pmatrix}2 & 5\\4 & 3\end{pmatrix}\begin{pmatrix}1 & -3\\2 & 5\end{pmatrix}$$

$$=\begin{pmatrix}2\times1+5\times2 & 2\times-3+5\times5\\4\times1+3\times2 & 4\times-3+3\times5\end{pmatrix}$$

$$=\begin{pmatrix}2+10 & -6+25\\4+6 & -12+15\end{pmatrix}$$

$$=\begin{pmatrix}12 & 19\\10 & 3\end{pmatrix}$$

$$BA=\begin{pmatrix}1 & -3\\2 & 5\end{pmatrix}\begin{pmatrix}2 & 5\\4 & 3\end{pmatrix}$$

$$=\begin{pmatrix}(1\times2)+(-3\times4) & 1\times5+(-3)\times3\\2\times2+5\times4 & 2\times5+5\times3\end{pmatrix}$$

$$=\begin{pmatrix}2-12 & 5-9\\4+20 & 10+15\end{pmatrix}$$

$$=\begin{pmatrix}-10 & -4\\24 & 25\end{pmatrix} \quad ....(2)$$

$\therefore$ From (1) & (2) $AB \neq BA$

5. Given that $A=\begin{pmatrix}1 & 3\\5 & -1\end{pmatrix}, B=\begin{pmatrix}1 & -1 & 2\\3 & 5 & 2\end{pmatrix},$ $C=\begin{pmatrix}1 & 3 & 2\\-4 & 1 & 3\end{pmatrix}$ verify that A(B + C) = AB + AC

Solution :

Given

$$A=\begin{pmatrix}1 & 3\\5 & -1\end{pmatrix}, B=\begin{pmatrix}1 & -1 & 2\\3 & 5 & 2\end{pmatrix}, C=\begin{pmatrix}1 & 3 & 2\\-4 & 1 & 3\end{pmatrix}$$

To verify : A (B + C) = AB + AC

LHS : A (B + C)

$$= \begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ -1 & 6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 2+3\times -1 & 1\times 2+3\times 6 & 1\times 4+3\times 5 \\ 5\times 2+(-1)\times(-1) & 5\times 2-1\times 6 & 5\times 4-1\times 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2-3 & 2+18 & 4+15 \\ 10+1 & 10-6 & 20-5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 20 & 19 \\ 11 & 4 & 15 \end{pmatrix} \quad \text{..........(1)}$$

RHS : AB + AC

$$= \begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+9 & -1+15 & 2+6 \\ 5-3 & -5-5 & 10-2 \end{pmatrix} + \begin{pmatrix} 1-12 & 3+3 & 2+9 \\ 5+4 & 15-1 & 10-3 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 14 & 8 \\ 2 & -10 & 8 \end{pmatrix} + \begin{pmatrix} -11 & 6 & 11 \\ 9 & 14 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 20 & 19 \\ 11 & 4 & 15 \end{pmatrix} \quad \text{...........(2)}$$

$\therefore$ From (1) & (2) LHS = RHS

6. Show that the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$ **satisfy commutative propertyAB = BA**

Solution :

Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & -2+2 \\ 3-3 & -6+1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & -2+2 \\ -3+3 & -6+1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$$

$\therefore$ AB = BA

$\therefore$ Commutative property is true.

7. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

Show that i) A(BC) = (AB)C

ii) (A − B) C = AC − BC

iii) $\mathbf{(A - B)^T = A^T - B^T}$

Solution :

Given $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

i) To show : A(BC) = (AB)C

$$BC = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8+0 & 0+0 \\ 2+5 & 0+10 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix} = \begin{pmatrix} 8+14 & 0+20 \\ 8+21 & 0+30 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 20 \\ 29 & 30 \end{pmatrix} \quad \text{........(1)}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4+2 & 0+10 \\ 4+3 & 0+15 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 10 \\ 7 & 15 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 6 & 10 \\ 7 & 15 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12+10 & 0+20 \\ 14+15 & 0+30 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 20 \\ 29 & 30 \end{pmatrix} \quad \text{...........(2)}$$

$\therefore$ From (1) & (2) LHS = RHS

ii) To show (A − B) C = AC − BC

(A − B) C

$$= \begin{pmatrix} -3 & 2 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -6+2 & 0+4 \\ 0-2 & 0-4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 4 \\ -2 & -4 \end{pmatrix} \quad \text{...........(1)}$$

$$AC = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2 & 0+4 \\ 2+3 & 0+6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 \\ 5 & 6 \end{pmatrix}$$

$$BC = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8+0 & 0+0 \\ 2+5 & 0+10 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix}$$

$$\therefore AC - BC = \begin{pmatrix} 4 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 4 \\ -2 & -4 \end{pmatrix} \quad \text{..........(2)}$$

$\therefore$ From (1) & (2), LHS = RHS

iii) To show : $(A - B)^T = A^T - B^T$

$$(A-B)^T = \begin{pmatrix} -3 & 2 \\ 0 & -2 \end{pmatrix}^T = \begin{pmatrix} -3 & 0 \\ 2 & -2 \end{pmatrix} \quad \text{........(1)}$$

$$A^T - B^T = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 2 & -2 \end{pmatrix} \quad \text{........(2)}$$

$\therefore$ From (1) & (2)

$(A - B)^T = A^T - B^T$

8. If $A = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}, B = \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix}$ **then show that $A^2 + B^2 = I$.**

Solution :

Given $A = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}, B = \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix}$

To show : $A^2 + B^2 = I$

$$A^2 = A \,.\, A$$

$$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + 0 & 0+0 \\ 0+0 & 0+\cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\theta + 0 & 0+0 \\ 0+0 & 0+\sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$\therefore A^2 + B^2 = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathrm{I}$$

Hence proved.

---

9. If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ prove that $AA^T = I$

Solution :

Given $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

To prove : $A.A^T = I$

LHS :

$$A.A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \mathrm{I}$$

Hence proved.

---

10. Verify that $A^2 = I$ when $A = \begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}$

Solution :

Given $A = \begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}$

To prove : $A^2 = I$

$$A^2 = A.A = \begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}\begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 25-24 & -20+20 \\ 30-30 & -24+25 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathrm{I}$$

Hence proved.

---

11. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ show that

$A^2 - (a + d) A = (bc - ad) I_2$.

Solution :

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

To prove : $A^2 - (a + d) A = (bc - ad) I$

$$A^2 = A\,.\,A$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$(a+d)A = (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2+ad & ab+bd \\ ca+cd & ad+d^2 \end{pmatrix}$$

$$A^2 - (a+d)A$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ca+cd & ad+d^2 \end{pmatrix}$$

$$= \begin{pmatrix} bc+ad & 0 \\ 0 & bc-ad \end{pmatrix}$$

$$= (bc-ad)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= (bc-ad)\,(\mathrm{I})$$

$$= \text{RHS}$$

Hence proved.

12. If $A=\begin{pmatrix}5&2&9\\1&2&8\end{pmatrix}, B=\begin{pmatrix}1&7\\1&2\\5&-1\end{pmatrix}$ **verify that $(AB)^T = B^T A^T$**

Solution :

Given $A=\begin{pmatrix}5&2&9\\1&2&8\end{pmatrix}, B=\begin{pmatrix}1&7\\1&2\\5&-1\end{pmatrix}$

To verify : $(AB)^T = B^T A^T$

$$AB=\begin{pmatrix}5&2&9\\1&2&8\end{pmatrix}\begin{pmatrix}1&7\\1&2\\5&-1\end{pmatrix}$$

$$=\begin{pmatrix}5\times1+2\times1+9\times5 & 5\times7+2\times2+9\times-1\\1\times1+2\times1+8\times5 & 1\times7+2\times2+8\times-1\end{pmatrix}$$

$$=\begin{pmatrix}5+2+45 & 35+4-9\\1+2+40 & 7+4-8\end{pmatrix}$$

$$=\begin{pmatrix}52&30\\43&3\end{pmatrix}$$

$$\therefore (AB)^T=\begin{pmatrix}52&43\\30&3\end{pmatrix} \quad ....(1)$$

$$B^T A^T=\begin{pmatrix}1&1&5\\7&2&-1\end{pmatrix}\begin{pmatrix}5&1\\2&2\\9&8\end{pmatrix}$$

$$=\begin{pmatrix}5+2+45 & 1+2+40\\35+4-9 & 7+4-8\end{pmatrix}$$

$$=\begin{pmatrix}52&43\\30&3\end{pmatrix} \quad ....(2)$$

$\therefore$ From (1) & (2), $(AB)^T = B^T A^T$

Hence proved.

13. If A = $\begin{pmatrix}3&1\\-1&2\end{pmatrix}$ **show that $A^2 - 5A + 7I_2 = 0$**

Solution :

Given A = $\begin{pmatrix}3&1\\-1&2\end{pmatrix}$

To verify : $A^2 - 5A + 7I_2 = 0$

$$A^2=\begin{pmatrix}3&1\\-1&2\end{pmatrix}\begin{pmatrix}3&1\\-1&2\end{pmatrix}$$

$$=\begin{pmatrix}9-1 & 3+2\\-3-2 & -1+4\end{pmatrix}$$

$$=\begin{pmatrix}8&5\\-5&3\end{pmatrix}$$

$$\therefore A^2-5A+7I_2$$

$$=\begin{pmatrix}8&5\\-5&3\end{pmatrix}-\begin{pmatrix}15&5\\-5&10\end{pmatrix}+7\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=\begin{pmatrix}8&5\\-5&3\end{pmatrix}-\begin{pmatrix}15&5\\-5&10\end{pmatrix}+\begin{pmatrix}7&0\\0&7\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$=0$$

EXERCISE 3.19

Multiple Choice Questions :

1. A system of three linear equations in three variables is inconsistent if their planes

(1) intersect only at a point

(2) intersect in a line

(3) coincides with each other

(4) do not intersect

Hint : **Ans : (4)**

System of equations is in consistent if their planes do not intersect.

2. The solution of the system x + y − 3x = −6, −7y + 7z = 7 , 3z = 9 is

(1) x = 1, y = 2, z = 3

(2) x = −1,y = 2,z = 3

(3) x = −1,y = −2,z = 3

(4) x = 1,y = 2,z = 3

Hint : **Ans : (1)**

(3) ⇒ 3z = 9 ⇒ z = 3

(2) ⇒ − y + z = 1 ⇒ − y + 3 = 1

⇒ − y = − 2

y = 2

(1) ⇒ x + y − 3z = − 6

⇒ x + 2 − 9 = − 6

⇒ x = 7 − 6 = 1

3. If (x − 6) is the HCF of $x^2 - 2x - 24$ and $x^2 - kx - 6$ then the value of k is

(1) 3 (2) 5 (3) 6 (4) 8

Hint : **Ans : (2)**

$x^2 - 2x - 24 = (x - 6)(x + 4)$

$x^2 - kx - 6 = (x - 6)(x + 1)$

(∵ x + 1 is the only possible factor)

$= x^2 - 5x - 6$

∴ k = 5

4. $\dfrac{3y-3}{y} \div \dfrac{7y-7}{3y^2}$ **is**

(1) $\dfrac{9y}{7}$ (2) $\dfrac{9y^3}{(21y-21)}$

(3) $\dfrac{21y^2-42y+21}{3y^3}$ (4) $\dfrac{7(y^2-2y+1)}{y^2}$

Hint : **Ans : (1)**

$$= \frac{3y-3}{y} \div \frac{7y-7}{3y^2}$$

$$= \frac{3(y-1)}{y} \div \frac{3y^2}{7(y-1)}$$

$$= \frac{9y}{7}$$

5. $y^2 + \dfrac{1}{y^2}$ **is not equal to**

(1) $\dfrac{y^4+1}{y^2}$ (2) $\left(y+\dfrac{1}{y}\right)^2$

(3) $\left(y+\dfrac{1}{y}\right)^2 + 2$ (4) $\left(y+\dfrac{1}{y}\right)^2 - 2$

Hint : **Ans : (2)**

$$y^2 + \frac{1}{y^2} \neq \left(y+\frac{1}{y}\right)^2$$

6. $\dfrac{x}{x^2-25} - \dfrac{8}{x^2+6x+5}$ **gives**

(1) $\dfrac{x^2-7x+40}{(x-5)(x+5)}$ (2) $\dfrac{x^2+7x+40}{(x-5)(x+5)(x+1)}$

(3) $\dfrac{x^2-7x+40}{(x^2-25)(x+1)}$ (4) $\dfrac{x^2+10}{(x^2-25)(x+1)}$

Hint : **Ans : (3)**

$$= \frac{x}{x^2-25} - \frac{8}{x^2+6x+5}$$

$$= \frac{x}{(x+5)(x-5)} - \frac{8}{(x+5)(x+1)}$$

$$= \frac{x(x+1) - 8(x-5)}{(x+5)(x-5)(x+1)}$$

$$= \frac{x^2 + x - 8x + 40}{(x+5)(x-5)(x+1)}$$

$$= \frac{x^2 - 7x + 40}{(x^2-25)(x+1)}$$

7. The square root of $\frac{256x^8y^4z^{10}}{25x^6y^6z^6}$ is equal to

(1) $\frac{16}{5}\left|\frac{x^2z^4}{y^2}\right|$ (2) $16\left|\frac{y^2}{x^2z^4}\right|$

(3) $\frac{16}{5}\left|\frac{y}{xz^2}\right|$ (4) $\frac{16}{5}\left|\frac{xz^2}{y}\right|$

Hint : **Ans : (4)**

$$= \sqrt{\frac{256x^8y^4z^{10}}{25x^6y^6z^6}}$$

$$= \frac{16}{5}\left|\frac{x^4y^2z^5}{x^3y^3z^3}\right|$$

$$= \frac{16}{5}\left|\frac{xz^2}{y}\right|$$

8. Which of the following should be added to make $x^2 + 64$ a perfect square

(1) $4x^2$ (2) $16x^2$

(3) $8x^2$ (4) $-8x^2$

Hint : **Ans : (2)**

$= x^4 + 64$

$= (x^2)^2 + 8^2$

$= (x^2)^2 + 8^2 + 2(x^2)(8)$

$= (x^2 + 8)^2$, perfect square

$\therefore$ $16x^2$ should be added

9. The solution of $(2x - 1)^2 = 9$ is equal to

(1) -1 (2) 2

(3) $-1, 2$ (4) None of these

Hint : **Ans : (3)**

$(2x-1)^2 = 9 \quad \Rightarrow \quad 2x - 1 = \pm 3$

$\Rightarrow \quad 2x = 4, 2x = -2$

$x = 2, x = -1$

10. The values of a and b if $4x^4 - 24x^3 + 76x^2 + ax + b$ is a perfect square are

(1) 100, 120 (2) 10, 12

(3) -120, 100 (4) 12, 10

Hint : **Ans : (3)**

$$\begin{array}{r|l}
 & 2x^2 - 6x + 10 \\
\hline
2x^2 & 4x^4 - 24x^3 + 76x^2 + ax + b \\
 & 4x^4 \\
\hline
4x^2 - 6x & -24x^3 + 76x^2 \\
 & -24x^3 + 36x^2 \\
\hline
4x^2 - 12x + 10 & \quad 40x^2 + ax + b \\
 & \quad 40x^2 - 120x + 100 \\
 & \qquad (+) \quad (-) \\
\hline
 & 0
\end{array}$$

($\because$ Perfect square)

$a + 120 = 0, \qquad b - 100 = 0$

$a = -120 \qquad b = 100$

11. If the roots of the equation $q^2x^2 + p^2x + r^2 = 0$ are the squares of the roots of the equation $qx^2 + px + r = 0$, then q, p, r are in _____

(1) A.P (2) G.P

(3) Both A.P and G.P (4) none of these

Hint : **Ans : (2)**

Roots of $q^2x^2 + p^2x + r^2 = 0$ are squares of roots of $qx^2 + px + r = 0$

$$\Rightarrow \qquad \alpha^2 + \beta^2 = \frac{-p^2}{q^2} \quad \Big| \quad \alpha^2\beta^2 = \frac{r^2}{q^2}$$

and $\alpha + \beta = \frac{-p}{q}, \alpha\beta = \frac{r}{q}$

$$\Rightarrow \quad (\alpha+\beta)^2 - 2\alpha\beta = \frac{-p^2}{q^2}$$

$$\Rightarrow \quad \frac{p^2}{q^2} - \frac{2r}{q} = \frac{-p^2}{q^2}$$

$$\Rightarrow \quad \frac{2r}{q} = \frac{2p^2}{q^2}$$

$$\Rightarrow \quad r = \frac{2p^2}{q^2}$$

$$\Rightarrow \quad p^2 = qr$$

$\therefore$ q, p, r are in G.P.

12. Graph of a linear polynomial is a

(1) straight line (2) circle

(3) parabola (4) hyperbola

Hint : **Ans : (1)**

Graph of a linear polynomial is a straight line.

13. The number of points of intersection of the quadratic polynomial $x^2 + 4x + 4$ with the X axis is

(1) 0 (2) 1 (3) 0 or 1 (4) 2

Hint : **Ans : (2)**

$(x + 2)^2 = 0$

$\Rightarrow \quad x + 2 = 0$

$\Rightarrow \quad x = -2$

$\therefore$ The polynomial will meet x - axis at $(-2, 0)$

No. of points of intersection = 1.

14. For the given matrix A= $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 9 & 11 & 13 & 15 \end{pmatrix}$ **the order of the matrix A^T is**

(1) 2×3 (2) 3×2

(3) 3×4 (4) 4×3

Hint : **Ans : (3)**

A has 3 rows & 4 columns

$\therefore$ A is of order 3×4

15. If A is a 2×3 matrix and B is a 3×4 matrix, how many columns does AB have?

(1) 3 (2) 4 (3) 2 (4) 5

Hint : **Ans : (2)**

$A \to 2 \times 3, \quad B \to 3 \times 4$

$\therefore$ AB is of order 2×4

$\therefore$ No.of columns of $A \times B$ is 4.

16. If number of columns and rows are not equal in a matrix then it is said to be a

(1) diagonal matrix

(2) rectangular matrix

(3) square matrix

(4) identity matrix

Hint : **Ans : (2)**

No. of rows $\neq$ No. of columns

$\Rightarrow$ Matrix is said to be rectangular

17. Transpose of a column matrix is

(1) unit matrix (2) diagonal matrix

(3) column matrix (4) row matrix

Hint : **Ans : (4)**

Transpose of a column matrix is a row matrix

Ex : If A = $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad A^T = (1 \quad 2 \quad 3)$

18. Find the matrix X if $2X+\begin{pmatrix}1&3\\5&7\end{pmatrix}=\begin{pmatrix}5&7\\9&5\end{pmatrix}$

(1) $\begin{pmatrix}-2&-2\\2&-1\end{pmatrix}$ (2) $\begin{pmatrix}2&2\\2&-1\end{pmatrix}$

(3) $\begin{pmatrix}1&2\\2&2\end{pmatrix}$ (4) $\begin{pmatrix}2&1\\2&2\end{pmatrix}$

Hint : **Ans : (2)**

$$2X+\begin{pmatrix}1&3\\5&7\end{pmatrix}=\begin{pmatrix}5&7\\9&5\end{pmatrix}$$

$$\Rightarrow \quad 2X=\begin{pmatrix}4&4\\4&-2\end{pmatrix}$$

$$\Rightarrow \quad X=\begin{pmatrix}2&2\\2&-1\end{pmatrix}$$

19. Which of the following can be calculated from the given matrices

$$A=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}, B=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

(i) A^2 (ii) B^2 (iii) AB (iv) BA

(1) (i) and (ii) only (2) (ii) and (iii) only

(3) (ii) and (iv) only (4) all of these

Hint : **Ans : (3)**

A is of order 3×2

i) A^2 is not possible $[(3 \times 2)\ (3 \times 2)$ is not possible$]$

B is of order 3×3

ii) B^2 is possible

iii) AB is not defined (no. of columns in A $\neq$ no. of rows in B)

iv) BA is of order 3×3

$\therefore$ BA is possible.

20. If $A=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$, $B=\begin{pmatrix}1&0\\2&-1\\0&2\end{pmatrix}$ and

0 1
2 5

and $C=\begin{pmatrix}0&1\\-2&5\end{pmatrix}$

Which of the following statements are correct?

(i) $AB+C=\begin{pmatrix}5&5\\5&5\end{pmatrix}$

(ii) $BC=\begin{pmatrix}0&1\\2&-3\\-4&10\end{pmatrix}$

(iii) $BA+C=\begin{pmatrix}2&5\\3&0\end{pmatrix}$

(iv) $(AB)C=\begin{pmatrix}-8&20\\-8&13\end{pmatrix}$

(1) (i) and (ii) only (2) (ii) and (iii) only
(3) (iii) and (iv) only (4) all of these

Hint :

$$i)\ AB+C=\begin{pmatrix}1+4+0&0-2+6\\3+4+0&0-2+2\end{pmatrix}+\begin{pmatrix}0&1\\-2&5\end{pmatrix}$$

$$=\begin{pmatrix}5&4\\7&0\end{pmatrix}+\begin{pmatrix}0&1\\-2&5\end{pmatrix}$$

$$=\begin{pmatrix}5&5\\5&5\end{pmatrix} \quad \text{Correct}$$

ii) B $\rightarrow 3 \times 2$, C $\rightarrow 2 \times 2$

$\therefore$ BC is of order 3×2

$$BC=\begin{pmatrix}0+0&1+0\\0+2&2-5\\0-4&0+10\end{pmatrix}=\begin{pmatrix}0&1\\2&-3\\-4&10\end{pmatrix} \quad \text{Correct}$$

$iii)\ BA+C \neq \begin{pmatrix} 2 & 5 \\ 3 & 0 \end{pmatrix}, \quad (\because BA \text{ is of order } 3\times 3)$

iv) $AB = \begin{pmatrix} 5 & 4 \\ 7 & 0 \end{pmatrix}$

$$\therefore ABC = \begin{pmatrix} 5 & 4 \\ 7 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0-8 & 5+20 \\ 0+0 & 7+0 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 25 \\ 0 & 7 \end{pmatrix} \neq \begin{pmatrix} -8 & 20 \\ -8 & 13 \end{pmatrix}$$

$\therefore$ (i) & (ii) only are correct

UNIT EXERCISE - 3

1. Solve

$$\frac{1}{3}(x+y-5) = y-z = 2x-11 = 9-(x+2z)$$

Solution :

Given

$$\frac{1}{3}(x+y-5) = y-z = 2x-11 = 9-(x+2z)$$

$$\Rightarrow \frac{1}{3}(x+y-5) = y-z$$

$\Rightarrow x + y - 5 = 3y - 3z$

$\Rightarrow x - 2y + 3z = 5$ .... (1)

$\Rightarrow y - z = 2x - 11$

$\Rightarrow 2x - y + z = 11$ .... (2)

Also, $2x - 11 = 9 - x - 2z$

$3x + 2z = 20$ .... (3)

(1) $\Rightarrow$ $x - 2y + 3z = 5$

(2) $\Rightarrow \times 2$ $4x - 2y + 2z = 22$

$(-)\ (+)\ (-)\ (-)$

$-3x + z = -17$ .... (4)

Solving (3) & (4)

$3x + 2z = 20$

$-3x + z = -17$

$(-)\ (+)\ (-)\ (-)$

$3z = 3$

$z = 1$

Sub $z = 1$ in (4)

$-3x + 1 = -17$

$-3x = -18$

$x = 6$

Sub $x = 6$, $z = 1$ in (1)

$6 - 2y + 3 = 5$

$\Rightarrow -2y = -4$

$y = 2$

$\therefore x = 6,\ y = 2,\ z = 1.$

2. One hundred and fifty students are admitted to a school. They are distrbuted over three sections A, B and C. If 6 students are shifted from section A to section C, the sections will have equal number of students. If 4 times of students of section C exceeds the number of students of section A by the number of students in section B, find the number of students in the three sections.

Solution :

Let the number of students in 3 sections.

A, B, C respectively be x, y, z

$\therefore$By data given, $x + y + z = 150$ — (1)

$x - 6 = z + 6$

$x - z = 12$ — (2)

Also, given $4z = x + y$

$\Rightarrow x + y - 4z = 0$ — (3)

Solving (1) & (2)

$$\begin{array}{rr} x + y + z = & 150 \\ x + y - 4z = & 0 \\ \hline \text{Subtracting,} \quad 5z = & 150 \\ z = & 30 \end{array}$$

Sub. $z = 30$ in (2)

$x - 30 = 12$

$x = 42$

Sub. $x = 42$, $z = 30$ in (1)

$42 + y + 30 = 150$

$\Rightarrow \quad y = 150 - 72$

$= 78$

$\therefore$ No. of students in Section A = 42

No. of students in Section B = 78

No. of students in Section C = 30

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3. In a three-digit number, when the tens and the hundreds digit are interchanged the new number is 54 more than three times the original number. If 198 is added to the number, the digits are reversed. The tens digit exceeds the hundreds digit by twice as that of the tens digit exceeds the unit digit. Find the original number.

Solution :

Let the original 3 digit number be

$100x + 10y + z$.

By data given

$100y + 10x + z = 3(100x + 10y + z) + 54$

$\Rightarrow \quad 100y + 10x + z = 300x + 30y + 3z + 54$

$\Rightarrow \quad 290x - 70y + 2z = -54$

$\Rightarrow \quad 145x - 35y + z = -27 \qquad \text{— (1)}$

When 198 is added to the number of digits are reversed

$\Rightarrow \quad (100x + 10y + z) + 198 = 100z + 10y + x$

$\Rightarrow \quad 99x - 99z = -198$

$\Rightarrow \quad x - z = -2$

$\Rightarrow \quad x = z - 2 \qquad \text{— (2)}$

10's digit exceeds 100's digit by twice as that of 10's digits exceed the unit's digit

$\Rightarrow \quad y - x = 2(y - z)$

$\Rightarrow \quad y - x = 2y - 2z$

$\Rightarrow \quad -x - y = -2z$

$\Rightarrow \quad x + y = 2z$

$\Rightarrow \quad z - 2 + y = 2z \qquad$ (from (2))

$y = 2z - z + 2$

$y = z + 2 \qquad \text{— (3)}$

Sub (2), (3) in (1)

$145(z - 2) - 35(z + 2) + z = -27$

$\Rightarrow \quad 145z - 290 - 35z - 70 + z = -27$

$111z = 333$

$z = 3$

Sub. $z = 3$ in (2) $\Rightarrow x = 3 - 2 = 1$

Sub. $z = 3$ in (3) $\Rightarrow y = 3 + 2 = 5$

$x = 1, y = 5, z = 3$

$\therefore$ The original number is $100x + 10y + z$

$= 100(1) + 10(5) + 3$

$= 100 + 50 + 3$

$= 153$

---

4. Find the least common multiple of xy $(k^2 + 1) + k(x^2 + y^2)$ and xy $(k^2 - 1) + k(x^2 - y^2)$.

Solution :

$xy(k^2 + 1) + k(x^2 + y^2)$

$= kx^2 + (k^2 + 1)xy + ky^2$

$= kx^2 + k^2xy + xy + ky^2$

$= kx(x + ky) + y(x + ky)$

$= (kx + y)(x + ky)$

$xy(k^2 - 1) + k(x^2 - y^2)$

$= kx^2 + (k^2 - 1)xy - ky^2$

$= kx^2 + k^2xy - xy - ky^2$

$= kx(x + ky) - y(x + ky)$

$= (x + ky)(kx - y)$

LCM $= (x + ky)(kx + y)(kx - y)$

$= (x + ky)(k^2x^2 - y^2)$

5. Find the GCD of the following by division algorithm $2x^4+13x^3+27x^2+23x+7$, $x^3 + 3x^2 + 3x + 1$, $x^2 + 2x + 1$

Solution :

$$\begin{array}{r|l} & x + 1 \\ \hline x^2 + 2x + 1 & x^3 + 3x^2 + 3x + 1 \\ & x^3 + 2x^2 + x \\ \hline & x^2 + 2x + 1 \\ & x^2 + 2x + 1 \\ \hline & 0 \end{array}$$

$$\begin{array}{r|l} & 2x^2 + 9x + 7 \\ \hline x^2 + 2x + 1 & 2x^4 + 13x^3 + 27x^2 + 23x + 7 \\ & 2x^4 + 4x^3 + 2x^2 \\ \hline & 9x^3 + 25x^2 + 23x \\ & 9x^3 + 18x^2 + 9x \\ \hline & 7x^2 + 14x + 7 \\ & 7x^2 + 14x + 7 \\ \hline & 0 \end{array}$$

$\therefore$ G.C.D. $= x^2 + 2x + 1$

6. Reduce the given Rational expressions to its lowest form

$(i)\ \dfrac{x^{3a} - 8}{x^2a + 2xa + 4}$ $\qquad (ii)\ \dfrac{10x^3 - 25x^2 + 4x - 10}{-4 - 10x^2}$

Solution :

$$(i)\ \frac{x^{3a} - 8}{x^2a + 2xa + 4}$$

$$= \frac{(x^a)^3 - 2^3}{x^{2a} + 2x^a + 4}$$

$$= \frac{(x^a - 2)(x^{2a} + 2x^a + 4)}{x^{2a} + 2x^a + 4}$$

$$= x^a - 2$$

$$(ii)\ \frac{10x^3 - 25x^2 + 4x - 10}{-4 - 10x^2}$$

$$= \frac{5x^2(2x - 5) + 2(2x - 5)}{-2(2 + 5x^2)}$$

$$= \frac{(5x^2 + 2)(2x - 5)}{-2(2 + 5x^2)}$$

$$= \frac{(2x - 5)}{-2}$$

$$= -x + \frac{5}{2}$$

7. Simplify

$$\frac{\frac{1}{p} + \frac{1}{q + r}}{\frac{1}{p} - \frac{1}{q + r}} \times \left(1 + \frac{q^2 + r^2 - p^2}{2qr}\right)$$

Solution :

$$= \frac{\frac{1}{p} + \frac{1}{q + r}}{\frac{1}{p} - \frac{1}{q + r}} \times \left(1 + \frac{q^2 + r^2 - p^2}{2qr}\right)$$

$$= \frac{\frac{q+r+p}{p(q+r)}}{\frac{q+r-p}{p(q+r)}} \times \left(\frac{q^2+r^2-p^2+2qr}{2qr}\right)$$

$$= \frac{q+r+p}{q+r-p} \times \frac{(q+r)^2-p^2}{2qr}$$

$$= \frac{q+r+p}{q+r-p} \times \frac{(q+r+p)(q+r-p)}{2qr}$$

$$= \frac{(p+q+r)^2}{2qr}$$

8. Arul, Ravi and Ram working together can clean a store in 6 hours. Working alone, Ravi takes twice as long to clean the store as Arul does. Ram needs three times as long as Arul does. How long would it take each if they are working alone?

Solution :

Let x, y, z be the working speed of Arul, Ravi and Ram respectively & W be the total work done.

$$x+y+z = \frac{W}{6} \qquad \text{— (1)}$$

By data given,

* Ravi takes twice as Arul does

$$\Rightarrow \frac{W}{y} = 2\left(\frac{W}{x}\right)$$

$$\Rightarrow \frac{1}{y} = \frac{2}{x}$$

$$\Rightarrow y = \frac{x}{2}$$

* Ram takes thrice as Arul does

$$\Rightarrow \frac{W}{z} = 3\left(\frac{W}{x}\right)$$

$$\Rightarrow \frac{1}{z} = \frac{3}{x}$$

$$\Rightarrow z = \frac{x}{3}$$

$$\therefore (1) \Rightarrow x + \frac{x}{2} + \frac{x}{3} = \frac{W}{6}$$

$$\Rightarrow \frac{6x+3x+2x}{6} = \frac{W}{6}$$

$$\Rightarrow W = 11x$$

$$\therefore x = \frac{W}{11}$$

$$\therefore y = \frac{x}{2} \Rightarrow y = \frac{W/11}{2} = \frac{W}{22} \text{ and}$$

$$z = \frac{x}{3} \Rightarrow z = \frac{W/11}{3} = \frac{W}{33}$$

$\therefore$ Arul alone does $= \dfrac{W}{x} = \dfrac{W}{W/11} = 11$ hours

Ravi alone does $= \dfrac{W}{y} = \dfrac{W}{W/22} = 22$ hours

Ram alone does $= \dfrac{W}{z} = \dfrac{W}{W/33} = 33$ hours

9 Find the square root of $289x^4 - 612x^3 + 970x^2 - 684x + 361$.

Solution :

	$17x^2 - 18x + 19$
$17x^2$	$289x^4 - 612x^3 + 970x^2 - 684x + 361$
	$289x^4$
$34x^2 - 18x$	$-612x^3 + 970x^2$
	$-612x^3 + 324x^2$
$34x^2 - 36x + 19$	$646x^2 - 684x + 361$
	$646x^2 - 684x + 361$
	0

$$\therefore \sqrt{289x^4 - 612x^3 + 970x^2 - 684x + 361}$$

$$= \left|17x^2 - 18x + 19\right|$$

10. Solve $\sqrt{y+1}+\sqrt{2y-5}=3$

Solution :

Given $\sqrt{y+1}+\sqrt{2y-5}=3$

$\Rightarrow \quad \sqrt{y+1} = 3-\sqrt{2y-5}$

Squaring on both sides

$$y+1=9+2y-5-6\sqrt{2y-5}$$

$\Rightarrow \quad -y-3=-6\sqrt{2y-5}$

$\Rightarrow \quad y+3= 6\sqrt{2y-5}$

$\Rightarrow \quad (y+3)^2=36(2y-5)$

$\Rightarrow \quad y^2+6y+9=72y-180$

$\Rightarrow \quad y^2-66y+189=0$

$\Rightarrow \quad (y-63)(y-3)=0$

$$y=63, 3$$

11. A boat takes 1.6 hours longer to go 36 kms up a river than down the river. If the speed of the water current is 4 km per hr, what is the speed of the boat in still water?

Solution :

Let the speed of the boat in still water be x km/hr.

Distance = 36 kms, Time difference = 1.6 hrs.

$\therefore$ By data given,

$$\frac{36}{x-4}-\frac{36}{x+4}=\frac{8}{5} \qquad (\because 1.6 \text{ hrs} = \frac{8}{5}\text{hrs})$$

$\Rightarrow 36\left(\frac{1}{x-4}-\frac{1}{x+4}\right)=\frac{8}{5}$

$\Rightarrow \frac{x+4-x+4}{x^2-16}=\frac{8}{36\times 5}$

$\Rightarrow \quad \frac{8}{x^2-16}=\frac{8}{180}$

$\Rightarrow \quad x^2-16=180$

$\Rightarrow \quad x^2=196$

$$\therefore x=14$$

$\therefore$ Speed of boat in still water = 14km / hr.

12. Is it possible to design a rectangular park of perimeter 320 m and area 4800 m²? If so find its length and breadth.

Solution :

Given perimeter of a rectangular park = 320 m

Area = 4800 m²

$\therefore$ 2 $(l + b) = 320$, $\quad lb = 4800$

$\Rightarrow \quad l + b = 160$, $\quad lb = 4800 \quad$ ...... (2)

$\therefore$ b + 160 − 2 $\quad$ .... (1)

Sub (1) in (2)

$$l\,(160 - l) = 4800$$

$\Rightarrow \quad 160l - l^2 = 4800$

$\Rightarrow \quad l^2 - 160l + 4800 = 0$

$\Rightarrow \quad (l - 120)(l - 40) = 0$

$\therefore \quad l = 120, l = 40$

$\therefore \quad l = 120$, (1) $\Rightarrow$ b = 160 − 120

= 40

$\therefore$ Length = 120 m

Breadth = 40 m

13. At t minutes past 2 pm, the time needed to 3 pm is 3 minutes less than $\frac{t^2}{4}$ Find t.

Solution :

Given time needed by the minutes hand show

$$\frac{t^2}{4} - 3.$$

$\therefore$ As per the data given,

$$\frac{t^2}{4}-3=60-t$$

$\Rightarrow \quad t^2-12=240-4t$

$\Rightarrow \quad t^2+4t-252=0$

$\Rightarrow (t+18)(t-14)=0$

$\therefore \quad t=-18, t=14$

$\therefore \quad t=14$ min.

14. The number of seats in a row is equal to the total number of rows in a hall. The total number of seats in the hall will increase by 375 if the number of rows is doubled and the number of seats in each row is reduced by 5. Find the number of rows in the hall at the beginning.

Solution :

Let the number of rows be x.

$\therefore$ Number of seats in each row = x

$\therefore$ Total number of seats in the hall = x^2

$\therefore$ By the data given,

$$2x \times (x - 5) = x^2 + 375$$

$$\Rightarrow \quad 2x^2 - 10x = x^2 + 375$$

$$\Rightarrow \quad x^2 - 10x - 375 = 0$$

$$\Rightarrow \quad (x - 25)(x + 15) = 0$$

$$\therefore \quad x = 25, -15$$

$\therefore$ No. of rows at the beginning = 25.

---

15. If α and β are the roots of the polynomial f(x) = $x^2 - 2x + 3$, find the polynomial whose roots are (i) α + 2, β + 2 (ii) $\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$

Solution :

Given α, β are the roots of f(x) = $x^2 - 2x + 3$

$\alpha + \beta = 2, \alpha\beta = 3$

i) To find the polynomial whose roots are $\alpha + 2, \beta + 2$

$$\text{Sum} = \alpha + \beta + 4 = 2 + 4 = 6$$

$$\text{Product} = (\alpha + 2)(\beta + 2) = \alpha\beta + 2\alpha + 2\beta + 4 = \alpha\beta + 2(\alpha + \beta) + 4 = 3 + 2(2) + 4 = 11$$

$\therefore$ The required polynomial is $x^2 - 6x + 11$.

ii) To find the polynomial whose roots are

$$\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$$

$$\text{Sum} = \frac{\alpha-1}{\alpha+1} + \frac{\beta-1}{\beta+1}$$

$$= \frac{\alpha\beta+\alpha-\beta-1+\alpha\beta-\alpha+\beta-1}{(\alpha+1)(\beta+1)}$$

$$= \frac{2\alpha\beta-2}{\alpha\beta+\alpha+\beta+1}$$

$$= \frac{2(3)-2}{3+2+1} = \frac{4}{6} = \frac{2}{3}$$

$$\text{Product} = \frac{\alpha-1}{\alpha+1} \times \frac{\beta-1}{\beta+1}$$

$$= \frac{\alpha\beta-\alpha-\beta-1}{\alpha\beta+\alpha+\beta+1} = \frac{\alpha\beta-(\alpha+\beta)+1}{\alpha\beta+\alpha+\beta+1}$$

$$= \frac{3-2+1}{3+2+1}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

$\therefore$ The required polynomial is

$$x^2 - \frac{2}{3}x + \frac{1}{3} = \frac{3x^2 - 2x + 1}{3}$$

and the equation is $3x^2 - 2x + 1 = 0$.

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16. If – 4 is a root of the equation $x^2+px-4=0$ and if the equation $x^2 + px + q = 0$ has equal roots, find the values of p and q.

Solution :

Given –4 is a root of $x^2 + px - 4 = 0$

$\therefore \alpha + \beta = -p,$	$\alpha\beta = -4$
$-4 + 1 = -p$	$\Rightarrow -4 \times \beta = -4$
$\therefore p = 3$	$\therefore \beta = 1$

Also $x^2 + px + q = 0$ has equal notes.

$\therefore$ a = 1, b = p, c = q

$\Rightarrow$ $b^2 - 4ac = 0$

$\Rightarrow$ $p^2 - 4q = 0$

$\Rightarrow$ $q - 4q = 0$

$\therefore$ $p = 3,\ q = \frac{9}{4}$

17. Two farmers Senthil and Ravi cultivates three varieties of grains namely rice, wheat and ragi. If the sale (in ₹) of three varieties of grains by both the farmers in the month of April is given by the matrix.

April sale in ₹

$$\begin{array}{cc} & \begin{array}{ccc} \text{rice} & \text{wheat} & \text{ragi} \end{array} \\ A = & \begin{pmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{pmatrix} \end{array} \begin{array}{c} \text{Senthil} \\ \text{Ravi} \end{array}$$

and the May month sale (in ₹) is exactly twice as that of the April month sale for each variety.

(i) What is the average sales of the months April and May.

(ii) If the sales continues to increase in the same way in the successive months, what will be sales in the month of August ?

Solution :

Given sale of April month is

$$A = \begin{pmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{pmatrix}$$

$\therefore$ By given data, sale of may month is

$$B = \begin{pmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{pmatrix}$$

i) Average sales of April & May

$$= \begin{pmatrix} \frac{1500}{2} & \frac{3000}{2} & \frac{4500}{2} \\ \frac{7500}{2} & \frac{4500}{2} & \frac{1500}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 750 & 1500 & 2250 \\ 3750 & 2250 & 750 \end{pmatrix}$$

ii) Sales in the month of August

$$= 16\begin{pmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{pmatrix}$$

$$= \begin{pmatrix} 8000 & 16000 & 24000 \\ 40000 & 24000 & 8000 \end{pmatrix}$$

18. If $\cos\theta\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + \sin\theta\begin{pmatrix} x & -\cos\theta \\ \cos\theta & x \end{pmatrix} = I_2$ **find x.**

Solution :

Given

$$\cos\theta\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + \sin\theta\begin{pmatrix} x & -\cos\theta \\ \cos\theta & x \end{pmatrix} = I_2$$

$$\Rightarrow \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} + \begin{pmatrix} x\sin\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & x\sin\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \cos^2\theta + x\sin\theta & 0 \\ 0 & \cos^2\theta + x\sin\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ $\cos^2\theta + x\sin\theta) = 1$

$\Rightarrow$ $x\sin\theta = 1 - \cos^2\theta$

$\Rightarrow$ $x\sin\theta = \sin^2\theta$

$\Rightarrow$ $x = \sin\theta$

19. Given $A = \begin{pmatrix} p & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$

and if BA = C^2, find p and q.

Solution : Given BA = C^2

$$\Rightarrow \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -2q \\ p & 0 \end{pmatrix} = \begin{pmatrix} 0 & -8 \\ 8 & 0 \end{pmatrix}$$

$\therefore$ p = 8, $-$ 2q = $-$ 8, q = 4

20. $A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 3 \\ 8 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 6 \\ 1 & 1 \end{pmatrix}$

find the matrix D, such that CD–AB = 0

Solution : Given CD $-$ AB = 0

$$\Rightarrow \begin{pmatrix} 3 & 6 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 6 & 3 \\ 8 & 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3a+6c & 3b+6d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 18 & 9 \\ 64 & 37 \end{pmatrix}$$

$\therefore$ 3a + 6c = 18 .... (1)

a + c = 64 .... (2)

(1) $\Rightarrow$ a + 2c = 6

(3) $\Rightarrow$ a + c = 64

c = $-$ 58

a $-$ 58 = 64

a = 122

3b + 6d = 9 .... (3)

b + d = 37 .... (4)

(3) $\Rightarrow$ b + 2d = 3

(4) $\Rightarrow$ b + d = 37

d = $-$ 34

b $-$ 34 = 37

b = 71

$\therefore$ a = 122, b = 71, c = $-$ 58, d = $-$ 34

$$\therefore D = \begin{pmatrix} 122 & 71 \\ -58 & -34 \end{pmatrix}$$

PROBLEMS FOR PRACTICE

1. Solve : 2x + 5y + 2z = $-$ 38, 3x $-$ 2y + 4z = 17, $-$ 6x + y $-$ 7z = 12

(Ans : x = 3, y = $-$ 8, z = $-$ 2)

2. Solve : 3x $-$ 9z = 33, 7x $-$ 4y $-$ z = $-$ 15, 4x + 6y + 5z = $-$ 6

(Ans : x = $-$ 1, y = 3, z = $-$ 4)

3. Solve : x+2y+z=7, x+3z = 11, 2x $-$ 3y = 1

(Ans : (2, 1, 3))

4. Solve : $x - \frac{y}{5} = 6$, $y - \frac{z}{7} = 8$, $z - \frac{x}{2} = 10$

(Ans : (8, 10, 14))

5. Solve : $\frac{1}{3}$(x + y $-$ 5) = y $-$ z = 2x $-$ 11 = 9 $-$ (x + 2z)

Ans : ((6, 2, 1))

6. Oviya, Sankee, Mithu have a total of \$ 89 in their wallets. Oviya has \$6. less than Mithu. Sankee has 3 times Mithu has Howmany does each have ?

(Ans : 13, 57, 19)

7. Sum of 3 numbers is 10. Sum of the first number, twice the second number and 3 times the third is 29 and the sum of first, four times the second and nine times the third is 43, Find the numbers.

(Ans : 4, 3, 3)

8. In a shop, the following items were sold on 3 days.

	Rice	Oil	Sugar
Day 1	25	10	10
Day 2	16	6	4
Day 3	30	12	6

If the total values sold were Rs.820, Rs.480 and Rs.912 respectively. Find the cost of 1 kg of each item.

(Ans : 12, 40, 12)

9. Find the GCD of the following :

i) $x^4 - 27a^3x, (x - 3a)^2$ **(Ans : x − 3a)**

ii) $x^3 + 8x^2 - x - 8, x^3 + x^2 - x - 1$ **(Ans : $x^2 - 1$)**

iii) $x^2 - x - 2, x^2 + x - 6, 3x^2 - 13x + 14$ **(Ans : x − 2)**

iv) $3(x^2 - 5x + 6), 4(x^2 - 4x + 3)$ **(Ans : x − 3)**

v) $6a^2 - 11a + 3, 12a^2 + 5a - 3$ **(Ans : 3a − 1)**

10. Find the LCM of the following :

i) $q^2 - 4, q^3 - 8, q^2 - 6q + 8$ **(Ans : $(q^2 - 4)(q - 4)(q^2 + 2q + 4)$)**

ii) $2(x - 1)^2, 3(x^2 - 1)$ **(Ans : $6(x - 1)^2(x + 1)$)**

iii) $2m^2 - 18n^2, 5m^2n + 15mn^2, m^3 + 27n^3$ **(Ans : $10mn(m - 3n)(m^2 + 27n^3)$)**

iv) $6b^2 - b - 1, 3b^2 + 7b + 2, 2b^2 + 3b - 2$ **(Ans : $(3b + 1)(2b - 1)(b + 2)$)**

11. Find the GCD by long division :

i) $3x^4+6x^3 -12x^2 -24x, 4x^4+14x^3+8x^2 - 8x$ **(Ans : $x(x^2 + 4x + 4)$)**

ii) $x^4 + x^3 + 4x^2 + 4x, x^3 - 3x^2 + 4x - 12$ **(Ans : $x^2 + 4$)**

iii) $3x^2 + 13x + 10, 3x^3 + 18x^2 + 33x + 18$ **(Ans : x + 1)**

12. If $(x + 3)(x - 2)$ is the GCD of $f(x) = (x + 3)(2x^2 - 3x + a)$ and $g(x) = (x - 2)(3x^2 + 7x - b)$, find a and b.

(Ans : a = −2, b = 6)

13. The HCF and LCM of 2 polynomials are $5x^2 + x$ and $(x^3 - 4x)(5x + 1)$ respectively. One of the polynomials is $5x^3 - 9x^2 - 2x$. Find the other.

(Ans : $x(x + 2)(5x + 1)$)

14. Find the LCM of the polynomials $2x^3 + 15x^2 + 2x - 35$, $x^3 + 8x^2 + 4x - 21$ whose GCD is x + 7.

(Ans $(x^3 + 8x^2 + 4x - 21)(2x^2 + x - 5)$)

15. Simplify :

i) $$\frac{2x^2+3x+1}{3x^2+4x+1} \times \frac{4x^2+5x+1}{5x^2+6x+1} \times \frac{15x^2+8x+1}{8x^2+6x+1}$$ **(Ans : 1)**

ii) $$\frac{x^4+x^2+1}{x^2+x+1}$$ **(Ans : $x^2 - x + 1$)**

iii) $$\frac{a^2-(b+c)^2}{c^2-(a+b)^2} \div \frac{b^2-(c+a)^2}{a^2+ab+ca}$$ **$\left(\text{Ans : } \frac{(a-b-c)a}{(c-a-b)(b-a-c)}\right)$**

16. Solve the following by factorisation :

i) $2x^2 + 6\sqrt{3}x - 60 = 0$ **(Ans : $2\sqrt{3}, -5\sqrt{3}$)**

ii) $\frac{4}{x} - 3 = \frac{5}{2x+3}$ **(Ans 1, − 2)**

iii) $9x^2 - 9(a + b)x + (2a^2 + 5ab + 2b^2) = 0$ **$\left(\text{Ans : } \frac{a+2b}{3}, \frac{2a+b}{3}\right)$**

iv) $(5x - 2)(x + 1) = 3x(3x - 1)$ **$\left(\text{Ans : } 1, \frac{1}{2}\right)$**

v) $\frac{2x+3}{x+8}=\frac{3(x-4)}{x-2}$ **(Ans : 5, − 18)**

17. Solve the following by completing the square.

i) $5x^2 - 6x - 2 = 0$ $\left(\textbf{Ans :} \frac{3 \pm \sqrt{19}}{5}\right)$

ii) $2x^2 + x - 4 = 0$ $\left(\textbf{Ans :} \frac{-1 \pm \sqrt{33}}{4}\right)$

iii) $4x^2 + 17x = 15$ $\left(\textbf{Ans :} \frac{3}{4}, -5\right)$

iv) $2\left(\frac{2x+1}{x+3}\right)-3\left(\frac{x+3}{2x-1}\right)=5$

$\left(\textbf{Ans :} -10, \frac{-1}{5}\right)$

18. Solve the following by using quadratic formula.

i) $\frac{1}{x+1}+\frac{2}{x+2}=\frac{4}{x+4}$ $\left(\textbf{Ans :} 2 \pm 2\sqrt{3}\right)$

ii) $9x^2-6ax+(a^2-b^2)=0$ $\left(\textbf{Ans :} \frac{a+b}{3}, \frac{a-b}{3}\right)$

iii) $a(x^2+1)=x(a^2+1)$ $\left(\textbf{Ans :} a, \frac{1}{a}\right)$

iv) $(x-2)(2x+3)=3(x-4)(x+8)$

(Ans : -6, -4)

19. Divide 18 into two parts such that twice the sum of their squares is 5 times their product.

(Ans : 6, 12)

20. In a music hall, the number of seats in each row is 10 less than the number of rows. If there are 704 seats in the hall. Find the number of rows.

(Ans : 32)

21. The perimeter of a right triangle is 60 cm. Its hypotenuse is 25 cm. Find the area of the triangle.

(Ans : 150 cm^2)

22. The speed of a boat in still water is 15 Km/hr. It goes 30 km upstream and return down stream to the original point in $4\frac{1}{2}$ hrs. Find the speed of the stream.

(Ans : 5 Km/hr)

23. An aeroplane left 30 min later than its scheduled time and in order to reach its destination 1500 Km away in time, it has to increase its speed by 250 Km/hr from its usual speed. Find the original speed.

(Ans : 750 Km/hr)

24. The numerator of a fraction is 3 less than its denominator. If 2 is added to both the numerator and the denominator, the the sum of new fraction and original fraction is $\frac{29}{20}$. Find the original fraction.

$\left(\textbf{Ans :} \frac{7}{10}\right)$

25. A flock of swans contained x^2 members. As the clouds gathered. 10x went lake, and $\frac{1}{8}$ x^2 flew away to a garden. The remaining three couples played about in the water. Howmany swans were there in that lake ?

(Ans : 144)

26. If one root of $x^2 - 3x + \phi = 0$ is twice the other. Find the value of ϕ.

(Ans : 2)

27. Find 'k' if the following equations have real & equal roots.

    i) $2x^2 - 10x + k = 0$ **(Ans : 25/2)**

    ii) $x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$

    (Ans : 2 (or) –10/ 9)

    iii) $(k + 4)x^2 + (k + 1)x + 1 = 0$

    (Ans : 5, – 3)

    iv) $(p + 1)x^2 - 6(p + 1)x + 3(p + q) = 0$

    (Ans : – 1, 3)

28. Show that the roots of the equation $3p^2x^2 - 2pq + q^2 = 0$ are not real.

29. If α and β are the roots of $3x^2 - 6x + 1 = 0$, form the equation whose roots are

    i) $\alpha^2\beta, \beta^2\alpha$ **(Ans : $27x^2 - 18x + 1 = 0$)**

    ii) $2\alpha + \beta, 2\beta + \alpha$

    (Ans : $3x^2 - 18x + 25 = 0$)

30. Draw the graph of $y = x^2 + 2x - 3 = 0$ and hence solve $x^2 - x - 6 = 0$ **(Ans : 3, – 2)**

31. Draw the graph of $y = x^2 + 3x + 2$ and hence solve $x^2 + 2x + 4 = 0$ **(Ans : not real)**

32. Draw the graph of $y = (2x + 3)(x - 2)$

33. Draw the graph of $y = 2x^2$ and hence solve $2x^2 + x - 6 = 0$. **(Ans : – 2, 3/2)**

34. If $A = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$, $B = (1 \quad 3 \quad -6)$, verify that $(AB)^T = B^T A^T$

35. Construct a 4 × 3 matri whose elements are

    $_{ij} \quad \frac{|2i \quad 3j|}{\ }$

36. If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$ does $(A + B)^2 = A^2 + 2AB + B^2$ hold ?

    (NO)

37. If $A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ Verify that $A(BC) = (AB)C$.

38. If $A = \begin{pmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{pmatrix}$, Verify $(A - B)^T = A^T - B^T$.

39. If $B.\ B^T = 9I$, where $B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$, find x, y.

    (Ans : –2, –1)

40. If $A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, show that $A^2 - 7A + 10I = 0$

OBJECTIVE TYPE QUESTIONS

1. The value of a, b, c, d

   $\begin{pmatrix} d+1 & 10+a \\ 3b-2 & a-4 \end{pmatrix} = \begin{pmatrix} 2 & 2a+1 \\ b-5 & 4c \end{pmatrix}$ if

$a)\ 11, 7, 3, 1$ $\quad b)\ 9, \frac{-3}{2}, 1, \frac{5}{4}$

$c) -9, \frac{3}{2}, \frac{-5}{4}, -1$ $\quad d)\ 9, \frac{-3}{2}, \frac{5}{4}, 1$

(Ans : (d))

2. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{pmatrix}$, then A^2 is

a) a null matrix b) a unit matrix
c) $-A$ d) A

(Ans : (b))

3. If $A = \begin{pmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $AB = I_3$, then x + y equals

a) 0 b) -1 c) 2 d) none

(Ans : (a))

4. If $A = \begin{pmatrix} 5 & x \\ y & 0 \end{pmatrix}$ and $A = A^T$, then

a) x = 0, y = 5 b) x + y = 5
c) x = y d) none

(Ans : (c))

5. Matrix $A = (a_{ij})_{m \times n}$ is a square matrix if
a) m < n b) m > n
c) m = 1 d) m = n

(Ans : (d))

6. If $1/\alpha$ is a root of $2x^2 - 5x + 7 = 0$, then the value of $7\alpha^2 - 5\alpha$ is
a) 2 b) -2 c) 5 d) -5

(Ans :(b))

7. If $x^2 + 1/x^2 = 23, x > 0$, then $x + 1/x$ is
a) 2 b) 3 c) 4 d) 5

(Ans : (d))

8. Which one of the following is a root of the equation $2x^4 - 5x^3 - 3x^2 + 13x + 9 = 0$
a) 1 b) -1 c) 2 d) 0

(Ans :(b))

9. For what value of k, will the system of equations 2x + 3y = k and (k – 1) x + (k + 2) y = 3k has infinite solutions ?
a) -7 b) 5 c) 7 d) 0

(Ans : c)

10. The discriminant Δ of the equation $3\sqrt{3}x^2 + 10x + \sqrt{3} = 0$
a) 64 b) – 64 c) 81 d) none

(Ans : (a))

11. The LCM of $6x^2y$, $9x^2yz$, $12x^2y^2z$ is
a) $36\ xy^2z^2$ b) $36\ x^2y^2z$
c) $36\ x^2y^2z^2$ d) $36\ xy^2z$

(Ans : (b))

12. On dividing $\frac{x^2 - 25}{x+3}$ by $\frac{x+5}{x^2-9}$ equal to
a) (x – 5) (x – 3) b) (x – 5) (x + 3)
c) (x + 5) (x – 3) d) (x + 5) (x + 3)

(Ans : (a))

13. The solution set of the equation $(x-3)^2 = 9$ is
a) {0, 3} b) {3, 3} c) {3, 6} d) {0, 6}

(Ans : (d))

14. The equation whose roots are b + c and b – c is
a) $x^2 + 2bx + (b^2 - c^2) = 0$
b) $x^2 - 2bx + (b^2 - c^2) = 0$
c) $x^2 - 2bx - (b^2 - c^2) = 0$
d) $x^2 + 2bx - (b^2 - c^2) = 0$ **(Ans : (b))**

15. The value of m if $10x^2 + mx - 10$ leaves a remainder 2 when divided by 2x – 3.

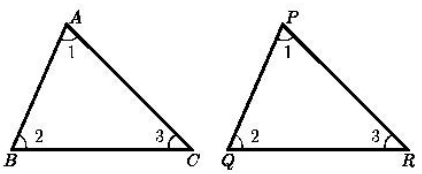
a) 7 b) – 7 c) $\frac{-80}{3}$ d) $\frac{80}{3}$

(Ans : (b))

CHAPTER 4 GEOMETRY

I. CONGRUENCY AND SIMILARITY OF TRIANGLES

Points to Remember



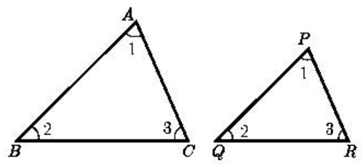
$\Delta ABC \cong \Delta PQR$

$\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R.$

$AB = PQ, BC = QR, CA = RP$

$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = 1$

Same shape and same size.



$\Delta ABC \sim \Delta PQR$

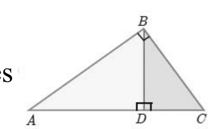
$\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R.$

$AB \neq PQ, BC \neq QR, CA \neq RP$

but $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} > 1$ or < 1

Same shape but not same size.

- ✓ **AA Criterion of similarity** If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar, because the third angle in both triangles must be equal. Therefore, AA similarity criterion is same as the AAA similarity criterion.
- ✓ **SAS Criterion of similarity.** If one angle of a triangle is equal to one angle of another triangle and if the sides including them are proportional then the two triangles are similar.
- ✓ **SSS Criterion of similarity.** If three sides of a triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.
- ✓ A perpendicular line drawn from the vertex of a right angled triangle divides two triangles similar to each other and also to original triangle.

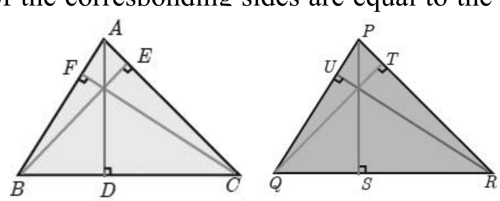


$\Delta ADB \sim \Delta BDC, \Delta ABC \sim \Delta ADB, \Delta ABC \sim \Delta BDC$

- ✓ If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of their corresponding altitudes.

i.e. if $\Delta ABC \sim \Delta PQR$ then

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = \frac{AD}{PS} = \frac{BE}{QT} = \frac{CF}{RU}$$



- ✓ If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of the corresponding perimeters.

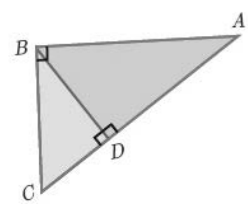
  $\Delta ABC \sim \Delta DEF$ then

  $$\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}=\frac{AB+BC+CA}{DE+EF+FD}$$

- ✓ The ratio of the area of two similar triangles are equal to the ratio of the squares of their corresponding sides.

  $$\frac{\text{area }(\Delta ABC)}{\text{area }(\Delta PQR)}=\frac{AB^2}{PQ^2}=\frac{BC^2}{QR^2}=\frac{AC^2}{PR^2}$$

- ✓ If two triangles have common vertex and their bases are on the same straight line, the ratio between their areas is equal to the ratio between the length of their bases.

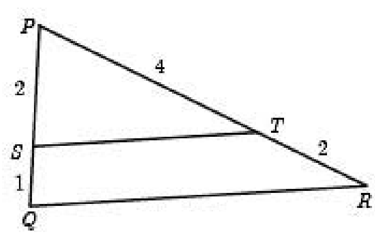
  Here, $\frac{\text{area }(\Delta ABD)}{\text{area }(\Delta BDC)}=\frac{AD}{DC}$

- ✓ Two triangles are said to be similar if their corresponding sides are proportional.
- ✓ The triangles are equiangular if the corresponding angles are equal.

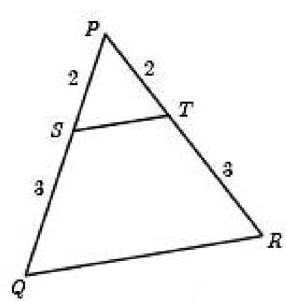
Example 4.1

Show that $\Delta PST \sim \Delta PQR$

(i)



(ii)



Solution :

i) In ΔPST and ΔPQR,

$$\frac{PS}{PQ}=\frac{2}{2+1}=\frac{2}{3},\ \frac{PT}{PR}=\frac{4}{4+2}=\frac{2}{3}$$

Thus, $\frac{PS}{PQ}=\frac{PT}{PR}$ and $\angle P$ is common

Therefore, by SAS similarity,

$\Delta PST \sim \Delta PQR$

ii) In ΔPST and ΔPQR,

$$\frac{PS}{PQ}=\frac{2}{2+3}=\frac{2}{5},\ \frac{PT}{PR}=\frac{2}{2+3}=\frac{2}{5}$$

Thus, $\frac{PS}{PQ}=\frac{PT}{PR}$ and $\angle P$ is common

Therefore, by SAS similarity,

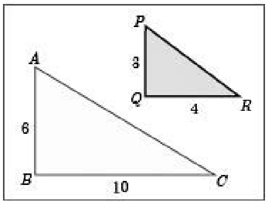
$\Delta PST \sim \Delta PQR$

Example 4.2

Is ΔABC ~ ΔPQR ?

Solution :

In ΔABC and ΔPQR,



$$\frac{PQ}{AB} = \frac{3}{6} = \frac{1}{2}, \frac{QR}{BC} = \frac{4}{10} = \frac{2}{5}$$

$$\text{since } \frac{1}{2} \neq \frac{2}{5}, \frac{PQ}{AB} \neq \frac{QR}{BC}$$

The corresponding sides are not proportional.

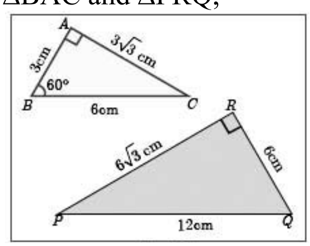
Therefore ΔABC is not similar to ΔPQR.

Example 4.3

Observe Fig.4.18 and find ∠P.

Solution :

In ΔBAC and ΔPRQ,



$$\frac{AB}{RQ} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{BC}{QP} = \frac{6}{12} = \frac{1}{2}; \frac{CA}{PR} = \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{1}{2}$$

$$\text{Therefore, } \frac{AB}{RQ} = \frac{BC}{QP} = \frac{CA}{PR}$$

By SSS similarity, we have ΔBAC ~ ΔQRP

∠P = ∠C (since the corresponding parts of similar triangle)

$\angle P = \angle C = 180^0 - (\angle A + \angle B) = 180^0 - (90^0 + 60^0)$

$\angle P = 180^0 - 150^0 = 30^0$

Example 4.4

A boy of height 90cm is walking away from the base of a lamp post at a speed of 1.2m/sec. If the lamp post is 3.6m above the ground, find the length of his shadow cast after 4 seconds.

Solution :

Given, speed = 1.2 m/s,

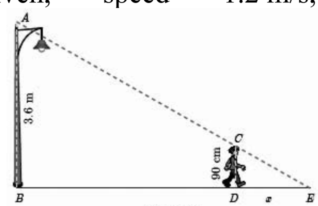


Fig. 4.19

time = 4 seconds

distance = speed × time

= 1.2 × 4 = 4.8 m

Let x be the length of the shadow after 4 seconds

$$\text{Since, } \Delta ABE \sim CDE, \frac{BE}{DE} = \frac{AB}{CD} \text{ gives } \frac{4.8+x}{x}$$

$$= \frac{3.6}{0.9} = 4 \text{ (since } 90cm = 0.9m)$$

48 = x = 4x gives 3x = 4.8 so, x = 1.6 m

The length of his shadow DE = 1.6 m

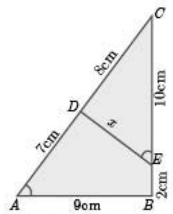
Example 4.5

In Fig. ∠A = ∠CED prove that ΔCAB ~ ΔCED. Also find the value of x.

Solution :

In ΔCAB and ΔCED, ∠C is common,

∠A = ∠CED



Therefore, ΔCAB ~ ΔCED (By AA similarity)

Hence

$$\frac{CA}{CE}=\frac{AB}{DE}=\frac{CB}{CD}$$

$$\frac{AB}{DE}=\frac{CB}{CD} \text{ gives } \frac{9}{x}=\frac{10+2}{8} \text{ so, } x=\frac{8\times 9}{12}=6 \text{ cm.}$$

Example 4.6

In Fig. QA and PB are perpendiculars to AB. If AO = 10 cm, BO = 6 cm and PB = 9 cm. Find AQ.

Solution :

In Δ AOQ and Δ BOP, ∠OAQ = ∠OBP = 90^0

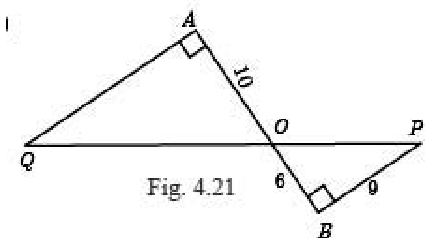


Fig. 4.21

∠AOQ = ∠BOP (Vertically opposite angles)

Therefore, by AA Criterion of similarity,

$$\Delta AOQ \sim \Delta BOP$$

$$\frac{AO}{BO}=\frac{OQ}{OP}=\frac{AQ}{BP}$$

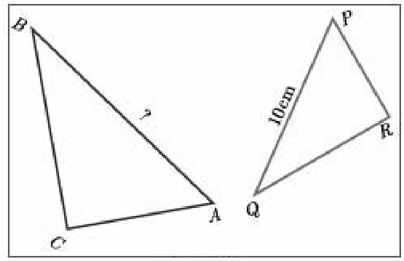
$$\frac{10}{6}=\frac{AQ}{9} \text{ gives } AQ=\frac{10\times 9}{6}=15cm$$

Example 4.7

The perimeters of two similar triangles ABC and PQR are respectively 36 cm and 24 cm. If PQ =10 cm, find AB.

Solution :

The ratio of the corresponding sides of similar triangles is same as the ratio of their perimeters,



Since ΔABC ~ ΔPQR,

$$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{AC}{PR}=\frac{36}{24}$$

$$\frac{AB}{PQ}=\frac{36}{24} \text{ gives } \frac{AB}{10}=\frac{36}{24}$$

$$AB=\frac{36\times 10}{24}=15cm$$

Example 4.8

If ΔABC is similar to ΔDEF such that BC = 3 cm, EF = 4 cm and area of ΔABC = 54 cm^2. Find the area of ΔDEF.

Solution :

Since the ratio of area of two similar triangles is equal to the ratio of the squares of any two corresponding sides, we have

$$\frac{Area\ (\Delta ABC)}{Area\ (\Delta DEF)}=\frac{BC^2}{EF^2} \text{ gives } \frac{54}{Area\ (\Delta DEF)}=\frac{3^2}{4^2}$$

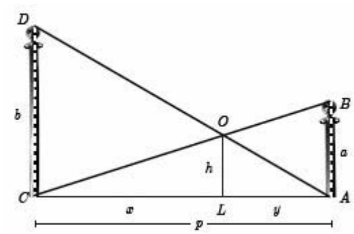
$$Area\ (\Delta DEF)=\frac{16\times 54}{9}=96\ cm^2$$

Example 4.9

Two poles of height 'a' metres and 'b' metres are 'p' metres apart. Prove that the height of the point of intersection of the lines joining the top of each pole to the foot of the opposite pole is given by $\frac{ab}{a+b}$ metres.

Solution :

Let AB and CD be two poles of height 'a' metres and 'b' metres respectively such that the poles are 'p' metres apart. That is AC = p metres. Suppose the lines AD and BC meet at O, such that OL = h metres



Let CL = x and LA = y.

Then, x + y = p

In ΔABC and ΔLOC, we have ∠CAB = ∠CLO [each equals to 90^0]

∠C = ∠C [C is common]

ΔCAB ~ ΔCLO [By AA similarity]

$$\frac{CA}{CL} = \frac{AB}{LO} \text{ gives } \frac{p}{x} = \frac{a}{h}$$

$$\text{so,} \quad x = \frac{ph}{a} \quad \text{.......... (1)}$$

In ΔALO and ΔACD, we have

∠ALO = ∠ACD [each equal to 90^0]

∠A = ∠A [A is common]

ΔALO ~ ΔACD [By AA similarity]

$$\frac{AL}{AC} = \frac{OL}{DC} \text{ gives}$$

$$\frac{y}{p} = \frac{h}{b} \text{ we get, } y = \frac{ph}{b} \quad \text{.......... (2)}$$

$$\text{(1) + (2) gives } x + y = \frac{ph}{a} + \frac{ph}{b}$$

$$p = ph\left(\frac{1}{a} + \frac{1}{b}\right) \quad (Since\ x + y = p)$$

$$1 = h\left(\frac{a+b}{ab}\right)$$

Therefore, $h = \dfrac{ab}{a+b}$

Hence, the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole is $\dfrac{ab}{a+b}$ metres.

Construction of similar triangles

Example 4.10

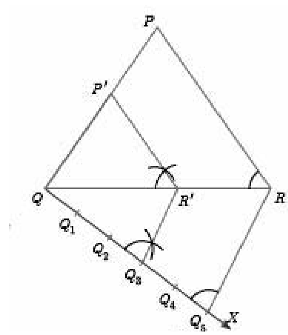
Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{3}{5}$ of the corresponding sides of the triangle PQR (scale factor $\frac{3}{5} < 1$)

Solution :

Given a triangle PQR we are required to construct another triangle whose sides are $\frac{3}{5}$ of the corresponding sides of the triangle PQR.

Steps of construction

1. Construct a ΔPQR with any measurement.



2. Draw a ray QX making an acute angle with QR on the side opposite to vertex P.

3. Locate 5 (the greater of 3 and 5 in $\frac{3}{5}$) points.

   Q_1, Q_2, Q_3, Q_4, and Q_5 on QX so that $QQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4Q_5$

4. Join Q_5R and draw a line through Q_3 (the third point, 3 being smaller of 3 and 5 in $\frac{3}{5}$) parallel to Q_5R to intersect QR at R'.
5. Draw line through R' parallel to the line RP to intersect QP at P'. Then, ΔP'QR' is the required triangle each of whose sides is three-fifths of the corresponding sides of ΔPQR.

Example 4.11

Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{7}{4}$ of the corresponding sides of the triangle PQR (scale factor $\frac{7}{4}>1$)

Solution :

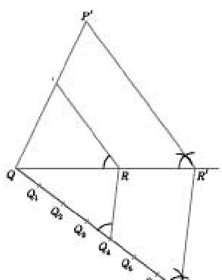


Fig. 4.26

Given a triangle PQR, we are required to construct another triangle whose sides are $\frac{7}{4}$ of the corresponding sides of the triangle PQR.

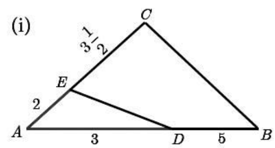
Steps of construction

1. Construct a ΔPQR with any measurement.
2. Draw a ray QX making an acute angle with QR on the side opposite to vertex P.
3. Locate 7 (the greater of 4 and 7 in $\frac{7}{4}$) points. $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ and Q_7 on QX so that $QQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4Q_5 = Q_5Q_6 = Q_6Q_7$
4. Join Q_4 (the 4th point, 4 being smaller of 4 and 7 in $\frac{7}{4}$) to R and draw a line through Q_7 parallel to Q_4R, intersecting the extended line segment QR at R'.
5. Draw a line through R' parallel to RP intersecting the extended line segment QP at P' Then ΔP'QR' is the required triangle each of whose sides is seven-fourths of the corresponding sides of ΔPQR.

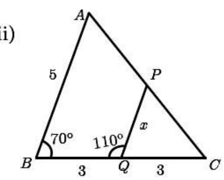
EXERCISE 4.1

1. Check whether the which triangles are similar and find the value of x.

(i)



(ii)



Solution:

$i)\quad \frac{AE}{AC} = \frac{2}{2+3.5} = \frac{2}{5.5} = \frac{4}{11}$

$\frac{AD}{AB} = \frac{3}{3+5} = \frac{3}{8}$

$\therefore \frac{AE}{AC} \neq \frac{AD}{AB}$

∴ the 2 triangles are not similar.

ii) Given ∠PQB=110⁰ ⇒ ∠PQC = 70⁰ = ∠QBA

∴ Corresponding angles are equal.

$\therefore \frac{CQ}{QB} = \frac{PQ}{AB}$

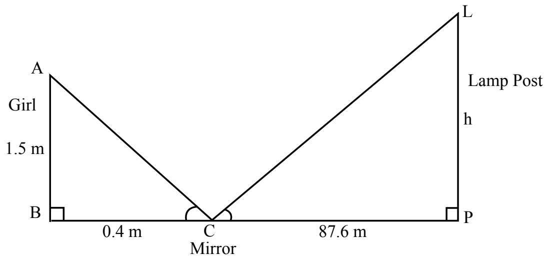
$\Rightarrow \frac{3}{3} = \frac{x}{5}$

$\Rightarrow 1 = \frac{x}{5}$

$\therefore x = 5$

2. A girl looks the reflection of the top of the lamp post on the mirror which is 66 m away from the foot of the lamppost. The girl whose height is 12.5 m is standing 2.5 m away from the mirror. Assuming the mirror is placed on the ground facing the sky and the girl, mirror and the lamp-post are in a same line, find the height of the lamp post.

Solution :



Given AB = height of girl = 1.5 m
BC = Dist. between girl & Mirror = 0.4 m
LP = height of lamp post = h
CP = dist. between Mirror & Post= 87.6 m

In ΔABC, ΔLPC,

$\angle B = \angle P = 90^0$

$\angle ACB = \angle LCP$ (angle of incidence & angle of reflection)

$\therefore$ ΔABC & ΔLPC are similar.

$$\therefore \frac{AB}{LP} = \frac{BC}{CP} \quad \text{(By } AA \text{ similarity)}$$

$$\Rightarrow \frac{1.5}{h} = \frac{0.4}{87.6}$$

$$\Rightarrow h = \frac{87.6 \times 1.5}{0.4}$$

$$= \frac{87.6 \times 3/2}{4/10}$$

$$= 43.8 \times 3 \times 5/2$$

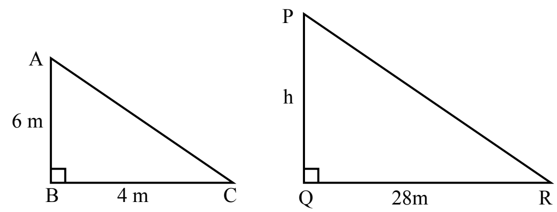
$$= 21.9 \times 15$$

$$= 328.5\, m$$

$\therefore$ Height of he lamp post = 328.5 m

3. A vertical stick of length 6 m casts a shadow 400 cm long on the ground and at the same time a tower casts a shadow 28 m long. Using similarity, find the height of the tower.

Solution :



In ΔABC and ΔPQR

$\angle B = \angle Q = 90^0$

$\angle C = \angle R$ (AC | | PR)

$\therefore$ By AA similarity, $\Delta ABC \sim \Delta PQR$

$$\therefore \frac{AB}{PQ} = \frac{BC}{QR}$$

$$\Rightarrow \frac{6}{h} = \frac{4}{28}$$

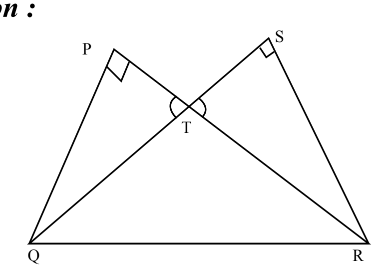
$$\Rightarrow \frac{6}{h} = \frac{1}{7}$$

$$\Rightarrow h = 42\, m$$

$\therefore$ Height of the tower = 42m.

4. Two triangles QPR and QSR, right angled at P and S respectively are drawn on the same base QR and on the same side of QR. If PR and SQ intersect at T, prove that PT × TR = ST × TQ.

Solution :



Consider ΔPQT and ΔSRT

i) $\angle P = \angle S = 90^0$

ii) $\angle PTQ = \angle STR$ (Vertically Opp.angle)

$\therefore$ By AA similarity,

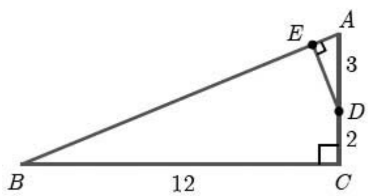
$$\Delta PQT \sim \Delta SRT$$

$$\therefore \frac{QT}{TR} = \frac{PT}{ST}$$

$$\Rightarrow PT \times TR = ST \times TQ$$

Hence proved.

5. In the adjacent figure, ΔABC is right angled at C and DE $\perp$ AB. Prove that $\Delta ABC \sim \Delta ADE$ and hence find the lengths of AE and DE.



Solution :

In ΔABC & ΔADE,

i) $\angle AED = \angle ACB = 90^0$

ii) $\angle A$ is common

$\therefore$ By AA similarity,

$$\Delta ABC \sim \Delta ADE$$

Also,

$AB^2 = AC^2 + BC^2$

$= 5^2 + 12^2$

$= 25 + 144$

$= 169$

$\therefore$ AB = 13

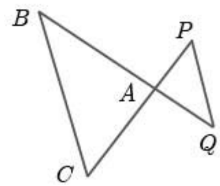
$\therefore$ By similarily,

$$\frac{AD}{AB} = \frac{ED}{BC} = \frac{AE}{AC}$$

$$\Rightarrow \frac{3}{13} = \frac{DE}{12} = \frac{AE}{5}$$

$$\therefore AE = \frac{15}{13}, DE = \frac{36}{13}$$

6. In the adjacent figure, $\Delta ACB \sim \Delta APQ$. If BC = 8 cm, PQ = 4 cm, BA = 6.5 cm and AP = 2.8 cm, find CA and AQ.



Solution :

Given Δ ACB ~ Δ APQ

$$\therefore \frac{BC}{PQ} = \frac{AC}{AP} = \frac{AB}{AQ}$$

$$\Rightarrow \frac{8}{4} = \frac{AC}{2.8} = \frac{6.5}{AQ}$$

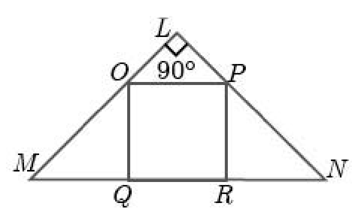
$$\therefore \frac{AC}{2.8} = 2$$

$$\Rightarrow AC = 5.6\ cm$$

$$\frac{6.5}{AQ} = 2$$

$$\Rightarrow AQ = \frac{6.5}{2} = 3.25\ cm$$

7. If figure OPRQ is a square and $\angle MLN = 90^0$. Prove that i) $\Delta LOP \sim \Delta QMO$ ii) $\Delta LOP \sim \Delta RPN$ iii) $\Delta QMO \sim \Delta RPN$ iv) $QR^2 = MQ \times RN$



Solution :

i) In ΔLOP, ΔQMO

a) $\angle OLP = \angle OQM = 90^0$

b) $\angle LOP = \angle OMQ$ (Corresponding angles)

$\therefore$ By AA similarity,

$$\Delta LOP \sim \Delta QMO$$

ii) In ΔLOP, ΔRPN

a) ∠OLP = ∠PRN = 90^0

b) ∠LPO = ∠PNR (Corresponding angles)

∴ By AA similarity,

ΔLOP ~ ΔRPN

∴ From (i) & (ii)

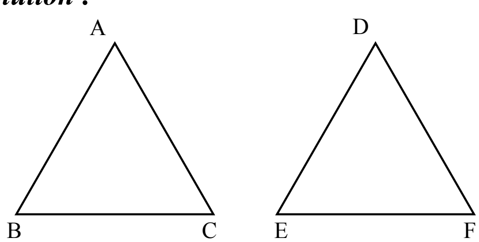
ΔQMO ~ ΔRPN

$$\therefore \frac{QM}{RP} = \frac{QO}{RN} \Rightarrow \frac{QM}{QR} = \frac{QR}{RN} \quad (\because \text{ Square})$$

$$\Rightarrow QR^2 = MQ \times RN$$

8. If ΔABC ~ ΔDEF such that area of ΔABC is 9cm² and the area of ΔDEF is 16cm² and BC = 2.1 cm. Find the length of EF.

Solution :



Given ΔABC ~ ΔDEF

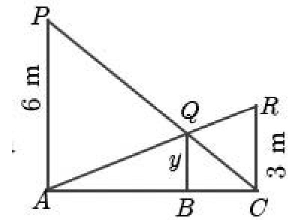
$$\therefore \frac{\text{Area of } \Delta\, ABC}{\text{Area of } \Delta\, DEF} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \frac{9}{16} = \frac{(2.1)^2}{EF^2}$$

$$\Rightarrow EF^2 = \frac{16 \times (2.1)^2}{9}$$

$$\therefore EF = \frac{4 \times 2.1}{3} = 2.8\, cm$$

9. Two vertical poles of heights 6 m and 3 m are erected above a horizontal ground AC. Find the value of y.



Solution :

From the fig.

ΔPAC ~ ΔQBC

∠PAC = ∠QBC (Corresponding angles)

∠C = ∠C

$$\therefore \frac{CB}{CA} = \frac{QB}{PA}$$

$$\Rightarrow \frac{CB}{CA} = \frac{y}{6} \quad \text{.......... (1)}$$

ΔRCA ~ ΔQBA

∠RCA = ∠QBA (Corresponding angles)

∠A = ∠A

$$\therefore \frac{AB}{AC} = \frac{BQ}{RC} \Rightarrow \frac{AB}{BC} = \frac{y}{3} \quad \text{....(2)}$$

Adding (1) & (2),

$$\frac{AB + BC}{AC} = \frac{y}{6} + \frac{y}{3}$$

$$\Rightarrow \frac{AC}{AC} = y\left(\tfrac{1}{6} + \tfrac{1}{3}\right)$$

$$\Rightarrow y\left(\frac{1+2}{6}\right) = 1$$

$$\Rightarrow y\left(\frac{1}{2}\right) = 1$$

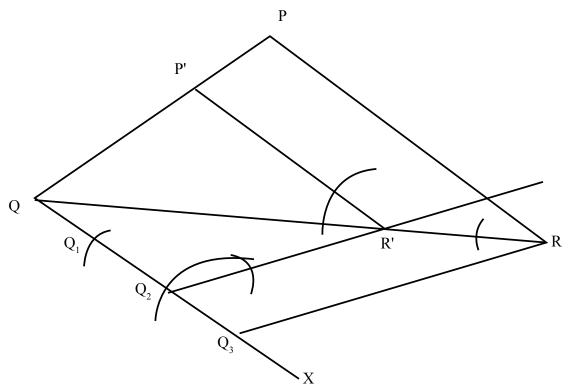
$$\therefore y = 2m$$

(or) Using formula $y = \frac{ab}{a+b}$

$$= \frac{6 \times 3}{6+3} = \frac{18}{9} = 2m$$

10. Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{2}{3}$ of the corresponding sides of the triangle PQR (scale factor $\frac{2}{3}$).

Solution :



Steps of construction

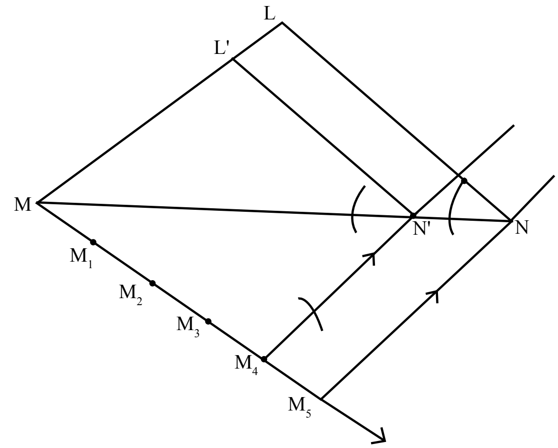
1. Construct a ΔPQR with any measurement.
2. Draw a ray QX making an acute angle with QR on the side opposite to vertex P.
3. Locate 3 points (greater of 2 and 3 in $\frac{2}{3}$) points.

   Q_1, Q_2, Q_3 on QX so that

   $QQ_1 = Q_1Q_2 = Q_2Q_3$
4. Join Q_3R and draw a line through Q_2 (3 being smaller of 2 and 3 in $\frac{2}{3}$) parallel to Q_3R to intersect QR at R'.
5. Draw line through R' parallel to the line RP intersecting the QP at P'. Then, ΔP'QR' is the required Δ.

11. Construct a triangle similar to a given triangle LMN with its sides equal to $\frac{4}{5}$ of the corresponding sides of the triangle LMN (scale factor $\frac{4}{5}$).

Solution :



Steps of construction

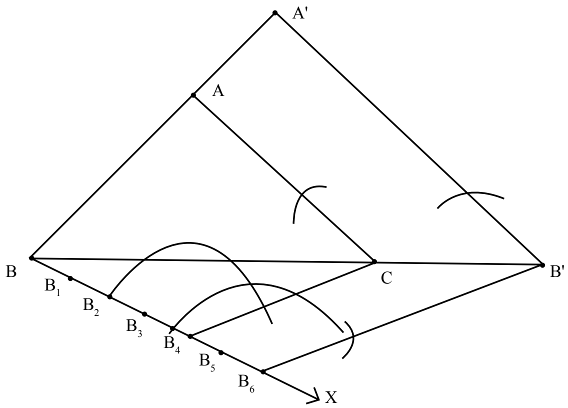
1. Construct a ΔLMN with any measurement.
2. Draw a ray MX making an acute angle with MN on the side opposite to vertex L.
3. Locate 5 points (greater of 4 and 5 in $\frac{4}{5}$) points.

   M_1, M_2, M_3, M_4 & M_5 so that $MM_1 = M_1M_2 = M_2M_3 = M_3M_4 = M_4M_5$,
4. Join M_5 to N and draw a line through M_4 (4 being smaller of 4 and 5 in $\frac{4}{5}$) parallel to M_5N to intersect MN at N'.
5. Draw line through N' parallel to the line LN intersecting line segment ML to L'.

   Then, L'M'N' is the required Δ.

12. Construct a triangle similar to a given triangle ABC with its sides equal to $\frac{6}{5}$ of the corresponding sides of the triangle ABC (scale factor $\frac{6}{5}$).

Solution :



Steps of construction

1. Construct a ΔABC with any measurement.

2. Draw a ray BX making an acute angle with BC on the side opposite to vertex A.

3. Locate 6 points (greater of 6 and 5 in $\frac{6}{5}$) points.

   B_1, B_2, ... B_6 on BX so that $BB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_3B_4 = B_4B_5 = B_5B_6$,

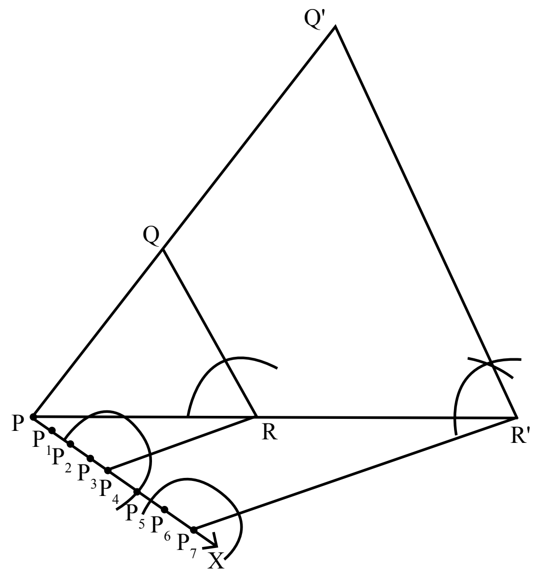
4. Join B_4 (4 being smaller of 4 and 6 in $\frac{6}{4}$) to C and draw a line through B_6 parallel to B_4C to intersecting the extended line segment BC at C'.

5. Draw line through C' parallel to CA intersect the extended line segment BA to A'.

   Then, $\Delta A'B'C'$ is the required Δ.

13. Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{7}{3}$ of the corresponding sides of the triangle PQR (scale factor $\frac{7}{3}$).

Solution :



Steps of construction

1. Construct a ΔPQR with any measurement.

2. Draw a ray PX making an acute angle with PR on the side opposite to vertex Q.

3. Locate 7 points (greater of 3 and 7 in $\frac{7}{3}$) points.

   P_1, P_2, ... P_7 on PX so that $PP_1 = P_1P_2 = P_2P_3$ .... $= P_6P_7$,

4. Join P_3R (3 being smaller of 3 and 7 in $\frac{7}{3}$) and draw a line through P_7 parallel to P_3R to intersecting the extended line segment PR at R'.

5. Draw line through R' parallel to QR intersect the extended line segment PQ to Q'.

   Then, $\Delta P'Q'R'$ is the required Δ.

II. Thales Theorem and Angle Bisector Theorem

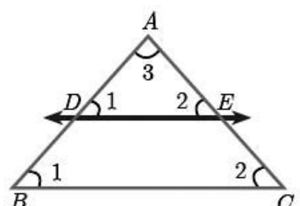
Theorem 1: Basic Proportionality Theorem (BPT) or Thales theorem

Statement

A straight line drawn parallel to a side of triangle intersecting the other two sides, divides the sides in the same ratio.

Proof

Given: In ΔABC, D is a point on AB and E is a point on AC.



To Prove : $\frac{AD}{DB} = \frac{AE}{EC}$

Construction : Draw a line DE | | BC

No.	Statement	Reason
1.	$\angle ABC = \angle ADE = \angle 1$	Corresponding angles are equal because DE || BC
2.	$\angle ACB = \angle AED = \angle 2$	Corresponding angles are equal because DE || BC
3.	$\angle DAE = \angle BAC = \angle 3$	Both triangles have a common angle
	$\Delta ABC \sim \Delta ADE$	By AAA similarity
	$\frac{AB}{AD} = \frac{AC}{AE}$	Corresponding sides are proportional
	$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$	Split AB and AC using th epoints D and E.
4.	$1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$	On simplification
	$\frac{DB}{AD} = \frac{EC}{AE}$	Cancelling 1 on both sides
	$\frac{AD}{DB} = \frac{AE}{EC}$	Taking reciprocals

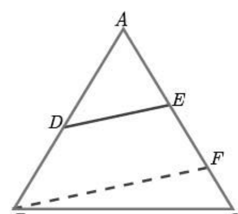
Hence proved

Theorem 2: Converse of Basic Proportionality Theorem

Statement

If a straight line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

Proof



Given : In $\Delta ABC, \frac{AD}{DB} = \frac{AE}{EC}$

To Prove : $DE \,||\, BC$

Construction : Draw BF | | DE

No.	Statement	Reason
1.	In Δ ABC, BF || DE	Construction
2.	$\frac{AD}{DB} = \frac{AE}{EC}$ ....(1)	Thales theorem (In ΔABC taking D in AB and E in AC).
3.	$\frac{AD}{DB} = \frac{AF}{FC}$ ....(2)	Thales theorem (In ΔABC taking F in AC)

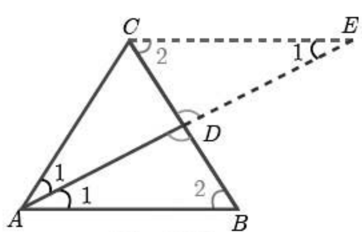
4.	$\frac{AE}{EC}=\frac{AF}{FC}$ $\frac{AE}{EC}+1=\frac{AF}{FC}+1$ $\frac{AE+EC}{EC}=\frac{AF+FC}{FC}$ $\frac{AC}{EC}=\frac{AC}{FC}$ $EC=FC$ Therefore, F = C Thus DE ∥ BC	From (1) and (2) Adding 1 to both sides Cancelling AC on both sides F lies between E and C. Hence proved

Theorem 3: Angle Bisector Theorem

Statement

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the corresponding sides containing the angle.

Proof



Given : In ΔABC, AD is the internal bisector

To Prove : $\frac{AB}{AC}=\frac{BD}{CD}$

Construction : Draw a line through C parallel to AB. Extend AD to meet line through C at E

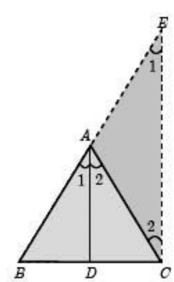
No.	Statement	Reason
1.	$\angle AEC = \angle BAE = \angle 1$	Two parallel lines cut by a transversal make alternate angles equal.
2.	ΔACE is isosceles AC = CE ....(1) ΔABD ~ ΔECD	In ΔACE, ∠CAE = ∠CEA
3.	$\frac{AB}{CE}=\frac{BD}{CD}$	By AA Similarity
4.	$\frac{AB}{AC}=\frac{BD}{CD}$	From (1) AC = CE. Hence proved.

Theorem 4: Converse of Angle Bisector Theorem

Statement

If a straight line through one vertex of a triangle divides the opposite side internally in the ratio of the other two sides, then the line bisects the angle internally at the vertex.

Proof



Given : ABC is a triangle. AD divides BC in the ratio of the sides containing the angles ∠A to meet BC at D.

That is $\frac{AB}{AC}=\frac{BD}{DC}$ .... (1)

To prove : AD bisects ∠A i.e. ∠1 = ∠2

Construction : Draw CE | | DA. Extend BA to meet at E.

No.	Statement	Reason
1.	$\angle BAD = \angle 1$ and $\angle DAC = \angle 2$	Assumption
2.	$\angle BAD = \angle AEC = \angle 1$	Since DA ‖ CE and AC is transversal, corresponding angles are equal
3.	$\angle DAC = \angle ACE = \angle 2$	Since DA ‖ CE and AC is transversal. Alternate angles are equal.
4.	$\dfrac{BA}{AE} = \dfrac{BD}{DC}$ ....(2)	In ΔBCE by Thales theorem
5.	$\dfrac{AB}{AC} = \dfrac{BD}{DC}$	From (1)
6.	$\dfrac{AB}{AC} = \dfrac{BA}{AE}$	From (1) and (2)
7.	AC = AE ....(3)	Cancelling AB
8.	$\angle 1 = \angle 2$	ΔACE is isosceles by (3)
9.	AD bisects $\angle A$	Since, $\angle 1 = \angle BAD = \angle 2 = \angle DAC$. Hence proved.

Example 4.12

In ΔABC if DE | | BC, AD = x, DB = x − 2, and EC = x − 1 then find the lengths of the sides AB and AC.

Solution :

In ΔABC we have DE | | BC

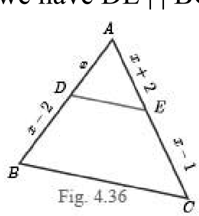


Fig. 4.36

By Thales theorem, we have $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

$\dfrac{x}{x-2} = \dfrac{x+2}{x-1}$ gives $x(x-1) = (x-2)(x+2)$

Hence, $x^2 - x = x^2 - 4$ so, x = 4

When x = 4, AD = 4, DB = x − 2 = 2,

AE = x + 2 = 6, EC = x − 1 = 3.

Hence, AB = AD + DB = 4 + 2 = 6,

AC = AE + EC = 6 + 3 = 9.

Therefore, AB = 6, AC = 9.

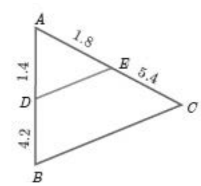
Example 4.13

D and E are respectively the points on the sides AB and AC of a Δ ABCsuch that AB = 5.6 cm, AD = 1.4 cm, AC = 7.2 cm and AE = 1.8 cm, show that DE | | BC.

Solution :

We have AB = 5.6 cm, AD = 1.4 cm, AC = 7.2 cm and AE = 1.8 cm

BD = AB – AD = 5.6 – 1.4 = 4.2 cm



and EC = AC – AE = 7.2 – 1.8 = 5.4 cm.

$$\frac{AD}{DB} = \frac{1.4}{4.2} = \frac{1}{3} \text{ and } \frac{AE}{EC} = \frac{1.8}{5.4} = \frac{1}{3}$$

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Therefore, by converse of Basic Proportionality Theorem, we have DE is parallel to BC.

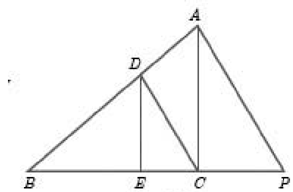
Hence proved.

Example 4.14

In the Fig. DE | | AC and DC | | AP. Prove that $\dfrac{BE}{EC} = \dfrac{BC}{CP}$.

Solution :

In ΔBPA, we have DC | | AP. By Basic Proportionality Theorem,



We have $\frac{BC}{CP} = \frac{BD}{DA}$ .... (1)

In ΔBCA, we have DE | | AC. By Basic Proportionality Theorem,

We have $\frac{BE}{EC} = \frac{BD}{DA}$ .... (2)

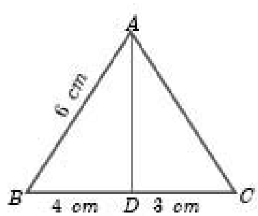
From (1) and (2) we get, $\frac{BE}{EC} = \frac{BC}{CP}$. Hence proved.

Example 4.15

In the Fig., AD is the bisector of $\angle A$. If BD = 4 cm, DC = 3 cm and AB = 6 cm, find AC.

Solution :

In Δ ABC, AD is the bisector of $\angle A$



Therefore by Angle Bisector Theorem

$$\frac{BD}{DC} = \frac{AB}{AC}$$

$$\frac{4}{3} = \frac{6}{AC} \text{ gives } 4AC = 18.$$

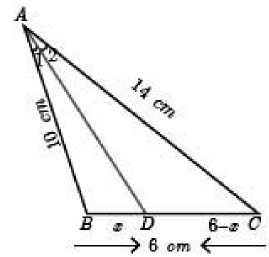
$$\text{Hence } AC = \frac{9}{2} = 4.5\ cm$$

Example 4.16

In the Fig. AD is the bisector of $\angle BAC$. If AB = 10 cm, AC = 14 cm and BC = 6 cm, find BD and DC.

Solution :

Let BD = x cm, then DC = (6 – x) cm



AD is the bisector of $\angle A$

Therefore by Angle Bisector Theorem

$$\frac{AB}{AC} = \frac{BD}{DC}$$

$$\frac{10}{14} = \frac{x}{6-x} \text{ gives } \frac{5}{7} = \frac{x}{6-x}$$

$$\text{So, } 12x = 30 \text{ we get, } x = \frac{30}{12} = 2.5\ cm$$

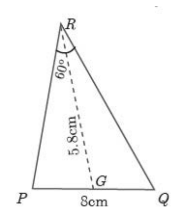
Therefore, BD = 2.5 cm, DC = 6 − x
= 6 − 2.5 = 3.5 cm

Construction of triangle

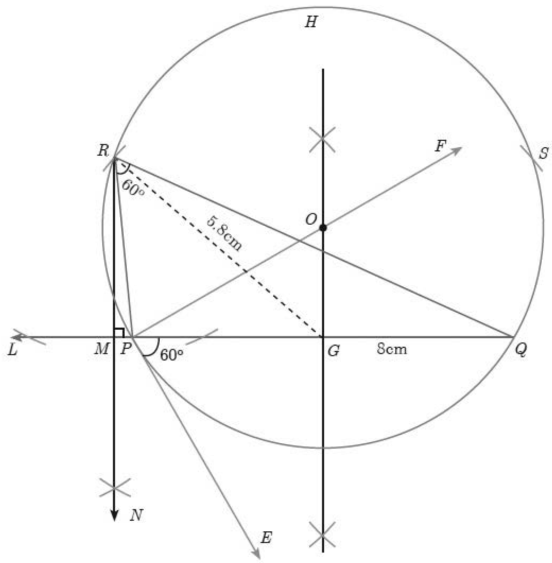
Example 4.17

Construct a ΔPQR in which PQ = 8 cm, $\angle R = 60^0$ and the median RG from R to PQ is 5.8 cm. Find the length of the altitude from R to PQ.

Solution :



Rough diagram



Construction

Step 1 : Draw a line segment PQ = 8cm.

Step 2 : At P, draw PE such that $\angle QPE = 60^0$.

Step 3 : At P, draw PF such that $\angle EPF = 90^0$.

Step 4 : Draw the perpendicular bisector to PQ, which intersects PF at O and PQ at G.

Step 5 : With O as centre and OP as radius draw a circle.

Step 6 : From G mark arcs of radius 5.8 cm on the circle. Mark them as R and S.

Step 7 : Join PR and RQ. Then ΔPQR is the required triangle .

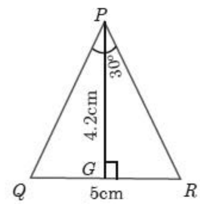
Step 8 : From R draw a line RN perpendicular to LQ. LQ meets RN at M

Step 9 : The length of the altitude is RM = 3.5 cm.

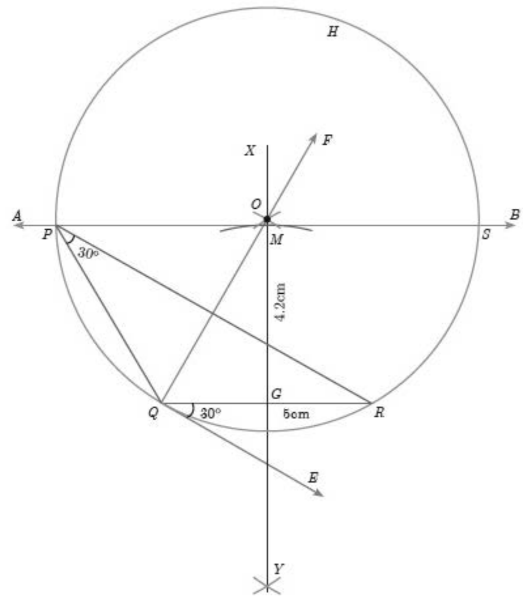
Example 4.18

Construct a triangle ΔPQR such that QR = 5 cm, $\angle P = 30^0$ and the altitude from P to QR is of length 4.2 cm.

Solution :



Rough diagram



Construction

Step 1 : Draw a line segment QR = 5cm.

Step 2 : At Q, draw QE such that $\angle RQE = 30^0$.

Step 3 : At Q, draw QF such that $\angle EQF = 90^0$.

Step 4 : Draw the perpendicular bisector XY to QR, which intersects QF at O and QR at G.

Step 5 : With O as centre and OQ as radius draw a circle.

Step 6 : From G mark an arc in the line XY at M, such that GM = 4.2 cm.

Step 7 : Draw AB through M which is parallel to QR.

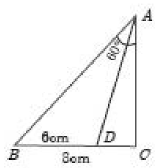
Step 8 : AB meets the circle at P and S.

Step 9 : Join QP and RP. Then ΔPQR is the required triangle.

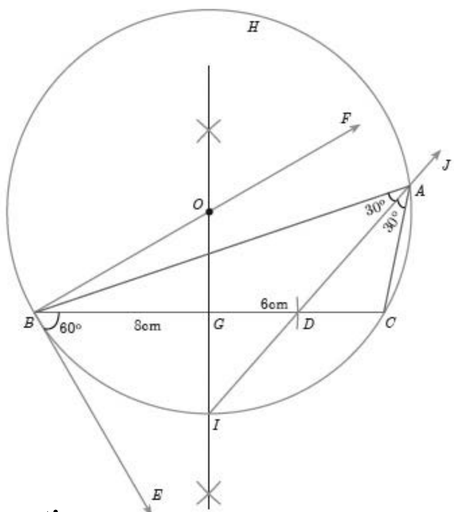
Example 4.19

Draw a triangle ABC of base BC = 8 cm, $\angle A = 60^0$ and the bisector of $\angle A$ meets BC at D such that BD = 6 cm.

Solution :



Rough diagram



Construction

Step 1 : Draw a line segment BC = 8cm.
Step 2 : At B, draw BE such that $\angle CBE = 60^0$.
Step 3 : At B, draw BF such that $\angle EBF = 90^0$.
Step 4 : Draw the perpendicular bisector to BC, which intersects BF at O and BC at G.
Step 5 : With O as centre and OB as radius draw a circle.
Step 6 : From B mark an arcs of 6 cm on BC at D.
Step 7 : The perpendicular bisector intersects the circle at I. Join ID.
Step 8 : ID produced meets the circle at A. Now join AB and AC. Then ΔABC is the required triangle.

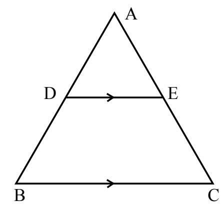
EXERCISE 4.2

1. In ΔABC, D and E are points on the sides AB and AC respectively such that DE || BC

(i) If $\frac{AD}{DB} = \frac{3}{4}$ and AC =15 cm find AE.

(ii) If AD = 8x − 7, DB = 5x − 3, AE = 4x − 3 and EC = 3x − 1, find the value of x.

Solution :



i) Given $\frac{AD}{DB} = -\frac{3}{4}, \; AC = 15$

Let $AE = x \Rightarrow EC = 15 - x$

$\therefore$ By $BPT \; \frac{AD}{DB} = \frac{AE}{EC}$

$$\Rightarrow \frac{3}{4} = \frac{x}{15 - x}$$

$$\Rightarrow 4x = 45 - 3x$$

$$\Rightarrow 7x = 45$$

$$x = 45/7 = 6.428 \square 6.43 \; cm$$

ii) Given AD = 8x – 7, DB = 5x – 3

AE = 4x – 3, EC = 3x – 1

By $BPT \qquad \frac{AD}{DB} = \frac{AE}{EC}$

$$\Rightarrow \frac{8x-7}{5x-3} = \frac{4x-3}{3x-1}$$

$$\Rightarrow (8x-7)(3x-1) = (4x-3)(5x-3)$$

$$\Rightarrow 24x^2 - 29x + 7 = 20x^2 - 27x + 9$$

$$\Rightarrow 4x^2 - 2x - 2 = 0$$

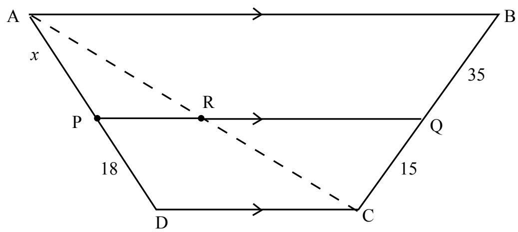
$$\Rightarrow 2x^2 - x - 1 = 0$$

$$\therefore \qquad x = 1, -1/2$$

$$x = 1$$

2. ABCD is a trapezium in which AB || DC and P,Q are points on AD and BC respectively, such that PQ || DC if PD = 18 cm, BQ = 35 cm and QC = 15 cm, find AD.

Solution :



In trapezium ABCD, AB | | DC | | PQ

Join AC, meet PQ at R.

In ΔACD, PR | | DC

$$\therefore \text{ By } BPT \quad \frac{AP}{PD} = \frac{AR}{RC}$$

$$\Rightarrow \quad \frac{x}{18} = \frac{AR}{RC} \qquad \text{..........} (1)$$

In ΔABC, RQ | | AB

$$\therefore \text{ By } ABT \quad \frac{BQ}{QC} = \frac{AR}{RC}$$

$$\Rightarrow \quad \frac{35}{15} = \frac{AR}{RC}$$

$$\Rightarrow \quad \frac{7}{3} = \frac{AR}{RC} \qquad \text{..........} (2)$$

∴ From (1) & (2)

$$\frac{x}{18} = \frac{7}{3}$$

$$\Rightarrow \quad x = 42$$

$$\therefore \quad AD = AP + PD$$

$$= 42 + 18$$

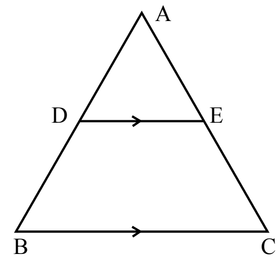
$$= 60\ m$$

3. In ΔABC, D and E are points on the sides AB and AC respectively. For each of the following cases show that DE | | BC

(i) AB = 12 cm, AD = 8 cm, AE = 12 cm and AC = 18 cm.

(ii) AB = 5.6 cm, AD = 1.4 cm, AC = 7.2 cm and AE = 1.8 cm.

Solution :



In ΔABC, To Prove : DE | | BC

$$i) \quad \frac{AD}{AB} = \frac{8}{12} = \frac{2}{3}$$

$$\frac{AE}{AC} = \frac{12}{18} = \frac{2}{3}$$

$$\frac{AD}{AB} = \frac{AE}{AC}$$

∴ By Converse of BPT DE | | BC

$$ii) \quad \frac{AD}{AB} = \frac{1.4}{5.6} = \frac{1}{4}$$

$$\frac{AE}{AC} = \frac{1.8}{7.2} = \frac{1}{4}$$

$$\therefore \frac{AD}{AB} = \frac{AE}{AC}$$

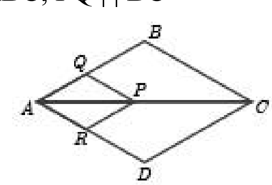
∴ By Converse of BPT, DE | | BC

4. In fig. if PQ | | BC and PR | | CD prove that

$$(i) \ \frac{AR}{AD} = \frac{AQ}{AB} \qquad (ii) \ \frac{QB}{AQ} = \frac{DR}{AR}$$

Solution :

i) In Δ ABC, PQ | | BC



$\therefore$ By BPT $\dfrac{AQ}{AB} = \dfrac{AP}{AC}$ .... (1)

In ΔADC, PR | | DC

$\therefore$ By BPT $\dfrac{AR}{AD} = \dfrac{AP}{AC}$ .... (2)

$\therefore$ From (1) & (2),

$$\frac{AQ}{AB} = \frac{AR}{AD}$$

ii) From (*i*) $\dfrac{AB}{AQ} = \dfrac{AD}{AR}$ (reciprocal)

$$\Rightarrow \quad \frac{AB}{AQ} - 1 = \frac{AD}{AR} - 1$$

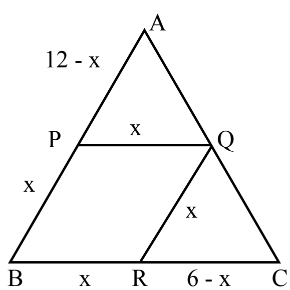
$$\Rightarrow \quad \frac{AB - AQ}{AQ} = \frac{AD - AR}{AR}$$

$$\Rightarrow \quad \frac{BQ}{AQ} = \frac{DR}{AR}$$

Hence proved.

5. Rhombus PQRB is inscribed in ΔABC such that $\angle$B is one of its angle. P, Q and R lie on AB, AC and BC respectively. If AB = 12 cm and BC = 6 cm, find the sides PQ, RB of the rhombus.

Solution :



Rhombus PQRS is inscribed in ΔABC.

Let the side of the rhombus be x.

$\therefore$ AB = 12 cm AP = 12 – x

BC = 6 cm RC = 6 – x

In ΔABC, PQ | | BC

$$\therefore \quad \frac{AP}{PB} = \frac{AQ}{QC} \quad .... (1)$$

In ΔABC, QR | | AB

$$\therefore \quad \frac{BR}{RC} = \frac{AQ}{QC} \quad .... (2)$$

$\therefore$ From (1) & (2)

$$\Rightarrow \quad \frac{AP}{PB} = \frac{BR}{RC}$$

$$\Rightarrow \quad \frac{12 - x}{x} = \frac{x}{6 - x}$$

$$\Rightarrow \quad x^2 = (6 - x)(12 - x)$$

$$\Rightarrow \quad x^2 = x^2 - 18x + 72$$

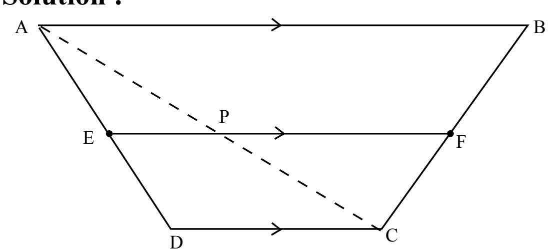
$$\Rightarrow \quad 18x = 72$$

$$\Rightarrow \quad x = 4\ cm$$

$$\therefore \quad PQ = RB = 4cm$$

6. In trapezium ABCD, AB | | DC, E and F are points on non-parallel sides AD and BC respectively, such that EF | | AB. Show that $\dfrac{AE}{ED} = \dfrac{BF}{FC}$.

Solution :



In trapezium ABCD, AB | | DC | | EF

Join AC to meet EF at P

In ΔADC, EP | | DC

$\therefore$ By BPT, $\dfrac{AE}{ED} = \dfrac{AP}{PC}$ .... (1)

In ΔABC, PR | | AB

$\therefore$ By $BPT, \dfrac{BF}{FC} = \dfrac{AP}{PC}$ .... (2)

From (1) & (2)

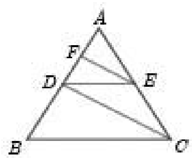
$$\frac{AE}{ED} = \frac{BF}{FC}$$

Hence proved.

7. In figure DE | | BC and CD | | EF. Prove that AD² = AB × AF

Solution :

In figure DE | | BC and CD | | EF



In Δ $ACD, by\ BPT, \dfrac{AF}{AD} = \dfrac{AE}{AC}$ .... (1)

In Δ $ABC, by\ BPT, \dfrac{AD}{AB} = \dfrac{AE}{AC}$ .... (2)

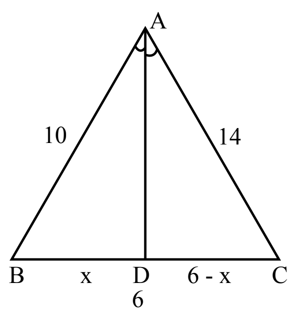
$\therefore$ From (1) & (2)

$$\frac{AF}{AD} = \frac{AD}{AB}$$

$\Rightarrow \quad AD^2 = AF \,.\, AB$

8. In ΔABC, AD is the bisector of ∠A meeting side BC at D, if AB = 10 cm, AC = 14 cm and BC = 6 cm, find BD and DC

Solution :



In ΔABC, AD is the bisector of ∠A.

$\therefore$ By $ABT, \quad \dfrac{AB}{AC} = \dfrac{BD}{DC}$

$\Rightarrow \quad \dfrac{10}{14} = \dfrac{x}{6-x}$

$\Rightarrow \quad \dfrac{5}{7} = \dfrac{x}{6-x}$

$\Rightarrow \quad 30 - 5x = 7x$

$\Rightarrow \quad 12x = 30$

$$x = \frac{5}{2} = 2.5$$

$\therefore$ BD = 2.5 cm and DC = $6 - x = 6 - 2.5$

DC = 3.5 cm

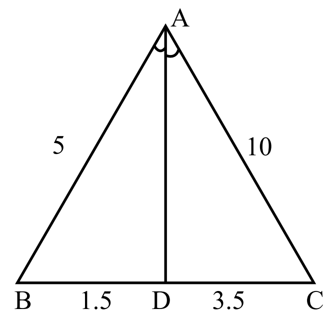
9. Check whether AD is bisector of ∠A of ΔABC in each of the following

(i) AB = 5 cm, AC = 10 cm, BD = 1.5 cm and CD = 3.5 cm.

(ii) AB = 4 cm, AC = 6 cm, BD = 1.6. cm and CD = 2.4 cm.

Solution :

i)

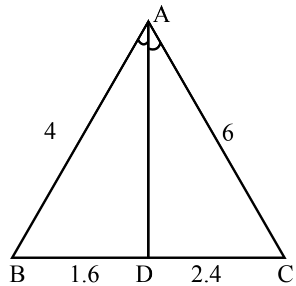


$$\frac{AB}{AC} = \frac{5}{10} = \frac{1}{2}, \quad \frac{BD}{DC} = \frac{1.5}{3.5} = \frac{3}{7}$$

$$\therefore \frac{AB}{AC} \neq \frac{BD}{DC}$$

$\therefore$ AD is not the bisector of ∠A.

ii)



$$\frac{AB}{AC}=\frac{4}{6}=\frac{2}{3}, \quad \frac{BD}{DC}=\frac{1.6}{2.4}=\frac{2}{3}$$

$$\therefore \frac{AB}{AC}=\frac{BD}{DC}$$

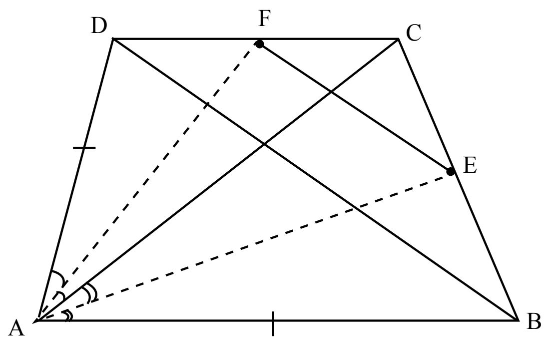
$\therefore$ By Converse of ABT,

AD is the bisector of $\angle$A.

---

11. ABCD is a quadrilateral in which AB = AD, the bisector of $\angle$BAC and $\angle$CAD intersect the sides BC and CD at the points E and F respectively. Prove that EF | | BD.

Solution :



In ΔACD, AF is the angle bisector

$\therefore$ By $ABT, \dfrac{AD}{AC}=\dfrac{DF}{FC}$ .... (1)

In ΔABC, AE is the angle bisector

$\therefore$ By $ABT, \dfrac{AB}{AC}=\dfrac{BE}{EC}$

$\Rightarrow \dfrac{AD}{AC}=\dfrac{BE}{EC}$ ... (2) (Given $AB = AD$)

$\therefore$ From (1) & (2),

$$\frac{BE}{EC}=\frac{DF}{FC}$$

$\therefore$ By Converse of BPT,

EF | | BD

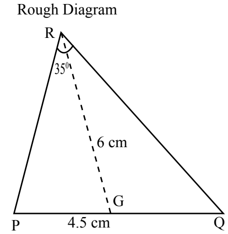
Hence proved.

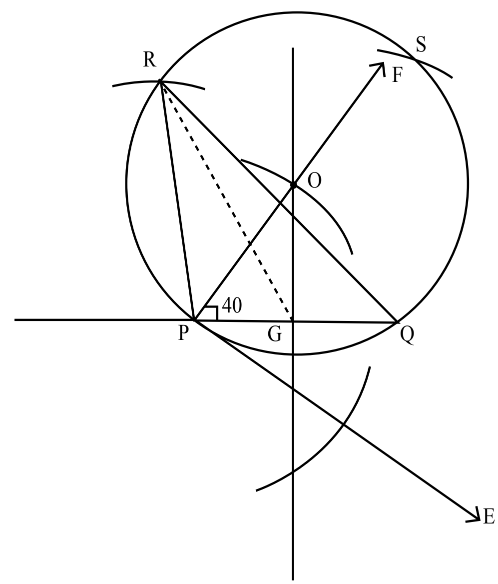
---

12. Construct a ΔPQR which the base PQ = 4.5 cm, $\angle$R = 35^0 and the median from R to PQ is 6 cm.

Solution :

Rough Diagram





Construction

Step 1 : Draw a line segment PQ = 4.5cm.

Step 2 : At P, draw PE such that $\angle QPE = 35^0$.

Step 3 : At P, draw PF such that $\angle EPF = 90^0$.

Step 4 : Draw the perpendicular bisector to PQ, meets PF at O and PQ at G.

Step 5 : With O as centre and OP as radius draw a circle.

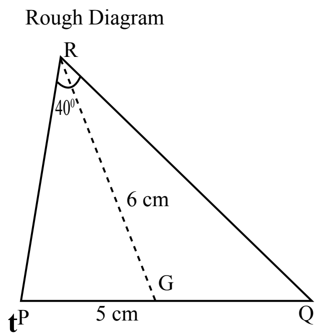
Step 6 : From G mark arcs of 6 cm on the circle at RAS.

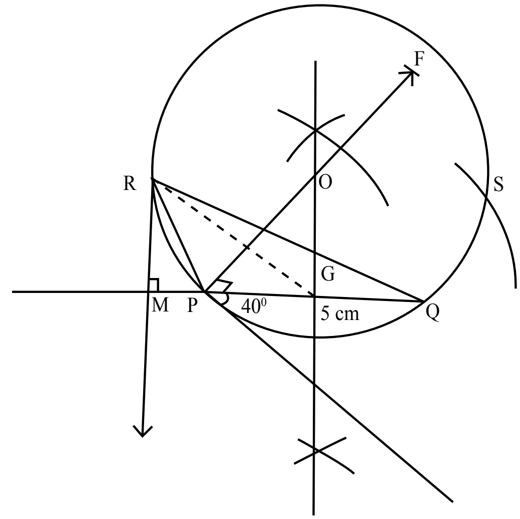
Step 7 : Join PR, RQ. Then ΔPQR is the required Δ.

Step 8 : Join RG, which is the median.

13. Construct a ΔPQR in which PQ = 5 cm, $\angle P = 40^0$ and the median PG from P to QR is 4.4 cm. Find the length of the altitude from P to QR.

Solution :





Construction

Step 1 : Draw a line segment PQ = 5 cm.

Step 2 : At P, draw PE such that $\angle QPE = 40^0$.

Step 3 : At P, draw PF such that $\angle EPF = 90^0$.

Step 4 : Draw the perpendicular bisector to PQ, meets PF at O and PQ at G.

Step 5 : With O as centre and OP as radius draw a circle.

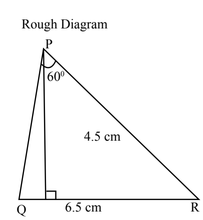
Step 6 : From G mark arcs of 4.4 cm on the circle radius 4.4m.

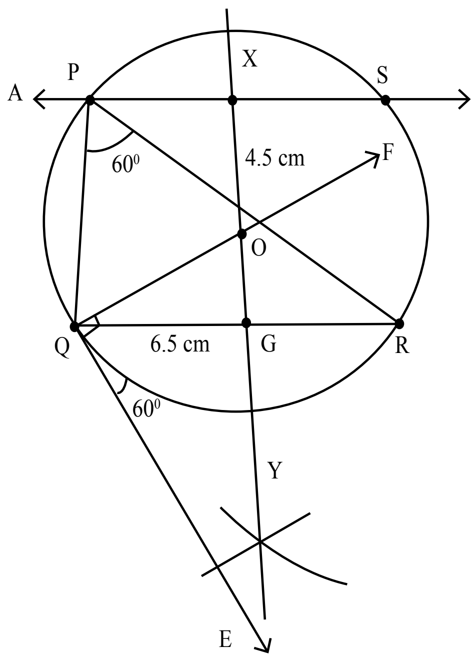
Step 7 : Join PR, RQ. Then ΔPQR is the required Δ.

Step 8 : Length of altitude is RM = 3 cm

14. Construct a ΔPQR such that QR = 6.5 cm, $\angle P = 60^0$ and the altitude from P to QR is of length 4.5 cm.

Solution :





Construction

Step 1 : Draw a line segment QR = 6.5 cm.

Step 2 : At Q, draw QE such that $\angle RQE = 60^0$.

Step 3 : At Q, draw QF such that $\angle EQF = 90^0$.

Step 4 : Draw the perpendicular bisector XY to QR intersects QF at O & QR at G.

Step 5 : With O as centre and OQ as radius draw a circle.

Step 6 : XY intersects QR at G. On XY, from G, mark arc M such that GM = 4.5 cm.

Step 7 : Draw AB, through M which is parallel to QR.

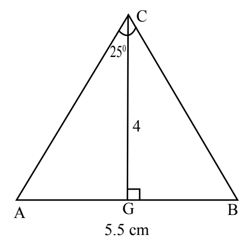
Step 8 : AB meets the circle at P and S.

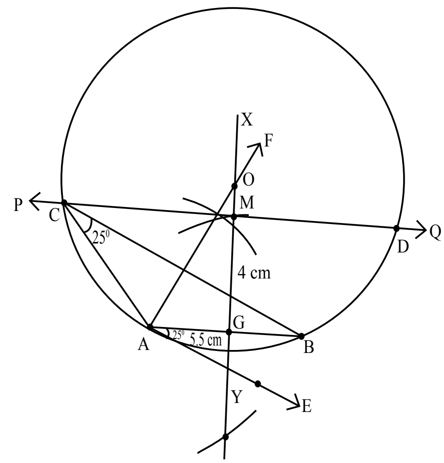
Step 9 : Join QP, RP. Then ΔPQR is the required Δ.

15. Construct a ΔABC such that AB = 5.5 cm, $\angle C = 25^0$ and the altitude from C to AB is 4 cm.

Solution :

Rough Diagram





Construction

Step 1 : Draw a line segment AB = 5.5 cm.

Step 2 : At A, draw AE such that $\angle BAE = 25^0$.

Step 3 : At A, draw AF such that $\angle EAF = 90^0$.

Step 4 : Draw the perpendicular bisector XY to AB intersects AF at O & AB at G.

Step 5 : With O as centre and OA as radius draw a circle.

Step 6 : XY intersects AB at G. On XY, from G, mark arc M such that GM = 4 cm.

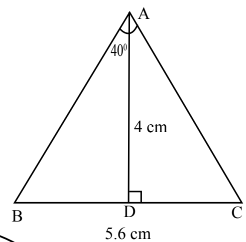
Step 7 : Draw PQ, through M parallel to AB meets the circle at C and D.

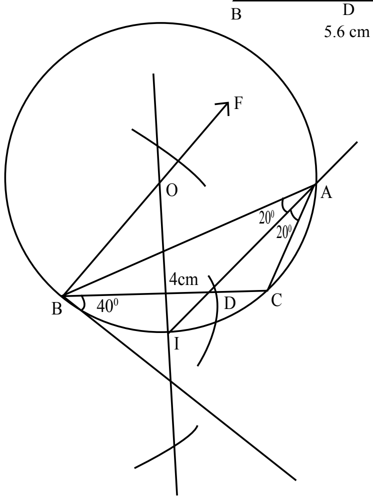
Step 8 : Join AC, BC. Then ΔABC is the required Δ.

16. Draw a triangle ABC of base BC = 5.6 cm, ∠A = 40⁰ and the bisector of ∠A meets BC at D such that CD = 4 cm.

Solution :

Rough Diagram





Construction

Step 1 : Draw a line segment BC = 5.6 cm.

Step 2 : At B, draw BE such that ∠CBE = 40^0.

Step 3 : At B, draw BF such that ∠CBF = 90^0.

Step 4 : Draw the perpendicular bisector to BC meets BF at O & BC at G.

Step 5 : With O as centre and OB as radius draw a circle.

Step 6 : From B, mark an arc of 4 cm on BC at D.

Step 7 : The ⊥r bisector meets the circle at I & Join ID.

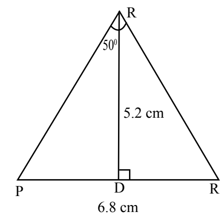
Step 8 : ID produced meets the circle at A. Join AB & AC.

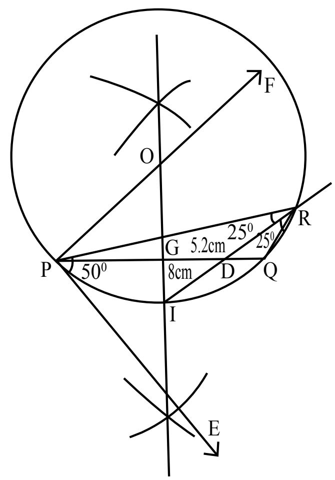
Step 9 : Then ΔABC is the required triangle.

17. Draw ΔPQR such that PQ = 6.8 cm, vertical angle is 50° and the bisector of the vertical angle meets the base at D where PD = 5.2 cm.

Solution :

Rough Diagram





Construction

Step 1 : Draw a line segment PQ = 6.8 cm.

Step 2 : At P, draw PE such that ∠QPE = 50^0.

Step 3 : At P, draw PF such that ∠QPF = 90^0.

Step 4 : Draw the perpendicular bisector to PQ meets PF at O and PQ at G.

Step 5 : With O as centre and OP as radius draw a circle.

Step 6 : From P mark an arc of 5.2 cm on PQ at D.

Step 7 : The perpendicular bisector meets the circle at R. Join PR and QR.

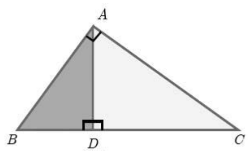
Step 8 : Then ΔPQR is the required triangle.

III. Pythagoras Theorem :

Theorem 5 : Pythagoras Theorem

Statement

In a right angle triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.



Proof

Given : In ΔABC, A = 90^0

To prove : $AB^2 + AC^2 = BC^2$

Construction : Draw AD ⊥ BC

No.	Statement	Reason
1.	Compare ΔABC and ΔABD ∠B is common ∠BAC = ∠BDA = 90^o	Given ∠BAC = 90^o and by cosntruction ∠BDA = 90^o
	Therefore, ΔABC ~ ΔABD $\frac{AB}{BD} = \frac{BC}{AB}$ $AB^2 = BC \times BD$ .... (1)	By AA similarity
2.	Compare ΔABC and ΔADC ∠C is common ∠BAC = ∠ADC = 90^o	Given ∠BAC = 90^o and by construction ∠CDA = 90^o
	Therefore ΔABC ~ ΔADC $\frac{BC}{AC} = \frac{AC}{DC}$ $AC^2 = BC \times DC$ .... (2)	By AA similarity

Converse of Pythagoras Theorem

Statement

If the square of the longest side of a triangle is equal to sum of squares of other two sides, then the triangle is a right angle triangle.

Example 4.20

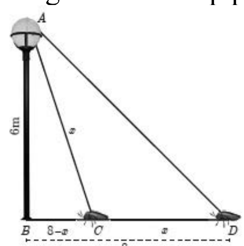
An insect 8 m away initially from the foot of a lamp post which is 6 m tall, crawls towards it moving through a distance. If its distance from the top of the lamp post is equal to the distance it has moved, how far is the insect away from the foot of the lamp post?

Solution :

Distance between the insect and the foot of the lamp post

BD = 8 m

The height of the lamp post, AB = 6 m



After moving a distance of x m, let the insect be at C

Let, AC = CD = x. Then BC = BD − CD = 8 − x

In ΔABC, ∠B = 90^0

$AC^2 = AB^2 + BC^2$ gives $x^2 = 6^2 + (8 - x)^2$

$x^2 = 36 + 64 - 16x + x^2$

$16x = 100$ then $x = 6.25$

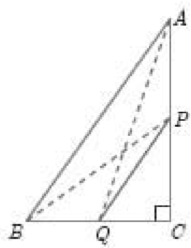
Then, BC = 8 − x = 8 − 6.25 = 1.75m

Therefore the insect is 1.75 m away from the foot of the lamp post.

Example 4.21

P and Q are the mid-points of the sides CA and CB respectively of a ΔABC, right angled at C. Prove that $4(AQ^2 + BP^2) = 54\,AB^2$.

Solution :



Since, ΔAQC is a right triangle at C,

$$AQ^2 = AC^2 + QC^2 \quad \text{........... (1)}$$

Also, ΔBPC is a right triangle at C,

$$BP^2 = BC^2 + CP^2 \quad \text{........... (2)}$$

From (1) and (2), $AQ^2 + BP^2 =$

$$AC^2 + QC^2 + BC^2 + CP^2$$

$$4(AQ^2 + BP^2) = 4AC^2 + 4QC^2 + 4BC^2 + 4CP^2$$

$$= 4AC^2 + (2QC)^2 + 4BC^2 + (2CP)^2$$

$$= 4AC^2 + BC^2 + 4BC^2 + AC^2$$

(Since P and Q are mid points)

$$= 5(AC^2 + BC^2)$$

$$4(AQ^2 + BP^2) = 5\,AB^2$$

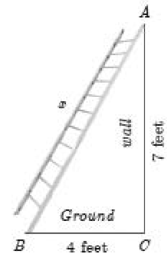
(By Pythagoras Theorem)

Example 4.22

What length of ladder is needed to reach a height of 7 ft along the wall when the base of the ladder is 4 ft from the wall? Round off your answer to the next tenth place.

Solution :

Let x be the length of the ladder. BC = 4 ft, AC = 7 ft.



By Pythagoras theorem we have, $AB^2 = AC^2 + BC^2$

$x^2 = 7^2 + 4^2$ gives $x^2 = 49 + 16$

$x^2 = 65$. Hence, $x = \sqrt{65}$

The number $\sqrt{65}$ is between 8 and 8.1.

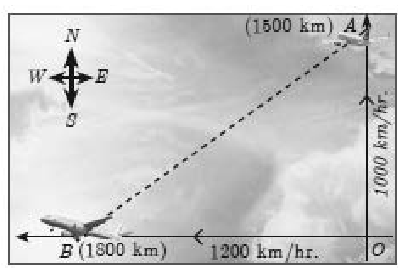
$$8^2 = 64 < 65 < 65.61 = 8.1^2$$

Therefore, the length of the ladder is approximately 8.1 ft.

Example 4.23

An Aeroplane leaves an airport and flies due north at a speed of 1000 km/hr. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km/hr. How far apart will be the two planes after 1½ hours?

Solution :



Let the first aeroplane starts from O and goes upto A towards north, (Distance = Speed × time)

$$\text{where } OA = \left(1000 \times \frac{3}{2}\right) \text{km} = 1500 \text{ km}$$

Let the second aeroplane starts from O at the same time and

goes upto B towards west,

where $OB = \left(1200 \times \frac{3}{2}\right) \text{km} = 1800 \text{ km}$

The required distance to be found is BA.

In right angled tirangle AOB, $AB^2 = OA^2 + OB^2$

$$AB^2 = (1500)^2 + (1800)^2 = 100^2 (15^2 + 18^2)$$

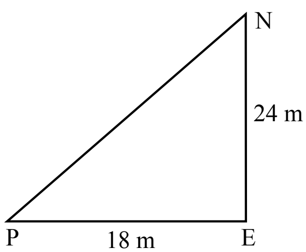
$$= 100^2 \times 549 = 100^2 \times 9 \times 61$$

$$AB = 100 \times 3 \times \sqrt{61} = 300\sqrt{61} \text{ kms.}$$

EXERCISE 4.3

1. A man goes 18 m due east and then 24 m due north. Find the distance of his current position from the starting point?

Solution:



P $\rightarrow$ Starting Point

By Pythagoras Theorem,

$$PN = \sqrt{18^2 + 24^2}$$

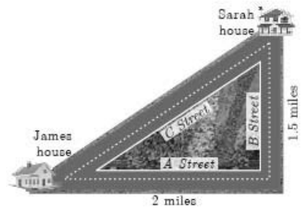
$$= \sqrt{324 + 576}$$

$$= \sqrt{900}$$

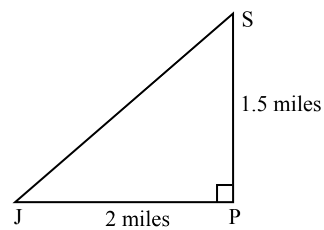
$$= 30\, m$$

$\therefore$ Distance of his current position from the starting point = 30 m

2. There are two paths that one can choose to go from Sarah's house to James house. One way is to take C street, and the other way requires to take A street and then B street. How much shorter is the direct path along C street? (Using figure).



Solution:



Path - 1 (Direct C Street)

$$SJ = \sqrt{(1.5)^2 + 2^2}$$

$$= \sqrt{2.25 + 4}$$

$$= \sqrt{6.25}$$

$$= 2.5 \text{ miles}$$

Path = 2 (B Street & then A Street)

$$SP + PJ = 1.5 + 2$$

$$= 3.5 \text{ miles}$$

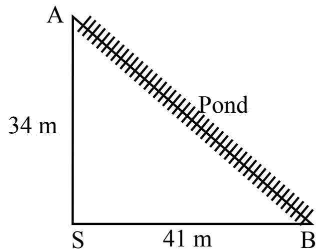
$$\therefore \text{Required} = 3.5 - 2.5$$

$$= 1 \text{ mile}$$

$\therefore$ 1 mile is shorter along C Street.

3. To get from point A to point B you must avoid walking through a pond. You must walk 34 m south and 41 m east. To the nearest meter, how many meters would be saved if it were possible to make a way through the pond?

Solution:



Path - 1 (Through pond)

$$AB = \sqrt{34^2 + 41^2}$$
$$= \sqrt{1156 + 1681}$$
$$= \sqrt{2837}$$
$$= 53.26 \text{ m}$$

Path – 2 (South & then East)

Total dist. covered

AB = AS + SB

= 34 + 41

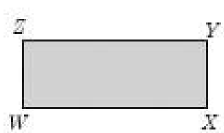
= 75 m

$\therefore$ Reqd. time saving = 75 – 53.26

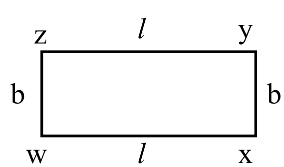
= 21.74 m

4. In the rectangle WXYZ, XY + YZ = 17 cm, and XZ + YW = 26 cm.

Calculate the length and breadth of the rectangle?



Solution:



Given $xy + yz = 17$

$\Rightarrow \quad l + b = 17 \quad$ .... (1)

$xz + yw = 26$

$\Rightarrow \sqrt{l^2 + b^2} + \sqrt{l^2 + b^2} = 26$

$\Rightarrow \quad 2\sqrt{l^2 + b^2} = 26$

$\therefore \quad l^2 + b^2 = 169$

$\Rightarrow \quad l^2 + (17 - l)^2 = 169 \qquad (From\ (1)$

$\Rightarrow \quad 2l^2 - 34l + 289 = 169$

$\Rightarrow \quad 2l^2 - 34l + 120 = 0$

$\Rightarrow \quad l^2 - 17l + 60 = 0$

$\Rightarrow \quad (l - 12)\,(l - 5) = 0$

$\therefore \quad l = 12,\ l = 5$

But $l = 12$ only

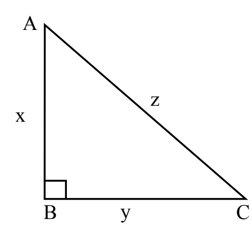
$\therefore$ (1) $\Rightarrow b = 17 - 12$

$= 5$

$\therefore$ Length = 12 cm, Breadth = 5m

5. The hypotenuse of a right triangle is 6 m more than twice of the shortest side. If the third side is 2 m less than the hypotenuse, find the sides of the triangle.

Solution:



Let x be the shortest side.

z be the hypotenuse & y be the 3rd side.

By data given, z = 2x + 6, y = z – 2

= 2x + 6 – 2

= 2x + 4

In ΔABC,

By Pythagoras theorem,

$$x^2 + y^2 = z^2$$

$\Rightarrow \quad x^2 + (2x+4)^2 = (2x+6)^2$

$\Rightarrow \quad x^2 + 4x^2 + 16x + 16 = 4x^2 + 24x + 36$

$\Rightarrow \quad x^2 - 8x - 20 = 0$

$\Rightarrow \quad (x-10)(x+2) = 0$

$\therefore \quad x = 10$ m

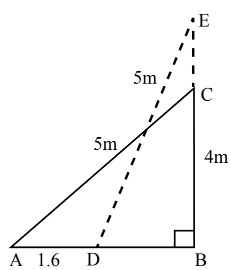
$\therefore \quad y = 2(10) + 4 = 24$ m

$z = 2(10) + 6 = 26$ m

$\therefore$ The length of 3 sides are 10m, 24m, 26m

6. 5 m long ladder is placed leaning towards a vertical wall such that it reaches the wall at a point 4m high. If the foot of the ladder is moved 1.6 m towards the wall, then find the distance by which the top of the ladder would slide upwards on the wall.

Solution:



In $\Delta\, ABC,\ AB = \sqrt{5^2 - 4^2}$

$= \sqrt{25-16}$

$= \sqrt{9}$

$= 3$

Given $AD = 1.6m$

$\Rightarrow \quad DB = 3 - 1.6$

$= 1.4\ m$

Now, In $\Delta\, DEB$,

$EB = \sqrt{5^2 - (1.4)^2}$

$= \sqrt{25 - 1.96}$

$= \sqrt{23.04}$

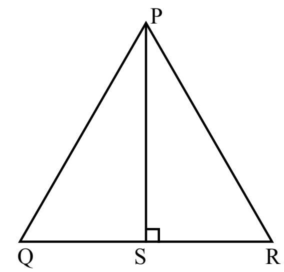
$= 4.8$

$\therefore \quad EC = EB - CB = 4.8 - 4 = 0.8\ m$

$\therefore$ When the foot of the ladder moves 1.6 m towards the wall, the top of the ladder will slide upwards at a dist of 0.8 m

7. The perpendicular PS on the base QR of a ΔPQR intersects QR at S, such that QS = 3 SR. Prove that $2PQ^2 = 2PR^2 + QR^2$.

Solution:



Given $\quad QS = 3 \,.\, SR$

To Prove : $2PQ^2 = 2PR^2 + QR^2$

$\therefore \quad QR = QS + SR$

$= 3SR + SR$

$QR = 4SR \Rightarrow SR = \frac{1}{4} QR$

In Δ PQS, $PQ^2 = PS^2 + QS^2$ ...... (1)

In Δ PRS, $PR^2 = PS^2 + SR^2$ ...... (2)

$(1) - (2) \Rightarrow PQ^2 - PR^2 = QS^2 - SR^2$

$= (3.SR)^2 - SR^2$

$= 8.SR^2$

$= 8\left(\frac{1}{4} QR\right)^2$

$= 8\left(\frac{1}{16} QR\right)^2$

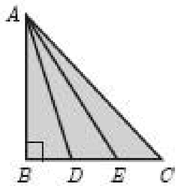
$$PQ^2 - PR^2 = \frac{QR^2}{2}$$

$\Rightarrow \quad 2PQ^2 - 2PR^2 = QR^2$

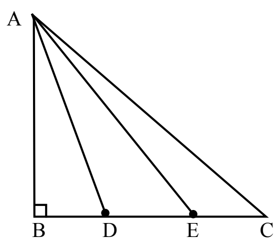
$\Rightarrow \quad 2PQ^2 = 2PR^2 + QR^2$

Hence proved.

8. In the adjacent figure, ABC is a right angled triangle with right angle at B and points D, E trisect BC. Prove that $8AE^2 = 3AC^2 + 5AD^2$



Solution:



Given D and E trisect BC.

$\therefore$ BD = DE = EC = x (take)

$\therefore$ BD = x, BE = 2x, BC = 3x

In Δ ABD, $AD^2 = AB^2 + BD^2$

$= AB^2 + x^2$

In Δ ABE, $AE^2 = AB^2 + BE^2$

$= AB^2 + (2x)^2$

$= AB^2 + 4x^2$

In Δ ABC, $AC^2 = AB^2 + BC^2$

$= AB^2 + (3x)^2$

$= AB^2 + 9x^2$

To prove : $8AE^2 = 3AC^2 + 5AD^2$

RHS :

$3AC^2 + 5AD^2 = 3\,(AB^2 + 9x^2) + 5\,(AB^2 + x^2)$

$= 3\,AB^2 + 27x^2 + 5AB^2 + 5x^2$

$= 8\,AB^2 + 32x^2$

$= 8\,(AB^2 + 4x^2)$

$= 8\,AE^2$

$=$ LHS

Hence proved.

IV. Circles and Tangents

Key Points

- ✓ If a line touches the given circle at only one point, then it is called tangent to the circle.
- ✓ A tangent at any point on a circle and the radius through the point are perpendicular to each other.
- ✓ No tangent can be drawn from an interior point of the circle.
- ✓ Only one tangent can be drawn at any point on a circle.
- ✓ Two tangents can be drawn from any exterior point of a circle.
- ✓ The lengths of the two tangents drawn from an exterior point to a circle are equal,
- ✓ If two circles touch externally the distance between their centers is equal to the sum of their radii.
- ✓ If two circles touch internally, the distance between their centers is equal to the difference of their radii.
- ✓ The two direct common tangents drawn to the circles are equal in length.

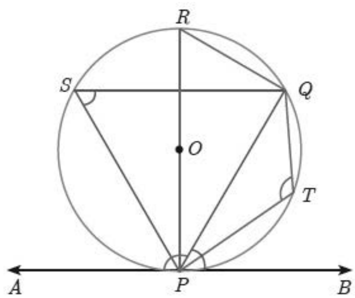
Theorem 6 : Alternate Segment theorem

Statement

If a line touches a circle and from the point of contact a chord is drawn, the angles between the tangent and the chord are respectively equal to the angles in the corresponding alternate segments.

Proof

Given : A circle with centre at O, tangent AB touches the circle at P and PQ is a chord. S and T are two points on the circle in the opposite sides of chord PQ.



To prove : (i) $\angle QPB = \angle PSQ$ and

(ii) $\angle QPA = \angle PTQ$

Construction : Draw the diameter POR. Draw QR, QS and PS.

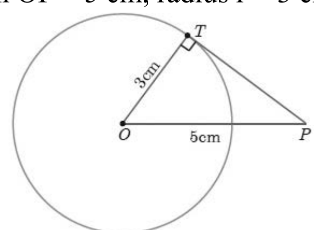
No.	Statement	Reason
1.	$\angle RPB = 90^\circ$ Now, $\angle RPQ + \angle QPB = 90^\circ$ ...(1)	Diameter RP is perpendicular to tangent AB.
2.	In ΔRPQ, $\angle PQR = 90^\circ$ ...(2)	Angle in a semicircle is 90°
3.	$\angle QRP + \angle RPQ = 90^\circ$ ....(3)	In a right angled triangle, sum of the two acute angles is 90°.
4.	$\angle RPQ + \angle QPB = \angle QRP + \angle RPQ$ $\angle QPB = \angle QRP$ ....(4)	From (1) and (3).
5.	$\angle QRP = \angle PSQ$ ...(5)	Angles in the same segment are equal.
6.	$\angle QPB = \angle PSQ$ ...(6)	From (4) and (5) ; Hence (i) is proved.
7.	$\angle QPB + \angle QPA = 180^\circ$ ....(7)	Linear pair of angles.
8.	$\angle PSQ + \angle PTQ = 180^\circ$ ....(8)	Sum of opposite angles of a cyclic quadrilateral is 180°.
9.	$\angle QPB + \angle QPA = \angle PSQ + \angle PTQ$	From (7) and (8).
10.	$\angle QPB + \angle QPA = \angle QPB + \angle PTQ$	$\angle QPB = \angle PSQ$ from (6)
11.	$\angle QPA = \angle PTQ$	Hence (ii) is proved. This completes the proof.

Example 4.24

Find the length of the tangent drawn from a point whose distance from the centre of a circle is 5 cm and radius of the circle is 3 cm.

Solution :

Given OP = 5 cm, radius r = 3 cm



To find the length of tangent PT.

In right angled ΔOTP,

$OP^2 = OT^2 + PT^2$ (by Pythagoras theorem)

$5^2 = 3^2 + PT^2$ gives $PT^2 = 25 - 9 = 16$

Length of the tangent PT = 4cm

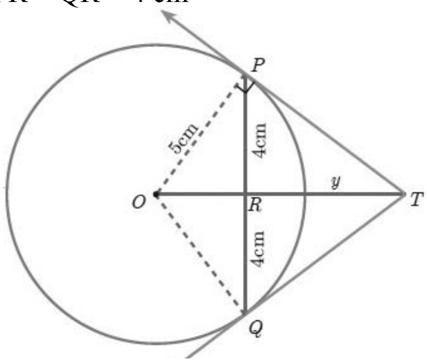
Example 4.25

PQ is a chord of length 8 cm to a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length of the tangent TP.

Solution :

Let TR = y. Since, OT is perpendicular bisector of PQ.

PR = QR = 4 cm



In ΔORP, $OP^2 = OR^2 + PR^2$

$OR^2 = OP^2 - PR^2$

$OR^2 = 5^2 - 4^2 = 25 - 16 = 9 \Rightarrow OR = 3$ cm

$OT = OR + RT = 3 + y$ .... (1)

In ΔPRT, $TP^2 = TR^2 + PR^2$ .... (2)

and Δ OPT we have, $OT^2 = TP^2 - OP^2$

$OT^2 = (TR^2 + PR^2) + OP^2$

(substitute for TP^2 from (2))

$(3+y)^2 = y^2+4^2+5^2$ (substitute for OT from (1))

$9 + 6y + y^2 = y^2 + 16 + 25$

Therefore $y = TR = \dfrac{16}{3}$

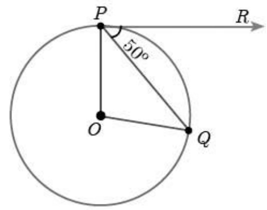
$6y = 41 - 9$ *we get* $y = \dfrac{16}{3}$

From (2), $TP^2 = TR^2 + PR^2$

$$TP^2 = \left(\frac{16}{3}\right)^2 + 4^2 = \frac{256}{9} + 16 = \frac{400}{9} \text{ so, } TP = \frac{20}{3} \text{ cm}$$

Example 4.26

In Figure O is the centre of a circle. PQ is a chord and the tangent PR at P makes an angle of 50° with PQ. Find ∠POQ



Solution :

$\angle OPQ = 90^0 - 50^0 = 40^0$ (angle between the radius and tangent is 90°)

OP = OQ (Radii of a circle are equal)

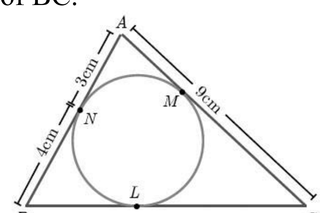
$\angle OPQ = \angle OQP = 40^0$ (ΔOPQ is isosceles)

$\angle POQ = 180^0 - \angle OPQ - \angle OQP$

$\angle POQ = 180^0 - 40^0 - 40^0 = 100^0$

Example 4.27

In Fig., Δ ABC is circumscribing a circle. Find the length of BC.



Solution :

AN = AM = 3 cm (Tangents drawn from same external point are equal)

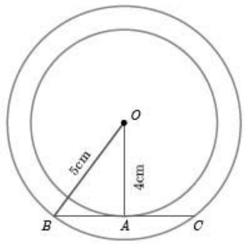
BN = BL = 4 cm

CL = CM = AC – AM = 9 – 3 = 6 cm

Gives BC = BL + CL = 4 + 6 = 10 cm

Example 4.28

If radii of two concentric circles are 4 cm and 5 cm then find the length of the chord of one circle which is a tangent to the other circle.



Solution :

OA = 4 cm, OB = 5 cm; also OA $\perp$ BC.

$OB^2 = OA^2 + AB^2$

$5^2 = 4^2 + AB^2$ gives $AB^2 = 9$

Therefore AB = 3 cm

BC = 2AB hence BC = $2 \times 3 = 6$ cm

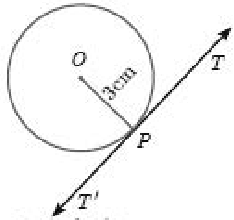
CONSTRUCTION OF A TANGENT TO A CIRCLE

Example 4.29

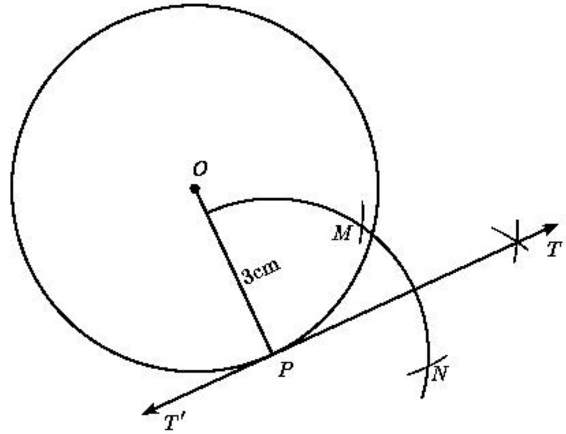
Draw a circle of radius 3 cm. Take a point P on this circle and draw a tangent at P.

Solution :

Given, radius r = 3 cm



Rough diagram



Construction

Step 1 : Draw a circle with centre at O of radius 3 cm.

Step 2 : Take a point P on the circle. Join OP.

Step 3 : Draw perpendicular line TT' to OP which passes through P.

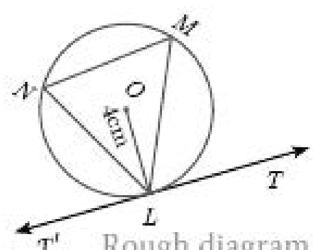
Step 4 : TT' is the required tangent.

Example 4.30

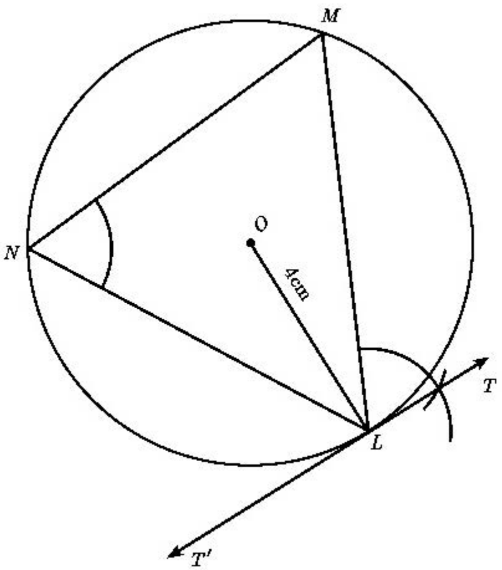
Draw a circle of radius 4 cm. At a point L on it draw a tangent to the circle using the alternate segment.

Solution :

Given, radius=4 cm



Rough diagram



Construction

Step 1 : With O as the centre, draw a circle of radius 4 cm.

Step 2 : Take a point L on the circle. Through L draw any chord LM.

Step 3 : Take a point M distinct from L and N on the circle, so that L, M and N are in anti-clockwise direction. Join LN and NM.

Step 4 : Through L draw a tangent TT' such that $\angle TLM = \angle MNL$.

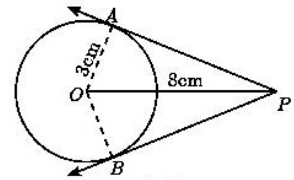
Step 5 : TT' is the required tangent.

Example 4.31

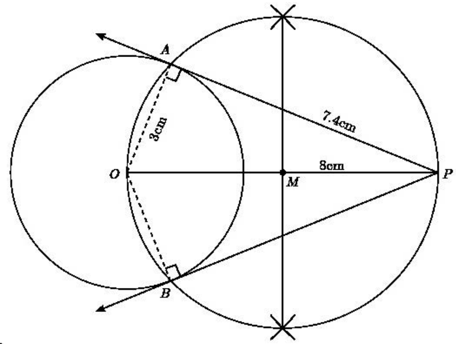
Draw a circle of diameter 6 cm from a point P, which is 8 cm away from its centre. Draw the two tangents PA and PB to the circle and measure their lengths.

Solution :

Given, diameter (d) = 6 cm, we find radius (r) = $\frac{6}{2}$ = 3 cm.



Rough diagram



Construction

Step 1 : With centre at O, draw a circle of radius 3 cm.

Step 2 : Draw a line OP of length 8 cm.

Step 3 : Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4 : With M as centre and MO as radius, draw a circle which cuts previous circle at A and B.

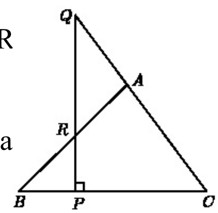
Step5 : Join AP and BP. AP and BP are the required tangents. Thus length of the tangents are PA = PB = 7.4 cm.

Verification : In the right angle triangle OAP , $PA^2 = OP^2 - OA^2 = 64 - 9 = 55$
$PA = \sqrt{55}$ = 7.4 cm (approximately) .

Concurrency Theorems

Key Points

- ✓ A cevian is a line segment that extends from one vertex of a triangle to the opposite side.
- ✓ A median is a cevian that divides the opposite side into two congruent(equal) lengths.
- ✓ An altitude is a cevian that is perpendicular to the opposite side.
- ✓ An angle bisector is a cevian that bisects the corresponding angle.
- ✓ Ceva's Theorem : Let ABC be a triangle and let D,E,F be points on lines BC, CA, AB respectively. Then the cevians AD, BE, CF are concurrent if and only if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$ where the lengths are directed.
- ✓ Menelaus Theorem : A necessary and sufficient condition for points P, Q, R on the respective sides BC, CA, AB (or their extension) of a triangle ABC to be collinear is that $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = 1$ where all segments in the formula are directed segments.

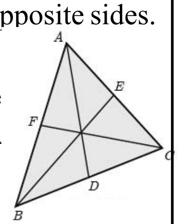


Example 4.32

Show that in a triangle, the medians are concurrent.

Solution :

Medians are line segments joining each vertex to the midpoint of the corresponding opposite sides.



Thus medians are the cevians where D, E, F are midpoints of BC, CA and AB respectively.

Since D is a mid point of

BC, BD = DC so $\frac{BD}{DC} = 1$ .... (1)

Since, E is a midpoint of

CA, CE = EA so $\frac{CE}{EA} = 1$ .... (2)

Since, F is a midpoint of AB,

AB, AF = FB so $\frac{AF}{FB} = 1$ .... (3)

Thus, multiplying (1), (2) and (3) we get,

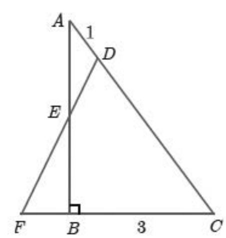
$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1 \times 1 \times 1 = 1$$

And so, Ceva's theorem is satisfied.

Hence the Medians are concurrent.

Example 4.33

In Fig., ABC is a triangle with $\angle B = 90^0$, BC = 3 cm and AB = 4 cm. D is point on AC such that AD = 1 cm and E is the midpoint of AB. Join D and E and extend DE to meet CB at F. Find BF.



Solution :

Consider ΔABC . Then D, E and F are respective points on the sides CA, AB and BC. By construction D, E, F are collinear.

By Menelaus' theorem $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CD}{DA} = 1$ .... (1)

By assumption, AE = EB = 2, DA = 1 and

FC = FB + BC = BF + 3

By Pythagoras theorem, $AC^2 = AB^2 + BC^2$

= 16 + 9 = 25. Therefore AC = 5 and So, CD = AC – AD = 5 – 1 = 4.

Substituting the values of FC, AE, EB, DA, CD in (1),

$$\text{we get, } \frac{2}{2}\times\frac{BF}{BF+3}\times\frac{4}{1}=1$$

4BF = BF + 3

4BF – BF = 3 therefore BF = 1

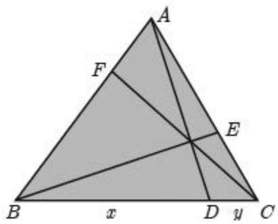
Example 4.34

Suppose AB, AC and BC have lengths 13, 14 and 15 respectively. If $\frac{AF}{FB}=\frac{2}{5}$ and $\frac{CE}{EA}=\frac{5}{8}$. Find BD and DC.

Solution :

Given that AB = 13 , AC = 14 and BC = 15

Let BD = x and DC = y



Using Ceva's theorem, we have,

$$\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{Af}{FB}=1 \qquad \text{............ (1)}$$

Substitute the values of $\frac{Af}{FB}$ and $\frac{CE}{EA}$ in (1),

$$\text{We have } \frac{BD}{DC}\times\frac{5}{8}\times\frac{2}{5}=1$$

$$\frac{x}{y}\times\frac{10}{40}=1 \text{ we get, } \frac{x}{y}\times\frac{1}{4}=1.$$

Hence, x = 4y .... (2)

$$BC = BD + DC = 15$$

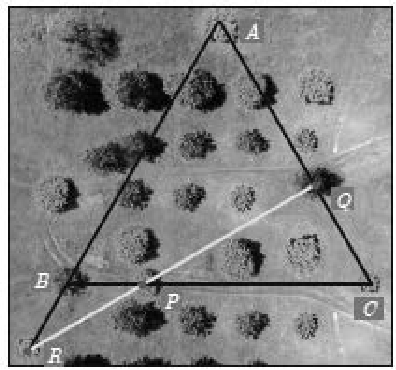
so, $x+y=15$ .... (3)

From (2), using x = 4y in (3) we get, 4y +y = 15 gives 5y = 15 then y = 3

Substitute y = 3 in (3) we get, x = 12 .
Hence BD = 12 , DC = 3.

Example 3.7

In a garden containing several trees, three particular trees P, Q, R are located in the following way, BP = 2 m, CQ = 3 m, RA = 10 m, PC = 6 m, QA = 5 m, RB = 2 m, where A, B, C are points such that P lies on BC, Q lies on AC and R lies on AB. Check whether the trees P, Q, R lie on a same straight line.



Solution :

By Meanlau's theorem, the trees P, Q, R will be collinear (lie on same straight line)

$$if\ \frac{BP}{PC}\times\frac{CQ}{QA}\times\frac{RA}{RB}=1 \qquad \text{........... (1)}$$

Given BP = 2 m, CQ =3 m, RA =10 m, PC =6 m, QA = 5 m and RB = 2 m

Substituting these values in (1) we get,

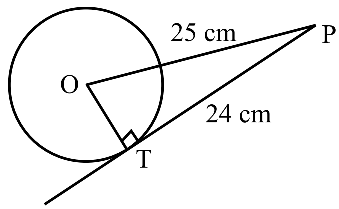
$$\frac{BP}{PC}\times\frac{CQ}{QA}\times\frac{RA}{RB}=\frac{2}{6}\times\frac{3}{5}\times\frac{10}{2}=\frac{60}{60}=1$$

Hence the trees P, Q, R lie on a same straight line.

EXERCISE 4.4

1. The length of the tangent to a circle from a point P, which is 25 cm away from the centre is 24 cm. What is the radius of the circle?

Solution:



Given OP = 25 cm, PT = 24cm

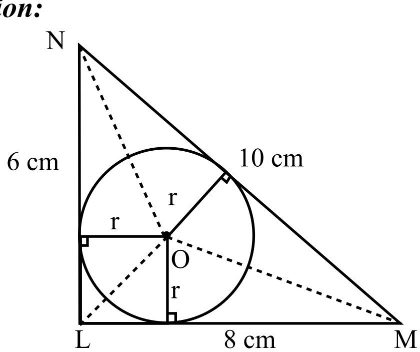
Radius & Tangent are Perpendicular

$$\therefore \; OT = \sqrt{OP^2 - PT^2}$$
$$= \sqrt{25^2 - 24^2}$$
$$= \sqrt{625 - 576}$$
$$= \sqrt{49}$$
$$= 7 \text{ cm}$$

$\therefore$ Radius = 7 cm

2. ΔLMN is a right angled triangle with $\angle L = 90°$. A circle is inscribed in it. The lengths of the sides containing the right angle are 6 cm and 8 cm. Find the radius of the circle.

Solution:



$$MN = \sqrt{ML^2 + LN^2}$$
$$= \sqrt{64 + 36}$$
$$= \sqrt{100}$$
$$= 10$$

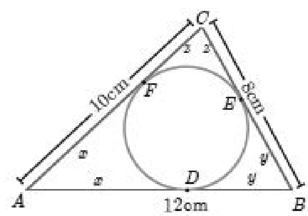
Area of ΔMLN = Area of ΔMOL + Area of ΔNOL + Area of ΔMON

$$\Rightarrow \frac{1}{2} \times ML \times LN = \frac{1}{2} \times LM \times r + \frac{1}{2} \times LN \times r + \frac{1}{2} \times MN \times r$$

$$\Rightarrow \quad 8 \times 6 = 8r + 6r + 10r$$
$$\Rightarrow \quad 24r = 48$$
$$r = 2cm$$

3. A circle is inscribed in ΔABC having sides 8 cm, 10 cm and 12 cm as shown in figure, Find AD, BE and CF.

Solution:



From the fig,

x + y = 12

y + z = 8

z + x = 10

Adding, 2 (x + y + z) = 30

$\Rightarrow$ x + y + z = 15

$\Rightarrow$ 12 + z = 15

$\therefore$ z = 3

$\Rightarrow$ y + 3 = 8

$\Rightarrow$ y = 5

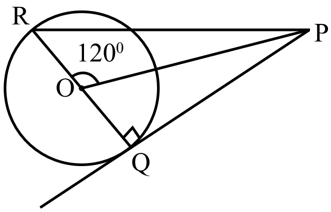
$\therefore$ x + 5 = 12

$\Rightarrow$ x = 7

$\therefore$ AD = 7 cm, BE = 5 cm, CF = 3 cm

4. PQ is a tangent drawn from a point P to a circle with centre O and QOR is a diameter of the circle such that $\angle POR = 120°$. Find $\angle OPQ$.

Solution:



Given $\angle POR = 120^0$

$\Rightarrow \angle POQ = 60^0$ (linear pair)

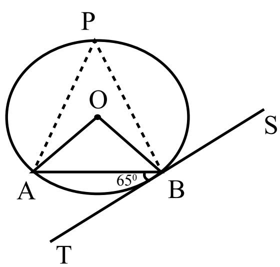
Also $\angle OQP = 90^0$ (Radius $\perp$ tangent)

$\therefore \angle OPQ = 90^0 - 60^0$

$= 30^0$

5. A tangent ST to a circle touches it at B. AB is a chord such that $\angle ABT = 65^0$. Find $\angle AOB$, where "O" is the centre of the circle.

Solution:



Given $\angle TBA = 65^0 \Rightarrow \angle APB = 65^0$

(angles in altemate segment).

$\therefore \angle AOB = 2\angle APB = 2(65^0) = 130^0$ circumference)

(Angle subtended at the centre is twice the angle subtended at any point on the remaining

6. In figure, O is the centre of the circle with radius 5 cm. T is a point such that OT = 13 cm and OT intersects the circle E, if AB is the tangent to the circle at E, find the lenght of AB.

Solution:

In the figure, given OP = 5, OT = 13

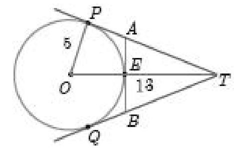
$\therefore PT = \sqrt{13^2 - 5^2}$

$= \sqrt{169 - 25}$

$= \sqrt{144}$

$= 12$

$=$ TQ



Also, OE = 5 $\Rightarrow$ ET = 13 – 5 = 8

Let AP = AE = x $\Rightarrow$ TA = 12 – x

$\therefore$ In ΔAET, $\angle AET = 90^0$

$\therefore \quad x^2 + 8^2 = (12 - x)^2$

$\Rightarrow \quad x^2 + 8^2 = 144 + x^2 - 24x$

$\Rightarrow \quad 64 = 144 - 24x$

$\Rightarrow \quad 24x = 80$

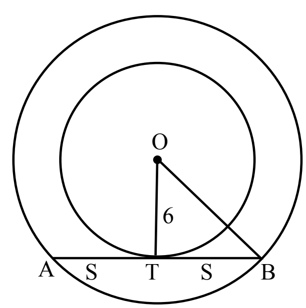
$$x = \frac{10}{3}$$

$\therefore \quad AB = 2x$

Length of tangent $AB = \dfrac{20}{3}$ cm

7. In two concentric circles, a chord of length 16 cm of larger circle becomes a tangent to the smaller circle whose radius is 6 cm. Find the radius of the larger circle.

Solution:



Given the chord AB of larger circle is a tangent for the smaller circle & OT is radius.

OT is perpendicular to AB.

$\therefore$ AT = TB = 8 cm, OT = 6 cm

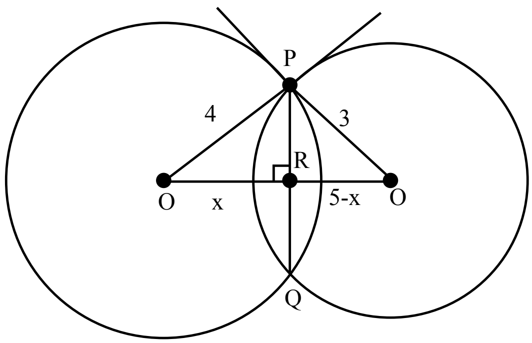
$\therefore$ In Δ OBT,

$$OB = \sqrt{8^2+6^2}$$
$$= \sqrt{64+36}$$
$$= \sqrt{100}$$
$$= 10\ cm$$

$\therefore$ Radius of the larger circle = 10 cm

8. Two circles with centres O and O' of radii 3 cm and 4 cm, respectively intersect at two points P and Q, such that OP and O'P are tangents to the two circles. Find the length of the common chord PQ.

Solution:



Given OP = 4 cm (radius of 1st circle)

O'P = 3 cm (radius of 2nd circle)

Clearly OP $\perp$ O'P (tangent & radius are $\perp$)

$$\therefore\ OO' = \sqrt{4^2+3^2}$$
$$= \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5$$

Let R be a point of PQ such that

OR = x & O'R = 5 – x

Also, ΔOPO' ~ ΔOQO' & ΔOPR ~ ΔOQR (by similarity)

$\therefore$ $\angle ORP = 90^0$

$\therefore$ In ΔORP, $PR^2 = 16 - x^2$

In ΔO'RP, $PR^2 = 9 - (5-x)^2$

$\therefore$ $16 - x^2 = 9 - (5-x)^2$

$\Rightarrow$ $10x = 32$

$$\therefore\ x = \frac{16}{5}$$

$$\therefore\ PR = \sqrt{16 - \frac{256}{25}}$$
$$= \sqrt{\frac{144}{25}}$$
$$= \frac{12}{5}$$
$$= 2.4$$

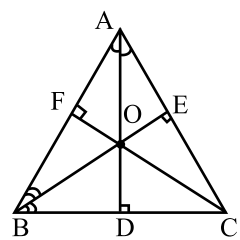
$\therefore$ PQ = 2 (PR)

= 2 (2.4)

= 4.8 cm

9. Show that the angle bisectors of a triangle are concurrent.

Solution:



Consider a Δ ABC & let the angular bisectors of A and B meet at 'O'.

From O, draw perpendicular OD, OE, OF to BC, CA, AB respectively.

Now ΔBOD $\equiv$ BOF

($\because$ $\angle ODB = \angle OFB = 90^0$

$\therefore$ OD = OF $\angle OBD = \angle OBF$)

IIIly in ΔOAE & ΔOAF, we can prove

OE = OF

$\therefore$ OD = OE = OF

Now, join OC,

Consider ΔOCD, ΔOCE

Here i) $\angle ODC = \angle OEC = 90^0$ & OC is common

ii) OD = OE

$\therefore$ $\Delta OCD \equiv \Delta OCE$

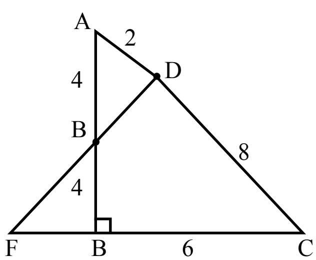
$\therefore$ $\angle OCD = \angle OCE$

CO is angle bisector of $\angle C$.

$\therefore$ Angle bisectors of a triangle are concurrent.

10. In ΔABC, with B = 90°, BC = 6 cm and AB = 8 cm, D is a point on AC such that AD = 2 cm and E is the midpoint of AB. Join D to E and extend it to meet at F. Find BF.

Solution:



Given In ΔABC, AB = 8 cm, BC = 6 cm

$$\therefore AC = \sqrt{64+36} = \sqrt{100} = 10$$

Also AD = 2 $\Rightarrow$ CD = 8 cm

E is the mid point of AB

$\Rightarrow$ AE = EB = 4 cm

By Menelaus Theorem,

$$\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CD}{DA} = 1$$

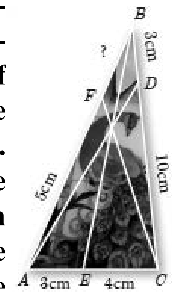
$$\Rightarrow \frac{\cancel{4}}{\cancel{4}} \times \frac{BF}{BF+6} \times \frac{8}{2} = 1$$

$$\Rightarrow 4BF = BF + 6$$

$$\Rightarrow 3BF = 6$$

$$\therefore BF = 2 \text{ cm}$$

11. An artist has created a triangular stained glass window and has one strip of small length left before completing the window. She needs to figure out the length of left out portion based on the lengths of the other sides as shown in the figure.



Solution:

By applying Ceva's theorem, the Cevians AD, BE and CF intersect at exactly one point if and only if

$$BD \times CE \times AF = DC \times EA \times FB$$

$$\Rightarrow 3 \times 4 \times 5 = 10 \times 3 \times FB$$

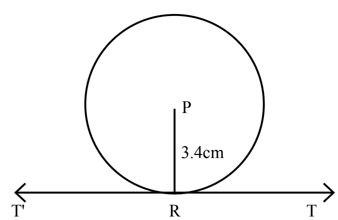
$$\Rightarrow 60 = 30 \times FB$$

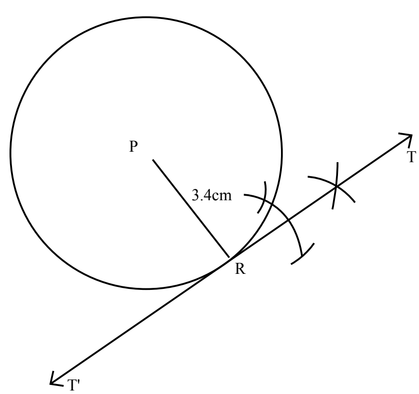
$$\therefore FB = 2 \text{ cm}$$

12. Draw a tangent at any point R on the circle of radius 3.4 cm and centre at P ?

Solution:

Rough Diagram





Construction

Step 1 : Draw a circle with centre at P of radius 3.4 cm.

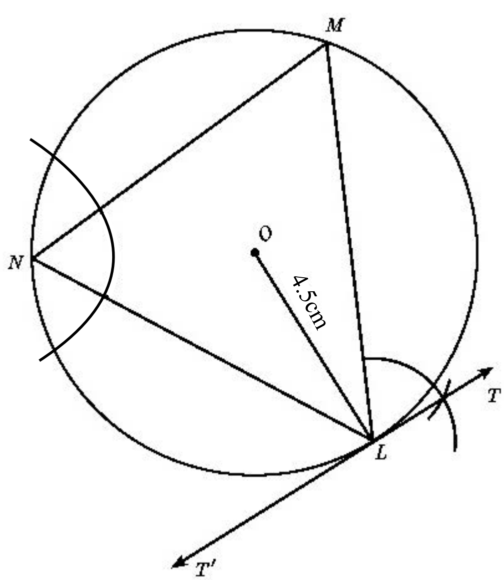
Step 2 : Take a point R on the circle and Join PR.

Step 3 : Draw perpendicular line TT' to PR which passes through R.

Step 4 : TT' is the required tangent.

13. Draw a circle of radius 4.5 cm. Take a point on the circle. Draw the tangent at that point using the alternate segment theorem.

Solution:



Construction

Step 1 : With O as the centre, draw a circle of radius 4.5 cm.

Step 2 : Take a point L on the circle. Through L draw any chord LM.

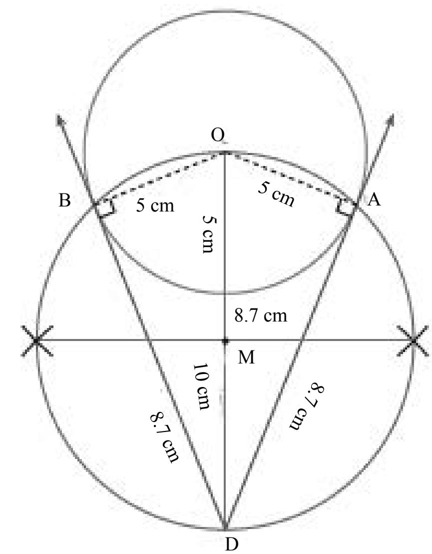
Step 3 : Take a point M distinct from L and N on the circle, so that L, M and N are in anti-clockwise direction. Join LN and NM.

Step 4 : Through L draw a tangent TT' such that $\angle TLM = \angle MNL$.

Step 5 : TT' is the required tangent.

14. Draw the two tangents from a point which is 10 cm away from the centre of a circle of radius 5 cm. Also, measure the lengths of the tangents.

Solution:



Construction

Step 1 : With centre at O, draw a circle of radius 5 cm.

Step 2 : Draw a line OP = 10 cm.

Step 3 : Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4 : With M as centre and MO as radius, draw a circle which cuts previous circle at A and B.

Step5 : Join AP and BP. AP and BP are the required tangents. Thus length of the tangents are PA = PB = 8.7 cm.

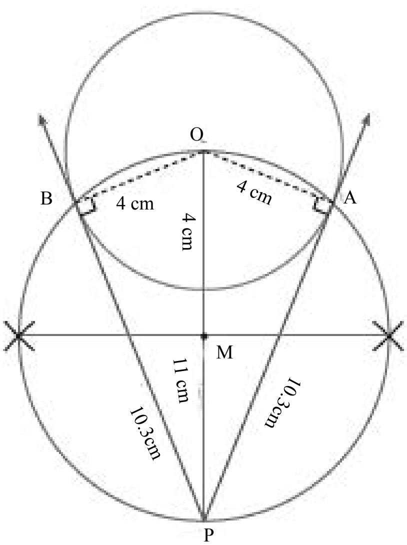
Verification : In the right angle triangle ΔOAP ,

$PA^2 = \sqrt{OP^2 - OA^2}$

$= \sqrt{100-25} = \sqrt{75} = 8.7$ cm

15. Take a point which is 11 cm away from the centre of a circle of radius 4 cm and draw the two tangents to the circle from that point.

Solution:



Construction

Step 1 : With centre at O, draw a circle of radius 4 cm.

Step 2 : Draw a line OP = 11 cm.

Step 3 : Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4 : With M as centre and MO as radius, draw a circle which cuts previous circle at A and B.

Step5 : Join AP and BP. They are the required tangents AP = BP = 10.3 cm.

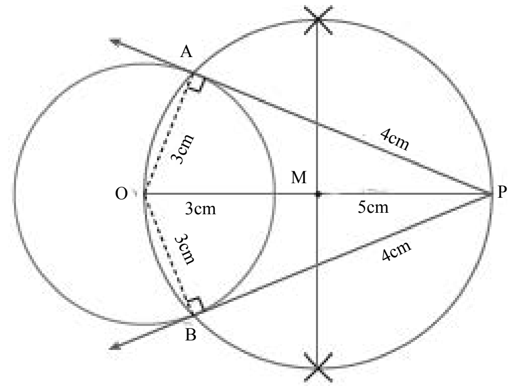
Verification : In the right angle triangle ΔOAP ,

$AP = \sqrt{OP^2 - OA^2}$

$= \sqrt{121-16} = \sqrt{105} = 10.3$ cm

16. Draw the two tangents from a point which is 5 cm away from the centre of a circle of diameter 6 cm. Also, measure the lengths of the tangents.

Solution:



Construction

Step 1 : With centre at O, draw a circle of radius 3 cm. with centre at O.

Step 2 : Draw a line OP = 5 cm.

Step 3 : Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4 : With M as centre and OM as radius, draw a circle which cuts previous circle at A and B.

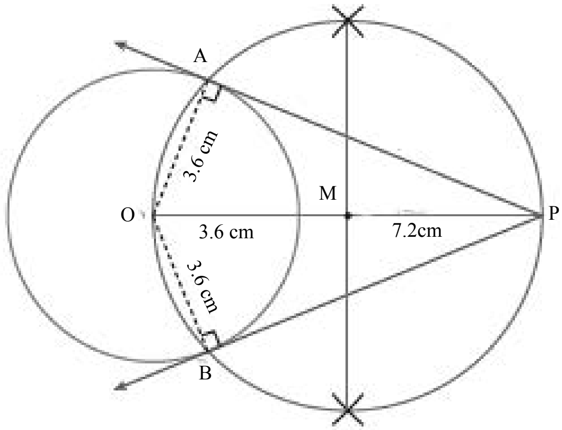
Step 5 : Join AP and BP. They are the required tangents AP = BP = 4 cm.

Verification :

$$AP = \sqrt{OP^2 - OA^2}$$
$$= \sqrt{5^2 - 3^2}$$
$$= \sqrt{25-9}$$
$$= \sqrt{16} = 4 \text{ cm}$$

17. Draw a tangent to the circle from the point P having radius 3.6 cm, and centre at O. Point P is at a distance 7.2 cm from the centre.

Solution:



Construction

Step 1 : Draw a circle of radius 3.6 cm. with centre at O.

Step 2 : Draw a line OP = 7.2 cm.

Step 3 : Draw a perpendicular bisector of OP, which cuts it M.

Step 4 : With M as centre and OM as radius, draw a circle which cuts previous circle at A and B.

Step5 : Join AP and BP. They are the required tangents AP = BP = 0.3 cm.

Verification :

$$AP = \sqrt{OP^2 - OA^2}$$
$$= \sqrt{(7.2)^2 - (3.6)^2}$$
$$= \sqrt{51.84 - 12.96}$$
$$= \sqrt{38.88} = 6.3 \text{ (approx)}$$

EXERCISE 4.5

Multiple choice questions

1. If in triangles ΔABC and ΔEDF, $\frac{AB}{DE} = \frac{BC}{FD}$ then they will be similar, when

1) $\angle B = \angle E$ 2) $\angle A = \angle D$

3) $\angle B = \angle D$ 4) $\angle A = \angle F$

Hint : **Ans : (3)**

$\Delta ABC \sim \Delta EDF$ if $\frac{AB}{DE} = \frac{BC}{FD}$ and

$\angle B = \angle D$

$\angle A = \angle F$

$\angle C = \angle F$

2. In ΔLMN, L = 60°, M = 50° . If ΔLMN ~ ΔPQR then the value of ∠R is

1) 40^0 2) 70^0

3) 30^0 4) 110^0

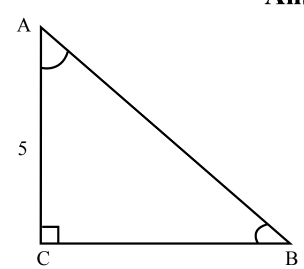
Hint : **Ans : (2)**

$$\angle R = 180^0 - (\angle L + \angle M)$$
$$= 180^0 - (60^0 + 50^0)$$
$$= 180^0 - 110^0$$
$$= 70^0$$

3. If ΔABC is an isosceles triangle with ∠C = 90° and AC = 5 cm, then AB is

1) 2.5 cm 2) 5 cm 3) 10 cm 4) $5\sqrt{2}$ cm

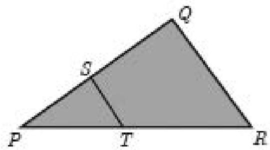
Hint : **Ans : (4)**



ΔABC is isosceles ⇒ $\angle B = \angle A = 25^0$

$$\therefore \sin 45^0 = \frac{5}{AB}$$
$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{5}{AB}$$
$$\Rightarrow AB = 5\sqrt{2} \text{ cm}$$

4. In a given figure ST | | QR, PS = 2 cm and SQ = 3 cm. Then the ratio of the area of ΔPQR to the area of ΔPST is



1) 25 : 4 2) 25 : 7 3) 25 : 11 4) 25 : 13

Hint : **Ans : (1)**

Where $PQ = PS + SQ$

$= 2 + 3$

$= 5$

$$\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta PST} = \frac{PQ^2}{PS^2}$$

$$= \frac{25}{4}$$

$\therefore$ Ratio = 25 : 4

5. The perimeters of two similar triangles ΔABC and ΔPQR are 36 cm and 24 cm respectively. If PQ = 10 cm, then the length of AB is

1) $6\frac{2}{3}$ cm 2) $\frac{10\sqrt{6}}{3}$ cm

3) $\frac{2}{3}$ cm 4) 15 cm

Hint : **Ans : (4)**

$$\frac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta PQR} = \frac{36}{24} = \frac{AB}{PQ}$$

$$\Rightarrow \quad \frac{3}{2} = \frac{AB}{10}$$

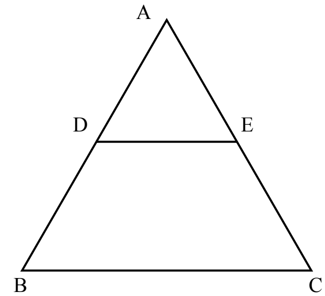
$$\Rightarrow \quad AB = 15 \text{ cm}$$

6. If in ΔABC, DE | | BC . AB = 3.6 cm, AC = 2.4 cm and AD = 2.1 cm then the length of AE is

1) 1.4 cm 2) 1.8 cm

3) 1.2 cm 4) 1.05 cm

Hint : **Ans : (1)**



By BPT, $\frac{AD}{AB} = \frac{AE}{AC}$

$$\Rightarrow \quad \frac{2.1}{3.6} = \frac{AE}{2.4}$$

$$\Rightarrow \quad AE = 2.4 \times \frac{2.1}{3.6}$$

$$= \frac{2}{3} \times 2.1$$

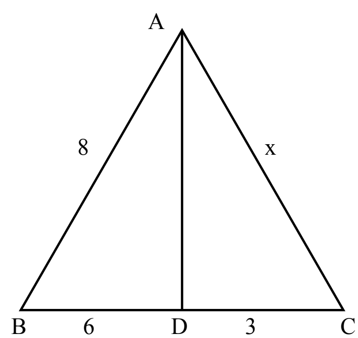
$$= 2 \times 0.7$$

$$= 1.4 \text{ cm}$$

7. In a ΔABC, AD is the bisector of ∠BAC . If AB = 8 cm, BD = 6 cm and DC = 3 cm. The length of the side AC is

1) 6 cm 2) 4 cm 3) 3 cm 4) 8 cm

Hint : **Ans : (2)**



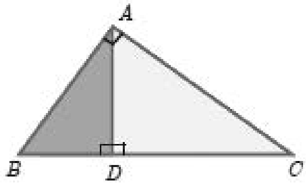
By ABT, $\frac{8}{x} = \frac{6}{3}$

$$\Rightarrow \quad \frac{8}{x} = 2$$

$$\Rightarrow \quad x = 4 \text{ cm}$$

8. In the adjacent figure ∠BAC = 90° and AD ⊥ BC then

1) BD . CD = BC^2 2) AB . AC = BC^2
3) BD . CD = AD^2 4) AB . AC = AD^2



Hint : **Ans : (3)**

$$\Delta\ DBA \sim \Delta DAC$$

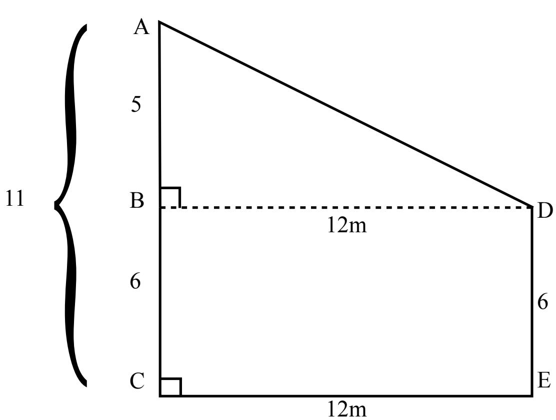
$$\therefore \quad \frac{BD}{AD} = \frac{AD}{CD}$$

$$\Rightarrow AD^2 = BD \times CD$$

9. Two poles of heights 6 m and 11 m stand vertically on a plane ground. If the distance between their feet is 12 m, what is the distance between their tops?

1) 13 m 2) 14 m 3) 15 m 4) 12.8 m

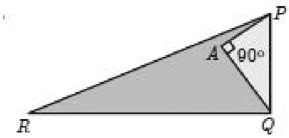
Hint : **Ans : (1)**



$$\therefore \quad AD = \sqrt{12^2 + 5^2} = \sqrt{169} = 13 \text{ cm}$$

10. In the given figure, PR = 26 cm, QR = 24 cm, ∠PAQ = 90°, PA = 6 cm and QA = 8 cm. Find ∠PQR

1) 80° 2) 85° 3) 75° 4) 90°



Hint : **Ans : (4)**

In PAQ, PA = 6, QA = 8

$$\Rightarrow \quad PQ = \sqrt{64+36}$$
$$= \sqrt{100}$$
$$= 10$$

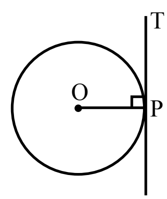
Also, in ΔPQR, $PQ^2 + QR^2 = 100 + 576$

$$= 676$$
$$= 26^2$$
$$= PR^2$$
$$\therefore Q = 90^0$$

11. A tangent is perpendicular to the radius at the

1) centre 2) point of contact
3) infinity 4) chord

Hint : **Ans : (2)**

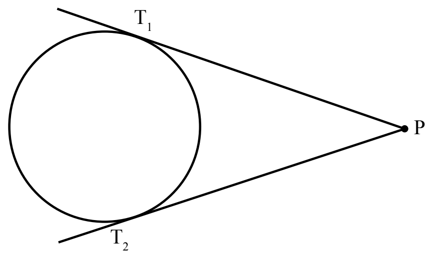


A tangent is perpendicular to the radius at the point of contact.

12. How many tangents can be drawn to the circle from an exterior point?

1) one 2) two 3) infinite 4) zero

Hint : **Ans : (2)**

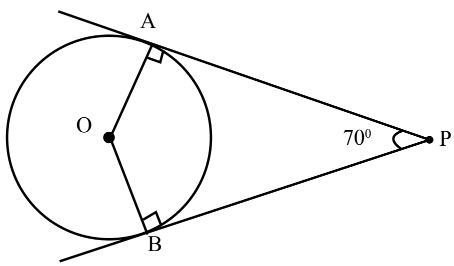


Two tangents can be drawn to the circle from an external point.

13. The two tangents from an external points P to a circle with centre at O are PA and PB. If ∠APB = 70° then the value of ∠AOB is

1) 100° 2) 110° 3) 120° 4) 130°

Hint : **Ans : (2)**



OA ⊥ AP, OB ⊥ BP

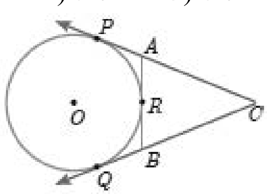
$\therefore \angle AOB + 90^0 + 90^0 + 70^0 = 360^0$

$\Rightarrow \angle AOB = 360^0 - 250^0$

$= 110^0$

14. In figure CP and CQ are tangents to a circle with centre at O. ARB is another tangent touching the circle at R. If CP = 11 cm and BC = 7 cm, then the length of BR is

1) 6 cm 2) 5 cm 3) 8 cm 4) 4 cm



Hint : **Ans : (4)**

CP = CQ = 11 cm, BC = 7

∴ BQ = 11 – 7

= 4

∴ BR = BQ = 4 cm

15. In figure if PR is tangent to the circle at P and O is the centre of the circle, then ∠POQ is

1) 120° 2) 100° 3) 110° 4) 90°

Hint : **Ans: (1)**

$\angle RPQ = 60^0 \Rightarrow QO'P = 60^0$ (O' is on the circle)

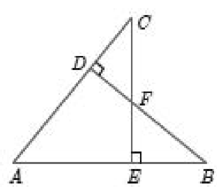
$\Rightarrow QOP = 2(60)$

$= 120^0$

UNIT EXERCISE - 4

1. In the figure, if BD ⊥ AC and CE ⊥ AB , prove that

(i) $\Delta\ AEC \sim \Delta\ ADB$ (ii) $\dfrac{CA}{AB} = \dfrac{CE}{DB}$



Solution :

In ΔAEC, ΔADB

i) ∠A is common

ii) $\angle AEC = \angle ADB = 90^0$

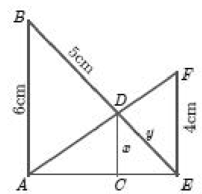
∴ By AA similarity, ΔAEC ~ ΔADB

∴ Corresponding sides are proportional.

$(ie)\ \dfrac{CA}{AB} = \dfrac{CE}{DB}$

Hence proved.

2. In the given figure AB | | CD | | EF . If AB = 6 cm, CD = x cm, EF = 4 cm, BD = 5 cm and DE = y cm. Find x and y.



Solution :

By example 4.9,

$$x = \frac{ab}{a+b} = \frac{6\times 4}{6+4} = \frac{24}{10} = \frac{12}{5}$$

In ΔABE,

$$\Rightarrow \quad \frac{ED}{EB} = \frac{CD}{AB}$$

$$\Rightarrow \quad \frac{y}{y+5} = \frac{x}{6}$$

$$\Rightarrow \quad \frac{y}{y+5} = \frac{12}{5\times 6}$$

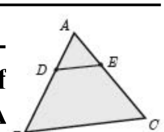
$$\Rightarrow \quad \frac{y}{y+5} = \frac{2}{5}$$

$$\Rightarrow \quad 5y = 2y + 10$$

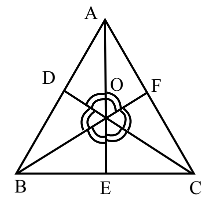
$$\Rightarrow \quad 3y = 10$$

$$\Rightarrow \quad y = \frac{10}{3}$$

3. O is any point inside a triangle ABC. The bisector of $\angle AOB$, $\angle BOC$ and $\angle COA$ meet the sides AB, BC and CA in point D, E and F respectively. Show that AD × BE × CF = DB × EC × FA.



Solution :



By using Ceva's Theorem,

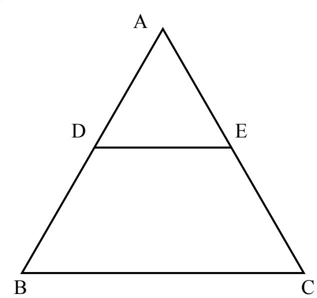
$$\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{AF}{FC} = 1$$

⇒ AD × BE × AF = DB × EC × FC

Hence proved.

4. In the figure, ABC is a triangle in which AB = AC . Points D and E are points on the side AB and AC respectively such that AD = AE. Show that the points B, C, E and D lie on a same circle.

Solution :



In ΔABC, AB = AC ⇒ $\angle ACB = \angle ABC$

⇒ $\angle ECB = \angle DBC$ .... (1)

Also given AD = AE

∴ BD = CE

∴ DE | | BC

∴ $\angle DBC + \angle BDE = 180^0$

(Sum of 2 adjacent angles is 180^0)

⇒ $\angle ECB + \angle BDE = 180^0$ (From (1))

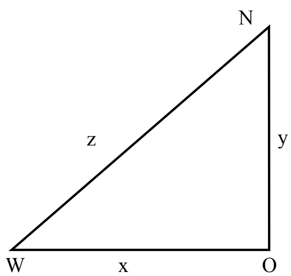
∴ Opposite angles are supplementary.

∴ The points B, C, E & D are concyclic.

Hence proved.

5. Two trains leave a railway station at the same time. The first train travels due west and the second train due north. The first train travels at a speed of 20 km/hr and the second train travels at 30 km/hr. After 2 hours, what is the distance between them?

Solution :



Given speed of 1st train = 20 Km/hr

$\therefore$ Speed of 2nd train = 30 Km/hr

$\therefore$ After 2 hrs, OW = 20 × 2 = 40 Km = x

ON = 30 × 2 = 60 Km = y

$\therefore$ Distance between 2 trains after 2 hrs.

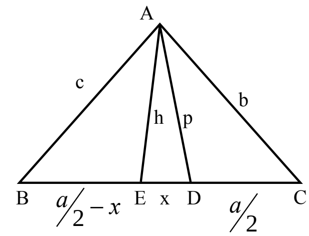
$$Z = \sqrt{x^2 + y^2}$$
$$= \sqrt{40^2 + 60^2}$$
$$= \sqrt{1600 + 3600}$$
$$= \sqrt{5200}$$
$$= \sqrt{40 \times 13}$$
$$= 20\sqrt{13} \text{ Km}$$

6. D is the mid point of side BC and AE $\perp$ BC. If BC = a, AC = b, AB = c, ED = x, AD = p and AE = h , prove that

$(i)\ b^2 = p^2 + ax + \frac{a^2}{4}$ $(ii)\ c^2 = p^2 - ax + \frac{a^2}{4}$

$(iii)\ b^2 + c^2 = 2p^2 + \frac{a^2}{2}$

Solution :



$i)\quad AC^2 = b^2 = AE^2 + EC^2$

$$= h^2 + \left(x + a/2\right)^2$$
$$= (p^2 - x^2) + \left(x^2 + a^2/4 + ax\right)$$
$$= p^2 + a^2/4 + ax$$

Hence proved

$ii)\quad AB^2 = c^2 = AE^2 + BE^2$

$$= h^2 + \left(a/2 - x\right)^2$$
$$= p^2 - x^2 + a^2/4 + x^2 - ax$$
$$= p^2 - ax + a^2/4$$

Hence proved

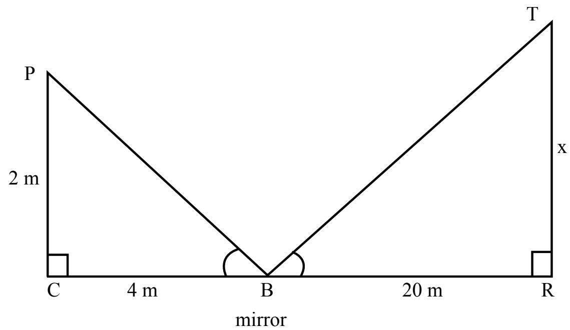
$iii)$ Adding (i) & (ii)

$$b^2 + c^2 = \left(p^2 + a^2/4 + ax\right) + \left(p^2 - ax + a^2/4\right)$$
$$= 2p^2 + 2\,a^2/4$$
$$= 2p^2 + a^2/2$$

Hence proved

7. A man whose eye-level is 2 m above the ground wishes to find the height of a tree. He places a mirror horizontally on the ground 20 m from the tree and finds that if he stands at a point C which is 4 m from the mirror B, he can see the reflection of the top of the tree. How height is the tree?

Solution :



Assume that the man & tree are standing up on a straight line

$\therefore \angle PBC = \angle TBR$

$\Delta PCB \sim \Delta TRB$

$\therefore$ Corresponding sides are proportional

$$(ie)\quad \frac{TR}{PC} = \frac{RB}{BC}$$

$$\Rightarrow \quad \frac{x}{2} = \frac{20}{4}$$

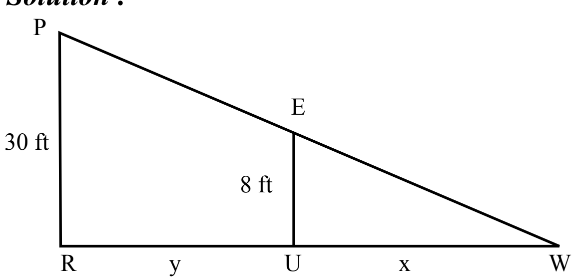
$$\Rightarrow \quad \frac{x}{2} = 5$$

$$\Rightarrow \quad x = 10\ m$$

$\therefore$ Height of the tree = 10 m

8. An emu which is 8 ft tall is standing at the foot of a pillar which is 30 ft high. It walks away from the pillar. The shadow of the emu falls beyond emu. What is the relation between the length of the shadow and the distance from the emu to the pillar?

Solution :



PR = 30 ft = height of pillar

RU = 8 ft = height of emu

UW = x = Shadow of emu,

y $\rightarrow$ distance between pillar & bird

$\Delta EUW \sim \Delta PWR$

$$\therefore \frac{x}{x+y} = \frac{\cancel{8}^{\,4}}{\cancel{30}_{\,15}} \quad \Rightarrow \quad 15x = 4x + 4y$$

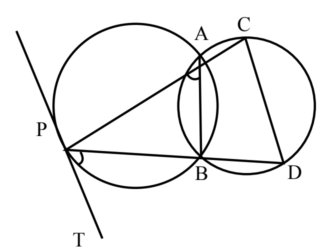
$$\Rightarrow \quad 11x = 4y$$

$$\Rightarrow \quad x = \frac{4}{11}y$$

$\therefore$ Length of the shadow = $\frac{4}{11}$ (Distance between pillar & emu)

9. Two circles intersect at A and B. From a point P on one of the circles lines PAC and PBD are drawn intersecting the second circle at C and D. Prove that CD is parallel to the tangent at P.

Solution :



PT is a tangent at P.

$\therefore \angle TPB = \angle PAB$ (angles in alternate segment)

$\angle PAB = \angle ACD$ (Exterior angle of a cyclic quad. ABCD is equal to interior opp. angles)

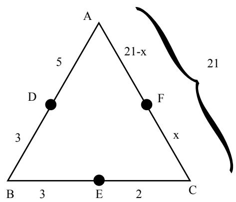
$\therefore \angle TPB = \angle ACD$

$\therefore$ Alternate interior angles are equal

$\therefore$ CD and PT are parallel.

10. Let ABC be a triangle and D,E,F are points on the respective sides AB, BC, AC (or their extensions). Let AD : DB = 5 : 3 , BE : EC = 3 : 2 and AC = 21 . Find the length of the line segment CF.

Solution :



By Ceva's theorem,

$$\frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CF}{FA} = 1$$

$$\Rightarrow \frac{5}{3}\times\frac{3}{2}\times\frac{x}{21-x} = 1$$

$$\Rightarrow \quad \frac{x}{21-x} = \frac{2}{5}$$

$$\Rightarrow \quad 5x = 42-2x$$

$$\Rightarrow \quad 7x = 42$$

$$\therefore \quad x=6$$

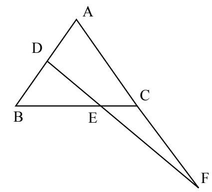
$$\therefore \quad CF = 6$$

PROBLEMS FOR PRACTICE

1. PQ is a diameter of a circle and PA is a chord. The tangent at A meets PQ produced at B. If $\angle QPA = 35^0$, find $\angle QBA$.
(Ans : 20^0)

2. The medians AD and BE of a ΔABC intersect at G. The line through G parallel to BD. intersects AC at K. Prove that AC = 6EK.

3. In the figure, $\angle BED = \angle BDE$ and E is the middle point of BC. Prove that $\frac{AF}{CF} = \frac{AD}{BE}$



4. In ΔABC, $B = 90^0$, AD, CE are two medians drawn from A and C respectively. If AC = 5 cm, AD = $\frac{3\sqrt{5}}{2}$ find CE.
(Ans : $2\sqrt{5}$ cm)

5. In ΔABC, $\angle C = 90^0$, P and Q are the points of the sides CA and CB respectively which divide the sides in the ratio 2 : 1. Prove that $9AQ^2 = 9AC^2 + 4BC^2$.

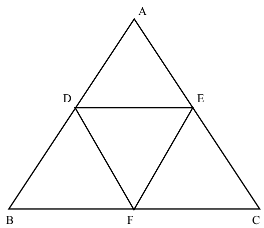
6. A vertical row of trees 12 m long cases a shadow 8 m long on the ground. At the same time, a tower casts the shadow 40 m long on the ground. Find the heightof the tower.
(Ans : 60m)

7. The area of 2 similar triangles ΔABC, ΔPQR are 25 cm^2, 49cm^2 respectively. If QR = 9.8cm, find BC.
(Ans : 7 cm)

8. In ΔABC, B = 90^0, D is the mid point of BC. Prove that $AC^2 = 4AD^2 - 3AB^2$.

9. Two isosceles triangles have equal vertical angles and their areas are in the ratio of 16 : 25. Find the ratio of their corresponding heights. **(Ans : 4 : 5)**

10. In the fig, AD = 3 cm, AE = 5 cm, BD = 4cm, CE = 4 cm, CF = 2 cm,BF = 2.5 cm, Find the pair of parallel lines & hence their lengths.



(Ans : $\frac{28}{9}$ cm ; 7 cm)

11. In an equilateral ΔABC, E is any point on BC such that BE = $\frac{1}{4}$ BC. Prove that 16 $AE^2 = 13\ AB^2$.

12. In ΔPQR, Qx is the bisector of $\angle Q$ meeting PR in x. If PQ : QR = 3 : 5, XR = 15 cm, Find PR.

(Ans : 29 cm)

13. $\Delta ABC \sim \Delta DEF$, If BC = 2 cm, EF = 5 cm and area of ΔDEF = 50, Find area of ΔABC

(Ans : 8)

14. In ΔABC, D is a point on AB, CD = 9 cm, DM = 6 cm, BC = 12 cm, $\angle CAB = \angle BCD$. Find the perimeter of ΔADC.

(Ans 45 cm)

15. A vertical pillar is bent at a height of 2.4 m and its upper end touches the ground at a distance of 1.8 m from its other end on the ground. Find the height of the pillar.

(Ans : 5.4 m)

16. Construct a ΔPQR, such that QR = 5.1 cm, $\angle P = 60^0$, attitude from P to QR is 3.2 cm.

17. Construct a ΔABC having base 6 cm, vertical angle 60^0 and median in rough the vertex is 4 cm.

18. Construct a Δ of base 5 cm and vertical angle 120^0 such that the bisector of the vertical angle meets the base at a point 3 cm.

19. Draw the tangents to a circle whose diameter is 10 cm from a point 13 cm from its centre.

20. Draw a circle of diameter 8 cm. Take a point Q on it. Without using the centre of the circle, draw a tangent to the circle at the point P.

CHAPTER 5 COORDINATE GEOMETRY

I. AREA OF A TRIANGLE AND QUADRILATERAL :

Key Points

- ✓ The area of Δ ABC is the absolute value of the expression

$$=\frac{1}{2}\{(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$$

- ✓ The vertices A(x_1, y_1), B (x_2, y_2) and C (x_3, y_3) of ΔABC are said to be "taken in order" if A, B, C are taken in counter-clock wise direction.
- ✓ The following pictorial representation helps us to write the above formula very easily.

$$\frac{1}{2}\left\{\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{array}\right\}$$

Area of ΔABC = $\frac{1}{2}\{(x_1y_2+x_2y_3+x_3y_1)-(x_2y_1+x_3y_2+x_1y_3)\}$ sq. units.

- ✓ Three distinct points A(x_1, y_1), B (x_2, y_2) and C (x_3, y_3) will be collinear if and only if area of ΔABC = 0 .
- ✓ Area of the quadrilateral ABCD = $\frac{1}{2}\{(x_1-x_3)(y_2-y_4)-(x_2-x_4)(y_1-y_3\}$ sq. units.

Example 5.1

Find the area of the triangle whose vertices are (-3,5) , (5,6) and (5,-2).

Solution :

Plot the points in a rough diagram and take them in counter-clockwise order.

Let the vertices be A(–3, 5) , B(5, –2), C(5,6)

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ (x_1, y_1) & (x_2, y_2) & (x_3, y_3) \end{array}$$

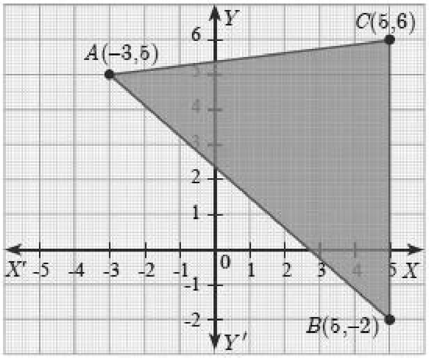
The area ΔABC is

$$=\frac{1}{2}\{(x_1y_2+x_2y_3+x_3y_1)-(x_2y_1+x_3y_2+x_1y_3)\}$$

$$=\frac{1}{2}\{(6+30+25)-(25-10-18)\}$$

$$=\frac{1}{2}\{61+3\}$$

$$=\frac{1}{2}(64)=32 \text{ sq. units}$$



Example 5.2

Show that the points P(–1.5, 3) , Q(6, –2) , R(–3, 4) are collinear.

Solution :

The points are P(– 1.5, 3) , Q(6, –2) , R(–3, 4)

Area of ΔPQR

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}$$

$$= \frac{1}{2}\{(3+24-9)-(18+6-6)\}$$

$$= \frac{1}{2}\{18-18\} = 0$$

Therefore, the given points are collinear.

Example 5.3

If the area of the triangle formed by the vertices A(–1, 2) , B(k , –2) and C(7, 4) (taken in order) is 22 sq. units, find the value of *k*.

Solution :

The vertices are A(–1, 2) , B(k , –2) and C(7, 4)

Area of triangle ABC is 22 sq.units

$$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 22$$

$$\frac{1}{2}\{(2+4k+14)-(2k-14-4)\} = 22$$

$$2k + 34 = 44$$

gives $2k = 10$ so $k = 5$

Example 5.4

If the points P(–1, – 4), Q (b, c) and R(5, –1) are collinear and if 2b + c = 4 , then find the values of b and c.

Solution :

Since the three points P(-1,-4), Q(b,c) and R(5, –1) are collinear,

Area of triangle PQR = 0

$$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 0$$

$$\frac{1}{2}\{(-c-b-20)-(-4b+5c+1)\} = 0$$

$$-c-b-20+4b-5c-1 = 0$$

b – 2c = 7 — (1)

Also, 2b + c = 4 — (2)

(from given information)

Solving (1) and (2) we get b = 3 , c = −2

Example 5.5

The floor of a hall is covered with identical tiles which are in the shapes of triangles. One such triangle has the vertices at (– 3, 2) , (–1, –1) and (1, 2) . If the floor of the hall is completely covered by 110 tiles, find the area of the floor.

Solution :

Vertices of one triangular tile are at (– 3, 2) , (–1, –1) and (1, 2)

Area of this tile

$$= \frac{1}{2}\{(3-2+2)-(-2-1-6)\} \text{ sq. units}$$

$$= \frac{1}{2}(12) = 6 \text{ sq. units}$$

Since the floor is covered by 110 triangle shaped identical tiles,

Area of floor = 110×6 = 660 sq.units

Example 5.6

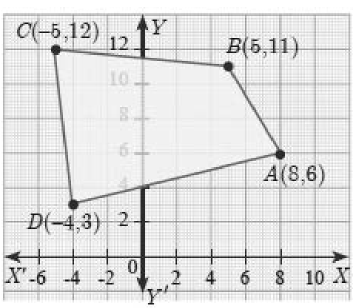
Find the area of the quadrilateral formed by the points (8, 6), (5, 11), (–5,12) and (– 4, 3) .

Solution :

Before determining the area of quadrilateral, plot the vertices in a graph.

Let the vertices be A(8,6), B(5,11), C(–5, 12) and D(– 4, 3)

Therefore, area of the quadrilateral ABCD



$$-\begin{Bmatrix}(x_1y_2+x_2y_3+x_3y_4+x_4y_1)\\-(x_2y_1+x_3y_2+x_4y_3+x_1y_4)\end{Bmatrix}$$

$$= -\{(88+60-15-24)-(30-55-48+24)\}$$

$$= -\{109+49\}$$

$$= -\{158\} = 79 \text{ sq. units}$$

Example 5.7

The given diagram shows a plan for constructing a new parking lot at a campus. It is estimated that such construction would cost ₹1300 per square feet. What will be the total cost for making the parking lot?

Solution :

The parking lot is a quadrilateral whose vertices are at A(2,2), B(5,5) , C(4,9) and D(1,7)

Therefore, Area of parking lot

$$= \frac{1}{2}\{(10+45+28+2)-(10+20+9+14)\}$$

$$= \frac{1}{2}\{85-53\}$$

$$\text{Use : } \frac{1}{2}\begin{Bmatrix}2 & 5 & 4 & 1 & 2\\ 2 & 5 & 9 & 7 & 2\end{Bmatrix}$$

$$= \frac{1}{2}(32) = 16 \text{ sq. units.}$$

So, area of parking lot = 16 sq. feets

Construction rate per square feet= ₹1300

Therefore, total cost for constructing the parking lot = 16 × 1300 = ₹20800

EXERCISE 5.1

1. Find the area of the triangle formed by the points

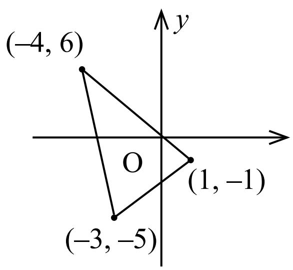
(i) (1, –1), (–4, 6) and (–3, –5)

(ii) (–10, – 4), (–8, –1) and (–3, –5)

Solution:

i) Given points are (1, –1), (–4, 6), (–3, –5)

Taking A, B, C in anti clockwise direction.



Let A (1, –1), B (–4, 6), C (–3, –5)

$\therefore$ Area of triangle ABC

$$= \frac{1}{2}\begin{Bmatrix} 1 & -4 & -3 & 3 \\ -1 & 6 & -5 & 1 \end{Bmatrix}$$

$$= \frac{1}{2}\{(6 + 20 + 3) - (4 - 18 - 5)]$$

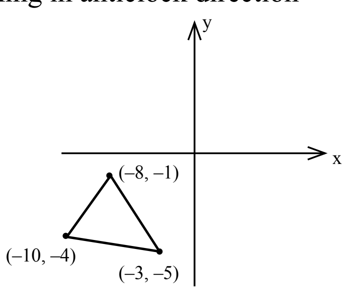
$$= \frac{1}{2}[29 + 19]$$

$$= \frac{1}{2}(48)$$

$$= 24 \text{ sq. units}$$

(ii) Given points are (–10, –4), (–8, –1), (–3, –3)

Taking in anticlock direction



Let A (–10, –4), B (–3, –5), C (–8, –1)

$\therefore$ Area of triangle ABC

$$= \frac{1}{2}\begin{Bmatrix} -10 & -3 & -8 & -10 \\ -4 & -5 & -1 & -4 \end{Bmatrix}$$

$$= \frac{1}{2}[(50+3+32)-(12+40+10)]$$

$$= \frac{1}{2}[85-62]$$

$$= \frac{23}{2}$$

$$= 11.5 \text{ sq.units}$$

---

2. Determine whether the sets of points are collinear ?

(i) $\left\{-\frac{1}{2}, 3\right\}$, (–5, 6) and (–8, 8)

(ii) (*a*, *b* + *c*), (*b*, *c* + *a*) and (*c*, *a* + *b*)

Solution :

Area of triangle formed by 3 points

$$= \frac{1}{2}\begin{bmatrix} -1/2 & -5 & -8 & -1/2 \\ 3 & 6 & 8 & 3 \end{bmatrix}$$

$$= \frac{1}{2}[(-3-40-24)-(-15-48-4)]$$

$$= \frac{1}{2}[-67-(67)]$$

$$= \frac{1}{2}(0) = 0$$

$\therefore$ The 3 points are collinear.

ii) Given points are A (*a, b* + *c*) , B (*b, c* + *a*), C (*c, a* + *b*)

Areaof tirangle by 3 points

$$= \frac{1}{2}\begin{bmatrix} a & b & c & a \\ b+c & c+a & a+b & b+c \end{bmatrix}$$

$$= \frac{1}{2}[(ac + a^2 + ab + b^2 + bc + c^2)$$
$$- (b^2 + bc + c^2 + ac + a^2 + ab)]$$
$$= \frac{1}{2}[(a^2 + b^2 + c^2 + ab + bc + ca)$$
$$- (a^2 + b^2 + c^2 + ab + bc + ca)]$$
$$= \frac{1}{2}[0]$$

∴ The 3 points are collinear.

3. Vertices of given triangles are taken in order and their areas are provided below. In each case, find the value of 'p'.

S.No.	Vertices	Area (sq. units)
(i)	(0, 0), (p, 8), (6, 2)	20
(ii)	(p, p), (5, 6), (5, –2)	32

Solution :

Area of triangle formed by 3 points

$$= \frac{1}{2}\begin{bmatrix} 0 & p & 6 & 0 \\ 0 & 8 & 2 & 0 \end{bmatrix}$$

$\Rightarrow$ [0 + 2p + 0] – [0 + 48 + 0] = 40

$\Rightarrow$ 2p – 48 = 40

$\Rightarrow$ 2p = 88

∴ p = 44

ii) Given vertices are (p, p), (5, 6), (5, –2), Area = 32 sq. units.

Area of triangle formed by 3 points

$$= \frac{1}{2}\begin{bmatrix} p & 5 & 5 & p \\ p & 6 & -2 & p \end{bmatrix} = 32$$

$\Rightarrow$ (6p – 10 + 5p) – (5p + 30 – 2p) = 64

$\Rightarrow$ (11p – 10) – (3p + 30) = 64

$\Rightarrow$ 8p = 104

p = $\frac{104}{8}$

p = 13

4. In each of the following, find the value of '*a*' for which the given points are collinear. (i) (2, 3), (4, *a*) and (6, –3) (ii) (*a*, 2 – 2*a*), (–a+1, 2*a*) and (– 4 – *a*, 6 – 2*a*)

Solution :

i) Given, 3 points (2, 3), (4, *a*), (6, –3) are collinear.

$$\text{ie, } \frac{1}{2}\begin{bmatrix} 2 & 4 & 6 & 2 \\ 3 & a & -3 & 3 \end{bmatrix} = 0$$

$\Rightarrow (2a - 12 + 18) - (12 + 6a - 6) = 0$

$\Rightarrow (2a + 6) - (6a + 6) = 0$

$\Rightarrow -4a = 0$

$\Rightarrow a = 0$

ii) Given 2 points $(a, 2 - 2a)$, $(-a + 1, 2a)$, $(-4 - a, 6 - 2a)$ are collinear.

∴ Area of triangle formed by 3 points is 0.

$$\text{ie, } \frac{1}{2}\begin{bmatrix} a & -a+1 & -4-a & a \\ 2-2a & 2a & 6-2a & 2-2a \end{bmatrix} = 0$$

$$\Rightarrow [2a^2 + (-a+1)(6-2a) + (-4-a)(2-2a)]$$
$$-[(2-2a)(-a+1) + 2a(-4-a) + a(6-2a)] = 0$$
$$\Rightarrow (2a^2 + 2a^2 - 8a + 6 + 2a^2 + 6a - 8)$$
$$-[2a^2 - 4a + 2 - 2a^2 - 8a + 6a - 2a^2] = 0$$
$$\Rightarrow (6a^2 - 2a - 2) - (-2a^2 - 6a + 2) = 0$$
$$\Rightarrow 8a^2 + 4a - 4 = 0$$
$$\Rightarrow 2a^2 + a - 1 = 0$$
$$\Rightarrow a = -1, \frac{1}{2}$$

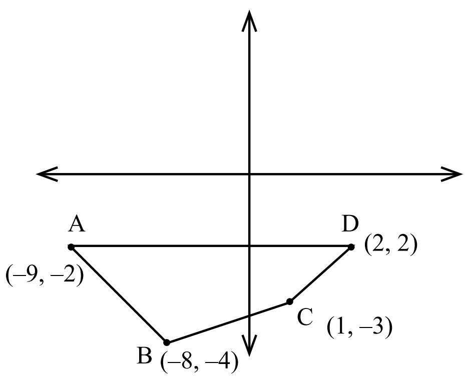
5. Find the area of the quadrilateral whose vertices are at
(i) (–9, –2), (–8, –4), (2, 2) and (1, –3) (ii) (–9, 0), (–8, 6), (–1, –2) and (–6, –3)

Solution :

(i) Given vertices of quadrilateral are (–9, –2), (–8, –4), (2, 2), (1, –3)

First we plot the points in the plane

Let A (–9, –2), B (–8, –4), C (1, –3), D (2, 2) ∴ Area of qadrilateral



$$=\frac{1}{2}\begin{bmatrix}-9 & -8 & 1 & 2 & -9\\ -2 & -4 & -3 & 2 & -2\end{bmatrix}$$

$$=\frac{1}{2}[(36+24+2-4)-(16-4-6-18)]$$

$$=\frac{1}{2}[58-(-12)]$$

$$=\frac{1}{2}[70]$$

$= 35$ sq. units

ii) Given vertices of quadrilateral are (–9, 0), (–8, 6), (–1, –2), (–6, –3)

First, we plot the points in the plane.

Let A (–8, 6), B (–9, 0), C (–6, –3), D (–1, –2)

Area of quadrilateral

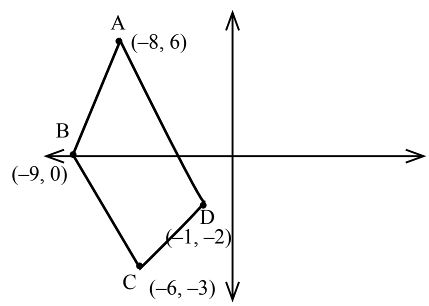
$$=\frac{1}{2}\begin{bmatrix}-8 & -9 & -6 & -1 & -8\\ 6 & 0 & -3 & -2 & 6\end{bmatrix}$$

$$=\frac{1}{2}[(0+27+12-6)-(-54+0+3+16)]$$

$$=\frac{1}{2}[33-(-35)]$$

$$=\frac{1}{2}[68]$$

$= 34$ sq. units



6. Find the value of k, if the area of a quadrilateral is 28 sq.units, whose vertices are (–4, –2), (–3, k), (3, –2) and (2, 3)

Solution :

$$=\frac{1}{2}\begin{bmatrix}-4 & -3 & 3 & 2 & -4\\ -2 & k & -2 & 3 & -2\end{bmatrix}=28$$

$\Rightarrow$ (–4k + 6 + 9 – 4) – (6 + 3k – 4 – 12) = 56

$\Rightarrow$ (11 – 4k) – (3k – 10) = 56

$\Rightarrow$ 21 – 7k = 56

∴ 7k = – 35

k = – 5

7. If the points A(-3, 9) , B (*a, b*) and C(4,-5) are collinear and if *a* + *b* = 1 , then find *a* and *b*.

Solution :

Given, A (– 3, 9), B (*a, b*), C (4, –5) are collinear and *a* + *b* = 1. — (1)

Area of triangle formed by 3 points = 0.

ie) $\frac{1}{2}\begin{bmatrix} -3 & a & 4 & -3 \\ 9 & b & -5 & 9 \end{bmatrix} = 0$

$\Rightarrow \quad (-3b - 5a + 36) - (9a + 4b + 15) = 0$

$\Rightarrow \quad -5a - 3b + 36 - 9a - 4b - 15 = 0$

$\Rightarrow \quad -14a - 7b + 21 = 0$

$\Rightarrow \quad 2a + b - 3 = 0$

$\Rightarrow \quad 2a + 1 - a - 3 = 0$ (from (1))

$\therefore \Rightarrow \quad a = 2 \quad b = -1$

---

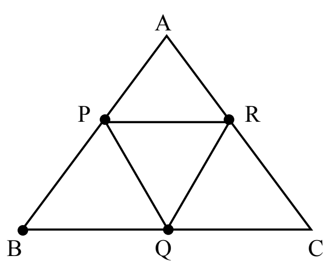
8. Let P(11,7), Q(13.5, 4) and R(9.5, 4) be the mid- points of the sides AB, BC and AC respectively of ΔABC. Find the coordinates of the vertices A, B and C. Hence find the area of ΔABC and compare this with area of ΔPQR.

Solution :

In ΔABC, given that P, Q, R are the mid points of AB, BC, CA respectively.

P (11, 7), Q (13.5, 4), R (9.5, 4)

Let A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) be the vertices.



$\therefore \frac{x_1 + x_2}{2} = 11, \ \frac{y_1 + y_2}{2} = 7$

$\Rightarrow x_1 + x_2 = 22, \quad y_1 + y_2 = 14$ .....(1)

$\frac{x_2 + x_3}{2} = \frac{27}{2}, \ \frac{y_2 + y_3}{2} = 4$

$\Rightarrow x_2 + x_3 = 27, \quad y_2 + y_3 = 8$ .....(2)

$\frac{x_1 + x_3}{2} = \frac{19}{2}, \ \frac{y_1 + y_3}{2} = 4$

$\Rightarrow x_1 + x_3 = 19, \quad y_1 + y_3 = 8$ .....(3)

$\therefore$ Adding (1), (2) and (3)

$2(x_1 + x_2 + x_3) = 68, \quad 2(y_1 + y_2 + y_3) = 30$

$x_1 + x_2 + x_3 = 34, \quad y_1 + y_2 + y_3 = 15$

$22 + x_3 = 34, \quad 14 + y_3 = 15$

$x_3 = 12 \quad y_3 = 1$

$\therefore$ C (12 , 1)

Also, (2) $\Rightarrow x_2 + 12 = 27, \ y_2 + 1 = 8$

$x_2 = 15 \quad y_2 = 7$

$\therefore$ B is (15, 7)

(1) $\Rightarrow x_1 + 15 = 22, \ y_1 + 7 = 14$

$x_1 = 7 \quad y_1 = 7$

$\therefore$ A is (7, 7)

Area of ΔABC

$= \frac{1}{2}\begin{bmatrix} 7 & 15 & 12 & 7 \\ 7 & 7 & 1 & 7 \end{bmatrix}$

$= \frac{1}{2}[(49 + 15 + 84) - (105 + 84 + 7)]$

$= \frac{1}{2}[148 - 196]$

$= \frac{1}{2}(-48)$

$= 24$ ($\because$ Area can't be –ve)

Area of ΔPQR

$$= \frac{1}{2}\begin{bmatrix} 11 & \frac{27}{2} & \frac{19}{2} & 11 \\ 7 & 4 & 4 & 7 \end{bmatrix}$$

$$= \frac{1}{2}[(44+54+66.5)-(94.5+38+44)]$$

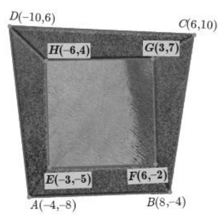
$$= \frac{1}{2}[164.5-176.5]$$

$$= \frac{1}{2}[-12]$$

$= 6$ (∵ Area can't be –ve)

∴ Area of ΔABC = 4 (Area of ΔPQR).

9. In the figure, the quadrilateral swimming pool shown is surrounded by concrete patio. Find the area of the patio.



Solution :

Required area of the patio = Area of portion ABCD – Area of portion EFGH

$$= \frac{1}{2}\begin{bmatrix} -4 & 8 & 6 & -10 & -4 \\ -8 & -4 & 10 & 6 & -8 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -3 & 6 & 3 & -6 & -3 \\ -5 & -2 & 7 & 4 & -5 \end{bmatrix}$$

$$= \frac{1}{2}[(16+80+36+80)-(-64-24-100-24)]$$

$$-\frac{1}{2}[(6+42+12+30)-(-30-6-42-12)]$$

$$= \frac{1}{2}[212-(-212)] - \frac{1}{2}[90-(-90)]$$

$$= \frac{1}{2}[424] - \frac{1}{2}[180]$$

10. A triangular shaped glass with vertices at A(-5,-4) , B(1,6) and C(7,-4) has to be painted. If one bucket of paint covers 6 square feet, how many buckets of paint will be required to paint the whole glass, if only one coat of paint is applied.

Solution :

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{bmatrix} -5 & 1 & 7 & -5 \\ -4 & 6 & -4 & -4 \end{bmatrix}$$

$$= \frac{1}{2}[(-30-4-28)-(-4+42+20)]$$

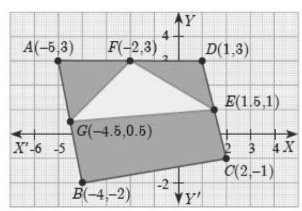
$$= \frac{1}{2}[-62-(58)]$$

$$= \frac{1}{2}[-120]$$

= 60 sq. units (Area can't be –ve).

∴ No. of paint cans needed = $\frac{60}{6}$ = 10

11. In the figure, find the area of (i) triangle AGF (ii) triangle FED (iii) quadrilateral BCEG.



Solution :

i) Area of ΔAGF

$$= \frac{1}{2}\begin{bmatrix} -5 & -9/2 & -2 & -5 \\ 3 & 1/2 & 3 & 3 \end{bmatrix}$$

$$= \frac{1}{2}[(-2.5-13.5-6)-(-13.5-1-15)]$$

$$= \frac{1}{2}[(-22)-(-29.5)]$$

$$= \frac{1}{2}[7.5]$$

= 3.75 sq. units

ii) Area of ΔFED

$$=\frac{1}{2}\begin{bmatrix}-2 & 3/2 & 1 & -2\\ 3 & 1 & 3 & 3\end{bmatrix}$$

$$=\frac{1}{2}[(-2+4.5+3)-(-4.5+1-6)]$$

$$=\frac{1}{2}[5.5-(-0.5)]$$

$$=\frac{1}{2}[6]$$

$= 3$ sq. units

iii) Area of quadrilateral BCEG

$$=\frac{1}{2}\begin{bmatrix}-4 & 2 & 3/2 & -9/2 & -4\\ -2 & -1 & 1 & 1/2 & -2\end{bmatrix}$$

$$=\frac{1}{2}[(4+2+0.75+9)-(-4-1.5-4.5-2)]$$

$$=\frac{1}{2}[15.75+12]$$

$$=\frac{27.75}{2}$$

$= 13.875$

□13.88 sq. units

II. INCLINATION AND SLOPE OF STRAIGHT LINE

Key Points

✓ The inclination of a line or the angle of inclination of a line is the angle which a straight line makes with the positive direction of X axis measured in the counter-clockwise direction to the part of the line above the X axis.

✓ The inclination of X axis and every line parallel to X axis is $0°$.

✓ The inclination of Y axis and every line parallel to Y axis is $90°$.

✓ If θ is the angle of inclination of a non-vertical straight line, then $\tan\theta$ is called the slope or gradient of the line and is denoted by *m*.

✓ The slope of the straight line is $m = \tan\theta,\ 0 \le 180°\ ,\ \theta \ne 90°$.

✓ The slope of the line through (x_1, y_1) and (x_2, y_2) with $x_1 \ne y_1$ is $\dfrac{y_2-y_1}{x_2-x_1}$.

✓ If $\theta = 0^o$, the line is parallel to the positive direction of X axis.

✓ If $0 < \theta < 90^o$, the line has positive slope.

✓ If $90^o < \theta < 180^o$, the line has negative slope.

✓ If $\theta = 180^o$, the line is parallel to the negative direction of X axis.

✓ If $\theta = 90^o$, the slope is undefined.

✓ Non vertical lines are parallel if and only if their slopes are equal.

✓ Two non-vertical lines with slopes m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$.

Example 5.8

(i) What is the slope of a line whose inclination is 30^o ?

(ii) What is the inclination of a line whose slope is $\sqrt{3}$?

Solution :

(i) Here $\theta = 30^o$

Slope $m = \tan\theta$

Therefore, slope $m = \tan 30^o = \frac{1}{\sqrt{3}}$

(ii) Given $m = \sqrt{3}$, let θ be the inclination of the line

$$\tan\theta = \sqrt{3}$$

We get, $\theta = 60^o$

Example 5.9

Find the slope of a line joining the given points

(i) $(-6, 1)$ and $(-3, 2)$ (ii) $\left(-\frac{1}{3}, \frac{1}{2}\right)$ and $\left(\frac{2}{7}, \frac{3}{7}\right)$ (iii) $(14, 10)$ and $(14, -6)$

Solution :

(i) $(-6, 1)$ and $(-3, 2)$

$$\text{The slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{-3+6} = \frac{1}{3}.$$

(ii) $\left(-\frac{1}{3}, \frac{1}{2}\right)$ and $\left(\frac{2}{7}, \frac{3}{7}\right)$

$$\text{The slope} = \frac{\frac{3}{7} - \frac{1}{2}}{\frac{2}{7} + \frac{1}{3}} = \frac{\frac{6-7}{14}}{\frac{6+7}{21}}$$

$$= -\frac{1}{14} \times \frac{21}{13} = -\frac{3}{26}.$$

(iii) $(14, 10)$ and $(14, -6)$

$$\text{The slope} = \frac{-6-10}{14-14} = \frac{-16}{0}.$$

The slope is undefined.

Example 5.10

The line r passes through the points $(-2, 2)$ and $(5, 8)$ and the line s passes through the points $(-8, 7)$ and $(-2, 0)$. Is the line r perpendicular to s ?

Solution :

The slope of line r is $m_1 = \frac{8 \quad 2}{5 \quad 2} \quad \frac{6}{7}$

The slope of line s is $m_2 = \frac{0-7}{-2+8} = \frac{-7}{6}$

The product of slopes $= \frac{6}{7} \times \frac{-7}{6} = -1$

That is, $m_1 m_2 = -1$

Therefore, the line r is perpendicular to line s.

Example 5.11

The line p passes through the points $(3, -2)$, $(12, 4)$ and the line q passes through the points $(6, -2)$ and $(12, 2)$. Is p parallel to q ?

Solution :

The slope of line p is $m_1 = \frac{4+2}{12-3} = \frac{6}{9} = \frac{2}{3}$

The slope of line q is $m_2 = \frac{2+2}{12-6} = \frac{4}{6} = \frac{2}{3}$

Thus, slope of line p = slope of line q.

Therefore, line p is parallel to the line q.

Example 5.12

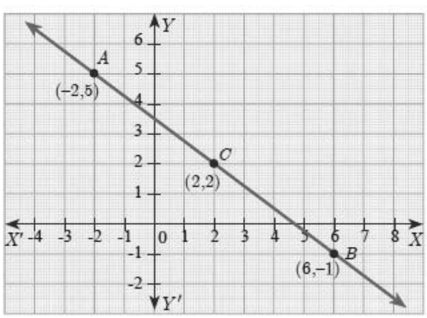
Show that the points $(-2, 5)$, $(6, -1)$ and $(2, 2)$ are collinear.

Solution :

The vertices are A $(-2,5)$, B $(6, -1)$ and C $(2, 2)$.

$$\text{Slope of AB} = \frac{-1-5}{6+2} = \frac{-6}{8} = \frac{-3}{4}$$

$$\text{Slope of BC} = \frac{2+1}{2-6} = \frac{3}{-4} = \frac{-3}{4}$$



We get, Slope of AB = Slope of BC.

Therefore, the points A, B, C all lie in a same straight line.

Hence the points A, B and C are collinear.

Example 5.13

Let A (1, –2), B (6, –2), C(5, 1) and D (2, 1) be four points.

(i) Find the slope of the line segments (a) AB (b) CD

(ii) Find the slope of the line segments (a) BC (b) AD

(iii) What can you deduce from your answer?

Solution :

(i) (a) Slope of AB $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2+2}{6-1} = 0$

(b) Slope of CD $= \dfrac{1-1}{2-5} = \dfrac{0}{-3} = 0$

(ii) (a) Slope of BC $= \dfrac{1+2}{5-6} = \dfrac{3}{-1} = -3$

(b) Slope of AD $= \dfrac{1+2}{2-1} = \dfrac{3}{1} = 3$

(iii) The slope of AB and CD are equal so AB, CD are parallel.

Similarly the lines AD and BC are not parallel, since their slopes are not equal.

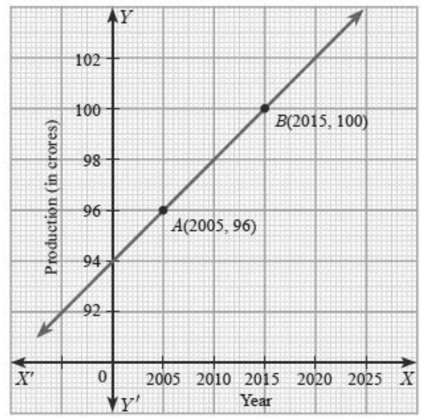
So, we can deduce that the quadrilateral ABCD is a trapezium.

Example 5.14

Consider the graph representing growth of population (in crores). Find the slope of the line AB and hence estimate the population in the year 2030?

Solution :

The points A(2005,96) and B(2015,100) are on the line AB.



Slope of AB $= \dfrac{100-96}{2015-2005} = \dfrac{4}{10} = \dfrac{2}{5}$

Let the growth of population in 2030 be k crores.

Assuming that the point C(2030,k) is on AB,

we have, slope of AC = slope of AB

$$\frac{k-96}{2030-2005} = \frac{2}{5} \text{ gives } \frac{k-96}{25} = \frac{2}{5}$$

$$k - 96 = 10$$

$$k = 106$$

Hence the estimated population in 2030 = 106 Crores.

Example 5.15

Without using Pythagoras theorem, show that the points (1,-4) , (2, –3) and (4, –7) form a right angled triangle.

Solution :

Let the given points be A(1, – 4) , B(2, –3) and C(4, –7) .

$$\text{The slope of AB} = \frac{-3+4}{2-1} = \frac{1}{1} = 1$$

$$\text{The slope of BC} = \frac{-7+3}{4-2} = \frac{-4}{2} = -2$$

$$\text{The slope of AC} = \frac{-7+4}{4-1} = \frac{-3}{3} = -1$$

Slope of AB slope of AC = (1)(−1) = −1

AB is perpendicular to AC. $\angle A = 90°$

Therefore, ΔABC is a right angled triangle.

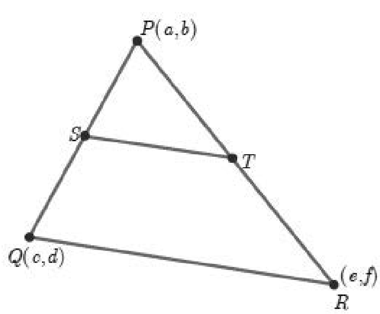
Example 5.16

Prove analytically that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is equal to half of its length.

Solution :

Let P (*a, b*) Q (*c, d*) and R (e, f) be the vertices of a triangle.

Let S be the mid-point of PQ and T be the mid-point of PR



$$\text{Therefore } S = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

$$\text{and T} = \left(\frac{a+e}{2}, \frac{b+f}{2}\right)$$

$$\text{Now, slope of ST} = \frac{\frac{b+f}{2} - \frac{b+d}{2}}{\frac{a+e}{2} - \frac{a+c}{2}} = \frac{f-d}{e-c}$$

$$\text{And slope of QR} = \frac{f-d}{e-c}$$

Therefore, ST is parallel to QR. (since, their slopes are equal)

Also

$$\text{ST} = \sqrt{\left(\frac{a+e}{2} - \frac{a+c}{2}\right)^2 + \left(\frac{b+f}{2} - \frac{b+d}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(e-c)^2 + (f-d)^2}$$

$$\text{ST} = \frac{1}{2}QR$$

Thus ST is parallel to QR and half of it.

EXERCISE 5.2

1. What is the slope of a line whose inclination with positive direction of *x*-axis is
(i) 90° **(ii) 0°**

Solution :

i) $\theta = 90°$ m = tan 90° = undefined.

ii) $\theta = 0°$ m = tan0° = 0

2. What is the inlination of a line whose slope is (i) 0 **(ii) 1**

Solution :

i) m = 0 $\Rightarrow$ tan θ = 0 $\therefore$ $\theta = 0°$

ii) m = 1 $\Rightarrow$ tan θ = 1 $\therefore$ $\theta = 45°$

3. Find the slope of a line joining the points
(i) (5, $\sqrt{5}$) with the origin
(ii) (sin θ, – cos θ) and (– sin θ, cos θ)

Solution :

i) Slope of the line joining (5, $\sqrt{5}$), (0, 0)

$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-\sqrt{5}}{0-5} = \frac{1}{\sqrt{5}}$$

$$\therefore \text{Slope} = \frac{1}{\sqrt{5}}$$

ii) Slope of line joining (sin θ, – cos θ) (– sin θ, cos θ)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\cos\theta + \cos\theta}{-\sin\theta - \sin\theta}$$
$$= \frac{2\cos\theta}{-2\sin\theta}$$
$$= -\cot\theta$$

4. What is the slope of a line perpendicular to the line joining A (5, 1) and P where P is the mid-point of the segment joining (4, 2) and (–6, 4).

Solution :

Given P is the midpoint of (4, 2), (–6, 4)

$$\Rightarrow P = \left(\frac{4-6}{2}, \frac{2+4}{2}\right)$$
= (–1, 3)

∴ Slope of the line joining A (5, 1), P (–1, 3)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-1-5}$$
$$= \frac{2}{-6}$$
$$= \frac{-1}{3}$$

∴ Slope of the line perpendicular to the line joining A and P is $\frac{-1}{m} = 3$

5. Show that the given points are collinear: (– 3, – 4), (7, 2) and (12, 5).

Solution :

Given points are A (–3, –4), B (7, 2), C (12, 5)

$$\text{Slope of AB} = \frac{2+4}{7+3}$$
$$= \frac{6}{10}$$
$$= \frac{3}{5}$$

$$\text{Slope of BC} = \frac{5-2}{12-7}$$
$$= \frac{3}{5}$$

∴ Slope of AB = Slope of BC

∴ AB and BC are parallel.

But B is the common point.

∴ A, B, C are collinear.

6. If the three points (3, –1), (*a*, 3) and (1, –3) are collinear, find the value of *a*.

Solution :

Given points A (3, –1), B (*a*, 3), C (1, –3) are collinear.

∴ Slope of AB = Slope of BC

$$\Rightarrow \frac{4}{a-3} = \frac{-6}{1-a}$$
$$\Rightarrow 4 - 4a = -6a + 18$$
$$\Rightarrow 2a = 14$$
$$a = 7$$

7. The line through the points (–2, *a*) and (9, 3) has slope $-\frac{1}{2}$. Find the value of *a*.

Solution :

Slope of the line joining (–2, *a*), (9, 3) = $-\frac{1}{2}$

$$\Rightarrow \frac{3-a}{9+2} = \frac{-1}{2}$$

$$\Rightarrow \frac{3-a}{11} = \frac{-1}{2}$$

$$\Rightarrow 6-2a = -11$$

$$\Rightarrow 2a = 17$$

$$\therefore a = \frac{17}{2}$$

8. The line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (*x*, 24). FInd the value of *x*.

Solution :

Slope of line joining (–2, 6), (4, 8)

$$m_1 = \frac{8-6}{4+2} = \frac{2}{6} = \frac{1}{3}$$

Slope of line joining (8, 12), (*x*, 24)

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since two lines are perpendicular,

$$\Rightarrow \frac{1}{3} \times \frac{12}{x-8} = -1$$

$$\Rightarrow \frac{4}{x-8} = -1$$

$$\Rightarrow -x+8 = 4$$

$$\Rightarrow x = 4$$

9. Show that the given points form a right angled triangle and check whether they satisfies pythagoras theorem

(i) A (1, – 4), B (2, –3) and C (4, – 7)

(ii) L (0, 5), M (9, 12) and N (3, 14)

Solution :

i) Given A (1, – 4), B (2, – 3), C (4, – 7)

$$\text{Slope of AB} = \frac{-3+4}{2-1} = \frac{1}{1} = 1$$

$$\text{Slope of BC} = \frac{-7+3}{4-2} = \frac{-4}{2} = -2$$

$$\text{Slope of CA} = \frac{-7+4}{4-1} = \frac{-3}{3} = -1$$

∴ Slope of AB × Slope of CA = 1 × – 1 = – 1

∴ AB is perpendicular to AC.

∴ ∠A = 90°

∴ ΔABC is a right angled triangle.

ii) Given L (0, 5), M (9, 12), N (3, 14)

$$\text{Slope of LM} = \frac{12-5}{9-0} = \frac{7}{9}$$

$$\text{Slope of MN} = \frac{14-12}{3-9} = \frac{2}{-6} = \frac{-1}{3}$$

$$\text{Slope of LN} = \frac{14-5}{3-0} = \frac{9}{3} = 3$$

∴ Slope of MN × Slope of LN

$$= \frac{-1}{3} \times 3$$

$$= -1$$

∴ MN is perpendicular to LN.

∴ ∠N = 90°

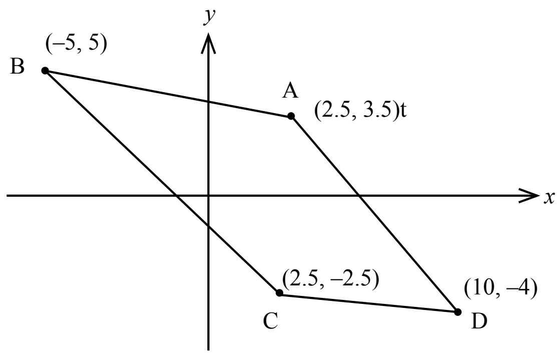
∴ ΔLMN is a right angled Δ.

10. Show that the given points form a parallelogram :

A (2.5, 3.5), B (10, – 4), C (2.5, – 2.5) and D (– 5, 5)

Solution :

Plot the points and taking in anticlockwise direction.



Let A (2.5, 3.5), B (– 5, 5), C (2.5, –2.5), D (10, –4)

$$\text{Slope of AB} = \frac{5-3.5}{-5-2.5} = \frac{1.5}{-7.5} = \frac{-1}{5}$$

$$\text{Slope of CD} = \frac{-4+2.5}{10-2.5} = \frac{-1.5}{7.5} = \frac{-1}{5}$$

$\therefore$ Slope of AB = Slope of CD.

$\therefore$ AB and CD are parallel.

Also,

$$\text{Slope of AD} = \frac{-4-3.5}{10-2.5} = \frac{-7.5}{7.5} = -1$$

$$\text{Slope of BC} = \frac{-2.5-5}{2.5+5} = \frac{-7.5}{7.5} = -1$$

$\therefore$ Slope of AD = Slope of BC.

$\therefore$ AD and BC are parallel.

$\therefore$ ABCD is a parallelogram.

11. If the points A (2, 2), B (–2, –3), C (1, –3) and D (*x*, *y*) form a parallelogram then find the value of *x* and *y*.

Solution :

Given A (2, 2), B (–2, –3), C (1, –3), D (x, y) form a parallelogram.

$\therefore$ Slope of AB = Slope of CD.

$$\Rightarrow \frac{-3-2}{-2-2} = \frac{y+3}{x-1}$$

$$\Rightarrow \frac{-5}{-4} = \frac{y+3}{x-1}$$

$$\Rightarrow \frac{5}{4} = \frac{y+3}{x-1}$$

$\Rightarrow 5x - 5 = 4y + 12$

$\Rightarrow 5x - 4y = 17$ — (1)

Also

Slope of AD = Slope of BC

$$\Rightarrow \frac{y-2}{x-2} = \frac{-3+3}{1+2}$$

$$\Rightarrow \frac{y-2}{x-2} = 0$$

$\Rightarrow y - 2 = 0$

$\Rightarrow \quad y = 2$

Sub. in (1)

$5x - 8 = 17$

$\Rightarrow 5x = 25$

$\therefore x = 5$

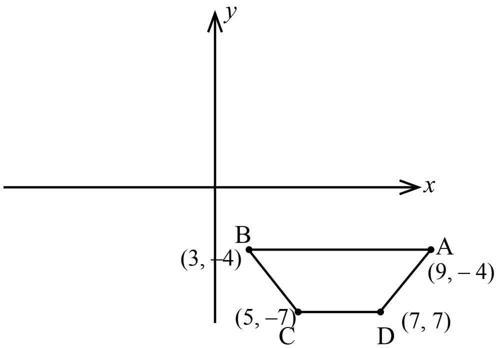
$\therefore x = 5, y = 2$

12. Let A (3, – 4), B (9, – 4), C (5, –7) and D (7, –7). Show that ABCD is a trapezium.

Solution :

Given points are (3, – 4), (9, –4), (5, –7), (7, –7)

Plotting the given points in a plane and taking in anti clockwise direction.



Let A (9, –4), B (3, –4), C (5, –7) D (7, –7)

Slope of AB = $\frac{-4+4}{3-9} = 0$

Slope of CD = $\frac{-7+7}{7-5} = 0$

∴ Slope of AB = Slope of CD.

∴ AB and CD are parallel.

Slope of AD = $\frac{-7+4}{7-9} = \frac{-3}{-2} = \frac{3}{2}$

Slope of BC = $\frac{-7+4}{5-3} = \frac{-3}{2}$

∴ Slope of AD ≠ Slope of BC

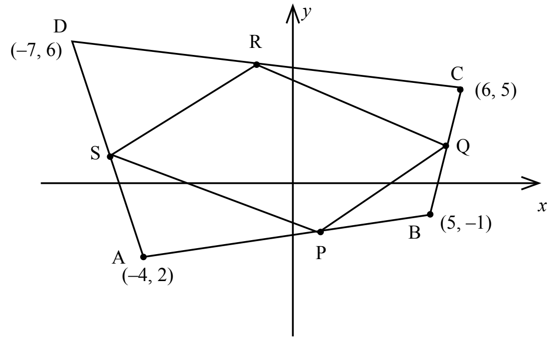
∴ One pair of opposite sides is equal.

∴ ABCD is a trapezium.

13. A quadrilateral has vertices at A (-4,-2) B(5, -1) , C(6, 5) and D(–7, 6). Show that the mid-points of its sides form a parallelogram.

Solution :

Given points are (– 4, – 2), (5, –1), (6, 5), (– 7, 6) which forms a quadrilateral.



Mid point of AB = $\left(\frac{-4+5}{2}, \frac{-2-1}{2}\right) = \left(\frac{1}{2}, \frac{-3}{2}\right)$P

Mid point of BC = $\left(\frac{5+6}{2}, \frac{-1+5}{2}\right) = \left(\frac{11}{2}, 2\right)$Q

Midpoint of CD = $\left(\frac{-7+6}{2}, \frac{6+5}{2}\right) = \left(\frac{-1}{2}, \frac{11}{2}\right)$R

Midpoint of AD = $\left(\frac{-7-4}{2}, \frac{6-2}{2}\right) = \left(\frac{-11}{2}, 2\right)$S

To prove : PQRS is a parallelogram.

$$\text{Slope of PQ} = \frac{2+\frac{3}{2}}{\frac{11}{2}-\frac{1}{2}} = \frac{7/2}{10/2} = \frac{7}{10}$$

$$\text{Slope of SR} = \frac{2-\frac{11}{2}}{\frac{-11}{2}+\frac{1}{2}} = \frac{-7/2}{-10/2} = \frac{7}{10}$$

$\therefore$ Slope of PQ = Slope of SR

$\therefore$ PQ and SR are parallel.

Also,

$$\text{Slope of PS} = \frac{2+\frac{3}{2}}{\frac{-11}{2}-\frac{1}{2}} = \frac{7/2}{-12/2} = \frac{-7}{12}$$

$$\text{Slope of QR} = \frac{11/2-2}{-1/2-11/2} = \frac{7/2}{-12/2} = \frac{-7}{12}$$

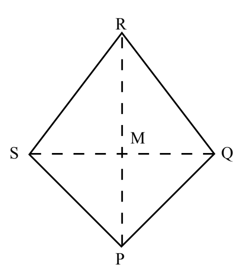
$\therefore$ Slope of PS = Slope of QR

$\therefore$ PS and QR are parallel.

$\therefore$ PQRS is a parallelogram.

14. PQRS is a rhombus. Its diagonals PR and QS intersect at the point M and satisfy QS = 2PR. If the coordinates of S and M are (1, 1) and (2, –1) respectively, find the coordinates of P.

Solution :



Given, In rhombus PQRS, diagonals PR and QS meet at M such that QS = 2PR.

Also, given S (1, 1), M (2, –1)

Let Q be (x, y)

Midpoint of SQ = M (∵ rhombus)

$$\left(\frac{x+1}{2}, \frac{y+1}{2}\right) = (2, -1)$$

$\Rightarrow x + 1 = 4 \qquad y + 1 = -2$

$\Rightarrow x = 3, \qquad y = -3$

$\therefore$ Q is (3, – 3)

Since QS = 2 PR

$QS^2 = 4 \,.\, PR^2$

$(3-1)^2 + (-3-1)^2 = 4 \,.\, PR^2$

$\therefore PR^2 = 5$

$\therefore PR = \sqrt{5}$

$\Rightarrow PM = \frac{\sqrt{5}}{2}, \qquad M(2, -1)$

Let P be (l, m)

III. EQUATIONS OF STRAIGHT LINES:

Key Points

- ✓ Equation of OY(Y axis) is $x = 0$.
- ✓ Equation of OX (X axis) is y = 0.
- ✓ Equation of a straight line parallel to X axis is $y = b$.
- ✓ If b > 0 , then the line y=b lies above the X axis, If b < 0 , then the line y=b lies below the X axis, If b = 0 , then the line y=b is the X axis itself.
- ✓ Equation of a Straight line parallel to the Y axis is $x = c$.

  If c > 0 , then the line x=c lies right to the side of the Y axis

  If c < 0 , then the line x=c lies left to the side of the Y axis

  If c = 0 , then the line x=c is the Y axis itself.
- ✓ Slope-Intercept Form A line with slope m and y intercept c can be expressed through the equation $y = mx + c$.
- ✓ Point-Slope form $y - y_1 = m(x - x_1)$.
- ✓ $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ is the equation of the line in two-point form.
- ✓ Intercept Form $\frac{x}{a}+\frac{y}{b}=1$.

Example 5.17

Find the equation of a straight line passing through (5,7) and is (i) parallel to X axis (ii) parallel to Y axis.

Solution :

(i) The equation of any straight line parallel to X axis is y=b.

Since it passes through (5,7), b = 7 .

Therefore, the required equation of the line is y=7.

(ii) The equation of any straight line parallel toY axis is x=c

Since it passes through (5,7), c = 5

Therefore, the required equation of the line is x = 5.

Example 5.18

Find the equation of a straight line whose (i) Slope is 5 and y intercept is – 9 (ii) Inclination is 45° and y intercept is 11

Solution :

(i) Given, Slope = 5, y intercept, c =–9

Therefore, equation of a straight line is

$y = mx + c$

y = 5x −9 gives 5x −y −9 = 0

(ii) Given, $\theta = 45°$, y intercept, $c = 11$

Slope m = tan θ = tan 45° = 1

Therefore, equation of a straight line is of the form $y = mx + c$

Hence we get, y = x +11 gives x −y +11 = 0.

Example 5.19

Calculate the slope and y intercept of the straight line $8x - 7y + 6 = 0$.

Solution :

Equation of the given straight line is

$8x - 7y + 6 = 0$

$7y = 8x + 6$ (bringing it to the form $y = mx + c$)

Slope $m = \frac{8}{7}$ and y intercept $c = \frac{6}{7}$

Example 5.20

The graph relates temperatures y (in Fahrenheit degree) to temperatures x (in Celsius degree) (a) Find the slope and y intercept (b) Write an equation of the line (c) What is the mean temperature of the earth in Fahrenheit degree if its mean temperature is 25° Celsius?

Solution :

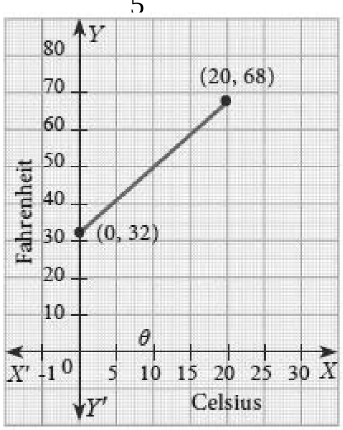
(a) From the figure, slope

$$= \frac{\text{change in } y \text{ coordinate}}{\text{change in } x \text{ coordinate}}$$

$$= \frac{68-32}{20-0} = \frac{36}{20} = \frac{9}{5} = 1.8$$

The line crosses the Y axis at (0, 32)

So the slope is $\frac{9}{5}$ and y intercept is 32.



(b) Use the slope and y intercept to write an equation

The equation is $y = \frac{9}{5}x + 32$.

(c) In Celsius, the mean temperature of the earth is 25°. To find the mean temperature in Fahrenheit, we find the value of y when $x = 25$.

$$y = \frac{9}{5}x + 32$$

$$y = \frac{9}{5}(25) + 32$$

$$y = 77$$

Therefore, the mean temperature of the earth is 77° F.

Example 5.21

Find the equation of a line passing through the point (3, – 4) and having slope $\frac{-5}{7}$.

Solution :

Given, $(x, y) = (3, -4)$ and $m = \frac{-5}{7}$

The equation of the point-slope form of the straight line is $y - y_1 = m(x - x_1)$

we write it as $y + 4 = -\frac{5}{7}(x - 3)$

gives us $5x + 7y + 13 = 0$.

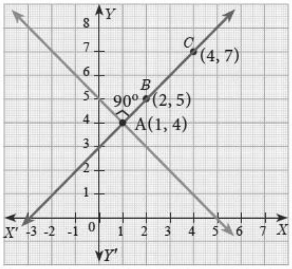
Example 5.22

Find the equation of a line passing through the point A(1, 4) and perpendicular to the line joining points (2,5) and (4,7) .

Solution :

Let the given points be A(1, 4) , B(2,5) and C(4,7) .

Slope of line BC = $\frac{7-5}{4-2} = \frac{2}{2} = 1$.



Let m be the slope of the required line.

Since the required line is perpendicular to BC,

$m \times 1 = -1$

$m = -1$

The required line also pass through the point A(1,4).

The equation of the required straight line is

$y - y_1 = m(x - x_1)$

$y - 4 = -1(x - 1)$

$y - 4 = -x + 1$

we get, $x + y - 5 = 0$.

Example 5.23

Find the equation of a straight line passing through (5, –3) and (7, – 4).

Solution :

The equation of a straight line passing through the two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Substituting the points we get,

$$\frac{y+3}{-4+3} = \frac{x-5}{7-5}$$

gives $2y + 6 = -x + 5$

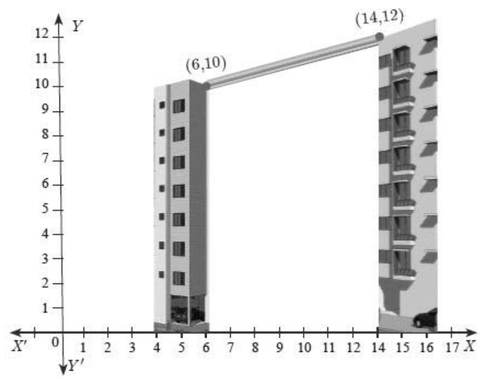
Therefore, $x + 2y + 1 = 0$.

Example 5.24

Two buildings of different heights are located at opposite sides of each other. If a heavy rod is attached joining the terrace of the buildings from (6, 10) to (14, 12), find the equation of the rod joining the buildings ?

Solution :

Let A(6,10) , B(14,12) be the points denoting the terrace of the buildings.



The equation of the rod is the equation of the straight line passing through A(6,10) and B(14,12)

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \text{ gives } \frac{y-10}{12-10} = \frac{x-6}{11-6}$$

$$\frac{y-10}{2} = \frac{x-6}{8}$$

Therefore, $x - 4y + 34 = 0$.

Hence, equation of the rod is $x - 4y + 34 = 0$.

Example 5.25

Find the equation of a line which passes through (5,7) and makes intercepts on the axes equal in magnitude but opposite in sign.

Solution :

Let the x intercept be ‘a’ and y intercept be ‘–a’.

The equation of the line in intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

gives $\frac{x}{a}+\frac{y}{-a}=1$ (Here $b=-a$)

Therefore, $x-y=a$ ....(1)

Since (1) passes through (5,7)

Therefore, $5-7=a$ gives $a=-2$

Thus the required equation of the straight line is $x-y=-2$; or $x-y+2=0$.

Example 5.26

Find the intercepts made by the line $4x-9y+36=0$ on the coordinate axes.

Solution :

Equation of the given line is $4x-9y+36=0$.

we write it as $4x-9y=-36$

(bringing it to the normal form)

Dividing by -36 we get, $\frac{x}{-9}+\frac{y}{4}=1$ ....(1)

Comparing (1) with intercept form, we get x intercept $a=-9$; y intercept $b=4$

Example 5.27

A mobile phone is put to use when the battery power is 100%. The percent of battery power 'y' (in decimal) remaining after using the mobile phone for x hours is assumed as $y=-0.25x+1$

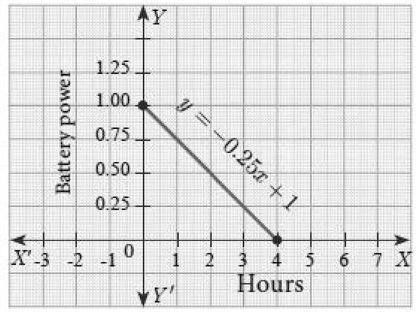
(i) Draw a graph of the equation.

(ii) Find the number of hours elapsed if the battery power is 40%.

(iii) How much time does it take so that the battery has no power?

Solution :

(i)



(ii) To find the time when the battery power is 40%, we have to take y = 0.40

$0.40=-0.25x+1$ gives $0.25x=0.60$

we get, $x=\frac{0.60}{0.25}=2.4$ hours.

(iii) If the battery power is 0 then y = 0

Therefore, $0=-0.25x+1$ gives $-0.25x=1$ hence $x=4$ hours.

Thus, after 4 hours, the battery of the mobile phone will have no power.

Example 5.28

A line makes positive intercepts on coordinate axes whose sum is 7 and it passes through (–3, 8). Find its equation.

Solution :

If a and b are the intercepts then $a+b=7$ or $b=7-a$

By intercept form $\frac{x}{a}+\frac{y}{b}=1$ ...(1)

We have $\frac{x}{a}+\frac{y}{7-a}=1$

As this line pass through the point (–3, 8), we have

$$\frac{-3}{a}+\frac{8}{7-a}=1$$

gives – 3 (7 – a) + 8a = a (7 – a).

– 21 + 3a + 8a = 7a – a^2

So, $a^2 + 4a - 21 = 0$.

Solving this equation (a – 3) (a + 7) = 0

a = 3 or a = – 7

Solving this equation (a – 3) (a + 7) = 0

a = 3 or a = – 7

Since a is positive, we have a = 3 and

$b = 7 - a = 7 - 3 = 4$.

Hence $\frac{x}{3}+\frac{y}{4}=1$

Therefore, $4x + 3y - 12 = 0$ is the required equation.

Example 5.29

A circular garden is bounded by East Avenue and Cross Road. Cross Road intersects North Street at D and East Avenue at E. AD is tangential to the circular garden at A(3, 10). Using the figure.

(a) Find the equation of

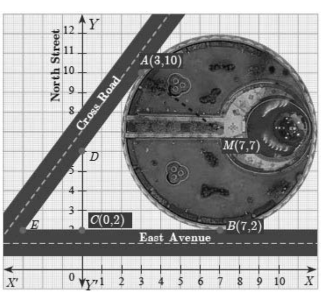
(i) East Avenue.

(ii) North Street

(iii) Cross Road

(b) Where does the Cross Road intersect the

(i) East Avenue ? (ii) North Street ?



Solution :

(a) (i) East Avenue is the straight line joining C(0, 2) and B(7, 2). Thus the equation of East Avenue is obtained by using two-point form which is

$$\frac{y-2}{2-2}=\frac{x-0}{7-0}$$

$$\frac{y-2}{0}=\frac{x}{7} \text{ gives } y=2$$

(ii) Since the point D lie vertically above C(0, 2). The x coordinate of D is 0.

Since any point on North Street has x coordinate value 0.

The equation of North Street is $x = 0$

(iii) To find equation of Cross Road.

Center of circular garden M is at (7, 7), A is (3, 10)

We first find slope of MA, which we call m_1

$$\text{Thus } m_1=\frac{10-7}{3-7}=\frac{-3}{4}$$

Since the Cross Road is perpendicular to MA, if m_2 is the slope of the Cross Road then,

$$m_1m_2=-1 \text{ gives } \frac{-3}{4}m_2=-1 \text{ so } m_2=\frac{4}{3}$$

Now, the cross road has slope $\frac{4}{3}$ and it passes through th epoint A (3, 10).

The equation of the Cross Road is

$$y - 10 = \frac{4}{3}(x - 3).$$

$3y - 30 = 4x - 12$

Hence, $4x - 3y + 18 = 0$

(b) (i) If D is (0, k) then D is a point on the Cross Road.

Therefore, substituting $x = 0$, $y = k$ in the equation of Cross Road,

we get, $0 - 3k + 18 = 0$

Value of $k = 6$

Therefore, D is (0, 6)

(ii) To find E, let E be (q,2)

Put y =2 in the equation of the Cross Road,

we get, $4q - 6 + 18 = 0$

$$4q = -12 \text{ gives } q = -3$$

Therefore, The point E is (-3,2)

Thus the Cross Road meets the North Street at D(0, 6) and East Avenue at E (-3,2) .

EXERCISE 5.3

1. Find the equation of a straight line passing through the mid-point of a line segment joining the points (1,-5), (4,2) and parallel to (i) X axis (ii) Y axis

Solution :

Mid point of the line joining the points (1, –5), (4, 2) is

$$= \left(\frac{1+4}{2}, \frac{-5+2}{2}\right)$$

$$= \left(\frac{5}{2}, \frac{-3}{2}\right)$$

i) Equation of straight line passing through $\left(\frac{5}{2}, \frac{-3}{2}\right)$ and

a) Parallel to x-axis is

$$y = -3/2 \Rightarrow 2y + 3 = 0$$

b) Parallel to y-axis is

$$x = 5/2 \Rightarrow 2x - 5 = 0$$

2. The equation of a straight line is $2(x - y) + 5 = 0$. Find its slope, inclination and intercept on the Y axis.

Solution :

Given equation of a straight line is

$2(x - y) + 5 = 0$

$\Rightarrow 2x - 2y + 5 = 0$ — (1)

i) Slope of the line $= \dfrac{-\text{coefficient of } x}{\text{coefficient of } y}$

$$= \frac{-2}{-2}$$

$$= 1$$

ii) Slope of the line = 1

$\tan\theta = 1$

$\therefore \theta = 45^o$.

iii) Interecept on y-axis

Put $x = 0$ in (1)

$-2y + 5 = 0$

$\Rightarrow -2y = -5$

$\Rightarrow y = 5/2$

$\therefore y$ - intercept $= 5/2$

3. Find the equation of a line whose inclination is 30° and making an intercept -3 on the Y axis.

Solution :

Given $\theta = 30^o \Rightarrow m = \tan 30^o = \dfrac{1}{\sqrt{3}}$ and

y-intercept = – 3

The required equation of the line is

$y = mx + c$

$$\Rightarrow y = \frac{1}{\sqrt{3}}x - 3$$

$$\Rightarrow \sqrt{3}y = x - 3\sqrt{3}$$

$$\Rightarrow x - \sqrt{3}y - 3\sqrt{3} = 0$$

4. Find the slope and y intercept of $\sqrt{3}x + (1 - \sqrt{3})y = 3$.

Solution :

Given line is $\sqrt{3}x + (1 - \sqrt{3})y - 3 = 0$

$$\Rightarrow (1-\sqrt{3})y = -\sqrt{3}x + 3$$

$$\Rightarrow y = \frac{-\sqrt{3}}{1-\sqrt{3}}x + \frac{3}{1-\sqrt{3}}$$

This is of the form $y = mx + c$.

$$m = \frac{-\sqrt{3}}{1-\sqrt{3}} \qquad c = \frac{3}{1-\sqrt{3}}$$

$$= \frac{\sqrt{3}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} \qquad = \frac{3}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{3\sqrt{3}+3}{2} \qquad = \frac{3+3\sqrt{3}}{-2}$$

$$\therefore \text{ Slope} = \frac{3+\sqrt{3}}{2}, \quad y-\text{intercept} = \frac{3+3\sqrt{3}}{-2}.$$

5. Find the value of 'a', if the line through (–2, 3) and (8, 5) is perpendicular to $y = ax + 2$.

Solution :

Slope of the line joining (– 2, 3), (8, 5).

$$= \frac{5-3}{8+2}$$

$$= \frac{2}{10}$$

$$m_1 = \frac{1}{5}$$

Slope of the line $y = ax + 2$ is $m_2 = a$.

Since the 2 lines are perpendicular.

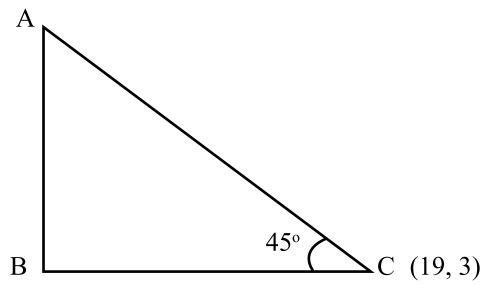
$$m_1 m_2 = -1$$

$$\Rightarrow \frac{1}{5}\times a = -1$$

$$\Rightarrow \quad a = -5$$

6. The hill in the form of a right triangle has its foot at (19, 3) . The inclination of the hill to the ground is 45°. Find the equation of the hill joining the foot and top.

Solution :



C – Foot of the hill.

$\therefore$ Slope of AC = m = tan 45° = 1

$\therefore$ Equation of AC whose slope 1 and passing through C (19, 3) is

$$y - y_1 = m\,(x - x_1)$$

$$\Rightarrow y - 3 = 1\,(x - 19)$$

$$\Rightarrow x - y - 16 = 0.$$

7. Find the equation of a line through the given pair of points

(i) $\left(2, \frac{2}{3}\right)$ and $\left(\frac{-1}{2}, -2\right)$

(ii) (2, 3) and (–7, –1)

Solution :

Given points are

$$\left(2, \frac{2}{3}\right), \left(\frac{-1}{2}, -2\right)$$

The equation of the line passing through 2 points

$$\frac{y - \frac{2}{3}}{-2 - \frac{2}{3}} = \frac{x-2}{\frac{-1}{2} - 2}$$

$$\Rightarrow \frac{3y-2}{-8} = \frac{x-2}{\frac{-5}{2}}$$

$$\Rightarrow \frac{3y-2}{-8} = \frac{2x-4}{-5}$$
$$\Rightarrow 15y-10 = 16x-32$$
$$\Rightarrow 16x-15y-22 = 0$$

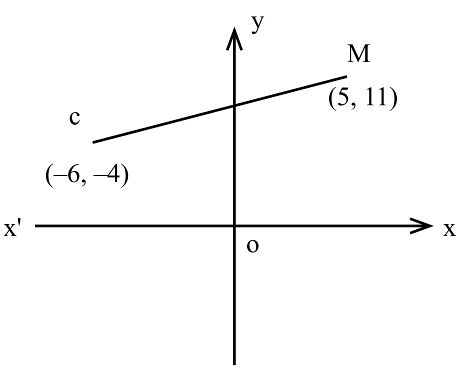
ii) Given points are (2, 3), (–7, –1)

Equation through 2 points is

$$\frac{y-3}{-1-3} = \frac{x-2}{-7-2}$$
$$\Rightarrow \frac{y-3}{-4} = \frac{x-2}{-9}$$
$$\Rightarrow 9y-27 = 4x-8$$
$$\Rightarrow 4x-9y+19 = 0$$

8. A cat is located at the point(-6,-4) in xy plane. A bottle of milk is kept at (5,11). The cat wish to consume the milk travelling through shortest possible distance. Find the equation of the path it needs to take its milk.

Solution :



C (– 6, –4) is the position of cat.

M (5, 11) is the position of milk.

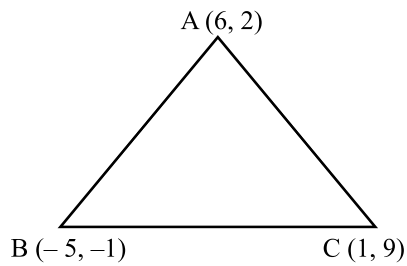
Equation of the path CM is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{y+4}{11+4} = \frac{x+6}{5+6}$$
$$\Rightarrow \frac{y+4}{15} = \frac{x+6}{11}$$
$$\Rightarrow 15x+90 = 11y+44$$
$$\Rightarrow 15x-11y+46-0$$

9. Find the equation of the median and altitude of ΔABC through A where the vertices are A(6,2) , B(-5,-1) and C(1,9)

Solution :



i) Equation of the median through A.

mid point of BC = $\left(\frac{-5+1}{2}, \frac{-1+9}{2}\right)$

= D (– 2, 4)

Equation of AD is [A (6, 2), D (– 2, 4).

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$
$$\Rightarrow \frac{y-2}{4-2} = \frac{x-6}{-2-6}$$
$$\Rightarrow \frac{y-2}{2} = \frac{x-6}{-8}$$
$$= \frac{y-2}{1} = \frac{x-6}{-4}$$
$$\Rightarrow x-6 = -4y+8$$
$$\Rightarrow x+4y-14 = 0$$

ii) Equation of altitude through 'A'

Slope of BC = $\frac{9+1}{1+5} = \frac{10}{6} = \frac{5}{3}$

Since AD ⊥ BC, slope of AD = $\frac{-3}{5}$

and A is (6, 2).

∴ Equation of altitude AD is

$y - y_1 = m(x - x_1)$

$\Rightarrow y - 2 = \frac{-3}{5}(x - 6)$

$\Rightarrow 5y - 10 = -3x + 18$

$\Rightarrow 3x + 5y - 28 = 0$

10. Find the equation of a straight line which has slope $\frac{-5}{4}$ and passing through the point (–1,2).

Solution :

Given slope of the line is $\frac{-5}{4}$ and (–1, 2) is a point on the line.

∴ its equation is $y - y_1 = m(x - x_1)$

$\Rightarrow y - 2 = \frac{-5}{4}(x + 1)$

$\Rightarrow 4y - 8 = -5x - 5$

$\Rightarrow 5x + 4y - 3 = 0$

11. You are downloading a song. The percent y (in decimal form) of mega bytes remaining to get downloaded in x seconds is given by y = −0.1x +1.

(i) graph the equation.

(ii) find the total MB of the song.

(iii) after how many seconds will 75% of the song gets downloaded?

(iv) after how many seconds the song will be downloaded completely?

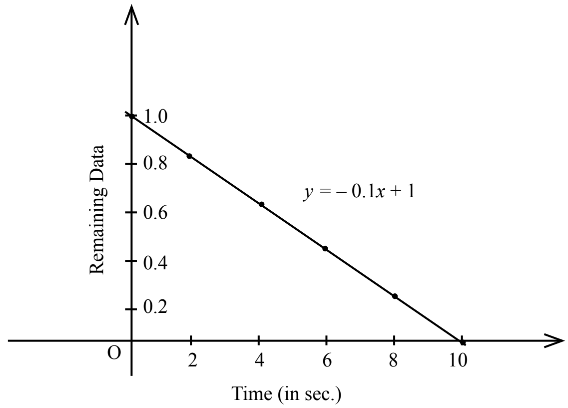
Solution :

Given $y = -0.1x + 1$.

where x - time (in sec.)

y - remaining data to be downloaded.

x	0	2	4	6	8	10
y	1	0.8	0.6	0.4	0.2	0



ii) $y = -0.1x + 1$

when $x = 0$, $y = 1$ (∵ x can't be –ve)

∴ Total MB of the song = 1 MB

iii) 75% of MB downloaded.

∴ 25% of MB to be downloaded

∴ Put y = 0.25 in (1)

$\Rightarrow 0.25 = -0.1x + 1$

$\Rightarrow 0.1x = 0.75$

$x = \frac{0.75}{0.10} = \frac{15}{2} = 7.5$

∴ Required time = 7.5 sec.

iv) Time when songs completely downloaded

Put $y = 0$ in (1)

ie, $0.1x = 1$

$x = 10$ sec.

∴ Songs will be downloaded completely after 10 sec.

12. Find the equation of a line whose intercepts on the x and y axes are given below.

(i) 4, – 6 (ii) – 5, $\frac{3}{4}$

Solution :

i) Given x-intercept = 4 = a

y - intercept = – 6 = b

Equation of line in intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \frac{x}{4}+\frac{y}{-6}=1$$

$$\Rightarrow \frac{x}{4}-\frac{y}{6}=1$$

$$\Rightarrow \frac{3x-2y}{12}=1$$

$$\Rightarrow 3x-2y-12=0$$

ii) Given x-intercept = – 5 = a

y-intercept = $\frac{3}{4}$ = b

Equation of line in intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \frac{x}{-5}+\frac{y}{3/4}=1$$

$$\Rightarrow \frac{x}{-5}-\frac{4y}{3}=1$$

$$\Rightarrow \frac{-3x+20y}{15}=1$$

$$\Rightarrow -3x+20y-15=0$$

$$\Rightarrow 3x-20y+15=0$$

13. Find the intercepts made by the following lines on the coordinate axes.

(i) 3x −2y − 6 = 0 (ii) 4x + 3y +12 = 0

Solution :

Method - 1 :

$3x – 2y = 6$

$$\Rightarrow \frac{3x}{6}-\frac{2y}{6}=1$$

$$\Rightarrow \frac{x}{2}+\frac{y}{-3}=1$$

$\therefore$ x-int = 2, y-int = – 3

Method - 2 :

$3x – 2y = 6$

Put y = 0 $\Rightarrow 3x = 6$

$\Rightarrow x = 2$ (x-int)

Put $x = 0 \Rightarrow – 2y = 6$

$\Rightarrow y = – 3$ (y - int)

ii) Given line is $4x + 3y + 12 = 0$

$$\Rightarrow 4x+3y=-12$$

$$\Rightarrow \frac{4x}{-12}+\frac{3y}{-12}=1$$

$$\Rightarrow \frac{x}{-3}+\frac{y}{-4}=1$$

$\therefore$ x-int = – 3, y-int = – 4.

14. Find the equation of a straight line

(i) passing through (1,-4) and has intercepts which are in the ratio 2:5

(ii) passing through (-8, 4) and making equal intercepts on the coordinate axes

Solution :

i) Required line is passing through (1, – 4) and has intercepts in the ratio 2 : 5.

Equation of line in intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \frac{x}{a}+\frac{y}{\frac{5a}{2}}=1$$

$$\Rightarrow \frac{x}{a}+\frac{2y}{5a}=1 \text{ ...(1)}$$

where $a : b = 2 : 5$

$$\Rightarrow \frac{a}{b}=\frac{2}{5}$$

$$\Rightarrow b=\frac{5a}{2}$$

Since (1) passes through (1, – 4)

$$\frac{1}{a}-\frac{8}{5a}=1$$

$$\Rightarrow \frac{5-8}{5a}=1$$

$$\Rightarrow \quad 5a=-3$$

$$a=\frac{-3}{5}$$

$$\therefore (1) \Rightarrow \frac{x}{\frac{-3}{5}}+\frac{2y}{-3}=1$$

$$\Rightarrow \frac{5x}{-3}+\frac{2y}{-3}=1$$

$$\Rightarrow 5x+2y=-3$$

$$\Rightarrow 5x+2y+3=0$$

ii) Required line is passing through (– 8, 4) and making equal intercepts on the axes.

Equation o fline in intercept form is

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{where} \quad a=b$$

$$\Rightarrow \frac{x}{a}+\frac{y}{a}=1$$

$$\Rightarrow x+y=a \qquad \text{....(1)}$$

Since (1) passes through (– 8, 4)

$-8 + 4 = a$

$a = -4$

$\therefore$ (1) $\Rightarrow x + y = -4 \qquad \Rightarrow x + y + 4 = 0$

IV. GENERAL FORM OF A STRAIGHT LINE

Key Points

- ✓ The equation of all lines parallel to the line ax +by +c = 0 can be put in the form $ax +by +k =0$ for different values of k.
- ✓ The equation of all lines perpendicular to the line $ax + by + c = 0$ can be written as $bx - ay+k=0$ for different values of k.
- ✓ Slope of a straight line $m=\dfrac{-\text{coefficient of } x}{\text{coefficient of } y}$
- ✓ y intercept = $\dfrac{-\text{ constant term}}{\text{coefficient of } y}$

Example 5.30

Find the slope of the straight line $6x+8y + 7 = 0$.

Solution :

Given $6x + 8y + 7 = 0$

$$\text{slope } m=\frac{-\text{coefficient of } x}{\text{coefficient of } y}=-\frac{6}{8}=-\frac{3}{4}$$

Therefore, the slope of the straight line is $-\dfrac{3}{4}$.

Example 5.31

Find the slope of the line which is

(i) parallel to $3x - 7y = 11$

(ii) perpendicular to $2x - 3y + 8 = 0$.

Solution :

(i) Given straight line is $3x - 7y = 11$

gives $3x - 7y - 11 = 0$

Slope $m = \frac{-3}{-7} = \frac{3}{7}$

Since parallel lines have same slopes, slope of any line parallel to

$3x - 7y = 11$ is $\frac{3}{7}$.

(ii) Given striaght line is $2x - 3y + 8 = 0$

Slope $m = \frac{-2}{-3} = \frac{2}{3}$

Since product of slopes is -1 for perpendicular lines, slope of any line perpendicular to $2x - 3y + 8 = 0$ is $\frac{-1}{\frac{2}{3}} = \frac{-3}{2}$

Example 5.32

Show that the straight lines $2x + 3y - 8 = 0$ and $4x + 6y + 18 = 0$ are parallel.

Solution :

Slope of the straight line $2x + 3y - 8 = 0$ is

$$m_1 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$$

$$m_1 = \frac{-2}{3}$$

Slope of the straight line $4x + 6y + 18 = 0$ is

$$m_2 = \frac{-4}{6} = \frac{-2}{3}$$

Here, $m_1 = m_2$

That is, slopes are equal. Hence, the two straight lines are parallel.

Example 5.33

Show that the straight lines $x - 2y + 3 = 0$ and $6x + 3y + 8 = 0$ are perpendicular.

Solution :

Slope of the straight line $x - 2y + 3 = 0$ is

$$m_1 = \frac{-1}{-2} = \frac{1}{2}$$

Slope of the straight line $6x + 3y + 8 = 0$ is

$$m_2 = \frac{-6}{3} = -2$$

Now, $m_1 \times m_2 = \frac{1}{2} \times (-2) = -1$

Hence, the two straight lines are perpendicular.

Example 5.34

Find the equation of a straight line which is parallel to the line $3x - 7y = 12$ and passing through the point $(6, 4)$.

Solution :

Equation of the straight line, parallel to $3x - 7y - 12 = 0$ is $3x - 7y + k = 0$

Since it passes through the point $(6,4)$

$3(6) - 7(4) + k = 0$

$k = 28 - 18 = 10$

Therefore, equation of the required straight line is $3x - 7y + 10 = 0$.

Example 5.35

Find the equation of a straight line perpendicular to the line $y = \frac{4}{3}x - 7$ and passing through the point $(7, -1)$.

Solution :

The equation $y = \frac{4}{3}x - 7$ can be written as $4x - 3y - 21 = 0$.

Equation of a straight line perpendicular to $4x - 3y - 21 = 0$ is $3x + 4y + k = 0$

Since it is passes through the point (7, –1),

$21 - 4 + k = 0$ we get, $k = -17$

Therefore, equation of the required straight line is $3x + 4y - 17 = 0$.

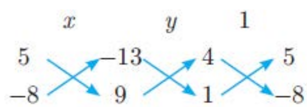
Example 5.36

Find the equation of a straight line parallel to Y axis and passing through the point of intersection of the lines $4x + 5y = 13$ and $x - 8y + 9 = 0$.

Solution :

Given lines $4x + 5y - 13 = 0$ ...(1)

$x - 8y + 9 = 0$ ...(2)



$$\frac{x}{45-104} = \frac{y}{-13-36} = \frac{1}{-32-5}$$

$$\frac{x}{-59} = \frac{y}{-49} = \frac{1}{-37}$$

$$x = \frac{59}{37}, y = \frac{49}{37}$$

Therefore, the point of intersection

$$(x, y) = \left(\frac{59}{37}, \frac{49}{37}\right)$$

The equation of line parallel to Y axis is x = c.

It passes through $(x, y) = \left(\frac{59}{37}, \frac{49}{37}\right)$.

Therefore, $c = \frac{59}{37}$

The equation of the line is $x = \frac{59}{37}$ gives $37x - 59 = 0$.

Example 5.37

The line joining the points A(0, 5) and B(4, 1) is a tangent to a circle whose centre C is at the point (4, 4) find

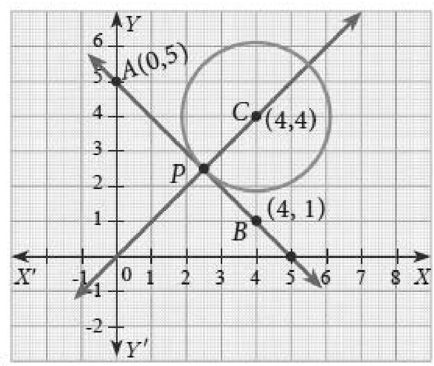
(i) the equation of the line AB.

(ii) the equation of the line through C which is perpendicular to the line AB.

(iii) the coordinates of the point of contact of tangent line AB with the circle.

Solution :

(i) Equation of line AB, A(0,5) and B(4,1)



$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y-5}{1-5} = \frac{x-0}{4-0}$$

$4(y - 5) = -4x$ gives $y - 5 = -x$

$x + y - 5 = 0$

(ii) The equation of a line which is perpendicular to the line AB : $x + y - 5 = 0$ is $x - y + k = 0$

Since it is passing through the point (4,4), we have

$4 - 4 + k = 0$ gives $k = 0$

The equation of a line which is perpendicular to AB and through C is

$x - y = 0$ ...(2)

(iii) The coordinate of the point of contact P of the tangent line AB with the circle is

$x + y - 5 = 0$ and $x - y = 0$

Solving, we get $x = \frac{5}{2}$ and $y = \frac{5}{2}$

Therefore, the coordinate of the point of contact is P $\left(\frac{5}{2}, \frac{5}{2}\right)$.

EXERCISE 5.4

1. Find the slope of the following straight lines

(i) 5y − 3 = 0 **(ii)** $7x - \frac{3}{17} = 0$

Solution :

i) Given line is $5y - 3 = 0$

$$\text{Slope} = \frac{\text{Co.eff. of } x}{\text{Co.eff. of } y}$$
$$= \frac{-0}{5}$$
$$= 0$$

ii) Given line is $7x - \frac{3}{17} = 0$

$$\text{Slope} = \frac{\text{Co.eff. of } x}{\text{Co.eff. of } y}$$
$$= \frac{-7}{0}$$
$$= \text{undefined}$$

2. Find the slope of the line which is

(i) parallel to y = 0.7*x* −11

(ii) perpendicular to the line *x* = −11.

Solution :

i) Given line is $y = 0.7x - 11$

whose slope is 0.7

$\therefore$ Slope of the line parallel to

$y = 0.7x - 11$ is also 0.7

ii) Given line is $x = -11$,

whose slope is 0

$\therefore$ Slope of the line perpendicular to

$x = -11$ is , $\frac{-1}{0}$

which is undefined.

3. Check whether the given lines are parellel or perpendicular

(i) $\frac{x}{3} + \frac{y}{4} + \frac{1}{7} = 0$ **and** $\frac{2x}{3} + \frac{y}{2} + \frac{1}{10} = 0$

(ii) 5*x* + 23*y* + 14 = 0 and 23*x* – 5*y* + 9 = 0

Solution :

i) Given pair of lines

$$\frac{x}{3} + \frac{y}{4} + \frac{1}{7} = 0, \quad \frac{2x}{3} + \frac{y}{2} + \frac{1}{10} = 0$$

Their slope,

$$m_1 = \frac{-1/3}{1/4} \quad \& \quad m_2 = \frac{-2/3}{1/2}$$
$$= -4/3 \qquad\qquad = -4/3$$

$\therefore m_1 = m_2$

$\Rightarrow$ the 2 lines are parallel

ii) Given lines are

$5x + 23y + 14 = 0, \quad 23x - 5y + 9 = 0$

Their slope,

$$m_1 = \frac{-5}{23} \qquad m_2 = \frac{23}{5}$$

$\therefore m_1 \times m_2 = -1$

$\therefore$ the 2 lines are perpendicular.

4. If the straight lines 12y = −(p + 3)*x* +12,

$12x - 7y = 16$ are perpendicular then find 'p'.

Solution :

Given lines

$12y = -(p+3)x + 12,$

$12x - 7y = 16$ are perpendicular

$\Rightarrow (p+3)x + 12y = 12$

$$m_1 = \frac{-(p+3)}{12} \qquad m_2 = \frac{12}{7}$$

Since 2 lines are perpendicular,

$m_1 \times m_2 = -1$

$$\Rightarrow \frac{-(p+3)}{12} \times \frac{12}{7} = -1$$

$$\Rightarrow -(p+3) = -7$$

$$\Rightarrow p+3 = 7$$

$$\Rightarrow p = 4$$

---

5. Find the equation of a straight line passing through the point P(-5,2) and parallel to the line joining the points Q(3,-2) and R(-5, 4) .

Solution :

The required line is passing through P (−5, 2) and parallel to the line joining the points Q (3, −2), R (−5, 4)

$$\text{Slope of } QR = \frac{4+2}{-5-3} = \frac{6}{-8} = \frac{-3}{4}$$

$\therefore$ Equn. of the line is

$y - y_1 = m(x - x_1)$

$$\Rightarrow y - 2 = {}^{-3}\!/_{4}(x+5)$$

$$\Rightarrow 4y - 8 = -3x - 15$$

$$\Rightarrow 3x + 4y + 7 = 0$$

---

6. Find the equation of a line passing through (6,–2) and perpendicular to the line joining the points (6,7) and (2,–3).

Solution :

The required line is passing through (6, −2) and perpendicular to the line joining (6, 7), (2,−3)

$\therefore$ Slope of the line joining (6, 7), (2, −3)

$$= \frac{-3-7}{2-6} = \frac{-10}{-4} = \frac{5}{2}$$

$\therefore$ Slope of the line perpendicular to it is $\frac{-2}{5}$

$\therefore$ Equation of the required line is

$$y + 2 = \frac{-2}{5}(x-4)$$

$$\Rightarrow 5y + 10 = -2x + 12$$

$$\Rightarrow 2x + 5y - 2 = 0$$

---

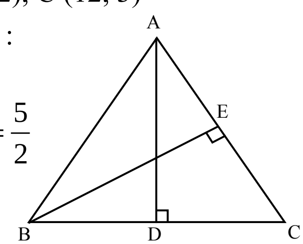
7. A(-3, 0) B(10,-2) and C(12, 3) are the vertices of ΔABC . Find the equation of the altitude through A and B.

Solution :

Given vertices of Δ are

A (−3, 0), B (10, −2), C (12, 3)

Equation of altitude AD :



$$\text{Slope of } BC = \frac{3+2}{12-10} = \frac{5}{2}$$

$$\therefore \text{Slope of } AD \text{ is} = \frac{-2}{5} \quad (AD \perp BC)$$

$\therefore$ Equation of AD is

$$y - 0 = {}^{-2}\!/_{5}(x+3)$$

$$5y = -2x - 6$$

$$\Rightarrow 2x + 5y + 6 = 0$$

$\therefore$ Equation of altitude BE is

$$\text{Slope of } AC = \frac{3-0}{12+3} = \frac{3}{15} = \frac{1}{5}$$

$\therefore$ Slope of $BE = -5 \quad (\because BE \perp AC)$

$\therefore$ Equation of BE is

$$y + 2 = -5\,(x - 10)$$

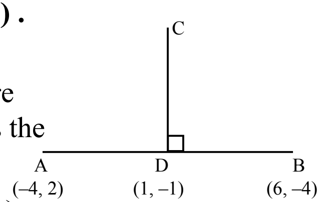
$$\Rightarrow \quad y + 2 = -5x + 50$$

$$\Rightarrow \quad 5x + y - 48 = 0$$

8. Find the equation of the perpendicular bisector of the line joining the points A(-4,2) and B(6,-4) .

Solution :

Given AB and CD are perpendicular & D is the midpoint of AB



$$\therefore D = \left(\frac{-4+6}{2}, \frac{2-4}{2}\right) = (1, -1)$$

$$\text{Slope of } AB = \frac{-4-2}{6+4} = \frac{-6}{10} = \frac{-3}{5}$$

$$\therefore \text{Slope of } CD = \frac{5}{3} \quad (\because CD \perp AB)$$

$\therefore$ Equation of perpendicular bisector CD is

$$y + 1 = \frac{5}{3}(x - 1)$$

$$\Rightarrow \quad 3y + 3 = 5x - 5$$

$$\Rightarrow \quad 5x - 3y - 8 = 0$$

9. Find the equation of a straight line through the intersection of lines 7x + 3y = 10 , 5x −4y = 1 and parallel to the line 13x + 5y +12 = 0.

Solution :

The required line is passing through the intersection of the lines

$7x + 3y = 10$ .... (1)

$5x - 4y = 1$ .... (2)

and parallel to the line $13x + 5y + 12 = 0$

Solving (1) & (2)

$$\begin{aligned} (1) \times 4 &\Rightarrow 28x + 12y = 40 \\ (2) \times 3 &\Rightarrow 15x - 12y = 3 \\ \hline &\quad\ 43x = 43 \end{aligned}$$

$$x = 1$$

Sub $x = 1$ in (1)

$$7(1) + 3y = 10$$

$$\Rightarrow \quad 3y = 3$$

$$y = 1$$

$\therefore$ The required line is

$$13x + 5y + k = 0$$

Since it passes through (1, 1)

$$13 + 5 + k = 0$$

$$k = -18$$

$$\therefore 13x + 5y - 18 = 0$$

10. Find the equation of a straight line through the intersection of lines 5x −6y = 2 , 3x + 2y = 10 and perpendicular to the line 4x −7y +13 = 0.

Solution :

Given lines are

$5x - 6y = 2$ .... (1)

$3x + 2y = 10$ .... (2)

$$\begin{aligned} (1) &\Rightarrow 5x - 6y = 2 \\ (2) \times 3 &\Rightarrow 9x + 6y = 30 \\ \hline &\quad\ 14x = 32 \end{aligned}$$

$$x = 16/7$$

Sub in (2)

$$\frac{48}{7} + 2y = 10 \Rightarrow 2y = 10 - \frac{48}{7}$$

$$\Rightarrow 2y = \frac{22}{7}$$

$$y = \frac{11}{7}$$

The required line is perpendicular to

$4x - 7y + 13 = 0$

Equation of the required line is

$7x + 4y + k = 0$

$\therefore$ Since it passes through

$\left(\frac{16}{7}, \frac{11}{7}\right)$

$$\Rightarrow 7\left(\frac{16}{7}\right) + 4\left(\frac{11}{7}\right) + k = 0$$

$$\Rightarrow 16 + \frac{44}{7} + k = 0$$

$$\Rightarrow k = -16 - \frac{44}{7}$$

$$\Rightarrow k = \frac{-156}{7}$$

$$\therefore 7x + 4y - \frac{156}{7} = 0$$

$$\Rightarrow 49x + 28y - 156 = 0$$

11. Find the equation of a straight line joining the point of intersection of $3x + y + 2 = 0$ and $x - 2y - 4 = 0$ to the point of intersection of $7x - 3y = -12$ and $2y = x + 3$.

Solution :

$$\begin{array}{lrl} (1) \times 2 & \Rightarrow & 6x + 2y = -4 \\ (2) & \Rightarrow & x - 2y = 4 \\ \hline & & 7x = 0 \end{array}$$

$x = 0$

$(1) \Rightarrow \quad y = -2$

$\therefore$ The point of int. of (1) & (2) is $(0, -2)$

Now, to find the point of int. of the lines

$7x - 3y = -12$ .... (3)

$x - 2y + 3 = 0$ .... (4)

$$\begin{array}{lrl} (3) & \Rightarrow & 7x - 3y = -12 \\ (4) \times 7 & \Rightarrow & 7x - 14y = -21 \\ \hline & & 11y = 9 \end{array}$$

$$y = \frac{9}{11}$$

Sub in (4)

$$x - \frac{18}{11} + 3 = 0$$

$$x = \frac{18}{11} - 3 = \frac{-15}{11}$$

The point of int. of (3) & (4) is

$\left(\frac{-15}{11}, \frac{9}{11}\right)$

The required equation of the line joining

$(0, -2), \left(\frac{-15}{11}, \frac{9}{11}\right)$

$$\frac{y+2}{\frac{9}{11}+2} = \frac{x-0}{\frac{-15}{11}}$$

$$\Rightarrow \frac{y+2}{31} = \frac{x}{-15}$$

$$\Rightarrow 31x = -15y - 30$$

$$\Rightarrow 31x + 15y + 30 = 0$$

12. Find the equation of a straight line through the point of intersection of the lines $8x + 3y = 18$, $4x + 5y = 9$ and bisecting the line segment joining the points $(5,-4)$ and $(-7,6)$.

Solution :

To find : The point of int. of

$8x + 3y = 18$ .... (1)

$4x + 5y = 9$ .... (2)

$$\begin{aligned} (1) \Rightarrow \quad & 8x + 3y = 18 \\ (2)\times 2 \Rightarrow \quad & 8x + 10y = 18 \\ \hline & -7y = 0 \\ & y = 0 \end{aligned}$$

$(2) \Rightarrow \quad 4x = 9$

$$\therefore x = \frac{9}{4}$$

$\therefore$ The point of int. of (1) & (2) in

$$\left(\frac{9}{4}, 0\right)$$

Mid point of the line joining

$(5, -4), (-7, 6)$

$$= \left(\frac{5-7}{2}, \frac{-4+6}{2}\right)$$

$$= (-1, 1)$$

Equation of the required line joining

$$\left(\frac{9}{5}, 0\right), (-1, 1)$$

$$\frac{y-0}{1} = \frac{x - \frac{9}{4}}{-1 - \frac{9}{4}}$$

$$y = \frac{4x-9}{-13}$$

$$\Rightarrow 4x + 13y - 9 = 0$$

EXERCISE 5.5

Multiple choice questions :

1. The area of triangle formed by the points (−5,0) , (0,−5) and (5,0) is

(1) 0 sq.units (2) 25 sq.units

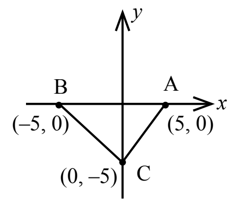
(3) 5 sq.units (4) none of these

Hint : **Ans : (2)**

$$\text{Area of ABC} = \frac{1}{2} \times b \times h$$

$$= \frac{1}{2} \times 10 \times 5$$

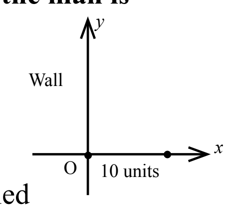
$$= 25 \text{ sq.units}$$



2. A man walks near a wall, such that the distance between him and the wall is 10 units. Consider the wall to be the Y axis. The path travelled by the man is

(1) x = 10 (2) y = 10

(3) x = 0 (4) y = 0



Hint : **Ans : (1)**

Equation of path travelled by the man is $x = 10$

3. The straight line given by the equation *x* = 11 is

(1) parallel to X axis

(2) parallel to Y axis

(3) passing through the origin

(4) passing through the point (0,11)

Hint : **Ans : (2)**

Equation x = C is a line parallel to y - axis

4. If (5,7), (3,p) and (6,6) are collinear, then the value of p is

(1) 3 (2) 6 (3) 9 (4) 12

Hint : **Ans : (3)**

A (5, 7), B (3, p), C (6, 6) are collinear

$\therefore$ Slope of AB = Slope of BC

$$\frac{p-7}{-2}=\frac{6-p}{3}$$

$$\Rightarrow \quad 3p-21=-12+2p$$

$$\Rightarrow \quad p=9$$

5. The point of intersection of $3x-y=4$ and $x+y=8$ is

(1) (5,3) (2) (2,4) (3) (3,5) (4) (4,4)

Hint : **Ans : (3)**

Substitute and check the point to satisfy the given lines.

6. The slope of the line joining (12, 3) , (4,a) is $\frac{1}{8}$. The value of 'a' is

(1) 1 (2) 4 (3) – 5 (4) 2

Hint : **Ans : (4)**

$$\text{Slope of } (12, 3), (4, a)=\frac{1}{8}$$

$$\Rightarrow \quad \frac{a-3}{-8}=\frac{1}{8}$$

$$\Rightarrow \quad a-3=-1$$

$$\Rightarrow \quad a=2$$

7. The slope of the line which is perpendicular to a line joining the points (0,0) and (–8,8) is

(1) –1 (2) 1 (3) $\frac{1}{3}$ (4) – 8

Hint : **Ans : (2)**

Slope of the line joining (0, 0), (– 8, 8)

$$=\frac{8-0}{-8-0}$$

$$=-1$$

$\therefore$ Slope of the line perpendicular to it = 1.

8. If slope of the line PQ is $\frac{1}{\sqrt{3}}$ then slope of the perpendicular bisector of PQ is

(1) $\sqrt{3}$ (2) $-\sqrt{3}$ (3) $\frac{1}{\sqrt{3}}$ (4) 0

Hint : **Ans : (2)**

$$\text{Slope of PQ}=\frac{+1}{\sqrt{3}}$$

$$\text{Slope of its perpendicular bisector}=-\sqrt{3}$$

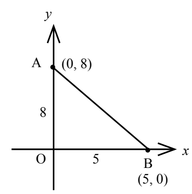
9. If A is a point on the Y axis whose ordinate is 8 and B is a point on the X axis whose abscissae is 5 then the equation of the line AB is

(1) 8x + 5y = 40 (2) 8x −5y = 40

(3) x = 8 (4) y = 5

Hint : **Ans : (1)**

Here a = 5, b = 8



$$\therefore \text{Equn. of the line is } \frac{x}{5}-\frac{y}{8}=1$$

$$\Rightarrow \quad 8x+5y-40=0$$

10. The equation of a line passing through the origin and perpendicular to the line 7x −3y + 4 = 0 is

(1) 7*x* −3y + 4 = 0 (2) 3*x* −7y + 4 = 0

(3) 3*x* + 7y = 0 (4) 7*x* −3y = 0

Hint : **Ans : (3)**

Equation of the line perpendicular to

7x −3y + 4 = 0 is

3x + 7y + k = 0

Since it passes through (0, 0), k = 0

$\therefore$ 3x + 7y = 0

11. Consider four straight lines

(i) $l_2 : 3y = 4x + 5$ (ii) $l_2 : 4y = 3x - 1$

(iii) $l_3 : 4y + 3x + 7$ (iv) $l_4 : 4x + 3y = 2$

Which of the following statement is true?

(1) l_1 and l_2 are perpendicular

(2) l_1 and l_4 are parallel

(3) l_2 and l_4 are perpendicular

(4) l_2 and l_3 are parallel

Hint : **Ans : (3)**

i) Slope of $l_1 = \frac{4}{3}$

ii) Slope of $l_2 = \frac{3}{4}$

iii) Slope of $l_3 = \frac{-3}{4}$

iv) Slope of $l_4 = \frac{-4}{3}$

Here l_1 and l_3 are perpendicular

l_2 and l_4 are perpendicular

But 3rd option is a contradiction

12. A straight line has equation 8y = 4x +21. Which of the following is true

(1) The slope is 0.5 and the y intercept is 2.6

(2) The slope is 5 and the y intercept is 1.6

(3) The slope is 0.5 and the y intercept is 1.6

(4) The slope is 5 and the y intercept is 2.6

Hint : **Ans : (1)**

Given equation is 8y = 4x + 21

$$\Rightarrow y = \frac{1}{2}x + \frac{21}{8}$$

$$\Rightarrow y = 0.5x + 2.6$$

$\therefore$ Slope = 0.5, y − int = 2.6

13. When proving that a quadrilateral is a trapezium, it is necessary to show

(1) Two sides are parallel.

(2) Two parallel and two non-parallel sides.

(3) Opposite sides are parallel.

(4) All sides are of equal length.

Hint : **Ans : (2)**

A quadrilateral is trapezoid if one pair of opposite sides are parallel and another pair is non parallel.

14. When proving that a quadrilateral is a parallelogram by using slopes you must find

(1) The slopes of two sides

(2) The slopes of two pair of opposite sides

(3) The lengths of all sides

(4) Both the lengths and slopes of two side

Hint : **Ans : (1)**

We should find the slopes of all the sides when proving a quadrilateral is a parallelogram.

15. (2, 1) is the point of intersection of two lines.

(1) $x - y - 3 = 0$; $3x - y - 7 = 0$

(2) $x + y = 3$; $3x + y = 7$

(3) $3x + y = 3$; $x + y = 7$

(4) $x + 3y - 3 = 0$; $x - y - 7 = 0$

Hint : **Ans : (2)**

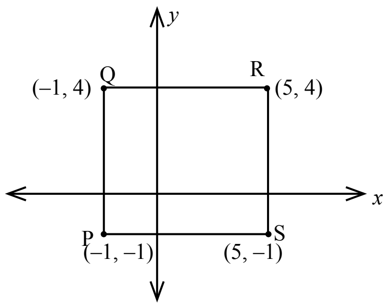
Substitute (2, 3) & check in all pair of lines.

UNIT EXERCISE - 5

1. PQRS is a rectangle formed by joining the points P(-1,-1), Q(-1, 4) ,R(5, 4) and S(5,-1) . A, B, C and D are the mid-points of PQ, QR, RS and SP respectively. Is the quadrilateral ABCD a square, a rectangle or a rhombus? Justify your answer.

Solution :

Given P = (−1, −1) Q (−1, 4), R (5, 4), S (5, −1)



$$A = \text{Mid point of } PQ = \left(\frac{-1-1}{2}, \frac{-1+4}{2}\right) = \left(-1, \frac{3}{2}\right)$$

$$B = \text{Mid point of } QR = \left(\frac{-1+5}{2}, \frac{4+4}{2}\right) = (2, 4)$$

$$C = \text{Mid point of } RS = \left(\frac{5+5}{2}, \frac{4-1}{2}\right) = \left(5, \frac{3}{2}\right)$$

$$D = \text{Mid point of } PS = \left(\frac{-1+5}{2}, \frac{-1-1}{2}\right) = (2, -1)$$

$$\text{Slope of } AB = \frac{4-\frac{3}{2}}{2+1} = \frac{5}{6}$$

$$\text{Slope of } BC = \frac{\frac{3}{2}-4}{5-2} = \frac{-\frac{5}{2}}{3} = \frac{-5}{6}$$

$$\text{Slope of } CD = \frac{-1-\frac{3}{2}}{2-5} = \frac{-\frac{5}{2}}{-3} = \frac{5}{6}$$

$$\text{Slope of } AD = \frac{-1-\frac{3}{2}}{2+1} = \frac{-5}{6}$$

∴ AB | | CD, BC | | AD

$$\text{Mid Point of } AC = \left(\frac{-1+5}{2}, \frac{\frac{3}{2}+\frac{3}{2}}{2}\right) = \left(2, \frac{3}{2}\right)$$

$$\text{Mid Point of } BD = \left(\frac{2+2}{2}, \frac{4-1}{2}\right) = \left(2, \frac{3}{2}\right)$$

∴ diagonals bisect each other & opposite sides are parallel.

ABCD is a rhombus.

---

2. The area of a triangle is 5 sq.units. Two of its vertices are (2,1) and (3, –2). The third vertex is (x, y) where $y = x + 3$. Find the coordinates of the third vertex.

Solution :

Given, area of triangle ABC is 5 sq.units and A (2, 1), B (3, −2), C (x, y) where y = x + 3

∴ Area of Δ

$$= \frac{1}{2}\begin{bmatrix} 2 & 3 & x & 2 \\ 1 & -2 & y & 1 \end{bmatrix} = 5$$

$$\Rightarrow \quad (-4 + 3y + x) - (3 - 2x + 2y) = 10$$

$$\Rightarrow \quad x + 3y - 4 - 3 + 2x - 2y = 10$$

$$\Rightarrow \quad 3x + y = 17 \quad \text{...... (1)}$$

$$\text{Also given,} \quad x - y = -3 \quad \text{...... (2)}$$

$$\text{Adding,} \quad 4x = 14$$

$$x = \frac{7}{2}$$

Sub, $x = \frac{7}{2}$ in (2)

$$\frac{7}{2} - y = -3$$

$$y = \frac{7}{2} + 3 = \frac{13}{2}$$

∴ Third vertex is $\left(\frac{7}{2}, \frac{13}{2}\right)$

3. Find the area of a triangle formed by the lines $3x + y - 2 = 0$, $5x + 2y - 3 = 0$ and $2x - y - 3 = 0$.

Solution :

Given lines are

$3x + y - 2 = 0$ .... (1)

$5x + 2y - 3 = 0$ .... (2)

$2x - y - 3 = 0$ .... (3)

Solving (1) & (2)

$$\begin{array}{rcl} (1)\times 2 & \Rightarrow & 6x + 2y = 4 \\ (2) & \Rightarrow & \underline{5x + 2y = 3} \\ & & x \quad = 1 \end{array}$$

Sub. in (1)

$3 + y - 2 = 0$

$y = -1$

$\therefore$ A $(1, -1)$

Solving (1) & (2)

$$\begin{array}{r} 3x + y = 2 \\ \underline{2x - y = 3} \\ 5x \quad = 5 \end{array}$$

$x = 1$

$\therefore y = -1$

$\therefore$ B is $(1, -1)$

Solving (2) & (3)

$$\begin{array}{rcl} (2) & \Rightarrow & 5x + 2y = 3 \\ (3)\times 2 & \Rightarrow & \underline{4x - 2y = 6} \\ & & 9x \quad = 9 \end{array}$$

$x = 1$

$\therefore (3) \Rightarrow 2 - y - 3 = 0$

$\Rightarrow \quad -y = 1$

$\therefore \quad y = -1$

$\therefore$ C is $(1, -1)$

$\therefore$ A $(1, -1)$, B $(1, -1)$, C $(1, -1)$

$\therefore$ All point line on the same line

$\therefore$ Area of $\Delta = 0$ sq. units

4. If vertices of a quadrilateral are at A(-5,7) , B(-4,k) , C(-1,-6) and D(4,5) and its area is 72 sq.units. Find the value of k.

Solution :

Given vertices of quadrilateral are

A $(-5, 7)$, B $(-4, k)$, C $(-1, -6)$, D $(4, 5)$ & its area = 72 sq.units

$$\Rightarrow \frac{1}{2}\begin{bmatrix} -5 & -4 & -1 & 4 & -5 \\ 7 & k & -6 & 5 & 7 \end{bmatrix} = 72$$

$(-5k+24-5+28) - (-28-k-24-25) = 144$

$(-5k + 47) - (-k - 77) = 144$

$-4k + 124 = 144$

$-4k = 20$

$k = -5$

5. Without using distance formula, show that the points (-2,-1) , (4, 0) , (3, 3) and (-3,2) are vertices of a parallelogram.

Solution :

Given vertices are

$(-2, -1)$ $(4, 0)$, $(3, 3)$, $(-3, 2)$

Let A $(-2, -1)$, B $(4, 0)$, C $(3, 3)$, D $(-3, 2)$

Slope of $AB = \dfrac{0+1}{4+2} = \dfrac{1}{6}$

Slope of $CD = \dfrac{3-2}{3+3} = \dfrac{1}{6}$

$\therefore$ AB & CD are parallel

Slope of $AD = \dfrac{2+1}{-3+2} = \dfrac{3}{-1} = -3$

Slope of $BC = \dfrac{0-3}{4-3} = \dfrac{-3}{1} = -3$

$\therefore$ AD & BC are parallel

$\therefore$ ABCD is a parallelogram

6. Find the equations of the lines, whose sum and product of intercepts are 1 and –6 respectively.

Solution :

Given, sum of intercepts = 1

$\Rightarrow$ a + b = 1

$\therefore$ b = 1 − a

Given, product of intercepts = − 6

$\Rightarrow$ ab = − 6

$\therefore$ ab = − 6 $\Rightarrow$ a (1 − a) = −6

$\Rightarrow$ a − a² = −6

$\Rightarrow$ a² − a − 6 = 0

$\Rightarrow$ (a − 3) (a + 2) = 0

$\therefore$ a = 3, −2

If a = 3, b = −2

If a = −2, b = 3

$$a = 3, b = -2 \Rightarrow \frac{x}{3} + \frac{y}{-2} = 1$$

$$\Rightarrow \frac{x}{3} - \frac{y}{2} = 1$$

$$\Rightarrow 2x - 3y - 6 = 0$$

$$a = -2, b = 3 \Rightarrow \frac{x}{-2} + \frac{y}{3} = 1$$

$$\Rightarrow \frac{-3x + 2y}{6} = 1$$

$$\Rightarrow -3x + 2y = 6$$

$$\Rightarrow 3x - 2y + 6 = 0$$

7. The owner of a milk store finds that, he can sell 980 litres of milk each week at ₹14/litre and 1220 litres of milk each week at ₹16/litre. Assuming a linear relationship-between selling price and demand, how many litres could he sell weekly at ₹17/litre?

Solution :

By data given,

the linear relationship between selling price per litre and demand is the equation of the line passing through the points

(14, 980) and (16, 1220) is

$$\therefore \quad \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - 980}{1220 - 980} = \frac{x - 14}{16 - 14}$$

$$\Rightarrow \frac{y - 980}{240} = \frac{x - 14}{2}$$

$$\Rightarrow y - 980 = 120(x - 14)$$

$$\Rightarrow \quad y = 120(x - 14) + 980 \quad \text{........ (1)}$$

When x = Rs.17 / litre

y = 120 (17 - 14) + 980

y = 120 (3) + 980

= 360 + 980

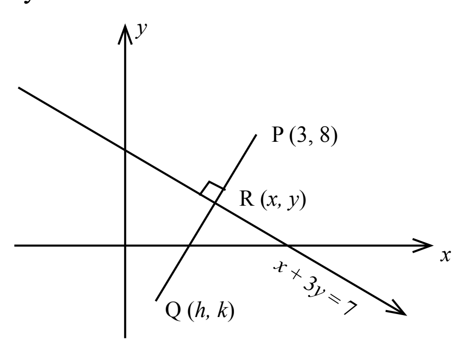
= 1340

$\therefore$ He can sell weekly 1340 litres at Rs.17/litre

8. Find the image of the point (3,8) with respect to the line x + 3y = 7 assuming the line to be a plane mirror.

Solution :

To find the image of (3, 8) w.r.to the line x + 3y = 7



Let Q (h, k) be the image of P (3, 8) about the line x + 3y = 7 .... (1)

Since the line is assumed as a plane mirror P & Q are equidistant from R (x, y)

$\therefore$ R is the midpoint and PQ is a perpendicular bisector of (1)

$$\therefore (x, y) = \left(\frac{h+3}{2}, \frac{k+8}{2}\right)$$

$$\therefore \quad x = \frac{h+3}{2}, \; y = \frac{k+8}{2}$$

Since R (x, y) is a point on (1)

$$\left(\frac{h+3}{2}\right) + 3\left(\frac{k+8}{2}\right) = 7$$

$\Rightarrow$ h + 3 + 3k + 24 = 14

$\Rightarrow$ h + 3k = − 13 .... (2)

Also, slope of PQ × Slope of (1) = −1

$$\Rightarrow \frac{k-8}{h-3} \times \frac{-1}{3} = -1$$

$$\Rightarrow \frac{k-8}{h-3} = 3$$

$$\Rightarrow k - 8 = 3h - 9$$

$$\Rightarrow 3h - k = 1 \quad ....(3)$$

Solving (2) & (3)

$$\begin{array}{rcl} (2) & \Rightarrow & h + 3k = -13 \\ (3) \times 3 & \Rightarrow & 9h - 3k = 3 \\ \hline & & 10h = -10 \\ & & h = -1 \end{array}$$

Sub in (2)

− 1 + 3k = − 13

3k = − 12

k = − 4

$\therefore$ Q is (− 1, − 4), which is the image of P (3, 8)

9. Find the equation of a line passing through the point of intersection of the lines 4x + 7y −3 = 0 and 2x −3y +1 = 0 that has equal intercepts on the axes.

Solution :

First, we find the point of intersection of the lines

4x + 7y = 3 .... (1)

2x − 3y = −1 .... (2)

$$\begin{array}{rcl} (1) & \Rightarrow & 4x + 7y = +3 \\ (2) \times 2 & \Rightarrow & 4x - 6y = -2 \\ \hline & & 13y = 5 \\ & & y = 5/13 \end{array}$$

Sub in (1)

$$4x + 35/13 = 3$$

$$\Rightarrow 4x = 3 - 35/13$$

$$\Rightarrow 4x = 4/13$$

$$\Rightarrow x = 1/13$$

$\therefore$ The point is $\left(1/13, 5/13\right)$

Equation of plane in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \text{where } a = b$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow x + y = a \quad ....(1)$$

Since (1) passes through $\left(1/13, 5/13\right)$

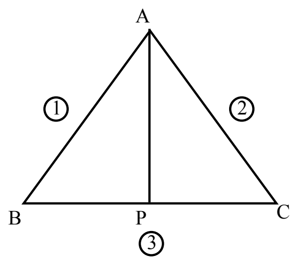
$$a = 1/13 + 5/13 = 6/13$$

$$\therefore x + y = 6/13$$

$$\Rightarrow \boxed{13x + 13y - 6 = 0}$$

10. A person standing at a junction (crossing) of two straight paths represented by the equations 2x −3y + 4 = 0 and 3x + 4y −5 = 0 seek to reach the path whose equation is 6x −7y + 8 = 0 in the least time. Find the equation of the path that he should follow.

Solution :



Given straight paths are

$2x - 3y + 4 = 0$ .... (1) (AB)

$3x + 4y - 5 = 0$ .... (2) (AC)

To reach the path

$6x - 7y + 8 = 0$ .... (3) (BC)

in the least time

To find : Equation of the path (AP)

A → Position of the person

Solving (1) & (2)

$$(1) \times 3 \Rightarrow 6x - 9y = -12$$
$$(2) \times 2 \Rightarrow 6x + 8y = 10$$
$$-17y = -22$$
$$\therefore y = \frac{22}{7}$$

Sub in (1)

$$2x - \frac{66}{17} = -4$$
$$\Rightarrow 2x = \frac{66}{17} - 4$$
$$\Rightarrow 2x = \frac{66-68}{17}$$
$$\Rightarrow 2x = \frac{-2}{7}$$
$$\Rightarrow x = \frac{-1}{7}$$
$$\therefore \text{ A is } \left(\frac{-1}{7}, \frac{22}{17}\right)$$

Also AP is perpendicular to BC, whose slope is $\frac{6}{7}$

$\therefore$ Slope of $AP = \frac{-7}{6}$

$\therefore$ Equation of the required path AP is

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - \frac{22}{17} = \frac{-7}{6}\left(x + \frac{1}{7}\right)$$
$$\Rightarrow \frac{17y-22}{17} = \frac{-7}{6}\left(\frac{7x+1}{7}\right)$$
$$\Rightarrow \frac{17y-22}{17} = \frac{-(7x+1)}{6}$$
$$\Rightarrow 6(17y-22) = -17(7x+1)$$
$$\Rightarrow 102y - 132 = -119x - 17$$
$$\Rightarrow 119x + 102y - 125 = 0$$

is the required path.

PROBLEMS FOR PRACTICE

1. If P $\left(\frac{a}{2}, 4\right)$ is the mid point of the line joining the points A (−6, 5), B (−2, 3), then find 'a'. **(Ans : a = −8)**
2. Find the area of ΔABC whose vertices are
   i) A (3, 8), B (− 4, 2), C (5, −1) **(Ans : 37.5)**
   ii) A (1, 2), B (−3, 4), C (−5, −6) **(Ans : 22)**
   iii) A (0, 1), B (2, 3), C (3, 4) **(Ans : 0)**
3. If the area of Δ is 12 sq. units with vertices (a, −3), (3, a) (−1, 5) find 'a'. **(Ans : 1, 3)**
4. If the area of Δ formed by (x, y), (1, 2), (2, 1) is 6 sq.units, prove that x + y = 15.
5. For what values of k, are the points (8, 1), (3, −2k), and (k, −5) are collinear.
   $\left(\text{Ans}: k = 2, \frac{11}{2}\right)$

6. Find the value of 'p' if the area of Δ formed by (p + 1, 2p − 2), (p − 1, p) and (p − 3, 2p − 6) is 0. **(Ans : p = 4)**

7. Find the area of quadrilateral whose vertices are,

   i) A (3, −1), B (9, −5), C (14, 0), D (9, 19) **(Ans : 132)**

   ii) P (−5,−3), Q (−4, −6), R (2, −3), S (1, 2) **(Ans : 28)**

   iii) E (−3,2), F (5, 4), G (7, −6), H (−5, −4) **(Ans : 80)**

   iv) A (−4, 5), B (0, 7), C (5,−5), D (−4, −2) **(Ans : 60.5)**

8. If (3, 3), (6, y), (x, 7) and (5, 6) are the vertices of a parallelogram taken in order, find x and y. **(Ans : x = 8, y = 4)**

9. If the points (p, q), (m, n), (p − m, q − n) are collinear, show that pn = qm.

10. Three vertices of a parallelogram ABCD are (1, 2), (4, 3), (6, 6). Find the 4th vertex D. **(Ans : 3, 5)**

11. The line joining A (0, 5) and B (4, 2) is perpendicular to the line joining C (−1, −2), D (5, b) find 'b'. **(Ans : b = 6)**

12. Find the equation of the line passing through (9, −1) having its x-intercept thrice as its y-intercept. **(Ans : x + 3y − 6 = 0)**

13. Find the slope and y-intercept of the line 10x + 15y + 6 = 0. $\left(\text{Ans}: m = \frac{-2}{5}, c = \frac{-2}{5}\right)$

14. Find whether the lines drawn through the two pair of points are parallel (or) perpendicular.

    i) (5, 2), (0, 5) and (0, 0), (−5, 3) **(Ans : parallel)**

    ii) (4, 5), (0, −2) and (−5, 1), (2, −3) **(Ans : perpendicular)**

15. A line passing through the points (2, 7) and (3, 6) is parallel to the line joining (9, a) and (11, 2) find 'a'. **(Ans : −2 (or) 4)**

16. Find the equation of a straight line whose slope is $\frac{2}{3}$ and passing through the point (5, −4) **(Ans : 2x − 3y − 22 = 0)**

17. Without using distance formula, show that the points P (3, 2), Q (0, −3), R (−3, −2) and S (0, 1) are the vertices of a parallelogram.

18. A triangle has vertices at (3, 4), (1, 2), (−5, −6). Find the slopes of the medians.

    $\left(\text{Ans}: \frac{6}{5}, \frac{3}{2}, \frac{9}{7}\right)$

19. Find the equation of altitude from A of a Δ ABC whose vertices are (1, −3), (−2, 5), (−3, 4). **(Ans : x + y + 2 = 0)**

20. Find the values of 'p' of the straight lines 8px + (2 − 3p)y + 1 = 0 and px + 8y − 7 = 0 are perpendicular to each other.

    (Ans : p = 1, 2)

21. Find the equation of the straight line passing through (1, 4) and having intercepts in the ratio 3 : 5 **(Ans : 5x + 3y = 17)**

22. Find the area of the triangle formed by sides x + 4y − 9 = 0, 9x + 10y + 23 = 0, 7x + 2y −11 = 0 **(Ans : 26 sq.units)**

23. Find the equation of the line through the point of intersection of the lines 2x + y − 5 = 0, x + y − 3 = 0 and bisecting the line segment joining the points (3, −2) (−5, 6)

    (Ans : x + 3y − 5 = 0)

24. Find the image of the point (−2, 3) w.r.to the line x + 2y − 9 = 0 **(Ans : 0, 7)**

25. The equation of the diagonals of a rectangle are 4x − 7y = 0, 8x − y = 26 and one of its sides is 2x + 3y = 0, find the equation of the other sides.

(Ans : 2x+3y−26=0, 3x−2y−13=0, 3x−2y = 0)

OBJECTIVE TYPE QUESTIONS

1. The point whose the line 3x - y + 6 = 0 meets the x - axis is

   a) (0, 6) b) (−2, 0)

   c) (−1, 3) d) (2, 0) **Ans : (b)**

2. The point of intersection of the lines 2x + y − 3 = 0, 5x + y − 6 = 0 lies in the quadrant

   a) I b) II c) IV d) III

   Ans : (a)

3. The value of 'k' if the lines 3x + 6y + 7 = 0 and 2x + ky − 5 = 0 are perpendicular is

   a) 1 b) −1 c) 2 d) $\frac{1}{2}$

   Ans : (b)

4. The slope of the line which is parallel to the line joining the points (0, 0) and (−5, 5) is

   a) 1 b) −1 c) 2 d) −2

   Ans : (a)

5. The area of triangle formed by (0, 4) (4, 0) and origin is

   a) 8 b) 16 c) 2 d) 4 **Ans : (a)**

6. The equation of a straight line which has the y-intercept 5 and slope 2 is

   a) 2x + y + 5 = 0 b) 2x − y + 5 = 0

   c) 2x − y − 5 = 0 d) 2x + y − 5 = 0

   Ans : (b)

7. If the point (a, a) lies on the line 3x + 4y − 14 = 0 then 'a' is

   a) 2 b) −2 c) 1 d) 0 **Ans : (a)**

8. Equation of the line parallel to y-axis and passing through (−2, 3) is

   a) x = 3 b) y = −2

   c) x = −2 d) y = 3 **Ans (c)**

9, The x-intercept of the line 3x − 2y + 12 = 0 is

   a) 6 b) −6 c) 4 d) −4

   Ans : (d)

10. AB is parallel to CD. If A and B are (2, 3) and (6, 9) the slope of CD is

    a) $\frac{4}{9}$ b) $\frac{3}{2}$ c) $\frac{2}{3}$ d) $\frac{9}{4}$

    Ans : (b)

11. If (1, 2), (4, 6), (x, 6) and (3, 2) are the vertices of a parallelogram taken in order, then x is

    a) 6 b) 2 c) 1 d) 3

    Ans : (a)

12. If the slope of a line is $-\sqrt{3}$, then the angle of inclination is

    a) 60^0 b) 30^0 c) 120^0 d) 150^0

    Ans : (c)

13. The value of a for which (−a, a) is collinear with the points (2, 0), (0, 1) is

    a) 1 b) 2 c) −2 d) −1

    Ans : (b)

14. The x - coordinates of the point of intersection of the lines x − 7y + 5 = 0, 3x + y = 0 is

    a) $\frac{15}{22}$ b) $\frac{5}{22}$ c) $\frac{-5}{22}$ d) $\frac{-10}{22}$

    Ans : (c)

15. The equations of the 4 sides of a rectangle are x = 1, y = 2, x = 4, y = 5. One vertex of the rectangle is at

    a) (2, 4) b) (5, 1)

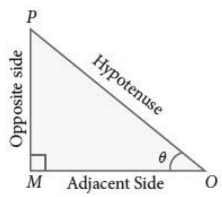
    c) (2, 5) d) (4, 2)

    Ans : (d)

CHAPTER 6 TRIGONOMETRY

TRIGONOMETRIC RATIOS

Key Points

✓ 

✓ $$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{MP}{OP}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{OM}{OP}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\text{cosec }\theta = \frac{1}{\sin\theta}; \sec\theta = \frac{1}{\cos\theta}$$

✓ $\sin(90^o - \theta) = \cos\theta$ $\cos(90^o - \theta) = \sin\theta$ $\tan(90^o - \theta) = \cot\theta$

✓ $\text{cosec}(90^o - \theta) = \sec\theta$ $\sec(90^o - \theta) = \text{cosec }\theta$ $\cot(90^o - \theta) = \tan\theta$

Trigonometric Ratio \ θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\text{cosec}\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

I. TRIGONOMETRIC IDENTITIES

Key Points

✓ $\sin^2\theta + \cos^2\theta = 1$	$\Rightarrow$	$\sin^2\theta = 1 - \cos^2\theta$ (or) $\cos^2\theta = 1 - \sin^2\theta$
✓ $1 + \tan^2\theta = \sec^2\theta$	$\Rightarrow$	$\tan^2\theta = \sec^2\theta - 1$ (or) $\sec^2\theta - \tan^2\theta = 1$
✓ $1 + \cot^2\theta = \text{cosec}^2\theta$	$\Rightarrow$	$\cot^2\theta = \text{cosec}^2\theta - 1$ (or) $\text{cosec}^2\theta - \cot^2\theta = 1$

Example 6.1

Prove that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

Solution :

$$\tan^2\theta - \sin^2\theta = \tan^2\theta - \frac{\sin^2\theta}{\cos^2\theta}.\cos^2\theta$$

$$= \tan^2\theta\,(1 - \cos^2\theta) = \tan^2\theta\sin^2\theta$$

Example 6.2

Prove that $\dfrac{\sin A}{1+\cos A} = \dfrac{1-\cos A}{\sin A}$

Solution :

$$\frac{\sin A}{1+\cos A} = \frac{\sin A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}$$

[multiply numerator and denominator by the conjugate of 1 + cosA]

$$= \frac{\sin A\,(1-\cos A)}{(1+\cos A)(1-\cos A)} = \frac{\sin A\,(1-\cos A)}{1-\cos^2 A}$$

$$= \frac{\sin A\,(1-\cos A)}{\sin^2 A} = \frac{1-\cos A}{\sin A}$$

Example 6.3

Prove that $1 + \dfrac{\cot^2\theta}{1+\text{cosec}\,\theta} = \text{cosec}\,\theta$

Solution :

$$= 1 + \frac{\cot^2\theta}{1+\text{cosec}\,\theta}$$

$$= 1 + \frac{\text{cosec}^2\theta - 1}{\text{cosec}\,\theta + 1} \quad \left[\text{since } \text{cosec}^2\theta - 1 = \cot^2\theta\right]$$

$$= 1 + \frac{(\text{cosec}\,\theta + 1)(\text{cosec}\,\theta - 1)}{\text{cosec}\,\theta + 1}$$

$$= 1 + (\text{cosec}\,\theta - 1) = \text{cosec}\,\theta$$

Example 6.4

Prove that $\sec\theta - \cos\theta = \tan\theta\sin\theta$

Solution :

$$\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta}{\cos\theta} \quad \left[\text{since } 1-\cos^2\theta = \sin^2\theta\right]$$

$$= \frac{\sin\theta}{\cos\theta} \times \sin\theta = \tan\theta\sin\theta$$

Example 6.5

Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \text{cosec}\,\theta + \cot\theta$

Solution :

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}}$$

[multiply numerator and denominator by the conjugate of 1 – cosθ]

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}}$$

$$[\text{since } \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{1+\cos\theta}{\sin\theta} = \text{cosec}\,\theta + \cot\theta$$

Example 6.6

Prove that $\dfrac{\sec\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \cot\theta$

Solution :

$$= \frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \cot\theta$$

Example 6.7

Prove that $\sin^2 A \cos^2 B + \cos^2 A \sin^2 B + \cos^2 A \cos^2 B + \sin^2 A \sin^2 B = 1$

Solution :

$= \sin^2 A \cos^2 B + \cos^2 A \sin^2 B + \cos^2 A \cos^2 B + \sin^2 A \sin^2 B$

$= \sin^2 A \cos^2 B + \sin^2 A \sin^2 B + \cos^2 A \sin^2 B + \cos^2 A \cos^2 B$

$= \sin^2 A (\cos^2 B + \sin^2 B) + \cos^2 A (\sin^2 B + \cos^2 B)$

$= \sin^2 A (1) + \cos^2 A (1)$ (since $\sin^2 B + \cos^2 B = 1$)

$= \sin^2 A + \cos^2 A = 1$

Example 6.8

If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Solution :

Now, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

Squaring both sides,

$(\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$

$\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$

$2\cos^2\theta - \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$

$\cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$

$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$

$$\cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta+\sin\theta} = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta}$$

$= \sqrt{2}\sin\theta$

[since $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$]

Therefore, $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Example 6.9

Prove that $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

Solution :

$(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta\sin^2\theta \times 1}{\sin^2\theta\cos^2\theta} = 1$$

Example 6.10

Prove that $\dfrac{\sin A}{1+\cos A} + \dfrac{\sin A}{1-\cos A} = 2\operatorname{cosec} A$

Solution :

$$= \frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A}$$

$$= \frac{\sin A(1-\cos A) + \sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$$

$$= \frac{\sin A - \sin A\cos A + \sin A + \sin A\cos A}{1-\cos^2 A}$$

$$= \frac{2\sin A}{1-\cos^2 A} = \frac{2\sin A}{\sin^2 A} = 2\operatorname{cosec} A$$

Example 6.11

If cosec θ + cot θ = P, then prove that

$$\cos\theta = \frac{P^2-1}{P^2+1}$$

Solution :

Given cosec θ + cot θ = P .... (1)

$\text{cosec}^2\,\theta - \cot^2\theta = 1$ (identity)

$$\text{cosec}\,\theta - \cot\theta = \frac{1}{\text{cosec}\,\theta + \cot\theta}$$

$$\text{cosec}\,\theta - \cot\theta = \frac{1}{P} \quad ....(2)$$

Adding (1) and (2) we get,

$$2\text{cosec}\,\theta = P + \frac{1}{P}$$

$$2\text{cosec}\,\theta = \frac{P^2+1}{P} \quad ....(3)$$

Subtracting (2) from (1), we get,

$$2\cot\theta = P - \frac{1}{P}$$

$$2\cot\theta = \frac{P^2-1}{P} \quad ....(4)$$

Dividing (4) by (3) we get,

$$\frac{2\cot\theta}{2\text{cosec}\,\theta} = \frac{P^2-1}{P} \times \frac{P}{P^2+1}$$

$$\text{gives, } \cos\theta = \frac{P^2-1}{P^2+1}$$

Example 6.12

Prove that $\tan^2 A - \tan^2 B = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$

Solution :

$\tan^2 A - \tan^2 B$

$$= \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{\sin^2 A\cos^2 B - \sin^2 B\cos^2 A}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A\left(1-\sin^2 B\right) - \sin^2 B\left(1-\sin^2 A\right)}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$$

Example 6.13

Prove that

$$\left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right) = 2\sin A\cos A$$

Solution :

$$= \left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right)$$

$$= \left(\frac{(\cos A - \sin A)(\cos^2 A + \sin^2 A + \cos A\sin A)}{\cos A - \sin A}\right) -$$

$$= \left(\frac{(\cos A + \sin A)(\cos^2 A + \sin^2 A - \cos A\sin A)}{\cos A + \sin A}\right)$$

$$\left[\begin{array}{l}\text{since } a^3 - b^3 = (a-b)(a^2+b^2+ab)\\ \qquad a^3 + b^3 = (a+b)(a^2+b^2-ab)\end{array}\right]$$

$$= (1 + \cos A\sin A) - (1 - \cos A\sin A)$$

$$= 2\cos A\sin A$$

Example 6.14

Prove that

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\text{cosec}A + \cot A - 1} = 1$$

Solution :

$$=\frac{\sin A}{\sec A+\tan A-1}+\frac{\cos A}{\text{cosec}A+\cot A-1}$$

$$=\frac{\sin A\left(\text{cosec}A+\cot A-1\right)+\cos A\left(\sec A+\tan A-1\right)}{\left(\sec A+\tan A-1\right)\left(\text{cosec}A+\cot A-1\right)}$$

$$=\frac{\sin A\text{cosec}\,A+\sin A\cot A-\sin A+\cos A\sec A+\cos A\tan A-\cos A}{\left(\sec A+\tan A-1\right)\left(\text{cosec}A+\cot A-1\right)}$$

$$=\frac{1+\cos A-\sin A+1+\sin A-\cos A}{\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}-1\right)\left(\frac{1}{\sin A}+\frac{\cos A}{\sin A}-1\right)}$$

$$=\frac{2}{\left(\frac{1+\sin A-\cos A}{\cos A}\right)\left(\frac{1+\cos A-\sin A}{\sin A}\right)}$$

$$=\frac{2\sin A\cos A}{\left(1+\sin A-\cos A\right)\left(1+\cos A-\sin A\right)}$$

$$=\frac{2\sin A\cos A}{\left[\left(1+\sin A-\cos A\right)\right]\left[\left(1-\sin A-\cos A\right)\right]}$$

$$=\frac{2\sin A\cos A}{1-(1-2\sin A\cos A)}$$

$$=\frac{2\sin A\cos A}{1-1+2\sin A\cos A}=\frac{2\sin A\cos A}{2\sin A\cos A}=1$$

Example 6.15

Show that $\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right)=\left(\frac{1-\tan A}{1-\cot A}\right)^2$

Solution :

LHS

$$\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right)=\frac{1+\tan^2 A}{1+\frac{1}{\tan^2 A}}$$

$$=\frac{1+\tan^2 A}{\frac{\tan^2 A+1}{\tan^2 A}}=\tan^2 A \qquad \text{...... (1)}$$

RHS

$$\left(\frac{1-\tan A}{1-\cot A}\right)^2=\left(\frac{1-\tan A}{1-\frac{1}{\tan A}}\right)$$

$$=\left(\frac{1-\tan A}{\frac{\tan A-1}{\tan A}}\right)^2=(-\tan A)^2=\tan^2 A \qquad \text{..... (2)}$$

From (1) and (2),

$$\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right)=\left(\frac{1-\tan A}{1-\cot A}\right)^2$$

Example 6.16

Prove that

$$\frac{(1+\cot A+\tan A)\,(\sin A-\cos A)}{\sec^3 A-\cos ec^3 A}=\sin^2 A\cos^2 A$$

Solution :

$$\frac{(1+\cot A+\tan A)\,(\sin A-\cos A)}{\sec^3 A-\cos ec^3 A}$$

$$=\frac{\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)}{(\sec A-\text{cosec}A)\,(\sec^2 A+\sec A\text{cosec}A+\text{cosec}^2 A)}$$

$$=\frac{\frac{(\sin A\cos A+\cos^2 A+\sin^2 A)\,(\sin A-\cos A)}{\sin A\cos A}}{(\sec A-\text{cosec}A)\left(\frac{1}{\cos^2 A}+\frac{1}{\cos A\sin A}+\frac{1}{\sin^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)\left(\frac{\sin A}{\sin A\cos A}-\frac{\cos A}{\sin A\cos A}\right)}{(\sec A-\text{cosec}A)\left(\frac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin^2 A\cos^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)\,(\sec A-\text{cosec}A)}{(\sec A-\text{cosec}A)\,(1+\sin A\cos A)}\times\sin^2 A\cos^2 A$$

$$=\sin^2 A\cos^2 A$$

Example 6.17

If $\frac{\cos^2\theta}{\sin\theta}=$ p and $\frac{\sin}{\cos}$ = q, then prove that

$p^2 q^2 (p^2 + q^2 +3) = 1$

Solution :

We have $\frac{\cos^2\theta}{\sin\theta}=$ p .... (1) and $\frac{\sin^2\theta}{\cos\theta}$ = q.... (2)

$p^2 q^2 (p^2 + q^2 +3) =$

$$\left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\times\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2+\left(\frac{\sin^2\theta}{\cos\theta}\right)^2+3\right]$$

[from (1) and (2)]

$$=\left(\frac{\cos^4\theta}{\sin^2\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right)\times\left[\left(\frac{\cos^4\theta}{\sin^2\theta}\right)+\left(\frac{\sin^4\theta}{\cos^2\theta}\right)+3\right]$$

$$=(\cos^2\theta\times\sin^2\theta)\times$$

$$\left[\left(\frac{\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)\right]$$

$$=\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta$$

$$=(\cos^2\theta)^3\times(\sin^2\theta)^3+3\sin^2\theta\cos^2\theta$$

$$=\left[(\cos^2\theta\sin^2\theta)^3-3\cos^2\theta\sin^2\theta(\cos^2\theta\sin^2\theta)\right]$$

$$+3\sin^2\theta\cos^2\theta$$

$$=1-3\cos^2\theta\sin^2\theta(1)+3\cos^2\theta\sin^2\theta=1$$

EXERCISE 6.1

1. Prove the following identities.

i) cot θ + tan θ = sec θ cosec θ

ii) $\tan^4\theta+\tan^2\theta=\sec^4\theta-\sec^2\theta$

Solution:

i) LHS

$$=\cot\theta+\tan\theta$$

$$=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}$$

$$=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta.\cos\theta}$$

$$=\frac{1}{\sin\theta.\cos\theta}$$

$$=\frac{1}{\cos\theta}\times\frac{1}{\sin\theta}$$

$$=\sec\theta.\operatorname{cosec}\theta$$

= RHS

ii) RHS

$$=\tan^4\theta+\tan^2\theta$$

$$=\tan^2\theta(\tan^2\theta+1)$$

$$=(\sec^2\theta-1).(\sec^2\theta)$$

$$=\sec^4\theta-\sec^2\theta$$

= RHS

2. Prove the following identities.

$$i)\ \frac{1-\tan^2\theta}{\cot^2\theta-1}=\tan^2\theta$$

$$ii)\ \frac{\cos\theta}{1+\sin\theta}=\sec\theta-\tan\theta$$

Solution :

i) LHS

$$=\frac{1-\tan^2\theta}{\cot^2\theta-1}$$

$$=\frac{1-\tan^2\theta}{\frac{1}{\tan^2\theta}-1}$$

$$=\frac{1-\tan^2\theta}{\frac{1-\tan^2\theta}{\tan^2\theta}}$$

$$=\tan^2\theta$$

= RHS

ii) LHS

$$=\frac{\cos\theta}{1+\sin\theta}$$

$$=\frac{\cos\theta}{1+\sin\theta}\times\frac{1-\sin\theta}{1-\sin\theta}$$

$$=\frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$$

$$=\frac{1-\sin\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\sec\theta-\tan\theta$$

= RHS

3. Prove the following identities.

$$i)\ \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\sec\theta+\tan\theta$$

$$ii)\ \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=2\sec\theta$$

Solution :

i) LHS

$$=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

$$=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}}$$

$$=\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$

$$=\frac{1+\sin\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}$$

$$=\sec\theta+\tan\theta$$

= RHS

ii) LHS

$$=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$=(\sec\theta+\tan\theta)+\frac{1}{\sec\theta+\tan\theta}$$

$$=(\sec\theta+\tan\theta)+(\sec\theta-\tan\theta)$$

$$=2\sec\theta$$

= RHS

4. Prove the following identities.

i) $\sec^6\theta=\tan^6\theta+3\tan^2\theta\sec^2\theta+1$

ii) $(\sin\theta+\sec\theta)^2+(\cos\theta+\text{cosec}\,\theta)^2=1+(\sec\theta+\text{cosec}\,\theta)^2$

Solution :

i) LHS

$=\sec^6\theta$

$=(\sec^2\theta)^3$

$=(1+\tan^2\theta)^3$

$(\because a+b)^3=a^3+b^3+3ab\,(a+b))$

$=1+\tan^6\theta+3(1)\tan^2\theta)\,(1+\tan^2\theta)$

$=1+\tan^6\theta+3\tan^2\theta\,.\,\sec^2\theta$

= RHS

ii) LHS

$=(\sin\theta+\sec\theta)^2+(\cos\theta+\text{cosec}\,\theta)^2$

$=\sin^2\theta+\sec^2\theta+2\sin\theta\,.\,\sec\theta+\cos^2\theta+\text{cosec}^2\theta+2\cos\theta\,.\,\text{cosec}\,\theta$

$$=(\sin^2\theta+\cos^2\theta)+\sec^2\theta+\frac{2\sin\theta}{\cos\theta}+\text{cosec}^2\theta+\frac{2\cos\theta}{\sin\theta}$$

$$=1+\sec^2\theta+\text{cosec}^2\theta+2\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$=1+\sec^2\theta+\text{cosec}^2\theta+2\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)$$

$$=1+\sec^2\theta+\text{cosec}^2\theta+2\sec\theta\,\text{cosec}\,\theta$$

$$=1+(\sec\theta\,\text{cosec}\,\theta)^2$$

= RHS

5. Prove the following identities.

i) $\sec^4\theta\,(1-\sin^4\theta)-2\tan^2\theta=1$

$$ii)\ \frac{\cot\theta-\cos\theta}{\cot\theta+\cos\theta}=\frac{\text{cosec}\,\theta-1}{\text{cosec}+1}$$

Solution :

i) LHS

$=\sec^4\theta\,(1-\sin^4\theta)-2\tan^2\theta$

$$= \frac{1}{\cos^4\theta}(1+\sin^2\theta).(1-\sin^2\theta) - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{(1+\sin^2\theta)\cos^2\theta}{\cos^4\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1+\sin^2\theta}{\cos^2\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1-\sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta}{\cos^2\theta}$$

$$= 1$$

= RHS

ii) LHS

$$= \frac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta}$$

$$= \frac{\frac{\cos\theta}{\sin\theta} - \cos\theta}{\frac{\cos\theta}{\sin\theta} + 1}$$

$$= \frac{\cos\theta\left(\frac{1}{\sin\theta} - 1\right)}{\cos\theta\left(\frac{1}{\sin\theta} + 1\right)}$$

$$= \frac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1}$$

= RHS

6. Prove the following identities.

$$i)\ \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$$

$$ii)\ \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$$

Solution :

i) LHS

$$= \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$= \frac{\left(\sin^2 A - \sin^2 B\right) + \left(\cos^2 A - \cos^2 B\right)}{\left(\cos A + \cos B\right).\left(\sin A + \sin B\right)}$$

$$= \frac{\left(\sin^2 A + \cos^2 A\right) - \left(\sin^2 B + \cos^2 B\right)}{\left(\cos A + \cos B\right).\left(\sin A + \sin B\right)}$$

$$= \frac{1-1}{\left(\cos A + \cos B\right).\left(\sin A + \sin B\right)}$$

$$= 0$$

= RHS

ii) LHS

$$= \frac{\sin^3 A - \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A + \cos^3 A}{\sin A - \cos A}$$

$$= \frac{\left(\sin A + \cos A\right).\left(\sin^2 A - \sin A\cos A + \cos^2 A\right)}{\sin A + \cos A}$$

$$+ \frac{\left(\sin A - \cos A\right).\left(\sin^2 A + \sin A\cos A + \cos^2 A\right)}{\sin A - \cos A}$$

$$= (1 - \sin A\cos A) + (1 + \sin A\cos A)$$

$$= 2$$

= RHS

7. i) If sin θ + cos θ = $\sqrt{3}$, then prove that tan θ + cot θ = 1

ii) If $\sqrt{3}$ sin θ – cos θ = 0, then show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Solution :

i) Given sin θ + cos θ = $\sqrt{3}$

⇒ $(\sin\theta + \cos\theta)^2 = 3$

⇒ $\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 3$

⇒ $1 + 2\sin\theta\cos\theta = 3$

⇒ $\sin\theta\cos\theta = 1$ .... (1)

TP : tan θ + cot θ = 1

LHS : tan θ + cot θ

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{1} \qquad \text{(from (1))}$$

$$= 1$$

= RHS

Hence proved.

ii) Given Given $\sqrt{3}$ sin θ – cos θ = 0

$$\Rightarrow \sqrt{3}\sin\theta = \cos\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^0$$

$$T.P: \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

LHS

tan3θ = tan 3 (30^0)

= tan 90^0

= undefined

RHS

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3\left(\frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{3}}\right)^3}{1 - 3\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{1-1}$$

$$= \frac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{0}$$

= undefined

= LHS = RHS

= Hence proved.

---

8. i) If $\frac{\cos\alpha}{\sin\beta}$ = m and $\frac{\cos\alpha}{\sin\beta}$ = n, then prove that ($m^2 + n^2$) $\cos^2\beta$ = n^2

ii) If cotθ + tanθ = x and secθ – cosθ = y, then prove that $(x^2y)^{\frac{2}{3}} - (xy^2)^{\frac{2}{3}} = 1$

Solution :

i) Given $\frac{\cos\alpha}{\sin\beta}$ = m, $\frac{\cos\alpha}{\sin\beta}$ = n

$$\therefore m^2 = \frac{\cos^2\alpha}{\cos^2\beta}, n^2 = \frac{\cos^2\alpha}{\sin^2\beta}$$

To Prove : ($m^2 + n^2$) $\cos^2\beta$ = n^2

LHS :

($m^2 + n^2$) $\cos^2\beta$

$$= \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta$$

$$= \cos^2\alpha\left(\frac{1}{\cos^2\beta} + \frac{1}{\sin^2\beta}\right).\cos^2\beta$$

$$= \cos^2\alpha\left(\frac{\sin^2\beta + \cos^2\beta}{\cos^2\beta.\sin^2\beta}\right).\cos^2\beta$$

$$= \frac{\cos^2\alpha}{\sin^2\beta}$$

$$= n^2$$

= RHS

ii) Given

$$x = \cot\theta + \tan\theta \qquad y = \sec\theta - \cos\theta$$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \qquad = \frac{1}{\cos\theta} - \cos\theta$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \qquad = \frac{1-\cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta.\cos\theta} \qquad = \frac{\sin^2\theta}{\cos\theta}$$

To Prove :

$$\left(x^2y\right)^{2/3} - \left(xy^2\right)^{2/3} = 1$$

LHS

$$= \left(\frac{1}{\sin^2\theta\cos^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right)^{2/3}$$

$$- \left(\frac{1}{\sin\theta.\cos\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right)^{2/3}$$

$$= \left(\frac{1}{\cos^3\theta}\right)^{2/3} - \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{2/3}$$

$$= \left(\sec^3\theta\right)^{2/3} - \left(\tan^3\theta\right)^{2/3}$$

$$= \sec^2\theta - \tan^2\theta$$

$$= 1$$

$$= 1$$

$$= \text{RHS}$$

9. i) If $\sin\theta + \cos\theta = p$ and $\sec\theta + \operatorname{cosec}\theta = q$, then prove that $q(p^2 - 1) = 2p$

ii) If $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$, then prove that $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

Solution :

i) Given $p = \sin\theta + \cos\theta$,

$q = \sec\theta + \operatorname{cosec}\theta$

To Prove : $q(p^2 - 1) = 2p$

LHS :

$q(p^2 - 1)$

$$(\sec\theta + \operatorname{cosec}\theta)\left[(\sin\theta + \cos\theta)^2 - 1\right]$$

$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)$$

$$\left[\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1\right]$$

$$= \frac{\sin\theta + \cos\theta}{\cos\theta.\sin\theta}\left[1 + 2\sin\theta\cos\theta - 1\right]$$

$$= \frac{\sin\theta + \cos\theta}{\cos\theta.\sin\theta} \times 2\sin\theta\cos\theta$$

$$= 2(\sin\theta\cos\theta)$$

$$= 2p$$

$$= \text{RHS}$$

ii) Given $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$

To Prove :

$\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

10. If $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1}{a}$, then prove that $\dfrac{a^2-1}{a^2+1} = \sin\theta$

Solution :

Given

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1}{a}$$

To Prove : $\dfrac{a^2-1}{a^2+1} = \sin\theta$

$$\therefore a = \frac{1+\sin\theta}{\cos\theta}$$

$$= (1+\sin\theta)\sec\theta$$

$$= \sec\theta + \tan\theta$$

$\therefore$ LHS :

$$= \frac{a^2 - 1}{a^2 + 1}$$

$$= \frac{(\sec\theta + \tan\theta)^2 - 1}{(\sec\theta + \tan\theta)^2 + 1}$$

$$= \frac{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta - 1}{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 1}$$

$$= \frac{2\tan^2\theta + 2\sec\theta\tan\theta}{2\sec^2\theta + 2\sec\theta\tan\theta}$$

$$= \frac{\cancel{2}\tan\theta\,\cancel{(\tan\theta + \sec\theta)}}{\cancel{2}\sec\theta\,\cancel{(\sec\theta + \tan\theta)}}$$

$$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$

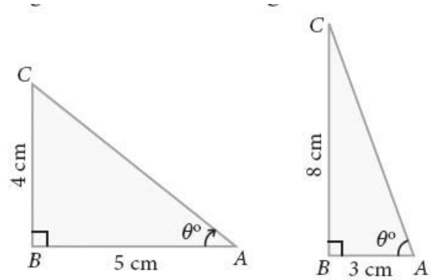
$$= \sin\theta$$

$$= \text{RHS}$$

II. PROBLEMS INVOLVING ANGLE OF ELEVATION

Example 6.18

Calculate the size of $\angle$BAC in the given triangles.



Solution :

(i) In right triangle ABC [see Fig.]

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{5}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right) = \tan^{-1}(0.8)$$

$$\theta = 38.7^0 \text{ (since } \tan 38.7^0 = 0.8011)$$

$$\angle BAC = 38.7^0$$

(ii) In right triangle ABC [see Fig.6.12(b)]

$$\tan\theta = \frac{8}{3}$$

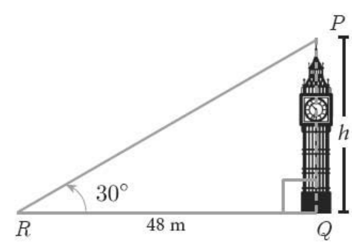
$$\theta = \tan^{-1}\left(\frac{8}{3}\right) = \tan^{-1}(2.66)$$

$$\theta = 69.4^0 \text{ (since } \tan 69.4^0 = 2.6604)$$

$$\angle BAC = 69.4^0$$

Example 6.19

A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is30°. Find the height of the tower.



Solution :

Let PQ be the height of the tower.

Take PQ = h and QR is the distance between the tower and the point R. In right triangle PQR, $\angle PRQ = 30^0$

$$\tan\theta = \frac{PQ}{QR}$$

$$\tan 30^0 = \frac{h}{48}$$

$$\text{gives, } \frac{1}{\sqrt{3}} = \frac{h}{48} \text{ so, } h = 16\sqrt{3}$$

Therefore the height of the tower is $16\sqrt{3}$ m

Example 6.20

A kite is flying at a height of 75 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

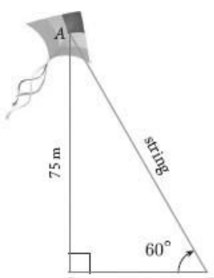


Fig. 6.14

Solution :

Let AB be the height of the kite above the ground. Then, AB = 75.

Let AC be the length of the string.

In right triangle ABC, $\angle ACB = 60^0$

$$\sin\theta = \frac{AB}{AC}$$

$$\sin 60^0 = \frac{75}{AC}$$

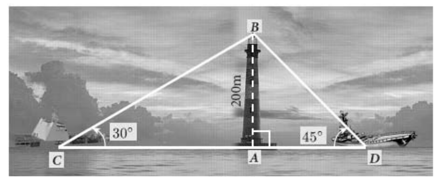
$$\text{gives, } \frac{\sqrt{3}}{2} = \frac{75}{AC} \text{ so, } AC = \frac{150}{\sqrt{3}} = 50\sqrt{3}$$

Hence, the length of the string is $50\sqrt{3}$ m

Example 6.21

Two ships are sailing in the sea on either sides of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are 30° and 45°respectively. If the lighthouse is 200 m high, find the distance between the two ships. ($\sqrt{3}$ = 1.732)

Solution :



Let AB be the lighthouse. Let C and D be the positions of the two ships.

Then, AB = 200 m.

$\angle ACB = 30^0$, $\angle ADB = 45^0$

In right triangle BAC,

$$\tan 30^0 = \frac{AB}{AC}$$

$$\frac{1}{\sqrt{3}} = \frac{200}{AC}$$

$$\text{gives, } AC = 200\sqrt{3} \quad \text{........ (1)}$$

In right triangle BAD,

$$\tan 45^0 = \frac{AB}{AD}$$

$$1 = \frac{200}{AD}$$

$$\text{gives, } AD = 200 \quad \text{........ (2)}$$

Now, CD $= AC + AD$

$= 200\sqrt{3} + 200$ [by (1) and (2)]

CD $= 200(\sqrt{3} + 1)$

$$= 200 \times 2.732 = 546.4$$

Distance between two ships is 546.4 m.

Example 6.22

From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 30 m high building are 45° and 60° respectively. Find the height of the tower. ($\sqrt{3}$ = 1.732)

Solution :

Fig. 6.16

Let AC be the height of the tower.

Let AB be the height of the building.

Then, AC = h metres, AB = 30 m

In right triangle CBP,

$\angle CPB = 60^0$

$$\tan\theta = \frac{BC}{BP}$$

$$\tan 60^0 = \frac{AB + AC}{BP}$$

$$so, \sqrt{3} = \frac{30+h}{BP} \quad ....(1)$$

In right triangle ABP,

$\angle APB = 45^0$

$$\tan\theta = \frac{AB}{BP}$$

$$\tan 45^0 = \frac{30}{BP}$$

gives, $BP = 30$ ....(2)

Substituting (2) in (1), we get

$$\sqrt{3} = \frac{30+h}{30}$$

$$h = 30\left(\sqrt{3}-1\right)$$

$$= 30\,(1.732-1)$$

$$= 30(0.732) = 21.96$$

Hence, the height of the tower is 21.96 m.

Example 6.23

A TV tower stands vertically on a bank of a canal. The tower is watched from a point on the other bank directly opposite to it. The angle of elevation of the top of the tower is 58°. From another point 20 m away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal. (tan58° = 1.6003)

Solution :

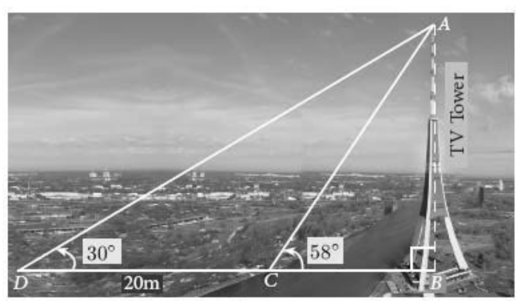


Fig. 6.17

Let AB be the height of the TV tower.

CD = 20 m.

Let BC be the width of the canal.

In right triangle ABC,

$$\tan 58^0 = \frac{AB}{BC}$$

$$1.6003 = \frac{AB}{BC} \qquad \text{..........} (1)$$

In right triangle ABD,

$$\tan 30^0 = \frac{AB}{BD} = \frac{AB}{BC+CD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC+20} \qquad \text{..........} (2)$$

Dividing (1) by (2) we get,

$$\frac{1.6003}{\frac{1}{\sqrt{3}}} = \frac{BC+20}{BC}$$

$$BC = \frac{20}{1.7791} = 11.24m \qquad \text{..........} (3)$$

$$1.6003 = \frac{AB}{11.24} \qquad \text{[from (1) and (3)]}$$

$$\text{AB} = 17.99$$

Hence, the height of the tower is 17.99 m and the width of the canal is 11.24 m.

Example 6.24

An aeroplane sets off from G on a bearing of 24° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 55° and a distance of 180 km away.

(i) How far is H to the North of G?

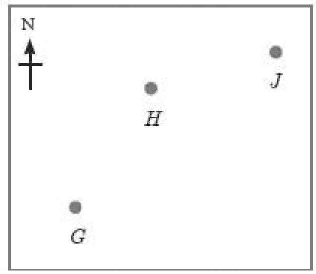
(ii) How far is H to the East of G?

(iii) How far is J to the North of H?

(iv) How far is J to the East of H?

$$\begin{pmatrix} \sin 24^0 = 0.4067 & \sin 11^0 = 0.1908 \\ \cos 24^0 = 0.9135 & \cos 11^0 = 0.9816 \end{pmatrix}$$

Solution :



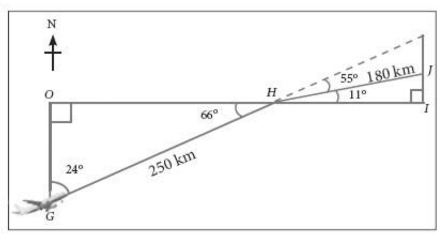
(i) In right triangle GOH,

$$\cos 24^0 = \frac{OG}{GH}$$

$$0.9135 = \frac{OG}{250}; \text{OG} = 228.38 \text{ km}$$

Distance of H to the North of G = 228.38 km

(ii) In right triangle GOH,



$$\sin 24^0 = \frac{OH}{GH}$$

$$0.4067 = \frac{OH}{250}; \text{OH} = 101.68$$

Distance of H to the East of G = 101.68 km

(iii) In right triangle HIJ,

$$\sin 11^0 = \frac{IJ}{HJ}$$

$$0.1908 = \frac{IJ}{180}; \text{ IJ} = 34.34 \text{ km}$$

Distance of J to the North of

H = 34.34 km

(iv) In right triangle HIJ,

$$\cos 11^0 = \frac{HI}{HJ}$$

$$0.9816 = \frac{HI}{180}; \text{HI} = 176.69 \text{ km}$$

Distance of J to the East of

H = 176.69 km

Example 6.25

Two trees are standing on flat ground. The angle of elevation of the top of both the trees from a point X on the ground is 40°. If the horizontal distance between X and the smaller tree is 8 m and the distance of the top of the two trees is 20 m, calculate

(i) the distance between the point X and the top of the smaller tree.

(ii) the horizontal distance between the two trees. (cos 40° = 0.7660)

Solution :

Let AB be the height of the bigger tree and CD be the height of the smaller tree and X is the point on the ground.

(i) In right triangle XCD,

$$\cos 40^0 = \frac{CX}{XD}$$

$$XD = \frac{8}{0.7660} = 10.44 \text{ km}$$

Therefore the distance between X and top of the smaller tree = XD = 10.44 m

(ii) In right triangle XAB,

$$\cos 40^0 = \frac{AX}{BX}$$

$$= \frac{AC + CX}{BD + DX}$$

$$= 0.7660 = \frac{AC + 8}{20 + 10.44}$$

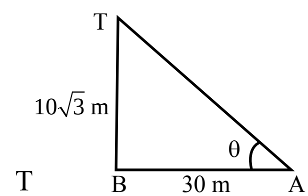
gives $AC = 23.32 - 8 = 15.32$ m

Therefore the horizontal distance between two trees = AC = 15.32 m

EXERCISE 6.2

1. Find the angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of a tower of height 10 $\sqrt{3}$ m.

Solution :



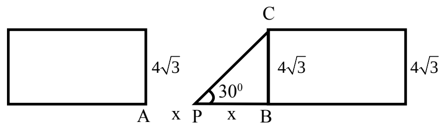
From the fig, $\tan \theta = \frac{10\sqrt{3}}{30}$

$$= \frac{1}{\sqrt{3}}$$

$$\therefore \theta = 30^0$$

2. A road is flanked on either side by continuous rows of houses of height $4\sqrt{3}$ m with no space in between them. A pedestrian is standing on the median of the road facing a row house. The angle of elevation from the pedestrian to the top of the house is 30°. Find the width of the road.

Solution :



Let AB = Width of theroad

P = Midpoint of AB

BC = height of the row houses

$= 4\sqrt{3}$ m

Let PB = PA = x m

In ΔPBC, $\tan 30^0 = \dfrac{BC}{PB}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{x}$$

$$\Rightarrow \quad x = 12 \text{ m}$$

$\therefore$ Width of the road = AB

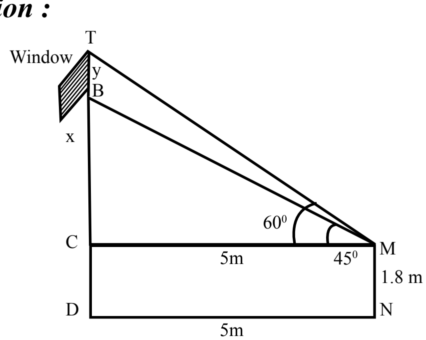
= 2x

= 2 (12)

= 24 m

3. To a man standing outside his house, the angles of elevation of the top and bottom of a window are 60° and 45° respectively. If the height of the man is 180 cm and if he is 5 m away from the wall, what is the height of the window? ($\sqrt{3}$ = 1.732)

Solution :



Let MN = 180 cm = 1.8 m

= height of the man

DN = 5 m = Dist. between Man & Wall

T, B $\rightarrow$ Top & Bottom of Window

BC = x, TB = y (Height of window)

In ΔCMB, $\tan 45^0 = \dfrac{x}{5}$

$$\Rightarrow 1 = \frac{x}{5}$$

$$\Rightarrow x = 5$$

In ΔCMT, $\tan 60^0 = \dfrac{x+y}{5}$

$$\Rightarrow \quad \sqrt{3} = \frac{5+y}{5}$$

$$\Rightarrow \quad 5 + y = 5\sqrt{3}$$

$$\Rightarrow \quad y = 5\sqrt{3} - 5$$

$$= 5\left(\sqrt{3} - 1\right)$$

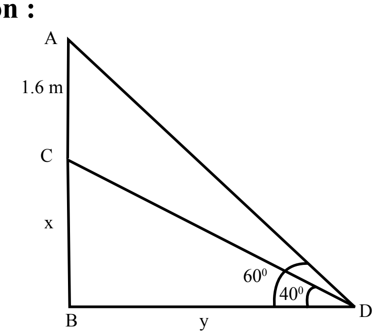
$$= 5\,(0.732)$$

$$= 3.66 \text{ m}$$

$\therefore$ Height of window = 3,66 m

4. A statue 1.6 m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 40°. Find the height of the pedestal. (tan 40° = 0.8391, ($\sqrt{3}$ = 1.732)

Solution :



Let D $\rightarrow$ Point of observation on the ground

BC = x m = height of Pedestal

AC = 1.6m = height of statue

$\angle BDC = 40^0$, $\angle BDA = 60^0$, BD = y m

In ΔBCD, $\tan 40^0 = \dfrac{x}{y}$

$$\Rightarrow y = \frac{x}{\tan 40^0}$$

$$= \frac{x}{0.8391} \qquad \text{.......... (1)}$$

In ΔBAD, $\tan 60^0 = \dfrac{AB}{BD}$

$$\Rightarrow \sqrt{3} = \frac{x+1.6}{y}$$

$$\Rightarrow y = \frac{x+1.6}{\sqrt{3}} \qquad \text{.......... (2)}$$

$\therefore$ From (1) & (2)

$$\frac{x}{0.8391} = \frac{x+1.6}{1.732}$$

$$\Rightarrow 1.732\,x = 0.8391\,x + 1.6\,(0.8391)$$

$$\Rightarrow 0.8929\,x = 1.343$$

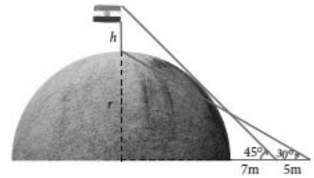
$$\Rightarrow x = \frac{1.343}{0.8929}$$

$$= 1.5 \text{ m}$$

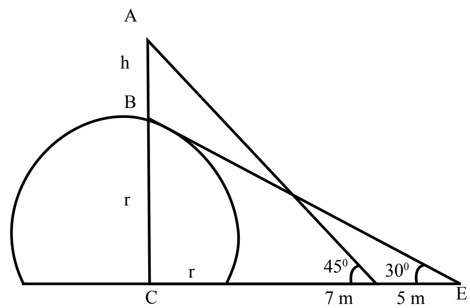
$\therefore$ Height of statue = 1.5 m

5. A flag pole 'h' metres is on the top of the hemispherical dome of radius 'r' metres. A man is standing 7 m away from the dome. Seeing the top of the pole at an angle 45° and moving 5 m away from the dome and seeing the bottom of the pole at an angle 30° . Find (i) the height of the pole (ii) radius of the dome.

($\sqrt{3}$ = 1.732)



Solution :



In ΔACD, $\tan 45^0 = \dfrac{AC}{CD}$

$$1 = \frac{h+r}{r+7}$$

$$\Rightarrow r + 7 = h + r$$

$$\Rightarrow h = 7$$

$\therefore$ Height of the pole $= 7$m

In ΔBCE, $\tan 30^0 = \dfrac{BC}{CE}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{r}{r+7+5}$$

$$\Rightarrow \sqrt{3}r = r + 12$$

$$\Rightarrow \sqrt{3}r - r = 12$$

$$\Rightarrow r\left(\sqrt{3}-1\right) = 12$$

$$\Rightarrow r = \frac{12}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{12\left(\sqrt{3}+1\right)}{2}$$

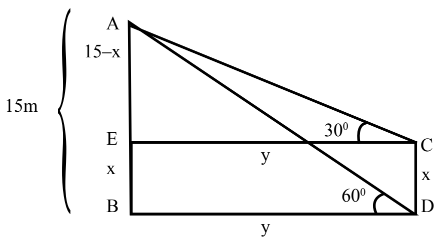
$$= 6\,(2.732)$$

$$= 16.392 \text{ m}$$

$\therefore$ Radius of dome = 16.39 m

6. The top of a 15 m high tower makes an angle of elevation of 60° with the bottom of an electronic pole and angle of elevation of 30° with the top of the pole. What is the height of the electric pole?

Solution :



Let AB = 15 m = Height of the tower

CD = x m = Height of the pole = BE

$\therefore$ AE = 15 – x

Let BD = EC = y

$$\text{In } \Delta ADB, \tan 60^0 = \frac{AB}{BD}$$

$$\sqrt{3} = \frac{15}{y}$$

$$\Rightarrow \quad y = \frac{15}{\sqrt{3}} = 5\sqrt{3} \quad \text{........ (1)}$$

$$\text{In } \Delta ACE, \tan 30^0 = \frac{AE}{EC}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{15-x}{y}$$

$$\Rightarrow \quad 5\sqrt{3} = (15-x)\sqrt{3} \quad \text{(From (1))}$$

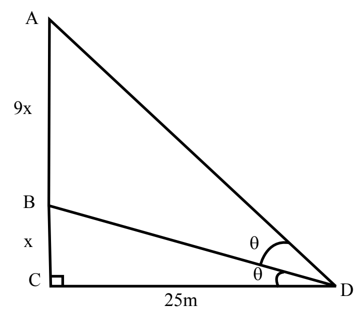
$$\Rightarrow \quad 5 = 15 - x$$

$$\Rightarrow \quad x = 10$$

$\therefore$ Height of the pole = 10m

7. A vertical pole fixed to the ground is divided in the ratio 1 : 9 by a mark on it with lower part shorter than the upper part. If the two parts subtend equal angles at a place on the ground, 25 m away from the base of the pole, what is the height of the pole?

Solution :



Let AC = height of the pole

B is a point on AC which divides it in the ratio 1 : 9 such that AB = 9x, BC = x

($\because$ lower part is shorter than upper part)

CD = 25 m = Dist. between pole and point of observation

$$\text{In } \Delta BCD, \tan\theta = \frac{BC}{CD} = \frac{x}{25}$$

$$\text{In } \Delta ACD, \tan 2\theta = \frac{AC}{CD} = \frac{10x}{25}$$

$$\Rightarrow \quad \tan 2\theta = \frac{10x}{25}$$

$$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} = \frac{10x}{25}$$

$$\Rightarrow \quad \frac{2\left(x/25\right)}{1-x^2/25} = \frac{2x}{5}$$

$$\Rightarrow \quad \frac{\frac{2x}{25}}{\frac{625-x^2}{625}} = \frac{2x}{5}$$

$$\Rightarrow \frac{\cancel{2x}}{25} \times \frac{625}{625-x^2} = \frac{\cancel{2x}}{5}$$

$$\Rightarrow \frac{25}{625-x^2} = \frac{1}{5}$$

$$\Rightarrow 625 - x^2 = 125$$

$$\Rightarrow x^2 = 500$$

$$\Rightarrow x = 10\sqrt{5}$$

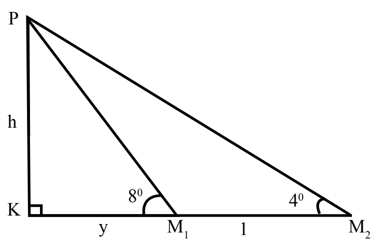
$$\therefore \text{Height of the pole} = 10x$$

$$= 10\left(10\sqrt{5}\right)$$

$$= 100\sqrt{5} \text{ m}$$

8. A traveler approaches a mountain on highway. He measures the angle of elevation to the peak at each milestone. At two consecutive milestones the angles measured are 4° and 8°. What is the height of the peak if the distance between consecutive milestones is 1 mile. (tan 4° = 0.0699, tan 8° = 0.1405)

Solution :



Let PK = h = height of mountain

M_1, M_2 = Mile stones, M_1, M_2 = 1 mile

Let KM_1 = y

In ΔPKM_1,

$$\tan 8^0 = \frac{h}{y}$$

$$\Rightarrow y = \frac{h}{\tan 8^0} \qquad \text{...........(1)}$$

In ΔPKM_2,

$$\tan 4^0 = \frac{h}{y+1}$$

$$\Rightarrow y+1 = \frac{h}{\tan 4^0}$$

$$\Rightarrow y = \frac{h}{\tan 4^0} - 1 \qquad \text{...........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{h}{\tan 8^0} = \frac{h}{\tan 4^0} - 1$$

$$\Rightarrow \frac{h}{\tan 4^0} - \frac{h}{\tan 8^0} = 1$$

$$\Rightarrow h\left[\frac{1}{\tan 4^0} - \frac{1}{\tan 8^0}\right] = 1$$

$$\Rightarrow h\left[\frac{\tan 8^0 - \tan 4^0}{\tan 8^0 \tan 4^0}\right] = 1$$

$$\Rightarrow h = \frac{\tan 8^0 \tan 4^0}{\tan 8^0 - \tan 4^0}$$

$$= \frac{0.14 \times 0.07}{0.14 - 0.07}$$

$$= \frac{0.0098}{0.07}$$

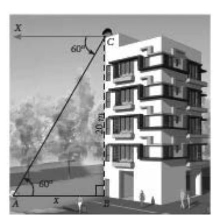
$$= 0.14 \text{ miles}$$

III. PROBLEMS INVOLVING ANGLE OF DEPRESSION

Example 6.26

A player sitting on the top of a tower of height 20 m observes the angle of depression of a ball lying on the ground as 60° . Find the distance between the foot of the tower and the ball. ($\sqrt{3}$ = 1.732)

Solution :



Let BC be the height of the tower and A be the position of the ball lying on the ground. Then,

BC = 20 m and ∠XCA= 60° = ∠CAB

Let AB = x metres.

In right triangle ABC,

$$\tan 60^0 = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{20}{x}$$

$$x = \frac{20\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{20\times 1.732}{3}$$

$$= 11.54\text{m}$$

Hence, the distance between the foot of the tower and the ball is 11.54 m.

Example 6.27

The horizontal distance between two buildings is 140 m. The angle of depression of the top of the first building when seen from the top of the second building is 30° . If the height of the first building is 60 m, find the height of the second building. ($\sqrt{3}$ = 1.732)

Solution :

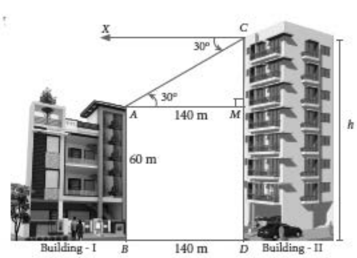


Fig. 6.22

The height of the first building AB = 60 m. Now, AB = MD = 60 m
Let the height of the second building CD = h. Distance BD = 140 m
Now, AM = BD = 140 m
From the diagram,

∠XCA = 30° = ∠CAM
In right triangle AMC,

$$\tan 30^0 = \frac{CM}{AM}$$

$$\frac{1}{\sqrt{3}} = \frac{CM}{140}$$

$$CM = \frac{140}{\sqrt{3}} = \frac{140\sqrt{3}}{3}$$

$$= \frac{140\times 1.732}{3}$$

$$CM = 80.78$$

Now, h = CD
= CM + MD
= 80.78 + 60 = 140.78

Therefore the height of the second building is 140.78 m

Example 6.28

From the top of a tower 50 m high, the angles of depression of the top and bottom of a tree are observed to be 30° and 45° respectively. Find the height of the tree. ($\sqrt{3}$ = 1.732)

Solution :

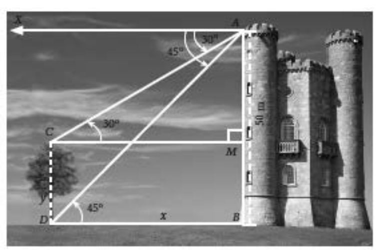


Fig. 6.23

The height of the tower AB = 50 m
Let the height of the tree
CD = y and BD = x
From the diagram,
∠XAC = 30° = ∠ACM and
∠XAD = 45° = ∠ADB
In right triangle ABD,

$$\tan 45^0 = \frac{AB}{BD}$$

$$1 = \frac{50}{x} \text{ gives } x = 50 \text{ m}$$

In right triangle AMC,

$$\tan 30^0 = \frac{AM}{CM}$$

$$\frac{1}{\sqrt{3}} = \frac{AM}{50} \quad [\text{since } DB = CM]$$

$$AM = \frac{50}{\sqrt{3}} = \frac{50\sqrt{3}}{3}$$

$$= \frac{50 \times 1.732}{3} = 28.85 \text{ m}$$

Therefore, height of the tree =

CD = MB = AB − AM

= 50 − 28.85 = 21.15 m

Example 6.29

As observed from the top of a 60 m high light house from the sea level, the angles of depression of two ships are 28° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. (tan 28° = 0.5317)

Solution :

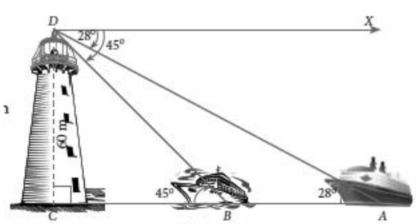


Fig. 6.24

Let the observer on the lighthouse CD be at D.

Height of the lighthouse CD = 60 m

From the diagram,

$\angle XDA = 28° = \angle DAC$ and

$\angle XDB = 45° = \angle DBC$

In right triangle DCB,

$$\tan 45^0 = \frac{DC}{BC}$$

$$1 = \frac{60}{BC} \text{ gives } BC = 60 \text{ m}$$

In right triangle DCA,

$$\tan 28^0 = \frac{DC}{AC}$$

$$0.5317 = \frac{60}{AC}$$

$$\text{gives } AC = \frac{60}{0.5317}$$

$$= 112.85$$

Distance between the two ships

AB = AC − BC = 52.85 m

Example 6.30

A man is watching a boat speeding away from the top of a tower. The boat makes an angle of depression of 60° with the man's eye when at a distance of 200 m from the tower. After 10 seconds, the angle of depression becomes 45°. What is the approximate speed of the boat (in km / hr), assuming that it is sailing in still water? ($\sqrt{3}$ = 1.732)

Solution :

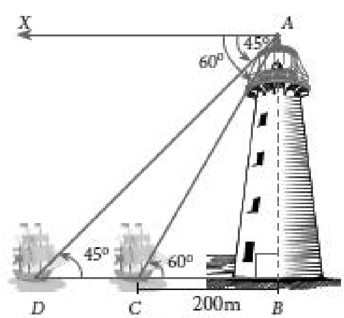


Fig. 6.25

Let AB be the tower.

Let C and D be the positions of the boat.

From the diagram,

$\angle XAC = 60° = \angle ACB$ and

$\angle XAD = 45° = \angle ADB$, BC = 200 m

In right triangle ABC, $\tan 60° = \dfrac{AB}{BC}$

gives $\sqrt{3} = \dfrac{AB}{200}$

we get $AB = 200\sqrt{3}$ .... (1)

In right triangle ABD, $\tan 45° = \dfrac{AB}{BD}$

gives $1 = \dfrac{200\sqrt{3}}{BD}$ [by (1)]

we get $BD = 200\sqrt{3}$

Now, $CD = BD - BC$

$CD = 200\sqrt{3} - 200$

$= 200(\sqrt{3} - 1) = 146.4$

It is given that the distance CD is covered in 10 seconds.

That is, the distance of 146.4 m is covered in 10 seconds.

Therefore, speed of the boat = $\dfrac{\text{distance}}{\text{time}}$

$= \dfrac{146.4}{10} = 14.64$ m/s

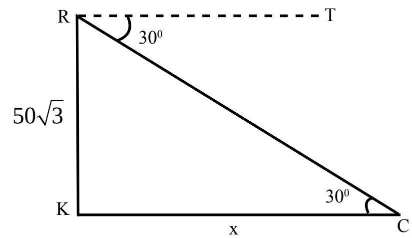
gives $14.64 \times \dfrac{3600}{1000}$ km/hr

$= 52.704$ km/hr

EXERCISE 6.3

1. From the top of a rock $50\sqrt{3}$ m high, the angle of depression of a car on the ground is observed to be 30°. Find the distance of the car from the rock.

Solution:



RK = $50\sqrt{3}$ m = height of the rock

C = Position of the car

KC = x m

$\angle TRC = \angle RCK = 30^0$

$\tan 30^0 = \dfrac{RK}{KC}$

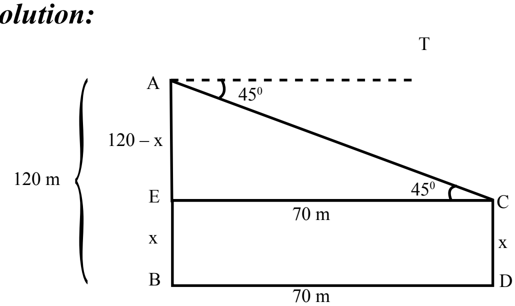
$\dfrac{1}{\sqrt{3}} = \dfrac{50\sqrt{3}}{x}$

$\Rightarrow \quad x = 150\text{m}$

$\therefore$ Dist. of the car from the rock = 150m

2. The horizontal distance between two buildings is 70 m. The angle of depression of the top of the first building when seen from the top of the second building is 45°. If the height of the second building is 120 m, find the height of the first building.

Solution:



Height of the 1st building = CD = x m

Height of the 2nd building = AB = 120 m

Distance between 2 buildings =

BD = EC = 70m

In Δ AEC,

$$\tan 45^0 = \frac{AE}{EC}$$

$$1 = \frac{120 - x}{70}$$

$$\Rightarrow \quad 70 = 120 - x$$

$$\Rightarrow \quad x = 120 - 70$$

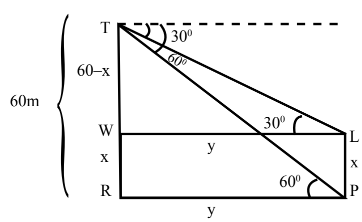
$$\Rightarrow \quad x = 50$$

$\therefore$ Height of 1st building = 50 m

---

3. From the top of the tower 60 m high the angles of depression of the top and bottom of a vertical lamp post are observed to be 38° and 60° respectively. Find the height of the lamp post. (tan 38° = 0.7813, $\sqrt{3}$ = 1.732)

Solution:



TR = Height of the tower = 60 m

LP = Height of the lamp post = x m = WR

$\therefore$ TW = 60 – x

Let RP = WL = y

In ΔTRP,

$$\tan 60^0 = \frac{TR}{RP}$$

$$\Rightarrow \sqrt{3} = \frac{60}{y}$$

$$\Rightarrow \quad y = \frac{60}{\sqrt{3}}$$

$$\Rightarrow \quad y = 20\sqrt{3} \quad ....(1)$$

In ΔTWL,

$$\tan 30^0 = \frac{TW}{WL}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{60 - x}{y}$$

$$\Rightarrow \quad y = \sqrt{3}\,(60 - x) ....(2)$$

$\therefore$ From (1) & (2)

$$\Rightarrow \quad 20\sqrt{3} = \sqrt{3}\,(60 - x)$$

$$\Rightarrow \quad 20 = 60 - x$$

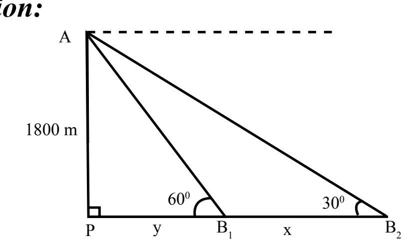
$$\Rightarrow \quad x = 60 - 20 \quad \Rightarrow \quad x = 40$$

$\therefore$ Height of the lamp post = 40m

---

4. An aeroplane at an altitude of 1800 m finds that two boats are sailing towards it in the same direction. The angles of depression of the boats as observed from the aeroplane are 60° and 30° respectively. Find the distance between the two boats. ($\sqrt{3}$ = 1.732)

Solution:



AP = 1800 m

= height of the plane from the ground

B_1, B_2 = Positions of 2 boats

Let B_1, B_2 = x m ; PB_1 = y

In ΔAPB_1,

$$\tan 60^0 = \frac{1800}{y}$$

$$\Rightarrow \quad \sqrt{3} = \frac{1800}{y}$$

$$\Rightarrow \quad y = \frac{1800}{\sqrt{3}} = 600\sqrt{3} ....(1)$$

In ΔAPB_2,

$$\tan 30^0 = \frac{1800}{y+x}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{1800}{y+x}$$

$$\Rightarrow \quad y+x = 1800\sqrt{3}$$

$$\Rightarrow \quad 600\sqrt{3} + x = 1800\sqrt{3}$$

(From (1))

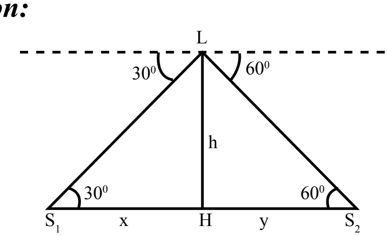
$$\Rightarrow \quad x = 1200\sqrt{3}$$

$$x = 1200\,(1.732)$$

$$= 2078.4 \text{ m}$$

5. From the top of a lighthouse, the angle of depression of two ships on the opposite sides of it are observed to be 30° and 60°. If the height of the lighthouse is h meters and the line joining the ships passes through the foot of the lighthouse, show that the distance between the ships is $\frac{4h}{\sqrt{3}}$ m.

Solution:



LH = h m = height of the light house

S_1, S_2 = Positions of 2 ships

S_1H = x m, S_2H = y m

To find : x + y

In ΔLS_1H,

$$\tan 30^0 = \frac{h}{x}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\Rightarrow \quad x = \sqrt{3}h \qquad \text{.........(1)}$$

In ΔLS_2H,

$$\tan 60^0 = \frac{h}{y}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{y}$$

$$\Rightarrow \quad y = \frac{h}{\sqrt{3}} \qquad \text{.........(2)}$$

$\therefore$ Adding (1) & (2)

$$x + y = \sqrt{3}h + \frac{h}{\sqrt{3}}$$

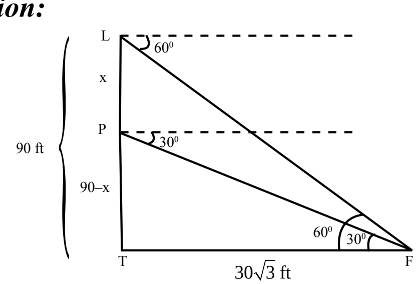
$$= \frac{3h+h}{\sqrt{3}} = \frac{4h}{\sqrt{3}}$$

$\therefore$ Distance between 2 ships

$$= \frac{4h}{\sqrt{3}} \text{ m}$$

6. A lift in a building of height 90 feet with transparent glass walls is descending from the top of the building. At the top of the building, the angle of depression to a fountain in the garden is 60°. Two minutes later, the angle of depression reduces to 30°. If the fountain is $30\sqrt{3}$ feet from the entrance of the lift, find the speed of the lift which is descending.

Solution:



LT= height of the lift = 90 ft

F = Position of the fountain

FT = Distance between fountain & lift

$= 30\sqrt{3}$ ft.

LP = x ft $\Rightarrow$ PT = $90^0 - x$

Time taken from L to P = 2 min.

In ΔPFT,

$$\tan 30^0 = \frac{90-x}{30\sqrt{3}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{90-x}{30\sqrt{3}}$$

$$\Rightarrow \quad 30 = 90 - x$$

$$\Rightarrow \quad x = 60 \text{ ft}$$

$$\therefore \text{ Speed of the lift} = \frac{\text{Dist.}}{\text{Time}} = \frac{60}{2}$$

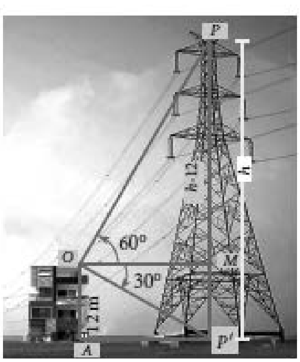
$$= 30 \text{ ft/min.}$$

PROBLEMS INVOLVING ANGLE OF ELEVATION AND DEPRESSION

Example 6.31

From the top of a 12 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 30°. Determine the height of the tower.

Solution :



As shown in Fig., OA is the building, O is the point of observation on the top of the building OA. Then, OA = 12 m.

PP' is the cable tower with P as the top and P' as the bottom.

Then the angle of elevation of P, $\angle MOP = 60°$.

And the angle of depression of P' , $\angle MOP' = 30°$

Suppose, height of the cable tower PP' = h metres.

Through O, draw OM $\perp$ PP'

MP = PP' − MP' = h −OA = h −12

In right triangle OMP, $\frac{MP}{OM} = \tan 60°$

$$\text{gives } \frac{h-12}{OM} = \sqrt{3}$$

$$\text{so, } OM = \frac{h-12}{\sqrt{3}} \quad \text{............ (1)}$$

In right triangle OMP', $\frac{MP'}{OM} = \tan 30°$

$$\text{gives } \frac{12}{OM} = \frac{1}{\sqrt{3}}$$

$$\text{so, } OM = 12\sqrt{3} \quad \text{............ (2)}$$

From (1) and (2) we have, $\frac{h-12}{\sqrt{3}} = 12\sqrt{3}$

gives, $h - 12 = 12\sqrt{3} \times \sqrt{3}$ we get, $h = 48$

Hence, the required height of the cable tower is 48 m.

Example 6.32

A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point 'A' on the ground is 60° and the angle of depression to the point 'A' from the top of the tower is 45°. Find the height of the tower. ($\sqrt{3}$ = 1.732)

Solution :

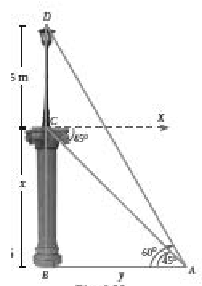


Fig: 6.28

Let BC be the height of the tower and CD be the height of the pole.

Let 'A' be the point of observation.

Let BC = x and AB = y.

From the diagram,

$\angle$BAD = 60° and $\angle$XCA = 45° = $\angle$BAC

In right triangle ABC, tan45° = $\dfrac{BC}{AB}$

$$\text{gives, } 1 = \frac{x}{y}\ so,\ x = y \qquad \text{...........} (1)$$

In right triangle ABD, tan60°

$$= \frac{BD}{AB} = \frac{BC+CD}{AB}$$

$$\text{gives, } \sqrt{3} = \frac{x+5}{y} \text{ so, } \sqrt{3}y = x+5$$

$$\text{we get, } \sqrt{3}x = x+5 \qquad \text{[From (1)]}$$

$$\text{so, } x = \frac{5}{\sqrt{3}-1} = \frac{5}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

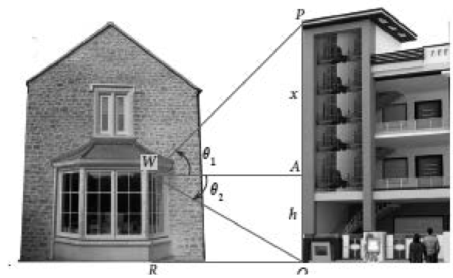
$$= \frac{5(1.732+1)}{2} = 6.83$$

Hence, height of the tower is 6.83 m.

Example6.33

From a window (h metres high above the ground) of a house in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are θ_1 and θ_2 respectively. Show that the height of the opposite house is $h\left(1+\dfrac{\cot\theta_2}{\cot\theta_1}\right)$

Solution :



Let W be the point on the window where the angles of elevation and depression are measured. Let PQ be the house on the opposite side.

Then WA is the width of the street.

Height of the window = h metres

= AQ (WR = AQ)

Let PA = x metres.

$$\text{In right triangle } PAW, \tan\theta_1 = \frac{AP}{AW}$$

$$\text{gives } \tan\theta_1 = \frac{x}{AW}$$

$$\text{so, } AW = \frac{x}{\tan\theta_1}$$

$$\text{we get, } AW = x\cot\theta \qquad \text{......} (1)$$

$$\text{In right triangle } QAW, \tan\theta_2 = \frac{AQ}{AW}$$

$$\text{gives } \tan\theta_2 = \frac{h}{AW}$$

$$\text{we get, } AW = h\cot\theta_2 \qquad \text{......} (2)$$

From (1) and (2) we get, $x\cot\theta_1 = h\cot\theta_2$

$$\text{gives, } \qquad x = h\frac{\cot\theta_2}{\cot\theta_1}$$

Therefore, height of the opposite house =

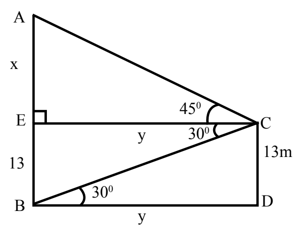
$$PA + AQ = x + h = h\frac{\cot\theta_2}{\cot\theta_1} + h = h\left(1+\frac{\cot\theta_2}{\cot\theta_1}\right)$$

Hence Proved.

EXERCISE 6.4

1. From the top of a tree of height 13 m the angle of elevation and depression of the top and bottom of another tree are 45° and 30° respectively. Find the height of the second tree. ($\sqrt{3}$ = 1.732)

Solution:



CD = 13m = height of tree 1

AB = x + 13 = height of tree 2

BD = EC = y m = dist. between 2 trees

In ΔBCD.

$$\tan 30^0 = \frac{CD}{BD}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{13}{y}$$

$$\Rightarrow \quad y = 13\sqrt{3}\text{ m}$$

In ΔACE,

$$\tan 45^0 = \frac{AE}{EC}$$

$$\Rightarrow \quad 1 = \frac{x}{y}$$

$$\Rightarrow \quad y = x$$

$$\Rightarrow \quad x = 13\sqrt{3}\text{ m}$$

∴ Height of 2nd tree = x + 13

$$= 13\sqrt{3} + 13$$

$$= 13\left(\sqrt{3}+1\right)$$

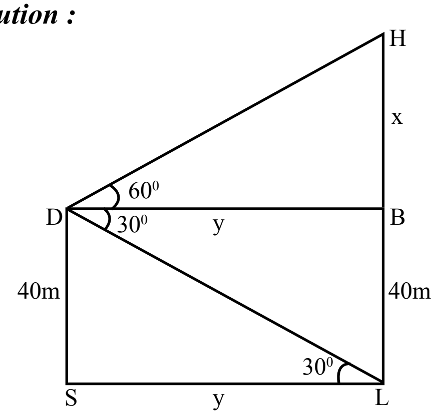
$$= 13 \times 2.732$$

$$= 35.516$$

$$\square 35.52\text{ m}$$

2. A man is standing on the deck of a ship, which is 40 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill. ($\sqrt{3}$ = 1.732)

Solution :



D → Deck of a ship

DS = 40m = Deck of ship from water level

HL = Height of the hill = x + 40

SL = DB = Dist. between ship & hill

In ΔDLS,

$$\tan 30^0 = \frac{DS}{SL}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{40}{y}$$

$$\Rightarrow \quad y = 40\sqrt{3} \qquad \text{..........(1)}$$

In ΔDHB,

$$\tan 60^0 = \frac{HB}{DB}$$

$$\Rightarrow \sqrt{3} = \frac{x}{y}$$

$$\Rightarrow \quad y = \frac{x}{\sqrt{3}} \qquad \text{..........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{x}{\sqrt{3}} = 40\sqrt{3}$$

$$\Rightarrow \quad x = 40\sqrt{3} \times \sqrt{3}$$

$$= 120\text{m}$$

$\therefore$ Height of the hill = x + 40

= 120 + 40

= 160 m and

Distance between ship & hill

$$y = 40\sqrt{3}$$

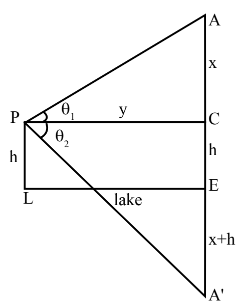
= 40 (1.732)

= 69.28 m

---

3. If the angle of elevation of a cloud from a point 'h' metres above a lake is θ_1 and the angle of depression of its reflection in the lake is θ_2. Prove that the height that the cloud is located from the ground is $\dfrac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 + \tan\theta_1}$

Solution :



LE $\rightarrow$ Surface of the lake

P $\rightarrow$ Point of observation 'h' mrs. from lake

A, A' $\rightarrow$ Positions of cloud & its reflection

AE = A'E, PL = CE = h m

To find :

$$EA = \frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$$

In ΔAPC,

$$\tan\theta_1 = \frac{x}{y}$$

$$\Rightarrow \quad y = \frac{x}{\tan\theta_1} \qquad \text{...........(1)}$$

In $\Delta PA'C$,

$$\tan\theta_2 = \frac{CA'}{PC}$$

$$\tan\theta_2 = \frac{x+2h}{y}$$

$$\Rightarrow \quad y = \frac{(x+2h)}{\tan\theta_2} \qquad \text{...........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{x}{\tan\theta_1} = \frac{x+2h}{\tan\theta_2}$$

$$\Rightarrow x\tan\theta_2 = x\tan\theta_1 + 2h\tan\theta_1$$

$$\Rightarrow x(\tan\theta_2 - \tan\theta_1) = 2h\tan\theta_1$$

$$\Rightarrow \qquad x = \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

$$\therefore AE = h + x$$

$$= h + \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

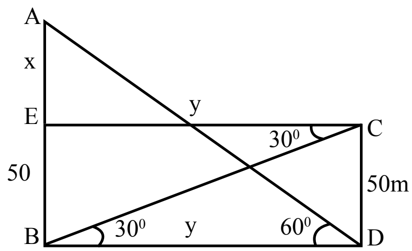
$$= h\left[1 + \frac{2\tan\theta_1}{\tan\theta_2 - \tan\theta_1}\right]$$

$$= h\left[\frac{\tan\theta_2 + \tan\theta_1}{\tan\theta_2 - \tan\theta_1}\right]$$

Hence proved.

4. The angle of elevation of the top of a cell phone tower from the foot of a high apartment is 60° and the angle of depression of the foot of the tower from the top of the apartment is 30° . If the height of the apartment is 50 m, find the height of the cell phone tower. According to radiations control norms, the minimum height of a cell phone tower should be 120 m. State if the height of the above mentioned cell phone tower meets the radiation norms.

Solution :



CD = 50m = height of the apartment = EB

AB = (x + 50) m = height of the cellphone tower

BD = EC = ym = dist. between tower & apartment

In ΔCDB,

$$\tan 30^0 = \frac{CD}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{50}{y}$$

$$\Rightarrow \quad y = 50\sqrt{3}\,m \qquad ....(1)$$

In ΔADB,

$$\tan 60^0 = \frac{AB}{BD}$$

$$\sqrt{3} = \frac{x+50}{y}$$

$$\Rightarrow \quad y = \frac{x+50}{\sqrt{3}}$$

$$\Rightarrow \quad 50\sqrt{3} = \frac{x+50}{\sqrt{3}} \qquad \text{(from (1))}$$

$$\Rightarrow \quad 150 = x+50$$

$$\therefore \; x = 100\text{m}$$

$\therefore$ Height of cell phone tower

= x + 50

= 150m > 120 m

$\therefore$ The tower does not meet the radiation norms.

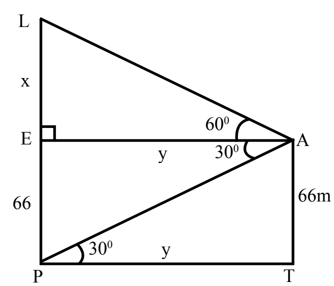
5. The angles of elevation and depression of the top and bottom of a lamp post from the top of a 66 m high apartment are 60° and 30° respectively. Find

(i) The height of the lamp post.

(ii) The difference between height of the lamp post and the apartment.

(iii) The distance between the lamp post and the apartment. ($\sqrt{3}$ = 1.732)

Solution :



AT = height of apartment = 66 m = EP

LP = height of the lamp post = x + 66

PT = EA = y m = dist. between post and apartment

In ΔAPT,

$$\tan 30^0 = \frac{AT}{y}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{66}{y}$$

$$\Rightarrow \quad y = 66\sqrt{3} \qquad \text{..........(1)}$$

In ΔALE,

$$\tan 60^0 = \frac{LE}{EA}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x}{y}$$

$$\Rightarrow \quad y = \frac{x}{\sqrt{3}} \qquad \text{..........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{x}{\sqrt{3}} = 66\sqrt{3}$$

$$\Rightarrow \quad x = 66 \times 3$$

$$= 198$$

i) Height of the lamp post = x + 66

= 198 + 64

= 264 mrs

ii) Difference between height of lamp post & apartment = 264 – 66 = 198 m

iii) Dist. between lamp post & the apartment

$$\Rightarrow \quad y = 66\sqrt{3}\text{ m}$$

$$= 66\,(1.732)$$

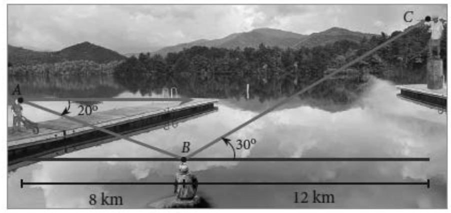
$$= 114.312\text{ m}$$

6. Three villagers A, B and C can see each other across a valley. The horizontal distance between A and B is 8 km and the horizontal distance between B and C is 12 km. The angle of depression of B from A is 20° and the angle of elevation of C from B is 30°. Calculate :

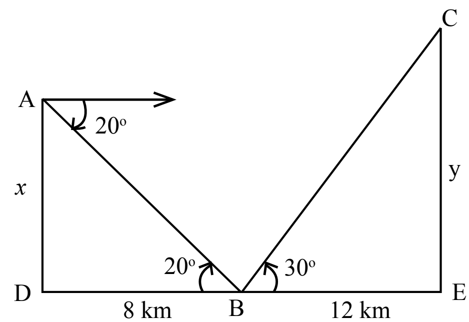
(i) the vertical height between A and B.

(ii) the vertical height between B and C.

(tan20° = 0.3640, ($\sqrt{3}$ = 1.732)



Solution :



A, B, C $\rightarrow$ Positions of 3 villagers

To find i) AD ii) CE

In ΔABD,

$$\tan 20^0 = \frac{x}{8}$$

$$\Rightarrow \quad x = 8\,.\tan 20^0$$

$$= 8\,(0.3640)$$

$$= 2.912$$

$$\therefore \quad AD = 2.912\text{ km}$$

In ΔCBE,

$$\tan 30^0 = \frac{y}{12}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{y}{12} \Rightarrow \quad y = \frac{12}{\sqrt{3}}$$

$$= 4\sqrt{3}$$

$$= 4\,(1.732)$$

$$= 6.928$$

$$\square 6.93$$

$$\therefore \quad CE = 6.93\text{ km}$$

EXERCISE 6.5

Multiple choice questions

1. The value of $\sin^2\theta + \frac{1}{1+\tan^2\theta}$ is equal to

(1) $\tan^2\theta$ (2) 1

(3) $\cot^2\theta$ (4) 0

Hint : **Ans : (2)**

$$= \sin^2\theta + \frac{1}{1+\tan^2\theta}$$
$$= \sin^2\theta + \frac{1}{\sec^2\theta}$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1$$

---

2. $\tan\theta \operatorname{cosec}^2\theta - \tan\theta$ is equal to

(1) $\sec\theta$ (2) $\cot^2\theta$

(3) $\sin\theta$ (4) $\cot\theta$

Hint : **Ans : (4)**

$= \tan\theta \,.\, \operatorname{cosec}^2\theta - \tan\theta$

$= \tan\theta(\operatorname{cosec}^2\theta - 1)$

$= \tan\theta \,.\, \cot^2\theta$

$$= \frac{1}{\cot\theta} \times \cot^2\theta$$

$= \cot\theta$

---

3. If $(\sin\alpha + \operatorname{cosec}\alpha)^2 + (\cos\alpha + \sec\alpha)^2 = k + \tan^2\alpha + \cot^2\alpha$, then the value of k is equal to

(1) 9 (2) 7 (3) 5 (4) 3

Hint : **Ans : (2)**

$= (\sin\alpha + \operatorname{cosec}\alpha)^2 + (\cos\alpha + \sec\alpha)^2$

$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow \sin^2\alpha + \operatorname{cosec}^2\alpha + 2\sin\alpha \,.\, \operatorname{cosec}\alpha + \cos^2\alpha + \sec^2\alpha + 2\cos\alpha\sec\alpha$

$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow 1 + 2 + 2 + \operatorname{cosec}^2\alpha + \sec^2\alpha$

$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow 5 + 1 + \cot^2\alpha + 1 + \tan^2\alpha$

$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow 7 + \cot^2\alpha + \tan^2\alpha = k + \tan^2\alpha + \cot^2\alpha$

$\therefore k = 7$

---

4. If $\sin\theta + \cos\theta = a$ and $\sec\theta + \operatorname{cosec}\theta = b$, then the value of $b(a^2 - 1)$ is equal to

(1) 2a (2) 3a

(3) 0 (4) 2ab

Hint : **Ans : (1)**

$b(a^2 - 1) = (\sec\theta + \operatorname{cosec}\theta)[(\sin\theta + \cos\theta)^2 - 1]$

$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)[2\sin\theta\cos\theta]$$

$= 2\sin\theta + 2\cos\theta$

$= 2(\sin\theta + \cos\theta)$

$= 2a$

---

5. If $5x = \sec\theta$ and $\frac{5}{x} = \tan\theta$, then $x^2 - \frac{1}{x^2}$ is equal to

(1) 25 (2) $\frac{1}{25}$ (3) 5 (4) 1

Hint : **Ans : (2)**

$$5x = \sec\theta, \ \frac{5}{x} = \tan\theta$$
$$\Rightarrow \sec^2\theta - \tan^2\theta = 1$$
$$\Rightarrow 25x^2 - \frac{25}{x^2} = 1$$
$$\Rightarrow 25\left(x^2 - \frac{1}{x^2}\right) = 1$$
$$\Rightarrow x^2 - \frac{1}{x^2} = \frac{1}{25}$$

6. If sin θ = cos θ, then 2 tan² θ + sin² θ − 1 is equal to

(1) $\frac{-3}{2}$ (2) $\frac{3}{2}$ (3) $\frac{2}{3}$ (4) $\frac{-2}{3}$

Hint : **Ans : (2)**

Given $\sin\theta = \cos\theta \Rightarrow \theta = 45^0$

$\therefore\ 2\tan^2\theta + \sin^2\theta - 1$

$= 2\tan^2 45^0 + \sin^2 45^0 - 1$

$$= 2(1) + \left(\frac{1}{\sqrt{2}}\right)^2 - 1$$

$$= 2 + \frac{1}{2} - 1$$

$$= \frac{3}{2}$$

7. If $x = a$ tan θ and y =b sec θ then

(1) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Hint : **Ans : (1)**

$$x = a\tan\theta,\ \ y = b\sec\theta$$

$$\therefore \tan\theta = \frac{x}{a},\ \ \sec\theta = \frac{y}{b}$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

8. (1+ tan θ + sec θ)(1+ cotθ − cosecθ) is equal to

(1) 0 (2) 1 (3) 2 (4) –1

Hint : **Ans : (3)**

$$= (1+\tan\theta+\sec\theta)\,(1+\cot\theta-\operatorname{cosec}\theta)$$

$$= \left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right).\left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)$$

$$= \left(\frac{(\cos\theta+\sin\theta)+1}{\cos\theta}\right)\left(\frac{(\sin\theta+\cos\theta)-1}{\sin\theta}\right)$$

$$= \left(\frac{(\cos\theta+\sin\theta)^2-1}{\cos\theta.\sin\theta}\right) = \frac{2\sin\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= 2$$

9. *a* cot θ + *b* cosec θ = *p* and *b* cot θ + *a* cosec θ = *q* then $p^2 - q^2$ is equal to

(1) $a^2 - b^2$ (2) $b^2 - a^2$

(3) $a^2 + b^2$ (4) $b - a$

Hint : **Ans : (2)**

$p^2 - q^2 = (a\cot\theta + b\operatorname{cosec}\theta)^2 - (b\cot\theta + a\operatorname{cosec}\theta)^2$

$= a^2\cot^2\theta + b^2\operatorname{cosec}^2\theta + 2ab\cot\theta\operatorname{cosec}\theta) - b^2\cot^2\theta + a^2\operatorname{cosec}^2\theta + 2ab\cot\theta\operatorname{cosec}\theta)$

$= a^2(\cot^2\theta - \operatorname{cosec}^2\theta) + b^2(\operatorname{cosec}^2 - \cot^2\theta)$

$= a^2(-1) + b^2(1)$

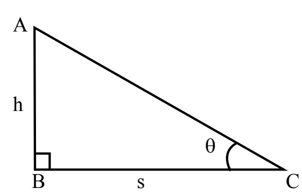
$= b^2 - a^2$

10. If the ratio of the height of a tower and the length of its shadow is $\sqrt{3}$: 1 , then the angle of elevation of the sun has measure

(1) 45° (2) 30°

(3) 90° (4) 60°

Hint : **Ans : (4)**

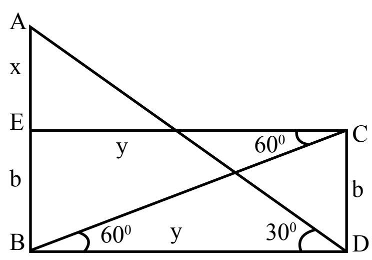


$$\tan\theta = \frac{h}{s} = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \theta = 60^0$$

11. The electric pole subtends an angle of 30° at a point on the same level as its foot. At a second point 'b' metres above the first, the depression of the foot of the tower is 60°. The height of the tower (in metres) is equal to

(1) $\sqrt{3}b$ (2) $\frac{b}{a}$ (3) $\frac{b}{2}$ (4) $\frac{b}{\sqrt{3}}$

Hint : **Ans : (4)**



$$\tan 30^0 = \frac{x+b}{y}$$
$$\frac{1}{\sqrt{3}} = \frac{x+b}{y}$$
$$\Rightarrow \quad y = \sqrt{3}\,(x+b)$$

$$\tan 60^0 = \frac{b}{y}$$
$$\sqrt{3} = \frac{b}{y}$$
$$\Rightarrow \quad y = \frac{b}{\sqrt{3}}$$

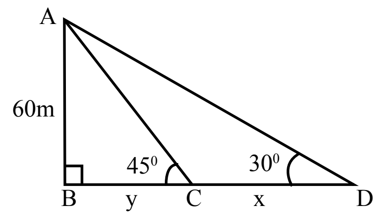
$$\therefore \sqrt{3}\,(x+b) = \frac{b}{\sqrt{3}}$$
$$\Rightarrow \quad 3(x+b) = b$$
$$\Rightarrow \quad b+x = \frac{b}{3}$$
$$\Rightarrow \quad \text{height of tower} = \frac{b}{3}\text{ mts}$$

12. A tower is 60 m height. Its shadow is x metres shorter when the sun's altitude is 45° than when it has been 30° , then x is equal to

(1) 41.92 m (2) 43.92 m

(3) 43 m (4) 45.6 m

Hint : **Ans : (3)**



In ΔABC,

$$\tan 45^0 = \frac{AB}{BC} = \frac{60}{y}$$
$$\Rightarrow \quad 1 = \frac{60}{y}$$
$$\Rightarrow \quad y = 60^0$$

In ΔABD,

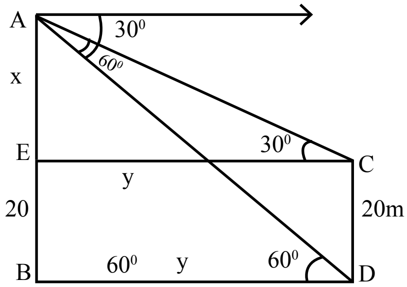
$$\tan 30^0 = \frac{AB}{BD}$$
$$\frac{1}{\sqrt{3}} = \frac{60}{x+y}$$
$$\Rightarrow \quad x+y = 60\sqrt{3}$$
$$\Rightarrow \quad x+60 = 60\sqrt{3}$$
$$\Rightarrow \quad x = 60\sqrt{3}-60$$
$$= 60\left(\sqrt{3}-1\right)$$
$$= 60 \times 0.732$$
$$= 43.92\text{ m}$$

13. The angle of depression of the top and bottom of 20 m tall building from the top of a multistoried building are 30° and 60° respectively. The height of the multistoried building and the distance between two buildings (in metres) is

(1) $20, 10\sqrt{3}$ (2) $30, 5\sqrt{3}$

(3) $20, 10$ (4) $30, 10\sqrt{3}$

Hint : **Ans : (4)**



In ΔACE,

$$\tan 30^0 = \frac{AE}{EC}$$

$$\frac{1}{\sqrt{3}} = \frac{x}{y}$$

$$\Rightarrow \quad y = \sqrt{3}x \quad \text{.........}(1)$$

In ΔADB,

$$\tan 60^0 = \frac{AB}{BD}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x+20}{y}$$

$$\Rightarrow \quad y = \frac{x+20}{\sqrt{3}} \quad \text{.........}(2)$$

$\therefore$ From (1) & (2)

$$\sqrt{3}\,x = \frac{x+20}{\sqrt{3}}$$

$$\Rightarrow \quad 3x = x+20$$

$$\Rightarrow \quad 2x = 20$$

$$\therefore x = 10$$

$\therefore$ Height of multistoried building

= x + 20

= 10 + 20

= 30 m

$\therefore$ Distance between 2 buildings

$$y = \frac{x+20}{\sqrt{3}} = \frac{30}{\sqrt{3}} = 10\sqrt{3}$$

$$= 10\,(1.732)$$

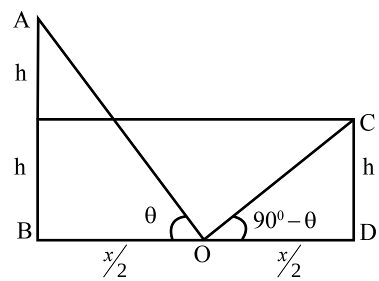
$$= 17.32 \text{ m}$$

---

14. Two persons are standing 'x' metres apart from each other and the height of the first person is double that of the other. If from the middle point of the line joining their feet an observer finds the angular elevations of their tops to be complementary, then the height of the shorter person (in metres) is

(1) $\sqrt{2}x$ (2) $\frac{x}{2\sqrt{2}}$ (3) $\frac{x}{\sqrt{2}}$ (4) $2x$

Hint : **Ans : (2)**



CD = h = height of shorter person

AB = 2h = height of taller person

BD = x mrs. $\Rightarrow$ BO = OD = $x/2$

In ΔAOB,

$$\tan\theta = \frac{2h}{x/2} = \frac{4h}{x} \quad \text{........}(1)$$

In ΔCOD,

$$\tan(90^0 - \theta) = \frac{h}{x/2}$$

$$\Rightarrow \quad \cot\theta = \frac{h}{x/2} = \frac{2h}{x}$$

$$\Rightarrow \quad \tan\theta = \frac{x}{2h} \quad \text{........ (2)}$$

$\therefore$ From (1) & (2)

$$\frac{4h}{x} = \frac{x}{2h}$$

$$\Rightarrow \quad 8h^2 = x^2$$

$$\Rightarrow \quad h^2 = \frac{x^2}{8}$$

$$\Rightarrow \quad h = \frac{x}{2\sqrt{2}}$$

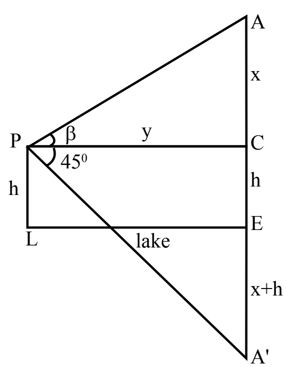
$\therefore$ Height of the shorter person $= \dfrac{x}{2\sqrt{2}}$ mrs.

15. The angle of elevation of a cloud from a point h metres above a lake is b . The angle of depression of its reflection in the lake is 45° . The height of location of the cloud from the lake is

(1) $\dfrac{h(1+\tan\beta)}{1-\tan\beta}$ (2) $\dfrac{h(1-\tan\beta)}{1+\tan\beta}$

(3) h tan(45° − β) (4) none of these

Hint : **Ans : (1)**



LE → Surface of the lake

P → Point of observation

PL = h mrs = CE

A, A' → Positions of cloud & its reflection

$\therefore$ AE = A'E = x + h

In ΔAPC,

$$\tan\beta = \frac{x}{y}$$

$$\Rightarrow \ y = \frac{x}{\tan\beta} \quad \text{......... (1)}$$

In ΔA'PC,

$$\tan 45 = \frac{x+2h}{y} \Rightarrow \frac{x+2h}{y} = 1$$

$$\Rightarrow \ y = x + 2h \quad \text{......... (2)}$$

$\therefore$ From (1) & (2),

$$x + 2h = \frac{x}{\tan\beta}$$

$$\Rightarrow \ 2h = \frac{x}{\tan\beta} - x$$

$$\Rightarrow \ 2h = x\left(\frac{1}{\tan\beta} - 1\right)$$

$$\Rightarrow \ 2h = x\left(\frac{1-\tan\beta}{\tan\beta}\right)$$

$$x = \frac{2h\tan\beta}{1-\tan\beta}$$

$\therefore$ Height of the cloud = h + x

$$= h + \frac{2h\tan\beta}{1-\tan\beta}$$

$$= h\left[1 + \frac{2\tan\beta}{1-\tan\beta}\right]$$

$$= h\left[\frac{1+\tan\beta}{1-\tan\beta}\right]$$

UNIT EXERCISE - 6

1. Prove that

(i) $\cot^2 A\left(\dfrac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\dfrac{\sin A-1}{1+\sec A}\right)=0$

(ii) $\dfrac{\tan^2\theta-1}{\tan^2\theta+1}=1-2\cos^2\theta$

Solution :

Solution :

i) LHS

$$\cot^2 A\left(\frac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\frac{\sin A-1}{1+\sec A}\right)$$

$$=\frac{1}{\tan^2 A}\left(\frac{\sec A-1}{1+\sin A}\right)-\frac{1}{\cos^2 A}\left(\frac{1-\sin A}{1+\sec A}\right)$$

$$=\frac{1}{(\sec A+1)\ \cancel{(\sec A-1)}}\cdot\left(\frac{\cancel{\sec A}-1}{1+\sin A}\right)-$$

$$\frac{1}{(1+\sin A)\ \cancel{(1-\sin A)}}\cdot\left(\frac{\cancel{1-\sin A}}{1+\sec A}\right)$$

$$=\frac{1}{(\sec A+1)(1+\sin A)}-\frac{1}{(1+\sin A)(1+\sec A)}$$

$=0$

$=$ RHS

ii) LHS

$$=\frac{\tan^2\theta-1}{\tan^2\theta+1}$$

$$=\frac{\tan^2\theta-1}{\sec^2\theta}$$

$$=\left(\tan^2\theta-1\right)\cos^2\theta$$

$$=\left(\frac{\sin^2\theta}{\cos^2\theta}-1\right)\cos^2\theta$$

$$=\sin^2\theta-\cos^2\theta$$

$$=1-\cos^2\theta-\cos^2\theta$$

$$=1-2\cos^2\theta$$

$=$ RHS

2. Prove that

$$\left(\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2=\frac{1-\cos\theta}{1+\cos\theta}$$

Solution :

LHS

$$=\left(\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2$$

Take $1+\sin\theta=a,\ \cos\theta=b$

$$\therefore\ \frac{(a-b)^2}{(a+b)^2}$$

$$=\frac{a^2+b^2-2ab}{a^2+b^2+2ab}$$

$$=\frac{(1+\sin\theta)^2+\cos^2\theta-2(1+\sin\theta)\cos\theta}{(1+\sin\theta)^2+\cos^2\theta+2(1+\sin\theta)\cos\theta}$$

$$=\frac{1+\sin^2\theta+2\sin\theta+\cos^2\theta-2(1+\sin\theta)\cos\theta}{1+\sin^2\theta+2\sin\theta+\cos^2\theta+2(1+\sin\theta)\cos\theta}$$

$$=\frac{2+2\sin\theta-2(1+\sin\theta)\cos\theta}{2+2\sin\theta+2(1+\sin\theta)\cos\theta}$$

$$=\frac{2(1+\sin\theta)-2(1+\sin\theta)\cos\theta}{2(1+\sin\theta)+2(1+\sin\theta)\cos\theta}$$

$$=\frac{\cancel{2(1+\sin\theta)}\,[1-\cos\theta]}{\cancel{2(1+\sin\theta)}\,[1+\cos\theta]}$$

$$=\frac{1-\cos\theta}{1+\cos\theta}$$

$=$ RHS

Hence proved.

3. If x $\sin^3\theta$ + y $\cos^3\theta$ = $\sin\theta\cos\theta$ and x $\sin\theta$ = y $\cos\theta$, then prove that $x^2+y^2=1$.

Solution :

Given $x\sin^3\theta+y\cos^3\theta=\sin\theta\cos\theta$,

$\Rightarrow x\sin\theta(\sin^2\theta)+y\cos\theta(\cos^2\theta)$

$=\sin\theta\cos\theta$

$\Rightarrow x\sin\theta(\sin^2\theta)+x\sin\theta(\cos^2\theta)$

$=\sin\theta\cos\theta$ (given $x\sin\theta=y\cos\theta$)

$\Rightarrow x \sin\theta (\sin^2\theta + \cos^2\theta) = \sin\theta \cos\theta$

$\Rightarrow \quad x = \cos\theta$

Also given $x \sin\theta = y \cos\theta$

$\Rightarrow \quad \cos\theta \cdot \sin\theta = y\cos\theta$

$\Rightarrow \quad y = \sin\theta$

$\therefore x^2 + y^2 = \cos^2\theta + \sin^2\theta$

$= 1$

Hence proved.

4. If a cos θ − b sin θ = c, then prove that (a sin θ + b cos θ) = $\pm\sqrt{a^2+b^2-c^2}$

Solution :

Given $a\cos\theta - b\sin\theta = c$

Squaring on both sides

$(a\cos\theta - b\sin\theta)^2 = c^2$

$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta = c^2$

$\Rightarrow a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) - 2ab\cos\theta\sin\theta = c^2$

$\Rightarrow a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\cos\theta\sin\theta = c^2$

$\Rightarrow a^2\sin^2\theta + b^2\cos^2\theta + 2ab\cos\theta\sin\theta + a^2 + b^2 - c^2$

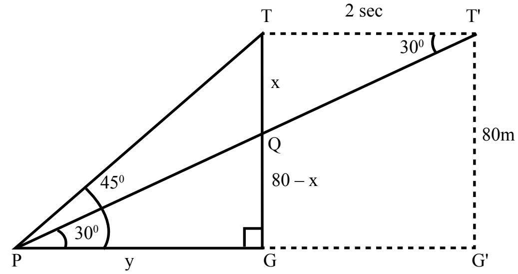
$(a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

$\therefore a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$

Hence proved.

5. A bird is sitting on the top of a 80 m high tree. From a point on the ground, the angle of elevation of the bird is 45°. The bird flies away horizontally in such away that it remained at a constant height from the ground. After 2 seconds, the angle of elevation of the bird from the same point is 30°. Determine the speed at which the bird flies. ($\sqrt{3}$ = 1.732)

Solution :



$P \rightarrow$ Point of observation

$T, T' \rightarrow$ Initial & final positions of the bird

TG = T'G' = 80m = height at which the bird is on the tree, from the ground.

$\angle QTG = \angle QT'T = 30^0$

In ΔPTG,

$$\tan 45^0 = \frac{TG}{PG}$$

$$\Rightarrow 1 = \frac{80}{y}$$

$$\Rightarrow y = 80 \quad \ldots\ldots\ldots\ldots (1)$$

In ΔPQG,

$$\tan 30^0 = \frac{80-x}{y}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{80-x}{80} \quad \text{(from (1))}$$

$$\Rightarrow 80 = 80\sqrt{3} - \sqrt{3}x$$

$$\Rightarrow \sqrt{3}x = 80\left(\sqrt{3}-1\right)$$

$$\therefore \quad x = \frac{80\left(\sqrt{3}-1\right)}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots (2)$$

In Δ TQT'

$$\tan 30^0 = \frac{TQ}{TT'}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{x}{TT'}$$

$$\Rightarrow TT' = \sqrt{3}x$$

$$\Rightarrow \quad = 80\left(\sqrt{3}-1\right) \quad \text{(from (2))}$$

Given, time taken by the bird from T to reach T' = 2 sec.

$\therefore$ Speed of the bird $= \dfrac{\text{Distance}}{\text{Time}}$

$= \dfrac{80\left(\sqrt{3}-1\right)}{2}$

$= 40\left(\sqrt{3}-1\right)$

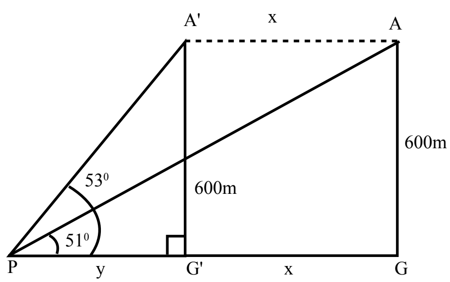
$= 40 \times 1.732$

$= 29.28$ m/sec.

---

6. An aeroplane is flying parallel to the Earth's surface at a speed of 175 m/sec and at a height of 600 m. The angle of elevation of the aeroplane from a point on the Earth's surface is 37° at a given point. After what period of time does the angle of elevation increase to 53° ? (tan 53° = 1.3270, tan 37° = 0.7536)

Solution :



P $\rightarrow$ Point of observation.

A, A' $\rightarrow$ Initial & final positions of the plane.

Speed of the plane = 175 m/sec.

AG = A'G' = 600 m

= height at which the plane is flying.

In ΔPAG,

$$\tan 37^0 = \frac{AG}{PG}$$

$$\Rightarrow \tan 37^0 = \frac{600}{y+x}$$

$$\Rightarrow y + x = \frac{600}{\tan 37^0}$$

$$\Rightarrow y = \frac{600}{\tan 37^0} - x \qquad \text{..........(1)}$$

In ΔPA' G',

$$\tan 53^0 = \frac{A'G'}{PG'}$$

$$\Rightarrow \tan 53^0 = \frac{600}{y}$$

$$\Rightarrow y = \frac{600}{\tan 53^0} \qquad \text{..........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{600}{\tan 53^0} = \frac{600}{\tan 37^0} - x$$

$$\Rightarrow x = 600\left(\frac{1}{\tan 37^0} - \frac{1}{\tan 53^0}\right)$$

$$\Rightarrow x = 600\left(\frac{\tan 53^0 - \tan 37^0}{\tan 53^0 \tan 37^0}\right)$$

$$= 600\left(\frac{\tan 53^0 - \tan 37^0}{\tan 53^0 \tan 37^0}\right) \qquad (\therefore \tan 53^0 = \cot 53^0 = 1)$$

$$= 600\left(\frac{\tan 53^0 - \tan 37^0}{1}\right)$$

$= 600\ (1.3270 - 0.7536)$

$= 600 \times 0.5734$

$= 344.04$ m

Time $= \dfrac{\text{Distance}}{\text{Speed}}$

$= \dfrac{344.04}{175}$

$= 1.9659$

≈ 1.97 sec.

$\therefore$ After 1.97 sec (approx), angle of elevation changes from 37^0 to 53^0.

7. A bird is flying from A towards B at an angle of 35° , a point 30 km away from A. At B it changes its course of flight and heads towards C on a bearing of 48° and distance 32 km away.

(i) How far is B to the North of A?

(ii) How far is B to the West of A?

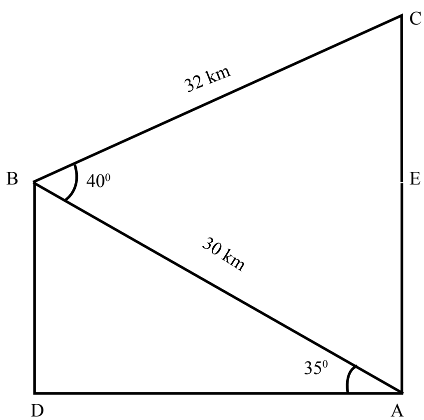
(iii) How far is C to the North of B?

(iv) How far is C to the East of B?

(sin 55° = 0.8192, cos 55° = 0.5736,

sin 42° = 0.6691, cos 42° = 0.7431)

Solution :



i) The distance of B to the north of A [BE]

In ΔBCE,

$$\Rightarrow \cos 48^0 = \frac{BE}{BC}$$

$$\Rightarrow \sin 42^0 = \frac{BE}{32}$$

$$\Rightarrow \quad BE = 32\,(0.6691)$$

$$= 21.4112$$

$$\simeq 21.41 \text{ km.}$$

ii) The distance of B to the west of A [BD]

In ΔBAD,

$$\Rightarrow \sin 35^0 = \frac{BD}{30}$$

$$\Rightarrow \cos 55^0 = \frac{BD}{30} \quad (\because \sin(90-\theta) = \cos\theta)$$

$$\Rightarrow \quad BD = 30^0 \,.\cos 55^0$$

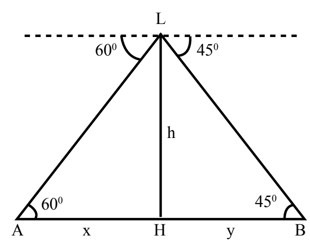
$$= 30\,(0.5736)$$

$$= 17.208$$

$$\simeq 17.21\,km.$$

8. Two ships are sailing in the sea on either side of the lighthouse. The angles of depression of two ships as observed from the top of the lighthouse are 60° and 45° respectively. If the distance between the ships is 200 $\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$ metres, find the height of the lighthouse.

Solution :



LH = h m = height of the light house

AB = Dist. between 2 ships =

$$x + y = 200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$$

In ΔLBH,

$$\tan 45^0 = \frac{h}{y}$$

$$\Rightarrow \quad 1 = \frac{h}{y}$$

$$\Rightarrow \quad y = h \quad \text{..........(1)}$$

In ΔLAH,

$$\tan 60^0 = \frac{h}{x}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow \quad x = \frac{h}{\sqrt{3}} \quad \text{..........} (2)$$

$\therefore$ Adding (1) & (2)

$$x + y = h + \frac{h}{\sqrt{3}}$$

$$\Rightarrow \quad 200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) = h\left(1 + \frac{1}{\sqrt{3}}\right)$$

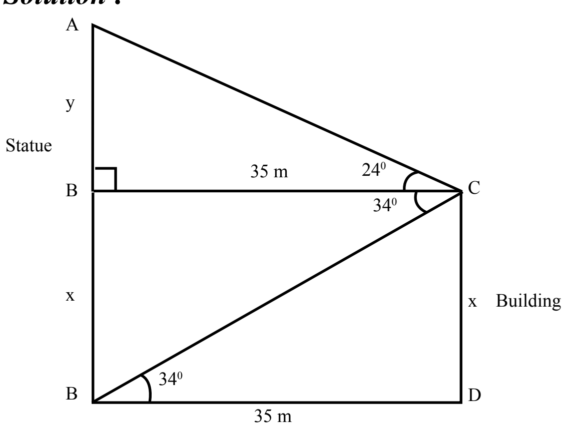
$$\Rightarrow \quad \frac{200(\sqrt{3}+1)}{\sqrt{3}} = h\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$$

$$\therefore \; h = 200$$

$\therefore$ Height of the light house = 200 m.

9. A building and a statue are in opposite side of a street from each other 35 m apart. From a point on the roof of building the angle of elevation of the top of statue is 24° and the angle of depression of base of the statue is 34° . Find the height of the statue.
(tan24° = 0.4452, tan 34° = 0.6745)

Solution :



Let CD = x m = height of the building = EB

AB = x + y m = height of the statue

In ΔCBD,

$$\tan 34^0 = \frac{CD}{BD}$$

$$\Rightarrow \quad \tan 34^0 = \frac{x}{35}$$

$$x = 35 \tan 34^0 \quad \text{...........} (1)$$

In ΔACE,

$$\tan 24^0 = \frac{AE}{EC}$$

$$\Rightarrow \quad \tan 24^0 = \frac{y}{35}$$

$$y = 35 \tan 24^0 \quad \text{...........} (2)$$

Height of the statue = x + y

= 35 ($\tan 34^0$ + $\tan 24^0$)

= 35 (0.6745 + 0.4452)

= 35 (1.1197)

= 39.1895

$\simeq$ 39.19 m

PROBLEMS FOR PRACTICE

1. If 7 $\sin^2\theta$ + 3 $\cos^2\theta$ = 4, show that $\tan\theta = 1/\sqrt{3}$.
2. Prove that $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta = 0$
3. Prove that $\dfrac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} = \dfrac{1 + \cos A}{\sin A}$
4. If $\tan\theta + \sin\theta = p$, $\tan\theta - \sin\theta = q$, prove that $p^2 - q^2 = 4\sqrt{pq}$
5. Prove that $\cot\theta - \tan\theta = \dfrac{2\cos^2\theta - 1}{\sin\theta . \cos\theta}$
6. If x = secA + sinA, y = secA – sinA, prove that $\left(\dfrac{2}{x+y}\right)^2 + \left(\dfrac{x-y}{2}\right)^2 = 1$

7. If $\sin\theta = \frac{15}{17}$, find the value of $\frac{3-4\sin^2\theta}{4\cos^2\theta-3}$ $\left(\text{Ans}: \frac{33}{611}\right)$

8. If $a = \sin\theta + \cos\theta$, $b = \sin^3\theta + \cos^3\theta$, then show that $(3a - 2b) = a^3$

9. If $\sec\theta = x + \frac{1}{4x}$ prove that $\sec\theta + \tan\theta = 2x$

10. Prove that
$$\frac{\sin\theta}{\cot\theta + \operatorname{cosec}\theta} = 2 + \frac{\sin\theta}{\cot\theta - \operatorname{cosec}\theta}$$

11. Prove that $(\tan\alpha + \operatorname{cosec}\beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha \cdot \cot\beta\,(\operatorname{cosec}\alpha + \sec\beta)$.

12. If $\sin\theta + \sin^2\theta = 1$, prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2 = 0$

13. Two vertical lamp - posts of equal height stand on either side of a roadway which is 50m wide. At a point in the road between lamp posts, elevations of the tops of the lamp posts are 60^0 and 30^0. Find the height of each lamp post. **(Ans : 21.65 m)**

14. The angle of elevation of a jet plane from a point P on the ground is 60^0. After 15 seconds the angle of elevation changes to 30^0. If the jet is flying at a speed of 720 km/h, find the height at which the jet is flying.
(Ans : 2.6 km)

15. From an aeroplane flying horizontally above a straight road, the angles of depression of two consecutive kilometer stones on the road are 45^0 and 30^0 respectively. Find the height of the aeroplane above the road when the km stones are i) on the same side of the vertical through aeroplane
ii) on the opposite sides.
(Ans : i) 1.366 kmii) 0.366 km)

Objective Type Qustions :

1. If $\cos\theta = a/b$, then $\operatorname{cosec}\theta$ is equal to
a) b/a b) $\frac{b}{\sqrt{b^2-a^2}}$
c) $\frac{\sqrt{b^2-a^2}}{b}$ d) $\frac{a}{\sqrt{b^2-a^2}}$
(Ans : (b))

2. If $\sin\theta - \cos\theta = 0$, then the value of $\sin^4\theta + \cos^4\theta$ is
a) $1/2$ b) $1/4$ c) $3/4$ d) 1
(Ans : (d))

3. The value of $\frac{11}{\cot^2\theta} - \frac{11}{\cos^2\theta}$ is
a) 11 b) 0 c) $1/11$ d) -11
(Ans : (d))

4. If $\tan\theta + \cot\theta = 5$, then the value of $\tan^2\theta + \cot^2\theta$ is
a) 23 b) 25 c) 27 d) 15
(Ans : (a))

5. The value of $(1 + \cot^2\theta) \cdot (1 + \cos\theta)(1 - \cos\theta)$ is
a) $\sin^2\theta$ b) $\operatorname{cosec}^2\theta$ c) 1
d) $\sec^2\theta$ **(Ans : (c))**

6. The value of $\tan 10^0 \cdot \tan 15^0 \cdot \tan 75^0 \tan 80^0$ is
a) -1 b) 0 c) 1 d) 4
(Ans : (c))

7. A pole 6m high cases a shadow $2\sqrt{3}$ m long on the ground, then the sun's elevation is
a) 60^0 b) 45^0 c) 30^0 d) 90^0
(Ans : (a))

8. The angle of elevation of the top of a tower from a point situated at a distance of 100m from the base of a tower is 30^0. The height of the tower is

a) $\frac{100}{\sqrt{3}}$m b) $100\sqrt{3}$m

c) $\frac{50}{\sqrt{3}}$m d) $50\sqrt{3}$m

(Ans : (a))

9. The angle of depression of a point on the horizontal from the top of a hill is 60^0. If one has towalk 300m to reach the top from this point, then the distance of this point from the base of the hill is

a) $300\sqrt{3}$m b) 150 m

c) $150\sqrt{3}$m d) $\frac{150}{\sqrt{3}}$ m

(Ans : (b))

10. The value of sinθ . cosecθ + cosθ secθ is

a) 1 b) 2 c) 0 d) $\frac{1}{2}$

(Ans : (b))

11. If 5 cos θ = 7sin θ, then the value of $\frac{7\sin\theta+5\cos\theta}{5\sin\theta+7\cos\theta}$ is

a) $\frac{37}{35}$ b) 1 c) $\frac{5}{7}$ d) $\frac{35}{37}$

(Ans : (d))

12. If $\sin A = \frac{1}{\sqrt{5}}$, then sec A is

a) $\frac{1}{\sqrt{5}}$ b) $\frac{2}{\sqrt{5}}$ c) $\frac{\sqrt{5}}{2}$ d) $\sqrt{5}$

(Ans : (c))

13. The acute angle 'θ' when $\sec^2\theta + \tan^2\theta = 3$ is

a) 30^0 b) 45^0 c) 60^0 d) 90^0

(Ans : (b))

14. If $\tan(20^0 - 3\alpha) = \cot(5\alpha - 20^0)$, then α is

a) 45^0 b) 30^0 c) 90^0

d) none of these **(Ans: (a))**

15. If $\tan\alpha = \sqrt{3}$, $\tan\beta = \frac{1}{\sqrt{3}}$, then $\cot(\alpha+\beta)$ is

a) $\sqrt{3}$ b) 0 c) $\frac{1}{\sqrt{3}}$ d) 1

(Ans : (b))

CHAPTER 7

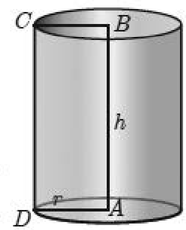
MENSURATION

I. SURFACE AREA

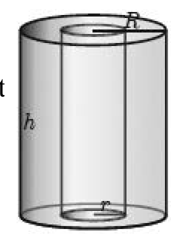
Key Points

Right Circular and Hollow Cylinder

- ✓ A right circular cylinder is a solid generated by the revolution of a rectangle about one of its sides as axis.

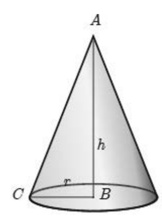


- ✓ If the axis is perpendicular to the radius then the cylinder is called a right circular cylinder.
- ✓ A solid cylinder is an object bounded by two circular plane surfaces and a curved surface.
- ✓ C.S.A. of a right circular cylinder = $2\pi rh$ sq. units.
- ✓ T.S.A. of a right circular cylinder = $2\pi r(h + r)$ sq. units

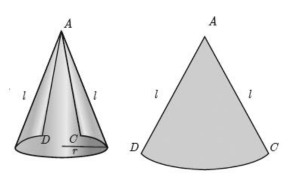


- ✓ An object bounded by two co-axial cylinders of the same height and different radii is called a 'hollow cylinder'.
- ✓ C.S.A of a hollow cylinder= $2\pi(R + r)h$ sq. units
- ✓ T.S.A. of a hollow cylinder = $2\pi(R + r)(R - r + h)$ sq. units

Right Circular and Hollow Cone

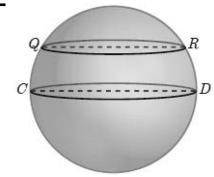


- ✓ A right circular cone is a solid generated by the revolution of a right angled triangle about one of the sides containing the right angle as axis.
- ✓ If the right triangle ABC revolves about AB as axis, the hypotenuse AC generates the curved surface of the cone.
- ✓ The height of the cone is the length of the axis AB, and the slant height is the length of the hypotenuse AC.

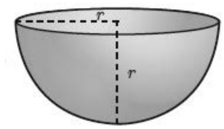


- ✓ Curved Surface Area of the cone = Area of the Sector = $\pi r l$ sq. units.
- ✓ T.S.A. of a right circular cone= $\pi r(l + r)$ sq. units, where $l = \sqrt{h^2 + r^2}$.

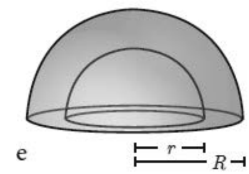
Sphere & Hemisphere



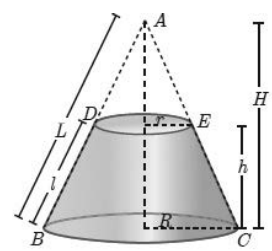
- ✓ A sphere is a solid generated by the revolution of a semicircle about its diameter as axis.
- ✓ Surface area of a sphere = $4\pi r^2$ sq.units.
- ✓ A section of the sphere cut by a plane through any of its great circle is a hemisphere.



- ✓ C.S.A. of a hemisphere = $2\pi r^2$ sq.units.
- ✓ T.S.A. of a hemisphere = $3\pi r^2$ sq.units.
- ✓ C.S.A. of a hollow hemisphere = $2\pi(R^2 + r^2)$ sq. units.
- ✓ T.S.A. of a hollow hemisphere = $\pi(3R^2 + r^2)$ sq. units.
- ✓ Thickness = R − r



Frustum of a Cone



- ✓ When a cone ABC is cut through by a plane parallel to its base, the portion of the cone DECB between the cutting plane and the base is called a frustum of the cone.
- ✓ C.S.A. of a frustum = $\pi(R + r)l$ sq. units, where, $l = \sqrt{h^2 + (R-r)^2}$.
- ✓ T.S.A. of a frustum = $\pi(R + r)l + \pi R^2 + \pi r^2$ sq. units.

Example 7.1

A cylindrical drum has a height of 20 cm and base radius of 14 cm. Find its curved surface area and the total surface area.

Solution :

Given that, height of the cylinder h = 20 cm ; radius r =14 cm

Now, C.S.A. of the cylinder = $2\pi rh$ sq. units

C.S.A. of the cylinder

$$= 2 \times \frac{22}{7} \times 14 \times 20 = 2\times22\times2\times20 = 1760 \text{ cm}^2$$

T.S.A. of the cylinder = $2\pi r$ (h + r) sq. units

$$= 2\times\frac{22}{7}\times14\times(20+14) = 2\times\frac{22}{7}\times14\times34$$

$= 2992$ cm^2

Therefore, C.S.A.

= 1760 cm^2 and T.S.A. = 2992 cm^2

Example 7.2

The curved surface area of a right circular cylinder of height 14 cm is 88 cm^2. Find the diameter of the cylinder.

Solution :

Given that, C.S.A. of the cylinder =88 sq. cm

$2\pi rh = 88$

$$= 2\times\frac{22}{7}\times r\times14 = 88 \ \ (\text{given } h = 14 \text{ cm})$$

$$2r = \frac{88\times7}{22\times14} = 2$$

Therefore, diameter = 2 cm

Example 7.3

A garden roller whose length is 3 m long and whose diameter is 2.8 m is rolled to level a garden. How much area will it cover in 8 revolutions?

Solution :

Given that, diameter d = 2.8 m and height = 3 m

radius r = 1.4 m

Fig. 7.6

Area covered in one revolution = curved surface area of the cylinder

= $2\pi rh$ sq. units

$$= 2 \times \frac{22}{7} \times 1.4 \times 3 = 26.4$$

Area covered in 1 revolution = 26.4 m^2

Area covered in 8 revolutions

= 8 × 26.4 = 211.2

Therefore, area covered is 211.2 m^2

Example 7.4

If one litre of paint covers 10 m^2, how many litres of paint is required to paint the internal and external surface areas of a cylindrical tunnel whose thickness is 2 m, internal radius is 6 m and height is 25 m.

Solution :

Given that, height h = 25 m; thickness = 2 m.

Internal radius r = 6 m

Now, external radius R = 6 + 2 = 8m

C.S.A. of the cylindrical tunnel

= C.S.A. of the hollow cylinder

C.S.A. of the hollow cylinder

= $2\pi (R + r) h$ sq. units

$$= 2 \times \frac{22}{7}(8+6) \times 25$$

Hence, C.S.A. of the cylindrical tunnel

= 2200 m^2

Area covered by one litre of paint = 10 m^2

Number of litres required to paint the tunnel

$$= \frac{2200}{10} = 220$$

Therefore, 220 litres of paint is needed to paint the tunnel.

Example 7.5

The radius of a conical tent is 7 m and the height is 24 m. Calculate the length of the canvas used to make the tent if the width of the rectangular canvas is 4 m?

Solution :

Let r and h be the radius and height of the cone respectively.

Given that, radius r =7 m and height h = 24 m

$$\text{Hence, } l = \sqrt{r^2 + h^2}$$

$$= \sqrt{49 + 576}$$

$$l = \sqrt{625} = 25\text{m}$$

C.S.A. of the conical tent = πrl sq. units

Area of the canvas = $\frac{22}{7} \times 7 \times 25 = 550\ m^2$

Now, length of the canvas =

$$\frac{\text{Area of canvas}}{\text{Width}} = \frac{550}{4} = 137.5\,m$$

Therefore, the length of the canvas is137.5 m

Example 7.6

If the total surface area of a cone of radius 7cm is 704 cm^2, then find its slant height.

Solution :

Given that, radius r = 7 cm

Now, total surface area of the cone

$= \pi r\,(l + r)$ sq. units

T.S.A. = 704 cm^2

$$704 = \frac{22}{7} \times 7\,(l+7)$$

$32 = l + 7$ implies $l = 25$ cm

Therefore, slant height of the cone is 25 cm.

Example 7.7

From a solid cylinder whose height is 2.4 cm and diameter 1.4 cm, a conical cavity of the same height and base is hollowed out (Fig.7.13). Find the total surface area of the remaining solid.

Solution :

Let h and r be the height and radius of the cone and cylinder.

Let l be the slant height of the cone.

Given that, h = 2.4 cm and d = 1.4 cm ;

r = 0.7 cm

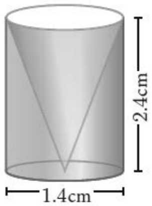


Fig. 7.13

Here, total surface area of the remaining solid $\}$ = C.S.A. of the cylinder + C.S.A. of the cone + area of the bottom

$= 2\pi rh + \pi rl + \pi r^2$ sq. units

Now,

$$l = \sqrt{r^2 + h^2} = \sqrt{0.49 + 5.76} = \sqrt{6.25} = 2.5cm$$

$l = 2.5$ cm

Area of the remaining solid

$= 2\pi rh + \pi rl + \pi r^2$ sq. units

$= \pi r\,(2h + l + r)$

$= \frac{22}{7} \times 0.7 \times [(2 \times 2.4) + 2.5 + 0.7]$

$= 17.6$

Therefore, total surface area of the remaining solid is 17.6 m^2

Example 7.8

Find the diameter of a sphere whose surface area is 154 m^2.

Solution :

Let r be the radius of the sphere.

Given that, surface area of sphere = 154 m^2

$4\pi r^2 = 154$

$$4 \times \frac{22}{7} \times r^2 = 154$$

gives $r^2 = 154 \times \frac{1}{4} \times \frac{7}{22}$

hence, $r^2 = \frac{49}{4}$ we get $r = \frac{7}{2}$

Therefore, diameter is 7 m

Example 7.9

The radius of a spherical balloon increases from 12 cm to 16 cm as air being pumped into it. Find the ratio of the surface area of the balloons in the two cases.

Solution :

Let r_1 and r_2 be the radii of the balloons.

Given that,

$$\frac{r_1}{r_2} = \frac{12}{16} = \frac{3}{4}$$

Now, ratio of C.S.A. of balloons =

$$\frac{4\pi r_1^2}{4\pi r_2^2} = \frac{r_1^2}{r_2^2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

Therefore, ratio of C.S.A. of balloons is 9:16.

Example 7.10

If the base area of a hemispherical solid is 1386 sq. metres, then find its total surface area?

Solution :

Let r be the radius of the hemisphere.

Given that, base area = πr^2 = 1386 sq. m

T.S.A. = $3\pi r^2$ sq.m

= 3 × 1386 = 4158

Therefore, T.S.A. of the hemispherical solid is 4158 m^2.

Example 7.11

The internal and external radii of a hollow hemispherical shell are 3 m and 5 m respectively. Find the T.S.A. and C.S.A. of the shell.

Solution :

Let the internal and external radii of the hemispherical shell be r and R respectively.

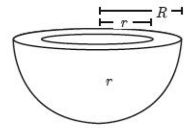


Fig. 7.19

Given that, R = 5 m, r = 3 m

C.S.A. of the shell = $2\pi(R^2 + r^2)$ sq. units

$$= 2 \times \frac{22}{7} \times (25 + 9) = 213.71$$

T.S.A. of the shell = $\pi(3R^2 + r^2)$ sq. units

$$= \frac{22}{7}(75 + 9) = 264$$

Therefore, C.S.A. = 213.71 m^2 and

T.S.A. = 264 m^2.

Example 7.12

A sphere, a cylinder and a cone (Fig.7.20) are of the same radius, where as cone and cylinder are of same height. Find the ratio of their curved surface areas.

Solution :

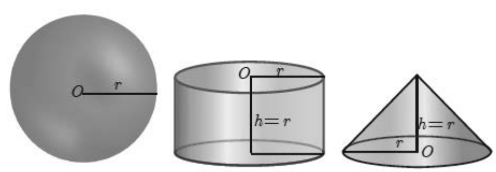


Fig. 7.20

Required Ratio = C.S.A. of the sphere: C.S.A. of the cylinder : C.S.A. of the cone

$$= 4\pi r^2 : 2\pi rh : \pi rl,$$

$$(l = \sqrt{r^2 + h^2} = \sqrt{2r^2} = \sqrt{2}r \text{ units})$$

$$= 4 : 2 : \sqrt{2} = 2\sqrt{2} : \sqrt{2} : 1$$

Example 7.13

The slant height of a frustum of a cone is 5 cm and the radii of its ends are 4 cm and 1 cm. Find its curved surface area.

Solution :

Let l, R and r be the slant height, top radius and bottom radius of the frustum.

Given that, l = 5 cm, R = 4 cm, r = 1 cm

Now, C.S.A. of the frustum

$$= \pi(R + r)\, l \text{ sq. units}$$

$$= \frac{22}{7} \times (4+1) \times 5$$

$$= \frac{550}{7}$$

Therefore, C.S.A. = 78.57 cm^2

Example 7.14

An industrial metallic bucket is in the shape of the frustum of a right circular cone whose top and bottom diameters are 10 m and 4 m and whose height is 4 m. Find the curved and total surface area of the bucket.

Solution :

Let h, l, R and r be the height, slant height, outer radius and inner radius of the frustum.

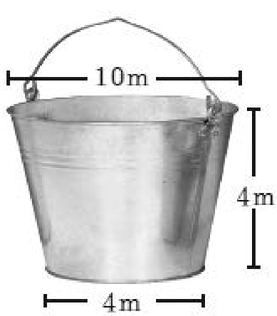


Fig. 7.24

Given that, diameter of the top =10 m; radius of the top R = 5 m. diameter of the bottom = 4 m; radius of the bottom r = 2 m, height h = 4 m

Now, $l = \sqrt{h^2 + (R-r)^2}$

$$= \sqrt{4^2 + (5-2)^2}$$

$$l = \sqrt{16+9} = \sqrt{25} = 5\text{m}$$

Here, C.S.A $= \pi (R+r)l$ sq.units

$$= \frac{22}{7}(5+2)\times 5 = 110 m^2$$

T.S.A. $= \pi(R+r)l + \pi R^2 + \pi r^2$ sq.units

$$= \frac{22}{7}[(5+2)\,5 + 25 + 4]$$

$$= \frac{1408}{7} = 201.14$$

Therefore, C.S.A. = 110 m^2 and

T.S.A. = 201.14 m^2

EXERCISE 7.1

1. **The radius and height of a cylinder are in the ratio 5 : 7 and its curved surface area is 5500 sq.cm. Find its radius and height.**

Solution:

Given r : h = 5 : 7

$$\Rightarrow \frac{r}{h} = \frac{5}{7}$$

$$\Rightarrow 7r = 5h$$

$$\Rightarrow h = \frac{7r}{5}$$

CSA of Cylinder = 5500

$$\Rightarrow 2\pi rh = 5500$$

$$\Rightarrow 2 \times \frac{22}{7} \times r \times \frac{7r}{5} = 5500$$

$$\Rightarrow \frac{44}{5} r^2 = 5500$$

$$\Rightarrow \quad r^2 = \frac{5500 \times 5}{44}$$

$$\Rightarrow \quad r = \frac{500 \times 5}{4}$$

$$= 125 \times 5$$

$$= 625$$

$$\therefore r = 25$$

$$\therefore h = \frac{7}{5} \times 25 = 35$$

$\therefore$ Radius = 25 cm, Height = 35 cm

2. A solid iron cylinder has total surface area of 1848 sq.m. Its curved surface area is five – sixth of its total surface area. Find the radius and height of the iron cylinder.

Solution :

Given total surface area of cylinder $= 1848\ m^2$ &

$$\text{CSA} = \frac{5}{6}(\text{TSA})$$

$$\Rightarrow 2\pi rh = \frac{5}{6} \times 1848$$

$$= 5 \times 308$$

$$2\pi rh = 1540 \qquad \text{..........(1)}$$

$$2\pi r\,(h + r) = 1848$$

$$\Rightarrow 2\pi rh + 2\pi r^2 = 1848$$

$$\Rightarrow 1540 + 2\pi r^2 = 1848$$

$$\Rightarrow 2\pi r^2 = 308$$

$$\Rightarrow 2 \times \frac{22}{7} \times r^2 = 308$$

$$\Rightarrow r^2 = \frac{\overset{\overset{7}{\cancel{14}}}{\cancel{308}} \times 7}{\cancel{2} \times \underset{1}{\cancel{22}}}$$

$$\Rightarrow r^2 = 49$$

$$r = 7\ m$$

Sub $r = 7$ in (1)

$$2 \times \frac{22}{7} \times 7 \times h = 1540$$

$$\Rightarrow \quad h = \frac{1540}{2 \times 22}$$

$$h = 35$$

$\therefore$ Radius = 7 m, Height = 35 m.

3. The external radius and the length of a hollow wooden log are 16 cm and 13 cm respectively. If its thickness is 4 cm then find its T.S.A.

Solution :

Given, external radius of

hollow cylinder = 16 cm = R

length of log = 13 cm = R

thickness = 4 cm = t

$$\therefore \quad t = R - r$$

$$\therefore \quad r = R - t$$

$$= 16 - 4$$

$$r = 12$$

$\therefore$ TSA of hollow cylinder

$$= 2\pi\,(R + r)\,(R - r + h)$$

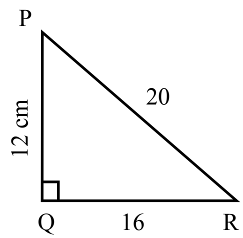
$$= 2 \times \frac{22}{7}(28)\,(4 + 13)$$

$$= 44 \times 4 \times 17$$

$$= 2992\ cm^2$$

4. A right angled triangle PQR where $\angle Q = 90^0$ is rotated about QR and PQ. If QR = 16 cm and PR = 20 cm, compare the curved surface areas of the right circular cones so formed by the triangle.

Solution :



$$\therefore PQ = \sqrt{400-256}$$
$$= \sqrt{144} = 12$$

i) When the Δ is rotated about PQ,

h = 12 cm, r = 16 cm

$\therefore$ CSA of cone $= \pi r l$

$= \pi \times 16 \times 20$

$= 320\pi$ cm^2

ii) When the Δ is rotated about QR,

h = 16 cm, r = 12 cm

$\therefore$ CSA of cone $= \pi r l$

$= \pi \times 16 \times 12$

$= 192\pi$ cm^2

$\therefore$ CSA of the cone when rotated about PQ is larger than that of QR.

5. 4 persons live in a conical tent whose slant height is 19 cm. If each person require 22 cm² of the floor area, then find the height of the tent.

Solution :

Given slant height of the cone l = 19 cm

Total floor area of 4 persons = 88 cm^2

$$\Rightarrow \quad \pi r^2 = 88$$
$$\Rightarrow \quad \frac{22}{7} \times r^2 = 88$$
$$\Rightarrow \quad r^2 = 28$$

$$\therefore h = \sqrt{l^2 - r^2}$$
$$= \sqrt{19^2 - 28}$$
$$= \sqrt{361-28}$$
$$= \sqrt{333}$$

height of cone $\simeq$ 18.25 cm.

6. A girl wishes to prepare birthday caps in the form of right circular cones for her birthday party, using a sheet of paper whose area is 5720 cm², how many caps can be made with radius 5 cm and height 12 cm.

Solution :

Given r = 5 cm, h = 12 cm in a cone

$$\therefore l = \sqrt{h^2 + r^2}$$
$$= \sqrt{144+25}$$
$$= \sqrt{169}$$
$$= 13$$

$\therefore$ CSA of cone $= \pi r l$

$$= \frac{22}{7} \times 5 \times 13$$
$$= \frac{110 \times 13}{7} \text{ cm}^2$$

Given, area of sheet of paper = 5720 cm^2

$$\therefore \text{Number of caps} = \frac{5720 \times 7}{110 \times 3}$$
$$= 28 \text{ caps}$$

7. The ratio of the radii of two right circular cones of same height is 1 : 3. Find the ratio of their curved surface area when the height of each cone is 3 times the radius of the smaller cone.

Solution :

Given $r_1 : r_2 = 1 : 3$ $h_1 = 3r_1, h_2 = 3r_1$

Let $r_1 = x$ $= 3x$, $= 3x$

$r_2 = 3x$

$$l_1 = \sqrt{h_1^2 + r_1^2}, \quad l_2 = \sqrt{h_2^2 + r_2^2}$$

$$= \sqrt{9x^2 + x^2} \quad = \sqrt{9x^2 + 9x^2}$$

$$= \sqrt{10x} \quad = 3\sqrt{2}x$$

$\therefore$ Ratio of their CSA

$$= \frac{\pi r_1 l_1}{\pi r_2 l_2}$$

$$= \frac{1}{3} \times \frac{\sqrt{10}}{3\sqrt{2}} = \frac{\sqrt{5}}{9}$$

$$\therefore \text{Ratio of their CSA} = \sqrt{5} : 9$$

8. The radius of a sphere increases by 25%. Find the percentage increase in its surface area.

Solution :

Le 'r' be the original radius of sphere

$\therefore$ Its surface area = $4\pi r^2$

If the radius increases by 25%

$$\text{New radius} = r + \frac{25}{100} r$$

$$= r + \frac{1}{4} r$$

$$= \frac{5}{4} r$$

$$\text{New surface area} = 4\pi \left(\frac{5}{4} r\right)^2$$

$$= 4\pi \times \frac{25}{16} r^2$$

$$= \frac{25\pi r^2}{4}$$

$$\therefore \text{Increment in SA} = \frac{25\pi r^2}{4} - 4\pi r^2$$

$$= \frac{9\pi r^2}{4}$$

$$\therefore \text{Percentage inc. in SA} = \frac{\frac{9\pi r^2}{4}}{4\pi r^2} \times 100$$

$$= \frac{9}{16} \times 100$$

$$= \frac{225}{4}$$

$$= 56.25\%$$

9. The internal and external diameters of a hollow hemispherical vessel are 20 cm and 28 cm respectively. Find the cost to paint the vessel all over at ₹ 0.14 per cm².

Solution :

Given in a hollow hemisphere

D = 28 cm, d = 20 cm

$\Rightarrow$ R = 14 cm, r = 10 cm

$\therefore$ TSA of hollow hemisphere

$$= \pi(3R^2 + r^2)$$

$$= \frac{22}{7}(588 + 100)$$

$$= \frac{22}{7} \times 688\, cm^2$$

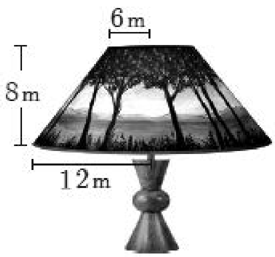
Given cost of painting = 0.14 / cm²

$\therefore$ Total cost of painting

$$= \frac{22}{7} \times 688 \times 0.14$$

$$= ₹\ 302.72$$

10. The frustum shaped outer portion of the table lamp has to be painted including the top part. Find the total cost of painting the lamp if the cost of painting 1 sq.cm is ₹ 2.



Solution :

Given in a frustum shaped lamp

R = 12m, r = 6m, h = 8m

$$l = \sqrt{(R-r)^2 + h^2}$$

$$= \sqrt{36+64}$$

$$= \sqrt{100} = 10 \text{ cm}$$

∴ Required portion to be painted =

CSA of frustum + πr^2

$$= \pi (R+r)\, l + \pi r^2$$

$$= \pi [18(10) + 36]$$

$$= \frac{22}{7} \times 216$$

$$= \frac{4752}{7}$$

$$= 678.86\, m^2$$

Given cost of painting = Rs.2/m^2

∴ Total cost = 678.86 × 2

= Rs.1357.72/-

II. VOLUME

Key Points

Right Circular and Hollow Cylinder

✓ Volume of a cylinder = $\pi r^2 h$ cu. units.

✓ Volume of a hollow cylinder = $\pi(R^2 - r^2)h$ cu. units.

✓ Volume of a cone = $\frac{1}{3}\pi r^2 h$ cu. units.

Sphere and Hemi-sphere

✓ Volume of a sphere = $\frac{4}{3}\pi r^3$ cu. units.

✓ Volume of a hollow sphere = $\frac{4}{3}\pi(R^3 - r^3)$ cu. units.

✓ Volume of a solid hemisphere = $\frac{2}{3}\pi r^3$ cu. units.

✓ Volume of a hollow hemisphere = $\frac{2}{3}\pi(R^3 - r^3)$

Frustum of a Cone

✓ Volume of a frustum = $\frac{\pi h}{3}(R^2 + Rr + r^2)$ cu. units.

Example 7.15

Find the volume of a cylinder whose height is 2 m and whose base area is 250 m^2.

Solution :

Let r and h be the radius and height of the cylinder respectively.

Given that, height h = 2 m,

base area = 250 m^2

Now, volume of a cylinder = $\pi r^2 h$ cu. units

= base area × h

= 250 × 2 = 500 m^3

Therefore, volume of the cylinder = 500 m^3

Example 7.16

The volume of a cylindrical water tank is 1.078×10^6 litres. If the diameter of the tank is 7 m, find its height.

Solution :

Let r and h be the radius and height of the cylinder respectively.

Given that, volume of the tank

= 1.078×10^6 = 1078000 litre

$$= 1078 \text{ m}^3 \quad \left(\text{since } 1\,l = \frac{1}{1000}\text{ m}^3\right)$$

diameter = 7m gives radius = $\frac{7}{2}$ m

volume of the tank = $\pi r^2 h$ cu. units

$$1078 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times h$$

Therefore, height of the tank is 28 m.

Example 7.17

Find the volume of the iron used to make a hollow cylinder of height 9 cm and whose internal and external radii are 21 cm and 28 cm respectively.

Solution :

Let r, R and h be the internal radius, external radius and height of the hollow cylinder respectively.

Given that, r = 21cm, R = 28 cm, h = 9 cm

Now, volume of hollow cylinder

= $\pi(R^2 - r^2)$ h cu. units

$$= \frac{22}{7}(28^2 - 21^2) \times 9$$

$$= \frac{22}{7}(784 - 441) \times 9 = 9702$$

Therefore, volume of iron used = 9702 cm^3.

Example 7.18

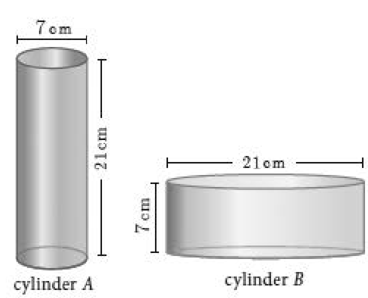
For the cylinders A and B

(i) find out the cylinder whose volume is greater.

(ii) verify whether the cylinder with greater volume has greater total surface area.

(iii) find the ratios of the volumes of the cylinders A and B.

Solution :



(i) Volume of cylinder = $\pi r^2 h$ cu. units

Volume of cylinder A

$$= \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 21$$
$$= 808.5 \text{ cm}^3$$

Volume of cylinder B

$$= \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 7$$
$$= 2425.5 \text{ cm}^3$$

Therefore, volume of cylinder B is greater than volume of cylinder A.

(ii) T.S.A. of cylinder = $2\pi r(h + r)$ sq. units

T.S.A. of cylinder

$$A = 2 \times \frac{22}{7} \times \frac{7}{2} \times (21 + 3.5) = 539 \text{ cm}^2$$

T.S.A. of cylinder

$$B = 2 \times \frac{22}{7} \times \frac{21}{2} \times (7 + 10.5) = 1155 \text{ cm}^2$$

Hence verified that cylinder B with greater volume has a greater surface area.

(iii) $\dfrac{\text{Volume of cylinder A}}{\text{Volume of cylinder B}} = \dfrac{808.5}{2425.5} = \dfrac{1}{3}$

Therefore, ratio of the volumes of cylinders A and B is 1:3.

Example 7.19

The volume of a solid right circular cone is 11088 cm^3. If its height is 24 cm then find the radius of the cone.

Solution :

Let r and h be the radius and height of the cone respectively.

Given that,

Volume of the cone = 11088 cm^3

$$\frac{1}{3}\pi r^2 h = 11088$$
$$\frac{1}{3} \times \frac{22}{7} \times r^2 \times 24 = 11088$$
$$r^2 = 441$$

Therefore, radius of the cone r = 21 cm

Example 7.20

The ratio of the volumes of two cones is 2 : 3. Find the ratio of their radii if the height of second cone is double the height of the first..

Solution :

Let r_1 and h_1 be the radius and height of the cone-I and let r_2 and h_2 be the radius and height of the cone-II.

$$\text{Given } h_2 = 2h_1 \text{ and } \frac{\text{Volume of the cone I}}{\text{Volume of the cone II}} = \frac{2}{3}$$

$$\frac{\frac{1}{3}\pi r_1^2 h_1}{\frac{1}{3}\pi r_2^2 h_2} = \frac{2}{3}$$

$$\frac{r_1^2}{r_2^2} \times \frac{h_1}{2h_1} = \frac{2}{3}$$

$$\frac{r_1^2}{r_2^2} = \frac{4}{3} \text{ gives } \frac{r_1}{r_2} = \frac{2}{\sqrt{3}}$$

Therefore, ratio of their radii = 2 : $\sqrt{3}$

Example 7.21

The volume of a solid hemisphere is 29106 cm^3. Another hemisphere whose volume is two-third of the above is carved out. Find the radius of the new hemisphere.

Solution :

Let r be the radius of the hemisphere.

Given that, volume of the hemisphere

$= 29106 \text{ cm}^3$

Now, volume of new hemisphere

$$= \frac{2}{3}(\text{Volume of original sphere})$$

$$= \frac{2}{3} \times 29106$$

Volume of new hemisphere = 19404 cm^3

$$\frac{2}{3}\pi r^3 = 19404$$

$$r^3 = \frac{19404 \times 3 \times 7}{2 \times 22} = 9261$$

$$r = \sqrt[3]{9261} = 21 \text{ cm}$$

Therefore, r = 21cm.

Example 7.22

Calculate the weight of a hollow brass sphere if the inner diameter is 14 cm and thickness is 1mm, and whose density is 17.3 g/ cm^3.

Solution :

Let r and R be the inner and outer radii of the hollow sphere.

Given that, inner diameter d = 14 cm; inner radius r = 7 cm ;

$$\text{thickness} = 1 \text{ mm} = \frac{1}{10} \text{ cm}$$

$$\text{Outer radius } R = 7 + \frac{1}{10} = \frac{71}{10} = 7.1 \text{ cm}$$

$$\text{Volume of hollow sphere} = \frac{4}{3}\pi(R^3 - r^3) \text{ cu.cm}$$

$$= \frac{4}{3} \times \frac{22}{7}(357.91 - 343)$$

$$= 62.48 \text{ cm}^3$$

But, weight of brass in 1 cm^3 = 17.3 gm

Total weight = 17.3 × 62.48 = 1080.90 gm

Therefore, total weight is 1080.90 grams.

Example 7.23

If the radii of the circular ends of a frustum which is 45 cm high are 28 cm and 7 cm, find the volume of the frustum.

Solution :

Let h, r and R be the height, top and bottom radii of the frustum.

Given that, h = 45 cm, R = 28 cm, r = 7 cm

Now,

$$\text{Volume} = \frac{1}{3}\pi[R^2 + Rr + r^2]h \text{ cu.units}$$

$$= \frac{1}{3} \times \frac{22}{7} \times [28^2 + (28 \times 7) + 7^2] \times 45$$

$$= \frac{1}{3} \times \frac{22}{7} \times 1029 \times 45 = 48510$$

Therefore, volume of the frustum is 48510 cm^3

EXERCISE 7.2

1. A 14 m deep well with inner diameter 10 m is dug and the earth taken out is evenly spread all around the well to form an embankment of width 5 m. Find the height of the embankment.

Solution :

Given radius of well, r = 5m

height of well, h = 14m

∴ Volume of earth taken out = $\pi r^2 h$

$$= \frac{22}{7} \times 25 \times 14$$

$$= 1100 \, m^3$$

Since it is spread to form embankment which is the form of hollow cylinder, Inner radius = 5 cm,

Outer radius = 5 + 5 (Given width = 5m)

= 10 m

$$\therefore \quad \pi h_1 (R^2 - r^2) = 1100$$

$$\Rightarrow \frac{22}{7} \times h_1 (10^2 - 5^2) = 1100$$

$$\Rightarrow \frac{22}{7} \times h_1 (75) = 1100$$

$$\Rightarrow h_1 = \frac{1100 \times 7}{22 \times 75}$$

$$h_1 = 4.67\ m$$

∴ height of embankment = 4.67 m

2. A cylindrical glass with diameter 20 cm has water to a height of 9 cm. A small cylindrical metal of radius 5 cm and height 4 cm is immersed it completely. Calculate the raise of the water in the glass?

Solution :

Cylindrical Glass	Cylindrical Metal
R = 10 cm	r = 5 cm
H = 9 cm	h = 4 cm

When cylindrical metal is immersed completely in glass,

Total volume = Vol. of glass + Vol. of metal

$= \pi[100 \times 9 + 25 \times 4] = 1000\pi\ cm^3$

Let h_1 be the height of water level

$\therefore \pi R^2 h_1 = 1000\pi$

$\Rightarrow \quad 100\ h_1 = 1000$

$\therefore \quad h_1 = 10$ cm

∴ Rise in water level = $h_1 - H$

= – 10 – 9 = 1 cm

3. If the circumference of a conical wooden piece is 484 cm then find its volume when its height is 105 cm.

Solution :

Given circumference of a cone = 484 cm

ie $\quad 2\pi r = 484$

$$\Rightarrow 2 \times \frac{22}{7} \times r = 484$$

$$\Rightarrow r = \frac{484 \times 7}{2 \times 22}$$

$$r = 77 \text{ cm}$$

Also given h = 105 cm

$$\therefore \text{Volume of cone} = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{\cancel{3}} \times \frac{22}{\cancel{7}} \times 77 \times 77 \times \overset{5}{\cancel{105}}$$

$$= 652190 \text{ cm}^3$$

4. A conical container is fully filled with petrol. The radius is 10m and the height is 15 m. If the container can release the petrol through its bottom at the rate of 25 cu. meter per minute, in how many minutes the container will be emptied. Round off your answer to the nearest minute.

Solution :

Given r = 10m, h = 15m in a cone

$$\therefore \text{Volume of the cone} = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{\cancel{3}} \times \frac{22}{\cancel{7}} \times 100 \times \overset{5}{\cancel{15}}$$

$$= \frac{11000}{7} \text{ cm}^3$$

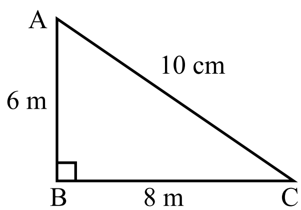
Rate of releasing the petrol = 25 cm^3/min

$\therefore$ Total time taken to empty the can

$$= \frac{11000}{7 \times 25}$$
$$= \frac{440}{7}$$
$$= 62.851 \text{ min}.$$
$$\square 63 \text{ min}.$$

5. A right angled triangle whose sides are 6 cm, 8 cm and 10 cm is revolved about the sides containing the right angle in two ways. Find the difference in volumes of the two solids so formed.

Solution :



i) When it revolves about AB = 6 cm

h = 6 cm, r = 8 cm

$\therefore$ Volume of the cone

$$= \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \frac{22}{7} \times 64 \times 6$$
$$= \frac{22 \times 64 \times 2}{7}$$
$$= \frac{2816}{7} \text{ cm}^3$$

i) When it revolves about BC = 8 cm

h = 8 cm, r = 6 cm

$\therefore$ Volume of the cone

$$= \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \frac{22}{7} \times 36 \times 8$$
$$= \frac{22 \times 12 \times 8}{7}$$
$$= \frac{2112}{7} \text{ cm}^3$$

$\therefore$ Difference in volumes

$$= \frac{2816}{7} - \frac{2112}{7}$$
$$= \frac{704}{7}$$
$$= 100.58 \text{ cm}^3$$

6. The volumes of two cones of same base radius are 3600 cm³ and 5040 cm³. Find the ratio of heights.

Solution :

Given volumes of 2 cones

= 3600 cm³ & 5040 cm³

& base radius are equal

$$\therefore \text{Ratio of volumes} = \frac{V_1}{V_2} = \frac{3600}{5040}$$
$$\Rightarrow \frac{\cancel{1/3}\,\pi r_1^2 h_1}{\cancel{1/3}\,\pi r_2^2 h_2} = \frac{3600}{5040}$$
$$\Rightarrow \frac{h_1}{h_2} = \frac{40}{56}$$
$$= \frac{5}{7}$$

$\therefore h_1 : h_2 = 5 : 7$

7. If the ratio of radii of two spheres is 4 : 7, find the ratio of their volumes.

Solution :

Given ratio of radii of 2 spheres = 4 : 7

$$ie\ \frac{r_1}{r_2} = \frac{4}{7}$$

$$\therefore \text{Ratio of their volumes} = \frac{V_1}{V_2}$$

$$= \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3}$$

$$= \left(\frac{r_1}{r_2}\right)^3$$

$$= \left(\frac{4}{7}\right)^3$$

$$= \frac{64}{343}$$

$\therefore$ Ratio of the volumes = 64 : 343

8. A solid sphere and a solid hemisphere have equal total surface area. Prove that the ratio of their volume is 3 $\sqrt{3}$: 4.

Solution :

Given TSA of a solid sphere

= TSA of a solid hemisphere

$$\Rightarrow \quad 4\pi R^2 = 3\pi r^2$$

$$\Rightarrow \quad \therefore \frac{R^2}{r^2} = \frac{3}{4}$$

$$\therefore \frac{R}{r} = \frac{\sqrt{3}}{2}$$

$$\therefore \text{Ratio of their volumes} = \frac{\frac{4}{3}\pi R^3}{\frac{2}{3}\pi r^3}$$

$$= \frac{2R^3}{r^3}$$

$$= 2\left[\frac{R}{r}\right]^3$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^3$$

$$= 2 \times \frac{3\sqrt{3}}{8}$$

$$= \frac{3\sqrt{3}}{4}$$

$$\therefore \text{Ratio of the volumes} = 3\sqrt{3} : 4$$

9. The outer and the inner surface areas of a spherical copper shell are 576π cm² and 324π cm² respectively. Find the volume of the material required to make the shell.

Solution :

Given $4\pi R^2 = 576\pi$	$4\pi r^2 = 324\pi$
$R^2 = 144$	$r^2 = 81$
$R = 12$ cm	$r = 9$ cm

$\therefore$ Volume of the material

$$= \frac{4}{3}\pi\left(R^3 - r^3\right)$$

$$= \frac{4}{3} \times \frac{22}{7}(1728 - 729)$$

$$= \frac{4}{3} \times \frac{22}{7} \times 999$$

$$= \frac{88 \times 333}{7} = 4186.29 \text{ cm}^3$$

10. A container open at the top is in the form of a frustum of a cone of height 16 cm with radii of its lower and upper ends are 8 cm and 20 cm respectively. Find the cost of milk which can completely fill a container at the rate of ₹ 40 per litre.

Solution :

Given R = 20cm, r = 8cm, h = 16cm

in frustum of a cone

$\therefore$ Volume of frustum of a cone

$$= \frac{\pi h}{3}\left(R^2 + Rr + r^2\right)$$

$$= \frac{22}{7} \times \frac{16}{3}(400 + 160 + 64)$$

$$= \frac{22}{\cancel{7}} \times \frac{16}{\cancel{3}} \times \overset{208}{\cancel{624}}$$

$$= \frac{73216}{7}$$

$$= 10,459.42 \text{ cm}^3$$

$$= \frac{10459.42}{1000} \text{ litres}$$

$$= 10.459 \text{ litres}$$

$\therefore$ Cost of milk at ₹ 40 / lr,

= 10.459 × 40

= ₹ 418.36

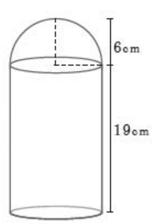
III. Volume and Surface Area of Combined Solids

Example 7.24

A toy is in the shape of a cylinder surmounted by a hemisphere. The height of the toy is 25 cm. Find the total surface area of the toy if its common diameter is 12 cm.

Solution :

Let r and h be the radius and height of the cylinder respectively.



Given that, diameter d = 12 cm, radius r = 6 cm

Total height of the toy is 25 cm

Therefore, height of the cylindrical portion

= 25 − 6 = 19 cm

T.S.A. of the toy = C.S.A. of the cylinder + C.S.A. of the hemisphere + Base Area of the cylinder

$$= 2\pi rh + 2\pi r^2 + \pi r^2$$

$$= \pi r(2h + 3r) \text{ sq.units}$$

$$= \frac{22}{7} \times 6 \times (38 + 18)$$

$$= \frac{22}{7} \times 6 \times 56 = 1056$$

Therefore, T.S.A. of the toy is 1056 cm^2.

Example 7.25

A jewel box (Fig. 7.39) is in the shape of a cuboid of dimensions 30 cm × 15 cm × 10 cm surmounted by a half part of a cylinder as shown in the figure. Find the volume and T.S.A. of the box.

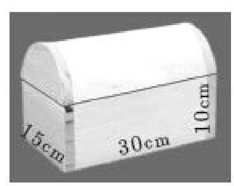


Fig. 7.39

Solution :

Let l, b and h_1 be the length, breadth and height of the cuboid. Also let us take r and h_2 be the radius and height of the cylinder.

Now, Volume of the box =

Volume of the cuboid +

$\frac{1}{2}$(Volume of cylinder)

$$= (l \times b \times h_1) + \frac{1}{2}(\pi r^2 h_2) \text{ cu. units}$$

$$= (30 \times 15 \times 10) + \frac{1}{2}\left(\frac{22}{7} \times \frac{15}{2} \times \frac{15}{2} \times 30\right)$$

$$= 4500 + 2651.79 = 7151.79$$

Therefore, Volume of the box = 7151.79 cm^3

Now, T.S.A. of the box = C.S.A. of the cuboid + $\frac{1}{2}$(C.S.A. of the cylinder)

$$= 2(l + b)\, h_1 + \frac{1}{2}(2\pi r h_2)$$

$$= 2\,(45 \times 10) + \left(\frac{22}{7} \times \frac{15}{2} \times 30\right)$$

$$= 900 + 707.14 = 1607.14$$

Therefore, T.S.A. of the box = 1607.14 cm^2

Example 7.26

Arul has to make arrangements for the accommodation of 150 persons for his family function. For this purpose, he plans to build a tent which is in the shape of cylinder surmounted by a cone. Each person occupies 4 sq. m of the space on ground and 40 cu. meter of air to breathe. What should be the height of the conical part of the tent if the height of cylindrical part is 8 m?.

Solution :

Let h_1 and h_2 be the height of cylinder and cone respectively.

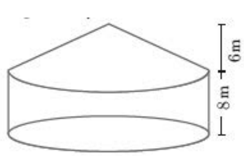


Fig. 7.40

Area for one person = 4 sq. m

Total number of persons = 150

Therefore total base area = 150 × 4

πr^2 = 600

$$r^2 = 600 \times \frac{7}{22} = \frac{2100}{11} \quad \text{........ (1)}$$

Volume of air required for 1 person = 40 m^3

Total Volume of air required for 150 persons

= 150 × 40 = 6000 m^3

$$\pi r^2 h_1 + \frac{1}{3}\pi r_2 h_2 = 6000$$

$$\pi r^2\left(h_1 + \frac{1}{3}h_2\right) = 6000$$

$$\frac{22}{7} \times \frac{2100}{11}\left(8 + \frac{1}{3}h_2\right) = 6000 \quad \text{[using (1)]}$$

$$8 + \frac{1}{3}h_2 = \frac{6000 \times 7 \times 11}{22 \times 2100}$$

$$\frac{1}{3}h_2 = 10 - 8 = 2$$

Therefore, the height of the conical tent h_2 is 6 m

Example 7.27

A funnel consists of a frustum of a cone attached to a cylindrical portion 12 cm long attached at the bottom. If the total height be 20 cm, diameter of the cylindrical portion be 12 cm and the diameter of the top of the funnel be 24 cm. Find the outer surface area of the funnel.

Solution :

Let R, r be the top and bottom radii of the frustum.

Let h_1, h_2 be the heights of the frustum and cylinder respectively.

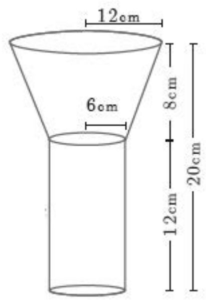


Fig. 7.41

Given that, R = 12 cm, r = 6 cm, h_2 = 12 cm

Now, h_1 = 20 – 12 = 8 cm

Here, Slant height of the frustum

$$l = \sqrt{(R-r)^2 + h_1^2} \text{ units}$$
$$= \sqrt{36+64}$$
$$l = 10 \text{ cm}$$

Outer surface area

$= 2\pi r h_2 + \pi(R + r)\, l$ sq. units

$= \pi[2rh_2 + (R + r)\, l]$

$= \pi[(2 \times 6 \times 12) + (18 \times 10)]$

$= \pi[144 + 180]$

$$= \frac{22}{7} \times 324 = 1018.28$$

Therefore, outer surface area of the funnel is 1018.28 cm^2

Example 7.28

A hemispherical section is cut out from one face of a cubical block (Fig.7.42) such that the diameter l of the hemisphere is equal to side length of the cube. Determine the surface area of the remaining solid.

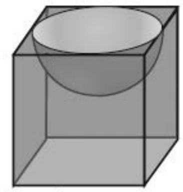


Fig. 7.42

Solution :

Let r be the radius of the hemisphere.

Given that, diameter of the hemisphere = side of the cube = l

Radius of the hemisphere = $\frac{l}{2}$

TSA of the remaining solid =

Surface area of the cubical part +
C.S.A. of the hemispherical part −
Area of the base of the hemispherical part

$= 6 \times (\text{Edge})^2 + 2\pi r^2 - \pi r^2$

$= 6 \times (\text{Edge})^2 + \pi r^2$

$$= 6 \times (l)^2 + \pi\left(\frac{l}{2}\right)^2 = \frac{1}{4}(24+\pi)\, l^2$$

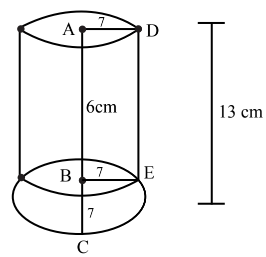
Total surface area of the remaining solid =

$$\frac{1}{4}(24+\pi)\, l^2 \text{ sq.units}$$

EXERCISE 7.3

1. A vessel is in the form of a hemispherical bowl mounted by a hollow cylinder. The diameter is 14 cm and the height of the vessel is 13 cm. Find the capacity of the vessel.

Solution:



Given AD = BE = 7 cm = radius of the vessel = BC

AC = 13 cm = height of the vessel

$\therefore$ AB = 13 – 7

= 6cm = height of cylindrical part

$\therefore$ Capacity of the vessel

= Capacity of cylinder + Capacity of HS

$$= \pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \pi r^2 \left[h + \frac{2}{3}r\right]$$

$$= \frac{22}{7} \times 49 \left[6 + \frac{14}{3}\right]$$

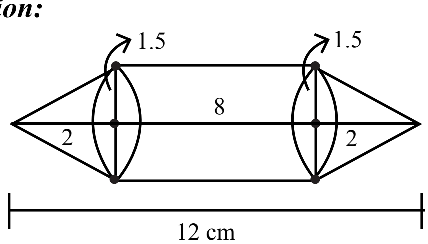
$$= 154 \times \frac{32}{3}$$

$$= 1642.67 \text{ cm}^3$$

---

2. Nathan, an engineering student was asked to make a model shaped like a cylinder with two cones attached at its two ends. The diameter of the model is 3 cm and its length is 12 cm. If each cone has a height of 2 cm, find the volume of the model that Nathan made.

Solution:



Cone	**Cylinder**
h = 2cm	H = 8cm
$r = 1.5 \text{ cm} = \frac{3}{2}$	$r = 1.5 \text{ cm} = \frac{3}{2}$

$\therefore$ Volume of the model =

2 (Vol. of Cone) + Vol. of Cylinder

$$= \frac{2}{3}\pi r^2 h + \pi r^2 H$$

$$= \pi r^2 \left[\frac{2h}{3} + H\right]$$

$$= \frac{22}{7} \times \frac{9}{4}\left[\frac{4}{3} + 8\right]$$

$$= \frac{11 \times 9}{7 \times 2}\left[\frac{28}{3}\right]$$

$$= \frac{11 \times 3 \times \cancel{14}^{2}}{\cancel{7}}$$

$$= 66 \text{ cm}^3$$

---

3. From a solid cylinder whose height is 2.4 cm and the diameter 1.4 cm, a cone of the same height and same diameter is carved out. Find the volume of the remaining solid to the nearest cm^3.

Solution:

Volume of the remaining solid

= Vol. of Cylinder – Vol. of Cone

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h$$

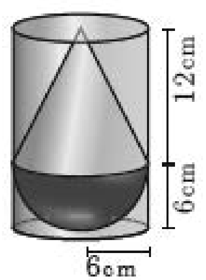
$$= \frac{2}{3}\pi r^2 h$$

$$= \frac{2}{3} \times \frac{22}{7} \times 0.7 \times 0.7 \times 2.4$$

$$= 2.46 \text{ cm}^3$$

---

4. A solid consisting of a right circular cone of height 12 cm and radius 6 cm standing on a hemisphere of radius 6 cm is placed upright in a right circular cylinder full of water such that it touches the bottom. Find the volume of the water displaced out of the cylinder, if the radius of the cylinder is 6 cm and height is 18 cm.



Solution:

Cone	Hemisphere	Cylinder
r = 6cm	r = 6cm	r = 6cm
h = 12cm		H = 18cm

Volume of water displaced out of cylinder =

Volumme of cone + Volume of HS

$$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \frac{1}{3}\pi r^2 [h + 2r]$$

$$= \frac{1}{3} \times \frac{22}{7} \times 36(12+12)$$

$$= \frac{22}{7} \times 12 \times 24$$

$$= 905.14\text{cm}^3$$

Note : When the conical hemisphere is completely submerged in water inside the cylinder,

Volume of water left in the cylinder.

= Volume of cylinder – [Volume of cone + Vol. of Hemi sphere)

$$= \pi r^2 H - \left[\frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3\right]$$

$$= \pi\left[36 \times 18 - \frac{1}{3} \times 36 \times 12 - \frac{2}{3} \times 216\right]$$

$$= \frac{22}{7}[648 - 144 - 144]$$

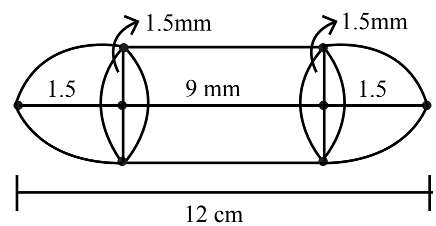
$$= \frac{22}{7} \times 360$$

$$= \frac{7920}{7}$$

$$= 1131.42 \text{ cm}^3$$

5. A capsule is in the shape of a cylinder with two hemisphere stuck to each of its ends. If the length of the entire capsule is 12 mm and the diameter of the capsule is 3 mm, how much medicine it can hold?

Solution:



Cylinder	Hemisphere
H = 9 mm	r = 1.5 mm = $3/2$
r = 1.5 mm = $3/2$	

∴ Volume of the Capsule =

Vol. of Cylinder + 2 (Vol. of hemisphere)

$$= \pi r^2 H + 2\left(\frac{2}{3}\pi r^3\right)$$

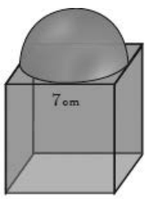
$$= \frac{22}{7}\left[\frac{9}{4} \times 9 + \frac{4}{3} \times \frac{27}{8}\right]$$

$$= \frac{22}{7}\left[\frac{81}{4} + \frac{9}{2}\right]$$

$$= \frac{22}{7}\left[\frac{81+18}{4}\right]$$

$$= \frac{22 \times 99}{28}$$
$$= \frac{11 \times 99}{14}$$
$$= 77.78 \text{ mm}^3$$

6. As shown in figure a cubical block of side 7 cm is surmounted by a hemisphere. Find the surface area of the solid.



Solution:

Given side of cube = 7 cm

radius of hemisphere = $7/2$ cm

Surface area of the solid = CSA of cube + CSA of hemisphere – Base area of HS

$$= 6a^2 + 2\pi r^2 - \pi r^2$$
$$= 6(49) + \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{2}} \times \frac{7}{2}$$
$$= 294 + \frac{77}{2}$$
$$= 294 + 38.5$$
$$= 332.5 \text{ cm}^2$$

7. A right circular cylinder just enclose a sphere of radius r units. Calculate

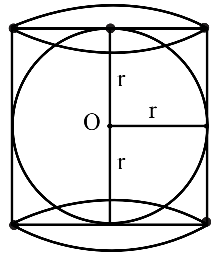
(i) the surface area of the sphere

(ii) the curved surface area of the cylinder

(iii) the ratio of the areas obtained in (i) and (ii).

Solution:

From the given fig, it is clear that h = 2r



i) Surface area of sphere

$= 4\pi r^2$ sq.units

ii) CSA of the cylinder = $2\pi rh$

$= 2\pi r\,(2r)$

$= 4\pi r^2$ sq. units

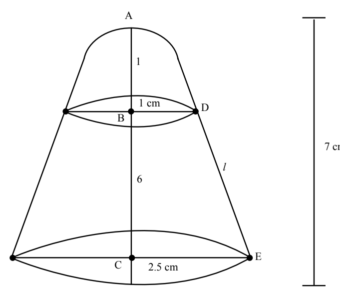
iii) Ratio of areas obtained in (i) & (ii)

$= 4\pi r^2 : 4\pi r^2$

$= 1 : 1$

8. A shuttle cock used for playing badminton has the shape of a frustum of a cone is mounted on a hemisphere. The diameters of the frustum are 5 cm and 2 cm. The height of the entire shuttle cock is 7 cm. Find its external surface area.

Solution:



AB = BD = radius of hemisphere = 1 cm

radius of frustum = r

AC = 7cm = Total length of cock

$\therefore$ BC = 7 – 1 = 6 cm = height of frustum

CE = 2.5 cm = R

$$\therefore l = \sqrt{h^2 + (R-r)^2} = \sqrt{26 + (1.5)^2} = 6.18$$

$\therefore$ External Surface Area =

CSA of Frustum + CSA of HS

$$= \pi(R+r)l + 2\pi r^2$$

$$= \pi[(2.5+1)\,6.18 + 2\times 1]$$

$$= \frac{22}{7}\left[\frac{7}{2}(6.1) + 2\right]$$

$$= \frac{22}{7}[21.35 + 2]$$

$$= \frac{22\times 23.35}{7}$$

$$= \frac{513.7}{7}$$

$$= 73.39 \text{ cm}^2$$

IV. Conversion of Solids

Example 7.29

A metallic sphere of radius 16 cm is melted and recast into small spheres each of radius 2 cm. How many small spheres can be obtained?

Solution :

Let the number of small spheres obtained be n.

Let r be the radius of each small sphere and R be the radius of metallic sphere.

Here, R = 16 cm, r = 2 cm

Now, n´(Volume of a small sphere)

= Volume of big metallic sphere

$$n\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$$

$$n\left(\frac{4}{3}\pi \times 2^3\right) = \frac{4}{3}\pi \times 16^3$$

8n = 4096 gives n = 512

Therefore, there will be 512 small spheres.

Example 7.30

A cone of height 24 cm is made up of modeling clay. A child reshapes it in the form of a cylinder of same radius as cone. Find the height of the cylinder.

Solution :

Let h_1 and h_2 be the heights of a cone and cylinder respectively.

Also, let r be the radius of the cone.

Given that, height of the cone h_1 = 24 cm; radius of the cone and cylinder r = 6 cm

Since, Volume of cylinder = Volume of cone

$$\pi^2 h_2 = \frac{1}{3}\pi r^2 h_1$$

$$h_2 = \frac{1}{3}\times h_1 \text{ gives } h_2 = \frac{1}{3}\times 24 = 8$$

Therefore, height of cylinder is 8 cm.

Example 7.31

A right circular cylindrical container of base radius 6 cm and height 15 cm is full of ice cream. The ice cream is to be filled in cones of height 9 cm and base radius 3 cm, having a hemispherical cap. Find the number of cones needed to empty the container.

Solution :

Let h and r be the height and radius of the cylinder respectively.

Given that, h = 15 cm, r = 6 cm

Volume of the container V = $\pi r^2 h$ cubic units.

$$= \frac{22}{7}\times 6\times 6\times 15$$

Let, r_1 = 3 cm, h_1 = 9 cm be the radius and height of the cone.

Also, r_1 = 3 cm is the radius of the hemispherical cap.

Volume of one ice cream cone = (Volume of the cone + Volume of the hemispherical cap)

$$= \frac{1}{3}\pi r_1^2 h_1 + \frac{2}{3}\pi r_1^3$$

$$= \frac{1}{3} \times \frac{22}{7} \times 3 \times 3 \times 9 + \frac{2}{3} \times \frac{22}{7} \times 3 \times 3 \times 3$$

$$= \frac{22}{7} \times 9(3+2) = \frac{22}{7} \times 45$$

Number of cones = $\dfrac{\text{Volume of the cylinder}}{\text{Volume of one ice cream cone}}$

Number of ice cream cones needed =

$$\frac{\frac{22}{7} \times 6 \times 6 \times 15}{\frac{22}{7} \times 45} = 12$$

Thus 12 ice cream cones are required to empty the cylindrical container.

EXERCISE 7.4

1. An aluminium sphere of radius 12 cm is melted to make a cylinder of radius 8 cm. Find the height of the cylinder.

Solution:

Given radius of sphere = 12 cm = R &

radius of cylinder = 8 cm = r

By the data given,

Volume of sphere = Volume of Cylinder

$$\Rightarrow \quad \frac{4}{3}\pi R^3 = \pi r^2 h$$

$$\Rightarrow \quad \frac{4}{3} \times 12 \times 12 \times 12 = 8 \times 8 \times h$$

$$\Rightarrow \quad h = 36 \text{ cm}$$

$\therefore$ Height of the cylinder = 36 cm

2. Water is flowing at the rate of 15 km per hour through a pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. Find the time in which the level of water in the tanks will rise by 21 cm.

Solution:

Cylindrical Pipe

Given, Speed of water in the pipe

= 15 Km/hr

H = 15000 m

Radius of pipe r = 7 cm = $\frac{7}{100}$ m

Rectangular Tank

$l = 50$ m $\quad b = 44\, m$

$h = 21\text{cm} = \frac{21}{100} m$

$$\therefore \text{Required time} = \frac{\text{Volume of tank}}{\text{Volume of pipe}}$$

$$= \frac{lbh}{\pi r^2 H}$$

$$= \frac{50 \times 44 \times 21/100}{\frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 15000}$$

= 2 hrs

3. A conical flask is full of water. The flask has base radius r units and height h units, the water poured into a cylindrical flask of base radius xr units. Find the height of water in the cylindrical flask.

Solution:

By the data given,

Volume of Cylindrical Flask =

Volume of Conical Flask

$$\Rightarrow \quad \pi(xr)^2 H = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad x^2 r^2 H = \frac{1}{3} r^2 h$$

$$\Rightarrow \quad H = \frac{h}{3x^2}$$

$\therefore$ Height of the Cylindrical Flask $= \dfrac{h}{3x^2}$ cm

4. A solid right circular cone of diameter 14 cm and height 8 cm is melted to form a hollow sphere. If the external diameter of the sphere is 10 cm, find the internal diameter.

Solution:

Right Circular Cone	**Hollow Sphere**
r = 7 cm	R = 5 cm
h = 8 cm	r = ?

By the problem,

Volume of Hollow Sphere = Vol. of Right Circular Cone

$$\Rightarrow \frac{4}{3}\pi\left(R^3 - r^3\right) = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow 4\,(125 - r^3) = 49 \times 8$$

$$\Rightarrow 125 - r^3 = 49 \times 2$$

$$\Rightarrow r^3 = 125 - 98$$

$$r^3 = 27$$

$$\therefore \quad r = 3$$

$\therefore$ Internal diameter of hollow sphere = 6 cm

5. Seenu's house has an overhead tank in the shape of a cylinder. This is filled by pumping water from a sump (underground tank) which is in the shape of a cuboid. The sump has dimensions 2 m × 1.5 m × 1 m. The overhead tank has its radius of 60 cm and height 105 cm. Find the volume of the water left in the sump after the overhead tank has been completely filled with water from the sump which has been full, initially.

Solution:

Over head tank	**Sump**
(Cylinder)	(Cuboid)
R = 60 cm	l = 2 m = 200 cm
H = 105 cm	b = 1.5 m = 150 cm
	h = 1 m = 100 cm

Volume of water left

= Volume of Sump – Volume of tank

= $l$bh – $\pi R^2 H$

$$= 200 \times 150 \times 100 - \frac{22}{\cancel{7}} \times 60 \times 60 \times \cancel{105}^{15}$$

$$= 3000000 - 1188000$$

$$= 2812000 \text{ cm}^3$$

6. The internal and external diameter of a hollow hemispherical shell are 6 cm and 10 cm respectively. If it is melted and recast into a solid cylinder of diameter 14 cm, then find the height of the cylinder.

Solution:

Hollow Hemisphere	**Solid Cylinder**
R = 5 cm	r = 7 cm
r = 3 cm	h = ?

$\therefore$ By the problem given,

Volume of Solid Cylinder = Volume of Hollow Hemisphere

$$\Rightarrow \quad \pi r^2 h = \frac{2}{3}\pi\,(R^3 - r^3)$$

$$\Rightarrow \quad 49 \times h = \frac{2}{3}(125 - 27)$$

$$\Rightarrow \quad h = \frac{2}{3} \times \frac{98}{49}$$

$$\therefore \quad h = \frac{4}{3} = 1.33\ cm$$

$\therefore$ Height of Solid Cylinder = 1.33 cm

7. A solid sphere of radius 6 cm is melted into a hollow cylinder of uniform thickness. If the external radius of the base of the cylinder is 5 cm and its height is 32 cm, then find the thickness of the cylinder.

Solution:

Solid Sphere	**Hollow Cylinder**
r = 6 cm	R = 5 cm
	H = 32 cm
	t = ?

By the problem given,

Volume of Hollow Cylinder = Volume of Solid Sphere

$$\Rightarrow \pi(R^2 - r^2)\,H = \frac{4}{3}\pi r^3$$

$$\Rightarrow (25 - r^2)32 = \frac{4}{\cancel{3}} \times \overset{2}{\cancel{6}} \times 6 \times 6$$

$$\Rightarrow 25 - r^2 = \frac{\cancel{4} \times \cancel{2} \times \overset{3}{\cancel{6}} \times \overset{3}{\cancel{6}}}{\underset{\underset{2}{\cancel{4}}}{\cancel{32}}}$$

$$\Rightarrow 25 - r^2 = 9$$

$$r^2 = 16$$

$$r^2 = 4$$

$\therefore$ Thickness = R – r

= 5 – 4

= 1 cm

8. A hemispherical bowl is filled to the brim with juice. The juice is poured into a cylindrical vessel whose radius is 50% more than its height. If the diameter is same for both the bowl and the cylinder then find the percentage of juice that can be transferred from the bowl into the cylindrical vessel.

Solution:

Hemisphere	**Cylinder**
Radius = r	Radius = r

$$= h + \frac{1}{2}h$$

$$r = \frac{3}{2}h$$

$\therefore$ Volume of hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3}\pi \times \left(\frac{3}{2}h\right)^3$$

$$= \frac{2}{3}\pi \times \frac{27}{8}h^3$$

$$= \frac{9}{4}\pi h^3$$

Volume of Cylinder $= \pi r^2 h$

$$= \pi \times \left(\frac{3}{2}h\right)^2 h$$

$$= \pi \times \frac{9}{4}h^2\, h$$

$$= \frac{9}{4}\pi h^3$$

$\therefore$ Vol. of Hemisphere = Vol. of Cylinder

$\therefore$ % of juice that can be trasnferred to the cylindrical vessel = 100 %

EXERCISE 7.5

Multiple choice questions.

1. The curved surface area of a right circular cone of height 15 cm and base diameter 16 cm is

(1) 60π cm^2 (2) 68π cm^2

(3) 120π cm^2 (4) 136π cm^2

Hint : **Ans : (4)**

h = 15 cm, r = 8 cm

$$\Rightarrow l = \sqrt{h^2 + r^2}$$
$$= \sqrt{225 + 64}$$
$$= \sqrt{289}$$
$$= 17$$

$\therefore$ CSA of Cone = $\pi r l$

$$= \pi \times 8 \times 17$$
$$= 136\ \pi \text{cm}^2$$

2. If two solid hemispheres of same base radius r units are joined together along their bases, then curved surface area of this new solid is

(1) $4\pi r^2$ sq. units (2) $6\pi r^2$ sq. units

(3) $3\pi r^2$ sq. units (4) $8\pi r^2$ sq. units

Hint : **Ans : (1)**

The CSA of the new solid is nothing but the CSA of a sphere = $4\pi r^2$ sq.units

3. The height of a right circular cone whose radius is 5 cm and slant height is 13 cm will be

(1) 12 cm (2) 10 cm

(3) 13 cm (4) 5 cm

Hint : **Ans : (1)**

r = 5cm, l = 13cm

$$\therefore\ h = \sqrt{l^2 - r^2}$$
$$= \sqrt{169 - 25}$$
$$= \sqrt{144}$$
$$= 12 \text{ cm}$$

4. If the radius of the base of a right circular cylinder is halved keeping the same height, then the ratio of the volume of the cylinder thus obtained to the volume of original cylinder is

(1) 1 : 2 (2) 1 : 4

(3) 1 : 6 (4) 1 : 8

Hint : **Ans : (2)**

$$\frac{\text{Volume of New Cylinder}}{\text{Volume of Original Cylinder}} = \frac{\pi R^2 h}{\pi r^2 h}$$

$$\text{where } R = \frac{r}{2}$$

$$= \frac{R^2}{r^2}$$

$$= \frac{\frac{r^2}{4}}{r^2}$$

$$= \frac{1}{4}$$

$\therefore V_1 : V_2 = 1 : 4$

5. The total surface area of a cylinder whose radius is $\frac{1}{3}$ of its height is

(1) $\frac{9\pi h^2}{8}$ sq.units (2) $24\pi h^2$ sq.units

(3) $\frac{8\pi h^2}{9}$ sq.units (4) $\frac{56\pi h^2}{9}$ sq.units

Hint : **Ans : (3)**

TSA of Cylinder = $2\pi r(h + r)$

$$\text{where } r = \frac{1}{3}h$$

$$= 2\pi \times \frac{h}{3}\left(h + \frac{h}{3}\right)$$

$$= 2\pi\frac{h}{3} \times \frac{4h}{3}$$

$$= \frac{8\pi h^2}{9} \text{ Sq. units}$$

6. In a hollow cylinder, the sum of the external and internal radii is 14 cm and the width is 4 cm. If its height is 20 cm, the volume of the material in it is

(1) $5600\,\pi$ cm^3 (2) $11200\,\pi$ cm^3

(3) 56π cm^3 (4) $3600\,\pi$ cm^3

Hint : **Ans : (2)**

$R + r = 14$ cm, $h = 20$ cm, $W = 4$ cm

$R - r = 4$ cm

Volume of hollow cylinder

$$= \pi h\,(R^2 - r^2)$$
$$= \pi h\,(R + r)\,(R - r)$$
$$= \pi \times 20 \times 14 \times 4$$
$$= 1120\pi \text{ cm}^3$$

7. If the radius of the base of a cone is tripled and the height is doubled then the volume is

(1) made 6 times (2) made 18 times

(3) made 12 times (4) unchanged

Hint : **Ans : (2)**

Volume of cone $= \frac{1}{3}\pi r^2 h$

When $r \to 3r$, $h \to 2h$

Volume of new cone

$$= \frac{1}{3}\pi \times 9r^2 \times 2h$$
$$= 18\left(\frac{1}{3}\pi r^2 h\right)$$
$$= 18 \text{ times}$$

8. The total surface area of a hemi-sphere is how much times the square of its radius.

(1) π (2) 4π

(3) 3π (4) 2π

Hint : **Ans : (3)**

TSA of a hemisphere $= 3\pi r^2$

$= 3\pi$ (square of its radius)

$= 3\pi$ times r^2

9. A solid sphere of radius x cm is melted and cast into a shape of a solid cone of same radius. The height of the cone is

(1) 3x cm (2) x cm

(3) 4x cm (4) 2x cm

Hint : **Ans : (3)**

Volume of sphere = Volume of Cone

$$\Rightarrow \quad \frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h$$
$$\Rightarrow \quad \frac{4}{3}\pi x^3 = \frac{1}{3}\pi x^2 \times h$$
$$\Rightarrow \quad h = \frac{\frac{4}{3}\pi x^3}{\frac{1}{3}\pi x^2} = 4x$$

10. A frustum of a right circular cone is of height 16cm with radii of its ends as 8cm and 20cm. Then, the volume of the frustum is

(1) 3328π cm^3 (2) 3228π cm^3

(3) 3240π cm^3 (4) 3340π cm^3

Hint : **Ans : (1)**

Volume of Frustum of a Cone

$$= \frac{\pi h}{3}(R^2 + Rr + r^2)$$
$$= \frac{\pi}{3} \times 16\,[400 + 160 + 64]$$
$$= \frac{16\pi}{3} \times 624$$
$$= 16\pi \times 208$$
$$= 3328\pi \text{ cm}^3$$

11. A shuttle cock used for playing badminton has the shape of the combination of

(1) a cylinder and a sphere

(2) a hemisphere and a cone

(3) a sphere and a cone

(4) frustum of a cone and a hemisphere

Hint : **Ans : (4)**

Frustum of a cone & a hemisphere

12. A spherical ball of radius r_1 units is melted to make 8 new identical balls each of radius r_2 units. Then $r_1 : r_2$ is

(1) 2:1 (2) 1:2

(3) 4:1 (4) 1:4

Hint : **Ans : (1)**

Volume of a sphere = 8 (Volume of new identical balls)

$$\frac{4}{3}\pi r_1^3 = 8\left(\frac{4}{3}\pi r_2^3\right)$$

$$\Rightarrow \frac{r_1^3}{r_2^3} = \frac{8}{1}$$

$\therefore r_1 : r_2 = 2 : 1$

13. The volume (in cm³) of the greatest sphere that can be cut off from a cylindrical log of wood of base radius 1 cm and height 5 cm is

(1) $\frac{4}{3}\pi$ (2) $\frac{10}{3}\pi$

(3) 5π (4) $\frac{20}{3}\pi$

Hint : **Ans : (1)**

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3 \text{ where } r = 1$$

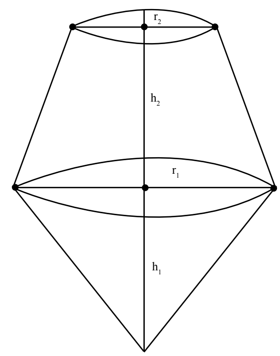
$$= \frac{4}{3}\pi$$

14. The height and radius of the cone of which the frustum is a part are h_1 units and r_1 units respectively. Height of the frustum is h_2 units and radius of the smaller base is r_2 units. If $h_2 : h_1 = 1 : 2$ then $r_2 : r_1$ is

(1) 1 : 3 (2) 1 : 2

(3) 2 : 1 (4) 3 : 1

Hint : **Ans : (2)**



Given $h_2 : h_1 = 1 : 2$

$$\Rightarrow h_2 = \frac{1}{2}h_1 \qquad \therefore \frac{r_2}{r_1} = \frac{1}{2}$$

15. The ratio of the volumes of a cylinder, a cone and a sphere, if each has the same diameter and same height is

(1) 1:2:3 (2) 2:1:3

(3) 1:3:2 (4) 3:1:2

Hint : **Ans : (4)**

Ratio of volumes of Cylinder, Cone, Sphere

$$= \pi r^2 h : \frac{1}{3}\pi r^2 h : \frac{4}{3}\pi r^3 h$$

with same height & same radius.

Since each of them has same diameter and same height, h = 2r

$$V_1 = \pi r^2(2r) = 2\pi r^3$$

$$V_2 = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$$

$$V_2 = \frac{4}{3}\pi r^3$$

$$\therefore V_1 : V_2 : V_3 = 2 : \frac{2}{3} : \frac{4}{3}$$

= 6 : 2 : 4

= 3 : 1 : 2

UNIT EXERCISE - 7

1. The barrel of a fountain-pen cylindrical in shape, is 7 cm long and 5 mm in diameter. A full barrel of ink in the pen will be used for writing 330 words on an average. Howmany words can be written using a bottle of ink containing one fifth of a litre?

Solution :

Given height of the pen = 7 cm = 70 mm

$$\text{radius} = \frac{5}{2}\text{ mm}$$

$\therefore$ Volume of the pen = $\pi r^2 h$

$$= \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{25}{\cancel{4}} \times \overset{\overset{5}{\cancel{10}}}{\cancel{70}}$$

$$= 1375 \text{ mm}^3$$

$$= 1.375 \text{ cm}^3$$

By data given

$1.375 \text{ cm}^3 \rightarrow 330$ words –

$$\frac{1}{5} \text{ of a litre} = \frac{1}{5}(1000 \text{ cm}^3)$$

$$\Rightarrow 200 \text{ cm}^3 \rightarrow x \text{ words}$$

$$\therefore x = \frac{200 \times 330}{1.375}$$

= 48000 words

2. A hemi-spherical tank of radius 1.75 m is full of water. It is connected with a pipe which empties the tank at the rate of 7 litre per second. How much time will it take to empty the tank completely?

Solution :

Radius of hemi-spherical tank = 1.75 m

$$r = \frac{7}{4}\text{ m}$$

$\therefore$ Volume of the tank

$$= \frac{2}{3}\pi r^3$$

$$= \frac{\cancel{2}}{3} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{7}}{\underset{2}{\cancel{4}}} \times \frac{7}{4} \times \frac{7}{4}$$

$$= \frac{539}{48}$$

$$= 11.229\ m^3$$

$$= 11.229 \times 1000 \text{ litres}$$

$$= 11229 \text{ litres}$$

Water falls at the rate of 7 lrs/second

$\therefore$ Time taken by the pipe to empty the tank

$$= \frac{11229}{7}\text{ Sec}$$

= 1604 sec (approx)

$$= \frac{1604}{60}\text{ min}$$

= 27 min (approx)

3. Find the maximum volume of a cone that can be carved out of a solid hemisphere of radius r units.

Solution :

Given radius of solid hemisphere = r

Volume of a cone that can be carved

Out of hemisphere

$$= \frac{2}{3}\pi r^3 - \frac{1}{3}\pi r^2 h$$

But volume is maximum (given)

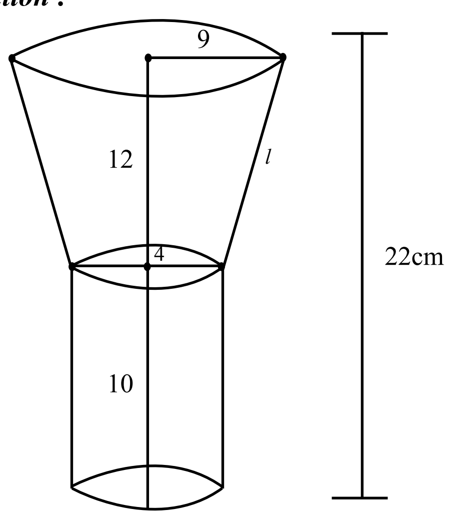
$\therefore$ h = r

$$\therefore \text{Required volume} = \frac{2}{3}\pi r^3 - \frac{1}{3}\pi r^3$$

$$= \frac{1}{3}\pi r^3$$

4. An oil funnel of tin sheet consists of a cylindrical portion 10 cm long attached to a frustum of a cone. If the total height is 22 cm, the diameter of the cylindrical portion be 8cm and the diameter of the top of the funnel be 18 cm, then find the area of the tin sheet required to make the funnel.

Solution :



Area of tin sheet required to make the funnel

where R = 9 cm r = 4 cm, H = 10 cm

$$l = \sqrt{(R-r)^2 + h^2}$$

$$= \sqrt{25+144}$$

$$= \sqrt{169}$$

$$= 13$$

= CSA of Frustum + CSA of Cylinder

= π (R + r) l + 2πrH

= π [13 × 13 + 2 × 4 × 10]

$$= \frac{22}{7}[169+80]$$

$$= \frac{22}{7} \times 249$$

$$= \frac{5478}{7}$$

$$= 782.57 \text{ cm}^3$$

5. Find the number of coins, 1.5 cm in diameter and 2 mm thick, to be melted to form a right circular cylinder of height 10 cm and diameter 4.5 cm.

Solution :

Given diameter of coin = 1.5 cm

(smaller cylinder)

$$\therefore \quad r = \frac{1.5}{2} = 0.75 \text{ cm}$$

$$h = 2 \text{ mm} = 0.2 \text{ cm}$$

Also, diameter of bigger cylinder = 4.5 cm

R = 2.25 cm

H = 10 cm

$\therefore$ Number of Coins =

$$\frac{\text{Volume of largest cylinder}}{\text{Volume of smallest cylinder}}$$

$$= \frac{\pi R^2 H}{\pi r^2 h}$$

$$= \frac{\frac{9}{4} \times \frac{9}{4} \times 10}{\frac{3}{4} \times \frac{3}{4} \times \frac{2}{10}}$$

$= 450$ coins

6. A hollow metallic cylinder whose external radius is 4.3 cm and internal radius is 1.1 cm and whole length is 4 cm is melted and recast into a solid cylinder of 12 cm long. Find the diameter of solid cylinder.

Solution :

Hollow Cylinder	**Solid Cylinder**
R = 4.3 cm	h = 12 cm
r = 1.1 cm	d = ?
H = 4 cm	

When hollow cylinder is melted & recast into a solid cylinder,

Volume of hollow cylinder = Volume of solid cylinder

$$\Rightarrow \cancel{\pi} H (R^2 - r^2) = \cancel{\pi} r^2 h$$

$$\Rightarrow 4[(4.3)^2 - (1.1)^2] = r^2 \times 12$$

$$\Rightarrow r^2 = \frac{4(17.28)}{12}$$

$$r^2 = \frac{17.28}{3}$$

$$= 5.76$$

$$r = 2.4$$

$\therefore$ Diameter of solid cylinder

$= 2r$

$= 4.8$ cm

7. The slant height of a frustum of a cone is 4 m and the perimeter of circular ends are 18 m and 16 m. Find the cost of painting its curved surface area at ₹100 per sq. m.

Solution :

In a frustum of a cone,

$l = 4$cm, $2\pi R = 18$, $2\pi r = 16$

$$\Rightarrow R = \frac{9}{\pi} \qquad r = \frac{8}{\pi}$$

$\therefore$ CSA of frustum of a cone

$$= \pi l (R + r)$$

$$= \pi \times 4 \left(\frac{9}{\pi} + \frac{8}{\pi}\right)$$

$$= 4 \times 17$$

$$= 68 \text{ m}^2$$

$\therefore$ Cost of painting at ₹ 100/m^2

$= 68 \times 100 =$ ₹ 6800

8. A hemi-spherical hollow bowl has material of volume $\frac{436\pi}{3}$ cubic cm. Its external diameter is 14 cm. Find its thickness.

Solution :

In a hollow hemisphere,

$$\text{Volume} = \frac{436\pi}{3} \text{ cm}^3$$

D = 14 cm, R = 7 cm, t = ?

$$\Rightarrow \frac{2}{3}\pi(R^3 - r^3) = \frac{436\pi}{3}$$

$$\Rightarrow 7^3 - r^3 = 218$$

$$\Rightarrow 343 - r^3 = 218$$

$$\therefore r^3 = 125$$

$$\therefore r = 5 \text{ cm}$$

$\therefore$ thickness, $t = R - r$

$= 7 - 5$

$= 2$ cm

9. The volume of a cone is $1005\frac{5}{7}$ cu. cm. The area of its base is $201\frac{1}{7}$ sq. cm. Find the slant height of the cone.

Solution :

Given volume of a cone = $1005\ \frac{5}{7}$ cm^3

& base area = $201\ \frac{1}{7}$ cm^2

$$\therefore \frac{1}{3}\pi r^2 h = \frac{7040}{7} \ \& \ \pi r^2 = \frac{1408}{7}$$

$$\Rightarrow \quad \frac{1}{3} \times \frac{1408}{\cancel{7}} \times h = \frac{7040}{\cancel{7}}$$

$$\Rightarrow \quad h = \frac{7040}{1408} \times 3$$

$$\Rightarrow \quad h = 5 \times 3$$

$$\Rightarrow \quad h = 15$$

$$\text{Also, } \pi r^2 = \frac{1408}{7}$$

$$\Rightarrow \quad \frac{22}{7} \times r^2 = \frac{1408}{7}$$

$$\Rightarrow \quad r^2 = \frac{1408}{7} = 64$$

$$\therefore \quad r = 8$$

$$\therefore l = \sqrt{h^2 + r^2} = \sqrt{15^2 + 8^2} = \sqrt{225 + 64}$$

$$= \sqrt{289} = 17 \text{ cm}$$

$\therefore$ Slant height = 17cm

10. A metallic sheet in the form of a sector of a circle of radius 21 cm has central angle of 216°. The sector is made into a cone by bringing the bounding radii together. Find the volume of the cone formed.

Solution :

Given radius of sector = 21 cm, θ = 216^0

ie R = 21 = l (slant height of cone)

When the sector is made into a cone by bringing the radii together.

Length of arc of the sector = Perimeter of base of cone

$$\Rightarrow \frac{\theta}{360} \times 2\pi R = 2\pi r$$

$$\Rightarrow \quad r = \frac{216}{360} \times 21$$

$$\Rightarrow \quad r = \frac{63}{5} = 12.6 \text{ cm}$$

$$\therefore \quad h = \sqrt{l^2 - r^2}$$

$$= \sqrt{21^2 - (12.6)^2}$$

$$= \sqrt{441 - 158.76}$$

$$= \sqrt{282.24}$$

$$= 16.8$$

$\therefore$ Volume of the cone

$$= \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{\cancel{3}} \times \frac{22}{\cancel{7}} \times \overset{4.2}{\cancel{12.6}} \times \overset{1.8}{\cancel{12.6}} \times 16.8$$

$$= 2794.176$$

$$\square 2794.18 \text{ cm}^3$$

PROBLEMS FOR PRACTICE

1. A girl empties a cylindrical bucket, full of sand of base radius 18 cm and height 32 cm on the floor, to form a conical heap of sand. If the height of this heap is 24 cm, find the slant height of cone. **(Ans : 43.27 cm)**

2. 12 cylindrical pillars of a building have to be cleaned. If the diameter of each pillar is 42 cm, height is 5m, What will be the cost of cleaning at the rate of Rs.5 per m^2. **(Ans : Rs.396/-)**

3. A conical tent of 56m base diameter requires 3080 m^2 of canvas for the cured surface. Find its height. **(Ans : 21m)**

4. A solid is in the form of a cylinder with hemispherical ends. The total height of the solid is 19cm and the diameter of the cylinder is 7 cm. Find the surface area of the solid. **(Ans 418 m^2)**

5. A farmer connects a pipe of internal diameter 20 cm from a Canal into a cylindrical tank which is 10m in diameter and 2m deep. If the water flows through the pipe at the rate of 4 Km/hr, in how much time will the tank be completely filled ?

(Ans : 1 hr, 15min)

6. A toy is in the form of a cone of base radius 3.5 cm mounted on a hemisphere of base diameter 7 cm. If the total height of the toy is 15.5 cm, find the total surface area of the toy. **(Ans : 214.5cm^2)**

7. A cylindrical glass tube with radius 10 cm has water upto a height of 9 cm. A metal cube of 8 cm edge is immersed completely. By how much, the water level rise in the tube ? **(Ans : 1.63 cm)**

8. A vessel is in the form of a cone. Its height is 8 cm and radius of its top which is open is 5 cm. It is filled with water up to the rim. When lead shots, each of which is a sphere of diameter 1cm are droppd into vessel, one fourth of the water flows out. Find the number of lead shots dropped into the vessel.

(Ans : (100))

9. A hollow spherical shell has an inner radius of 8cm. If the volume of material is $\frac{1952\pi}{3}$ C.C, Find the thickness of the shell.

(Ans : 2cm)

10. Find the length of arc of the sector formed by opening out a cone of base radius 8cm. What is the central angle if the height of the cone is 6cm.

$$\left(Ans: 280^o,\ 50\frac{2}{7}\text{ cm}\right)$$

11. Find the capacity of a bucket having the radius of the top as 36cm and that of the bottom as 12cm, depth is 35cm.

(Ans 68640 cm^3)

12. Water flows through a cylindrical pipe of internal radius 3.5 cm at 5m per sec. Find the volume of water in litres discharged by the pipe in 1 min. **(Ans : 1155 litres)**

13. A rectangular sheet of metal foil with dimension 66 cm × 12 cm is rolled to form a cylinder of height 12cm. Find the volume of the cylinder. **(Ans : 4158 cm^3)**

14. Using clay, a student made a right circular cone of height 48cm and base radius 12cm. Another student reshapes it in the form of a sphere. Find the radius of sphere.

(Ans : 12 cm)

15. A solid sphere of diameter 28 cm is melted and recast into smaller solid cones each of diameter $4\frac{2}{3}$ cm and height 3cm. Find the number of cones so formed.

(Ans : 672)

OBJECTIVE TYPE QUESTIONS

1. The radii of 2 cylinders are in the ratio 2 : 3 and their heights are in the ratio 5 : 3. Then the ratio of their volumes is

a) 10 : 9 b) 20 : 27 c) 7 : 6

d) 5 : 2 **(Ans : (b))**

2. If the volume of sphere is $\frac{9}{16}$ cm^3, its radius is

   a) $\frac{4}{3}$ cm b) $\frac{3}{4}$ cm

   c) $\frac{3}{2}$ cm d) $\frac{2}{3}$ cm

   (Ans : (b))

3. The height of a cone whose slant height 26cm and the diameter of the base is 10cm, is

   a) 24.5 cm b) 26.5 cm c) 25.5 cm

   d) 27.5 cm **(Ans : (c))**

4. The total surface area of a cylinder whose height is half the radius is

   a) $6\pi r^2$ b) $8\pi r^2$ c) $2\pi r^2$

   d) $3\pi r^2$ **(Ans : (d))**

5. The base area of a cone is 80cm^2. If its height is 9cm, then its volume is

   a) 720cm^3 b) 720π cm^3

   c) 240 cm^3 d) none **(Ans : (c))**

6. A well of diameter 2.1m is dug to a depth of 4m. The volume of the earth removed is

   a) 4.4π m^3 b) 44.1 m^3

   c) 0.441 m^3 d) 4.41 πm^3

   (Ans : (d))

7. The volume of a hemisphere is 18π. Its radius is

   a) 4cm b) 3cm c) 2cm

   d) 6cm **(Ans : (b))**

8. If a rectangle ABCD is folded by bringing AB and CD together to form a cylinder, then the height of the cylinder is

   a) BC b) AD c) AB

   d) none **(Ans : (c))**

9. The CSA of a solid hemisphere if the TSA of the solid hemisphere is 12π cm^2, is

   a) 8π b) 36π c) 6π d) 24π

   (Ans : (a))

10. Ths CSA of a cone whose radius x cm, height y cm is

    a) πrl b) $\pi r\sqrt{x^2+y^2}$

    c) $\pi y\sqrt{x^2+y^2}$ d) $\pi x\sqrt{x^2+y^2}$

    (Ans : (d))

CHAPTER 8 STATISTICS AND PROBABILITY

I. MEASURES OF DISPERSION

Key Points

- ✓ Measures of Variation (or) Dispersion of a data provide an idea of how observations spread out (or) scattered throughout the data.
- ✓ Different Measures of Dispersion are

  1. Range 2. Mean deviation 3. Quartile deviation 4. Standard deviation 5. Variance
  6. Coefficient of Variation
- ✓ Range R = L– S
- ✓ Coefficient of range = $\frac{L-S}{L+S}$ where L - Largest value; S - Smallest value.
- ✓ The mean of the squares of the deviations from the mean is called Variance. It is denoted by σ^2.

  Variance $\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$
- ✓ The positive square root of Variance is called Standard deviation.

  Standard deviation $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}}$
- ✓ Formula Table for Standard Deviation (σ).

Data Type	Direct Method	Mean Method	Assumed mean method	Step Deviation method
Ungrouped Data	$\sqrt{\frac{\Sigma x^2}{n}-\left(\frac{\Sigma x}{n}\right)^2}$	$\sqrt{\frac{\Sigma d^2}{n}}$	$\sqrt{\frac{\Sigma d^2}{n}-\left(\frac{\Sigma d}{n}\right)^2}$	$\sqrt{\frac{\Sigma d^2}{n}-\left(\frac{\Sigma d}{n}\right)^2}\times C$
Grouped Data	–	$\sqrt{\frac{\Sigma fd^2}{N}}$ $N=\Sigma f$	$\sqrt{\frac{\Sigma fd^2}{N}-\left(\frac{\Sigma fd}{N}\right)^2}$ $N=\Sigma f$	$\sqrt{\frac{\Sigma fd^2}{N}-\left(\frac{\Sigma fd}{N}\right)^2}\times C$ $N=\Sigma f$

✓ Standard deviation of first 'n' natural numbers

$$\sigma = \sqrt{\frac{n^2-1}{12}}$$

✓ The value of SD will not be changed if we add (or) subtract some fixed constant to all the values.

✓ When we multiply each value of a data by a constant, the value of SD is also multiplied by the same constant.

Example 8.1

Find the range and coefficient of range of the following data: 25, 67, 48, 53, 18, 39, 44.

Solution :

Largest value L = 67; Smallest value S =18

Range R= L − S = 67 −18 = 49

Coefficient of range = $\dfrac{L-S}{L+S}$

Coefficient of range = $\dfrac{67-18}{67+18} = \dfrac{49}{85} = 0.576$

Example 8.2

Find the range of the following distribution.

Age (in years)	16-18	18-20	20-22	22-24	24-26	26-28
Number of students	0	4	6	8	2	2

Solution :

Here Largest value L = 28

Smallest value S = 18

Range R = L −S

R = 28 −18 = 10Years.

Example 8.3

The range of a set of data is 13.67 and the largest value is 70.08. Find the smallest value.

Solution :

Range R = 13.67

Largest value L = 70.08

Range R = L −S

13.67 = 70.08 −S

S = 70.08 −13.67 = 56.41

Therefore, the smallest value is 56.41.

Example 8.4

The number of televisions sold in each day of a week are 13, 8, 4, 9, 7, 12, 10. Find its standard deviation.

Solution :

x_i	x_i^2
13	169
8	64
4	16
9	81
7	49
12	144
10	100
$\sum x_i = 63$	$\sum x_i^2 = 623$

Standard deviation

$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$= \sqrt{\frac{623}{7} - \left(\frac{63}{7}\right)^2}$$

$$= \sqrt{89-81} = \sqrt{8}$$

Hence, σ □2.83

Example 8.5

The amount of rainfall in a particular season for 6 days are given as 17.8 cm, 19.2 cm, 16.3 cm, 12.5 cm, 12.8 cm and 11.4 cm. Find its standard deviation.

Solution :

Arranging the numbers in ascending order we get, 11.4, 12.5, 12.8, 16.3, 17.8, 19.2. |Number of observations n = 6

$$\text{Mean} = \frac{11.4+12.5+12.8+16.3+17.8+19.2}{6}$$

$$= \frac{90}{6} = 15$$

x_i	$d_i = x_i - \bar{x}$ $= x - 15$	d_i^2
11.4	–3.6	12.96
12.5	–2.5	6.25
12.8	–2.2	4.84
16.3	1.3	1.69
17.8	2.8	7.84
19.2	4.2	17.64
		$\Sigma d_i^2 = 51.22$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$$

$$= \sqrt{\frac{51.22}{6}} = \sqrt{8.53}$$

Hence, σ □2.9

Example 8.6

The marks scored by 10 students in a class test are 25, 29, 30, 33, 35, 37, 38, 40, 44, 48. Find the standard deviation.

Solution :

The mean of marks is 35.9 which is not an integer. Hence we take assumed mean, A = 35, n = 10.

x_i	$d_i = x_i - A$ $d_i = x_i - 35$	d_i^2
25	–10	100
29	–6	36
30	–5	25
33	–2	4
35	0	0
37	2	4
38	3	9
40	5	25
44	9	81
48	13	169
	$\Sigma d_i = 9$	$\Sigma d_i^2 = 453$

Example 8.7

The amount that the children have spent for purchasing some eatables in one day trip of a school are 5, 10, 15, 20, 25, 30, 35, 40. Using step deviation method, find the standard deviation of the amount they have spent.

Solution :

We note that all the observations are divisible by 5. Hence we can use the step deviation method. Let the Assumed mean A = 20, n = 8.

x_i	$d_i = x_i - A$ $d_i = x_i - 20$	$d_i = \frac{x_i - A}{c}$ c = 5	d_i^2
5	–15	–3	9
10	–10	–2	4
15	–5	–1	1
20	0	0	0
25	5	1	1
30	10	2	4
35	15	3	9
40	20	4	16
		$\Sigma d_i = 4$	$\Sigma d_i^2 = 44$

Standard Deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} \times c$$

$$= \sqrt{\frac{44}{8} - \left(\frac{4}{8}\right)^2} \times 5 = \sqrt{\frac{11}{2} - \frac{1}{4}} \times 5$$

$$= \sqrt{5.5 - 0.25} \times 5 = 2.29 \times 5$$

$$\sigma \square 11.45$$

Example 8.8

Find the standard deviation of the following data 7, 4, 8, 10, 11. Add 3 to all the values then find the standard deviation for the new values.

Solution :

Arranging the values in ascending order we get, 4, 7, 8, 10, 11 and n = 5

x_i	x_i^2
4	16
7	49
8	64
10	100
11	121
$\Sigma x_i = 40$	$\Sigma x_i^2 = 350$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{350}{5} - \left(\frac{40}{5}\right)^2}$$

$$\sigma = \sqrt{6} \square 2.45$$

When we add 3 to all the values, we get the new values as 7,10,11,13,14.

x_i	x_i^2
7	9
10	100
11	121
13	169
14	196
$\Sigma x_i = 55$	$\Sigma x_i^2 = 635$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{635}{5} - \left(\frac{55}{5}\right)^2}$$

$$\sigma = \sqrt{6} \square 2.45$$

From the above, we see that the standard deviation will not change when we add some fixed constant to all the values.

Example 8.9

Find the standard deviation of the data 2, 3, 5, 7, 8. Multiply each data by 4. Find the standard deviation of the new values.

Solution :

Given, n = 5

x_i	x_i^2
2	49
3	9
5	25
7	49
8	64
$\Sigma x_i = 25$	$\Sigma x_i^2 = 151$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{151}{5} - \left(\frac{25}{5}\right)^2}$$

$$= \sqrt{30.2 - 25}$$

$$= \sqrt{5.2} \square 2.28$$

When we multiply each data by 4, we get the new values as 8, 12, 20, 28, 32.

x_i	x_i^2
8	64
12	144
20	400
28	784
32	1024
$\Sigma x_i = 100$	$\Sigma x_i^2 = 2416$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{2416}{5} - \left(\frac{100}{5}\right)^2}$$

$$= \sqrt{483.2 - 400}$$

$$= \sqrt{83.2}$$

$$\sigma = \sqrt{16 \times 5.2}$$

$$= 4\sqrt{5.2} \square 9.12$$

From the above, we see that when we multiply each data by 4 the standard deviation also get multiplied by 4.

Example 8.10

Find the mean and variance of the first n natural numbers.

Solution :

$$\text{Mean } \bar{x} = \frac{\text{Sum of all the observations}}{\text{Number of observations}}$$

$$= \frac{\sum x_i}{n} = \frac{1+2+3+....+n}{n} = \frac{n(n+1)}{2\times n}$$

$$\text{Mean } \bar{x} = \frac{n+1}{2}$$

$$\text{Variance } \sigma^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

$$= \begin{bmatrix} \sum x_i^2 = 1^2+2^2+3^2+....+n^2 \\ (\sum x_i)^2 = (1+2+3+....+n)^2 \end{bmatrix}$$

$$= \frac{n(n+1)(2n+1)}{6\times n} - \left[\frac{n(n+1)}{2\times n}\right]^2$$

$$= \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$\text{Variance } \sigma^2 = \frac{4n^2+6n+2-3n^2-6n-3}{12}$$

$$= \frac{n^2-1}{12}$$

Example 8.11

48 students were asked to write the total number of hours per week they spent on watching television. With this information find the standard deviation of hours spent for watching television.

x	6	7	8	9	10	11	12
f	3	6	9	13	8	5	4

Solution :

x_i	f_i	x_if_i	$d_i = x_i - \bar{x}$	d_i^2	$f_id_i^2$
6	3	18	–3	9	27
7	6	42	–2	4	24
8	9	72	–1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	N=48	$\sum x_i f_i = 432$	$\sum d_i = 0$		$\sum f_i d_i^2 = 124$

$$\text{Mean } \bar{x} = \frac{\sum x_i f_i}{N} - \frac{432}{48} = 9 \ \ (\text{Since } N = \sum f_i)$$

Standard deviation

$$\sigma = \sqrt{\frac{\sum f_i d_i^2}{N}} = \sqrt{\frac{124}{48}} = \sqrt{2.58}$$

σ □1.6

Example 8.12

The marks scored by the students in a slip test are given below.

x	4	6	8	10	12
f	7	3	5	9	5

Find the standard deviation of their marks.

Solution :

Let the assumed mean, A = 8

x_i	f_i	$d_i = x_i - A$	f_id_i	$f_id_i^2$
4	7	– 4	–28	112
6	3	– 2	– 6	12
8	5	0	0	0
10	9	2	18	36
12	5	4	20	80
	N=29		$\sum f_i d_i = 4$	$\sum f_i d_i^2 = 240$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{240}{29} - \left(\frac{4}{29}\right)^2} = \sqrt{\frac{240 \times 29 - 16}{29 \times 29}}$$

$$\sigma = \sqrt{\frac{6944}{29 \times 29}} \;; \sigma \square 2.87$$

Example 8.13

Marks of the students in a particular subject of a class are given below.

Marks	0-10	10-20	20-30	30-40
Number of students	8	12	17	14
Marks	40-50	50-60	60-70	-
Number of students	9	7	4	-

Find its standard deviation.

Solution :

Let the assumed mean, A = 35, c = 10

Marks	Mid value (x_i)	f_i	$d_i = x_i - A$	$d_i = \frac{x_i - A}{c}$	$f_i d_i$	$f_i d_i^2$
0-10	5	8	–30	–3	–24	72
10-20	15	12	–20	–2	–24	48
20-30	25	17	–10	–1	–17	17
30-40	35	14	0	0	0	0
40-50	45	9	10	1	9	9
50-60	55	7	20	2	14	28
60-70	65	4	30	3	12	36
		N=71			$\Sigma f_i d_i = -30$	$\Sigma f_i d_i^2 = 210$

Standard deviation $\sigma = c \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$

$$\sigma = 10 \times \sqrt{\frac{210}{71} - \left(-\frac{30}{71}\right)^2} = 10 \times \sqrt{\frac{210}{71} - \frac{900}{5041}}$$

$$= 10 \times \sqrt{2.779} \;;\; \sigma \square 16.67$$

Example 8.14

The mean and standard deviation of 15 observations are found to be 10 and 5 respectively. On rechecking it was found that one of the observation with value 8 was incorrect. Calculate the correct mean and standard deviation if the correct observation value was 23?

Solution :

n = 15 , $\bar{x} = 10$, $\sigma = 5$;

$$\bar{x} = \frac{\Sigma x}{n}; \; \Sigma x = 15 \times 10 = 150$$

Wrong observation value = 8,

Correct observation value = 23.

Correct total = 150 − 8 + 23 = 165

$$\text{Correct mean } \bar{x} = \frac{165}{15} = 11$$

Standard deviation $\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$

Incorrect value of σ = $5 = \sqrt{\frac{\Sigma x^2}{15} - (10)^2}$

$$25 = \frac{\Sigma x^2}{15} - 100 \text{ gives } \frac{\Sigma x^2}{15} = 125$$

Incorrect value of Σx^2 = 1875

Correct value of $\Sigma x^2 = 1875 - 8^2 + 23^2 = 2340$

Correct standard deviation $\sigma = \sqrt{\frac{2340}{15} - (11)^2}$

$$\sigma = \sqrt{156 - 121} = \sqrt{35} \qquad \sigma \square 5.9$$

EXERCISE 8.1

1. Find the range and coefficient of range of the following data.

(i) 63, 89, 98, 125, 79, 108, 117, 68

(ii) 43.5, 13.6, 18.9, 38.4, 61.4, 29.8

Solution:

i) Range = L – S

= 125 – 63

= 62

$$\text{Coefficient of range} = \frac{L-S}{L+S} = \frac{125-63}{125+63} = \frac{62}{185} = 0.33$$

ii) Range = L – S

= 61.4 – 13.6

= 47.8

$$\text{Coefficient of range} = \frac{L-S}{L+S} = \frac{61.4-13.6}{61.4+13.6} = \frac{47.8}{75} = 0.64$$

2. If the range and the smallest value of a set of data are 36.8 and 13.4 respectively, then find the largest value.

Solution :

Given ; range = 36.8

Smallest value = 13.4

$\therefore$ R = L – S

$\therefore$ L = R + S

= 36.8 + 13.4

= 50.2

3. Calculate the range of the following data.

Income	400-450	450-500	500-550
Number of workers	8	12	30
Income	550-600	600-650	-
Number of workers	21	6	-

Solution :

Here, Largest value = L = 650

Smallest value = S = 450

$\therefore$ Range = L – S

= 650 – 450

= 200

4. A teacher asked the students to complete 60 pages of a record note book. Eight students have completed only 32, 35, 37, 30, 33, 36, 35 and 37 pages. Find the standard deviation of the pages yet to be completed by them.

Solution :

The pages yet to be completed by them are

60-32, 60-35, 60-37, 60-30, 60-33, 60-36, 60-35, 60-37, 60-37

= 28, 25, 23, 30, 27, 24, 25, 23

To find the SD of the data 28, 25, 23, 30, 27, 24, 25, 23

Arrange them in ascending order

x	$d = x - A$	d^2
23	– 2	4
23	– 2	4
24	– 1	1
25	0	0
25	0	0

27	2	4
28	3	9
30	5	25
	5	47
	$\sum d = 5$	$\sum d^2 = 47$

$$\therefore \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

$$= \sqrt{\frac{47}{8} - \left(\frac{5}{8}\right)^2}$$

$$= \sqrt{\frac{47}{8} - \frac{25}{64}}$$

$$= \sqrt{\frac{376-25}{64}}$$

$$= \frac{\sqrt{351}}{8}$$

$$= \frac{18.74}{8}$$

$$= 2.34$$

5. Find the variance and standard deviation of the wages of 9 workers given below:
₹310, ₹290, ₹320, ₹280, ₹300, ₹290, ₹320, ₹310, ₹280.

Solution :

Given wages of a workers are ₹310, ₹290, ₹320, ₹280, ₹300, ₹290, ₹320, ₹310, ₹280

To find the variance and SD, arrange them in ascending order.

x	$d = \frac{x-300}{10}$	d^2
280	–2	4
280	–2	4
290	–1	1
290	–1	1
300	0	0
310	1	1
310	1	1
320	2	4
320	2	4
	0	20
	$\sum d = 0$	$\sum d^2 = 20$

$$d = \frac{x - A}{c}$$

A - Assumed Mean   C - Common divisor

$$\sigma^2 = \frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2 \times c^2$$

$$= \frac{20}{9} - 0 \times 100$$

$$= \frac{2000}{9}$$

$$\sigma^2 = 222.2$$

$\therefore$ Variance = 222.2

$\therefore$ S.D = $\sqrt{222.2}$

$= 14.906$

$\square$14.91

6. A wall clock strikes the bell once at 1 o' clock, 2 times at 2 o' clock, 3 times at 3 o' clock and so on. How many times will it strike in a particular day. Find the standard deviation of the number of strikes the bell make a day.

Solution :

A clock strikes bell at 1 o' clock once

twice at 2 o' clock,

3 times at 3 o' clock ....

$\therefore$ Number of times it strikes in a particular day

$= 2\ (1 + 2 + 3 + \ldots\ldots\ldots\ 12)$

$= 2\left(\frac{12 \times 13}{2}\right)$

$= 156$ times

To find the S.D of 2 (1, 2, 3, ...12)

$= 2\left[\sqrt{\frac{n^2 - 1}{12}}\right]$

$= 2\left[\sqrt{\frac{144 - 1}{12}}\right]$

$= 2\sqrt{\frac{143}{12}} = 2\sqrt{11.91}$

$= 2(3.45)$

$= 6.9$

7. Find the standard deviation of first 21 natural numbers.

Solution :

SD of first 21 natural numbers

$= \sqrt{\frac{n^2 - 1}{12}}$

$= \sqrt{\frac{441 - 1}{12}}$

$= \sqrt{\frac{440}{12}}$

$= \sqrt{36.66}$

$= 6.0547$

□6.05

8. If the standard deviation of a data is 4.5 and if each value of the data is decreased by 5, then find the new standard deviation.

Solution :

Given, S.D of a data = 4.5

Since each value is decreased by 5, then the new SD = 4.5

($\because$ S.D will not be changed when we add (or) subtract fixed constant to all the values of the data).

9. If the standard deviation of a data is 3.6 and each value of the data is divided by 3, then find the new variance and new standard deviation.

Solution :

Given, S.D of a data = 3.6

Since each value is divided by 3 then the new S.D = $\frac{3.6}{3}$

$= 1.2$

New Variance $= (1.2)^2$

$= 1.44$

10. The rainfall recorded in various places of five districts in a week are given below.

Rainfall (in mm)	**45**	**50**	**55**	**60**	**65**	**70**
Number of places	**5**	**13**	**4**	**9**	**5**	**4**

Find its standard deviation.

Solution :

x	f	$d = \frac{x-60}{5}$	d^2	$f.d$	$f.d^2$
45	5	– 3	9	–15	45
50	13	– 2	4	–26	52
55	4	–1	1	–4	4
60	9	0	0	0	0
65	5	1	1	5	5
70	4	2	4	8	16
				–32	122

$\sum f = N = 40, \quad \sum fd = -32 \quad \sum f.d^2 = 122$

$c = 5$

$$\therefore \sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \times c$$

$$= \sqrt{\frac{122}{40} - \left(\frac{-32}{40}\right)^2} \times 5$$

$$= \sqrt{\frac{122}{140} - \frac{1024}{1600}} \times 5$$

$$= \sqrt{\frac{4880 - 1024}{1600}} \times 5$$

$$= \frac{\sqrt{3856}}{40} \times 5$$

$$= \frac{62.096}{8} = 7.76$$

$\therefore$ S.D = 7.76

11. In a study about viral fever, the number of people affected in a town were noted as

Age in years	0-10	10-20	20-30	30-40
Number of people affected	3	5	16	18
Age in years	40-50	50-60	60-70	-
Number of people affected	12	7	4	

Solution :

C.I	mid value (x)	f	$d = \frac{x-35}{10}$	d^2	$f.d$	$f.d^2$
0-10	5	3	–3	9	–9	27
10-20	15	5	–2	4	–10	20
20-30	25	16	–1	1	–16	16
30-40	35-A	18	0	0	0	0
40-50	45	12	1	1	12	12
50-60	55	7	2	4	14	28
60-70	65	4	3	9	12	36
					3	139

$\therefore \sum f = 65, \sum fd = 3, \sum fd^2 = 139$ and $c = 10$

$$\therefore \sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \times c$$

$$= \sqrt{\frac{139}{65} - \left(\frac{3}{65}\right)^2} \times 10$$

$$= \sqrt{\frac{139}{65} - \frac{9}{65^2}} \times 10$$

$$= \sqrt{\frac{9035 - 9}{65^2}} \times 10$$

$$= \frac{\sqrt{9026}}{65} \times 10$$

$$= \frac{95.005}{65} \times 10$$

$$= 1.46 \times 10$$

$$= 14.6$$

12. The measurements of the diameters (in cms) of the plates prepared in a factory are given below. Find its standard deviation.

Diameter (cm)	21-24	25-28	29-32	33-36	37-40	41-44
Number of plates	15	18	20	16	8	7

Solution :

C.I	mid value (x)	f	$d = \frac{x-34.5}{4}$	d^2	$f.d$	$f.d^2$
21-24	22.5	15	–3	9	–45	135
25-28	26.5	18	–2	4	–36	72
29-32	30.5	20	–1	1	–20	20
33-36	34.5	16	0	0	0	0
37-40	38.5	8	1	1	8	8
41-44	42.5	7	2	4	14	28
		84			–79	263

$\therefore \sum f = 94, \sum fd = -79, \sum fd^2 = 263, c = 4$

$$\therefore \sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \times c$$

$$= \sqrt{\frac{263}{84} - \left(\frac{-79}{84}\right)^2} \times 4$$

$$= \sqrt{\frac{263}{84} - \frac{6241}{84^2}} \times 4$$

$$= \sqrt{\frac{22092 - 6241}{84^2}} \times 4$$

$$= \frac{\sqrt{15851}}{84} \times 4$$

$$= \frac{125.9}{21}$$

$$= 5.995$$

□6

13. The time taken by 50 students to complete a 100 meter race are given below. Find its standard deviation

Time taken (sec)	**8.5-9.5**	**9.5-10.5**	**10.5-11.5**	**11.5-12.5**	**12.5-13.5**
Number of students	**6**	**8**	**17**	**10**	**9**

Solution :

C.I	mid value (x)	f	$d = x - 11$	d^2	$f.d$	$f.d^2$
8.5-9.5	9	6	–2	4	–12	24
9.5-10.5	10	8	–1	1	–8	8
10.5-11.5	11	17	0	0	0	0
11.5-12.5	12	10	1	1	10	10
12.5-13.5	13	9	2	4	18	36
		50			8	78

$\therefore \sum f = 50, \sum fd = 8, \sum fd^2 = 78$ and $c = 1$

$$\therefore \sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

$$= \sqrt{\frac{78}{50} - \left(\frac{8}{50}\right)^2}$$

$$= \sqrt{\frac{78}{50} - \frac{64}{50^2}}$$

$$= \sqrt{\frac{3900 - 64}{50^2}}$$

$$= \frac{\sqrt{3836}}{50}$$

$$= \frac{61.935}{65}$$

$$= 1.238$$

□1.24

14. For a group of 100 candidates the mean and standard deviation of their marks were found to be 60 and 15 respectively. Later on it was found that the scores 45 and 72 were wrongly entered as 40 and 27. Find the correct mean and standard deviation.

Solution :

Given $n = 100, \bar{x} = 60, \sigma = 15$

$$\therefore \frac{\sum x}{n} = 60$$

$$\Rightarrow \frac{\sum x}{100} = 60$$

$$\Rightarrow \sum x = 6000$$

$\therefore$ Corrected $\sum x = 6000 - (40 + 27) + (45 + 72)$

$= 6000 - 67 + 117$

$= 6050$

$\therefore$ Corrected mean $= \dfrac{6050}{100}$

$= 60.5$

$$\text{Variance} = \sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$225 = \frac{\sum x^2}{100} - 60^2$$

$$\therefore \frac{\sum x^2}{100} = 3825$$

$$\Rightarrow \sum x^2 = 382500$$

$\therefore$ The correct $\sum x^2 = 382500$

$\therefore$ Corrected $\sum x^2$

$= \text{Incorrect } \sum x^2 - 40^2 - 27^2 + 45^2 + 72^2$

$= 382500 - 1600 - 729 + 2025 + 5184$

$= 387380$

$$\therefore \text{Corrected } \sigma^2 = \frac{\text{Corrected } \sum x^2}{n} - (\text{Corr. mean})^2$$

$$= \frac{387380}{100} - (60.5)^2$$

$$= 3873.80 - 3660.25$$

$$= 213.55$$

$\therefore$ Corrected SD $= \sqrt{213.55}$

$= 14.6$

15. The mean and variance of seven observations are 8 and 16 respectively. If five of these are 2, 4, 10, 12 and 14, then find the remaining two observations.

Solution :

Given n = 7, $\bar{x} = 8$, $\sigma^2 = 16$

5 of the observerations are 2, 4, 10, 12, 14

Let the emaining 2 observations be *a, b*.

$$\therefore \bar{x} = 8 \Rightarrow \frac{\sum x}{n} = 8$$

$$\Rightarrow \frac{42 + a + b}{7} = 8$$

$$\Rightarrow a + b = 56 - 42$$

$$\Rightarrow a + b = 14 \qquad ....(1)$$

Also, $\sigma^2 = 16$

$$\Rightarrow \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = 16$$

$$\Rightarrow \frac{\sum x^2}{7} - 8^2 = 16$$

$$\Rightarrow \frac{\sum x^2}{7} = 80$$

$$\Rightarrow \sum x^2 = 560$$

$$\Rightarrow 2^2 + 4^2 + 10^2 + 12^2 + 10^2 + a^2 + b^2 = 560$$

$$\Rightarrow 460 + a^2 + b^2 = 560$$

$$\Rightarrow a^2 + b^2 = 100$$

$$\Rightarrow a^2 + (14 - a)^2 = 100 \qquad \text{(from (1))}$$

$$\Rightarrow a^2 + 196 + a^2 - 28a = 100$$

$$\Rightarrow 2a^2 - 28a + 96 = 0$$

$$\Rightarrow a^2 - 14a + 48 = 0$$

$$\Rightarrow (a - 8)(a - 6) = 0$$

$$a = 8, \qquad a = 6$$

$$\therefore b = 6, \qquad b = 8$$

II. COEFFICIENT OF VARIATION :

Key Points

- ✓ Coefficient of variation, CV = $\frac{\sigma}{\overline{x}} \times 100$.
- ✓ If the C.V value is less, then the observations of $\overline{x}$ corresponding data are consistent.
- ✓ If the C.V value is more, then the observations of corresponding data are inconsistent.

Example 8.15

The mean of a data is 25.6 and its coefficient of variation is 18.75. Find the standard deviation.

Solution :

Mean $\overline{x} = 25.6$,

Coefficient of variation, C.V. = 18.75

Coefficient of variation, C.V. = $\frac{\sigma}{\overline{x}} \times 100\%$

$18.75 = \frac{\sigma}{25.6} \times 100$; $\sigma = 4.8$

Example 8.16

The following table gives the values of mean and variance of heights and weights of the 10th standard students of a school.

	Height	Weight
Mean	155 cm	46.50 kg^2
Variance	72.25 cm^2	28.09 kg^2

Which is more varying than the other?

Solution :

For comparing two data, first we have to find their coefficient of variations

Mean $\overline{x}_1 = 155$cm, variance $\sigma_1^2 = 72.25\text{cm}^2$

Therefore standard deviation $\sigma_1 = 8.5$

Coefficient of variantion $\text{C.V}_1 = \frac{\sigma_1}{\overline{x}_1} \times 100\%$

$\text{C.V}_1 = \frac{8.5}{155} \times 100\% = 5.48\%$ (for heights)

Mean $\overline{x}_2 = 46.50$kg, variance $\sigma_2^2 = 28.09\,\text{kg}^2$

Standard deviation $\sigma_2 = 5.3$kg

Coefficient of variantion $\text{C.V}_2 = \frac{\sigma_2}{\overline{x}_2} \times 100\%$

$\text{C.V}_2 = \frac{5.3}{46.50} \times 100\% = 11.40\%$ (for weights)

$\text{C.V}_1 = 5.48\%$ and $\text{C.V}_2 = 11.40\%$

Since $\text{C.V}_2 > \text{C.V}_1$, the weight of the students is more varying than the height.

Example 8.17

The consumption of number of guava and orange on a particular week by a family are given below.

Number of Guavas	3	5	6	4	3	5	4
Number of Oranges	1	3	7	9	2	6	2

Which fruit is consistenly consumed by the family?

Solution :

First we find the coefficient of variation for guavas and oranges separately.

x_i	x_i^2
3	9
5	25
6	36
4	16
3	9
5	25
4	16
$\sum x_i = 30$	$\sum x_i^2 = 136$

Number of guavas, n = 7

$$\text{Mean } \bar{x}_1 = \frac{30}{7} = 4.29$$

$$\text{Standard deviation } \sigma_1 = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\sigma_1 = \sqrt{\frac{136}{7} - \left(\frac{30}{7}\right)^2} = \sqrt{19.43 - 18.40} \;\square 1.01$$

Coefficient of variation for guavas

$$\text{C.V}_1 = \frac{\sigma_1}{\bar{x}_1} \times 100\% = \frac{1.01}{4.29} \times 100\% = 23.54\%$$

x_i	x_i^2
1	1
3	9
7	49
9	81
2	4
6	36
2	4
$\sum x_i = 30$	$\sum x_i^2 = 184$

Number of oranges n = 7

$$\text{Mean } \bar{x}_2 = \frac{30}{7} = 4.29$$

$$\text{Standard deviation } \sigma_1 = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\sigma_2 = \sqrt{\frac{184}{7} - \left(\frac{30}{7}\right)^2} = \sqrt{26.29 - 18.40} = 2.81$$

Coefficient of variation for oranges :

$$\text{C.V}_2 = \frac{\sigma_2}{\bar{x}_2} \times 100\% = \frac{2.81}{4.29} \times 100\% = 65.50\%$$

$\text{C.V}_1 = 23.54\%$ and $\text{C.V}_2 = 65.50\%$

Since $\text{C.V}_1 < \text{C.V}_2$, we can conclude that the consumption of guavas is more consistent than oranges.

EXERCISE 8.2

1. The standard deviation and mean of a data are 6.5 and 12.5 respectively. Find the coefficient of variation.

Solution :

Given $\sigma = 6.5$, $\bar{x} = 12.5$

$$\therefore \text{C.V} = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{6.5}{12.5} \times 100$$

$$= \frac{13/2}{25/2} \times 100$$

$$= 13 \times 4$$

$$= 52\%$$

2. The standard deviation and coefficient of variation of a data are 1.2 and 25.6 respectively. Find the value of mean.

Solution :

Given $\sigma = 1.2$, C.V = 25.6

$$\therefore \text{C.V} = \frac{\sigma}{\bar{x}} \times 100$$

$$25.6 = \frac{1.2}{\bar{x}} \times 100$$

$$\Rightarrow \bar{x} = \frac{120}{25.6}$$

$$\bar{x} = 4.69$$

3. If the mean and coefficient of variation of a data are 15 and 48 respectively, then find the value of standard deviation.

Solution :

Given $\bar{x} = 15$, CV = 48, σ = ?

$$\therefore \text{C.V} = \frac{\sigma}{\overline{x}} \times 100$$

$$\Rightarrow 48 = \frac{\sigma}{15} \times 100$$

$$\sigma = \frac{15 \times 48}{100} = \frac{720}{100} = 7.2$$

4. If n = 5 , $\overline{x} = 6$, $\sum x^2 = 765$, then calculate the coefficient of variation.

Solution :

Given n = 5, $\overline{x} = 6$, $\sum x^2 = 765$, CV = ?

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$= \sqrt{\frac{765}{5} - (6)^2}$$

$$= \sqrt{\frac{765 - 180}{5}}$$

$$= \sqrt{\frac{585}{5}}$$

$$= \sqrt{117}$$

$$= 10.82$$

$$\therefore \text{C.V} = \frac{\sigma}{\overline{x}} \times 100$$

$$= \frac{10.82}{6} \times 100$$

$$= \frac{1082}{6}$$

$$= 180.33\%$$

5. Find the coefficient of variation of 24, 26, 33, 37, 29, 31.

Solution :

Given data is 24, 26, 33, 37, 29, 31.

$$\therefore \text{C.V} = \frac{\sigma}{\overline{x}} \times 100$$

$$\overline{x} = \frac{24 + 26 + 33 + 37 + 29 + 31}{6}$$

$$= \frac{180}{6}$$

$$= 30$$

To find σ_1 arrange them in ascending order.

x	$d = x - 31$	d^2
24	– 7	49
26	– 5	25
29	– 2	4
31	0	0
33	2	4
37	6	36
	– 6	118

$$\therefore \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

$$= \sqrt{\frac{118}{6} - \left(\frac{-6}{6}\right)^2}$$

$$= \sqrt{\frac{118}{6} - 1}$$

$$= \sqrt{\frac{112}{6}}$$

$$= \sqrt{18.6}$$

$$\sigma = 4.31$$

$$\therefore \text{C.V} = \frac{4.31}{30} \times 100$$

$$= \frac{43.1}{3}$$

$$= 14.36$$

$$\square 14.4\%$$

6. The time taken (in minutes) to complete a homework by 8 students in a day are given by 38, 40, 47, 44, 46, 43, 49, 53. Find the coefficient of variation.

Solution :

Given data is 38, 40, 47, 44, 46, 43, 49, 53.

$$\bar{x} = \frac{38+40+47+44+46+43+49+53}{8}$$
$$= \frac{360}{8}$$
$$= 45$$

To find σ, arrange them in ascending order.

x	$d = x - 46$	d^2
38	– 8	64
40	– 6	36
43	– 3	9
44	– 2	4
46	0	0
47	1	1
49	3	9
53	7	49
	– 8	172

$$\therefore \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$
$$= \sqrt{\frac{172}{8} - \left(\frac{-8}{8}\right)^2}$$
$$= \sqrt{\frac{172}{8} - 1} = \sqrt{\frac{164}{8}} = \sqrt{20.5} = 4.53$$

$$\therefore \text{C.V} = \frac{4.53}{45} \times 100$$
$$= \frac{453}{45}$$
$$= 10.07\%$$

7. The total marks scored by two students Sathya and Vidhya in 5 subjects are 460 and 480 with standard deviation 4.6 and 2.4 respectively. Who is more consistent in performance?

Solution :

Sathya	**Vidhya**
$\sum x_1 = 460$	$\sum x_2 = 480$
$n = 5$	$n = 5$
$\therefore \bar{x}_1 = \frac{460}{5}$	$\therefore \bar{x}_2 = \frac{480}{5}$
$= 92$	$= 96$
$\sigma_1 = 4.6$	$\sigma_2 = 2.4$

$$\therefore \text{C.V}_1 = \frac{\sigma_1}{\bar{x}_1} \times 100$$
$$= \frac{4.6}{92} \times 100$$
$$= \frac{460}{92}$$
$$= 5$$

$$\therefore \text{C.V}_2 = \frac{\sigma_2}{\bar{x}_2} \times 100$$
$$= \frac{2.4}{96} \times 100$$
$$= \frac{240}{96}$$
$$= 2.5$$

∴ $\text{C.V}_2 < \text{C.V}_1$

∴ Vidhya is more consistent than Sathya.

8. The mean and standard deviation of marks obtained by 40 students of a class in three subjects Mathematics, Science and Social Science are given below.

Subject	Mean	SD
Mathematics	56	12
Science	65	14
Social Science	60	10

Which of the three subjects shows highest variation and which shows lowest variation in marks?

Solution :

$$C.V = \frac{\sigma}{\bar{x}} \times 100$$

For Maths, C.V = $\frac{12}{56} \times 100 = 21.428$

For Science, C.V = $\frac{14}{65} \times 100 = 21.538$

For Social Science, C.V = $\frac{10}{60} \times 100 = 16.67$

Highest variation in Science.

Lowest variation in Social Science.

9. The temperature of two cities A and B in a winter season are given below.

Temperature of city A (in degree Celsius)	18	20	22	24	26
Temperature of city B (in degree Celsius)	11	14	15	17	18

Find which city is more consistent in temperature changes?

Solution :

Temperature of City 'A' :
18, 20, 22, 24, 26

$$\bar{x} = \frac{110}{5} = 22$$

x	$d = \frac{x-22}{2}$	d^2
18	– 2	4
20	– 1	1
22	0	0
24	1	1
26	2	4
	0	10

$$\therefore \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \times C$$

$$= \sqrt{\frac{10}{5} - 0} \times 2$$

$$= 2\sqrt{2}$$

$$\therefore \text{CV for city A} = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{2\sqrt{2}}{22} \times 100$$

$$= \frac{100 \times 1.414}{22}$$

$$= 6.427$$

Temperature of Ciy B 11, 14, 15, 17, 18

$$\bar{x} = \frac{75}{5} = 15$$

x	$d = x - 15$	d^2
11	– 4	16
14	– 1	1
15	0	0
17	2	4
18	3	9
	0	30

$$\therefore \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \times C$$

$$= \sqrt{\frac{30}{5} - 0}$$

$$= \sqrt{6}$$

$\therefore$ CV for city B $= \frac{\sigma}{\bar{x}} \times 100$

$= \frac{\sqrt{6}}{15} \times 100$

$= \frac{2.45}{15} \times 100$

$= 16.34$

$\therefore$ CV for City A < CV for City B.

$\therefore$ City A is more consistent in temperature changes.

III. PROBABILITY :

Key Points

- ✓ A **random experiment** is an experiment in which

  (i) The set of all possible outcomes are known (ii)Exact outcome is not known.
- ✓ The set of all possible outcomes in a random experiment is called a **sample space.** It is generally denoted by S.
- ✓ Each element of a sample space is called a **sample point.**
- ✓ In a random experiment, each possible outcome is called an **event.**
- ✓ An event will be a subset of the sample space.
- ✓ If an event E consists of only one outcome then it is called an **elementary event.**
- ✓ $P(E) = \frac{n(E)}{n(S)}$
- ✓ $P(S) = \frac{n(S)}{n(S)} = 1$. The probability of sure event is 1.
- ✓ $P(\phi) = \frac{n(\phi)}{n(s)} = \frac{0}{n(s)} = 0$. The probability of impossible event is 0.
- ✓ E is a subset of S and ϕ is a subset of any set.

  $\phi \subseteq E \subseteq S$ $\quad$ $P(\phi) \leq P(E) \leq P(S)$ $\quad$ $0 \leq P(E) \leq 1$
- ✓ The complement event of E is $\overline{E}$.
- ✓ $P(E) + P(\overline{E}) = 1$.

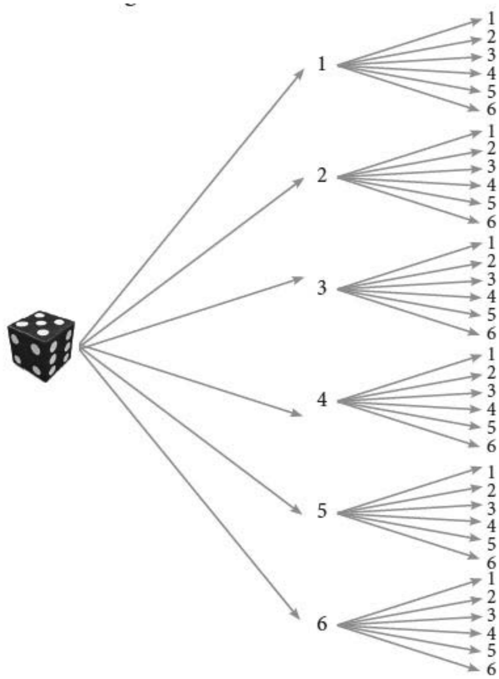
Example 8.18

Express the sample space for rolling two dice using tree diagram.

Solution :

When we roll two dice, since each die contain 6 faces marked with 1,2,3,4,5,6 the tree diagram will look like

Hence, the sample space can be written as



S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

Example 8.19

A bag contains 5 blue balls and 4 green balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is (i) blue (ii) not blue.

Solution :

Total number of possible outcomes

n(S) = 5 + 4 = 9

(i) Let A be the event of getting a blue ball.
Number of favourable outcomes for the event A. Therefore, n(A) = 5
Hence we get, y = x +11 gives
x − y +11 = 0.
Probability that the ball drawn is blue.
Therefore, P(A) = $\frac{n(A)}{n(S)} = \frac{5}{9}$

(ii) $\overline{A}$ will be the event of not getting a blue ball.
So P($\overline{A}$) = 1 – P(A) = 1 – $\frac{5}{9} = \frac{4}{9}$

Example 8.20

Two dice are rolled. Find the probability that the sum of outcomes is (i) equal to 4 (ii) greater than 10 (iii) less than 13

Solution :

When we roll two dice, the sample space is given by

S = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) };
n (S) = 36

(i) Let A be the event of getting the sum of outcome values equal to 4.
Then A = {(1,3),(2,2),(3,1)}; n (A) = 3 .
Probability of getting the sum of outcomes equal to 4 is

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(ii) Let B be the event of getting the sum of outcome values greater than 10.

Then B = {(5,6),(6,5),(6,6)}; n (B) = 3

Probability of getting the sum of outcomes greater than 10 is

$$P(B)=\frac{n(B)}{n(S)}=\frac{3}{36}=\frac{1}{12}$$

(iii) Let C be the event of getting the sum of outcomes less than 13. Here all the outcomes have the sum value less than 13. Hence C = S .

Therefore, n(C) = n(S) = 36

Probability of getting the total value less than 13 is

$$P(C)=\frac{n(C)}{n(S)}=\frac{36}{36}=1$$

Example 8.21

Two coins are tossed together. What is the probability of getting different faces on the coins?

Solution :

When two coins are tossed together, the sample space is

S = {HH,HT,TH,TT}; n(S) = 4

Let A be the event of getting different faces on the coins.

A = {HT,TH}; n(A) = 2

Probability of getting different faces on the coins is

$$P(A)=\frac{n(A)}{n(S)}=\frac{2}{4}=\frac{1}{2}$$

Example 8.22

From a well shuffled pack of 52 cards, one card is drawn at random. Find the probability of getting (i) red card (ii) heart card (iii) red king (iv) face card (v) number card

Solution :

n (S) = 52

(i) Let A be the event of getting a red card.

n(A) = 26

Probability of getting a red card is

$$P(A)=\frac{26}{52}=\frac{1}{2}$$

(ii) Let B be the event of getting a heart card.

n (B) = 13

Probability of getting a heart card is

$$P(B)=\frac{n(B)}{n(S)}=\frac{13}{52}=\frac{1}{4}$$

(iii) Let C be the event of getting a red king card. A red king card can be either a diamond king or a heart king.

n(C) = 2

Probability of getting a red king card is

$$P(C)=\frac{n(C)}{n(S)}=\frac{2}{52}=\frac{1}{26}$$

(iv) Let D be the event of getting a face card. The face cards are Jack (J), Queen (Q), and King (K).

n(D) = 4 × 3 = 12

Probability of getting a face card is

$$P(D)=\frac{n(D)}{n(S)}=\frac{12}{52}=\frac{3}{13}$$

(v) Let E be the event of getting a number card. The number cards are 2, 3, 4, 5, 6, 7, 8, 9 and 10.

n(E) = 4 × 9 = 36

Probability of getting a number card is

$$P(E)=\frac{n(E)}{n(S)}=\frac{36}{52}=\frac{9}{13}$$

Example 8.23

What is the probability that a leap year selected at random will contain 53 saturdays.

(Hint: 366 = 52×7 +2)

Solution :

A leap year has 366 days. So it has 52 full weeks and 2 days. 52 Saturdays must be in 52 full weeks.

The possible chances for the remaining two days will be the sample space.

S = {(Sun-Mon, Mon-Tue, Tue-Wed, Wed-Thu, Thu-Fri, Fri-Sat, Sat-Sun)}

n(S) = 7

Let A be the event of getting 53[rd] Saturday.

Then A = {Fri-Sat, Sat-Sun}; n(A) = 2

Probability of getting 53 Saturdays in a leap year is

$$P(A)=\frac{n(A)}{n(S)}=\frac{2}{7}$$

Example 8.24

A die is rolled and a coin is tossed simultaneously. Find the probability that the die shows an odd number and the coin shows a head.

Solution :

Sample space

S = {1H,1T,2H,2T,3H,3T,4H,4T,5H,5T,6H,6T};

n (S) = 12

Let A be the event of getting an odd number and a head.

A = {1H, 3H, 5H}; n (A)= 3

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{12}=\frac{1}{4}$$

Example 8.25

A bag contains 6 green balls, some black and red balls. Number of black balls is as twice as the number of red balls. Probability of getting a green ball is thrice the probability of getting a red ball. Find (i) number of black balls (ii) total number of balls.

Solution :

Number of green balls is n (G) = 6

Let number of red balls is n (R) = x

Therefore, number of black balls is n (B) = $2x$

Total number of balls n(S) = $6 + x + 2x = 6 + 3x$

It is given that, P(G) = 3 × P(R)

$$\frac{6}{6+3x}=3\times\frac{x}{6+3x}$$

$3x = 6$ gives, $x = 2$.

(i) Number of black balls = 2×2 = 4

(ii) Total number of balls = 6 +(3×2) = 12

Example 8.26

A game of chance consists of spinning an arrow which is equally likely to come to rest pointing to one of the numbers 1, 2, 3, …12. What is the probability that it will point to (i) 7 (ii) a prime number (iii) a composite number?

Solution :

Sample space

S = {1,2,3,4,5,6,7,8,9,10,11,12}; n(S) = 12

(i) Let A be the event of resting in 7. n(A)=1

$$P(A)=\frac{n(A)}{n(S)}=\frac{1}{12}$$

(ii) Let B be the event that the arrow will come to rest in a prime number.

B = {2,3,5,7,11}; n (B) = 5

$$P(B)=\frac{n(B)}{n(S)}=\frac{5}{12}$$

(iii) Let C be the event that arrow will come to rest in a composite number.

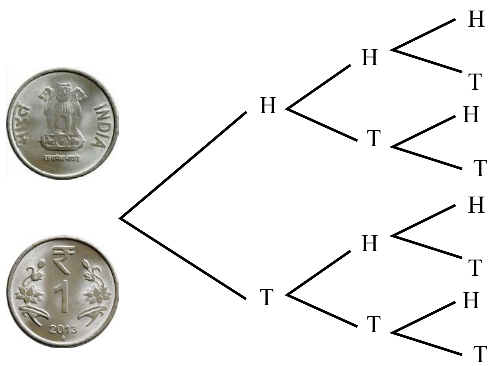
C = {4,6,8,9,10,12}; n (C)=6

$$P(C)=\frac{n(C)}{n(S)}=\frac{6}{12}=\frac{1}{2}$$

EXERCISE 8.3

1. Write the sample space for tossing three coins using tree diagram.

Solution :

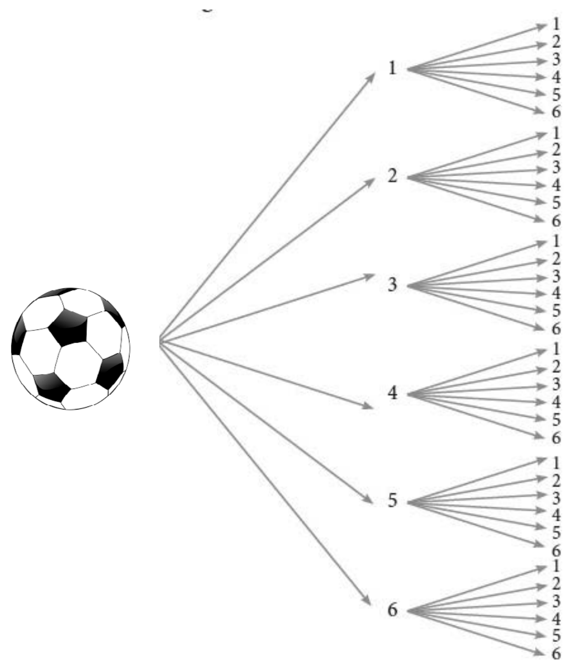


Sample space = {(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)}

2. Write the sample space for selecting two balls from a bag containing 6 balls numbered 1 to 6 (using tree diagram).

Solution :

S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}



3. If A is an event of a random experiment such that P(A) : P $(\overline{A})$=17:15 and n(S) = 640 then find (i) P $(\overline{A})$ (ii) n(A).

Solution :

Given P(A) : P$(\overline{A})$ = 17 : 15

$$\Rightarrow \frac{1-P(\overline{A})}{P(A)}=\frac{17}{15}$$

$$\Rightarrow 15-15P(A)=17P(\overline{A})$$

$$\Rightarrow 32P(\overline{A})=15$$

$$\Rightarrow P(A)=\frac{17}{32}$$

$$\Rightarrow \frac{n(A)}{n(S)}=\frac{17}{32}$$

$$\Rightarrow n(A)=\frac{17}{32}\times 640=340$$

4. A coin is tossed thrice. What is the probability of getting two consecutive tails?

Solution :

When a coin is tossed thrice,

S = {(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)}

n(S) = 8

Let A bet the event of getting 2 tails continuously,

A = {(HTT), (TTH), (TTT)}

n(A) = 3

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{8}$$

---

5. At a fete, cards bearing numbers 1 to 1000, one number on one card are put in a box. Each player selects one card at random and that card is not replaced. If the selected card has a perfect square number greater than 500, the player wins a prize. What is the probability that (i) the first player wins a prize (ii) the second player wins a prize, if the first has won?

Solution :

n(S) = 1000

i) Let A be the event of getting perfect squares between 500 and 1000

A = {23^2, 24^2, 25^2, 26^2 ...... 31^2}

n(A) = 9

$$P(A)=\frac{9}{1000}$$

is the probablity for the 1st player to win a prize.

ii) When the card which was taken first is not replaced.

n(S) = 999

n(B) = 8

$$P(B)=\frac{8}{999}$$

---

6. A bag contains 12 blue balls and *x* red balls. If one ball is drawn at random (i) what is the probability that it will be a red ball? (ii) If 8 more red balls are put in the bag, and if the probability of drawing a red ball will be twice that of the probability in (i), then find *x*.

Solution :

Total number of balls in the bag

= x + 12. ($x \to$ red 12 $\to$ black)

i) Let A be the event of getting red balls

$$P(A)=\frac{n(A)}{n(S)}=\frac{x}{x+12}$$

ii) If 8 more red balls are added in the bag.

n(S) = x + 20

By the problem, $\frac{x+8}{x+20}=2\left(\frac{x}{x+12}\right)$

$$\Rightarrow (x+8)(x+12)=2x^2+40x$$

$$\Rightarrow x^2+20x+96=2x^2+40x$$

$$\Rightarrow x^2+20x-96=0$$

$$\Rightarrow (x+24)(x-4)=0$$

$$\therefore x=-24,\ 4$$

$$\therefore x=4$$

$$\therefore P(A)=\frac{4}{16}=\frac{1}{4}$$

---

7. Two unbiased dice are rolled once. Find the probability of getting

(i) a doublet (equal numbers on both dice)

(ii) the product as a prime number

(iii) the sum as a prime number

(iv) the sum as 1

Solution :

S ={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

i) Let A be the event of getting a doublet

A = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

n(A) = 6

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

ii) Let B the event of getting the product as a prime number.

B = {(1, 2), (1, 3), (1, 5), (2, 1), (3, 1), (5, 1)}

n(B) = 6

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

iii) Let C be the event of getting the sum of numbers on the dice is prime.

C = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (4, 1), (4, 3), (5, 2), (5, 6), (6, 1), (6, 5)}

n(C) = 14

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{7}{36}$$

iv) Let D be the event of getting sum of numbers is 1.

n(D) = 0

P(D) = 0

8. Three fair coins are tossed together. Find the probability of getting

(i) all heads (ii) atleast one tail

(iii) atmost one head (iv) atmost two tails

Solution :

When 3 fair coins are tossed,

S = {(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)}

n(S) = 8

i) Let A be the event of getting all heads.

A = {(HHH)}

n(A) = 1

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$$

ii) Let B be the event of getting atleast one tail.

B = {(HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)}

n(B) = 7

$$P(B) = \frac{7}{8}$$

iii) Let C be the event of getting at most one head.

C = {(HTT), (THT), (TTH), (TTT)}

n(C) = 4

$$P(C) = \frac{4}{8} = \frac{1}{2}$$

iv) Let D - atmost 2 tails

D = {(HHH), (HHT), (HTT), (HTH), (THH), (THT), (TTH)}

n(D) = 7

$$P(D) = \frac{7}{8}$$

9. Two dice are numbered 1,2,3,4,5,6 and 1,1,2,2,3,3 respectively. They are rolled and the sum of the numbers on them is noted. Find the probability of getting each sum from 2 to 9 separately.

Solution :

S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

i) Let A - Sum of 2

n(A) = 2

$\therefore P(A) = \frac{2}{36}$

ii) Let B - Sum of 3

n(B) = 4

$P(B) = \frac{4}{36}$

iii) Let C - Sum of 4

n(C) = 6

$P(C) = \frac{6}{36}$

iv) Let D - Sum of 5

n(D) = 6

$P(D) \quad \frac{}{36}$

v) Let E - Sum of 6

n(E) = 6

$P(E) = \frac{6}{36}$

vi) Let F - Sum of 7

n(F) = 6

$P(F) = \frac{6}{36}$

vii) Let G - Sum of 8

n(G) = 4

$P(G) = \frac{4}{36}$

viii) Let H - Sum of 9

n(H) = 2

$P(H) = \frac{2}{36}$

$P(H) = \frac{2}{36}$

---

10. A bag contains 5 red balls, 6 white balls, 7 green balls, 8 black balls. One ball is drawn at random from the bag. Find the probability that the ball drawn is (i) white (ii) black or red (iii) not white (iv) neither white nor black

Solution :

S = {5R, 6W, 7G, 8B}

i) Let A - White ball

n(A) = 6

$P(A) = \frac{6}{26} = \frac{3}{13}$

ii) Let B - Black (or) red

n(B) = 5 + 8 = 13

$P(B) = \frac{13}{26} = \frac{1}{2}$

iii) Let C - not white

n(C) = 20

$P(C) = \frac{20}{26} = \frac{10}{13}$

iv) Let D - Neither white nor black

n(D) = 12

$P(D) = \frac{12}{26} = \frac{6}{13}$

11. In a box there are 20 non-defective and some defective bulbs. If the probability that a bulb selected at random from the box found to be defective is $\frac{3}{8}$ then, find the number of defective bulbs.

Solution :

Let x be the number of defective bulbs.

$\therefore$ n(S) = $x + 20$

Let A be the event of selecting defective balls

$\therefore$ n(A) = x

$$P(A) = \frac{x}{x+20}$$

Given $\frac{x}{x+20} = \frac{3}{8}$

$\Rightarrow$ $8x = 3x + 60$

$\Rightarrow$ $5x = 60$

$x = 12$

$\therefore$ Number of defective balls = 12.

12. The king and queen of diamonds, queen and jack of hearts, jack and king of spades are removed from a deck of 52 playing cards and then well shuffled. Now one card is drawn at random from the remaining cards. Determine the probability that the card is (i) a clavor (ii) a queen of red card (iii) a king of black card

Solution :

Solution :

By the data given,

n(S) = 52 – 2 – 2 – 2 = 46

i) Let A be the event of selecting clubber card.

n(A) = 13

$$P(A) = \frac{13}{46}$$

ii) Let B - queen of red card.

n(B) = 0

P(B) = 0

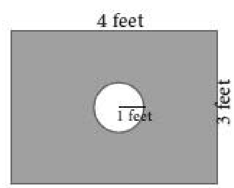
(queen diamond and heart are included in S)

iii) Let C - King of black cards

n(C) = 1 (encluding spade king)

$$\therefore P(C) = \frac{1}{46}$$

13. Some boys are playing a game, in which the stone thrown by them landing in a circular region (given in the figure) is considered as win and landing other than the circular region is considered as loss. What is the probability to win the game?



Solution :

Area of the rectangular region $= 4 \times 3$

$= 12\text{ft}^2$

Area of the circular region $= \pi r^2$

$= \pi \times 1^2$

$= \pi\text{ft}^2$

$\therefore$ Probability to win the game $= \frac{\pi}{12}$

$$= \frac{3.14}{12}$$

$$= \frac{314}{1200}$$

$$= \frac{157}{600}$$

14. Two customers Priya and Amuthan are visiting a particular shop in the same week (Monday to Saturday). Each is equally likely to visit the shop on any one day as on another day. What is the probability that both will visit the shop on

(i) the same day (ii) different days

(iii) consecutive days?

Solution :

Given n(S) = 6. (Monday - Saturday)

i) Prob. that both of them will visit the shop on the same day = $\frac{1}{6}$

ii) Prob. that both of them will visit the shop in different days = $\frac{5}{6}$.

($\because$ if one visits on Monday, other one visit the shop out of remaining 5 days).

iii) Prob. that both of them will visit the shop in consecutive days.

A = {(Mon, Tue), (Tue, Wed), (Wed, Thu), (Thu, Fri), (Fri, Sat)}

n(A) = 5

$P(A) = \frac{5}{6}$

15. In a game, the entry fee is ₹150. Th e game consists of tossing a coin 3 times. Dhana bought a ticket for entry . If one or two heads show, she gets her entry fee back. If she throws 3 heads, she receives double the entry fees. Otherwise she will lose. Find the probability that she (i) gets double entry fee (ii) just gets her entry fee (iii) loses the entry fee.

Solution :

S = {(HHH), (HHT), (HTH), (THH), (HTT), (THT), (HTT), (TTT)}

n(S) = 8

i) P (gets double entry fee) = $\frac{1}{8}$ ($\because$ 3 heads)

ii) P (just gets for her entry fee) = $\frac{6}{8} = \frac{3}{4}$

($\because$ 1 (or) 2 heads)

iii) P (loses the entry fee) = $\frac{1}{8}$

($\because$ 3 no heads (TTT) only)

IV. ALGEBRA OF EVENTS:

Key Points

- ✓ $A \cap \overline{A} = \phi \quad A \cup \overline{A} = S$
- ✓ If A, B are mututally exclusive events, the $P(A \cup B) = P(A) + P(B)$.
- ✓ P (Union of mutually exclusive events) = $\sum$ (Probability of events)

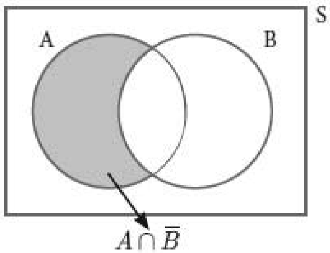
Theorem 1

If A and B are two events associated with a random experiment, then prove that

(i) $P(A \cap \overline{B})$ = P(only A) = $P(A) - P(A \cap B)$

(ii) $P(\overline{A} \cap B)$ = P(only B) = $P(B) - P(A \cap B)$

Proof



(i) By Distributive property of sets,

1. $(A\cap B)\cup(A\cap \bar{B}) = A\cap(B\cup \bar{B}) = A\cap S = A$

2. $(A\cap B)\cap(A\cap \bar{B}) = A\cap(B\cap \bar{B}) = A\cap\phi = \phi$

Therefore, $P(A) = P[(A\cap B)\cup(A\cap B)]$

$P(A) = P(A\cap B) + P(A\cap \bar{B})$

Therefore, $P(A\cap \bar{B}) = P(A) - P(A\cap B)$

That is, $P(A\cap \bar{B})$ = P (oly A) = $P(A) - P(A\cap B)$

(ii) By Distributive property of sets,

1. $(A\cap B) \cup (\bar{A}\cap B) = (A\cup \bar{A}) \cap B = S\cap B = B$

2. $(A\cap B) \cap (\bar{A}\cap B) = (A\cap \bar{A}) \cap B = \phi\cap B = \phi$

Therefore, the events $A\cap B$ and $\bar{A}\cap B$ are mutually exclusive whose union is B.

$P(B) = P[(A\cap B)\cup(\bar{A}\cap B)]$

$P(B) = P(A\cap B) + P(\bar{A}\cap B)$

Therefore, $P(\bar{A}\cap B) = P(B) - P(A\cap B)$

That is, $P(\bar{A}\cap B)$ = P (only B) = $P(B) - P(A\cap B)$

Theorem 2

(i) If A and B are any two events then

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

(ii) If A, B and C are any three events then

$P(A\cup B\cup C) = P(A) + P(B) + P(C)$

$- P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)$

Proof

(i) Let A and B be any two events of a random experiment with sample space S.

From the Venn diagram, we have the events only A, $A \cap B$ and only B are mutually exclusive and their union is $A \cup B$

Therefore,

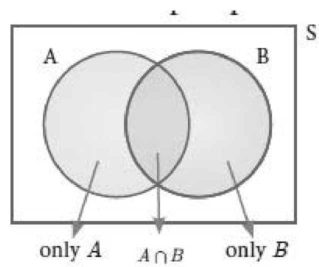
$P(A \cup B) = P[(\text{only } A) \cup (A\cap B) \cup (\text{only } B)]$

$= P(\text{only } A) + P(A\cap B) + P(\text{only } B)$

$= [P(A)-P(A\cap B)]+P(A\cap B) + [P(B) - P(A\cap B)]$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

(ii) Let A, B, C are any three events of a random experiment with sample space S.



Let $D = B\cup C$

$P(A\cup B\cup C) = P(A\cup D)$

$= P(A) + P(D) - P(A\cap D)$

$= P(A) + P(B \cup C) - P[A\cap(B \cup C)]$

$=P(A) + P(B) + P(C) - P(B\cap C) - P[(A\cap B)\cup(A\cap C)]$

$= P(A) + P(B) + P(C) - P(B\cap C) - P[(A\cap B) - P(A\cap C) + P[(A\cap B)\cap(A\cap C)]$

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

Example 8.27

If P(A) = 0.37 , P(B) = 0.42 , P(A∩B) = 0.09 then find P(A∪B) .

Solution :

P(A) = 0.37 , P(B) = 0.42 , P(A∩B) = 0.09

P(A∪B) = P(A)+P(B)−P(A∩B)

P(A∪B) = 0.37 + 0.42 − 0.09 = 0.7

Example 8.28

What is the probability of drawing either a king or a queen in a single draw from a well shuffled pack of 52 cards?

Solution :

Total number of cards = 52

Number of king cards = 4

Probability of drawing a king card = $\frac{4}{52}$

Number of queen cards = 4

Probability of drawing a queen card = $\frac{4}{52}$

Both the events of drawing a king and a queen are mutually exclusive

⇒ P(A∪B) = P(A)+P(B)

Therefore, probability of drawing either a king or a queen = $\frac{4}{52}+\frac{4}{52}=\frac{2}{13}$

Example 8.29

Two dice are rolled together. Find the probability of getting a doublet or sum of faces as 4.

Solution :

When two dice are rolled together, there will be 6×6 = 36 outcomes. Let S be the sample space. Then n(S) = 36

Let A be the event of getting a doublet and B be the event of getting face sum 4.

Then A = {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)}

B = {(1,3),(2,2),(3,1)}

Therefore, A ∩ B = {(2,2)}

Then, n(A) = 6 , n(B) = 3 , n(A∩B) = 1 .

$$P(A)=\frac{n(A)}{n(S)}=\frac{6}{36}$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{3}{36}$$

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{1}{36}$$

Therefore,

P (getting a doublet or a total of 4) = P(A∪B)

P(A∪B) = P(A) + P(B) – P(A∩B)

$$=\frac{6}{36}+\frac{3}{36}-\frac{1}{36}=\frac{8}{36}=\frac{2}{9}$$

Hence, the required probability is $\frac{2}{9}$

Example 8.30

If A and B are two events such that P(A)=$\frac{1}{4}$, P(B) = $\frac{1}{2}$ and P (A and B) = $\frac{1}{8}$, find

(i) P(A or B) (ii) P (not A and not B).

Solution :

(i) P(A or B) = P(A∪B)

= P(A) + P(B) – P(A∩B)

$$P(A \text{ or } B)=\frac{1}{4}+\frac{1}{2}-\frac{1}{8}=\frac{5}{8}$$

(ii) P (not A and not B) = $P(\overline{A}\cap\overline{B})$

= $P(\overline{A\cup B})$

= 1 – P(A∪B)

P (not A and not B) = 1 – $\frac{5}{8}=\frac{3}{8}$

Example 8.31

A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution :

Total number of cards = 52; n (S) = 52

Let A be the event of getting a king card.

n(A) = 4

$$P(A)=\frac{n(A)}{n(S)}=\frac{4}{52}$$

Let B be the event of getting a heart card.

n(B) =13

$$P(B)=\frac{n(B)}{n(S)}=\frac{13}{52}$$

Let C be the event of getting a red card.

n (C) = 26

$$P(C)=\frac{n(C)}{n(S)}=\frac{26}{52}$$

$P(A\cap B)$ = P (getting heart king) = $\frac{1}{52}$

$P(B\cap C)$ = P (getting red and heart) = $\frac{13}{52}$

$P(A\cap C)$ = P (getting red king) = $\frac{2}{52}$

$P(A\cap B\cap C)$=P (getting heart, king which is red)

$$=\frac{1}{52}$$

Therefore, required probability is

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$$=\frac{4}{52}+\frac{13}{52}+\frac{26}{52}-\frac{1}{52}-\frac{13}{52}-\frac{2}{52}+\frac{1}{52}=\frac{28}{52}=\frac{7}{13}$$

Example 8.32

In a class of 50 students, 28 opted for NCC, 30 opted for NSS and 18 opted both NCC and NSS. One of the students is selected at random. Find the probability that

(i) The student opted for NCC but not NSS.

(ii) The student opted for NSS but not NCC.

(iii) The student opted for exactly one of them.

Solution:

Total number of students n(S)= 50.

Let A and B be the events of students opted for NCC and NSS respectively.

$n(A) = 28$, $n(B) = 30$, $n(A\cap B) = 18$

$$P(A)=\frac{n(A)}{n(S)}=\frac{28}{50}$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{30}{50}$$

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{18}{50}$$

(i) Probability of the students opted for NCC but not NSS

$$P(A\cap\overline{B}) = P(A) - P(A\cap B) = \frac{28}{50}-\frac{18}{50}=\frac{1}{5}$$

(ii) Probability of the students opted for NSS but not NCC.

$$P(A\cap\overline{B}) = P(B) - P(A\cap B) = \frac{30}{50}-\frac{18}{50}=\frac{6}{25}$$

(iii) Probability of the students opted for exactly one of them

$$= P\,[(A\cap\overline{B})\cup(\overline{A}\cap B)]$$

$$= P(A\cap\overline{B}) + P(\overline{A}\cap B) = \frac{1}{5}+\frac{6}{25}=\frac{11}{25}$$

(Note that $(A\cap\overline{B})$, $(\overline{A}\cap B)$ are mutually exclusive events)

Example 8.33

A and B are two candidates seeking admission to IIT. The probability that A getting selected is 0.5 and the probability that both A and B getting selected is 0.3. Prove that the probability of B being selected is atmost 0.8.

Solution:

$P(A) = 0.5$, $P(A \cap B) = 0.3$

We have $P(A \cup B) \leq 1$

$$P(A) + P(B) - P(A \cap B) \leq 1$$

$$0.5 + P(B) - 0.3 \leq 1$$

$$P(B) \leq 1 - 0.2$$

$$P(B) \leq 0.8$$

Therefore, probability of B getting selected is atmost 0.8.

EXERCISE 8.4

1. If $P(A) = \frac{2}{3}$, $P(B) = \frac{2}{5}$, $P(A \cup B) = \frac{1}{3}$ then find $P(A \cap B)$.

Solution :

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{2}{3} + \frac{2}{5} - \frac{1}{3}$$

$$= \frac{10 + 6 - 5}{15}$$

$$= \frac{11}{15}$$

2. A and B are two events such that, P(A) = 0.42, P(B) = 0.48 , and $P(A \cap B) = 0.16$. Find (i) P,(not A) (ii) P,(not B) (iii) P,(A or B)

Solution :

a) $P(\text{not } A) = P(\overline{A}) = 1 - P(A)$

$= 1 - 0.42$

$= 0.58$

b) $P(\text{not } B) = P(\overline{B}) = 1 - P(B)$

$= 1 - 0.48$

$= 0.52$

c) $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) + P(A \cap B)$

$= 0.42 + 0.48 - 0.16$

$= 0.74$

3. If A and B are two mutually exclusive events of a random experiment and P(not A) = 0.45, $P(A \cup B)$=0.65, then find P(B).

Solution :

Given A and B are mutually exclusive events

$P(A \cap B) = 0$

Also, $P(\text{not } A) = 0.45$

$\therefore P(\overline{A}) = 0.45$

$1 - P(A) = 0.45$

$P(A) = 0.55$

$P(A \cup B) = P(A) + P(B)$

$\therefore P(B) = P(A \cup B) - P(A)$

$= 0.65 - 0.55$

$= 0.10$

4. The probability that atleast one of A and B occur is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\overline{A}) + P(\overline{B})$.

Solution :

Given $P(A \cup B) = 0.6$, $P(A \cap B) = 0.2$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.6 = P(A) + P(B) - 0.2$

$\therefore P(A) + P(B) = 0.8$

$\therefore P(\overline{A}) + P(\overline{B})$

$= 1 - P(A) + 1 - P(B)$

$= 2 - (P(A) + P(B))$

$= 2 - 0.8$

$= 1.2$

5. The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then find the probability that neither A nor B happen.

Solution :

Given $P(A) = 0.5$, $P(B) = 0.3$, $P(A \cap B) = 0$

P (neither A nor B)

$= P(\overline{A} \cap \overline{B})$

$= P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - (0.8)$

$= 0.2$

6. Two dice are rolled once. Find the probability of getting an even number on the first die or a total of face sum 8.

Solution :

S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

$n(S) = 36$

Let A be the event of getting even number on the 1st die.

A = {(2, 1),(2, 2),(2, 3),(2, 4),(2, 5),(2, 6)
(4, 1),(4, 2),(4, 3),(4, 4),(4, 5),(4, 6)
(6, 1),(6, 2),(6, 3),(6, 4),(6, 5),(6, 6)}

$n(A) = 18$

$P(A) = \frac{18}{36}$

Let B - Total of face sum as 8.

B = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

$n(B) = 5,\ P(B) = \frac{5}{36}$

$A \cap B$ = {(2, 6), (4, 4), (6, 2)}

$n(A \cap B) = 3$

$P(A \cap B) = \frac{3}{36}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{18}{36} + \frac{5}{36} - \frac{3}{36}$

$= \frac{20}{36}$

$= \frac{5}{9}$

7. From a well-shuffled pack of 52 cards, a card is drawn at random. Find the probability of it being either a red king or a black queen.

Solution :

$n(S) = 52$

Let A - Red King

$n(A) = 2$

$P(A) = \frac{2}{52}$

Let B - Black Queen

$n(B) = 2$

$P(B) = \frac{2}{52}$

Here A and B are mutually enclusive

$\therefore P(A \cup B) = P(A) + P(B)$

$= \frac{4}{52}$

$= \frac{1}{13}$

8. A box contains cards numbered 3, 5, 7, 9, … 35, 37. A card is drawn at random from the box. Find the probability that the drawn card have either multiples of 7 or a prime number.

Solution :

$S = \{3, 5, 7, 9, \text{...........} 35, 37\}$

$n(S) = 18$

Let A - multiple of 7.

$A = \{7, 14, 21, 28, 35\}$

$n(A) = 5$

$P(A) = \frac{5}{18}$

Let B - a prime number

$B = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$

$n(B) = 11$

$P(B) = \frac{11}{18}$

Here $A \cap B = \{7\}$

$n(A \cap B) = 1$

$P(A \cap B) = \frac{1}{18}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cup B)$

$= \frac{5}{18} + \frac{11}{18} - \frac{1}{18}$

$= \frac{15}{18}$

$= \frac{5}{6}$

9. Three unbiased coins are tossed once. Find the probability of getting atmost 2 tails or atleast 2 heads.

Solution :

$S = \{(HHH), (HHT), (HTH), (THH), (HTT), (THT), (TTH), (TTT)\}$

$n(S) = 8$

Let A - at most 2 tails

$A = \{(HHT), (HTH), (THH), (HTT), (THT), (TTH), (HHH)\}$

$n(A) = 7$

$P(A) = \frac{7}{8}$

Let B - atleast 2 heads

$B = \{(HHH), (HHT), (HTH), (THH)\}$

$n(B) = 4$

$P(B) = \frac{4}{8}$

$\therefore A \cap B = \{(HHH), (HHT), (HTH), (THH)\}$

$n(A \cap B) = 4$, $P(A \cap B) = \frac{4}{8}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{7}{8} + \frac{4}{8} - \frac{4}{8}$

$= \frac{7}{8}$

10. The probability that a person will get an electrification contract is $\frac{3}{5}$ and the probability that he will not get plumbing contract is $\frac{5}{8}$. The probability of getting atleast one contract is $\frac{5}{7}$. What is the probability that he will get both?

Solution :

Let A - electrification contract

B - not plumbing contract

Given

$$P(A)=\frac{3}{5},\ P(\overline{B})=\frac{5}{8},\ P(A\cup B)=\frac{5}{7}$$

$$\Rightarrow P(B)=1-\frac{5}{8}$$

$$=\frac{3}{8}$$

$$\therefore P(A\cap B)=P(A)+P(B)-P(A\cup B)$$

$$=\frac{3}{5}+\frac{3}{8}-\frac{5}{7}$$

$$=\frac{168+105-200}{280}$$

$$=\frac{73}{280}$$

11. In a town of 8000 people, 1300 are over 50 years and 3000 are females. It is known that 30% of the females are over 50 years. What is the probability that a chosen individual from the town is either a female or over 50 years?

Solution :

Let A - Female

B - Over 50 years

Given n(S) = 8000, n(A) = 3000,

n(B) = 1300 and $n(A\cap B) = \frac{30}{100} \times 3000 = 900$

$$\therefore P(A) = \frac{3000}{8000},\ P(B) = \frac{1300}{8000},\ P(A\cap B) = \frac{900}{8000}$$

$\therefore$ P (either a female (or) over 50 years)

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$=\frac{3000+1300-900}{8000}$$

$$=\frac{3400}{8000}$$

$$=\frac{34}{80}$$

$$=\frac{17}{40}$$

---

12. A coin is tossed thrice. Find the probability of getting exactly two heads or atleast one tail or two consecutive heads.

Solution :

S = {(HHH), (HHT), (HTH), (THH), (TTH), (THT), (HTT), (TTT)}

n(S) = 8

Let A - exactly 2 heads

A = {(HHT), (HTH), (THH)}

n(A) = 3

$$P(A) = \frac{3}{8}$$

Let B - atleast one tail

B = {(HHT), (HTH), (THH), (TTH), (THT), (HTT), (TTT)}

n(B) = 7

$$P(B) = \frac{7}{8}$$

Let C - Consecutively 2 heads

C = {(HHH), (HHT), (THH)}

n(C) = 3

$$P(C) = \frac{3}{8}$$

$A\cap B = \{(HHT), (HTH), (THH)\}$

$n(A\cap B) = 3$

$P(A\cap B) = \frac{3}{8}$

$B\cap C = \{(HHT), (THH)\}$

$n(B\cap C) = 2$

$P(B\cap C) = \frac{2}{8}$

$C\cap A = \{(HHT), (THH)\}$

$n(C\cap A) = 2$

$P(C\cap A) = \frac{2}{8}$

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$$=\frac{3}{8}+\frac{7}{8}+\frac{3}{8}-\frac{3}{8}-\frac{2}{8}-\frac{2}{8}+\frac{2}{8}$$

$$=\frac{8}{8}=1$$

---

13. If A, B, C are any three events such that probability of B is twice as that of probability of A and probability of C is thrice as that of probability of A and if $P(A\cap B)=\frac{1}{6}$, $P(B\cap C)=\frac{1}{4}$, $P(A\cap C)=\frac{1}{8}$, $P(A\cup B\cup C) = \frac{9}{10}$, $P(A\cap B\cap C) = \frac{1}{15}$, then find P(A),P(B) and P(C) ?

Solution :

Given $P(B) = 2.\, P(A)$, $P(C) = 3\, .\, P(A)$

$P(A\cap B)=\frac{1}{6}$, $P(B\cap C)=\frac{1}{4}$, $P(A\cap C)=\frac{1}{8}$,

$P(A\cup B\cup C) = \frac{9}{10}$, $P(A\cap B\cap C) = \frac{1}{15}$

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$$\Rightarrow \frac{9}{10} = P(A)+2.P(A)+3.P(A)-\frac{1}{6}-\frac{1}{4}-\frac{1}{8}+\frac{1}{15}$$

$$\Rightarrow 6\, .\, P(A) = \frac{9}{10}+\frac{1}{6}+\frac{1}{4}+\frac{1}{8}-\frac{1}{15}$$

$$\Rightarrow 6.P(A) = \frac{108+20+30+15-8}{120}$$

$$\Rightarrow 6.P(A) = \frac{165}{120}$$

$$\Rightarrow P(A) = \frac{165}{720} = \frac{11}{48}$$

$$\therefore P(A) = \frac{11}{48}$$

$$\therefore P(B) = 2\, .\, P(A) = 2\times\frac{11}{48} = \frac{11}{24}$$

$$P(C) = 3\, .\, P(A) = 3\times\frac{11}{48} = \frac{11}{16}$$

---

14. In a class of 35, students are numbered from 1 to 35. The ratio of boys to girls is 4:3. The roll numbers of students begin with boys and end with girls. Find the probability that a student selected is either a boy with prime roll number or a girl with composite roll number or an even roll number.

Solution :

Given n(S) = 35 and ratio of boys and girls=4:3

No. of boys = $\frac{4}{7} \times 35 = 20$

No. of boys = $\frac{3}{7} \times 35 = 15$

Let A - a boy with prime roll no

$A = \{2, 3, 5, 7, 11, 13, 19\}$ ($\because$ only 20 boys)

$n(A) = 7$

$P(A) = \frac{7}{35}$

Let B - a girl with composite roll no.

B={21,22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35}

n(B) = 12

$\therefore$ P(B) = $\frac{12}{25}$

Let C - even roll no.

B={2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34}

n(C) = 17

$\therefore$ P(C) = $\frac{17}{35}$

$A\cap B = \{ \ \ \}$, $n(A\cap B) = 0$, $P(A\cap B) = 0$

$B\cap C = \{22, 24, 26, 28, 30, 32, 34\}$

$\therefore n(B\cap C) = 7 \Rightarrow P(B\cap C) = \frac{7}{35}$

$C\cap A = \{2\} \Rightarrow n(C\cap A) = 1$

$P(C\cap A) = \frac{1}{35}$

at $P(A\cap B\cap C) = 0$

$$\therefore P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$$

$$= \frac{7}{35} + \frac{12}{35} + \frac{17}{35} - 0 - \frac{7}{35} - \frac{1}{35} + 0$$

$$= \frac{28}{35}$$

$$= \frac{4}{5}$$

EXERCISE 8.5

Multiple choice questions :

1. Which of the following is not a measure of dispersion?

(1) Range (2) Standard deviation

(3) Arithmetic mean (4) Variance

Ans : (3)

Hint :

A.M is not a measure of dispersion and it is a measure of central tendency.

2. The range of the data 8, 8, 8, 8, 8. . . 8 is

(1) 0 (2) 1 (3) 8 (4) 3

Ans : (1)

Hint :

Range $= L - S$

$= 8 - 8 = 0$

3. The sum of all deviations of the data from its mean is

(1) Always positive

(2) always negative

(3) zero

(4) non-zero integer

Ans : (3)

Hint :

Sum of all deviations of the data from the mean = 0

ie $\sum (x - \bar{x}) = 0$

4. The mean of 100 observations is 40 and their standard deviation is 3. The sum of squares of all deviations is

(1) 40000 (2) 160900

(3) 160000 (4) 30000

Ans : (2)

Hint :

$\overline{x} = 40,\ n = 100,\ \sigma = 3$

$$\sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$9 = \frac{\sum x^2}{100} - (40)^2$$

$$\frac{\sum x^2}{100} = 1609$$

$$\Rightarrow \sum x^2 = 160900$$

5. **Variance of first 20 natural numbers is**

(1) 32.25 (2) 44.25 (3) 33.25 (4) 30

Ans : (3)

Hint :

Variance for first 20 natural numbers

$$\sigma^2 = \frac{n^2 - 1}{12}$$

$$= \frac{400 - 1}{12}$$

$$= \frac{399}{12}$$

$$= 33.25$$

6. **The standard deviation of a data is 3. If each value is multiplied by 5 then the new variance is**

(1) 3 (2) 15 (3) 5 (4) 225

Ans : (4)

Hint :

$\sigma = 3$ of a data.

If each value is multiplied by 5,

then the new SD = 15

$\therefore$ Variance = $(SD)^2$

$= 15^2$

$= 225$

7. **If the standard deviation of x, y, z is p then the standard deviation of $3x + 5$, $3y + 5$, $3z + 5$ is**

(1) 3p + 5 (2) 3p (3) p + 5 (4) 9p +15

Ans : (2)

Hint :

SD of x, y, z = p

$\Rightarrow$ SD of $3x, 3y, 3z$ = 3p

$\Rightarrow$ SD of $3x + 5, 3y + 5, 3z + 5$ = 3p.

8. **If the mean and coefficient of variation of a data are 4 and 87.5% then the standard deviation is**

(1) 3.5 (2) 3 (3) 4.5 (4) 2.5

Ans : (1)

Hint :

$\overline{x} = 4$, CV = 87.5, $\sigma = ?$

$$CV = \frac{\sigma}{\overline{x}} \times 100$$

$$87.5 = \frac{\sigma}{4} \times 100$$

$$\therefore \sigma = \frac{87.5}{25}$$

$$= 3.5$$

9. **Which of the following is incorrect?**

(1) P (A) > 1 (2) $0 \le P(A) \le 1$

(3) $P(\phi) = 0$ (4) $P(A) + P(\overline{A}) = 1$

Ans : (1)

Hint :

P(A) > 1 is incorrect.

since $0 \le P(A) \le 1$

10. The probability a red marble selected at random from a jar containing *p* red, *q* blue and *r* green marbles is

(1) $\dfrac{q}{p+q+r}$ (2) $\dfrac{p}{p+q+r}$

(3) $\dfrac{p+q}{p+q+r}$ (4) $\dfrac{p+r}{p+q+r}$

Ans : (2)

Hint :

n (Red) = p, n(S) = $p + q + r$

Required probability = $\dfrac{p}{p+q+r}$

11. A page is selected at random from a book. The probability that the digit at units place of the page number chosen is less than 7 is

(1) $\dfrac{3}{10}$ (2) $\dfrac{7}{10}$ (3) $\dfrac{3}{9}$ (4) $\dfrac{7}{9}$

Hint : **Ans : (2)**

P (digit at unit's place of the page is less than 7) = $\dfrac{7}{10}$

($\because$ n(S) = 10, A = {0, 1, 2, 3, 4, 5, 6}, n(A) = 7)

12. The probability of getting a job for a person is $\dfrac{x}{3}$ If the probability of not getting the job is $\dfrac{2}{3}$ then the value of *x* is

(1) 2 (2) 1 (3) 3 (4) 1.5

Hint : **Ans : (2)**

Given P(A) = $\dfrac{x}{3}$, P($\overline{A}$) = $\dfrac{2}{3}$

P(A) + P($\overline{A}$) = 1

$\Rightarrow \dfrac{x+2}{3} = 1$

$\Rightarrow x + 2 = 3$

$\Rightarrow x = 1$

13. Kamalam went to play a lucky draw contest. 135 tickets of the lucky draw were sold. If the probability of Kamalam winning is $\dfrac{1}{9}$, then the number of tickets bought by Kamalam is

(1) 5 (2) 10 (3) 15 (4) 20

Hint : **Ans : (3)**

$n(S) = 135 \quad n(A) = x$

$\therefore P(A) = \dfrac{x}{135} = \dfrac{1}{9}$ (given)

$\Rightarrow x = \dfrac{135}{9} = 15$

14. If a letter is chosen at random from the English alphabets {a, b,...., z}, then the probability that the letter chosen precedes *x*

(1) $\dfrac{12}{13}$ (2) $\dfrac{1}{13}$ (3) $\dfrac{23}{26}$ (4) $\dfrac{3}{26}$

Hint : **Ans : (3)**

n(S) = 26 n(A) = 23 ($\because$ 26 – 3)

P(A) = $\dfrac{23}{26}$

15. A purse contains 10 notes of ₹2000, 15 notes of ₹500, and 25 notes of ₹200. One note is drawn at random. What is the probability that the note is either a ₹500 note or ₹200 note?

(1) $\dfrac{1}{5}$ (2) $\dfrac{3}{10}$ (3) $\dfrac{2}{3}$ (4) $\dfrac{4}{5}$

Hint : **Ans : (4)**

n(S) = 50, n(A) = 10, n(B) = 15, n(C) = 25

$P(B \cup C) = P(B) + P(C)$ ($\because$ B & C are mutually exclusive)

$= \dfrac{15}{50} + \dfrac{25}{50}$

$= \dfrac{40}{50}$

$= \dfrac{4}{5}$

UNIT EXERCISE - 8

1. The mean of the following frequency distribution is 62.8 and the sum of all frequencies is 50. Compute the missing frequencies f_1 and f_2.

Class Interval	0-20	20 - 40	40 - 60	60-80	80-100	100-120
Frequency	5	f_1	10	f_2	7	8

Solution :

Given $\bar{x} = 62.8$, $\Sigma f = 50$

$$\Rightarrow f_1 + f_2 + 30 = 50$$

$$\Rightarrow f_1 + f_2 = 20$$

$$\Rightarrow f_2 = 20 - f_1$$

C.I.	x	f	$d = \frac{x-70}{20}$	fd
0-20	10	5	–3	– 15
20-40	30	f_1	–2	$-2f_1$
40-60	50	10	–1	– 10
60-80	70	$20-f_1$	0	0
80-100	90	7	1	7
100-120	110	8	2	16
		50		$-2f_1-2$

$$\bar{x} = A + \left(\frac{\Sigma fd}{\Sigma f} \times c\right)$$

$$62.8 = 70 + \left(\frac{-2f_1 - 2}{50} \times 20\right)$$

$$\Rightarrow 62.8 = 70 + \left(\frac{-4f_1 - 4}{5}\right)$$

$$\Rightarrow 314 = 350 - 4f_1 - 4 \Rightarrow -4f_1 = -32$$

$$f_1 = \frac{32}{4} = 8$$

$$\therefore f_1 = 8, \quad f_2 = 20 - f_1$$

$$= 20 - 8 = 12.$$

2. The diameter of circles (in mm) drawn in a design are given below.

Diameters	33-36	37-40	41-44	45-48	49-52
Number of circles	15	17	21	22	25

Calculate the standard deviation.

Solution :

C.I.	x	f	$d = \frac{x-42.5}{4}$	d^2	$f.d$	$f.d^2$
32.5-36.5	34.5	15	– 2	4	–30	60
36.5-40.5	38.5	17	– 1	1	–17	17
40.5-44.5	42.5	21	0	0	0	0
44.5-48.5	46.5	22	1	1	22	22
48.5-52.5	50.5	25	2	4	50	100
		100			25	199

$\therefore \Sigma f = 100$, $\Sigma fd = 25$, $\Sigma fd^2 = 199$

$$\therefore \sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2} \times 4$$

$$= \sqrt{\frac{199}{100} - \left(\frac{25}{100}\right)^2} \times 4$$

$$= \sqrt{\frac{19900 - 625}{100^2}} \times 4$$

$$= \frac{\sqrt{19275}}{100} \times 4$$

$$= \frac{138.83}{25}$$

$$= 5.55$$

$\therefore$ S.D $= 5.55$

3. The frequency distribution is given below.

x	k	$2k$	$3k$	$4k$	$5k$	$6k$
f	2	1	1	1	1	1

In the table, k is a positive integer, has a variance of 160. Determine the value of k.

Solution :

x	f	$d = \frac{x-A}{k}$	d^2	$f.d$	$f.d^2$
k	2	– 3	9	– 6	18
$2k$	1	– 2	4	– 2	4
$3k$	1	– 1	1	– 1	1
$4k$	1	0	0	0	0
$5k$	1	1	1	1	1
$6k$	1	2	4	2	4
	7			– 6	28

Given variance = 160

$$\therefore k^2\left(\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2\right) = 160$$

$$\Rightarrow \quad k^2\left[\frac{28}{7} - \left(\frac{-6}{7}\right)^2\right] = 160$$

$$\Rightarrow \quad k^2\left[4 - \frac{36}{49}\right] = 160$$

$$\Rightarrow \quad k^2\left[\frac{160}{49}\right] = 160$$

$$\Rightarrow \quad k^2 = \frac{16 \times 40}{16}$$

$$\Rightarrow \quad k^2 = 49$$

$\therefore k = 7$ $\quad$ ($\because k$ is positive)

4. The standard deviation of some temperature data in degree celsius (°C) is 5. If the data were converted into degree Farenheit (°F) then what is the variance?

Solution : Given $\sigma_c = 5$

$$F = \frac{9c}{5} + 32$$

$$\Rightarrow \sigma_F = \frac{9}{5}\sigma_c$$

$$= \frac{9}{5} \times 5$$

$$= 9$$

($\because$ Add (or) subtract the value to a data won't effect the SD)

$$\therefore \sigma_F{}^2 = 9^2 = 81.$$

5. If for a distribution, $\sum(x - 5) = 3$, $\sum(x-5)^2 = 43$, and total number of observations is 18, find the mean and standard deviation.

Solution :

Given $\sum(x-5) = 3$,	$\sum(x-5)^2 = 43$, n = 18
$\Rightarrow \sum x - \sum 5 = 3$	$\Rightarrow \sum(x^2 - 10x + 25) = 43$
$\Rightarrow \sum x - 5.\sum 1 = 3$	$\Rightarrow \sum x^2 - 10.\sum x + 25\sum 1 = 43$
$\Rightarrow \sum x - 5(18) = 3$	$\Rightarrow \sum x^2 - 10(93) + 25(18) = 43$
$\Rightarrow \sum x = 93$	$\Rightarrow \sum x^2 = 523$

i) Mean :

$$\bar{x} = \frac{\sum x}{n} = \frac{93}{18} = 5.17$$

ii) SD :

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{523}{18} - \left(\frac{93}{18}\right)^2}$$

$$= \sqrt{\frac{523}{18} - \frac{8649}{324}}$$

$$= \sqrt{\frac{9414 - 8649}{324}}$$

$$= \frac{\sqrt{765}}{18} = \frac{27.65}{18} = 1.536$$

6. Prices of peanut packets in various places of two cities are given below. In which city, prices were more stable?

Prices in City A	**20**	**22**	**19**	**23**	**16**
Prices in city B	**10**	**20**	**18**	**12**	**15**

Solution :

CV for prices in City A

Given data is 20, 22, 19, 23, 16

$$\therefore \bar{x} = \frac{100}{5} = 20$$

To find σ_1 arrange them is ascending order.

x	$d = x - 20$	d^2
16	– 4	16
19	– 1	1
20	0	0
22	2	4
23	3	9
	0	30

$$\therefore \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

$$= \sqrt{\frac{30}{5}}$$

$$= \sqrt{6}$$

$$= 2.44$$

$$\therefore C.V = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{2.44}{20} \times 100$$

$$= 12.24$$

CV for prices in City B

Given data is 10, 20, 18, 12, 15

$$\therefore \bar{x} = \frac{75}{5} = 15$$

To find σ arrange them is ascending order.

x	$d = x - 15$	d^2
10	– 5	25
12	– 3	9
15	0	0
18	3	0
20	5	25
	0	68

$$\therefore \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

$$= \sqrt{\frac{68}{5}}$$

$$= \sqrt{13.6}$$

$$= 3.68$$

$$\therefore C.V = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{3.68}{15} \times 100$$

$$= 24.53$$

$\therefore$ C.V for price in City A < City B

$\therefore$ Prices are very stable in City A.

7. If the range and coefficient of range of the data are 20 and 0.2 respectively, then find the largest and smallest values of the data.

Solution :

Given range = 20, Co.eff. of range = 0.2

$\Rightarrow L - S = 20$ ...(1)

$$\frac{L-S}{L+S} = 0.2$$

$$\Rightarrow \frac{20}{L+S} = 0.2$$

$$\Rightarrow L + S = 100 \quad ...(2)$$

Solviong (1) and (2)

L = 60, S = 40

8. If two dice are rolled, then find the probability of getting the product of face value 6 or the difference of face values 5.

Solution :

S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

n(S) = 36

Let A - Product of face value is 6.

A = {(1, 6), (2, 3), (3, 2), (6, 1)}
n(A) = 4

$P(A) = \frac{4}{36}$

Let B - Difference of face value is 5.

B = {(6, 1)}

n(B) = 1

$P(B) = \frac{1}{36}$

$A \cap B = \{(6, 1)\}$

$n(A \cap B) = 1$

$P(A \cap B) = \frac{1}{36}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{4}{36} + \frac{1}{36} - \frac{1}{36} = \frac{4}{36} = \frac{1}{9}$

9. In a two children family, find the probability that there is at least one girl in a family.

Solution :

S = {(BB), (BG), (GB), (GG)}

n(S) = 4

Let A be the event of getting atleast one girl.

A = {(BG), (GB), (GG)}

$\therefore$ n(A) = 3

$\therefore P(A) = \frac{3}{4}$

10. A bag contains 5 white and some black balls. If the probability of drawing a black ball from the bag is twice the probability of drawing a white ball then find the number of black balls.

Solution :

Given n(S) = 5 + x, 5 white balls
x black balls

By daa given,
P(B) = 2. P(W)

$\Rightarrow \frac{x}{5+x} = 2.\left(\frac{5}{5+x}\right)$

$\Rightarrow x = 10$

$\therefore$ No. of black balls = 10

11. The probability that a student will pass the final examination in both English and Tamil is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Tamil examination?

Solution :

Given $P(E \cap T) = 0.5$; $P(\bar{E} \cap \bar{T}) = 0.1$

& P(E) = 0.75 $\Rightarrow P(\overline{E \cup T}) = 0.1$

$\Rightarrow P(E \cup T) = 1 - 01$

$= 0.9$

$P(E \cup T) = P(E) + P(T) - P(E \cap T)$

$0.9 = 0.75 + P(T) - 0.5$

$P(T) = 0.9 - 0.25$

$= 0.65$

$= \frac{65}{100}$

$= \frac{13}{20}$

12. The King, Queen and Jack of the suit spade are removed from a deck of 52 cards. One card is selected from the remaining cards. Find the probability of getting (i) a diamond (ii) a queen (iii) a spade (iv) a heart card bearing the number 5.

Solution :

n(S) = 52 – 3 = 49

i) Let A - a diamond card

n(A) = 13

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{13}{49}$$

ii) Let B - a queen card

n(B) = 3 (except spade queen out of 4)

$$\therefore P(B) = \frac{3}{49}$$

iii) Let C - a spade card

n(C) = 10 (13 – 3 = 10)

$$\therefore P(C) = \frac{10}{49}$$

iv) Let D - 5 of heart

n(D) = 1

$$\therefore P(D) = \frac{1}{49}$$

PROBLEMS FOR PRACTICE

1. Find the SD of the data

i) 45, 60, 62, 60, 50, 65, 58, 68, 44, 48

ii) 8, 10, 15, 20, 22

iii) 18, 11, 10, 13, 17, 20, 12, 19

(Ans: (i) 8.14, (ii) 5.44, (iii) 3.67)

2. Find the variance of the wages :

Rs.210, Rs.190, Rs.220, Rs.180, Rs.200, Rs. 190, Rs.200, Rs.210, Rs.180

(Ans : 172.8)

3. Find the range of the heights of 12 girls in a class given in cm.

120, 110, 150, 100, 130, 145, 150, 100, 140, 150, 135, 125

(Ans : 50)

4. The variance of 5 values is 36. If each value is doubled, find the SD of new values.

(Ans : 12)

5. For a group of 200 students, the mean and SD of scores were found to be 40 and 15 respectively. Later on, it was found that scores 43, 35 were misread as 34, 53 respectively. Find the correct mean, SD.

(Ans : 30.955, 14.995)

6. Mean of 100 items is 48 and their S.D is 10. Find the sum of all the items and the sum of the squares of all the items.

(Ans : 4800, 240400)

7. If the coefficient of variation of a collection of data is 57 and its SD is 6.84, find the mean.

(Ans : 12)

8. Calculate S.D from the data :

Marks :	10	20	30	40	50	60
No. of students :	8	12	20	10	7	3

(Ans : 13.45)

9. Find the SD for the data.

Age (in years) :	18	22	21	23	19
No. of students:	100	120	140	150	80

(Ans : 1.84)

10. The following table gives the distribution of income of 100 families in a village. Find the variance.

Income :	0-1000	1000-2000	2000-3000	3000-4000	4000-5000	5000-6000
No. of families :	18	26	30	12	10	4

(Ans : 1827600)

11. Find the coefficient of variation :

20, 18, 32, 24, 26

(Ans : 20.412)

12. Find the coefficient of variation of the data

Size (in cms) :	10-15	15-20	20-25	25-30	30-35	35-40
No. if items :	2	8	20	35	20	15

(Ans : 21.86)

13. Which of the following cricketers A or B is more consistent player, who scored runs in a cricket season.

A :	58	59	60	54	65	66	52	75	69	52
B :	84	56	92	65	86	78	44	54	78	68

(Ans : Player 'A')

14. Find the missing frequencies of the distribution whose mean is 28.2.

C.V	0-10	10-20	20-30	30-40	40-50
f :	5	f_1	15	f_2	6

(Ans : $f_1 = 8, f_2 = 16$)

15. A number is selected at random from 1 to 100. Find the probability that it is a perfect cube.

(Ans : 1/25)

16. A two digit number is formed of the digits 2, 5 and 9. Find the probability that it is divisible by 2 (or) 5, without repetition.

(Ans : 2/3)

17. From a set of whole numbers less than 40, find the probability of getting a number not divisible by 5 or 7.

(Ans : 12/41)

18. Two dices are thrown together. What is the probability that only odd numbers turn upon both the dices.

(Ans : 5/6)

19. What is the probability that a leap year to contain 53 sundays ?

(Ans : 2/7)

20. The probability that A, B and C can solve a problem are 4/5, 2/3, 3/7 respectively. The probability of the problem being solved by A and B is 8/15, B and C is 2/7, A and C is 12/35. The probability of the problem being solved by all the 3 is 8/35. Find the probability that the problem can be solved by atleast one of them.

(Ans : 101/105)

OBJECTIVE TYPE QUESTIONS

1. The range of first 20 whole numbers is
   a) 19 b) 38 c) 20 d) 19.5
   Ans : (c)
2. Variance of 1, 2, 3 is
   a) 2/3 b) 2 c) 0 d) $\sqrt{2}/3$
   Ans : (a
3. The sum of the squares deviations for 10 observations taken from their mean 50 is 250. The coefficient of variation is
   a) 10% b) 40% c) 50% d) 15%
   Ans : (a)
4. If A and B are mutually exclusive and S is the sample space such that P(A) = 1/3, P(B) and S = A$\cup$B, their P(A) is
   a) 1/4 b) 1/2 c) 4/3 d) 3/2
   Ans : (a)
5. If the first 10 positive integers, in which we multiply each no. by – 1 and then add 1 to each, the variance of the numbers so obtained is
   a) 8.25 b) 6.5 c) 3.87 d) 8.25
   Ans : (c)
6. The variance of 15 observations is 4. If each observation is increased by 9, the variance of the new data is
   a) 13 b) 36 c) 4 d) 16
   Ans : (c)
7. Consider the numbers from 1 to 10. If 1 is added to each number, the variance of the numbers so obtained is
   a) 6.5 b) 2.87 c) 3.87 d) 8.25
   Ans : (c)
8. The probability of drawing neither an ace nor a king is
   a) 2/13 b) 11/13 c) 4/13 d) 8/13
   Ans : (b)
9. A number is chosen from 40 to 75. Find the prob. that it is divisible by 7 and 11.
   a) 2/9 b) 1/9 c) 3/9 d) 4/9
   Ans : (a)
10. The probability of selecting a rotten apple randomly from a heap of 900 apples is 0.18. The number of rotten apples is
    a) 0.872 b) 1620 c) 162 d) 172
    Ans : (c)
11. In a family of 3 children, probability of having atleast one boy is
    a) 1/3 b) 7/8 c) 3/8 d) 1/2
    Ans : (b)
12. If a card is drawn at random from 30 cards, the probability that the number on the card is not divisible by 3 is
    a) 2/3 b) 1/3 c) 27/30 d) none
    Ans : (a)
13. 3 digit numbers are made using the digits 4, 5, 9 without repetition. If a number is selected at random, the prob. that the number will be ended with 9 is
    a) 1/3 b) 5/9 c) 1/2 d) none
    (Ans : (a)
14. A letter of english alphabet is chosen at random. The prob. that the letter chosen is a consonant.
    a) 5/26 b) 21/26 c) 7/13 d) 1
    Ans : (b)
15. To prob. of a card to be a club card when it is taken from 52 cards where all red face cards are removed is
    a) 3/23 b) 13/46 c) 10/23 d) 13/40
    Ans : (b)