CHAPTER 4

FOURIER SERIES

§ 1. Consider the following trigonometric series

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots\ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots$$

where the a's and b's are constants and x a variable.

We see that every term except the first term has a period of 2π and consequently any function represented by a series of the above form in an interval of length 2π. If the series converges in any closed interval, say $\lambda \le x \le \lambda + 2\pi$, then the series is converget for every real value of x since the series represented by the function is periodic.

§ 2. Suppose that a given function $f(x)$ can be expressed as a trigonometric series as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots\ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots \qquad \ldots \qquad (1)$$

Let us assume that the series is uniformly converget in the interval

$$\lambda \le x \le \lambda + 2\pi.$$

Then the series can be integrated term by term. To determine the a's and b's in the series, the following identities have to be used :

(*i*) $\displaystyle\int_{\lambda}^{\lambda+2\pi} \cos nx \, dx = 0$ where n *is an integer.*

(*ii*) $\displaystyle\int_{\lambda}^{\lambda+2\pi} \sin nx \, dx = 0$ where n is an integer.

(*iii*) $\displaystyle\int_{\lambda}^{\lambda+2\pi} \cos mx \cos nx \, dx = 0$ if $m \ne n$ and

m and n are integers.

(iv) $\displaystyle\int_{\lambda}^{\lambda+2\pi} \sin mx \sin nx\, dx = 0$ if $m \neq n$ and

m and n are integers.

(v) If $m = n$ and m and n are integers, then

$$\int_{\lambda}^{\lambda+2\pi} \cos mx \cos nx\, dx = \int_{\lambda}^{\lambda+2\pi} \cos^2 mx\, dx = \pi.$$

$$\int_{\lambda}^{\lambda+2\pi} \sin mx \sin nx\, dx = \int_{\lambda}^{\lambda+2\pi} \sin^2 mx\, dx = \pi.$$

$$\int_{\lambda}^{\lambda+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_{\lambda}^{\lambda+2\pi} \sin(2mx)\, dx = 0.$$

If we integrate both sides of equation (1), we have

$$\int_{\lambda}^{\lambda+2\pi} f(x)\, dx = \int_{\lambda}^{\lambda+2\pi} \frac{a_0}{2}\, dx = \pi a_0 .$$

$$\therefore\ a_0 = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\, dx \quad \ldots\ldots \quad (2)$$

If both sides of the equation (1) are multiplied by $\cos nx$ and integrating term by term from λ to $\lambda + 2\pi$, we see that all the terms on the right side vanish except the term containing a_n.

$$\therefore \text{ We have } \int_{\lambda}^{\lambda+2\pi} f(x) \cos nx\, dx = a_n \pi.$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x) \cos nx\, dx \quad \ldots\ldots \quad (3)$$

Similarly, multiplying both sides of the equation (1) by $\sin nx$ and integrating we have

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\,dx \qquad \cdots\cdots \quad (4)$$

In (3), if $n = 0$ is substituted, a_0 is obtained.

Hence we have the result that if $f(x)$ can be expressed as a trigonometric series of the form.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

then $$a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\,dx \quad (n = 0, 1, 2, \ldots\ldots)$$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\,dx \quad (n = 1, 2, \ldots\ldots)$$

Note : (1) The constant term in the series is taken as $a_0/2$ instead of a_0, for the formula for finding a_n is valid when $n = 0$ as well as when n is a positive integer.

(2) If we construct the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

from $f(x)$, by means of these relations, then the series is called a *Fourier series* for $f(x)$.

(3) We cannot conclude that the Fourier series for $f(x)$ will converge to and represent $f(x)$. What our analysis has shown is merely that if $f(x)$ has an expression of the form (1), then the coefficients of the terms in the series are given by the formulae.

(4) The convergence of the Fourier series and if convergent under what conditions it will represent the function which generates it are broad questions under investigation.

(5) B richlet has formulated certain conditions known as Dirichlet cond under which certain functions possess valid Fourier expansions.

These c itions guarantee that the Fourier expansion of $f(x)$ will converge to $f(x)$ at all points of continuity. The conditions are :

(i) it $f(x)$ must never become infinite in the defined interval.

(ii) must be single-valued.

(iii) have at most finite n f maxima and e Interval of de

(iv) e at most, a finite number of discontinuities es) in the interval of definition.

Note : are not *necessary* ; but it is not easy to give more gen a deep study of the subject.

(6) It c at any point of discontinuity (say, $x=a$) where a fun sented by a Fourier series, the value of the Fourier series is

$$\frac{1}{2}[f(a+0)+f(a-0)].$$

The function is discontinuous at $x=a$. If $f(x)$ is expressed at a Fourier series, the value of the series at $x=a$ is $\frac{1}{2}$(PB + PC).

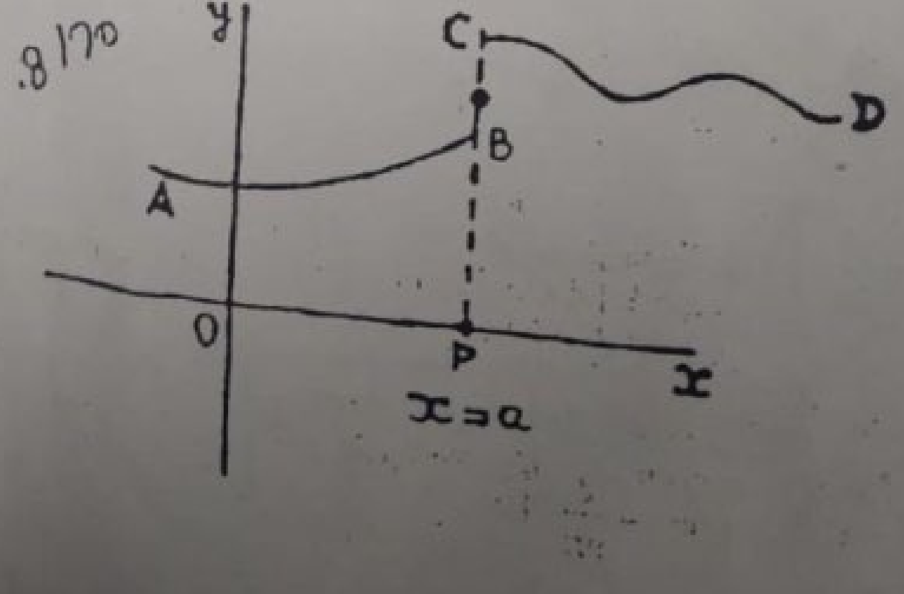

Fig. 19

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(7) Generally $f(x)$ is expanded in the interval from 0 to 2π or in the interval $-\pi$ to π.

Putting $\lambda = 0$, in the interval $\lambda \le x \le \lambda + 2\pi$, we get the interval

$$0 \le x \le 2\pi.$$

Putting $\lambda = -\pi$, in the interval $\lambda \le x \le \lambda + 2\pi$, we get the interval

$$-\pi \le x \le \pi.$$

Examples.

Ex. 1. Show that $x^2 = \dfrac{\pi^2}{3} + 4 \displaystyle\sum_{n=1}^{\infty} (-1)^n \dfrac{\cos nx}{n^2}$ in the interval $(-\pi < x < \pi)$. (Anc. '75)

Deduce that (i) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots\ldots = \dfrac{\pi^2}{12}$

(ii) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\ldots = \dfrac{\pi^2}{6}$

(iii) $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}.$

Let $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

Then $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x^2\,dx = \dfrac{1}{\pi}\left[\dfrac{x^3}{3}\right]_{-\pi}^{\pi} = \dfrac{2\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$\int u\,dv$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{x \sin nx}{n}\,dx$$

$$= -\frac{2}{n\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n}\,dx\right\}$$

$$= \frac{4}{n^2}\cos n\pi = \frac{(-1)^n 4}{n^2}.$$

When n is odd, $a_n = \frac{-4}{n^2}$. When n is even, $a_n = \frac{4}{n^2}$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\, dx\right\}$$

$$= \frac{2}{n\pi}\int_{-\pi}^{\pi} x\cos nx\, dx$$

$$= \frac{2}{n\pi}\left\{\left[\frac{x\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\, dx\right\}$$

$= 0.$ (This could have been inferred as the integrand is an odd function and

$\int_{-\pi}^{\pi} f(x)\,dx = 0$ where $f(x)$ is odd.)

$$\therefore x^2 = \frac{\pi^2}{3} + \sum \frac{(-1)^n 4\cos nx}{n^2}$$

$$= \frac{\pi^2}{3} + 4\sum\frac{(-1)^n\cos nx}{n^2}$$

When $x = 0$, we have

$$0 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}$$

$$= \frac{\pi^2}{3} + 4\left\{-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\dots\right\}.$$

$$\therefore \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\dots\dots = \frac{\pi^2}{12} \qquad \dots (1)$$

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Put $x = \pi$, we have

$$\pi^2 = \frac{\pi^2}{3} + 4\sum \frac{(-1)^n \cos n\pi}{n^2}$$

$$= \frac{\pi^2}{3} + 4\sum \frac{1}{n^2}.$$

$\pi^2 = \frac{\pi^2}{12}$

$\pi^2 -$

$\frac{12\pi^2 -}{12}$

$$\therefore \sum \frac{1}{n^2}, \text{ i.e., } \frac{1}{1^2} + \frac{1}{2^2} + \ldots\ldots = \frac{\pi^2}{6} \qquad \ldots \qquad (2)$$

Additing (1) and (2) and dividing it by 2, we get the result

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

6

Ex. 2. Express $f(x) = \frac{1}{2}(\pi - x)$ as a Fourier series with period 2π, to be valid in the interval 0 to 2π. (Anc. '73)

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= -\frac{1}{4\pi}\left[(\pi - x)^2\right]_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left\{\left[\frac{(\pi - x)\sin nx}{n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\sin nx}{n}\,dx\right\}$$

$= 0.$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-(\pi-x)\cos nx}{n}\right]_0^{2\pi} - \int_0^{2\pi}\frac{\cos nx}{n}\,dx\right\}$$

$$- \frac{(\pi-2\pi)\cos n2\pi}{n}$$

$$= \frac{1}{n}.$$

$$\therefore\ \frac{1}{2}(\pi - x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\ldots$$

In this series if we put $x = \frac{\pi}{2}$, we get the well-known result

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots = \frac{\pi}{4}.$$

10 **Ex. 3.** A function $f(x)$ is defined within the range $(0, 2\pi)$ by the relations

$f(x) = x$ in the range $(0, \pi)$

$f(x) = 2\pi - x$ in the range $(\pi, 2\pi)$.

Express $f(x)$ as a Fourier series in the range $(0, 2\pi)$.

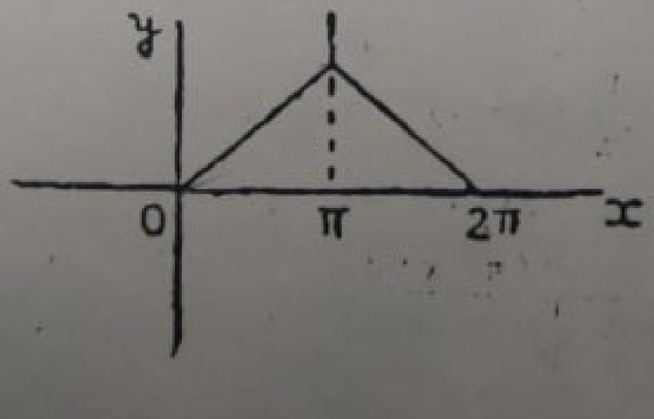

Fig. 20

If we draw the curve $f(x)$ in the range $(0, 2\pi)$, the shape of the curve is as shown in the figure.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_0^{2\pi} (2\pi - x)\,dx\right\}$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} - \frac{1}{\pi}\left[\frac{(2\pi - x)^2}{2}\right]_{\pi}^{2\pi}$$

$$= \pi.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$+ \frac{1}{\pi}\left\{\left[\frac{(2\pi - x)\sin nx}{n}\right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$= -\frac{1}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\frac{\sin nx}{n}\,dx$$

$$= \frac{1}{\pi^2 n}\left[\cos nx\right]_0^{\pi} - \frac{1}{n^2\pi}\left[\cos nx\right]_{\pi}^{2\pi}$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1) - \frac{1}{n^2\pi}(1-\cos n\pi)$$

$$= \frac{2\cos n\pi}{n^2\pi} - \frac{2}{n^2\pi}$$

$$= \frac{2(-1)^n - 2}{n^2\pi}$$

Similarly, it can be shown that $b_n = 0$.

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n - 2}{n^2\pi} \cos nx$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos nx$$

When n is even, $1-(-1)^n = 0$ and

when n is odd, $1-(-1)^n = 2$.

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$$

When $x=0, f(x)=0$.

$$\therefore \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right).$$

$$\therefore \quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

If we put $x = \pi$, we get the same result.

Ex. 4. Find in the range $-\pi$ to π, a Fourier series for

$$y = 1 + x, \qquad 0 < x < \pi$$

$$y = -1 + x, \qquad -\pi < x < 0.$$

Let $y = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} y \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (-1+x) \, dx + \frac{1}{\pi} \int_{0}^{\pi} (1+x) \, dx$$

$$= \frac{1}{\pi} \left[\frac{(x-1)^2}{2} \right]_{-\pi}^{0} + \frac{1}{\pi} \left[\frac{(1+x)^2}{2} \right]_{0}^{\pi}$$

$$= 0.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (-1+x) \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} (1+x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left\{ \left[\frac{(x-1) \sin nx}{n} \right]_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\sin nx}{n} \, dx \right\}$$

$$+ \frac{1}{\pi} \left\{ \left[\frac{(x+1) \sin nx}{n} \right]_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin nx}{n} \, dx \right\}$$

$$= \frac{1}{n^2} [\cos nx]_{-\pi}^{0} + \frac{1}{n^2} [\cos nx]_{0}^{\pi}$$

$$= 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1+x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1+x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{-(x-1)\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\right\}$$

$$+ \frac{1}{\pi}\left\{\left[\frac{-(1+x)\cos nx}{n}\right]_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos nx\,dx\right\}$$

$$= \frac{2}{n\pi} - \frac{2(\pi+1)}{n\pi}\cos n\pi.$$

$$= \frac{2}{n\pi} - \frac{2(\pi+1)(-1)^n}{n\pi}$$

When n is even, $b_n = \frac{2}{n\pi} - \frac{2(\pi+1)}{n\pi} = -\frac{2}{n}$

When n is odd, $b_n = \frac{2}{n\pi} + \frac{2(\pi+1)}{n\pi}$

$$= \frac{2(\pi+2)}{n\pi}$$

$$\therefore y = \frac{2(\pi+2)}{\pi}\sin x - \frac{2}{2}\sin 2x + \frac{2(\pi+2)}{3\pi}\sin 3x$$

$$-\frac{2}{4}\sin 4x + \frac{2(\pi+2)}{5\pi}\sin 5x - \ldots\ldots\ldots$$

When $x = \frac{\pi}{2}$, $y = 1 + \frac{\pi}{2} = \frac{2+\pi}{2}$.

When $x = \frac{\pi}{2}$, the right side of the series becomes

$$\frac{2(\pi+2)}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\right).$$

$\therefore$ (1/1 - 1/3 + 1/5 - 1/7 ...

$$\therefore \quad \left(1-\frac{1}{3}+\frac{1}{5}\,\cdots\cdots\right)\frac{2(\pi+2)}{\pi}=\frac{2+\pi}{2}.$$

$$\therefore \quad 1-\frac{1}{3}+\frac{1}{5}\ldots=\frac{\pi}{4}.$$

Exercises 34.

1. Determine the Fourier expansion of the following functions in the intervals noted against them :-

(i) $f(x) = x \qquad -\pi<x<\pi.$

(ii) $f(x)=\pi^2-x^2 \qquad -\pi<x<\pi.$

(iii) $f(x)=\dfrac{(\pi-x)^2}{4} \qquad 0<x<2\pi.$

2. Show that in the range 0 to 2π, the expansion of e^x as a Fourier series is

$$e^x=\frac{e^{2\pi}-1}{\pi}\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+1}-\sum_{n=1}^{\infty}\frac{n\sin nx}{n^2+1}\right\}.$$

3. Snow that in the range $-\pi$ to π, e^x as a Fourier series is

$$e^x=\frac{\sinh\pi}{\pi}\left\{1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx-n\sin nx)\right\}$$

Deduce from this that $\dfrac{\pi}{\sinh\pi}=1+2\displaystyle\sum_{2}^{\infty}\frac{(-1)^n}{n^2+1}$

4. If $f(x)=x(2\pi-x)$ in $0<x<2\pi$, prove that

$$f(x)=\frac{2\pi^2}{3}-4\left(\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\right)$$

5. If $f(x)=x+x^2\ (-\pi<x<\pi)$, prove that

$$f(x)=\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}\ldots\ldots\right)$$
$$+2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\right)$$

§ 3. Even and odd functions.

If $f(x)=f(-x)$, then $f(x)$ is said to be an even function. If $f(x)=-f(-x)$, then $f(x)$ is said to be an odd function.

The functions $x^2, x^4+3x^2+2, \cos x$, ... are examples of even functions and $x^3, 2x^3+3x, \sin 2x$, ... are examples of odd functions. If we actually draw the graphs of some odd functions and some even functions, we will note that graphs of even functions are symmetrical with respect to the y-axis and the graphs of odd functions are symmetrical with respect to the origin.

§ 3.1. Properties of odd and even functions.

(i) $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd.

(ii) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x) + \int_{0}^{a} f(x)\,dx.$$

In the first integral on the right side, put $x=-y$.

$$\text{Then} \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y)(-dy) = \int_{0}^{a} f(-y)\,dy.$$

$$= \int_{0}^{a} f(-x)\,dx.$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x) + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} [f(-x)+f(x)]\,dx$$

If $f(x)$ is odd, $f(-x) = -f(x)$.

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Hence, if $f(x)$ odd, $\int_{-a}^{a} f(x)\, dx = 0.$

If $f(x)$ is even, $f(-x) = f(x)$.

$$\therefore \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ if } f(x) \text{ is even.}$$

§ 3.2 These properties of odd and even functions can be used to shorten the computation when we have to find the Fourier series of either an even or odd function for the interval $-\pi < x < \pi$.

If $f(x)$ be expanded as a Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

We have $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots\ldots).$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots\ldots).$$

Case (i) $f(x)$ is an odd function, then $f(x)\cos nx$ is also an odd function.

$$\therefore \int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0.$$

Hence $a_n = 0$.

$f(x)\sin nx$ is an even function.

$$\therefore \int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2\int_{0}^{\pi} f(x)\sin nx\,dx.$$

Hence $b_n = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(x)\sin nx\,dx.$

Case (ii) If $f(x)$ is an even function, then $f(x)\sin nx$ is an odd function and hence $\int_{-\pi}^{\pi} f(x)\sin nx = 0.$

$$\therefore\ b_n = 0.$$

$f(x)\cos nx$ is an even function.

$$\therefore \int_{-\pi}^{\pi} f(x)\ \cos nx\,dx = 2\int_{0}^{\pi} f(x)\ \cos ns\,dx\,.$$

$$\therefore\ a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\ \cos\ nx\ dx.$$

Hence, we get the results that

(*i*) If $f(x)$ is an even function, $f(x)$ can be expanded as a series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ in the interval $(-\pi < x < \pi)$ where

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\ \cos nx\,dx\ (n = 0, 1, 2, \ldots\ldots);$$

(*ii*) If $f(x)$ is an odd function, $f(x)$ can be expanded as a series of the form $\sum_{n=1}^{\infty} b_n\ \sin nx$ in the interval $(-\pi < x < \pi)$ where

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\ \sin\ nx\,dx.$$

Examples.

Ex. 1. Express $f(x) = x\ (-\pi < x < \pi)$ as a Fourier series with period 2π. (Anc. '74)

$f(x) = x$ is an odd function.

Hence in the expansion, the cosine terms are absent.

= $\frac{2}{\pi}$ [-π

= ...

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$$\therefore x = \sum b_n \sin nx$$

$\frac{\cos n\pi}{n} + \frac{0 . \cos 0}{0}$ where

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$u = x$ $\quad dv = \sin nx \, dx$

$du = dx$ $\quad \int dv = v = \frac{-\cos nx}{n}$

$+ \left\{\frac{\sin nx}{n^2}\right\}_0^{\pi}$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx$$

$\int u\,dv = uv - \int v\,du$

$\pi \frac{\cos n\pi}{n}$

$$= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n$$

$(-1)^n$

$= -\frac{2}{n}$

$$= \frac{(-1)^{n+1}2}{n}$$

$$\therefore \quad x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\ldots\right)$$

Ex. 2. If $f(x) = -x$ in $-\pi < x < 0$

$$= x \quad \text{in} \quad 0 < x < \pi$$

expand $f(x)$ as a Fourier series in the interval $-\pi$ to π (Anc. '75)

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots$

We easily see that $f(x)$ is an even function. By drawing the graph of the function and noting that it is symmetrical with respect to the y-axis.

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots\ldots)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi.$$

$$a_n = \frac{2}{\pi}\int_0^\pi x \cos nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{x \sin nx}{n}\right]_0^\pi - \int_0^\pi \frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{2}{n^2\pi}\left[\cos nx\right]_0^\pi = \frac{2}{n^2\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}$$

When n is odd, $a_n = -\frac{4}{n^2\pi}$

When n is even, $a_n = 0$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

When $x = 0$, $f(x) = 0$

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

§ 4. Half range fourier series

It is often convenient to obtain a Fourier expansion of a function to hold for a range which is half the period of the Fourier series, that is to expand $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π. In the half range $f(x)$ can be expanded as a series containing cosines alone or sines alone.

The following identities are very useful in this connection :-

(i) $\int_0^{\pi} \cos mx \, dx = 0$ if m is an integer.

(ii) $\int_0^{\pi} \cos mx \cos nx \, dx = 0$ if $m \neq n$ and m and n are integers.

(iii) $\int_0^{\pi} \sin mx \sin nx \, dx = 0$ if $m \neq n$ and m and n are integers.

(iv) $\int_0^{\pi} \cos mx \cos nx \, dx = \int_0^{\pi} \cos^2 mx \, dx$ if $m = n$

$$= \frac{\pi}{2}$$

(v) $\int_0^{\pi} \sin mx \sin nx \, dx = \int_0^{\pi} \sin^2 mx \, dx$ if $m = n$.

$$= \frac{\pi}{2}$$

§ 5.1 Development in cosine series.

Let $f(x)$ be expanded as a series containing cosines only and let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{....} \quad (1)$$

If we integrate both sides of (1) between limits 0 and π, then

$$\int_0^{\pi} f(x) \, dx = \int_0^{\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} a_n \int_0^{\pi} \cos nx \, dx$$

$$= \frac{a_0 \pi}{2}.$$

$$\therefore a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$$

If we multiply both sides of the equation (1) by $\cos nx$ and integrate between 0 and π, then

$$\int_0^{\pi} f(x)\cos nx\,dx = a_n\frac{\pi}{2}.$$

Since all the terms except the term containing a_n vanish

$$\therefore\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

§ 5.2 Development in sine series

Let $f(x)$ be expanded as a series containing sines only and let

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Multiply both sides of the above equation by $\sin nx$ and integrate from 0 to π.

Then $\int_0^{\pi} f(x)\sin nx\,dx = b_n\frac{\pi}{2}$ since all the terms except the term containing b_n vanish.

$$\therefore\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Examples

Ex. 1. Find a sine series for $f(x) = c$ in the range 0 to π.

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where}\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} c \sin nx\, dx$$

$$= \frac{2c}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2c}{n\pi}(1-\cos n\pi)$$

$$= \frac{2c}{n\pi}[1-(-1)^n]$$

When n is even, $b_n = 0$.

When n is odd, $b_n = \dfrac{4c}{n\pi}$

Hence $c = \dfrac{4c}{\pi}\left(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x \ldots\ldots\right)$

Putting $x = \dfrac{\pi}{2}$, $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots\ldots$

Ex. 2. If $f(x) = x$ when $0 < x < \dfrac{\pi}{2}$

$= \pi - x$ when $x > \dfrac{\pi}{2}$

expand $f(x)$ as a sine series in the interval $(0, \pi)$.

Let $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi/2} x \sin nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos nx}{n}\,dx\right\}$$

$$+\frac{2}{\pi}\left\{\left[\frac{-(\pi-x)\cos nx}{n}\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\frac{\cos nx}{n}\,dx\right\}$$

$$= \frac{2}{n^2\pi}[\sin nx]_0^{\pi/2} - \frac{2}{n^2\pi}[\sin nx]_{\pi/2}^{\pi}$$

$$= \frac{2}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$

$$= \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

When n is even, $b_n = 0$.

When n is odd and is of the form $4p+1$, $b_n = \frac{4}{n^2\pi}$

When n is odd and is of the form $4p-1$, $b_n = -\frac{4}{n^2\pi}$

$$\therefore\ b_2 = b_4 = b_6 = \dots\dots\dots = 0.$$

$$b_1 = \frac{4}{1^2\pi},\ b_5 = \frac{4}{5^2\pi},\ b_9 = \frac{4}{9^2\pi}, \dots\dots\dots$$

$$b_3 = \frac{-4}{3^2\pi},\ b_7 = -\frac{4}{7^2\pi},$$

$$\therefore\ f(x) = \frac{4}{\pi}\left\{\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2}\dots\dots\right\}.$$

3. Find a cosine series in the range 0 to π for

$$f(x) = x \qquad 0 < x < \frac{\pi}{2}$$

$$= \pi - x \qquad \frac{\pi}{2} < x < \pi.$$

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$$\text{Let} f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx$$

$$\text{where} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$\therefore \quad a_0 = \frac{2}{\pi}\int_0^{\pi/2} x\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2} - \frac{2}{\pi}\left[\frac{(\pi-x)^2}{2}\right]_{\pi/2}^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} x\cos nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi/2} - \frac{1}{n}\int_0^{\pi/2} \sin nx\, dx\right\}$$

$$+ \frac{2}{\pi}\left\{\left[\frac{(\pi - x)\sin nx}{\pi}\right]_{\pi/2}^{\pi} + \frac{1}{n}\int_{\pi/2}^{\pi} \sin nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi/2}\right\}$$

$$+ \frac{2}{\pi}\left\{-\frac{\pi}{2n}\sin\frac{n\pi}{2} - \frac{1}{n^2}\left[\cos nx\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{n^2} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2}\cos n\pi + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1-(-1)^n + 2\cos\frac{n\pi}{2}}{n^2}\right\}.$$

When n is odd, $a_n = 0$.