$\textcircled{4} \Rightarrow \textcircled{1}$.

07.09.2016 Hence proved.

THEOREM-2:

Statement

1

A metric space M is connected iff there does not exist a continuous functn f from M onto the discrete metric space $\{0,1\}$

Pf:

Let M be connected.

Suppose $\exists$ a continuous functn f from M onto $\{0,1\}$.

Since $\{0,1\}$ is discrete. [$\because$ Every singleton in discrete m.s. is open]

We have $\{0\}$ & $\{1\}$ are open.

$A = f^{-1}(\{0\})$ & $B = f^{-1}(\{1\})$

Also A & B are open in M since f is continuous. [$\because$ Continuous $\Rightarrow$ Inverse image of open set is open] Also A & B are non-empty.

Clearly $A \cap B = \phi$ & $A \cup B = M$.

Then $M = A \cup B$ where A & B are disjoint non-empty open sets.

$\therefore$ M is not connected. which is a $\Rightarrow\Leftarrow$

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There does not exist a continuous functn f from M onto a [discrete m.s $\{0,1\}$ ~~Continuous~~ functn f from ~~M onto {0,1}~~].

Hence proved. (2)

Conversely, let there does not exist a continuous functn f from M onto $\{0,1\}$.

To P.T. M is connected.

Suppose M is not connected, then $\exists$ disjoint non-empty open sets A & B in M $\ni$

$M = A \cup B$.

We define $f : M \to \{0,1\}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B. \end{cases}$$

Clearly f is onto.

Also, $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = A$.

$f^{-1}(\{1\}) = B$, $f^{-1}(\{0,1\}) = M$.

Thus the inverse image of every openset in $\{0,1\}$ is open in M.

Hence f is continuous.

Thus $\exists$ a continuous functn $f : M$ onto $\{0,1\}$. which is a $\Rightarrow\Leftarrow$.

Hence M is not connected.

Hence proved.

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The above thm can be stated as follows

NOTE

M is connected iff every continuous functn $f : M \to \{0,1\}$ is not onto

THEOREM-3

Statement

Let M be a metric space. Let A be a connected subset of M. If B is subset of M $\ni: A \subseteq B \subseteq \bar{A}$ then B is connected

Pf:

Given A is connected & B is subset of M $\ni: A \subseteq B \subseteq \bar{A}$

To P.T. B is connected.

Suppose B is not connected.

Then $B = B_1 \cup B_2$ where $B_1 \neq \phi$, $B_2 \neq \phi$ & $B_1 \cap B_2 = \phi$ & B_1 & B_2 are open sets

Since B_1 & B_2 are open sets $\exists$ G_1 & G_2 $\ni: B_1 = G_1 \cap B$, $B_2 = G_2 \cap B$.

$B = B_1 \cup B_2$.

$B = (G_1 \cap B) \cup (G_2 \cap B)$

$= (G_1 \cup G_2) \cap B$.

$B \subseteq G_1 \cup G_2$.

W.K.T $A \subseteq B$.

$\therefore A \subseteq G_1 \cup G_2$

Then $A = (G_1 \cup G_2) \cap A$

$= (G_1 \cap A) \cup (G_2 \cap A)$

$\therefore$ Now $(G_1 \cap A)$ & $(G_2 \cap A)$ are open in A.

[we need to see the disjoint ∴ we find the intersection]

Also, $(G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A$

$= (G_1 \cap G_2) \cap B$ $\quad (\because A \subseteq B)$

$= (G_1 \cap B) \cap (G_2 \cap B) = B_1 \cap B_2 = \phi$

$(G_1 \cap A) \cap (G_2 \cap A) = \phi$.

Let us assume either $G_1 \cap A = \phi$ (or)

$G_2 \cap A = \phi$.

Clearly we assume $G_1 \cap A = \phi$.

Then $A \subseteq G_1^c$

Since G_1 is open G_1^c is closed.

Also G_1^c is a closed set

Containing A we know that $\bar{A}$ is the

Smallest closed set Containing A.

$\therefore \bar{A} \subseteq G_1^c$.

$(A \subseteq \bar{A})$ $\Downarrow$ $\bar{A}$ is smaller $\therefore \bar{A} \subseteq G_1^c$

$G_1 \cap \bar{A} \subseteq G_1 \cap G_1^c$

$G_1 \cap \bar{A} \subseteq \phi$.

$G_1 \cap \bar{A} = \phi$.

Since $B \subseteq \bar{A}$. We have, $G_1 \cap B = \phi$

$B_1 = \phi$ which is a $\Rightarrow\Leftarrow$.

Since $B_1 \neq \phi$

$\therefore B$ is connected

Hence proved.

09.09.2016

THEOREM - 4

Statement *

Let A & B are connected subset of a m.s. M of iff $A \cap B \neq \phi$. Then P.T. $A \cup B$ is connected.

Proof:

Given A & B are connected. & $A \cap B \neq \phi$.

To P.T. $A \cup B$ is connected.

Suppose $A \cup B$ is not connected. Then $\exists$ a continuous onto functn $f : A \cup B \to \{0, 1\}$. [From Note]

Since $A \cap B \neq \phi$ we have $x_0 \in A \cap B$.

Then $x_0 \in A \cup B$.

$\therefore f(x_0) = 0$ (or) 1.

Case(i):

Let $f(x_0) = 0$.

Consider the restricted map.

(f restricted A)

$f/A : A \to \{0, 1\}$. (do not change codomain)

Since f is continuous.

Given A is connected.

(6)

$\therefore f|A$ is not onto.

$\therefore f|A(x)=0 \ \forall \ x \in A$ (or)

$f|A(x)=1 \ \forall \ x \in A$.

Let $x_0 \in A$ & $f(x_0)=0$.

$\therefore f|A(x)=0 \ \forall \ x \in A$.

$\therefore f(x)=0 \ \forall \ x \in A$ ——— (1).

$f|B: B \to \{0,1\}$ is continuous.

Proceeding like above,

$f(x)=0 \ \forall \ x \in B$ ——— (2).

From (1) & (2),

$\Rightarrow f(x)=0 \ \forall \ x \in A \cup B$.

$\therefore f$ is not onto, which is a $\Rightarrow\Leftarrow$.

$\therefore A \cup B$ is connected.

Case (ii):

Let $f(x_0)=1$.

By similar argument as abv we can get,

$f(x)=1 \ \forall \ x \in A \cup B$.

$\therefore$ It is not onto which is a $\Rightarrow\Leftarrow$

$\therefore A \cup B$ is connected. Hence Proved.

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15-09-2016

THEOREM-5

Statement

P.T. A subspace of R is connected iff it is an interval.

Proof

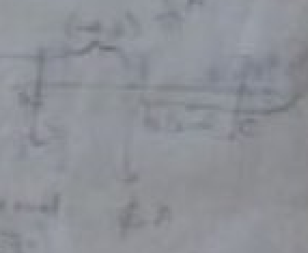
Let A be a connected subspace of R.

To P.T A is an interval

Suppose A is not an interval then $\exists$ $a, b, c \in R$ $\ni$ $a < b < c$ & $a, c \in A$ and $b \notin A$

Let $A_1 = (-\infty, b) \cap A$ and

$A_2 = (b, \infty) \cap A$



Since $(-\infty, b)$ and (b, ∞) are open in R, we have A_1 and A_2 are open sets in A

Also $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = A$

Further $a \in A_1$ and $c \in A_2$

Hence $A_1 \neq \phi$ and $A_2 \neq \phi$

Thus A is the union of two disjoint non-empty open sets A_1 and A_2.

$\therefore$ A is not connected. which is a $\Rightarrow\Leftarrow$

$\therefore$ A is an interval.

Hence proved.

Conversely, let A be an interval.

To P.T. A is connected. (8)

Suppose A is not connected

Then $A = A_1 \cup A_2$ where $A_1 \neq \phi$, $A_2 \neq \phi$, $A_1 \cap A_2 = \phi$ and A_1 and A_2 are closed sets.

Choose $x \in A_1$ and $z \in A_2$.

Since $A_1 \cap A_2 = \phi$ we have,

$$x \neq z$$

We assume that $x < z$.

Since A is an interval we have,

$$[x, z] \subseteq A.$$

$$[x, z] \subseteq A_1 \cup A_2$$

$\therefore$ Every element of ~~closed inter~~ $[x, z]$ is either in A_1 (or) A_2

Now, $y = \text{lub}\{[x, z] \cap A_1\}$

$= \sup$ (only for our ref)

($\because$ If S is a non-empty set of all Real numbers then M is said to be the lub of S if $a \leq M \ \forall \ a \in S$.)

$$x \in A_1, \ x \in [x, z].$$

$$\therefore x \in [x, z] \cap A_1$$

Since y is the lub we have,

$$x \leq y.$$

$\text{ly}\ \ y \le x.$

ie) $x \le y \le z.$

Hence, $y \in A$

27-09-2016

Let $\epsilon > 0$ be given then by the defn of lub $\exists\ t \in [x,z] \cap A_1 \ni: y-\epsilon < t < y$

Since $\epsilon > 0$ we have,

$\cancel{y+\epsilon < y}\quad y+\epsilon > y$

$\therefore\ y-\epsilon < t < y < y+\epsilon$ where $t \in (y-\epsilon, y+\epsilon)$

Then, $t \in (y-\epsilon, y+\epsilon) \cap ([x,z] \cap A_1)$

$(y-\epsilon, y+\epsilon) \cap ([x,z] \cap A_1) \neq \phi$

$B(y,\epsilon) \cap ([x,z] \cap A_1) \neq \phi$

By the thm which states that

$x \in \bar{A}$ iff $B(x,r) \cap A \neq \phi$

$\therefore\ y \in \overline{[x,z] \cap A_1}$

$y \in [x,z] \cap A_1 \qquad [\because A_1 = \bar{A_1}]$

ie) $y \in A_1$

By the defn of y,

$y+\epsilon \in A_2 \ni:\ y+\epsilon \le z$

Every open ball of y contains a pt of A_2 different from y.

$(y-\epsilon, y+\epsilon) \cap A_2 - \{y\} \neq \phi$

$(y-\epsilon, y+\epsilon) \cap A_2 \neq \phi$

$B(y,\epsilon) \cap A_2 \neq \phi$

$y \in \overline{A_2}$

$y \in A_2$ $[\because A_2 = \bar{A}]$ (10)

$\therefore y \in A_1 \cap A_2$ which is a $\Rightarrow \Leftarrow$

$A_1 \cap A_2 = \phi$.

$\therefore$ A is connected. Hence proved.

THEOREM-6:

Statement:

P.T. R is connected (write converse part only in thm-6)

Proof:

$R = (-\infty, \infty)$ is an interval

By the prev thm R is connected.

Hence proved.

THEOREM-7:

Statement

If A and B are connected subset

CONNECTEDNESS AND CONTINUITY

THEOREM-7:

Statement

Let M_1 be a connected m.s. Let M_2 be any m.s. Let $f: M_1 \rightarrow M_2$ be a continuous functn. Then P.T. $f(M_1)$ is a connected subset of M_2

(or)

P.T. any continuous image of a connected set is connected.

Proof:

Let $f(M_1) = A$. (11)

Let $f: M_1 \to M_2$ is continuous.

To P.T A is connected.

Suppose A is not connected.

Then $\exists$ a proper non-empty subset B of A which is both open and closed.

Since B is open and closed and f is continuous we have, [∵ From thm-1]

$f^{-1}(B)$ is a proper non-empty subset of M_1 which is both open and closed in M_1

Hence M_1 is not connected. which is a $\Rightarrow\Leftarrow$

∴ A is connected.

∴ $f(M_1)$ is connected. Hence proved.

THEOREM-8:

INTERMEDIATE VALUE THEOREM

Statement

Let f be a real valued continuous functn defined on an interval I. Then f takes every value b/w any two values it assumes.

Proof:

Let $a, b \in I$.

28.09.16 Let $f(a) \neq f(b)$.

without lose of generality we assume that

$f(a) < f(b)$. 12

Let c be $\ni$: $f(a) < c < f(b)$

Since I is an interval it is connected.

Then $f(I)$ is a connected subset of R.

$\therefore$ $f(I)$ is an interval.

Also $f(a), f(b) \in f(I)$.

Hence, $[f(a), f(b)] \subseteq f(I)$

$\therefore$ $c \in f(I)$. $[\because f(a) < c < f(b)]$

$\therefore$ $c = f(x)$

For some $x \in I$ $\therefore$ f takes every value b/w any 2 values it assume Hence proved.

PROBLEMS:

1) P.T. if f is a non-constant real valued continuous functn on R then the range of f is uncountable.

Sol:

W.K.T R is connected.

Since f is continuous on R, $f(R)$ is connected.

[$f(R)$ is an interval.] 13

$\therefore f(R)$ is an interval.

Since f is a non-constant functn, the interval $f(R)$ contains more than one pt.

$\therefore f(R)$ is uncountable. ie) the range of f is uncountable. Hence proved.

2) Give an example to S.T a subspace of a connected m.s. need not be connected.

Sol:

W.k.T. R is connected.

$A = [1,2] \cup [3,4]$ is a subspace of R which is not connected.

Hence proved.

3) Prove or disprove if A and C are connected subsets of a m.s. M and if $A \subseteq B \subseteq C$ then B is connected.

Sol:

We disprove this statement by an example.

Let $A = [1,2]$, $B = [1,2] \cup [3,4]$, ~~and [2,~~
$C = R$.

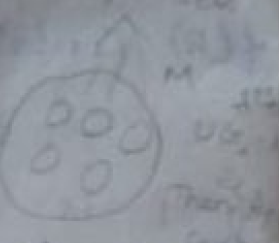
Clearly $A \subseteq B \subseteq C$.

Here A and C are connected and B is not connected. Hence it is disproved.

(14)

UNIT-5

COMPACTNESS



Defn:

OPEN COVER:

→ Let M be a m.s.. A family of open sets $\{G_\alpha\}$ in M is called an open cover for M if $\cup G_\alpha = M$.

→ A sub family of $\{G_\alpha\}$ which itself is an open cover is called a sub cover. (some sets are dropped) (take finitely)

COMPACT

Defn:

A m.s M, is said to be compact if every open cover for M has a finite subcover.

ie) for each family of open sets $\{G_\alpha\}$ $\ni$: $\cup G_\alpha = M$ $\exists$ a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ $\ni$: $\bigcup_{i=1}^{n} G_{\alpha_i} = M$

eg:

i) P.T. R with usual metric is not compact.

Sol:

Consider the family of open interval $\{(-n,n) / n \in N\}$.

(15)

This is a family ~~ope~~ of open sets in R.

Clearly, $\bigcup_{n=1}^{\infty} (-n,n) = R$ $[\because R = (-\infty, \infty)]$

$\therefore$ ~~Colle~~ $\{(-n,n) / n \in N\}$ is an open cover for R. & this open cover has no finite sub cover.

$\therefore$ R is not compact.

ii) P.T. $(0,1)$ with usual metric is not Compact.

Sol:

Consider the ~~follow~~ family of open intervals $\{(1/n, 1) / n = 2, 3, \ldots\}$.

Clearly, $\bigcup_{n=2}^{\infty} (1/n, 1) = (0,1)$

$\{(1/n, 1) / n = 2, 3, \ldots\}$ is an open cover for $(0,1)$ and this open cover has no finite subcover.

$\therefore (0,1)$ is not compact.

iii) P.T. $[0, \infty)$ with usual metric is not Compact.

Sol:

Consider the family of intervals $\{[0,n) / n \in N\}$.

$[0,n)$ is open in $[0, \infty)$.

Also $\bigcup_{n=1}^{\infty} [0,n) = [0, \infty)$

$\therefore \{[0,n) / n \in N\}$ is an open cover for $[0,\infty)$ and this open cover has no finite sub cover.

16

Hence $[0,\infty)$ is not compact.

iv) Let M be an infinite set with discrete metric then P.T. M is not compact.

Sol:

Let $x \in M$.

Since M is a discrete m.s. $\{x\}$ is open in M.

Also, $\bigcup_{x \in M} \{x\} = M$.

Hence, $\{\{x\} / x \in M\}$ is an open cover for M

Since M is ~~infu~~ infinite this open cover has no finite subcover

Hence M is not compact.

THEOREM-1:

Statement

Let M be a m.s, let $A \subseteq M$. P.T. A is compact iff given a family of open sets $\{G_\alpha\}$ in M $\ni: \bigcup G_\alpha \supseteq A$ $\exists$ a sub family $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ $\ni: \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

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Proof:

Let A be a compact subset of M.

Let $\{G_\alpha\}$ be a family of open sets in M $\ni$ $\cup G_\alpha \supseteq A$.

To P.T. $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

$\cup G_\alpha \supseteq A \Rightarrow \cup G_\alpha \cap A = A$

$\cup(G_\alpha \cap A) = A$.

Also $G_\alpha \cap A$ is open in A.

$\therefore$ The family $\{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact this open cover has a finite subcover $\ni$ $\bigcup_{i=1}^{n}(G_{\alpha_i} \cap A) = A$

$\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A = A$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$. Hence proved.

Conversely,

To P.T. A is compact

Let $\{H_\alpha\}$ be an open cover for A.

ie) Each H_α is open in A.

$\therefore H_\alpha = G_\alpha \cap A$ where G_α is open in M.

$\cup H_\alpha = A$.

$\cup(G_\alpha \cap A) = A$

$(\cup G_\alpha) \cap A = A$

$\cup G_{\alpha} \supseteq A$.

By hypothesis,

$$\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$$

$$\bigcup_{i=1}^{n} G_{\alpha_i} \cap A = A.$$

$$\bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A.$$

$$\bigcup_{i=1}^{n} H_{\alpha_i} = A.$$

$\therefore \{H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}\}$ is a finite Subcover. for A.

$\therefore$ A is compact. Hence proved.

03.10.2016

THEOREM-2:

Statement

v.v. IMportant points (combine thm-2&3)

P.T. any Compact subset A of a m.s M is bounded.

Proof:

Let A be a compact subset of a m.s. M.

To P.T. A is bounded.

Let $x_0 \in A$.

→ family of open sets (many sets) (countable)

Consider $\{B(x_0, n) / n \in N\}$.

Clearly $\bigcup_{n=1}^{\infty} B(x_0, n) = M$

Point x_0 radius n

$\therefore \bigcup_{n=1}^{\infty} B(x_0, n) \supseteq A$.

Since A is compact $\exists$ a finite subfamily

s: $\bigcup_{i=1}^{k} B(x_0, n_i) \supseteq A$. (19)

Let $n_0 = \max\{n_1, n_2, \dots, n_k\}$.

Then $\bigcup_{i=1}^{k} B(x_0, n_i) = B(x_0, n_0)$

$\therefore B(x_0, n_0) \supseteq A$ [$\because$ Every open ball is bounded set]

W.K.T. $B(x_0, n_0)$ is a bounded set.

and a subset of a bounded set is bounded.

$\therefore A$ is bounded. Hence proved.

NOTE:

The converse of the abv thm is not true.

eg:

$(0,1)$ is a bounded subset of R but it is not compact. (In ex. ii)

THEOREM-3

Statement

P.T. any compact subset A of a m.s M is closed.

Proof:

Let A be a compact subset of M.

To p.T. A is closed

i.e) to p.T. A^c is open.

Let $y \in A^c$ and $x \in A$

Then; $x \neq y$. (distinct elements A and A^c) (distinct)

Then $d(x,y) = r_x > 0$.

To P.T $B(x, \frac{1}{2}r_x) \cap B(y, \frac{1}{2}r_x) = \phi$ (20)

Suppose $B(x, \frac{1}{2}r_x) \cap B(y, \frac{1}{2}r_x) \neq \phi$

Let $z \in B(x, \frac{1}{2}r_x) \cap B(y, \frac{1}{2}r_x)$

Then $z \in B(x, \frac{1}{2}r_x)$ and $z \in B(y, \frac{1}{2}r_x)$

$d(x,z) < \frac{1}{2}r_x$ and $d(y,z) < \frac{1}{2}r_x$.

By triangle inequality.

$d(x,y) \leq d(x,z) + \overset{d(z,y)}{\cancel{d(y,z)}}$. [M is m.s]

$\leq d(x,z) + d(y,z)$

$< \frac{1}{2}r_x + \frac{1}{2}r_x$.

$d(x,y) < r_x$ which is a $\Rightarrow\Leftarrow$

Since $d(x,y) = r_x$

$\therefore B(x, \frac{1}{2}r_x) \cap B(y, \frac{1}{2}r_x) = \phi$

Consider $\{B(x, \frac{1}{2}r_x) / x \in A\}$.

Clearly $\cup B(x, \frac{1}{2}r_x) \supseteq A$

Since A is compact $\exists$ a finite Sub family $\ni$: $\bigcup_{i=1}^{n} B(x_i, \frac{1}{2}r_{x_i}) \supseteq A$ ——— (1)

Let $V_y = \bigcap_{i=1}^{n} B(y, \frac{1}{2}r_{x_i})$

(centre is y)

Cleary V_y is an open set containing y.

$\because B(x, \frac{1}{2} r_x) \cap B(y, \frac{1}{2} r_x) = \phi$ we have,

$V_y \cap B(x, \frac{1}{2} r_{x_i}) = \phi$

(21)

$V_y \cap \left[\bigcup_{i=1}^{n} B(x_i, \frac{1}{2} r_{x_i}) \right] = \phi$

$V_y \cap A = \phi$ $(\because$ by ①$)$.

$V_y \subseteq A^c$

$\cancel{\bigcup_{y \in A^c} V_y = A^c}$.

$\bigcup_{y \in A^c} V_y = A^c$.

~~$\therefore A$~~ $\because V_y$ is open, A^c is open.

Hence A is closed. Hence proved.

NOTE:

i) The converse of the abv thm is not true.

eg:

$[0, \infty)$ is a closed set but it is not compact.

THEOREM-4:

State: ii) P.T. any compact subset of a m.s is closed and bounded.

Pf:

Write the abv 2 thms

(combined them).

THEOREM-5:

Statement

P.T. A closed subset of a Compact m.s is compact.

Pf:

Let M be a compact m.s.

Let A be a closed subset of M.

To P.T A is compact.

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in M $\ni$: $\cup G_\alpha \supseteq A$.

$$A^c \cup \left(\bigcup_{\alpha \in I} G_\alpha\right) = M$$

$A \cup A^c$

Since A is closed, A^c is open.

$\therefore \{G_\alpha / \alpha \in I\} \cup A^c$ is an open cover for M.

Since M is compact it has a finite subcover $\ni$: $\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cup A^c = M$.

$$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$$

$\therefore A$ is compact. Hence proved.

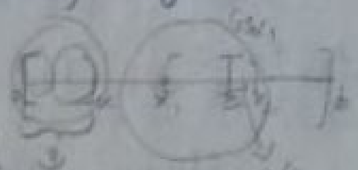
THEOREM-6: HEINE BOREL THEOREM

Statement

Any closed interval $[a,b]$ is a compact subset of R.

Proof:

Let $\{G_\alpha / \alpha \in I\}$ be a family of open sets in $R \ni$: $\bigcup_{\alpha \in I} G_\alpha \supseteq [a,b]$.



Define $S = \{x / x \in [a,b]$ & $[a,x]$ can be covered by a finite no. of G_α's$\}$

Clearly $a \in S$ hence $S \neq \phi$.

Also S is bounded above by b

Let c denote the ~~lub{f(S)}~~ lub of S. (upper limit is b)

Clearly $c \in [a,b]$.

$\therefore c \in G_{\alpha_1}$ for some $\alpha_1 \in I$.

Since G_{α_1} is open $\exists_1 \ \varepsilon > 0 \ \ni$: $B(c,\varepsilon) \subseteq G_{\alpha_1}$

$\Rightarrow (c-\varepsilon, c+\varepsilon) \subseteq G_{\alpha_1}$.

Choose $x_1 \in [a,b] \ni$:

$x_1 < c$ & $[x_1, c] \subseteq G_{\alpha_1}$

Since $x_1 < c$ we have $[a, x_1]$ can be covered by a finite no. of G_α.

These finite no. of G_α together with G_{α_1} covers $[a,c]$.

04.10.2016

By defn of S, $c \in S$.

We claim that $c = b$.

Suppose $c \neq b$, then choose $x_2 \in [a,b]$

$\exists\ x_2 > c$ & $[c, x_2] \subseteq G_{\alpha_1}$.

As before $[a, x_2]$ can be covered by a finite no. of G_α.

Hence $x_2 \in S$. But $x_2 > c$ which is a contradiction. Since c is the lub of S.

Hence $c = b$.

$[a, b]$ can be covered by a finite no. of G_α.

$\therefore [a, b]$ is compact. Hence proved.

THEOREM-7:

Statement

P.T. a subset A of R is compact iff it is closed and bounded.

Proof:

Let A be a compact subset of R.

To P.T. A is closed and bounded.

By the prev. thms (3 & 4), A is bounded and closed. Hence proved.
(write the thms)

Let A be a closed and bounded subset of R.

To P.T. A is compact.

Since A is bounded, we can find a closed interval $[a, b]$ $\exists$: $A \subseteq [a, b]$

W.K.T. $[a,b]$ is compact. (25)

Since A is a closed subset of a compact m.s. $[a,b]$ we have A is compact.

[∵ From thm-5]

Hence proved.

Defn:

script f

* A family $\mathcal{F}$ of subset of a set M is said to have finite intersection property if any finite members of $\mathcal{F}$ have non-empty intersection.

(F.C)-7

THEOREM-8:

Statement:

P.T. a m s M is compact iff any family of closed sets with finite intersection property has non-empty intersection.

Proof:

Let M is compact.

Let $\{A_\alpha\}$ be a family of closed subsets of M with finite intersection property.

To P.T. $\cap A_\alpha \neq \phi$

Suppose $\cap A_\alpha = \phi$

$(\cap A_\alpha)^c = \phi^c$

$(\cap A_\alpha)^c = M$

$\cup A_\alpha^c = M$

Since each A_α is closed A_α^c is open

$\therefore \{A_\alpha^c\}$ is an open cover for M.

Since M is compact, this open cover has a finite sub cover $\ni \bigcup_{i=1}^{n} A_{\alpha_i}^c = M$

$$\left[\bigcup_{i=1}^{n} A_{\alpha i}^c\right]^c = M^c$$

$\bigcap_{i=1}^{n} A_{\alpha_i} = \phi$. which is a $\Rightarrow\Leftarrow$

to the finite intersection property.

$\therefore \cap A_\alpha \neq \phi$. Hence proved.

Conversely,

Suppose that each family of closed sets in M with finite intersection property has non-empty intersection. $[\because \bigcap_{i=1}^{n} G_\alpha^c \neq \phi \Leftrightarrow \cap G_\alpha \neq \phi$

To P.T. M is compact.

Let $\{G_\alpha / \alpha \in I\}$ be an open cover for M.

ie) $\bigcup_{\alpha \in I} G_\alpha = M$

$$\left[\bigcup_{\alpha \in I} G_\alpha\right]^c = M^c$$

$$\bigcap_{\alpha \in I} G_\alpha^c = \phi$$

Since G_α is open G_α^c is closed.

$\therefore \{G_\alpha^c\}$ is a family of closed sets whose intersection is empty

By hypothesis, this family of closed sets does not have finite intersection property. (27)

Hence $\exists$ a ~~subset~~ subcollection of G_{α}^{c} $\ni$: $\bigcap_{i=1}^{n} G_{\alpha_i}^{c} = \phi$.

$$\left[\bigcap_{i=1}^{n} G_{\alpha_i}^{c}\right]^{c} = \phi^{c}$$

$$\bigcup_{i=1}^{n} G_{\alpha_i} = M$$

$\therefore$ M is Compact. Hence proved.

05-10-2016

Defn

* A m.s. M is said to be totally bounded if for every $\varepsilon>0$ $\exists$ a finite no. of elements $x_1, x_2, \ldots, x_n \in M$ $\ni$: $B(x_1,\varepsilon) \cup B(x_2,\varepsilon) \ldots \cup B(x_n,\varepsilon) = M$

THEOREM-9

Statement

P.T any compact m.s is totally bounded.

Proof

Let M be a Compact m.s.

Then $\{B(x,\varepsilon) / x \in M\}$ is an open cover for M.

Since M is compact this open cover has a finite sub cover.

ie) $B(x_1,\varepsilon) \cup B(x_2,\varepsilon) \ldots \cup B(x_n,\varepsilon) = M$.

$\therefore$ M is totally bounded. Hence proved.

V.V.I (Its converse not true)

THEOREM-10

Statement

Let A be a subset of a m.s M. If A is totally bounded then P.T A is bounded.

Proof:

Let A be a totally bounded subset of M.

To P.T. A is bounded.

Since A is totally bounded we have,

$$B(x_1, \epsilon) \cup B(x_2, \epsilon) \dots \cup B(x_n, \epsilon) = A.$$

W.K.T. every open ball is ~~an~~ a bounded set.

Also union of finite no. of bounded sets is bounded.

$\therefore$ A is bounded. Hence proved.

NOTE

The converse of the abv. thm is not true.

eg:

Let M be an infinite set with discrete metric.

ie) $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$\therefore d(x,y) \leq 1$

$\therefore$ M is bounded.

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$B(x, 1/2) = \{ y \in M / d(x,y) < 1/2 \}$

If $x = y$.

Then $d(x,y) = 0$

$\therefore 0 < 1/2$ is true.

If $x \neq y$.

Then $d(x,y) = 1$.

$\therefore 1 < 1/2$ is not true.

$\therefore B(x, 1/2) = \{x\}$

(This ball contains only one pt)

Since M is infinite, M can't be written as the union of finite no. of open balls.

$\therefore M$ is not totally bounded.

Defn:

* Let (x_n) be a sequence in a m.s. M. Let $n_1 < n_2 \ldots < n_k \ldots$ be an increasing sequence of +ve integers then, (x_{n_k}) is called a Subsequence of (x_n).

(P.B)-10

* A m.s. M is said to be Sequentially compact if every sequence in M has a convergent subsequence.

THEOREM-1 (P.B)-9

Statement

Let (x_n) be a cauchy sequence in a m.s. M. if (x_n) has a subsequence (x_{n_k}) converging to x then p.T. $(x_n) \to x$.

If $B(x_1, \epsilon) = M$, then we can say that M is totally bounded.

If $B(x_1, \epsilon) \neq M$, then choose $x_2 \in M - B(x_1, \epsilon)$ so that $d(x_1, x_2) \geq \epsilon$.

If $B(x_1, \epsilon) \cup B(x_2, \epsilon) = M$.

Then, $\therefore$ M is totally bounded.

If not then choose ~~$x_3 \in M - B$~~ $x_3 \in M - B(x_1, \epsilon) \cup B(x_2, \epsilon)$ and so on.

Suppose, this process does not stop at finite stage then we obtain a sequence $x_1, x_2, \ldots, x_n, \ldots \ni : d(x_n, x_m) \geq \epsilon$.

Clearly, this (x_n) can not have a cauchy ~~seq~~ subsequence which is a $\Rightarrow\Leftarrow$ to our hypothesis.

Hence the abv process stop at a finite stage & hence we get a finite no. of pts $x_1, x_2, \ldots, x_n \ni :$
$B(x_1, \epsilon) \cup B(x_2, \epsilon) \ldots \cup B(x_n, \epsilon) = M$.

$\therefore$ M is totally bounded. Hence proved.

7-10-2016

Conversely,

Suppose M is totally bounded.

To p.T. every sequence in M has a

Cauchy subsequence

Let $S_1 = \{x_{i_1}, x_{i_2}, \dots, x_{i_n}, \dots\}$ be a seq. in M.

If one term of the seq is infinitely repeated, then S_1 contains a const seq, (repeated terms) which is obviously a cauchy seq.

Hence we assume that no term of S_1 in infinitely repeated.

Since M is totally bounded, M can be covered by a finite no. of open balls of radius $1/2$.

Hence atleast one of these balls must contain an infinite no. of terms of the seqns.

$\therefore S_1$ contains a subsequences

$S_2 = \{x_{21}, x_{22}, \dots, x_{2n}, \dots\}$

All the terms of S_2 lie within the ball of radius $1/2$.

IIIly S_2 contains a subsequence

$S_3 = \{x_{31}, x_{32}, \dots, x_{3n}, \dots\}$.

All the terms of S_3 lie within the ball of radius $1/3$.

We repeat the process & we take the diagonal seqns.

$S = \{x_{11}, x_{22}, \dots, x_{nn}, \dots\}$.

We claim that S is a cauchy subsequence of S_1.

If $m > n$, both x_{mm} & x_{nn} lie within the open ball of radius $1/n$.

$\therefore d(x_{mm}, x_{nn}) < 2/n$.

$d(x_{mm}, x_{nn}) < \epsilon$.

$\therefore S$ is a cauchy subsequence of S_1.

Thus every seq in M has a cauchy subseq. Hence proved.

NOTE:

A non-empty subset of a totally bounded ~~set~~ is totally bounded.

Proof:

Let A be a totally bounded subset of M.

Let B be a non-empty subset of A.

Let seq (x_n) be a seq. in B.

Then (x_n) is a seq. in A also.

Since A is totally bounded (x_n) has a cauchy subseq.

$\therefore B$ is totally bounded. Hence proved.

THEOREM:- 13:

Statement

In a m.s. M, P.T. the following are equivalent. i) M is compact.

ii) Any infinite subset of M has a limit pt.
iii) M is sequentially compact.
iv) M is totally bounded & complete.

Proof:

(i) ⇒ (ii)

Given M is compact

To P.T. any infinite subset of M has a limit pt

Let A be a infinite subset of M.

Suppose A has no limit pt in M.

Let $x \in M$, Since x is not a limit pt of A ∃ an open ball ∋:

$$B(x, r_x) \cap A - \{x\} = \phi.$$

$$\therefore B(x, r_x) \cap A = \begin{cases} \{x\} & \text{if } x \in A \\ \phi & \text{if } x \notin A \end{cases}$$

$\{B(x, r_x) / x \in M\}$ is an open cover for M. Also each $B(x, r_x)$ covers atmost one pt of the infinite set of A.

∴ This open cover cannot have a sub cover which is a ⇒⇐.

Since M is compact.

∴ A has a limit pt.

(ii) ⇒ (iii)

Let A be an infinite subset of M which has a limit pt.

To P.T. M is sequentially compact.

Let (x_n) be a seq, in M.

If one term of the seqns is infinitely repeated, then (x_n) contains a constant subsequences which is convergent. (35)

Otherwise (x_n) has a infinite no. of terms. By hypothesis, this infinite set has a limit pt x.

For any radius $r>0$, the open ball $B(x,r)$ contains infinite no. of terms of the (x_n).

Choose a +ve integer $n_1 \ni: x_{n_1} \in B(x,1)$.

Then choose $n_2 > n_1 \ni: x_{n_2} \in B(x, 1/2)$.

In general, $x_{n_k} \in B(x, 1/k)$.

Hence (x_{n_k}) is a subseq, of (x_n).

$x_{n_k} \in B(x, 1/k)$.

$\therefore d(x_{n_k}, x) < 1/k < \epsilon$.

$\therefore (x_{n_k}) \to x$.

Hence (x_{n_k}) is a convergent subseq, of (x_n).

$\therefore M$ is sequentially compact.

(iii) $\Rightarrow$ (iv)

Let M be a sequentially compact.

To P.T. M is ~~tot~~ totally complete & bounded.

By the defn, of ~~su~~ subsequentially

Compact, every sequences in M has a Converg. Subsequence. (3)

Also, W.K.T. every convergence seq is a cauchy seq.

By the thm, which states that M is totally bounded iff every seqns in M has a cauchy subseq,

$\therefore$ M is totally bounded.

To prove M is complete.

Let (x_n) be a cauchy seq, in M.

By hypothesis (x_n) has a convergent Subseq, (x_{n_k}).

Let $(x_{n_k}) \to x$ then (x_n) also converges to x.

$\therefore$ M is complete.

(iv) $\Rightarrow$ (i)

Let M is totally bounded & Complete.

To P.T. M is compact.

Suppose M is not Compact.

Thus $\exists$ an open cover $\{G_\alpha\}$ for M which has no finite Subcover.

Let $r_n = 1/2^n$.

Since M is totally bounded M can be covered by a finite no. of open balls of radius r_1.

Since M is not compact. M can't

be covered by a finite no of G_α's

i.e) Atleast one of these balls say $B(x_1, r_1)$ cannot be covered by a finite no. of G_α's.

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$B(x_1, r_1)$ is totally bounded. Hence we can find $x_2 \in B(x_1, r_1) \ni B(x_2, r_2)$ cannot be covered by a finite no. of G_α's

Proceeding like this we obtain a seq (x_n) in $M \ni B(x_n, r_n)$ cannot be covered by finite no. of G_α's and $x_{n+1} \in B(x_n, r_n)$

$$d(x_n, x_{n+p}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+(p-1)}, x_{n+p})$$

$$\le r_n + r_{n+1} + \cdots + r_{n+p-1}$$

$$\le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}}$$

$$\le \frac{1}{2^{n-1}}\left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^p}\right]$$

$$< \frac{1}{2^{n-1}} < \varepsilon.$$

$\therefore (x_n)$ is a cauchy sequence.

Since M is complete $(x_n) \to x$.

1.10.14

Let $x \in G_\alpha$ for some α.

Since G_α is open, we get $B(x, \varepsilon) \subseteq G_\alpha$ —①

Since $(x_n) \to x$, we get $d(x_n, x) < \varepsilon/2$

and $r_n = \frac{1}{2^n} \to 0$.

We claim that $B(x_n, r_n) \subset B(x, \epsilon)$.

Let $y \in B(x_n, r_n)$

$d(x_n, y) < r_n < \epsilon/2$.

$d(x,y) \leq d(x, x_n) + d(x_n, y)$

$< \epsilon/2 + \epsilon/2$.

$d(x, y) < \epsilon$

$\therefore y \in B(x, \epsilon)$.

$\therefore B(x_n, r_n) \subseteq B(x, \epsilon)$.

By ①, $B(x_n, r_n) \subseteq G_\alpha$

(ie) $B(x_n, r_n)$ is covered by a single set G_α

which is $\Rightarrow\Leftarrow$

Since $B(x_n, r_n)$ cannot be covered by a finite no. of G_α's.

Hence M is compact. Hence proved.

13.10.2016.

THEOREM-14:

State:

P.T. R with usual metric is complete

Pf:

Let (x_n) be a cauchy seq in R. [Every Cauchy seq is bounded]

Then, (x_n) is a bounded seq. and hence it is contained in a closed interval $[a, b]$.

W.K.T. $[a, b]$ is compact. (Heine Borel thm)

By the prev. thm $[a, b]$ is totally bounded and complete.

$a[(x_n)]b$

$\therefore [a, b]$ is complete

Hence $(x_n) \to x$.

Thus every Cauchy seq. (x_n) in R converges.

Hence R is Complete. Hence proved.

THEOREM-15:

State

** P.T. the closure of a totally bounded set is totally bounded.

Pf:

Let A be a totally bounded subset of a m.s. M.

To P.T. $\bar{A}$ is totally bounded.

We shall P.T. $\bar{A}$ contains a Cauchy subseq.

Let (x_n) be a seq. in $\bar{A}$.

Since $(x_n) \in \bar{A}$, we have,

$$B(x_n, \varepsilon/3) \cap A \neq \phi.$$

Choose $y_n \in B(x_n, \varepsilon/3) \cap A$.

$$d(x_n, y_n) < \varepsilon/3 \text{ --- } ①$$

Let (y_n) be a seq in A

Since A is totally bounded (y_n) contains a Cauchy subsequence (y_{n_k})

ie) $d(y_{n_i}, y_{n_j}) < \varepsilon/3$ --- ②

$$d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, y_{n_j}) + d(y_{n_j}, x_{n_j})$$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ ($\because$ By ① & ②).

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$< \varepsilon$

$d(x_{n_i}, x_{n_j}) < \varepsilon$

$\therefore (x_{n_k})$ is a cauchy subseq. of (x_n).

ie) Every seq in $\bar{A}$ contains a cauchy subseq

$\therefore \bar{A}$ is totally bounded. Hence proved.

(Continuation in D.E. Note)

Name: S.K. KANCHANA

Proof:

Let (x_n) be a cauchy seq. in M.

Let $\epsilon > 0$ be given. ~~then~~

Then, $d(x_n, x_m) < \epsilon/2$.

Also given $(x_{n_k}) \to x$.

Then, $d(x_{n_k}, x) < \epsilon/2$.

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$$

$$< \epsilon/2 + \epsilon/2.$$

$$d(x_n, x) < \epsilon$$

$\therefore (x_n) \to x$. Hence proved.

THEOREM- 12.

State

P.T. a m.s (M,d) is totally bounded iff every seq in M has a Cauchy Subsequence.

Pf:

Suppose every seq, in M has a Cauchy Subsequence.

To P.T. M is totally bounded.

Let $\epsilon > 0$ be given.

Choose $x_1 \in M$.