Fifth Edition

Essential Mathematics for Economic Analysis

Knut Sydsæter, Peter Hammond
Arne Strøm & Andrés Carvajal

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ESSENTIAL MATHEMATICS FOR

ECONOMIC ANALYSIS

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To Knut Sydsæter (1937–2012), an inspiring mathematics teacher, as well as wonderful friend and colleague, whose vision, hard work, high professional standards, and sense of humour were all essential in creating this book.

—Arne, Peter and Andrés

To Else, my loving and patient wife.

—Arne

To the memory of my parents Elsie (1916–2007) and Fred (1916–2008), my first teachers of Mathematics, basic Economics, and many more important things.

—Peter

To Yeye and Tata, my best ever students of "matemáquinas", who wanted this book to start with "Once upon a time …"

—Andrés

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PREFACE

Once upon a time there was a sensible straight line who was hopelessly in love with a dot. 'You're the beginning and the end, the hub, the core and the quintessence,' he told her tenderly, but the frivolous dot wasn't a bit interested, for she only had eyes for a wild and unkempt squiggle who never seemed to have anything on his mind at all. All of the line's romantic dreams were in vain, until he discovered . . . angles! Now, with newfound self-expression, he can be anything he wants to be — a square, a triangle, a parallelogram . . . And that's just the beginning!
—Norton Juster (*The Dot and the Line: A Romance in Lower Mathematics* 1963)

I came to the position that mathematical analysis is not one of many ways of doing economic theory: It is the only way. Economic theory is mathematical analysis. Everything else is just pictures and talk.
—R. E. Lucas, Jr. (2001)

Purpose

The subject matter that modern economics students are expected to master makes significant mathematical demands. This is true even of the less technical "applied" literature that students will be expected to read for courses in fields such as public finance, industrial organization, and labour economics, amongst several others. Indeed, the most relevant literature typically presumes familiarity with several important mathematical tools, especially calculus for functions of one and several variables, as well as a basic understanding of multivariable optimization problems with or without constraints. Linear algebra is also used to some extent in economic theory, and a great deal more in econometrics.

The purpose of *Essential Mathematics for Economic Analysis*, therefore, is to help economics students acquire enough mathematical skill to access the literature that is most relevant to their undergraduate study. This should include what some students will need to conduct successfully an undergraduate research project or honours thesis.

As the title suggests, this is a book on *mathematics*, whose material is arranged to allow progressive learning of mathematical topics. That said, we do frequently emphasize economic applications, many of which are listed on the inside front cover. These not only help motivate particular mathematical topics; we also want to help prospective economists acquire mutually reinforcing intuition in both mathematics and economics. Indeed, as the list of examples on the inside front cover suggests, a considerable number of economic concepts and ideas receive some attention.

We emphasize, however, that this is not a book about economics or even about mathematical economics. Students should learn economic theory systematically from other courses, which use other textbooks. We will have succeeded if they can concentrate on the economics in these courses, having already thoroughly mastered the relevant mathematical tools this book presents.

Special Features and Accompanying Material

Virtually all sections of the book conclude with exercises, often quite numerous. There are also many review exercises at the end of each chapter. Solutions to almost all these exercises are provided at the end of the book, sometimes with several steps of the answer laid out.

There are two main sources of supplementary material. The first, for both students and their instructors, is via MyMathLab. Students who have arranged access to this web site for our book will be able to generate a practically unlimited number of additional problems which test how well some of the key ideas presented in the text have been understood. More explanation of this system is offered after this preface. The same web page also has a "student resources" tab with access to a *Solutions Manual* with more extensive answers (or, in the case of a few of the most theoretical or difficult problems in the book, the only answers) to problems marked with the special symbol SM.

The second source, for instructors who adopt the book for their course, is an *Instructor's Manual* that may be downloaded from the publisher's Instructor Resource Centre.

In addition, for courses with special needs, there is a brief online appendix on trigonometric functions and complex numbers. This is also available via MyMathLab.

Prerequisites

Experience suggests that it is quite difficult to start a book like this at a level that is really too elementary.[1] These days, in many parts of the world, students who enter college or university and specialize in economics have an enormous range of mathematical backgrounds and aptitudes. These range from, at the low end, a rather shaky command of elementary algebra, up to real facility in the calculus of functions of one variable. Furthermore, for many economics students, it may be some years since their last formal mathematics course. Accordingly, as mathematics becomes increasingly essential for specialist studies in economics, we feel obliged to provide as much quite elementary material as is reasonably possible. Our aim here is to give those with weaker mathematical backgrounds the chance to get started, and even to acquire a little confidence with some easy problems they can really solve on their own.

[1] In a recent test for 120 first-year students intending to take an elementary economics course, there were 35 different answers to the problem of expanding $(a + 2b)^2$.

To help instructors judge how much of the elementary material students really know before starting a course, the *Instructor's Manual* provides some diagnostic test material. Although each instructor will obviously want to adjust the starting point and pace of a course to match the students' abilities, it is perhaps even more important that each individual student appreciates his or her own strengths and weaknesses, and receives some help and guidance in overcoming any of the latter. This makes it quite likely that weaker students will benefit significantly from the opportunity to work through the early more elementary chapters, even if they may not be part of the course itself.

As for our economic discussions, students should find it easier to understand them if they already have a certain very rudimentary background in economics. Nevertheless, the text has often been used to teach mathematics for economics to students who are studying elementary economics at the same time. Nor do we see any reason why this material cannot be mastered by students interested in economics before they have begun studying the subject in a formal university course.

Topics Covered

After the introductory material in Chapters 1 to 3, a fairly leisurely treatment of single-variable differential calculus is contained in Chapters 4 to 8. This is followed by integration in Chapter 9, and by the application to interest rates and present values in Chapter 10. This may be as far as some elementary courses will go. Students who already have a thorough grounding in single-variable calculus, however, may only need to go fairly quickly over some special topics in these chapters such as elasticity and conditions for global optimization that are often not thoroughly covered in standard calculus courses.

We have already suggested the importance for budding economists of multivariable calculus (Chapters 11 and 12), of optimization theory with and without constraints (Chapters 13 and 14), and of the algebra of matrices and determinants (Chapters 15 and 16). These six chapters in some sense represent the heart of the book, on which students with a thorough grounding in single-variable calculus can probably afford to concentrate. In addition, several instructors who have used previous editions report that they like to teach the elementary theory of linear programming, which is therefore covered in Chapter 17.

The ordering of the chapters is fairly logical, with each chapter building on material in previous chapters. The main exception concerns Chapters 15 and 16 on linear algebra, as well as Chapter 17 on linear programming, most of which could be fitted in almost anywhere after Chapter 3. Indeed, some instructors may reasonably prefer to cover some concepts of linear algebra before moving on to multivariable calculus, or to cover linear programming before multivariable optimization with inequality constraints.

Satisfying Diverse Requirements

The less ambitious student can concentrate on learning the key concepts and techniques of each chapter. Often, these appear boxed and/or in colour, in order to emphasize their importance. Problems are essential to the learning process, and the easier ones should definitely be attempted. These basics should provide enough mathematical background for the student to be able to understand much of the economic theory that is embodied in applied work at the advanced undergraduate level.

Students who are more ambitious, or who are led on by more demanding teachers, can try the more difficult problems. They can also study the material in smaller print. The latter is intended to encourage students to ask why a result is true, or why a problem should be tackled in a particular way. If more readers gain at least a little additional mathematical insight from working through these parts of our book, so much the better.

The most able students, especially those intending to undertake postgraduate study in economics or some related subject, will benefit from a fuller explanation of some topics than we have been able to provide here. On a few occasions, therefore, we take the liberty of referring to our more advanced companion volume, *Further Mathematics for Economic Analysis* (usually abbreviated to FMEA). This is written jointly with our colleague Atle Seierstad in Oslo. In particular, FMEA offers a proper treatment of topics like second-order conditions for optimization, and the concavity or convexity of functions of more than two variables—topics that we think go rather beyond what is really "essential" for all economics students.

Changes in the Fourth Edition

We have been gratified by the number of students and their instructors from many parts of the world who appear to have found the first three editions useful.[2] We have accordingly been encouraged to revise the text thoroughly once again. There are numerous minor changes and improvements, including the following in particular:

1. The main new feature is MyMathLab Global,[3] explained on the page after this preface, as well as on the back cover.
2. New exercises have been added for each chapter.
3. Some of the figures have been improved.

Changes in the Fifth Edition

The most significant change in this edition is that, tragically, we have lost the main author and instigator of this project. Our good friend and colleague Knut Sydsæter died suddenly on 29th September 2012, while on holiday in Spain with his wife Malinka Staneva, a few days before his 75th birthday.

The Department of Economics at the University of Oslo has a web page devoted to Knut and his memory.[4] There is a link there to an obituary written by Jens Stoltenberg, at that

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[2] Different English versions of this book have been translated into Albanian, French, German, Hungarian, Italian, Portuguese, Spanish, and Turkish.
[3] Superseded by MyMathLab for this fifth edition.
[4] See http://www.sv.uio.no/econ/om/aktuelt/aktuelle-saker/sydsaeter.html.

time the Prime Minister of Norway, which includes this tribute to Knut's skills as one of his teachers:

> With a small sheet of paper as his manuscript he introduced me and generations of other economics students to mathematics as a tool in the subject of economics. With professional weight, commitment, and humour, he was both a demanding and an inspiring lecturer. He opened the door into the world of mathematics. He showed that mathematics is a language that makes it possible to explain complicated relationships in a simple manner.

There one can also find Peter's own tribute to Knut, with some recollections of how previous editions of this book came to be written.

Despite losing Knut as its main author, it was clear that this book needed to be kept alive, following desires that Knut himself had often expressed while he was still with us. Fortunately, it had already been agreed that the team of co-authors should be joined by Andrés Carvajal, a former colleague of Peter's at Warwick who, at the time of writing, has just joined the University of California at Davis. He had already produced a new Spanish version of the previous edition of this book; he has now become a co-author of this latest English version. It is largely on his initiative that we have taken the important step of extensively rearranging the material in the first three chapters in a more logical order, with set theory now coming first.

The other main change is one that we hope is invisible to the reader. Previous editions had been produced using the "plain TEX" typesetting system that dates back to the 1980s, along with some ingenious macros that Arne had devised in collaboration with Arve Michaelsen of the Norwegian typesetting firm Matematisk Sats. For technical reasons we decided that the new edition had to be produced using the enrichment of plain TEX called LATEX that has by now become the accepted international standard for typesetting mathematical material. We have therefore attempted to adapt and extend some standard LATEX packages in order to preserve as many good features as possible of our previous editions.

Other Acknowledgements

Over the years we have received help from so many colleagues, lecturers at other institutions, and students, that it is impractical to mention them all.

At the time when we began revising the textbook, Andrés Carvajal was visiting the Fundaçao Getulio Vargas in Brazil. He was able to arrange assistance from Cristina Maria Igreja, who knows both TEX and LATEX from her typesetting work for Brazil's most prestigious academic economics journal, the *Revista Brasileira de Economia*. Her help did much to expedite the essential conversion from plain TEX to LATEX of the computer files used to produce the book.

In the fourth edition of this book, we gratefully acknowledged the encouragement and assistance of Kate Brewin at Pearson. While we still felt Kate's welcome support in the background, our more immediate contact for this edition was Caitlin Lisle, who is Editor for Business and Economics in the Higher Education Division of Pearson. She was always very helpful and attentive in answering our frequent e-mails in a friendly and encouraging way,

and in making sure that this new edition really is getting into print in a timely manner. Many thanks also to Carole Drummond, Helen MacFadyen, and others associated with Pearson's editing team, for facilitating the process of transforming our often imperfect LaTeX files into the well designed book you are now reading.

On the more academic side, very special thanks go to Prof. Dr Fred Böker at the University of Göttingen. He is not only responsible for translating several previous editions of this book into German, but has also shown exceptional diligence in paying close attention to the mathematical details of what he was translating. We appreciate the resulting large number of valuable suggestions for improvements and corrections that he has continued to provide, sometimes at the instigation of Dr Egle Tafenau, who was also using the German version of our textbook in her teaching.

To these and all the many unnamed persons and institutions who have helped us make this text possible, including some whose anonymous comments on earlier editions were forwarded to us by the publisher, we would like to express our deep appreciation and gratitude. We hope that all those who have assisted us may find the resulting product of benefit to their students. This, we can surely agree, is all that really matters in the end.

Andrés Carvajal, Peter Hammond, and Arne Strøm

Davis, Coventry, and Oslo, February 2016

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1 ESSENTIALS OF LOGIC AND SET THEORY

Everything should be made as simple as possible, but not simpler.
—Albert Einstein[1]

Arguments in mathematics require tight logical reasoning; arguments in economic analysis are no exception to this rule. We therefore present some basic concepts from logic. A brief section on mathematical proofs might be useful for more ambitious students.

A short introduction to set theory precedes this. This is useful not just for its importance in mathematics, but also because of the role sets play in economics: in most economics models, it is assumed that, following some specific criterion, economic agents are to choose, optimally, from a feasible *set* of alternatives.

The chapter winds up with a discussion of mathematical induction. Very occasionally, this is used directly in economic arguments; more often, it is needed to understand mathematical results which economists often use.

1.1 Essentials of Set Theory

In daily life, we constantly group together objects of the same kind. For instance, we refer to the faculty of a university to signify all the members of the academic staff. A garden refers to all the plants that are growing in it. We talk about all Scottish firms with more than 300 employees, all taxpayers in Germany who earned between €50 000 and €100 000 in 2004. In all these cases, we have a collection of objects viewed as a whole. In mathematics, such a collection is called a *set*, and its objects are called its *elements*, or its *members*.

How is a set specified? The simplest method is to list its members, in any order, between the two braces { and }. An example is the set $S = \{a, b, c\}$ whose members are the first three letters in the English alphabet. Or it might be a set consisting of three members represented by the letters a, b, and c. For example, if $a = 0$, $b = 1$, and $c = 2$, then $S = \{0, 1, 2\}$. Also,

[1] Attributed; circa 1933.

$S = \{a, b, c\}$ denotes the set of roots of the cubic equation $(x - a)(x - b)(x - c) = 0$ in the unknown x, where a, b, and c are any three real numbers.

Two sets A and B are considered *equal* if each element of A is an element of B and each element of B is an element of A. In this case, we write $A = B$. This means that the two sets consist of exactly the same elements. Consequently, $\{1, 2, 3\} = \{3, 2, 1\}$, because the order in which the elements are listed has no significance; and $\{1, 1, 2, 3\} = \{1, 2, 3\}$, because a set is not changed if some elements are listed more than once.

Alternatively, suppose that you are to eat a meal at a restaurant that offers a choice of several main dishes. Four choices might be feasible—fish, pasta, omelette, and chicken. Then the *feasible set*, F, has these four members, and is fully specified as

$$F = \{\text{fish, pasta, omelette, chicken}\}$$

Notice that the order in which the dishes are listed does not matter. The feasible set remains the same even if the order of the items on the menu is changed.

The symbol "$\varnothing$" denotes the set that has no elements. It is called the *empty set*.[2]

Specifying a Property

Not every set can be defined by listing all its members, however. For one thing, some sets are infinite—that is, they contain infinitely many members. Such infinite sets are rather common in economics. Take, for instance, the *budget set* that arises in consumer theory. Suppose there are two goods with quantities denoted by x and y. Suppose one unit of these goods can be bought at prices p and q, respectively. A consumption bundle (x, y) is a pair of quantities of the two goods. Its value at prices p and q is $px + qy$. Suppose that a consumer has an amount m to spend on the two goods. Then the *budget constraint* is $px + qy \leq m$ (assuming that the consumer is free to underspend). If one also accepts that the quantity consumed of each good must be nonnegative, then the *budget set*, which will be denoted by B, consists of those consumption bundles (x, y) satisfying the three inequalities $px + qy \leq m$, $x \geq 0$, and $y \geq 0$. (The set B is shown in Fig. 4.4.12.) Standard notation for such a set is

$$B = \{(x, y) : px + qy \leq m,\ x \geq 0,\ y \geq 0\} \tag{1.1.1}$$

The braces { } are still used to denote "the set consisting of". However, instead of listing all the members, which is impossible for the infinite set of points in the triangular budget set B, the specification of the set B is given in two parts. To the left of the colon, (x, y) is used to denote the typical member of B, here a consumption bundle that is specified by listing the respective quantities of the two goods. To the right of the colon, the three properties that these typical members must satisfy are all listed, and the set thereby specified. This is an example of the general specification:

$$S = \{\text{typical member : defining properties}\}$$

[2] Note that it is *the*, and not *an*, empty set. This is so, following the principle that a set is completely defined by its elements: there can only be one set that contains no elements. The empty set is the same, whether it is being studied by a child in elementary school or a physicist at CERN—or, indeed, by an economics student in her math courses!

Note that it is not just infinite sets that can be specified by properties—finite sets can also be specified in this way. Indeed, some finite sets almost *have* to be specified in this way, such as the set of all human beings currently alive.

Set Membership

As we stated earlier, sets contain members or elements. There is some convenient standard notation that denotes the relation between a set and its members. First,

$$x \in S$$

indicates that *x is an element of S*. Note the special "belongs to" symbol $\in$ (which is a variant of the Greek letter ε, or "epsilon").

To express the fact that x is *not* a member of S, we write $x \notin S$. For example, $d \notin \{a, b, c\}$ says that d is not an element of the set $\{a, b, c\}$.

For additional illustrations of set membership notation, let us return to the main dish example. Confronted with the choice from the set $F = \{\text{fish, pasta, omelette, chicken}\}$, let s denote your actual selection. Then, of course, $s \in F$. This is what we mean by "feasible set"—it is possible only to choose some member of that set but nothing outside it.

Let A and B be any two sets. Then A is a *subset* of B if it is true that every member of A is also a member of B. Then we write $A \subseteq B$. In particular, $A \subseteq A$. From the definitions we see that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Set Operations

Sets can be combined in many different ways. Especially important are three operations: *union*, *intersection*, and the *difference* of sets, as shown in Table 1.1.

Table 1.1 Elementary set operations

Notation	Name	The set that consists of:
$A \cup B$	*A union B*	The elements that belong to at least one of the sets A and B
$A \cap B$	*A intersection B*	The elements that belong to both A and B
$A \setminus B$	*A minus B*	The elements that belong to set A, but not to B

Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

EXAMPLE 1.1.1 Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 6\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$.[3]

Solution: $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \{3\}$, $A \setminus B = \{1, 2, 4, 5\}$, $B \setminus A = \{6\}$.

[3] Here and throughout the book, we strongly suggest that when reading an example, you first attempt to solve the problem, while covering the solution, and then gradually reveal the proposed solution to see if you are right.

An economic example can be obtained by considering workers in Utopia in 2001. Let A be the set of all those workers who had an income of at least 15 000 Utopian dollars and let B be the set of all who had a net worth of at least 150 000 dollars. Then $A \cup B$ would be those workers who earned at least 15 000 dollars or who had a net worth of at least 150 000 dollars, whereas $A \cap B$ are those workers who earned at least 15 000 dollars and who also had a net worth of at least 150 000 dollars. Finally, $A \setminus B$ would be those who earned at least 15 000 dollars but who had less than 150 000 dollars in net worth.

If two sets A and B have no elements in common, they are said to be *disjoint*. Thus, the sets A and B are disjoint if and only if $A \cap B = \varnothing$.

A collection of sets is often referred to as a *family* of sets. When considering a certain family of sets, it is often natural to think of each set in the family as a subset of one particular fixed set Ω, hereafter called the *universal set*. In the previous example, the set of all Utopian workers in 2001 would be an obvious choice for a universal set.

If A is a subset of the universal set Ω, then according to the definition of difference, $\Omega \setminus A$ is the set of elements of Ω that are not in A. This set is called the *complement* of A in Ω and is sometimes denoted by A^c, so that $A^c = \Omega \setminus A$.[4] When finding the complement of a set, it is *very* important to be clear about which universal set Ω is being used.

EXAMPLE 1.1.2 Let the universal set Ω be the set of all students at a particular university. Moreover, let F denote the set of female students, M the set of all mathematics students, C the set of students in the university choir, B the set of all biology students, and T the set of all tennis players. Describe the members of the following sets: $\Omega \setminus M$, $M \cup C$, $F \cap T$, $M \setminus (B \cap T)$, and $(M \setminus B) \cup (M \setminus T)$.

Solution: $\Omega \setminus M$ consists of those students who are not studying mathematics, $M \cup C$ of those students who study mathematics and/or are in the choir. The set $F \cap T$ consists of those female students who play tennis. The set $M \setminus (B \cap T)$ has those mathematics students who do not both study biology and play tennis. Finally, the last set $(M \setminus B) \cup (M \setminus T)$ has those students who either are mathematics students not studying biology or mathematics students who do not play tennis. Do you see that the last two sets are equal?[5]

Venn Diagrams

When considering the relationships between several sets, it is instructive and extremely helpful to represent each set by a region in a plane. The region is drawn so that all the elements belonging to a certain set are contained within some closed region of the plane. Diagrams constructed in this manner are called *Venn diagrams*. The definitions discussed in the previous section can be illustrated as in Fig. 1.1.1.

By using the definitions directly, or by illustrating sets with Venn diagrams, one can derive formulas that are universally valid regardless of which sets are being considered. For example, the formula $A \cap B = B \cap A$ follows immediately from the definition of the

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[4] Other ways of denoting the complement of A include $\complement A$ and $\tilde{A}$.

[5] For arbitrary sets M, B, and T, it is true that $(M \setminus B) \cup (M \setminus T) = M \setminus (B \cap T)$. It will be easier to verify this equality after you have read the following discussion of Venn diagrams.

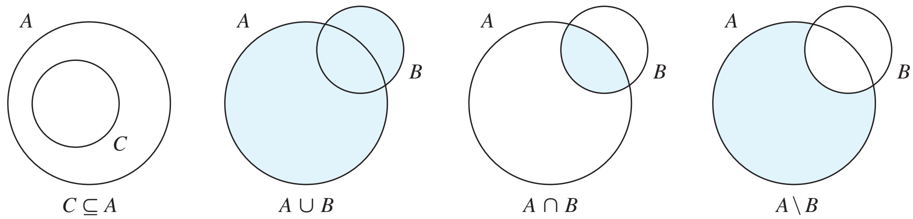

Figure 1.1.1 Venn diagrams

intersection between two sets. It is somewhat more difficult to verify directly from the definitions that the following relationship is valid for all sets A, B, and C:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{$*$}$$

With the use of a Venn diagram, however, we easily see that the sets on the right- and left-hand sides of the equality sign both represent the shaded set in Fig. 1.1.2. The equality in $(*)$ is therefore valid.

It is important that the three sets A, B, and C in a Venn diagram be drawn in such a way that all possible relations between an element and each of the three sets are represented. In other words, as in Fig. 1.1.3, the following eight different sets all should be nonempty:

1. $(A \cap B) \setminus C$
2. $(B \cap C) \setminus A$
3. $(C \cap A) \setminus B$
4. $A \setminus (B \cup C)$
5. $B \setminus (C \cup A)$
6. $C \setminus (A \cup B)$
7. $A \cap B \cap C$
8. $(A \cup B \cup C)^c$

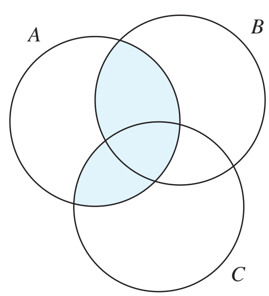

Figure 1.1.2 Venn diagram for $A \cap (B \cup C)$

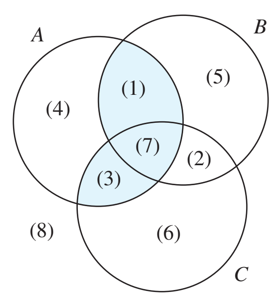

Figure 1.1.3 Venn diagram for three sets

Notice, however, that this way of representing sets in the plane becomes unmanageable if four or more sets are involved, because then there would have to be at least $2^4 = 16$ regions in any such Venn diagram.

From the definition of intersection and union, or by the use of Venn diagrams, it easily follows that $A \cup (B \cup C) = (A \cup B) \cup C$ and that $A \cap (B \cap C) = (A \cap B) \cap C$. Consequently, it does not matter where the parentheses are placed. In such cases, the parentheses can be dropped and the expressions written as $A \cup B \cup C$ and $A \cap B \cap C$. Note, however, that the parentheses cannot generally be moved in the expression $A \cap (B \cup C)$, because this set is not always equal to $(A \cap B) \cup C$. Prove this fact by considering the case where $A = \{1, 2, 3\}$, $B = \{2, 3\}$, and $C = \{4, 5\}$, or by using a Venn diagram.

Cantor

The founder of set theory is Georg Cantor (1845–1918), who was born in St Petersburg but then moved to Germany at the age of eleven. He is regarded as one of history's great mathematicians. This is not because of his contributions to the development of the useful, but relatively trivial, aspects of set theory outlined above. Rather, Cantor is remembered for his profound study of infinite sets. Below we try to give just a hint of his theory's implications.

A collection of individuals are gathering in a room that has a certain number of chairs. How can we find out if there are exactly as many individuals as chairs? One method would be to count the chairs and count the individuals, and then see if they total the same number. Alternatively, we could ask all the individuals to sit down. If they all have a seat to themselves and there are no chairs unoccupied, then there are exactly as many individuals as chairs. In that case each chair corresponds to an individual and each individual corresponds to a chair — i.e., there is a *one-to-one correspondence* between individuals and chairs.

Generally we say that two sets of elements have the same *cardinality*, if there is a one-to-one correspondence between the sets. This definition is also valid for sets with an infinite number of elements. Cantor struggled for three years to prove a surprising consequence of this definition—that there are as many points in a square as there are points on one of the edges of the square, in the sense that the two sets have the same cardinality. In a letter to Richard Dedekind dated 1877, Cantor wrote of this result: "I see it, but I do not believe it."

EXERCISES FOR SECTION 1.1

1. Let $A = \{2, 3, 4\}$, $B = \{2, 5, 6\}$, $C = \{5, 6, 2\}$, and $D = \{6\}$.

(a) Determine which of the following statements are true: $4 \in C$; $5 \in C$; $A \subseteq B$; $D \subseteq C$; $B = C$; and $A = B$.

(b) Find $A \cap B$; $A \cup B$; $A \setminus B$; $B \setminus A$; $(A \cup B) \setminus (A \cap B)$; $A \cup B \cup C \cup D$; $A \cap B \cap C$; and $A \cap B \cap C \cap D$.

2. Let F, M, C, B, and T be the sets in Example 1.1.2.

(a) Describe the following sets: $F \cap B \cap C$, $M \cap F$, and $((M \cap B) \setminus C) \setminus T$.

(b) Write the following statements in set terminology:

(i) All biology students are mathematics students.

(ii) There are female biology students in the university choir.

(iii) No tennis player studies biology.

(iv) Those female students who neither play tennis nor belong to the university choir all study biology.

3. A survey revealed that 50 people liked coffee and 40 liked tea. Both these figures include 35 who liked both coffee and tea. Finally, ten did not like either coffee or tea. How many people in all responded to the survey?

4. Make a complete list of all the different subsets of the set $\{a, b, c\}$. How many are there if the empty set and the set itself are included? Do the same for the set $\{a, b, c, d\}$.

5. Determine which of the following formulas are true. If any formula is false, find a counter example to demonstrate this, using a Venn diagram if you find it helpful.

   (a) $A \setminus B = B \setminus A$

   (b) $A \cap (B \cup C) \subseteq (A \cap B) \cup C$

   (c) $A \cup (B \cap C) \subseteq (A \cup B) \cap C$

   (d) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$

6. Use Venn diagrams to prove that: (a) $(A \cup B)^c = A^c \cap B^c$; and (b) $(A \cap B)^c = A^c \cup B^c$

7. If A is a set with a finite number of elements, let $n(A)$ denote its *cardinality*, defined as the number of elements in A. If A and B are arbitrary finite sets, prove the following:

   (a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

   (b) $n(A \setminus B) = n(A) - n(A \cap B)$

8. A thousand people took part in a survey to reveal which newspaper, A, B, or C, they had read on a certain day. The responses showed that 420 had read A, 316 had read B, and 160 had read C. These figures include 116 who had read both A and B, 100 who had read A and C, and 30 who had read B and C. Finally, all these figures include 16 who had read all three papers.

   (a) How many had read A, but not B?

   (b) How many had read C, but neither A nor B?

   (c) How many had read neither A, B, nor C?

   (d) Denote the complete set of all people in the survey by Ω (the universal set). Applying the notation in Exercise 7, we have $n(A) = 420$ and $n(A \cap B \cap C) = 16$, for example. Describe the numbers given in the previous answers using the same notation. Why is $n(\Omega \setminus (A \cup B \cup C)) = n(\Omega) - n(A \cup B \cup C)$?

9. [HARDER] The equalities proved in Exercise 6 are particular cases of the *De Morgan's Laws*. State and prove these two laws:

   (a) The complement of the union of any family of sets equals the intersection of all the sets' complements.

   (b) The complement of the intersection of any family of sets equals the union of all the sets' complements.

1.2 Some Aspects of Logic

Mathematical models play a critical role in the empirical sciences, especially in modern economics. This has been a useful development in these sciences, but requires practitioners to work with care: errors in mathematical reasoning are easy to make. Here is a typical example of how a faulty attempt to use logic could result in a problem being answered incorrectly.

EXAMPLE 1.2.1 Suppose that we want to find *all* the values of x for which the following equality is true: $x + 2 = \sqrt{4 - x}$.

Squaring each side of the equation gives $(x + 2)^2 = (\sqrt{4 - x})^2$, and thus $x^2 + 4x + 4 = 4 - x$. Rearranging this last equation gives $x^2 + 5x = 0$. Cancelling x results in $x + 5 = 0$, and therefore $x = -5$.

According to this reasoning, the answer should be $x = -5$. Let us check this. For $x = -5$, we have $x + 2 = -3$. Yet $\sqrt{4 - x} = \sqrt{9} = 3$, so this answer is incorrect.[6]

This example highlights the dangers of routine calculation without adequate thought. It may be easier to avoid similar mistakes after studying the structure of logical reasoning.

Propositions

Assertions that are either true or false are called statements, or *propositions*. Most of the propositions in this book are mathematical ones, but other kinds may arise in daily life. "All individuals who breathe are alive" is an example of a true proposition, whereas the assertion "all individuals who breathe are healthy" is a false proposition. Note that if the words used to express such an assertion lack precise meaning, it will often be difficult to tell whether it is true or false. For example, the assertion "67 is a large number" is neither true nor false without a precise definition of "large number".

Suppose an assertion, such as "$x^2 - 1 = 0$", includes one or more variables. By substituting various real numbers for the variable x, we can generate many different propositions, some true and some false. For this reason we say that the assertion is an *open proposition*. In fact, the proposition $x^2 - 1 = 0$ happens to be true if $x = 1$ or -1, but not otherwise. Thus, an open proposition is not simply true or false. Instead, it is neither true nor false until we choose a particular value for the variable.

Implications

In order to keep track of each step in a chain of logical reasoning, it often helps to use "implication arrows". Suppose P and Q are two propositions such that whenever P is true, then Q is necessarily true. In this case, we usually write

$$P \Rightarrow Q \tag{$*$}$$

This is read as "P implies Q"; or "if P, then Q"; or "Q is a consequence of P". Other ways of expressing the same implication include "Q if P"; "P only if Q"; and "Q is an implication of P". The symbol $\Rightarrow$ is an *implication arrow*, and it points in the direction of the logical implication.

[6] Note the wisdom of checking your answer whenever you think you have solved an equation. In Example 1.2.4, below, we explain how the error arose.

EXAMPLE 1.2.2 Here are some examples of correct implications:

(a) $x > 2 \Rightarrow x^2 > 4$ (b) $xy = 0 \Rightarrow$ either $x = 0$ or $y = 0$[7]

(c) S is a square $\Rightarrow$ S is a rectangle (d) She lives in Paris $\Rightarrow$ She lives in France.

In certain cases where the implication (∗) is valid, it may also be possible to draw a logical conclusion in the other direction: $Q \Rightarrow P$. In such cases, we can write both implications together in a single *logical equivalence*:

$$P \Leftrightarrow Q$$

We then say that "P is equivalent to Q". Because we have both "P if Q" and "P only if Q", we also say that "P if and only if Q", which is often written as "P iff Q" for short. Unsurprisingly, the symbol $\Leftrightarrow$ is called an *equivalence arrow*.

In Example 1.2.2, we see that the implication arrow in (b) could be replaced with the equivalence arrow, because it is also true that $x = 0$ or $y = 0$ implies $xy = 0$. Note, however, that no other implication in Example 1.2.2 can be replaced by the equivalence arrow. For even if x^2 is larger than 4, it is not necessarily true that x is larger than 2 (for instance, x might be -3); also, a rectangle is not necessarily a square; and, finally, the fact that a person is in France does not mean that she is in Paris.

EXAMPLE 1.2.3 Here are some examples of correct equivalences:

(a) $(x < -2$ or $x > 2) \Leftrightarrow x^2 > 4$ (b) $xy = 0 \Leftrightarrow (x = 0$ or $y = 0)$

(c) $A \subseteq B \Leftrightarrow (a \in A \Rightarrow a \in B)$

Necessary and Sufficient Conditions

There are other commonly used ways of expressing that proposition P implies proposition Q, or that P is equivalent to Q. Thus, if proposition P implies proposition Q, we state that P is a "sufficient condition" for Q—after all, for Q to be true, it is sufficient that P be true. Accordingly, we know that if P is satisfied, then it is certain that Q is also satisfied. In this case, we say that Q is a "necessary condition" for P, for Q must necessarily be true if P is true. Hence,

> P is a *sufficient condition* for Q means: $P \Rightarrow Q$
> Q is a *necessary condition* for P means: $P \Rightarrow Q$

The corresponding verbal expression for $P \Leftrightarrow Q$ is, simply, that P *is a necessary and sufficient condition for* Q.

[7] It is important to notice that the word "or" in mathematics is *inclusive*, in the sense that the statement "P or Q" allows for the possibility that P and Q are *both* true.

It is worthwhile emphasizing the importance of distinguishing between the propositions "P is a necessary condition for Q", "P is a sufficient condition for Q", and "P is a necessary and sufficient condition for Q". To emphasize the point, consider the propositions:

Living in France is a necessary condition for a person to live in Paris.[8]

and

Living in Paris is a necessary condition for a person to live in France.

The first proposition is clearly true. But the second is false,[9] because it is possible to live in France, but outside Paris. What is true, though, is that

Living in Paris is a sufficient condition for a person to live in France.

In the following pages, we shall repeatedly refer to necessary and sufficient conditions. Understanding them, and the difference between them, is a necessary condition for understanding much of economic analysis. It is not a sufficient condition, alas!

EXAMPLE 1.2.4 In finding the solution to Example 1.2.1, why was it necessary to test whether the values we found were actually solutions? To answer this, we must analyse the logical structure of our analysis. Using implication arrows marked by letters, we can express the "solution" proposed there as follows:

$$\begin{aligned} x+2=\sqrt{4-x} &\overset{(a)}{\Longrightarrow} (x+2)^2 = 4-x \\ &\overset{(b)}{\Longrightarrow} x^2+4x+4 = 4-x \\ &\overset{(c)}{\Longrightarrow} x^2+5x = 0 \\ &\overset{(d)}{\Longrightarrow} x(x+5) = 0 \\ &\overset{(e)}{\Longrightarrow} [x = 0 \text{ or } x = -5] \end{aligned}$$

Implication (a) is true, because $a = b \Rightarrow a^2 = b^2$ and $(\sqrt{a})^2 = a$. It is important to note, however, that the implication cannot be replaced by an equivalence: if $a^2 = b^2$, then either $a = b$ or $a = -b$; it need not be true that $a = b$. Implications (b), (c), (d), and (e) are also all true; moreover, all could have been written as equivalences, though this is not necessary in order to find the solution. Therefore, a chain of implications has been obtained that leads from the equation $x + 2 = \sqrt{4-x}$ to the proposition "$x = 0$ or $x = -5$".

Because the implication (a) cannot be reversed, there is no corresponding chain of implications going in the opposite direction. We have verified that if the number x satisfies $x + 2 = \sqrt{4-x}$, then x must be either 0 or -5; no other value can satisfy the given equation. However, we have not yet shown that either 0 or -5 really satisfies the equation.

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8 Unless the person lives in Paris, Texas.

9 As is the proposition *Living in France is equivalent to living in Paris*.

Only after we try inserting 0 and -5 into the equation do we see that $x = 0$ is the only solution.[10]

Looking back at Example 1.2.1, we now realize that two errors were committed. Firstly, the implication $x^2 + 5x = 0 \Rightarrow x + 5 = 0$ is wrong, because $x = 0$ is also a solution of $x^2 + 5x = 0$. Secondly, it is logically necessary to check if 0 or -5 really satisfies the equation.

EXERCISES FOR SECTION 1.2

1. There are many other ways to express implications and equivalences, apart from those already mentioned. Use appropriate implication or equivalence arrows to represent the following propositions:

(a) The equation $2x - 4 = 2$ is fulfilled only when $x = 3$.

(b) If $x = 3$, then $2x - 4 = 2$.

(c) The equation $x^2 - 2x + 1 = 0$ is satisfied if $x = 1$.

(d) If $x^2 > 4$, then $|x| > 2$, and conversely.

2. Determine which of the following formulas are true. If any formula is false, find a counter example to demonstrate this, using a Venn diagram if you find it helpful.

(a) $A \subseteq B \Leftrightarrow A \cup B = B$ (b) $A \subseteq B \Leftrightarrow A \cap B = A$ (c) $A \cap B = A \cap C \Rightarrow B = C$

(d) $A \cup B = A \cup C \Rightarrow B = C$ (e) $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$

3. In each of the following implications, where x, y, and z are numbers, decide: (i) if the implication is true; and (ii) if the converse implication is true.

(a) $x = \sqrt{4} \Rightarrow x = 2$ (b) $(x = 2 \text{ and } y = 5) \Rightarrow x + y = 7$

(c) $(x - 1)(x - 2)(x - 3) = 0 \Rightarrow x = 1$ (d) $x^2 + y^2 = 0 \Rightarrow x = 0 \text{ or } y = 0$

(e) $(x = 0 \text{ and } y = 0) \Rightarrow x^2 + y^2 = 0$ (f) $xy = xz \Rightarrow y = z$

4. Consider the proposition $2x + 5 \geq 13$.

(a) Is the condition $x \geq 0$ necessary, or sufficient, or both necessary and sufficient for the inequality to be satisfied?

(b) Answer the same question when $x \geq 0$ is replaced by $x \geq 50$.

(c) Answer the same question when $x \geq 0$ is replaced by $x \geq 4$.

5. [HARDER] If P is a statement, the *negation* of P is denoted by $\neg P$. If P is true, then $\neg P$ is false, and vice versa. For example, the negation of the statement $2x + 3y \leq 8$ is $2x + 3y > 8$. For each of the following six propositions, state the negation as simply as possible.

(a) $x \geq 0$ and $y \geq 0$.

(b) All x satisfy $x \geq a$.

---

[10] Note that in this case, the test we have suggested not only serves to check our calculations, but is also a logical necessity.

(c) Neither x nor y is less than 5.

(d) For each $\varepsilon > 0$, there exists a $\delta > 0$ such that B is satisfied.

(e) No one can help liking cats.

(f) Everyone loves somebody some of the time.

1.3 Mathematical Proofs

In every branch of mathematics, the most important results are called *theorems*. Constructing logically valid proofs for these results often can be very complicated. For example, the "four-colour theorem" states that any map in the plane needs at most four colours in order that all adjacent regions can be given different colours. Proving this involved checking hundreds of thousands of different cases, a task that was impossible without a sophisticated computer program.

In this book, we often omit formal proofs of theorems. Instead, the emphasis is on providing a good intuitive grasp of what the theorems tell us. That said, it is still useful to understand something about the different types of proof that are used in mathematics.

Every mathematical theorem can be formulated as one or more implications of the form

$$P \Rightarrow Q \tag{$*$}$$

where P represents a proposition, or a series of propositions, called *premises* ("what we know"), and Q represents a proposition or a series of propositions that are called the *conclusions* ("what we want to know").

Usually, it is most natural to prove a result of the type $(*)$ by starting with the premises P and successively working forward to the conclusions Q; we call this a *direct proof*. Sometimes, however, it is more convenient to prove the implication $P \Rightarrow Q$ by an *indirect* or *contrapositive proof*. In this case, we begin by supposing that Q is not true, and on that basis demonstrate that P cannot be true either. This is completely legitimate, because we have the following equivalence:

THE CONTRAPOSITIVE PRINCIPLE

The statement $P \Rightarrow Q$ is equivalent to the statement

$$\textit{not } Q \Rightarrow \textit{not } P$$

It is helpful to consider how this rule of logic applies to a concrete example: "If it is raining, the grass is getting wet" asserts precisely the same thing as "If the grass is not getting wet, then it is not raining".

EXAMPLE 1.3.1 Use the two methods of proof to show that $-x^2 + 5x - 4 > 0 \Rightarrow x > 0$.

Solution:

(a) *Direct proof*: Suppose $-x^2 + 5x - 4 > 0$. Adding $x^2 + 4$ to each side of the inequality gives $5x > x^2 + 4$. Because $x^2 + 4 \geq 4$, for all x, we have $5x > 4$, and so $x > 4/5$. In particular, $x > 0$.

(b) *Indirect proof*: Suppose $x \leq 0$. Then $5x \leq 0$ and so $-x^2 + 5x - 4$, as a sum of three nonpositive terms, is itself nonpositive.

The method of indirect proof is closely related to an alternative one known as *proof by contradiction* or by *reductio ad absurdum*. In this method, in order to prove that $P \Rightarrow Q$, one assumes that P is true and Q is not, and develops an argument that concludes something that *cannot* be true. So, since P and the negation of Q lead to something absurd, it must be that whenever P holds, so does Q.

In the last example, let us assume that $-x^2 + 5x - 4 > 0$ and $x \leq 0$ are true simultaneously. Then, as in the first step of the direct proof, we have $5x > x^2 + 4$. But since $5x \leq 0$, as in the first step of the indirect proof, we are forced to conclude that $0 > x^2 + 4$. Since the latter cannot possibly be true, we have proved that $-x^2 + 5x - 4 > 0$ and $x \leq 0$ cannot be both true, so that $-x^2 + 5x - 4 > 0 \Rightarrow x > 0$, as desired.

Deductive and Inductive Reasoning

The two methods of proof just outlined are all examples of *deductive reasoning*—that is, reasoning based on consistent rules of logic. In contrast, many branches of science use *inductive reasoning*. This process draws general conclusions based only on a few (or even many) observations. For example, the statement that "the price level has increased every year for the last n years; therefore, it will surely increase next year too" demonstrates inductive reasoning. This inductive approach is of fundamental importance in the experimental and empirical sciences, despite the fact that conclusions based upon it never can be absolutely certain. Indeed, in economics, such examples of inductive reasoning (or the implied predictions) often turn out to be false, with hindsight.

In mathematics, inductive reasoning is not recognized as a form of proof. Suppose, for instance, that students in a geometry course are asked to show that the sum of the angles of a triangle is always 180 degrees. If they painstakingly measure as accurately as possible, say, one thousand different triangles, demonstrating that in each case the sum of the angles is 180 degrees, it would not serve as proof for the assertion. It would represent a very good indication that the proposition is true, but it is not a mathematical proof. Similarly, in business economics, the fact that a particular company's profits have risen for each of the past 20 years is no guarantee that they will rise once again this year.

EXERCISES FOR SECTION 1.3

1. Which of the following statements are equivalent to the (dubious) statement: "If inflation increases, then unemployment decreases"?
   (a) For unemployment to decrease, inflation must increase.
   (b) A sufficient condition for unemployment to decrease is that inflation increases.
   (c) Unemployment can only decrease if inflation increases.
   (d) If unemployment does not decrease, then inflation does not increase.
   (e) A necessary condition for inflation to increase is that unemployment decreases.

2. Analyse the following epitaph, using logic: *Those who knew him, loved him. Those who loved him not, knew him not.* Might this be a case where poetry is better than logic?

3. Use the contrapositive principle to show that if x and y are integers and xy is an odd number, then x and y are both odd.

1.4 Mathematical Induction

Proof by induction is an important technique for verifying formulas involving natural numbers. For instance, consider the sum of the first n odd numbers. We observe that

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$

This suggests a general pattern, with the sum of the first n odd numbers equal to n^2:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \qquad (*)$$

To prove that this is generally valid, we can proceed as follows. Suppose that the formula in $(*)$ is correct for a certain natural number $n = k$, so that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

By adding the next odd number $2k + 1$ to each side, we get

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$$

But this is the formula $(*)$ with $n = k + 1$. Hence, we have proved that if the sum of the first k odd numbers is k^2, then the sum of the first $k + 1$ odd numbers equals $(k + 1)^2$. This implication, together with the fact that $(*)$ *is* valid for $n = 1$, implies that $(*)$ is valid in general. For we have just shown that if $(*)$ is true for $n = 1$, then it is true for $n = 2$; that if it is true for $n = 2$, then it is true for $n = 3$; ...; that if it is true for $n = k$, then it is true for $n = k + 1$; and so on.

A proof of this type is called a *proof by induction*. It requires showing first that the formula is indeed valid for $n = 1$, and second that, *if* the formula is valid for $n = k$, then it is also valid for $n = k + 1$. It follows by induction that the formula is valid for all natural numbers n.

EXAMPLE 1.4.1 Prove by induction that, for all positive integers n,

$$3 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 3) \qquad (**)$$

Solution: For $n = 1$, both sides are 3. Suppose $(**)$ is true for $n = k$. Then

$$3 + 3^2 + 3^3 + 3^4 + \cdots + 3^k + 3^{k+1} = \frac{1}{2}(3^{k+1} - 3) + 3^{k+1} = \frac{1}{2}(3^{k+2} - 3)$$

which is $(**)$ for $n = k + 1$. Thus, by induction, $(**)$ is true for all n.

On the basis of these examples, the general structure of an induction proof can be explained as follows: We want to prove that a mathematical formula $A(n)$ that depends on n is valid for all natural numbers n. In the two previous examples $(*)$ and $(**)$, the respective statements $A(n)$ were

$$A(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

$$A(n) : 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 3)$$

The steps required in each proof are as follows: First, verify that $A(1)$ is valid, which means that the formula is correct for $n = 1$. Then prove that for each natural number k, if $A(k)$ is true, it follows that $A(k + 1)$ must be true. Here $A(k)$ is called *the induction hypothesis*, and the step from $A(k)$ to $A(k + 1)$ is called *the induction step* in the proof. When the induction step is proved for an arbitrary natural number k, then, by induction, statement $A(n)$ is true for all n.

The general principle can now be formulated:

THE PRINCIPLE OF MATHEMATICAL INDUCTION

For each natural number n, let $A(n)$ denote a statement that depends on n. Suppose that:

(a) $A(1)$ is true; and

(b) for each natural number k, if the induction hypothesis $A(k)$ is true, then $A(k + 1)$ is true.

Then, $A(n)$ is true for all natural numbers n.

The principle of mathematical induction seems intuitively evident. If the truth of $A(k)$ for each k implies the truth of $A(k+1)$, then because $A(1)$ is true, $A(2)$ must be true, which, in turn, means that $A(3)$ is true, and so on.[11]

The principle of mathematical induction can easily be generalized to the case in which we have a statement $A(n)$ for each integer greater than or equal to an arbitrary integer n_0. Suppose we can prove that $A(n_0)$ is valid and moreover that, for each $k \geq n_0$, if $A(k)$ is true, then $A(k+1)$ is true. If follows that $A(n)$ is true for all $n \geq n_0$.

EXERCISES FOR SECTION 1.4

1. Prove by induction that for all natural numbers n,

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1) \tag{$*$}$$

2. Prove by induction that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \tag{$**$}$$

3. Noting that $1^3 + 2^3 + 3^3 = 36$ is divisible by 9, prove by induction that the sum $n^3 + (n+1)^3 + (n+2)^3$ of three consecutive cubes is always divisible by 9.

4. Let $A(n)$ be the statement:

Any collection of n people in one room all have the same income

Find what is wrong with the following "induction argument":

$A(1)$ is obviously true. Suppose $A(k)$ is true for some natural number k. We will then prove that $A(k+1)$ is true. So take any collection of $k+1$ people in one room and send one of them outside. The remaining k people all have the same income by the induction hypothesis. Bring the person back inside and send another outside instead. Again the remaining people will have the same income. But then all the $k+1$ people will have the same income. By induction, this proves that all n people have the same income.

REVIEW EXERCISES

1. Let $A = \{1, 3, 4\}$, $B = \{1, 4, 6\}$, $C = \{2, 4, 3\}$, and $D = \{1, 5\}$. Find $A \cap B$; $A \cup B$; $A \setminus B$; $B \setminus A$; $(A \cup B) \setminus (A \cap B)$; $A \cup B \cup C \cup D$; $A \cap B \cap C$; and $A \cap B \cap C \cap D$.

2. Let the universal set be $\Omega = \{1, 2, 3, 4, \ldots, 11\}$, and define $A = \{1, 4, 6\}$ and $B = \{2, 11\}$. Find $A \cap B$; $A \cup B$; $\Omega \setminus B$; A^c.

[11] An analogy: Consider a ladder with an infinite number of steps. Suppose you can climb the first step and suppose, moreover, that after each step, you can always climb the next. Then you are able to climb up to any step.

SM 3. A liberal arts college has one thousand students. The numbers studying various languages are: English 780; French 220; and Spanish 52. These figures include 110 who study English and French, 32 who study English and Spanish, 15 who study French and Spanish. Finally, all these figures include ten students taking all three languages.

(a) How many study English and French, but not Spanish?

(b) How many study English, but not French?

(c) How many study no languages?

SM 4. Let x and y be real numbers. Consider the following implications and decide in each case: (i) if the implication is true; and (ii) if the converse implication is true.

(a) $x = 5$ and $y = -3 \Rightarrow x + y = 2$

(b) $x^2 = 16 \Rightarrow x = 4$

(c) $(x - 3)^2(y + 2)$ is a positive number $\Rightarrow y$ is greater than -2

(d) $x^3 = 8 \Rightarrow x = 2$

5. [HARDER] Let the symbol $\geq$ denote the relation "at least as great as". Prove that, for all x:

(a) $(1 + x)^2 \geq 1 + 2x$;

(b) if $x \geq -3$, then $(1 + x)^3 \geq 1 + 3x$;

(c) for all natural numbers n, if x is greater than or equal to -1, then

$$(1 + x)^n \geq 1 + nx$$

This result is known as *Bernoulli's inequality*.

2 ALGEBRA

God made the integers, all else is the work of Man.
—Leopold Kronecker[1]

This chapter deals with elementary algebra, but we also briefly consider a few other topics that you might find that you need to review. Indeed, tests reveal that even students with a good background in mathematics often benefit from a brief review of what they learned in the past. These students should browse through the material and do some of the less simple exercises. Students with a weaker background in mathematics, or who have been away from mathematics for a long time, should read the text carefully and then do most of the exercises. Finally, those students who have considerable difficulties with this chapter should turn to a more elementary book on algebra.

2.1 The Real Numbers

We start by reviewing some important facts and concepts concerning numbers. The basic numbers are the natural numbers:

$$1, 2, 3, 4, \dots$$

also called positive integers. Of these 2, 4, 6, 8, ... are the even numbers, whereas 1, 3, 5, 7, ... are the odd numbers. Though familiar, such numbers are in reality rather abstract and advanced concepts. Civilization crossed a significant threshold when it grasped the idea that a flock of four sheep and a collection of four stones have something in common, namely "fourness". This idea came to be represented by symbols such as the primitive :: (still used on dominoes or playing cards), the Roman numeral IV, and eventually the modern 4. This key notion is grasped and then continually refined as young children develop their mathematical skills.

[1] Attributed; circa 1886.

The positive integers, together with 0 and the negative integers $-1, -2, -3, -4, \ldots$, make up the integers, which are

$$0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$$

They can be represented on a *number line* like the one shown in Fig. 2.1.1, where the arrow gives the direction in which the numbers increase.

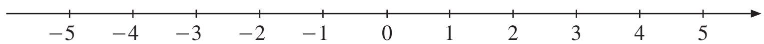

Figure 2.1.1 The number line

The *rational numbers* are those like $3/5$ that can be written in the form a/b, where a and b are both integers. An integer n is also a rational number, because $n = n/1$. Other examples of rational numbers are

$$\frac{1}{2}, \frac{11}{70}, \frac{125}{7}, -\frac{10}{11}, 0 = \frac{0}{1}, -19, -1.26 = -\frac{126}{100}$$

The rational numbers can also be represented on the number line. Imagine that we first mark $1/2$ and all the multiples of $1/2$. Then we mark $1/3$ and all the multiples of $1/3$, and so forth. You can be excused for thinking that "finally" there will be no more places left for putting more points on the line. But in fact this is quite wrong. The ancient Greeks already understood that "holes" would remain in the number line even after all the rational numbers had been marked off. For instance, there are no integers p and q such that $\sqrt{2} = p/q$. Hence, $\sqrt{2}$ is not a rational number.[2]

The rational numbers are therefore insufficient for measuring all possible lengths, let alone areas and volumes. This deficiency can be remedied by extending the concept of numbers to allow for the so-called irrational numbers. This extension can be carried out rather naturally by using decimal notation for numbers, as explained below.

The way most people write numbers today is called the *decimal system*, or the *base 10 system*. It is a positional system with 10 as the base number. Every natural number can be written using only the symbols, $0, 1, 2, \ldots, 9$, which are called *digits*.[3] The positional system defines each combination of digits as a sum of powers of 10. For example,

$$1\,984 = 1 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10^1 + 4 \cdot 10^0$$

Each natural number can be uniquely expressed in this manner. With the use of the signs $+$ and $-$, all integers, positive or negative, can be written in the same way. Decimal points also enable us to express rational numbers other than natural numbers. For example,

$$3.1415 = 3 + 1/10^1 + 4/10^2 + 1/10^3 + 5/10^4$$

Rational numbers that can be written exactly using only a finite number of decimal places are called *finite decimal fractions*.

[2] Euclid proved this fact in around the year 300 BCE.

[3] You may recall that a digit is either a finger or a thumb, and that most humans have ten digits.

Each finite decimal fraction is a rational number, but not every rational number can be written as a finite decimal fraction. We also need to allow for *infinite decimal fractions* such as

$$100/3 = 33.333\ldots$$

where the three dots indicate that the digit 3 is repeated indefinitely.

If the decimal fraction is a rational number, then it will always be *recurring* or *periodic*—that is, after a certain place in the decimal expansion, it either stops or continues to repeat a finite sequence of digits. For example,

$$11/70 = 0.1\underbrace{571428}\underbrace{571428}5\ldots$$

with the sequence of six digits 571428 repeated infinitely often.

The definition of a real number follows from the previous discussion. We define a *real number* as an arbitrary infinite decimal fraction. Hence, a real number is of the form $x = \pm m.\alpha_1\alpha_2\alpha_3\ldots$, where m is a nonnegative integer, and for each natural number n, α_n is a digit in the range 0 to 9.

We have already identified the periodic decimal fractions with the rational numbers. In addition, there are infinitely many new numbers given by the nonperiodic decimal fractions. These are called *irrational numbers*. Examples include $\sqrt{2}$, $-\sqrt{5}$, π, $2^{\sqrt{2}}$, and $0.12112111211112\ldots$.[4]

We mentioned earlier that each rational number can be represented by a point on the number line. But not all points on the number line represent rational numbers. It is as if the irrational numbers "close up" the remaining holes on the number line after all the rational numbers have been positioned. Hence, an unbroken and endless straight line with an origin and a positive unit of length is a satisfactory model for the real numbers. We frequently state that there is a *one-to-one correspondence* between the real numbers and the points on a number line. Often, too, one speaks of the "real line" rather than the "number line".

The set of rational numbers as well as the set of irrational numbers are said to be "dense" on the number line. This means that between any two different real numbers, irrespective of how close they are to each other, we can always find both a rational and an irrational number—in fact, we can always find infinitely many of each.

When applied to the real numbers, the four basic arithmetic operations always result in a real number. The only exception is that we cannot divide by 0: in the words of American comedian Steven Wright, "Black holes are where God divided by zero."

DIVISION BY ZERO

The ratio $p/0$ is *not* defined for any real number p.

This is very important and should not be confused with the fact that $0/a = 0$, for all $a \neq 0$. Notice especially that $0/0$ is not defined as any real number. For example, if a

[4] In general, it is very difficult to decide whether a given number is rational or irrational. It has been known since the year 1776 that π is irrational and since 1927 that $2^{\sqrt{2}}$ is irrational. However, there are many numbers about which we still do not know whether they are irrational or not.

car requires 60 litres of fuel to go 600 kilometres, then its fuel consumption is $60/600 =$ 10 litres per 100 kilometres. However, if told that a car uses 0 litres of fuel to go 0 kilometres, we know nothing about its fuel consumption; $0/0$ is completely undefined.

EXERCISES FOR SECTION 2.1

1. Which of the following statements are true?

(a) 1984 is a natural number.

(b) -5 is to the right of -3 on the number line.

(c) -13 is a natural number.

(d) There is no natural number that is not rational.

(e) 3.1415 is not rational.

(f) The sum of two irrational numbers is irrational.

(g) $-3/4$ is rational.

(h) All rational numbers are real.

2. Explain why the infinite decimal expansion

$$1.01001000100001000001\ldots$$

is not a rational number.

2.2 Integer Powers

You should recall that we often write 3^4 instead of the product $3 \cdot 3 \cdot 3 \cdot 3$, that $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ can be written as $\left(\frac{1}{2}\right)^5$, and that $(-10)^3 = (-10)(-10)(-10) = -1000$. If a is any number and n is any natural number, then a^n is defined by

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

The expression a^n is called the *n-th power* of a; here a is the *base*, and n is the *exponent*. We have, for example, $a^2 = a \cdot a$, $x^4 = x \cdot x \cdot x \cdot x$, and

$$\left(\frac{p}{q}\right)^5 = \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q} \cdot \frac{p}{q}$$

where $a = p/q$, and $n = 5$. By convention, $a^1 = a$, a "product" with only one factor.

We usually drop the multiplication sign if this is unlikely to create misunderstanding. For example, we write abc instead of $a \cdot b \cdot c$, but it is safest to keep the multiplication sign in $1.05^3 = 1.05 \cdot 1.05 \cdot 1.05$.

For any real number $a \neq 0$, we define, further, $a^0 = 1$. Thus, $5^0 = 1$; $(-16.2)^0 = 1$; and $(x \cdot y)^0 = 1$, if $x \cdot y \neq 0$. But if $a = 0$, we do *not* assign a numerical value to a^0: the expression 0^0 is *undefined*.

We also need to define powers with negative exponents. What do we mean by 3^{-2}? It turns out that the sensible definition is to set 3^{-2} equal to $1/3^2 = 1/9$. In general,

$$a^{-n} = \frac{1}{a^n}$$

whenever n is a natural number and $a \neq 0$. In particular, $a^{-1} = 1/a$. In this way we have defined a^x for all integers x.[5]

Properties of Powers

There are some rules for powers that you really must not only know by heart, but understand why they are true. The two most important are:

PROPERTIES OF POWERS

For any real number a, and any integer numbers r and s:

(i) $a^r \cdot a^s = a^{r+s}$ (ii) $(a^r)^s = a^{rs}$

Note carefully what these rules say. According to rule (i), powers with the same base are multiplied by *adding* the exponents. For example,

$$a^3 \cdot a^5 = \underbrace{a \cdot a \cdot a}_{\text{3 times}} \cdot \underbrace{a \cdot a \cdot a \cdot a \cdot a}_{\text{5 times}} = \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}_{3+5=8 \text{ times}} = a^8 = a^{3+5}$$

Here is an example of rule (ii):

$$(a^2)^4 = \underbrace{\underbrace{a \cdot a}_{\text{2 times}} \cdot \underbrace{a \cdot a}_{\text{2 times}} \cdot \underbrace{a \cdot a}_{\text{2 times}} \cdot \underbrace{a \cdot a}_{\text{2 times}}}_{\text{4 times}} = \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}_{4 \cdot 2 = 8 \text{ times}} = a^8 = a^{2 \cdot 4}$$

Division of two powers with the same non-zero base goes like this:

$$a^r \div a^s = \frac{a^r}{a^s} = a^r \frac{1}{a^s} = a^r \cdot a^{-s} = a^{r-s}$$

Thus we divide two powers with the same base by *subtracting* the exponent in the denominator from that in the numerator.[6] For example, $a^3 \div a^5 = a^{3-5} = a^{-2}$.

Finally, note that

$$(ab)^r = \underbrace{ab \cdot ab \cdot \cdots \cdot ab}_{r \text{ times}} = \underbrace{a \cdot a \cdot \cdots \cdot a}_{r \text{ times}} \cdot \underbrace{b \cdot b \cdot \cdots \cdot b}_{r \text{ times}} = a^r b^r$$

---

[5] Calculators usually have a power key, denoted by $\boxed{y^x}$ or $\boxed{a^x}$, which can be used to compute powers. Make sure you know how to use it by computing 2^3, which is 8; 3^2, which is 9; and 25^{-3}, which is 0.000064.

[6] An important motivation for introducing the definitions $a^0 = 1$ and $a^{-n} = 1/a^n$ is that we want the rules for powers to be valid for negative and zero exponents as well as for positive ones. For example, we want $a^r \cdot a^s = a^{r+s}$ to be valid when $r = 5$ and $s = 0$. This requires that $a^5 \cdot a^0 = a^{5+0} = a^5$, so we must choose $a^0 = 1$. If $a^n \cdot a^m = a^{n+m}$ is to be valid when $m = -n$, we must have $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$. Because $a^n \cdot (1/a^n) = 1$, we *must* define a^{-n} to be $1/a^n$.

and

$$\left(\frac{a}{b}\right)^r = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdot \cdots \cdot \frac{a}{b}}_{r \text{ times}} = \frac{\overbrace{a \cdot a \cdot \cdots \cdot a}^{r \text{ times}}}{\underbrace{b \cdot b \cdot \cdots \cdot b}_{r \text{ times}}} = \frac{a^r}{b^r} = a^r b^{-r}$$

These rules can be extended to cases where there are several factors. For instance,

$$(abcde)^r = a^r b^r c^r d^r e^r$$

We saw that $(ab)^r = a^r b^r$. What about $(a+b)^r$? One of the most common errors committed in elementary algebra is to equate this to $a^r + b^r$. For example, $(2+3)^3 = 5^3 = 125$, but $2^3 + 3^3 = 8 + 27 = 35$. Thus, in general, $(a+b)^r \neq a^r + b^r$.

EXAMPLE 2.2.1 Simplify the expressions:

(a) $x^p x^{2p}$ (b) $t^s \div t^{s-1}$

(c) $a^2 b^3 a^{-1} b^5$ (d) $\dfrac{t^p t^{q-1}}{t^r t^{s-1}}$

Solution:

(a) $x^p x^{2p} = x^{p+2p} = x^{3p}$

(b) $t^s \div t^{s-1} = t^{s-(s-1)} = t^{s-s+1} = t^1 = t$

(c) $a^2 b^3 a^{-1} b^5 = a^2 a^{-1} b^3 b^5 = a^{2-1} b^{3+5} = a^1 b^8 = ab^8$

(d) $\dfrac{t^p \cdot t^{q-1}}{t^r \cdot t^{s-1}} = \dfrac{t^{p+q-1}}{t^{r+s-1}} = t^{p+q-1-(r+s-1)} = t^{p+q-1-r-s+1} = t^{p+q-r-s}$ ▮

EXAMPLE 2.2.2 If $x^{-2}y^3 = 5$, compute $x^{-4}y^6$, x^6y^{-9}, and $x^2y^{-3} + 2x^{-10}y^{15}$.

Solution: In computing $x^{-4}y^6$, how can we make use of the assumption that $x^{-2}y^3 = 5$? A moment's reflection might lead you to see that $(x^{-2}y^3)^2 = x^{-4}y^6$, and hence $x^{-4}y^6 = 5^2 = 25$. Similarly,

$$x^6y^{-9} = (x^{-2}y^3)^{-3} = 5^{-3} = 1/125$$

and

$$x^2y^{-3} + 2x^{-10}y^{15} = (x^{-2}y^3)^{-1} + 2(x^{-2}y^3)^5 = 5^{-1} + 2 \cdot 5^5 = 6250.2$$ ▮

EXAMPLE 2.2.3 It is easy to make mistakes when dealing with powers. The following examples highlight some common sources of confusion.

(a) There is an important difference between $(-10)^2 = (-10)(-10) = 100$, and $-10^2 = -(10 \cdot 10) = -100$. The square of minus 10 is not equal to minus the square of 10.

(b) Note that $(2x)^{-1} = 1/(2x)$. Here the product $2x$ is raised to the power of -1. On the other hand, in $2x^{-1}$ only x is raised to the power -1, so $2x^{-1} = 2 \cdot (1/x) = 2/x$.

(c) The volume of a ball with radius r is $\frac{4}{3}\pi r^3$. What will the volume be if the radius is doubled? The new volume is

$$\frac{4}{3}\pi(2r)^3 = \frac{4}{3}\pi(2r)(2r)(2r) = \frac{4}{3}\pi 8r^3 = 8\left(\frac{4}{3}\pi r^3\right)$$

so the volume is 8 times the initial one. If we made the mistake of "simplifying" $(2r)^3$ to $2r^3$, the result would imply only a doubling of the volume; this should be contrary to common sense.

Compound Interest

Powers are used in practically every branch of applied mathematics, including economics. To illustrate their use, recall how they are needed to calculate compound interest.

Suppose you deposit \$1 000 in a bank account paying 8% interest at the end of each year.[7] After one year you will have earned $\$1\,000 \cdot 0.08 = \$80$ in interest, so the amount in your bank account will be \$1 080. This can be rewritten as

$$1000 + \frac{1000 \cdot 8}{100} = 1000\left(1 + \frac{8}{100}\right) = 1000 \cdot 1.08$$

Suppose this new amount of \$1 080 is left in the bank for another year at an interest rate of 8%. After a second year, the extra interest will be $\$1\,000 \cdot 1.08 \cdot 0.08$. So the total amount will have grown to

$$1000 \cdot 1.08 + (1000 \cdot 1.08) \cdot 0.08 = 1000 \cdot 1.08(1 + 0.08) = 1000 \cdot (1.08)^2$$

Each year the amount will increase by the factor 1.08, and we see that at the end of t years it will have grown to $\$1\,000 \cdot (1.08)^t$.

If the original amount is $\$K$ and the interest rate is $p\%$ per year, by the end of the first year, the amount will be $K + K \cdot p/100 = K(1 + p/100)$ dollars. The growth factor per year is thus $1 + p/100$. In general, after t (whole) years, the original investment of $\$K$ will have grown to an amount

$$K\left(1 + \frac{p}{100}\right)^t$$

when the interest rate is $p\%$ per year and interest is added to the capital every year—that is, there is compound interest.

This example illustrates a general principle:

EXPONENTIAL GROWTH

A quantity K which increases by $p\%$ per year will have increased to

$$K\left(1 + \frac{p}{100}\right)^t$$

after t years. Here $1 + p/100$ is called the *growth factor* for a growth of $p\%$.

[7] Remember that 1% means one in a hundred, or 0.01. So 23%, for example, is $23 \cdot 0.01 = 0.23$. To calculate 23% of 4000, we write $4000 \cdot 23/100 = 920$ or $4000 \cdot 0.23 = 920$.

If you see an expression like $(1.08)^t$ you should immediately be able to recognize it as the amount to which \$1 has grown after t years when the interest rate is 8% per year. How should you interpret $(1.08)^0$? You deposit \$1 at 8% per year, and leave the amount for 0 years. Then you still have only \$1, because there has been no time to accumulate any interest, so that $(1.08)^0$ must necessarily equal 1.[8]

EXAMPLE 2.2.4 A new car has been bought for \$15 000 and is assumed to decrease in value (depreciate) by 15% per year over a six-year period. What is its value after six years?

Solution: After one year its value is down to

$$15\,000 - \frac{15\,000 \cdot 15}{100} = 15\,000 \cdot \left(1 - \frac{15}{100}\right) = 15\,000 \cdot 0.85 = 12\,750$$

After two years its value is $\$15\,000 \cdot (0.85)^2 = \$10\,837.50$, and so on. After six years we realize that its value must be $\$15\,000 \cdot (0.85)^6 \approx \$5\,657$.

This example illustrates a general principle:

EXPONENTIAL DECLINE

A quantity K which decreases by $p\%$ per year will have decreased to

$$K\left(1 - \frac{p}{100}\right)^t$$

after t years. Here $1 - p/100$ is called the *growth factor* for a decline of $p\%$ a year. (Note that a growth factor that is less than 1 indicates shrinkage.)

Do We Really Need Negative Exponents?

How much money should you have deposited in a bank five years ago in order to have \$1 000 today, given that the interest rate has been 8% per year over this period? If we call this amount x, the requirement is that $x \cdot (1.08)^5$ must equal \$1 000, or that $x \cdot (1.08)^5 = 1000$. Dividing by 1.08^5 on both sides yields

$$x = \frac{1000}{(1.08)^5} = 1000 \cdot (1.08)^{-5}$$

which is approximately \$681. Thus, $\$(1.08)^{-5}$ is what you should have deposited five years ago in order to have \$1 today, given the constant interest rate of 8%.

In general, $\$P(1 + p/100)^{-t}$ *is what you should have deposited t years ago in order to have \$P today, if the interest rate has been p% every year.*

[8] Note that $1000 \cdot (1.08)^5$ is the amount you will have in your account after five years if you invest \$1 000 at 8% interest per year. Using a calculator, you find that you will have approximately \$1 469.33. A rather common mistake is to put $1000 \cdot (1.08)^5 = (1000 \cdot 1.08)^5 = (1080)^5$. This is a trillion ($10^{12}$) times the right answer!

EXERCISES FOR SECTION 2.2

1. Compute the following numbers: (a) 10^3; (b) $(-0.3)^2$; (c) 4^{-2}; and (d) $(0.1)^{-1}$.

2. Write as powers of 2 the following numbers: (a) 4; (b) 1; (c) 64; and (d) 1/16.

3. Write as powers the following numbers:

(a) $15 \cdot 15 \cdot 15$ (b) $\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$ (c) $\frac{1}{10}$ (d) 0.0000001

(e) $t\,t\,t\,t\,t\,t$ (f) $(a-b)(a-b)(a-b)$ (g) $a\,a\,b\,b\,b\,b$ (h) $(-a)(-a)(-a)$

4. Expand and simplify the following expressions:

(a) $2^5 \cdot 2^5$ (b) $3^8 \cdot 3^{-2} \cdot 3^{-3}$ (c) $(2x)^3$ (d) $(-3xy^2)^3$

(e) $\dfrac{p^{24}p^3}{p^4 p}$ (f) $\dfrac{a^4 b^{-3}}{(a^2 b^{-3})^2}$ (g) $\dfrac{3^4(3^2)^6}{(-3)^{15}3^7}$ (h) $\dfrac{p^{\gamma}(pq)^{\sigma}}{p^{2\gamma+\sigma}q^{\sigma-2}}$

5. Expand and simplify the following expressions:

(a) $2^0 \cdot 2^1 \cdot 2^2 \cdot 2^3$ (b) $\left(\dfrac{4}{3}\right)^3$ (c) $\dfrac{4^2 \cdot 6^2}{3^3 \cdot 2^3}$

(d) x^5x^4 (e) $y^5y^4y^3$ (f) $(2xy)^3$

(g) $\dfrac{10^2 \cdot 10^{-4} \cdot 10^3}{10^0 \cdot 10^{-2} \cdot 10^5}$ (h) $\dfrac{(k^2)^3k^4}{(k^3)^2}$ (i) $\dfrac{(x+1)^3(x+1)^{-2}}{(x+1)^2(x+1)^{-3}}$

6. The surface area of a sphere with radius r is $4\pi r^2$.

(a) By what factor will the surface area increase if the radius is tripled?

(b) If the radius increases by 16%, by how many % will the surface area increase?

7. Suppose that a and b are positive, while m and n are integers. Which of the following equalities are true and which are false?

(a) $a^0 = 0$ (b) $(a+b)^{-n} = 1/(a+b)^n$ (c) $a^m \cdot a^m = a^{2m}$

(d) $a^m \cdot b^m = (ab)^{2m}$ (e) $(a+b)^m = a^m + b^m$ (f) $a^n \cdot b^m = (ab)^{n+m}$

8. Complete the following implications:

(a) $xy = 3 \Rightarrow x^3y^3 = \dots$ (b) $ab = -2 \Rightarrow (ab)^4 = \dots$

(c) $a^2 = 4 \Rightarrow (a^8)^0 = \dots$ (d) n integer implies $(-1)^{2n} = \dots$

9. Compute the following: (a) 13% of 150; (b) 6% of 2400; and (c) 5.5% of 200.

10. Give economic interpretations to each of the following expressions and then use a calculator to find the approximate values: (a) $50 $\cdot (1.11)^8$; (b) €10 000 $\cdot (1.12)^{20}$; and (c) £5 000 $\cdot (1.07)^{-10}$.

11. A box containing five balls costs €8.50. If the balls are bought individually, they cost €2.00 each. How much cheaper is it, in percentage terms, to buy the box as opposed to buying five individual balls?

12. (a) £12 000 is deposited in an account earning 4% interest per year. What is the amount after 15 years?

    (b) If the interest rate is 6% each year, how much money should you have deposited in a bank five years ago to have £50 000 today?

13. A quantity increases by 25% each year for three years. How much is the combined percentage growth p over the three-year period?

14. A firm's profit increased from 2010 to 2011 by 20%, but it decreased by 17% from 2011 to 2012.

    (a) Which of the years 2010 and 2012 had the higher profit?

    (b) What percentage decrease in profits from 2011 to 2012 would imply that profits were equal in 2010 and 2012?

2.3 Rules of Algebra

You are certainly already familiar with the most common rules of algebra. We have already used some in this chapter. Nevertheless, it may be useful to recall those that are most important.

RULES OF ALGEBRA

If a, b, and c are arbitrary numbers, then:

(i) $a + b = b + a$ (ii) $(a + b) + c = a + (b + c)$

(iii) $a + 0 = a$ (iv) $a + (-a) = 0$

(v) $ab = ba$ (vi) $(ab)c = a(bc)$

(vii) $1 \cdot a = a$ (viii) $aa^{-1} = 1$ for $a \neq 0$

(ix) $(-a)b = a(-b) = -ab$ (x) $(-a)(-b) = ab$

(xi) $a(b + c) = ab + ac$ (xii) $(a + b)c = ac + bc$

EXAMPLE 2.3.1 These rules are used in the following equalities:

(a) $5 + x^2 = x^2 + 5$ (b) $(a + 2b) + 3b = a + (2b + 3b) = a + 5b$

(c) $x\frac{1}{3} = \frac{1}{3}x$ (d) $(xy)y^{-1} = x(yy^{-1}) = x$

(e) $(-3)5 = 3(-5) = -(3 \cdot 5) = -15$ (f) $(-6)(-20) = 120$

(g) $3x(y + 2z) = 3xy + 6xz$ (h) $(t^2 + 2t)4t^3 = t^2 4t^3 + 2t4t^3 = 4t^5 + 8t^4$

The algebraic rules can be combined in several ways to give:

$$a(b-c) = a[b+(-c)] = ab + a(-c) = ab - ac$$

$$x(a+b-c+d) = xa + xb - xc + xd$$

$$(a+b)(c+d) = ac + ad + bc + bd$$

Figure 2.3.1 provides a geometric argument for the last of these algebraic rules for the case in which the numbers a, b, c, and d are all positive. The area $(a+b)(c+d)$ of the large rectangle is the sum of the areas of the four small rectangles.

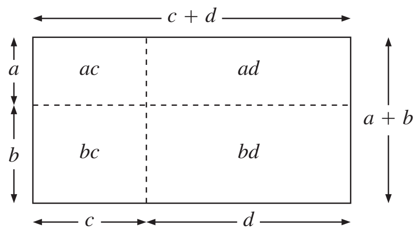

Figure 2.3.1 $(a+b)(c+d) = ac + ad + bc + bd$

Recall the following three "quadratic identities", which are so important that you should definitely memorize them.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

The last of these is called the *difference-of-squares formula*. The proofs are very easy. For example, $(a+b)^2$ means $(a+b)(a+b)$, which equals $aa + ab + ba + bb = a^2 + 2ab + b^2$.

EXAMPLE 2.3.2 Expand the following expressions:

(a) $(3x+2y)^2$ (b) $(1-2z)^2$ (c) $(4p+5q)(4p-5q)$

Solution:

(a) $(3x+2y)^2 = (3x)^2 + 2(3x)(2y) + (2y)^2 = 9x^2 + 12xy + 4y^2$

(b) $(1-2z)^2 = 1 - 2 \cdot 1 \cdot 2 \cdot z + (2z)^2 = 1 - 4z + 4z^2$

(c) $(4p+5q)(4p-5q) = (4p)^2 - (5q)^2 = 16p^2 - 25q^2$

We often encounter parentheses with a minus sign in front. Because $(-1)x = -x$,

$$-(a+b-c+d) = -a - b + c - d$$

In words: *When removing a pair of parentheses with a minus in front, change the signs of* all *the terms within the parentheses — do not leave any out*.

We saw how to multiply two factors, $(a+b)$ and $(c+d)$. How do we compute such products when there are several factors? Here is an example:

$$\begin{aligned}(a+b)(c+d)(e+f) &= [(a+b)(c+d)](e+f) \\ &= (ac+ad+bc+bd)(e+f) \\ &= (ac+ad+bc+bd)e+(ac+ad+bc+bd)f \\ &= ace+ade+bce+bde+acf+adf+bcf+bdf\end{aligned}$$

EXAMPLE 2.3.3 Expand the expression $(r+1)^3$. Use the solution to compute by how much the volume of a ball with radius r metres expands if the radius increases by one metre.

Solution:

$$(r+1)^3 = [(r+1)(r+1)]\,(r+1) = (r^2+2r+1)(r+1) = r^3+3r^2+3r+1$$

A ball with radius r metres has a volume of $\frac{4}{3}\pi r^3$ cubic metres. If the radius increases by one metre, its volume expands by

$$\frac{4}{3}\pi(r+1)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(r^3+3r^2+3r+1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2+3r+1)$$

Algebraic Expressions

Expressions involving letters such as $3xy - 5x^2y^3 + 2xy + 6y^3x^2 - 3x + 5yx + 8$ are called *algebraic expressions*. We call $3xy$, $-5x^2y^3$, $2xy$, $6y^3x^2$, $-3x$, $5yx$, and 8 the *terms* in the expression that is formed by adding all the terms together. The numbers 3, -5, 2, 6, -3, and 5 are the *numerical coefficients* of the first six terms. Two terms where only the numerical coefficients are different, such as $-5x^2y^3$ and $6y^3x^2$, are called *terms of the same type*. In order to simplify expressions, we collect terms of the same type. Then within each term, we put numerical coefficients first and place the letters in alphabetical order. Thus,

$$3xy - 5x^2y^3 + 2xy + 6y^3x^2 - 3x + 5yx + 8 = x^2y^3 + 10xy - 3x + 8$$

EXAMPLE 2.3.4 Expand and simplify the expression:

$$(2pq - 3p^2)(p+2q) - (q^2 - 2pq)(2p - q)$$

Solution: The expression equals

$$\begin{aligned}&2pqp + 2pq2q - 3p^3 - 6p^2q - (q^22p - q^3 - 4pqp + 2pq^2) \\ &\quad = 2p^2q + 4pq^2 - 3p^3 - 6p^2q - 2pq^2 + q^3 + 4p^2q - 2pq^2 \\ &\quad = -3p^3 + q^3\end{aligned}$$

Factoring

When we write $49 = 7 \cdot 7$ and $672 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7$, we have factored these numbers. Algebraic expressions can often be factored in a similar way: to *factor an expression* means to express it as a product of simpler factors. For example, $6x^2y = 2 \cdot 3 \cdot x \cdot x \cdot y$ and $5x^2y^3 - 15xy^2 = 5 \cdot x \cdot y \cdot y(xy - 3)$.[9]

EXAMPLE 2.3.5 Factor each of the following expressions:

(a) $5x^2 + 15x$ (b) $-18b^2 + 9ab$

(c) $K(1 + r) + K(1 + r)r$ (d) $\delta L^{-3} + (1 - \delta)L^{-2}$

Solution:

(a) $5x^2 + 15x = 5x(x + 3)$

(b) $-18b^2 + 9ab = 9ab - 18b^2 = 3 \cdot 3b(a - 2b)$

(c) $K(1 + r) + K(1 + r)r = K(1 + r)(1 + r) = K(1 + r)^2$

(d) $\delta L^{-3} + (1 - \delta)L^{-2} = L^{-3}[\delta + (1 - \delta)L]$

The "quadratic identities" can often be used in reverse for factoring. They sometimes enable us to factor expressions that otherwise appear to have no factors.

EXAMPLE 2.3.6 Factor each of the following expressions:

(a) $16a^2 - 1$ (b) $x^2y^2 - 25z^2$

(c) $4u^2 + 8u + 4$ (d) $x^2 - x + \frac{1}{4}$

Solution:

(a) $16a^2 - 1 = (4a + 1)(4a - 1)$

(b) $x^2y^2 - 25z^2 = (xy + 5z)(xy - 5z)$

(c) $4u^2 + 8u + 4 = 4(u^2 + 2u + 1) = 4(u + 1)^2$

(d) $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2$

Sometimes one has to show a measure of inventiveness to find a factoring:

$$\begin{aligned} 4x^2 - y^2 + 6x^2 + 3xy &= (4x^2 - y^2) + 3x(2x + y) \\ &= (2x + y)(2x - y) + 3x(2x + y) \\ &= (2x + y)(2x - y + 3x) \\ &= (2x + y)(5x - y) \end{aligned}$$

[9] Note that $9x^2 - 25y^2 = 3 \cdot 3 \cdot x \cdot x - 5 \cdot 5 \cdot y \cdot y$ does *not* factor $9x^2 - 25y^2$. A correct factoring is $9x^2 - 25y^2 = (3x - 5y)(3x + 5y)$.

Although it might be difficult, sometimes even impossible, to find a factoring, it is very easy to verify that an algebraic expression has been factored correctly by simply multiplying the factors. For example, we check that

$$x^2 - (a+b)x + ab = (x-a)(x-b)$$

by expanding $(x-a)(x-b)$.

Most algebraic expressions *cannot* be factored. For example, there is no way to write $x^2 + 10x + 50$ as a product of simpler factors.[10]

EXERCISES FOR SECTION 2.3

1. Expand and simplify the following expressions:

(a) $-3 + (-4) - (-8)$ (b) $(-3)(2-4)$ (c) $(-3)(-12)(-\frac{1}{2})$

(d) $-3[4-(-2)]$ (e) $-3(-x-4)$ (f) $(5x-3y)9$

(g) $2x\left(\dfrac{3}{2x}\right)$ (h) $0 \cdot (1-x)$ (i) $-7x\dfrac{2}{14x}$

2. Expand and simplify the following expressions:

(a) $5a^2 - 3b - (-a^2 - b) - 3(a^2 + b)$ (b) $-x(2x-y) + y(1-x) + 3(x+y)$

(c) $12t^2 - 3t + 16 - 2(6t^2 - 2t + 8)$ (d) $r^3 - 3r^2s + s^3 - (-s^3 - r^3 + 3r^2s)$

3. Expand and simplify the following expressions:

(a) $-3(n^2 - 2n + 3)$ (b) $x^2(1+x^3)$ (c) $(4n-3)(n-2)$

(d) $6a^2b(5ab - 3ab^2)$ (e) $(a^2b - ab^2)(a+b)$ (f) $(x-y)(x-2y)(x-3y)$

(g) $(ax+b)(cx+d)$ (h) $(2-t^2)(2+t^2)$ (i) $(u-v)^2(u+v)^2$

SM 4. Expand and simplify the following expressions:

(a) $(2t-1)(t^2-2t+1)$ (b) $(a+1)^2 + (a-1)^2 - 2(a+1)(a-1)$

(c) $(x+y+z)^2$ (d) $(x+y+z)^2 - (x-y-z)^2$

5. Expand the following expressions:

(a) $(x+2y)^2$ (b) $\left(\frac{1}{x} - x\right)^2$

(c) $(3u-5v)^2$ (d) $(2z-5w)(2z+5w)$

6. Complete the following expressions:

(a) $201^2 - 199^2 =$ (b) If $u^2 - 4u + 4 = 1$, then $u =$ (c) $\dfrac{(a+1)^2 - (a-1)^2}{(b+1)^2 - (b-1)^2} =$

7. Compute $1000^2/(252^2 - 248^2)$ without using a calculator.

---

[10] Unless we introduce "complex" numbers, that is.

8. Verify the following cubic identities, which are occasionally useful:

(a) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (b) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

(c) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ (d) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

9. Factor the following expressions:

(a) $21x^2y^3$ (b) $3x - 9y + 27z$ (c) $a^3 - a^2b$

(d) $8x^2y^2 - 16xy$ (e) $28a^2b^3$ (f) $4x + 8y - 24z$

(g) $2x^2 - 6xy$ (h) $4a^2b^3 + 6a^3b^2$ (i) $7x^2 - 49xy$

(j) $5xy^2 - 45x^3y^2$ (k) $16 - b^2$ (l) $3x^2 - 12$

10. Factor the following expressions:

(a) $a^2 + 4ab + 4b^2$ (b) $K^2L - L^2K$ (c) $K^{-4} - LK^{-5}$

(d) $9z^2 - 16w^2$ (e) $-\frac{1}{5}x^2 + 2xy - 5y^2$ (f) $a^4 - b^4$

11. Factor the following expressions:

(a) $x^2 - 4x + 4$ (b) $4t^2s - 8ts^2$ (c) $16a^2 + 16ab + 4b^2$

(d) $5x^3 - 10xy^2$ (e) $5x + 5y + ax + ay$ (f) $u^2 - v^2 + 3v + 3u$

(g) $P^3 + Q^3 + Q^2P + P^2Q$ (h) $K^3 - K^2L$ (i) $KL^3 + KL$

(j) $L^2 - K^2$ (k) $K^2 - 2KL + L^2$ (l) $K^3L - 4K^2L^2 + 4KL^3$

2.4 Fractions

Recall that

$$a \div b = \frac{a}{b} \begin{array}{l} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{array}$$

For example, $5 \div 8 = \frac{5}{8}$. For typographical reasons we often write $5/8$ instead of $\frac{5}{8}$. Of course, $5 \div 8 = 0.625$, in which case we have written the fraction as a decimal number. The fraction $5/8$ is called a *proper fraction* because 5 is less than 8. The fraction $19/8$ is an *improper fraction* because the numerator is larger than (or equal to) the denominator. An improper fraction can be written as a *mixed number*:[11]

$$\frac{19}{8} = 2 + \frac{3}{8} = 2\frac{3}{8}$$

The most important properties of fractions are listed below, with simple numerical examples. It is absolutely essential for you to master these rules, so you should carefully check that you know each of them.

[11] Here $2\frac{3}{8}$ means 2 *plus* $\frac{3}{8}$. On the other hand, $2 \cdot \frac{3}{8} = \frac{2 \cdot 3}{8} = \frac{3}{4}$ (by the rules reviewed in what follows). Note, however, that $2\frac{x}{8}$ also means $2 \cdot \frac{x}{8}$; the notation $\frac{2x}{8}$ or $2x/8$ is obviously preferable in this case. Indeed, $\frac{19}{8}$ or $19/8$ is probably better than $2\frac{3}{8}$ because it also helps avoid ambiguity.

PROPERTIES OF FRACTIONS

Let a, b and c be any numbers, with the proviso that $b \neq 0$ and $c \neq 0$ whenever they appear in a denominator. Then,

(i) $\dfrac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \dfrac{a}{b}$

(ii) $\dfrac{-a}{-b} = \dfrac{(-a)\cdot(-1)}{(-b)\cdot(-1)} = \dfrac{a}{b}$

(iii) $-\dfrac{a}{b} = (-1)\dfrac{a}{b} = \dfrac{(-1)a}{b} = \dfrac{-a}{b}$

(iv) $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$

(v) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a \cdot d + b \cdot c}{b \cdot d}$

(vi) $a + \dfrac{b}{c} = \dfrac{a \cdot c + b}{c}$

(vii) $a \cdot \dfrac{b}{c} = \dfrac{a \cdot b}{c}$

(viii) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{a \cdot c}{b \cdot d}$

(ix) $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{a \cdot d}{b \cdot c}$

EXAMPLE 2.4.1 The following expressions illustrate the properties of fractions, one by one:

(a) $\dfrac{21}{15} = \dfrac{7 \cdot \cancel{3}}{5 \cdot \cancel{3}} = \dfrac{7}{5}$

(b) $\dfrac{-5}{-6} = \dfrac{5}{6}$

(c) $-\dfrac{13}{15} = (-1)\dfrac{13}{15} = \dfrac{(-1)13}{15} = \dfrac{-13}{15}$

(d) $\dfrac{5}{3} + \dfrac{13}{3} = \dfrac{18}{3} = 6$

(e) $\dfrac{3}{5} + \dfrac{1}{6} = \dfrac{3 \cdot 6 + 5 \cdot 1}{5 \cdot 6} = \dfrac{23}{30}$

(f) $5 + \dfrac{3}{5} = \dfrac{5 \cdot 5 + 3}{5} = \dfrac{28}{5}$

(g) $7 \cdot \dfrac{3}{5} = \dfrac{21}{5}$

(h) $\dfrac{4}{7} \cdot \dfrac{5}{8} = \dfrac{4 \cdot 5}{7 \cdot 8} = \dfrac{\cancel{4} \cdot 5}{7 \cdot 2 \cdot \cancel{4}} = \dfrac{5}{14}$

(i) $\dfrac{3}{8} \div \dfrac{6}{14} = \dfrac{3}{8} \cdot \dfrac{14}{6} = \dfrac{\cancel{3} \cdot \cancel{2} \cdot 7}{\cancel{2} \cdot 2 \cdot 2 \cdot 2 \cdot \cancel{3}} = \dfrac{7}{8}$

Property (i) is very important. It is the rule used to reduce fractions by factoring the numerator and the denominator, then cancelling *common factors*—that is, dividing both the numerator and denominator by the same nonzero quantity.[12]

EXAMPLE 2.4.2 Simplify the expressions:

(a) $\dfrac{5x^2yz^3}{25xy^2z}$ (b) $\dfrac{x^2 + xy}{x^2 - y^2}$ (c) $\dfrac{4 - 4a + a^2}{a^2 - 4}$

[12] When we use property (i) in reverse, we are *expanding* the fraction. For example, $5/8 = 5 \cdot 125/8 \cdot 125 = 625/1000 = 0.625$.

Solution:

(a) $\dfrac{5x^2yz^3}{25xy^2z} = \dfrac{\cancel{5}\cdot\cancel{x}\cdot x\cdot\cancel{y}\cdot\cancel{z}\cdot z\cdot z}{\cancel{5}\cdot 5\cdot\cancel{x}\cdot\cancel{y}\cdot y\cdot\cancel{z}} = \dfrac{xz^2}{5y}$

(b) $\dfrac{x^2+xy}{x^2-y^2} = \dfrac{x(x+y)}{(x-y)(x+y)} = \dfrac{x}{x-y}$

(c) $\dfrac{4-4a+a^2}{a^2-4} = \dfrac{(a-2)(a-2)}{(a-2)(a+2)} = \dfrac{a-2}{a+2}$

EXAMPLE 2.4.3 When we simplify fractions, only *common* factors can be removed. A frequently occurring error is illustrated in the following expression:

$$\textit{Wrong!} \quad \rightarrow \quad \frac{2\cancel{x}+3y}{\cancel{x}y} = \frac{2+3\cancel{y}}{\cancel{y}} = \frac{2+3}{1} = 5$$

In fact, the numerator and the denominator in the fraction $(2x+3y)/xy$ do not have any common factors. But a correct simplification is this: $(2x+3y)/xy = 2/y + 3/x$.

Another common error is:

$$\textit{Wrong!} \quad \rightarrow \quad \frac{x}{x^2+2x} = \frac{x}{x^2} + \frac{x}{2x} = \frac{1}{x} + \frac{1}{2}$$

A correct way of simplifying the fraction is to cancel the common factor x, which yields the fraction $1/(x+2)$.

Properties (iv) to (vi) are those used to add fractions. Note that (v) follows from (i) and (iv). Then we see:

$$\frac{a}{b} + \frac{c}{d} = \frac{a\cdot d}{b\cdot d} + \frac{c\cdot b}{d\cdot b} = \frac{a\cdot d + b\cdot c}{b\cdot d}$$

and we see easily that, for example,

$$\frac{a}{b} - \frac{c}{d} + \frac{e}{f} = \frac{adf}{bdf} - \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf - cbf + ebd}{bdf} \tag{$*$}$$

If the numbers b, d, and f have common factors, the computation carried out in $(*)$ involves unnecessarily large numbers. We can simplify the process by first finding the least common denominator (LCD) of the fractions. To do so, factor each denominator completely; the LCD is the product of all the distinct factors that appear in any denominator, each raised to the highest power to which it gets raised in any denominator. The use of the LCD is demonstrated in the following example.

EXAMPLE 2.4.4 Simplify the following expressions:

(a) $\dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{6}$ (b) $\dfrac{2+a}{a^2b} + \dfrac{1-b}{ab^2} - \dfrac{2b}{a^2b^2}$ (c) $\dfrac{x-y}{x+y} - \dfrac{x}{x-y} + \dfrac{3xy}{x^2-y^2}$

Solution:

(a) The LCD is 6 and so $\frac{1}{2} - \frac{1}{3} + \frac{1}{6} = \frac{1 \cdot 3}{2 \cdot 3} - \frac{1 \cdot 2}{2 \cdot 3} + \frac{1}{2 \cdot 3} = \frac{3 - 2 + 1}{6} = \frac{2}{6} = \frac{1}{3}$

(b) The LCD is a^2b^2 and so

$$\begin{aligned}\frac{2+a}{a^2b} + \frac{1-b}{ab^2} - \frac{2b}{a^2b^2} &= \frac{(2+a)b}{a^2b^2} + \frac{(1-b)a}{a^2b^2} - \frac{2b}{a^2b^2} = \frac{2b + ab + a - ba - 2b}{a^2b^2} \\ &= \frac{a}{a^2b^2} = \frac{1}{ab^2}\end{aligned}$$

(c) The LCD is $(x+y)(x-y)$ and so

$$\begin{aligned}\frac{x-y}{x+y} - \frac{x}{x-y} + \frac{3xy}{x^2-y^2} &= \frac{(x-y)(x-y)}{(x-y)(x+y)} - \frac{(x+y)x}{(x+y)(x-y)} + \frac{3xy}{(x-y)(x+y)} \\ &= \frac{x^2 - 2xy + y^2 - x^2 - xy + 3xy}{(x-y)(x+y)} \\ &= \frac{y^2}{x^2-y^2}\end{aligned}$$

The expression $1 - \frac{5-3}{2}$ means that from the number 1, we subtract the number $\frac{5-3}{2} = \frac{2}{2} = 1$, so $1 - \frac{5-3}{2} = 0$. Alternatively,

$$1 - \frac{5-3}{2} = \frac{2}{2} - \frac{(5-3)}{2} = \frac{2-(5-3)}{2} = \frac{2-5+3}{2} = \frac{0}{2} = 0$$

In the same way,

$$\frac{2+b}{ab^2} - \frac{a-2}{a^2b}$$

means that we subtract $(a-2)/a^2b$ from $(2+b)/ab^2$:

$$\frac{2+b}{ab^2} - \frac{a-2}{a^2b} = \frac{(2+b)a}{a^2b^2} - \frac{(a-2)b}{a^2b^2} = \frac{(2+b)a - (a-2)b}{a^2b^2} = \frac{2(a+b)}{a^2b^2}$$

It is a good idea first to enclose in parentheses the numerators of the fractions, as in the next example.

EXAMPLE 2.4.5 Simplify the expression:

$$\frac{x-1}{x+1} - \frac{1-x}{x-1} - \frac{-1+4x}{2(x+1)}$$

Solution:

$$\begin{aligned}\frac{x-1}{x+1} - \frac{1-x}{x-1} - \frac{-1+4x}{2(x+1)} &= \frac{(x-1)}{x+1} - \frac{(1-x)}{x-1} - \frac{(-1+4x)}{2(x+1)} \\ &= \frac{2(x-1)^2 - 2(1-x)(x+1) - (-1+4x)(x-1)}{2(x+1)(x-1)} \\ &= \frac{2(x^2-2x+1) - 2(1-x^2) - (4x^2-5x+1)}{2(x+1)(x-1)} \\ &= \frac{x-1}{2(x+1)(x-1)} = \frac{1}{2(x+1)}\end{aligned}$$

We prove property (ix) by writing $(a/b) \div (c/d)$ as a ratio of fractions:[13]

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{b \cdot d \cdot \frac{a}{b}}{b \cdot d \cdot \frac{c}{d}} = \frac{\frac{\cancel{b} \cdot d \cdot a}{\cancel{b}}}{\frac{b \cdot \cancel{d} \cdot c}{\cancel{d}}} = \frac{d \cdot a}{b \cdot c} = \frac{a \cdot d}{b \cdot c} = \frac{a}{b} \cdot \frac{c}{d}$$

When we deal with fractions of fractions, we should be sure to emphasize which is the fraction line of the dominant fraction. For example,

$$\frac{a}{\frac{b}{c}} = a \div \frac{b}{c} = \frac{ac}{b} \tag{$*$}$$

whereas

$$\frac{\frac{a}{b}}{c} = \frac{a}{b} \div c = \frac{a}{bc} \tag{$**$}$$

Of course, it is safer to write $\frac{a}{b/c}$ or $a/(b/c)$ in the first case, and $\frac{a/b}{c}$ or $(a/b)/c$ in the second case.[14]

EXERCISES FOR SECTION 2.4

1. Simplify the following expressions:

(a) $\frac{3}{7} + \frac{4}{7} - \frac{5}{7}$ (b) $\frac{3}{4} + \frac{4}{3} - 1$ (c) $\frac{3}{12} - \frac{1}{24}$ (d) $\frac{1}{5} - \frac{2}{25} - \frac{3}{75}$

(e) $3\frac{3}{5} - 1\frac{4}{5}$ (f) $\frac{3}{5} \cdot \frac{5}{6}$ (g) $\left(\frac{3}{5} \div \frac{2}{15}\right) \cdot \frac{1}{9}$ (h) $\left(\frac{2}{3} + \frac{1}{4}\right) \Big/ \left(\frac{3}{4} + \frac{3}{2}\right)$

2. Simplify the following expressions:

(a) $\frac{x}{10} - \frac{3x}{10} + \frac{17x}{10}$ (b) $\frac{9a}{10} - \frac{a}{2} + \frac{a}{5}$ (c) $\frac{b+2}{10} - \frac{3b}{15} + \frac{b}{10}$

(d) $\frac{x+2}{3} + \frac{1-3x}{4}$ (e) $\frac{3}{2b} - \frac{5}{3b}$ (f) $\frac{3a-2}{3a} - \frac{2b-1}{2b} + \frac{4b+3a}{6ab}$

3. Cancel common factors in the following expressions:

(a) $\frac{325}{625}$ (b) $\frac{8a^2b^3c}{64abc^3}$ (c) $\frac{2a^2 - 2b^2}{3a + 3b}$ (d) $\frac{P^3 - PQ^2}{(P+Q)^2}$

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[13] Illustration: You buy half a litre of a soft drink. Each sip is one fiftieth of a litre. How many sips? Answer: $(1/2) \div (1/50) = 25$. One easily becomes thirsty reading this stuff!

[14] As a numerical example of $(*)$ and $(**)$,

$$\frac{1}{\frac{3}{5}} = \frac{5}{3}, \quad \text{whereas} \quad \frac{\frac{1}{3}}{5} = \frac{1}{15}$$

4. If $x = 3/7$ and $y = 1/14$, find the simplest forms of the following fractions:

(a) $x + y$ (b) x/y (c) $(x - y)/(x + y)$ (d) $13(2x - 3y)/(2x + 1)$

SM 5. Simplify the following expressions:

(a) $\dfrac{1}{x-2} - \dfrac{1}{x+2}$ (b) $\dfrac{6x+25}{4x+2} - \dfrac{6x^2+x-2}{4x^2-1}$ (c) $\dfrac{18b^2}{a^2-9b^2} - \dfrac{a}{a+3b} + 2$

(d) $\dfrac{1}{8ab} - \dfrac{1}{8b(a+2)}$ (e) $\dfrac{2t-t^2}{t+2} \cdot \left(\dfrac{5t}{t-2} - \dfrac{2t}{t-2}\right)$ (f) $2 - \dfrac{a\left(1 - \frac{1}{2a}\right)}{0.25}$

SM 6. Simplify the following expressions:

(a) $\dfrac{2}{x} + \dfrac{1}{x+1} - 3$ (b) $\dfrac{t}{2t+1} - \dfrac{t}{2t-1}$ (c) $\dfrac{3x}{x+2} - \dfrac{4x}{2-x} - \dfrac{2x-1}{x^2-4}$

(d) $\dfrac{1/x + 1/y}{1/xy}$ (e) $\dfrac{1/x^2 - 1/y^2}{1/x^2 + 1/y^2}$ (f) $\dfrac{a/x - a/y}{a/x + a/y}$

7. Verify that $x^2 + 2xy - 3y^2 = (x + 3y)(x - y)$, and then simplify the expression:

$$\frac{x-y}{x^2+2xy-3y^2} - \frac{2}{x-y} - \frac{7}{x+3y}$$

SM 8. Simplify the following expressions:

(a) $\left(\dfrac{1}{4} - \dfrac{1}{5}\right)^{-2}$ (b) $n - \dfrac{n}{1 - \frac{1}{n}}$ (c) $\dfrac{1}{1+x^{p-q}} + \dfrac{1}{1+x^{q-p}}$

(d) $\dfrac{\dfrac{1}{x-1} + \dfrac{1}{x^2-1}}{x - \dfrac{2}{x+1}}$ (e) $\dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$ (f) $\dfrac{\dfrac{10x^2}{x^2-1}}{\dfrac{5x}{x+1}}$

2.5 Fractional Powers

In textbooks and research articles on economics, we constantly see powers with fractional exponents such as $K^{1/4}L^{3/4}$ and $Ar^{2.08}p^{-1.5}$. How do we define a^x when x is a rational number? Of course, we would like the usual rules for powers still to apply.

You probably know the meaning of a^x if $x = 1/2$. In fact, if $a \geq 0$ and $x = 1/2$, then we define $a^x = a^{1/2}$ as equal to $\sqrt{a}$, the *square root* of a. Thus, $a^{1/2} = \sqrt{a}$ is defined as the nonnegative number that when multiplied by itself gives a. This definition makes sense because $a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1 = a$. Note that a real number multiplied by itself must always be nonnegative, whether that number is positive, negative, or zero. Hence, if $a \geq 0$,

$$a^{1/2} = \sqrt{a}$$

For example, $\sqrt{16} = 16^{1/2} = 4$ because $4^2 = 16$ and $\sqrt{1/25} = 1/5$ because

$$\frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$$

PROPERTIES OF SQUARE ROOTS

(i) If a and b are nonnegative numbers, then

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

(ii) If $a \geq 0$ and $b > 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Of course, these rules can also be written $(ab)^{1/2} = a^{1/2}b^{1/2}$ and $(a/b)^{1/2} = a^{1/2}/b^{1/2}$. For example, $\sqrt{16 \cdot 25} = \sqrt{16} \cdot \sqrt{25} = 4 \cdot 5 = 20$, and $\sqrt{9/4} = \sqrt{9}/\sqrt{4} = 3/2$.

Note that formulas (i) and (ii) are not valid if a or b or both are negative. For example, $\sqrt{(-1)(-1)} = \sqrt{1} = 1$, whereas $\sqrt{-1} \cdot \sqrt{-1}$ is not defined (unless one uses complex numbers).

It is important to recall that, in general, $(a+b)^r \neq a^r + b^r$. For $r = 1/2$, this implies that we generally have[15]

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

Note also that $(-2)^2 = 4$ and $2^2 = 4$, so both $x = -2$ and $x = 2$ are solutions of the equation $x^2 = 4$. Therefore we have $x^2 = 4$ if and only if $x = \pm\sqrt{4} = \pm 2$. Note, however, that the symbol $\sqrt{4}$ means *only* 2, not -2.

By using a calculator, we find that $\sqrt{2} \div \sqrt{3} \approx 0.816$. Without a calculator, the division $\sqrt{2} \div \sqrt{3} \approx 1.414 \div 1.732$ would be tedious. But if we expand the fraction by rationalizing the denominator—that is, if we multiply both numerator and denominator by the same term in order to remove expressions with roots in the denominator—it becomes easier:

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{2 \cdot 3}}{3} = \frac{\sqrt{6}}{3} \approx \frac{2.448}{3} = 0.816$$

[15] The following observation illustrates just how frequently this fact is overlooked: During an examination in a basic course in mathematics for economists, 22% of 190 students simplified $\sqrt{1/16 + 1/25}$ incorrectly and claimed that it was equal to $1/4 + 1/5 = 9/20$. The correct answer is $\sqrt{41/400} = \sqrt{41}/20$.

Sometimes the difference-of-squares formula of Section 2.3 can be used to eliminate square roots from the denominator of a fraction:

$$\frac{1}{\sqrt{5}+\sqrt{3}} = \frac{\sqrt{5}-\sqrt{3}}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)} = \frac{\sqrt{5}-\sqrt{3}}{5-3} = \frac{1}{2}\left(\sqrt{5}-\sqrt{3}\right)$$

The *n*-th Root

What do we mean by $a^{1/n}$, where n is a natural number, and a is positive? For example, what does $5^{1/3}$ mean? If the rule $(a^r)^s = a^{rs}$ is still to apply in this case, we must have $(5^{1/3})^3 = 5^1 = 5$. This implies that $5^{1/3}$ must be a solution of the equation $x^3 = 5$. Using the intermediate value theorem 7.10.1, this equation can be shown to have a unique positive solution, denoted by $\sqrt[3]{5}$, the *cube root* of 5. Therefore, we must define $5^{1/3}$ as $\sqrt[3]{5}$.

In general, $(a^{1/n})^n = a^1 = a$. Thus, $a^{1/n}$ is a solution of the equation $x^n = a$. Using theorem 7.10.1 again, this equation can be shown to have a unique positive solution denoted by $\sqrt[n]{a}$, the *n-th root* of a:

$$a^{1/n} = \sqrt[n]{a}$$

THE *n*-th ROOT

If a is positive and n is a natural number, then $\sqrt[n]{a}$ is the unique positive number that, raised to the n-th power, gives a—that is,

$$\left(\sqrt[n]{a}\right)^n = a$$

EXAMPLE 2.5.1 Compute the following numbers:

(a) $\sqrt[3]{27}$ (b) $(1/32)^{1/5}$ (c) $(0.0001)^{0.25} = (0.0001)^{1/4}$

Solution:

(a) $\sqrt[3]{27} = 3$, because $3^3 = 27$

(b) $(1/32)^{1/5} = 1/2$ because $(1/2)^5 = 1/32$

(c) $(0.0001)^{1/4} = 0.1$ because $(0.1)^4 = 0.0001$

EXAMPLE 2.5.2 An amount \$5 000 in an account has increased to \$10 000 in 15 years. What constant yearly interest rate p has been used?

Solution: After 15 years the amount of \$5 000 has grown to $5000(1 + p/100)^{15}$, so we have the equation

$$5000\left(1+\frac{p}{100}\right)^{15} = 10\,000$$

or

$$\left(1+\frac{p}{100}\right)^{15}=2$$

In general, $(a^t)^{1/t}=a^1=a$ for $t\neq 0$. Raising each side to the power of 1/15 yields

$$1+\frac{p}{100}=2^{1/15}$$

or $p=100(2^{1/15}-1)$. With a calculator we find $p\approx 4.73$.

We proceed to define $a^{p/q}$ whenever p is an integer, q is a natural number, and $a>0$. Consider first $5^{2/3}$. We have already defined $5^{1/3}$. For the second property of powers that we saw in Section 2.2, namely that $(a^r)^s=a^{rs}$, to apply, we must have $5^{2/3}=(5^{1/3})^2$. So we must define $5^{2/3}$ as $\left(\sqrt[3]{5}\right)^2$. In general, for $a>0$, we define

$$a^{p/q}=\left(a^{1/q}\right)^p=\left(\sqrt[q]{a}\right)^p$$

when p is an integer and q a natural number. From the properties of exponents,

$$a^{p/q}=\left(a^{1/q}\right)^p=\left(a^p\right)^{1/q}=\sqrt[q]{a^p}$$

Thus, to compute $a^{p/q}$, we could either first take the q-th root of a and raise the result to p, or first raise a to the power p and then take the q-th root of the result. We obtain the same answer either way.[16] For example,

$$4^{7/2}=(4^7)^{1/2}=16384^{1/2}=128=2^7=(4^{1/2})^7$$

EXAMPLE 2.5.3 Compute the numbers:

(a) $16^{3/2}$ (b) $16^{-1.25}$ (c) $(1/27)^{-2/3}$

Solution:

(a) $16^{3/2}=(16^{1/2})^3=4^3=64$

(b) $16^{-1.25}=16^{-5/4}=\frac{1}{16^{5/4}}=\frac{1}{\left(\sqrt[4]{16}\right)^5}=\frac{1}{2^5}=\frac{1}{32}$

(c) $(1/27)^{-2/3}=27^{2/3}=\left(\sqrt[3]{27}\right)^2=3^2=9$

[16] Tests reveal that many students, while they are able to handle quadratic identities, nevertheless make mistakes when dealing with more complicated powers. Here are examples taken from recent tests:

(a) $(1+r)^{20}$ is *not* equal to $1^{20}+r^{20}$.
(b) If $u=9+x^{1/2}$, it does *not* follow that $u^2=81+x$; instead $u^2=81+18x^{1/2}+x$.
(c) $(e^x-e^{-x})^p$ is *not* equal to $e^{xp}-e^{-xp}$ (unless $p=1$).

EXAMPLE 2.5.4 Simplify the following expressions so that the answers contain only positive exponents:

(a) $\dfrac{a^{3/8}}{a^{1/8}}$ (b) $(x^{1/2}x^{3/2}x^{-2/3})^{3/4}$ (c) $\left(\dfrac{10p^{-1}q^{2/3}}{80p^2q^{-7/3}}\right)^{-2/3}$

Solution:

(a) $\dfrac{a^{3/8}}{a^{1/8}} = a^{3/8-1/8} = a^{2/8} = a^{1/4} = \sqrt[4]{a}$

(b) $(x^{1/2}x^{3/2}x^{-2/3})^{3/4} = (x^{1/2+3/2-2/3})^{3/4} = (x^{4/3})^{3/4} = x$

(c) $\left(\dfrac{10p^{-1}q^{2/3}}{80p^2q^{-7/3}}\right)^{-2/3} = (8^{-1}p^{-1-2}q^{2/3-(-7/3)})^{-2/3} = 8^{2/3}p^2q^{-2} = 4\dfrac{p^2}{q^2}$

If q is an odd number and p is an integer, $a^{p/q}$ can be defined even when $a < 0$. For example, $(-8)^{1/3} = \sqrt[3]{-8} = -2$, because $(-2)^3 = -8$. However, in defining $a^{p/q}$ when $a < 0$, q must be odd. If not, we could get contradictions such as "$-2 = (-8)^{1/3} = (-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$".

When computing $a^{p/q}$ it is often easier to find $\sqrt[q]{a}$ first and then raise the result to the p-th power. For example, $(-64)^{5/3} = (\sqrt[3]{-64})^5 = (-4)^5 = -1024$.

EXERCISES FOR SECTION 2.5

1. Compute the following numbers:

(a) $\sqrt{9}$ (b) $\sqrt{1600}$ (c) $(100)^{1/2}$ (d) $\sqrt{9+16}$

(e) $(36)^{-1/2}$ (f) $(0.49)^{1/2}$ (g) $\sqrt{0.01}$ (h) $\sqrt{1/25}$

2. Let a and b be positive numbers. Decide whether each "?" should be replaced by $=$ or $\neq$. Justify your answer.

(a) $\sqrt{25 \cdot 16}$? $\sqrt{25} \cdot \sqrt{16}$ (b) $\sqrt{25+16}$? $\sqrt{25} + \sqrt{16}$

(c) $(a+b)^{1/2}$? $a^{1/2} + b^{1/2}$ (d) $(a+b)^{-1/2}$? $(\sqrt{a+b})^{-1}$

3. Solve for x the following equalities:

(a) $\sqrt{x} = 9$ (b) $\sqrt{x} \cdot \sqrt{4} = 4$ (c) $\sqrt{x+2} = 25$

(d) $\sqrt{3} \cdot \sqrt{5} = \sqrt{x}$ (e) $2^{2-x} = 8$ (f) $2^x - 2^{x-1} = 4$

4. Rationalize the denominator and simplify the following expressions:

(a) $\dfrac{6}{\sqrt{7}}$ (b) $\dfrac{\sqrt{32}}{\sqrt{2}}$ (c) $\dfrac{\sqrt{3}}{4\sqrt{2}}$ (d) $\dfrac{\sqrt{54}-\sqrt{24}}{\sqrt{6}}$

(e) $\dfrac{2}{\sqrt{3}\sqrt{8}}$ (f) $\dfrac{4}{\sqrt{2y}}$ (g) $\dfrac{x}{\sqrt{2x}}$ (h) $\dfrac{x\left(\sqrt{x}+1\right)}{\sqrt{x}}$

SM **5.** Simplify the following expressions by making the denominators rational:

(a) $\dfrac{1}{\sqrt{7}+\sqrt{5}}$ (b) $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ (c) $\dfrac{x}{\sqrt{3}-2}$

(d) $\dfrac{x\sqrt{y}-y\sqrt{x}}{x\sqrt{y}+y\sqrt{x}}$ (e) $\dfrac{h}{\sqrt{x+h}-\sqrt{x}}$ (f) $\dfrac{1-\sqrt{x+1}}{1+\sqrt{x+1}}$

6. Compute, without using a calculator, the following numbers:

(a) $\sqrt[3]{125}$ (b) $(243)^{1/5}$ (c) $(-8)^{1/3}$ (d) $\sqrt[3]{0.008}$

(e) $81^{1/2}$ (f) $64^{-1/3}$ (g) $16^{-2.25}$ (h) $\left(\dfrac{1}{3^{-2}}\right)^{-2}$

7. Using a calculator, find approximations to:

(a) $\sqrt[3]{55}$ (b) $(160)^{1/4}$ (c) $(2.71828)^{1/5}$ (d) $(1+0.0001)^{10\,000}$

8. The population of a nation increased from 40 million to 60 million in 12 years. What is the yearly percentage rate of growth p?

9. Simplify the following expressions:

(a) $\left(27x^{3p}y^{6q}z^{12r}\right)^{1/3}$ (b) $\dfrac{(x+15)^{4/3}}{(x+15)^{5/6}}$ (c) $\dfrac{8\sqrt[3]{x^2}\sqrt[4]{y}\sqrt{1/z}}{-2\sqrt[3]{x}\sqrt{y^5}\sqrt{z}}$

10. Simplify the following expressions, so that each contains only a single exponent:

(a) $\{[(a^{1/2})^{2/3}]^{3/4}\}^{4/5}$ (b) $a^{1/2}\cdot a^{2/3}\cdot a^{3/4}\cdot a^{4/5}$

(c) $\{[(3a)^{-1}]^{-2}(2a^{-2})^{-1}\}/a^{-3}$ (d) $\dfrac{\sqrt[3]{a}\cdot a^{1/12}\cdot\sqrt[4]{a^3}}{a^{5/12}\cdot\sqrt{a}}$

SM **11.** Which of the following equations are valid for all x and y?

(a) $(2^x)^2 = 2^{x^2}$ (b) $3^{x-3y} = \dfrac{3^x}{3^{3y}}$

(c) $3^{-1/x} = \dfrac{1}{3^{1/x}}$, with $x \neq 0$ (d) $5^{1/x} = \dfrac{1}{5^x}$, with $x \neq 0$

(e) $a^{x+y} = a^x + a^y$ (f) $2^{\sqrt{x}}\cdot 2^{\sqrt{y}} = 2^{\sqrt{xy}}$ with x and y positive

12. If a firm uses x units of input in process A, it produces $32x^{3/2}$ units of output. In the alternative process B, the same input produces $4x^3$ units of output. For what levels of input does process A produce more than process B?

2.6 Inequalities

The real numbers consist of the positive numbers, 0, and the negative numbers. If a is a positive number, we write $a > 0$ (or $0 < a$), and we say that a is greater than zero. If the number c is negative, we write $c < 0$ (or $0 > c$).

A fundamental property of the positive numbers is that:

$$(a > 0 \text{ and } b > 0) \Rightarrow (a + b > 0 \text{ and } a \cdot b > 0) \tag{2.6.1}$$

In general, we say that *the number a is greater than the number b*, and we write $a > b$ (or $b < a$), if $a - b$ is positive. Thus, $4.11 > 3.12$ because $4.11 - 3.12 = 0.99 > 0$, and $-3 > -5$ because $-3 - (-5) = 2 > 0$. On the number line, Fig. 2.1.1, $a > b$ means that a lies to the right of b.

When $a > b$, we often say that *a is strictly greater than b* in order to emphasize that $a = b$ is ruled out. If $a > b$ or $a = b$, then we write $a \geq b$ (or $b \leq a$) and say that a is *greater than or equal to b*. Thus, $a \geq b$ means that $a - b \geq 0$. For example, $4 \geq 4$ and $4 \geq 2$.[17] We call $>$ and $<$ *strict* inequalities, whereas $\geq$ and $\leq$ are *weak* inequalities. The difference is often very important in economic analysis.

One can prove a number of important properties of $>$ and $\geq$. For example, for any number c

$$a > b \Rightarrow a + c > b + c \tag{2.6.2}$$

The argument is simple: one has $(a + c) - (b + c) = a + c - b - c = a - b$ for all numbers a, b, and c. Hence $a - b > 0$ implies $(a + c) - (b + c) > 0$, so the conclusion follows. On the number line shown in Fig. 2.6.1, this implication is self-evident (here c is chosen to be negative):

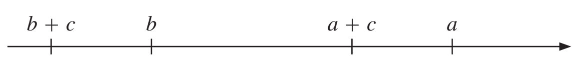

Figure 2.6.1 $a > b \Rightarrow a + c > b + c$

Dealing with more complicated inequalities involves using the following properties:

PROPERTIES OF INEQUALITIES

Let a, b, c, and d be numbers

$$(a > b \text{ and } b > c) \Rightarrow a > c \tag{2.6.3}$$

$$(a > b \text{ and } c > 0) \Rightarrow ac > bc \tag{2.6.4}$$

$$(a > b \text{ and } c < 0) \Rightarrow ac < bc \tag{2.6.5}$$

$$(a > b \text{ and } c > d) \Rightarrow a + c > b + d \tag{2.6.6}$$

All four properties remain valid when each $>$ is replaced by $\geq$, and each $<$ by $\leq$. The properties all follow easily from (2.6.1). For example, property (2.6.5) is proved as

[17] Note in particular that it *is* correct to write $4 \geq 2$, because $4 - 2$ *is* positive or 0.

follows: Suppose $a > b$ and $c < 0$. Then $a - b > 0$ and $-c > 0$, so, according to (2.6.1), $(a - b)(-c) > 0$. Hence $-ac + bc > 0$, implying that $ac < bc$.

According to (2.6.4) and (2.6.5):

(a) If the two sides of an inequality are multiplied by a positive number, the direction of the inequality is preserved.

(b) If the two sides of an inequality are multiplied by a negative number, the direction of the inequality is reversed.

It is important that you understand these rules, and realize that they correspond to everyday experience. For instance, (2.6.4) can be interpreted this way: given two rectangles with the same base, the one with the larger height has the larger area.

EXAMPLE 2.6.1 Find what values of x satisfy $3x - 5 > x - 3$.

Solution: Adding 5 to both sides of the inequality yields $3x - 5 + 5 > x - 3 + 5$, or $3x > x + 2$. Adding $(-x)$ to both sides yields $3x - x > x - x + 2$, so $2x > 2$, and after dividing by the positive number 2, we get $x > 1$. The argument can obviously be reversed, so the solution is $x > 1$.

Sign Diagrams

EXAMPLE 2.6.2 Check whether the inequality $(x - 1)(3 - x) > 0$ is satisfied for $x = -3$, $x = 2$, and $x = 5$. Then, find all the values x that satisfy the same inequality.

Solution: For $x = -3$, we have $(x - 1)(3 - x) = (-4) \cdot 6 = -24 < 0$; for $x = 2$, we have $(x - 1)(3 - x) = 1 \cdot 1 = 1 > 0$; and for $x = 5$, we have $(x - 1)(3 - x) = 4 \cdot (-2) = -8 < 0$. Hence, the inequality is satisfied for $x = 2$, but not for $x = -3$ or $x = 5$.

To find the entire solution set, we use a *sign diagram*. The sign variation for each factor in the product is determined. For example, the factor $x - 1$ is negative when $x < 1$, is 0 when $x = 1$, and is positive when $x > 1$. This sign variation is represented in Fig. 2.6.2.

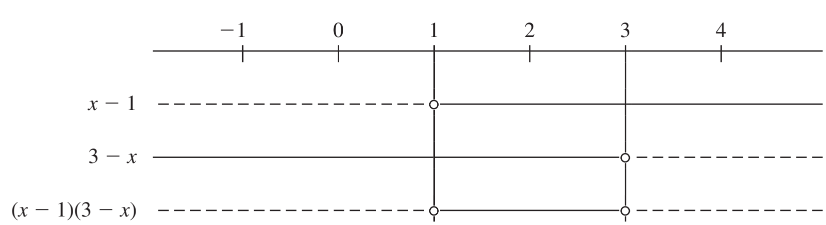

Figure 2.6.2 Sign diagram for $(x - 1)(3 - x)$

The upper dashed line to the left of the vertical line $x = 1$ indicates that $x - 1 < 0$ if $x < 1$; the small circle indicates that $x - 1 = 0$ when $x = 1$; and the solid line to the right of $x = 1$ indicates that $x - 1 > 0$ if $x > 1$. In a similar way, we represent the sign variation for $3 - x$. The sign variation of the product is obtained as follows. If $x < 1$, then $x - 1$ is negative and $3 - x$ is positive, so the product is negative. If $1 < x < 3$, both factors are positive, so the product is positive. If $x > 3$, then $x - 1$ is positive and $3 - x$ is negative, so the product is negative. In conclusion: The solution set consists of those x that are greater than 1, but less than 3. So $(x - 1)(3 - x) > 0 \Leftrightarrow 1 < x < 3$.

EXAMPLE 2.6.3 Find all values of p that satisfy the inequality:

$$\frac{2p - 3}{p - 1} > 3 - p$$

Solution: It is tempting to begin by multiplying each side of the inequality by $p - 1$. However, then we must distinguish between the two cases, $p - 1 > 0$ and $p - 1 < 0$, because if we multiply through by $p - 1$ when $p - 1 < 0$, we have to reverse the inequality sign. There is an alternative method, which makes it unnecessary to distinguish between two different cases. We begin by adding $p - 3$ to both sides. This yields

$$\frac{2p - 3}{p - 1} + p - 3 > 0$$

Making $p - 1$ the common denominator gives

$$\frac{2p - 3 + (p - 3)(p - 1)}{p - 1} > 0$$

Because $2p - 3 + (p - 3)(p - 1) = 2p - 3 + p^2 - 4p + 3 = p^2 - 2p = p(p - 2)$, substituting this in the numerator gives

$$\frac{p(p - 2)}{p - 1} > 0$$

To find the solution set, we again use a sign diagram. On the basis of the sign variations for p, $p - 2$, and $p - 1$, the sign variation for $p(p - 2)/(p - 1)$ is determined. For example, if $0 < p < 1$, then p is positive and $(p - 2)$ is negative, so $p(p - 2)$ is negative. But $p - 1$ is also negative on this interval, so $p(p - 2)/(p - 1)$ is positive. Arguing this way for all the relevant intervals leads to the sign diagram shown in Fig. 2.6.3.[18]

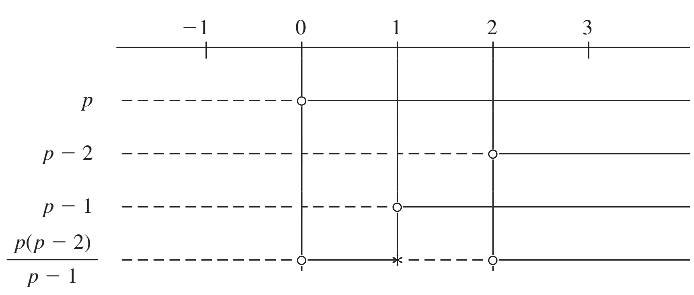

Figure 2.6.3 Sign diagram for $\dfrac{p(p - 2)}{p - 1}$

So the original inequality is satisfied if and only if $0 < p < 1$ or $p > 2$.

[18] The original inequality has no meaning when $p = 1$.

Two notes of warning are in order. First, note the most common error committed in solving inequalities, which is precisely that indicated in Example 2.6.3: if we multiply by $p - 1$, the inequality is preserved *only* if $p - 1$ is positive—that is, if $p > 1$. Second, it is vital that you really understand the method of sign diagrams. Another common error is illustrated by the following example.

EXAMPLE 2.6.4 Find all values of x that satisfy the inequality:

$$\frac{(x-2)+3(x+1)}{x+3} \leq 0$$

"Solution": Suppose we construct the inappropriate sign diagram shown in Fig. 2.6.4.

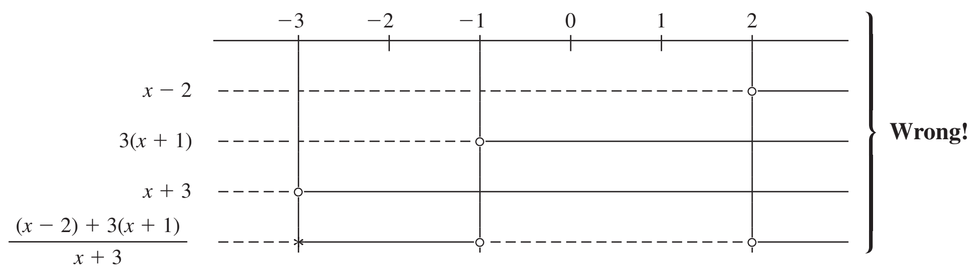

Figure 2.6.4 Wrong sign diagram for $\dfrac{(x-2)+3(x+1)}{x+3}$

According to this diagram, the inequality should be satisfied for $x < -3$ and for $-1 \leq x \leq 2$. However, for $x = -4$ (< -3), the fraction reduces to 15, which is positive. What went wrong? Suppose $x < -3$. Then $x - 2 < 0$ and $3(x+1) < 0$ and, therefore, the numerator $(x-2)+3(x+1)$ is negative. Because the denominator $x + 3$ is also negative for $x < -3$, the fraction is positive. The sign variation for the fraction in the diagram is, therefore, completely wrong. The product of two negative numbers is positive, but their sum is negative, and not positive as the wrong sign diagram suggests.

We obtain a correct solution to the given problem by first collecting terms in the numerator so that the inequality becomes $(4x+1)/(x+3) \leq 0$. A sign diagram for this inequality reveals the correct answer, which is $-3 < x \leq -1/4$.

Double Inequalities

Two inequalities that are valid simultaneously are often written as a *double inequality*. If, for example, $a \leq z$ and moreover $z < b$, it is natural to write $a \leq z < b$. (On the other hand, if $a \leq z$ and $z > b$, but we do not know which is the larger of a and b, then we cannot write $a \leq b < z$ or $b \leq a \leq z$, and we do *not* write $a \leq z > b$.)

EXAMPLE 2.6.5 One day, the lowest temperature in Buenos Aires was 50°F, and the highest was 77°F. What is the corresponding temperature variation in degrees Celsius? (Recall that if F denotes degrees Fahrenheit and C denotes degrees Celsius, then $F = \frac{9}{5}C + 32$.)

Solution: We have

$$50 \le \frac{9}{5}C + 32 \le 77$$

Subtracting 32 from each term yields

$$50 - 32 \le \frac{9}{5}C \le 77 - 32$$

or

$$18 \le \frac{9}{5}C \le 45$$

Dividing these inequalities by $9/5$ yields $10 \le C \le 25$. The temperature thus varies between 10°C and 25°C.

EXERCISES FOR SECTION 2.6

1. Decide which of the following inequalities are true:

(a) $-6.15 > -7.16$ (b) $6 \ge 6$ (c) $(-5)^2 \le 0$ (d) $-\frac{1}{2}\pi < -\frac{1}{3}\pi$

(e) $\dfrac{4}{5} > \dfrac{6}{7}$ (f) $2^3 < 3^2$ (g) $2^{-3} < 3^{-2}$ (h) $\dfrac{1}{2} - \dfrac{2}{3} < \dfrac{1}{4} - \dfrac{1}{3}$

2. Find what values of x satisfy the following inequalities:

(a) $-x - 3 \le 5$ (b) $3x + 5 < x - 13$

(c) $3x - (x - 1) \ge x - (1 - x)$ (d) $\dfrac{2x - 4}{3} \le 7$

(e) $\dfrac{1}{3}(1 - x) \ge 2(x - 3)$ (f) $\dfrac{x}{24} - (x + 1) + \dfrac{3x}{8} < \dfrac{5}{12}(x + 1)$

SM 3. Solve the following inequalities:

(a) $2 < \dfrac{3x + 1}{2x + 4}$ (b) $\dfrac{120}{n} + 1.1 \le 1.85$ (c) $g^2 - 2g \le 0$

(d) $\dfrac{1}{p - 2} + \dfrac{3}{p^2 - 4p + 4} \ge 0$ (e) $\dfrac{-n - 2}{n + 4} > 2$ (f) $x^4 < x^2$

4. Solve the following inequalities:

(a) $\dfrac{x + 2}{x - 1} < 0$ (b) $\dfrac{2x + 1}{x - 3} > 1$ (c) $5a^2 \le 125$

(d) $(x - 1)(x + 4) > 0$ (e) $(x - 1)^2(x + 4) > 0$ (f) $(x - 1)^3(x - 2) \le 0$

(g) $(5x - 1)^{10}(x - 1) < 0$ (h) $(5x - 1)^{11}(x - 1) < 0$ (i) $\dfrac{3x - 1}{x} > x + 3$

(j) $\dfrac{x - 3}{x + 3} < 2x - 1$ (k) $x^2 - 4x + 4 > 0$ (l) $x^3 + 2x^2 + x \le 0$

5. Solve the following inequalities:

(a) $1 \leq \frac{1}{3}(2x-1) + \frac{8}{3}(1-x) < 16$ (b) $-5 < \frac{1}{x} < 0$ (c) $\frac{(1/x)-1}{(1/x)+1} \geq 1$

SM 6. Fill in the blanks with "⇒", "⇐", or "⇔", when this results in a true statement:

(a) $x(x+3) < 0$ ____ $x > -3$ (b) $x^2 < 9$ ____ $x < 3$

(c) $x^2 > 0$ ____ $x > 0$ (d) $x > y^2$ ____ $x > 0$

7. Decide whether the following inequalities are valid for all x and y:

(a) $x + 1 > x$ (b) $x^2 > x$ (c) $x + x > x$ (d) $x^2 + y^2 \geq 2xy$

8. Recall the formula to convert Celsius to Fahrenheit, from Example 2.6.5.

(a) The temperature for storing potatoes should be between 4°C and 6°C. What are the corresponding temperatures in degrees Fahrenheit?

(b) The freshness of a bottle of milk is guaranteed for seven days if it is kept at a temperature between 36°F and 40°F. Find the corresponding temperature variation in degrees Celsius.

SM 9. If a and b are two positive numbers, define their *arithmetic*, *geometric*, and *harmonic means*, respectively, by $m_A = \frac{1}{2}(a+b)$, $m_G = \sqrt{ab}$ and

$$m_H = 2\left(\frac{1}{a} + \frac{1}{b}\right)^{-1}$$

Prove that $m_A \geq m_G \geq m_H$, with strict inequalities unless $a = b$.[19]

2.7 Intervals and Absolute Values

Let a and b be any two numbers on the real line. Then we call the set of all numbers that lie between a and b an *interval*. In many situations, it is important to distinguish between the intervals that include their end points and the intervals that do not. When $a < b$, there are four different intervals that all have a and b as end points, as shown in Table 2.1.

Note that an open interval includes neither of its end points, but a closed interval includes both of its end points. A half-open interval contains one of its end points, but not both. All four intervals, however, have the same length, $b - a$. We usually illustrate intervals on the number line as in Fig. 2.7.1, with included end points represented by solid dots, and excluded end points at the tips of arrows.

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[19] You should first test these inequalities by choosing some specific numbers, using a calculator if you wish. To show that $m_A \geq m_G$, start with the obvious inequality $(\sqrt{a} - \sqrt{b})^2 \geq 0$, and then expand. To show that $m_G \geq m_H$, start by showing that $\sqrt{xy} \leq \frac{1}{2}(x+y)$. Then let $x = 1/a$, $y = 1/b$.

Table 2.1 Intervals on the real line

Notation	Name	Consists of all x satisfying:
(a, b)	The *open* interval from a to b	$a < x < b$
$[a, b]$	The *closed* interval from a to b	$a \leq x \leq b$
$(a, b]$	A *half-open* interval from a to b	$a < x \leq b$
$[a, b)$	A *half-open* interval from a to b	$a \leq x < b$

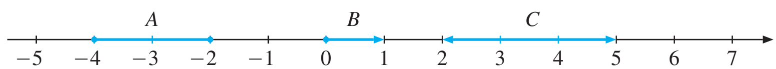

Figure 2.7.1 $A = [-4, -2]$, $B = [0, 1)$, and $C = (2, 5)$

The intervals mentioned so far are all *bounded*. We also use the word "interval" to signify certain unbounded sets of numbers. For example, in set notation: $[a, \infty) = \{x : x \geq a\}$ consists of all numbers $x \geq a$; and $(-\infty, b) = \{x : x < b\}$ contains all numbers with $x < b$. Here, "∞" is the common symbol for infinity. This symbol is not a number at all, and therefore the usual rules of arithmetic do not apply to it. In the notation $[a, \infty)$, the symbol ∞ is only intended to indicate that we are considering the collection of *all* numbers larger than or equal to a, without any upper bound on the size of the number. Similarly, $(-\infty, b)$ has no lower bound. From the preceding, it should be apparent what we mean by (a, ∞) and $(-\infty, b\,]$. The collection of all real numbers is also denoted by the symbol $(-\infty, \infty)$.

Absolute Value

Let a be a real number and imagine its position on the real line. The distance between a and 0 is called the *absolute value* of a. If a is positive or 0, then the absolute value is the number a itself; if a is negative, then because distance must be positive, the absolute value is equal to the positive number $-a$. That is:

ABSOLUTE VALUE

The *absolute value* of the number a is the number $|a|$ defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \tag{2.7.1}$$

For example, $|13| = 13$, $|-5| = -(-5) = 5$, $|-1/2| = 1/2$, and $|0| = 0$. Note in particular that $|-a| = |a|$.[20]

[20] It is a common fallacy to assume that a must denote a positive number, even if this is not explicitly stated. Similarly, on seeing $-a$, many students are led to believe that this expression is always

EXAMPLE 2.7.1 Compute $|x-2|$ for $x=-3$, $x=0$, and $x=4$. Then, rewrite $|x-2|$ using the definition of absolute value.

Solution: Using the definition, (2.7.1), we have that $|x-2| = |-3-2| = |-5| = 5$, for $x=-3$. For $x=0$, $|x-2| = |0-2| = |-2| = 2$. Similarly, for $x=4$, $|x-2| = |4-2| = |2| = 2$.

According, again, to (2.7.1), $|x-2| = x-2$ if $x-2 \geq 0$, that is, $x \geq 2$. However, $|x-2| = -(x-2) = 2-x$ if $x-2<0$, that is, $x<2$. Hence,

$$|x-2| = \begin{cases} x-2, & \text{if } x \geq 2 \\ 2-x, & \text{if } x < 2 \end{cases}$$

Let x_1 and x_2 be two arbitrary numbers. The *distance* between x_1 and x_2 on the number line is $x_1 - x_2$ if $x_1 \geq x_2$, and $-(x_1 - x_2)$ if $x_1 < x_2$. Therefore, we have:

DISTANCE BETWEEN NUMBERS

The *distance* between x_1 and x_2 on the number line is

$$|x_1 - x_2| = |x_2 - x_1| \tag{2.7.2}$$

In Fig. 2.7.2 we have indicated geometrically that the distance between 7 and 2 is 5, whereas the distance between -3 and -5 is equal to 2, because $|-3-(-5)| = |-3+5| = |2| = 2$.

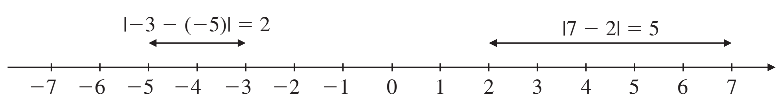

Figure 2.7.2 The distances between 7 and 2 and between -3 and -5.

Suppose $|x| = 5$. What values can x have? There are only two possibilities: either $x=5$ or $x=-5$, because no other numbers have absolute value equal to 5. Generally, if a is greater than or equal to 0, then $|x| = a$ means that $x=a$ or $x=-a$. Because $|x| \geq 0$ for all x, the equation $|x| = a$ has no solution when $a<0$.

If a is a positive number and $|x| < a$, then the distance from x to 0 is less than a. Furthermore, when a is nonnegative, and $|x| \leq a$, the distance from x to 0 is less than or equal to a. In symbols:

$$|x| < a \Leftrightarrow -a < x < a \tag{2.7.3}$$

$$|x| \leq a \Leftrightarrow -a \leq x \leq a \tag{2.7.4}$$

negative. Observe, however, that the number $-a$ is positive when a itself is negative. For example, if $a=-5$, then $-a = -(-5) = 5$. Nevertheless, it is often a useful convention in economics to define variables so that, as far as possible, their values are positive rather than negative.

EXAMPLE 2.7.2 Check first to see if the inequality $|3x - 2| \leq 5$ holds for $x = -3$, $x = 0$, $x = 7/3$, and $x = 10$. Then, find all the x such that the inequality holds.

Solution: For $x = -3$, $|3x - 2| = |-9 - 2| = 11$; for $x = 0$, $|3x - 2| = |-2| = 2$; for $x = 7/3$, $|3x - 2| = |7 - 2| = 5$; and for $x = 10$, $|3x - 2| = |30 - 2| = 28$. Hence, the given inequality is satisfied for $x = 0$ and $x = 7/3$, but not for $x = -3$ and $x = 10$.

Now, from (2.7.4) the inequality $|3x - 2| \leq 5$ means that $-5 \leq 3x - 2 \leq 5$. Adding 2 to all three expressions gives

$$-5 + 2 \leq 3x - 2 + 2 \leq 5 + 2$$

or $-3 \leq 3x \leq 7$. Dividing by 3 gives $-1 \leq x \leq 7/3$.

EXERCISES FOR SECTION 2.7

1. (a) Calculate $|2x - 3|$ for $x = 0$, $1/2$, and $7/2$.

(b) Solve the equation $|2x - 3| = 0$.

(c) Rewrite $|2x - 3|$ by using the definition of absolute value.

2. (a) Calculate $|5 - 3x|$ for $x = -1$, $x = 2$, and $x = 4$.

(b) Solve the equation $|5 - 3x| = 5$.

(c) Rewrite $|5 - 3x|$ by using the definition of absolute value.

3. Determine x such that the following expressions hold true:

(a) $|3 - 2x| = 5$ (b) $|x| \leq 2$ (c) $|x - 2| \leq 1$

(d) $|3 - 8x| \leq 5$ (e) $|x| > \sqrt{2}$ (f) $|x^2 - 2| \leq 1$

4. A 5-metre iron bar is to be produced. The bar may not deviate by more than 1 mm from its stated length. Write a specification for the bar's length x in metres: (a) by using a double inequality; (b) with the aid of an absolute-value sign.

2.8 Summation

Economists often make use of census data. Suppose, for instance, that a country is divided into six regions. Let N_i denote the population in region i. Then the total population is given by

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6$$

It is convenient to have an abbreviated notation for such lengthy expressions. The capital Greek letter sigma, Σ, is conventionally used as a *summation symbol*, and the sum is written as

$$\sum_{i=1}^{6} N_i$$

This reads "the sum, from $i = 1$ to $i = 6$, of N_i". If there are n regions, then one possible notation for the total population is

$$N_1 + N_2 + \cdots + N_n \tag{$*$}$$

where $\cdots$ indicates that the obvious previous pattern continues, but comes to an end just before the last term N_n. In summation notation, we write

$$\sum_{i=1}^{n} N_i$$

The summation notation tells us to form the sum of all the terms that result when we substitute successive integers for i, starting with its lower limit $i = 1$ and ending with the upper limit $i = n$, respectively, in the example. The symbol i is called the *index of summation*. It is a "dummy variable" that can be replaced by any other letter (which has not already been used for something else). Thus, both $\sum_{j=1}^{n} N_j$ and $\sum_{k=1}^{n} N_k$ represent the same sum as ($*$).

The upper and lower limits of summation can both vary. For example,

$$\sum_{i=30}^{35} N_i = N_{30} + N_{31} + N_{32} + N_{33} + N_{34} + N_{35}$$

is the total population in the six regions numbered from 30 to 35. More generally, suppose p and q are integers with $q \geq p$. Then,

$$\sum_{i=p}^{q} a_i = a_p + a_{p+1} + \cdots + a_q$$

denotes the sum that results when we substitute successive integers for i, starting with $i = p$ and ending with $i = q$. If the upper and lower limits of summation are the same, then the "sum" reduces to one term. And if the upper limit is less than the lower limit, then there are no terms at all, so the usual convention is that the "sum" reduces to zero.

EXAMPLE 2.8.1 Compute the following summations:

(a) $\sum_{i=1}^{5} i^2$ (b) $\sum_{k=3}^{6}(5k - 3)$ (c) $\sum_{j=0}^{2} \dfrac{(-1)^j}{(j+1)(j+3)}$

Solution:

(a) $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$

(b) $\sum_{k=3}^{6}(5k - 3) = (5 \cdot 3 - 3) + (5 \cdot 4 - 3) + (5 \cdot 5 - 3) + (5 \cdot 6 - 3) = 78$

(c) $\sum_{j=0}^{2} \dfrac{(-1)^j}{(j+1)(j+3)} = \dfrac{1}{1 \cdot 3} + \dfrac{-1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} = \dfrac{40 - 15 + 8}{120} = \dfrac{33}{120} = \dfrac{11}{40}$

Sums and the summation notation occur frequently in economics, so it is important to be able to interpret such sums. Often, there are several variables or parameters in addition to the summation index.

EXAMPLE 2.8.2 Expand the following expressions:

(a) $\sum_{i=1}^{n} p_t^{(i)} q^{(i)}$ (b) $\sum_{j=-3}^{1} x^{5-j} y^j$ (c) $\sum_{i=1}^{N} (x_{ij} - \bar{x}_j)^2$

Solution:

(a) $\sum_{i=1}^{n} p_t^{(i)} q^{(i)} = p_t^{(1)} q^{(1)} + p_t^{(2)} q^{(2)} + \cdots + p_t^{(n)} q^{(n)}$

(b) $\sum_{j=-3}^{1} x^{5-j} y^j = x^{5-(-3)} y^{-3} + x^{5-(-2)} y^{-2} + x^{5-(-1)} y^{-1} + x^{5-0} y^0 + x^{5-1} y^1$

(c) $\sum_{i=1}^{N} (x_{ij} - \bar{x}_j)^2 = (x_{1j} - \bar{x}_j)^2 + (x_{2j} - \bar{x}_j)^2 + \cdots + (x_{Nj} - \bar{x}_j)^2$

Note that t is *not* an index of summation in (a), and that j is *not* one in (c).

EXAMPLE 2.8.3 Write the following sums using summation notation:

(a) $1 + 3 + 3^2 + 3^3 + \cdots + 3^{81}$

(b) $a_i^6 + a_i^5 b_j + a_i^4 b_j^2 + a_i^3 b_j^3 + a_i^2 b_j^4 + a_i b_j^5 + b_j^6$

Solution:

(a) This is easy if we note that $1 = 3^0$ and $3 = 3^1$, so that the sum can be written as $3^0 + 3^1 + 3^2 + 3^3 + \cdots + 3^{81}$. The general term is 3^i, and we have

$$1 + 3 + 3^2 + 3^3 + \cdots + 3^{81} = \sum_{i=0}^{81} 3^i$$

(b) This is more difficult. Note, however, that the indices i and j never change. Also, the exponent for a_i decreases step by step from 6 to 0, whereas that for b_j increases from 0 to 6. The general term has the form $a_i^{6-k} b_j^k$, where k varies from 0 to 6. Thus,

$$a_i^6 + a_i^5 b_j + a_i^4 b_j^2 + a_i^3 b_j^3 + a_i^2 b_j^4 + a_i b_j^5 + b_j^6 = \sum_{k=0}^{6} a_i^{6-k} b_j^k$$

EXAMPLE 2.8.4 **(Price Indices).** In order to summarize the overall effect of price changes for several different goods within a country, a number of alternative *price indices* have been suggested. Considering a "basket" of n commodities, define, for $i = 1, \ldots, n$: $q^{(i)}$ as the number of units of good i in the basket; $p_0^{(i)}$ as the price per unit of good i in year 0; and $p_t^{(i)}$ as the price per unit of good i in year t. Then,

$$\sum_{i=1}^{n} p_0^{(i)} q^{(i)} = p_0^{(1)} q^{(1)} + p_0^{(2)} q^{(2)} + \cdots + p_0^{(n)} q^{(n)}$$

is the cost of the basket in year 0, whereas

$$\sum_{i=1}^{n} p_t^{(i)} q^{(i)} = p_t^{(1)} q^{(1)} + p_t^{(2)} q^{(2)} + \cdots + p_t^{(n)} q^{(n)}$$

is the cost of the basket in year t. A price index for year t, with year 0 as the base year, is defined as

$$\frac{\sum_{i=1}^{n} p_t^{(i)} q^{(i)}}{\sum_{i=1}^{n} p_0^{(i)} q^{(i)}} \cdot 100 \qquad \text{(price index)}$$

If the cost of the basket is 1032 in year 0 and the cost of the same basket in year t is 1548, then the price index is $(1548/1032) \cdot 100 = 150$.

In the case where the quantities $q^{(i)}$ are levels of consumption in the base year 0, this is called the *Laspeyres price index*. But if the quantities $q^{(i)}$ are levels of consumption in the year t, this is called the *Paasche price index*.

EXERCISES FOR SECTION 2.8

1. Evaluate the following sums:

(a) $\sum_{i=1}^{10} i$ (b) $\sum_{k=2}^{6} (5 \cdot 3^{k-2} - k)$ (c) $\sum_{m=0}^{5} (2m+1)$

(d) $\sum_{l=0}^{2} 2^{2^l}$ (e) $\sum_{i=1}^{10} 2$ (f) $\sum_{j=1}^{4} \frac{j+1}{j}$

2. Expand the following sums:

(a) $\sum_{k=-2}^{2} 2\sqrt{k+2}$ (b) $\sum_{i=0}^{3} (x+2i)^2$

(c) $\sum_{k=1}^{n} a_{ki} b^{k+1}$ (d) $\sum_{j=0}^{m} f(x_j)\, \Delta x_j$

SM 3. Express the following sums in summation notation:

(a) $4 + 8 + 12 + 16 + \cdots + 4n$ (b) $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$

(c) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1}$ (d) $a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$

(e) $3x + 9x^2 + 27x^3 + 81x^4 + 243x^5$ (f) $a_i^3 b_{i+3} + a_i^4 b_{i+4} + \cdots + a_i^p b_{i+p}$

(g) $a_i^3 b_{i+3} + a_{i+1}^4 b_{i+4} + \cdots + a_{i+p}^{p+3} b_{i+p+3}$ (h) $81\,297 + 81\,495 + 81\,693 + 81\,891$

4. Compute the price index in Example 2.8.4, for $n = 3$,when:

$$p_0^{(1)} = 1,\ p_0^{(2)} = 2,\ p_0^{(3)} = 3,\ p_t^{(1)} = 2,\ p_t^{(2)} = 3,\ p_t^{(3)} = 4,\ q^{(1)} = 3,\ q^{(2)} = 5,\ \text{and } q^{(3)} = 7$$

5. Insert the appropriate limits of summation in the right-hand side of the following sums:

(a) $\sum_{k=1}^{10} (k-2)t^k = \sum_{m=}\quad mt^{m+2}$ (b) $\sum_{n=0}^{N} 2^{n+5} = \sum_{j=}\quad 32 \cdot 2^{j-1}$

6. As of early 2016, the European Economic Area has 31 nations, and officially there is a long-run goal of free mobility of labour throughout the area. For the year 2025, let c_{ij} denote an estimate of the number of persons who will move from nation i to nation j, $i \neq j$. If, say, $i = 25$ and $j = 10$, then we write $c_{25,10}$ for c_{ij}. Explain the meaning of the sums: (a) $\sum_{j=1}^{31} c_{ij}$, and (b) $\sum_{i=1}^{31} c_{ij}$.

SM **7.** Decide which of the following equalities are generally valid.

(a) $\sum_{k=1}^{n} ck^2 = c\sum_{k=1}^{n} k^2$

(b) $\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2$

(c) $\sum_{j=1}^{n} b_j + \sum_{j=n+1}^{N} b_j = \sum_{j=1}^{N} b_j$

(d) $\sum_{k=3}^{7} 5^{k-2} = \sum_{k=0}^{4} 5^{k+1}$

(e) $\sum_{i=0}^{n-1} a_{i,j}^2 = \sum_{k=1}^{n} a_{k-1,j}^2$

(f) $\sum_{k=1}^{n} \dfrac{a_k}{k} = \dfrac{1}{k}\sum_{k=1}^{n} a_k$

2.9 Rules for Sums

The following properties of the summation notation are helpful when manipulating sums:

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{2.9.1}$$

and

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \tag{2.9.2}$$

These properties are known, respectively, as *additivity* and *homogeneity*. Their proofs are straightforward. For example, (2.9.2) is proved by noting that

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{i=1}^{n} a_i$$

The homogeneity property states that a constant factor can be moved outside the summation sign. In particular, if $a_i = 1$ for all i, then

$$\sum_{i=1}^{n} c = nc \tag{2.9.3}$$

which just states that a constant c summed n times is equal to n times c.

The summation rules can be applied in combination, to give formulas like

$$\sum_{i=1}^{n}(a_i + b_i - c_i + d) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} c_i + nd$$

EXAMPLE 2.9.1 Evaluate the sum

$$\sum_{m=2}^{n} \frac{1}{(m-1)m} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n}$$

by using the identity

$$\frac{1}{(m-1)m} = \frac{1}{m-1} - \frac{1}{m}$$

Solution:

$$\begin{aligned}\sum_{m=2}^{n}\frac{1}{m(m-1)} &= \sum_{m=2}^{n}\left(\frac{1}{m-1}-\frac{1}{m}\right)\\ &= \sum_{m=2}^{n}\frac{1}{m-1}-\sum_{m=2}^{n}\frac{1}{m}\\ &= \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}\right)-\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right)\\ &= 1-\frac{1}{n}\end{aligned}$$

To derive the last equality, note that most of the terms cancel pairwise. The only exceptions are the first term within the first parentheses and the last term within the last parentheses. This powerful trick is commonly used to calculate some special sums of this kind. See Exercise 4, below.

EXAMPLE 2.9.2 The *arithmetic mean* (or *mean*), μ_x, of T numbers $x_1, x_2, \ldots, x_T$ is their average, defined as the sum of all the numbers divided by the number of terms, T. That is,

$$\mu_x = \frac{1}{T}\sum_{i=1}^{T} x_i$$

Prove that $\sum_{i=1}^{T}(x_i-\mu_x)=0$ and $\sum_{i=1}^{T}(x_i-\mu_x)^2=\sum_{i=1}^{T}x_i^2-T\mu_x^2$.

Solution: The difference $x_i-\mu_x$ is the deviation between x_i and the mean. We prove first that the sum of these deviations is 0, using the foregoing definition of μ_x:

$$\sum_{i=1}^{T}(x_i-\mu_x)=\sum_{i=1}^{T}x_i-\sum_{i=1}^{T}\mu_x=\sum_{i=1}^{T}x_i-T\mu_x=T\mu_x-T\mu_x=0$$

Furthermore, the sum of the squares of the deviations is

$$\begin{aligned}\sum_{i=1}^{T}(x_i-\mu_x)^2 &= \sum_{i=1}^{T}(x_i^2-2\mu_x x_i+\mu_x^2)=\sum_{i=1}^{T}x_i^2-2\mu_x\sum_{i=1}^{T}x_i+\sum_{i=1}^{T}\mu_x^2\\ &= \sum_{i=1}^{T}x_i^2-2\mu_x T\mu_x+T\mu_x^2=\sum_{i=1}^{T}x_i^2-T\mu_x^2\end{aligned}$$

Dividing by T, the mean square deviation, $(1/T)\sum_{i=1}^{T}(x_i-\mu_x)^2$, is therefore equal to the mean square, $(1/T)\sum_{i=1}^{T}x_i^2$, minus the square of the mean, μ_x^2.

Useful Formulas

A (very) demanding teacher once asked his students to sum

$$81\,297+81\,495+81\,693+\cdots+100\,899$$

There are one hundred terms and the difference between successive terms is constant and equal to 198. Carl Gauss (1777–1855), later one of the world's leading mathematicians, was in the class, and (at age nine!) is reputed to have given the right answer in only a few minutes. You already took a key step toward finding the solution to this question in Exercise 1.4.1, using mathematical induction. Applied to that easier problem of finding the sum $x = 1 + 2 + \cdots + n$, Gauss' argument was probably different, as follows: First, write the sum x in two ways

$$x = 1 + 2 + \cdots + (n-1) + n$$
$$x = n + (n-1) + \cdots + 2 + 1$$

Summing vertically term by term gives

$$\begin{aligned} 2x &= (1+n) + [2 + (n-1)] + \cdots + [(n-1) + 2] + (n+1) \\ &= (1+n) + (1+n) + \cdots + (1+n) + (1+n) \\ &= n(1+n) \end{aligned}$$

Thus, solving for x gives the result:

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1) \tag{2.9.4}$$

The following two summation formulas are occasionally useful in economics.[21] Exercise 1 below asks you to provide their proofs.

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1) \tag{2.9.5}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2 = \left(\sum_{i=1}^{n} i\right)^2 \tag{2.9.6}$$

EXERCISES FOR SECTION 2.9

1. Prove formulas (2.9.5) and (2.9.6), using the principle of mathematical induction seen in Section 1.4.

2. Use results (2.9.1) to (2.9.5) to find $\sum_{k=1}^{n}(k^2 + 3k + 2)$.

3. Prove the summation formula for an *arithmetic series*:

$$\sum_{i=0}^{n-1}(a + id) = na + \frac{n(n-1)d}{2}$$

   Apply the result to find the sum Gauss is supposed to have calculated at age 9.

---

[21] Check to see if they are true for $n = 1, 2, 3$.

4. (a) Prove that $\sum_{k=1}^{n}(a_{k+1} - a_k) = a_{n+1} - a_1$.

(b) Use the result in (a) to compute the following:

(i) $\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ (ii) $\sum_{k=1}^{12}(3^{k+1} - 3^k)$ (iii) $\sum_{k=1}^{n}(ar^{k+1} - ar^k)$

2.10 Newton's Binomial Formula

We all know that $(a+b)^1 = a + b$ and $(a+b)^2 = a^2 + 2ab + b^2$. Using the latter equality and writing $(a+b)^3 = (a+b)(a+b)^2$ and $(a+b)^4 = (a+b)(a+b)^3$, we find that

$$\begin{aligned}
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

The corresponding formula for $(a+b)^m$, where m is any natural number, can be expressed as follows:

NEWTON'S BINOMIAL FORMULA

$$(a+b)^m = a^m + \binom{m}{1}a^{m-1}b + \cdots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m \tag{2.10.1}$$

This formula involves the *binomial coefficients* $\binom{m}{k}$, which are defined, for $m = 1, 2, \ldots$ and for $k = 0, 1, 2, \ldots, m$, by

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

where $k!$, read as "k factorial", is standard notation for the product $1 \cdot 2 \cdot 3 \ldots (k-1) \cdot k$ of the first k natural numbers, with the conventions that $0! = 1$, $\binom{m}{0} = 1$, $\binom{m}{1} = m$, and $\binom{m}{m} = 1$.

When $m = 5$, for example, we have

$$\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10, \qquad \binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10, \qquad \binom{5}{4} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} = 5$$

Then (2.10.1) gives $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

The coefficients occurring in the expansions for successive powers of $(a+b)$ form the following pattern, called *Pascal's triangle*:[22]

[22] Though it was known in China by about the year 1100, long before French mathematician Blaise Pascal (1623–1662) was born.

$$
\begin{array}{ccccccccccccccccccc}
&&&&&&&&&1&&&&&&&&& \\
&&&&&&&&1&&1&&&&&&&& \\
&&&&&&&1&&2&&1&&&&&&& \\
&&&&&&1&&3&&3&&1&&&&&& \\
&&&&&1&&4&&6&&4&&1&&&&& \\
&&&&1&&5&&10&&10&&5&&1&&&& \\
&&&1&&6&&15&&20&&15&&6&&1&&& \\
&&1&&7&&21&&35&&35&&21&&7&&1&& \\
&1&&8&&28&&56&&70&&56&&28&&8&&1& \\
1&&9&&36&&84&&126&&126&&84&&36&&9&&1
\end{array}
$$

This table can be continued indefinitely. The numbers in this triangle are indeed the binomial coefficients. For instance, the numbers in row 6 (when the first row is numbered 0) are

$$\binom{6}{0},\ \binom{6}{1},\ \binom{6}{2},\ \binom{6}{3},\ \binom{6}{4},\ \binom{6}{5},\ \binom{6}{6}$$

Note first that the numbers are symmetric about the middle line. This symmetry can be expressed as

$$\binom{m}{k} = \binom{m}{m-k} \tag{2.10.2}$$

For example, $\binom{6}{2} = 15 = \binom{6}{4}$. Second, apart from the 1 at both ends of each row, each number is the sum of the two adjacent numbers in the row above. For instance, 56 in the eighth row is equal to the sum of 21 and 35 in the seventh row. In symbols,

$$\binom{m+1}{k+1} = \binom{m}{k} + \binom{m}{k+1} \tag{2.10.3}$$

In Exercise 2 you are asked to prove these two properties.

EXERCISES FOR SECTION 2.10

1. Use Newton's binomial formula to find $(a+b)^6$.

2. (a) Prove that $\binom{5}{3} = \dfrac{5!}{2!\,3!}$, and, in general, that

$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!} \tag{2.10.4}$$

(b) Verify, by direct computation, that $\binom{8}{3} = \binom{8}{8-3}$ and $\binom{8+1}{3+1} = \binom{8}{3} + \binom{8}{3+1}$.

(c) Use (2.10.4) to verify (2.10.2) and (2.10.3).

2.11 Double Sums

Often one has to combine several summation signs. Consider, for example, the following rectangular array of numbers:

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{matrix} \tag{2.11.1}$$

The array can be regarded as a *spreadsheet*. A typical number in the array is of the form a_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$.[23] There are $n \cdot m$ numbers in all. Let us find the sum of all the numbers in the array by first finding the sum of the numbers in each of the m rows and then adding all these row sums. The m different row sums can be written in the form $\sum_{j=1}^{n} a_{1j}, \sum_{j=1}^{n} a_{2j}, \ldots, \sum_{j=1}^{n} a_{mj}$.[24] The sum of these m sums is equal to $\sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}$, which can be written as $\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)$. If instead we add the numbers in each of the n columns first and then add these sums, we get

$$\sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots + \sum_{i=1}^{m} a_{in} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right)$$

In both these cases, we have calculated the sum of all the numbers in the array.[25] For this reason, we must have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$$

where, according to usual practice, we have deleted the parentheses. This formula says that *in a (finite) double sum, the order of summation is immaterial*. It is important to note that the summation limits for i and j are independent of each other.[26]

EXAMPLE 2.11.1 Compute $\sum_{i=1}^{3} \sum_{j=1}^{4} (i + 2j)$.

Solution:

$$\sum_{i=1}^{3} \sum_{j=1}^{4} (i + 2j) = \sum_{i=1}^{3} [(i + 2) + (i + 4) + (i + 6) + (i + 8)]$$

$$= \sum_{i=1}^{3} (4i + 20) = 24 + 28 + 32 = 84$$

You should check that the result is the same by summing over i first instead.

---

[23] For example, a_{ij} may indicate the total revenue of a firm from its sales in region i in month j.

[24] In our example, these row sums are the total revenues in each region summed over all the n months.

[25] How do you interpret this sum in our economic example?

[26] Otherwise, changing the order in a double sum like $\sum_{j=1}^{n} \sum_{i=1}^{j} a_{ij}$ to obtain $\sum_{i=1}^{j} \sum_{j=1}^{n} a_{ij}$ results in an expression that makes little sense.

EXERCISES FOR SECTION 2.11

SM 1. Expand and compute the following double sums:

(a) $\sum_{i=1}^{3}\sum_{j=1}^{4} i \cdot 3^j$ (b) $\sum_{s=0}^{2}\sum_{r=2}^{4}\left(\frac{rs}{r+s}\right)^2$

(c) $\sum_{i=1}^{m}\sum_{j=1}^{n}(i+j^2)$ (d) $\sum_{i=1}^{m}\sum_{j=1}^{2} i^j$

2. Consider a group of individuals each having a certain number of units of m different goods. Let a_{ij} denote the number of units of good i owned by person j, for $i = 1, \dots, m$ and $j = 1, \dots, n$. Explain in words the meaning of the following sums:

(a) $\sum_{j=1}^{n} a_{ij}$ (b) $\sum_{i=1}^{m} a_{ij}$ (c) $\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}$

3. Prove that the sum of all the numbers in the triangular table

$$\begin{array}{lllll} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ a_{31} & a_{32} & a_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{array}$$

can be written as $\sum_{i=1}^{m}\left(\sum_{j=1}^{i} a_{ij}\right)$ and also as $\sum_{j=1}^{m}\left(\sum_{i=j}^{m} a_{ij}\right)$.

SM 4. [HARDER] Consider the $m \cdot n$ numbers a_{ij} in the rectangular array (2.11.1). Denote the arithmetic mean of them all by $\bar{a}$, and the mean of all the numbers in the j-th column by $\bar{a}_j$, so that

$$\bar{a} = \frac{1}{mn}\sum_{r=1}^{m}\sum_{s=1}^{n} a_{rs}$$

and

$$\bar{a}_j = \frac{1}{m}\sum_{r=1}^{m} a_{rj}$$

Prove that $\bar{a}$ is the mean of the column sums $\bar{a}_j$ $(j = 1, \dots, n)$ and that

$$\sum_{r=1}^{m}\sum_{s=1}^{m}(a_{rj} - \bar{a})(a_{sj} - \bar{a}) = m^2(\bar{a}_j - \bar{a})^2 \tag{$*$}$$

REVIEW EXERCISES

1. (a) What is three times the difference between 50 and x?

(b) What is the quotient between x and the sum of y and 100?

(c) If the price of an item is a including 20% VAT (value added tax), what is the price before VAT?

(d) A person buys x_1, x_2, and x_3 units of three goods whose prices per unit are respectively p_1, p_2, and p_3. What is the total expenditure?

(e) A rental car costs F dollars per day in fixed charges and b dollars per kilometre. How much must a customer pay to drive x kilometres in one day?

(f) A company has fixed costs of F dollars per year and variable costs of c dollars per unit produced. Find an expression for the total cost per unit (total average cost) incurred by the company if it produces x units in one year.

(g) A person has an annual salary of \$$L$ and then receives a raise of $p\%$ followed by a further increase of $q\%$. What is the person's new salary?

2. Express as single real numbers, in decimal notation:

(a) 5^3 (b) 10^{-3} (c) $\dfrac{1}{3^{-3}}$ (d) $\dfrac{-1}{10^{-3}}$

(e) $3^{-2}3^3$ (f) $(3^{-2})^{-3}$ (g) $-\left(\dfrac{5}{3}\right)^0$ (h) $\left(-\dfrac{1}{2}\right)^{-3}$

3. Which of the following expressions are defined, and what are their values?

(a) $(0+2)^0$ (b) 0^{-2} (c) $\dfrac{(10)^0}{(0+1)^0}$ (d) $\dfrac{(0+1)^0}{(0+2)^0}$

4. Simplify the following expressions:

(a) $(2^3 2^{-5})^3$ (b) $\left(\dfrac{2}{3}\right)^{-1} - \left(\dfrac{4}{3}\right)^{-1}$

(c) $(3^{-2} - 5^{-1})^{-1}$ (d) $(1.12)^{-3}(1.12)^3$

SM **5.** Simplify the following expressions:

(a) $(2x)^4$ (b) $(2^{-1} - 4^{-1})^{-1}$ (c) $\dfrac{24x^3y^2z^3}{4x^2yz^2}$

(d) $\left[-(-ab^3)^{-3}(a^6b^6)^2\right]^3$ (e) $\dfrac{a^5 \cdot a^3 \cdot a^{-2}}{a^{-3} \cdot a^6}$ (f) $\left[\left(\dfrac{x}{2}\right)^3 \cdot \dfrac{8}{x^{-2}}\right]^{-3}$

6. Complete the following statements:

(a) $x^{-1}y^{-1} = 3 \Rightarrow x^3y^3 = \cdots$ (b) $x^7 = 2 \Rightarrow (x^{-3})^6(x^2)^2 = \cdots$

(c) $\left(\dfrac{xy}{z}\right)^{-2} = 3 \Rightarrow \left(\dfrac{z}{xy}\right)^6 = \cdots$ (d) $a^{-1}b^{-1}c^{-1} = 1/4 \Rightarrow (abc)^4 = \cdots$

7. Compute the following numbers: (a) 12% of 300; (b) 5% of 2000; and (c) 6.5% of 1500.

8. Give economic interpretations to each of the following expressions and then use a calculator to find their approximate values: (a) $100 \cdot (1.01)^8$€; (b) £$50\,000 \cdot (1.15)^{10}$; and (c) \$$6\,000 \cdot (1.03)^{-8}$.

9. (a) \$100 000 is deposited into an account earning 8% interest per year. What is the amount after ten years?

(b) If the interest rate is 8% each year, how much money should you have deposited in a bank six years ago to have \$25 000 today?

SM **10.** Expand and simplify the following expressions:

(a) $a(a-1)$ (b) $(x-3)(x+7)$ (c) $-\sqrt{3}\left(\sqrt{3}-\sqrt{6}\right)$

(d) $\left(1-\sqrt{2}\right)^2$ (e) $(x-1)^3$ (f) $(1-b^2)(1+b^2)$

(g) $(1+x+x^2+x^3)(1-x)$ (h) $(1+x)^4$

11. Factor the following expressions:

(a) $25x-5$ (b) $3x^2-x^3y$ (c) $50-x^2$ (d) $a^3-4a^2b+4ab^2$

SM **12.** Factor the following expressions:

(a) $5(x+2y)+a(x+2y)$ (b) $(a+b)c-d(a+b)$ (c) $ax+ay+2x+2y$

(d) $2x^2-5yz+10xz-xy$ (e) p^2-q^2+p-q (f) $u^3+v^3-u^2v-v^2u$

13. Compute the following numbers, without using a calculator:

(a) $16^{1/4}$ (b) $243^{-1/5}$ (c) $5^{1/7}\cdot 5^{6/7}$ (d) $(4^8)^{-3/16}$

(e) $64^{1/3}+\sqrt[3]{125}$ (f) $(-8/27)^{2/3}$ (g) $(-1/8)^{-2/3}+(1/27)^{-2/3}$ (h) $\dfrac{1000^{-2/3}}{\sqrt[3]{5^{-3}}}$

14. Solve the following equations for x:

(a) $2^{2x}=8$ (b) $3^{3x+1}=1/81$ (c) $10^{x^2-2x+2}=100$

15. Find the unknown x in each of the following equations:

(a) $25^5\cdot 25^x=25^3$ (b) $3^x-3^{x-2}=24$ (c) $3^x\cdot 3^{x-1}=81$

(d) $3^5+3^5+3^5=3^x$ (e) $4^{-6}+4^{-6}+4^{-6}+4^{-6}=4^x$ (f) $\dfrac{2^{26}-2^{23}}{2^{26}+2^{23}}=\dfrac{x}{9}$

SM **16.** Simplify the following expressions:

(a) $\dfrac{s}{2s-1}-\dfrac{s}{2s+1}$ (b) $\dfrac{x}{3-x}-\dfrac{1-x}{x+3}-\dfrac{24}{x^2-9}$ (c) $\dfrac{\dfrac{1}{x^2y}-\dfrac{1}{xy^2}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$

SM **17.** Reduce the following fractions:

(a) $\dfrac{25a^3b^2}{125ab}$ (b) $\dfrac{x^2-y^2}{x+y}$ (c) $\dfrac{4a^2-12ab+9b^2}{4a^2-9b^2}$ (d) $\dfrac{4x-x^3}{4-4x+x^2}$

18. Solve the following inequalities:

(a) $2(x-4)<5$ (b) $\dfrac{1}{3}(y-3)+4\geq 2$ (c) $8-0.2x\leq\dfrac{4-0.1x}{0.5}$

(d) $\dfrac{x-1}{-3}>\dfrac{-3x+8}{-5}$ (e) $|5-3x|\leq 8$ (f) $|x^2-4|\leq 2$

19. Using a mobile phone costs \$30 per month, and an additional \$0.16 per minute of use.

(a) What is the cost for one month if the phone is used for a total of x minutes?

(b) What are the smallest and largest numbers of *hours* you can use the phone in a month if the monthly telephone bill is to be between \$102 and \$126?

20. If a rope could be wrapped around the Earth's surface at the equator, it would be approximately circular and about 40 million metres long. Suppose we wanted to extend the rope to make it 1 metre above the equator at every point. How many more metres of rope would be needed? (The circumference of a circle with radius r is $2\pi r$.)

21. (a) Prove that

$$a + \frac{a \cdot p}{100} - \frac{\left(a + \frac{a \cdot p}{100}\right) \cdot p}{100} = a\left[1 - \left(\frac{p}{100}\right)^2\right]$$

(b) An item initially costs \$2 000 and then its price is increased by 5%. Afterwards the price is lowered by 5%. What is the final price?

(c) An item initially costs a dollars and then its price is increased by $p\%$. Afterwards the (new) price is lowered by $p\%$. What is the final price of the item? (After considering this exercise, look at the expression in part (a).)

(d) What is the result if one first *lowers* a price by $p\%$ and then *increases* it by $p\%$?

22. (a) If $a > b$, is it necessarily true that $a^2 > b^2$?

(b) Show that if $a + b > 0$, then $a > b$ implies $a^2 > b^2$.

23. (a) If $a > b$, use numerical examples to check whether $1/a > 1/b$, or $1/a < 1/b$.

(b) Prove that if $a > b$ and $ab > 0$, then $1/b > 1/a$.

24. Prove that, for all real numbers a and b:

(a) $|ab| = |a| \cdot |b|$ (b) $|a + b| \leq |a| + |b|$

The inequality in (b) is called the *triangle inequality*.

SM **25.** Consider an equilateral triangle, and let P be an arbitrary point within the triangle. Let h_1, h_2, and h_3 be the shortest distances from P to each of the three sides. Show that the sum $h_1 + h_2 + h_3$ is independent of where point P is placed in the triangle. (*Hint*: Compute the area of the triangle as the sum of three triangles.)

26. Evaluate the following sums:

(a) $\sum_{i=1}^{4} \frac{1}{i(i+2)}$ (b) $\sum_{j=5}^{9} (2j-8)^2$ (c) $\sum_{k=1}^{5} \left(\frac{k-1}{k+1}\right)$

(d) $\sum_{n=2}^{5} (n-1)^2(n+2)$ (e) $\sum_{k=1}^{5} \left(\frac{1}{k} - \frac{1}{k+1}\right)$ (f) $\sum_{i=-2}^{3} (i+3)^i$

27. Express the following sums in summation notation:

(a) $3 + 5 + 7 + \cdots + 199 + 201$

(b) $\dfrac{2}{1} + \dfrac{3}{2} + \dfrac{4}{3} + \cdots + \dfrac{97}{96}$

(c) $4 \cdot 6 + 5 \cdot 7 + 6 \cdot 8 + \cdots + 38 \cdot 40$

(d) $\dfrac{1}{x} + \dfrac{1}{x^2} + \cdots + \dfrac{1}{x^n}$

(e) $1 + \dfrac{x^2}{3} + \dfrac{x^4}{5} + \dfrac{x^6}{7} + \cdots + \dfrac{x^{32}}{33}$

(f) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots - \dfrac{1}{80} + \dfrac{1}{81}$

28. Which of these equalities are always right and which of them are sometimes wrong?

(a) $\sum_{i=1}^{n} a_i = \sum_{j=3}^{n+2} a_{j-2}$

(b) $\sum_{i=1}^{n} (a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2$

(c) $\sum_{k=0}^{n} 5a_{k+1,j} = 5 \sum_{k=1}^{n+1} a_{k,j}$

(d) $\sum_{i=1}^{3} \dfrac{a_i}{b_i} = \dfrac{\sum_{i=1}^{3} a_i}{\sum_{i=1}^{3} b_i}$

SM 29. Find the sums:

(a) $3 + 5 + 7 + \cdots + 197 + 199 + 201$

(b) $1001 + 2002 + 3003 + \cdots + 8008 + 9009 + 10\,010$

3 SOLVING EQUATIONS

The true mathematician is not a juggler of numbers, but of concepts.
—Ian Stewart (1975)

Virtually all applications of mathematics involve equations that have to be solved. Economics is no exception, so this chapter considers some types of equation that appear frequently in economic models.

Many students are used to dealing with algebraic expressions and equations involving *only one* variable, usually denoted by x. Often they have difficulties, at first, in dealing with expressions involving several variables with a wide variety of names, and denoted by different letters. For economists, however, it is very important to be able to handle with ease such algebraic expressions and equations.

3.1 Solving Equations

To *solve* an equation means to find all values of the variables for which the equation is satisfied. Consider the following simple example

$$3x + 10 = x + 4$$

which contains the *variable* x. In order to isolate x on one side of the equation, we add $-x$ to both sides. This gives $2x + 10 = 4$. Adding -10 to both sides of this equation yields $2x = 4 - 10 = -6$. Dividing by 2 we get the solution $x = -3$.

This procedure was probably already familiar to you. The method is summed up next, noting that two equations that have exactly the same solutions are called *equivalent*.

EQUIVALENT EQUATIONS

To get an equivalent equation, do either of the following on both sides of the equality sign:

(i) add (or subtract) the same number;

(ii) multiply (or divide) by the same number different from 0.

When faced with more complicated equations involving parentheses and fractions, we usually begin by multiplying out the parentheses, and then we multiply both sides of the equation by the lowest common denominator for all the fractions.

EXAMPLE 3.1.1 Solve the equation

$$6p - \frac{1}{2}(2p - 3) = 3(1 - p) - \frac{7}{6}(p + 2)$$

Solution: First multiply out the parentheses: $6p - p + \frac{3}{2} = 3 - 3p - \frac{7}{6}p - \frac{7}{3}$. Second, multiply both sides by the lowest common denominator: $36p - 6p + 9 = 18 - 18p - 7p - 14$. Third, gather terms: $55p = -5$. Thus $p = -5/55 = -1/11$.

If a value of a variable makes an expression in an equation undefined, that value is not allowed. For instance, the choice of value 5 for variable z is not allowed in any equation that involves the expression

$$\frac{z}{z - 5}$$

because $5/0$ is undefined. As we shall show in the next example, this fact has implications for the existence of a solution to an equation.

EXAMPLE 3.1.2 Solve the equation

$$\frac{z}{z - 5} + \frac{1}{3} = \frac{-5}{5 - z}$$

Solution: We now know that z cannot be 5. Remembering this restriction, multiply both sides by $3(z - 5)$. This gives $3z + z - 5 = 15$ which has the unique solution $z = 5$. Because we had to assume $z \neq 5$, we must conclude that no solution exists for the original equation.

The next example shows, again, that sometimes a surprising degree of care is needed to find the right solutions.

EXAMPLE 3.1.3 Solve the equation

$$\frac{x+2}{x-2} - \frac{8}{x^2-2x} = \frac{2}{x}$$

Solution: Since $x^2 - 2x = x(x-2)$, the common denominator is $x(x-2)$. We see that $x = 2$ and $x = 0$ both make the equation absurd, because then at least one of the denominators becomes 0. If $x \neq 0$ and $x \neq 2$, we can multiply both sides of the equation by the common denominator $x(x-2)$ to obtain

$$\frac{x+2}{x-2} \cdot x(x-2) - \frac{8}{x(x-2)} \cdot x(x-2) = \frac{2}{x} \cdot x(x-2)$$

Cancelling common factors, this reduces to $(x+2)x - 8 = 2(x-2)$ or $x^2 + 2x - 8 = 2x - 4$, and so $x^2 = 4$. Equations of the form $x^2 = a$, where $a > 0$, have two solutions $x = \sqrt{a}$ and $x = -\sqrt{a}$. In our case, $x^2 = 4$ has solutions $x = 2$ and $x = -2$. But $x = 2$ makes the original equation absurd, so *only* $x = -2$ is a solution.

Often, solving a problem in economic analysis requires formulating an appropriate *algebraic* equation.

EXAMPLE 3.1.4 A firm manufactures a commodity that costs \$20 per unit to produce. In addition, the firm has fixed costs of \$2 000. Each unit is sold for \$75. How many units must be sold if the firm is to meet a profit target of \$14 500?

Solution: If the number of units produced and sold is denoted by Q, then the revenue of the firm is $75Q$ and the total cost of production is $20Q + 2000$. Because profit is the difference between total revenue and total cost, it can be written as $75Q - (20Q + 2000)$. Since the profit target is \$14 500, the equation

$$75Q - (20Q + 2000) = 14\,500$$

must be satisfied. It is now easy to find the solution: $Q = 16\,500/55 = 300$ units.

EXERCISES FOR SECTION 3.1

1. Solve each of the following equations:

   (a) $2x - (5 + x) = 16 - (3x + 9)$ (b) $-5(3x - 2) = 16(1 - x)$

   (c) $4x + 2(x - 4) - 3 = 2(3x - 5) - 1$ (d) $(8x - 7)5 - 3(6x - 4) + 5x^2 = x(5x - 1)$

   (e) $x^2 + 10x + 25 = 0$ (f) $(3x - 1)^2 + (4x + 1)^2 = (5x - 1)(5x + 1) + 1$

2. Solve each of the following equations:

   (a) $3x = \frac{1}{4}x - 7$ (b) $\dfrac{x-3}{4} + 2 = 3x$ (c) $\dfrac{1}{2x+1} = \dfrac{1}{x+2}$ (d) $\sqrt{2x+14} = 16$

SM 3. Solve each of the following equations:

   (a) $\dfrac{x-3}{x+3} = \dfrac{x-4}{x+4}$ (b) $\dfrac{3}{x-3} - \dfrac{2}{x+3} = \dfrac{9}{x^2-9}$ (c) $\dfrac{6x}{5} - \dfrac{5}{x} = \dfrac{2x-3}{3} + \dfrac{8x}{15}$

4. Solve the following problems, by first formulating an equation in each case:

(a) The sum of three successive natural numbers is 10 more than twice the smallest of them. Find the numbers.

(b) Jane receives double pay for every hour she works over and above 38 hours per week. Last week, she worked 48 hours and earned a total of \$812. What is Jane's regular hourly wage?

(c) James has invested £15 000 at an annual interest rate of 10%. How much additional money should he invest at the interest rate of 12%, if he wants the total interest earned by the end of the year to equal £2 100?

(d) When Mr Barnes passed away, 2/3 of his estate was left to his wife, 1/4 was shared by his children, and the remainder, \$100 000, was donated to a charity. How big was Mr Barnes's estate?

SM 5. Solve the following equations:

(a) $\dfrac{3y-1}{4} - \dfrac{1-y}{3} + 2 = 3y$ (b) $\dfrac{4}{x} + \dfrac{3}{x+2} = \dfrac{2x+2}{x^2+2x} + \dfrac{7}{2x+4}$

(c) $\dfrac{2 - z/(1-z)}{1+z} = \dfrac{6}{2z+1}$ (d) $\dfrac{1}{2}\left(\dfrac{p}{2} - \dfrac{3}{4}\right) - \dfrac{1}{4}\left(1 - \dfrac{p}{3}\right) - \dfrac{1}{3}(1-p) = -\dfrac{1}{3}$

6. Ms. Preston has y euros to spend on apples, bananas, and cherries. She decides to spend the same amount of money on each kind of fruit. The prices per kilo are 3€ for apples, 2€ for bananas, and 6€ for cherries. What is the total weight of fruit she buys, and how much does she pay per kilo of fruit?[1]

3.2 Equations and Their Parameters

Economists use mathematical models to describe different economic phenomena. These models enable them to explain the interdependence of different economic variables. Macroeconomic models, for instance, are designed to explain the broad outlines of a country's economy; in these models, examples of such variables include the total production of the economy (or gross domestic product), its total consumption, and its total investment.

The simplest kind of relationship between two variables occurs when the response of one of them to a change of one unit in the other one is always the same. In this case, the relationship can be described by a *linear equation*, such as $y = 10x$, $y = 3x + 4$, or

$$y = -\frac{8}{3}x - \frac{7}{2}$$

These three equations have a common structure. This makes it possible to write down a general linear equation covering all the special cases where x and y are the variables:

$$y = ax + b \tag{3.2.1}$$

[1] This is an example of "dollar cost" averaging, which we will encounter again in Exercise 11.5.4.

Here a and b are real numbers. For example, letting $a = 3$ and $b = 4$ yields the particular case where $y = 3x + 4$. To accommodate straight lines of the form $x = c$ where c is a constant, it may be necessary to interchange the variables x and y.

The numbers a and b are called *parameters*, as they take on different, but "fixed" values.[2] In economics, parameters often have interesting interpretations.

EXAMPLE 3.2.1 Consider the basic macroeconomic model

$$\text{(i)}\ \ Y = C + \bar{I} \qquad \text{and} \qquad \text{(ii)}\ \ C = a + bY \tag{$*$}$$

where Y is the gross domestic product (GDP), C is consumption, and $\bar{I}$ is total investment, which is treated as fixed. Equation (i) says that GDP is, by definition, the sum of consumption and total investment. Equation (ii) says that consumption is a linear function of GDP. Here, a and b are positive parameters of the model, with $b < 1$.[3] Solve the model for Y in terms of $\bar{I}$ and the parameters.

Solution: Substituting $C = a + bY$ into (i) gives

$$Y = a + bY + \bar{I}$$

Now, we rearrange this equation so that all the terms containing Y are on the left-hand side. This can be done by adding $-bY$ to both sides, thus cancelling the bY term on the right-hand side, to give

$$Y - bY = a + \bar{I}$$

Notice that the left-hand side is equal to $(1 - b)Y$, so $(1 - b)Y = a + \bar{I}$. Dividing both sides by $1 - b$, so that the coefficient of Y becomes 1, gives the answer:

$$Y = \frac{a}{1-b} + \frac{1}{1-b}\bar{I} \tag{$**$}$$

This solution is a linear equation expressing Y in terms of $\bar{I}$ and the parameters a and b.

Note the power of the approach used here: the model is solved only once, and then numerical answers are found simply by substituting appropriate numerical values for the parameters of the model. For instance, if $\bar{I} = 100$, $a = 500$, $b = 0.8$, then $Y = 3000$.

Economists usually call the equations in ($*$) the *structural form* of the model, whereas ($**$) is called its *reduced form*. In the reduced form, the number $1/(1 - b)$ is itself a parameter. This is known as the *investment multiplier*, as it measures the response in income to an "exogenous" increase in investment.

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[2] Linear equations are studied in more detail in Section 4.4.

[3] Parameter a is often referred to as *autonomous consumption*, as it represents the part of consumption that is *not* determined by the economy's income. The increase of consumption caused by an increase of one unit in income is measured by b; this parameter is, hence, known as *marginal propensity to consume*. Special cases of the model are obtained by choosing particular numerical values for the parameters, such as $\bar{I} = 100$, $a = 500$, $b = 0.8$, or $\bar{I} = 150$, $a = 600$, $b = 0.9$. Thus, $Y = C + 100$ and $C = 500 + 0.8Y$; or $Y = C + 150$ and $C = 600 + 0.9Y$.

Of course, we often need to solve more complicated "non-linear" equations, which often involve "strange" letters denoting their parameters and variables.

EXAMPLE 3.2.2 Suppose that the total demand for money in an economy is given by

$$M = \alpha Y + \beta (r - \gamma)^{-\delta}$$

where M is the quantity of money in circulation, Y is national income and r is the interest rate, while α, β, γ, and δ are positive parameters.

(a) Solve the model for the interest rate, r.

(b) For the USA during the period 1929–1952, the parameters have been estimated as $\alpha = 0.14$, $\beta = 76.03$, $\gamma = 2$, and $\delta = 0.84$. Show that r is then given by

$$r = 2 + \left(\frac{76.03}{M - 0.14Y}\right)^{25/21}$$

Solution:

(a) It follows easily from the given equation that $(r-\gamma)^{-\delta} = (M - \alpha Y)/\beta$. Then, raising each side to the power $-1/\delta$ and adding γ on both sides yields

$$r = \gamma + \left(\frac{\beta}{M - \alpha Y}\right)^{1/\delta} \qquad (*)$$

where we used the fact that $(a/b)^{-p} = (b/a)^p$.

(b) In this case $1/\delta = 1/0.84 = 100/84 = 25/21$, and the required formula follows immediately from $(*)$.

EXERCISES FOR SECTION 3.2

1. Find the value of Y for the case when $Y = C + 150$ and $C = 600 + 0.9Y$ in the model of Example 1. Verify that formula $(**)$ gives the same result.

SM 2. Solve the following equations for x:

(a) $\dfrac{1}{ax} + \dfrac{1}{bx} = 2$ (b) $\dfrac{ax+b}{cx+d} = A$ (c) $\frac{1}{2}px^{-1/2} - w = 0$

(d) $\sqrt{1+x} + \dfrac{ax}{\sqrt{1+x}} = 0$ (e) $a^2x^2 - b^2 = 0$ (f) $(3 + a^2)^x = 1$

3. Solve the following equations for the indicated variables:

(a) $q = 0.15p + 0.14$ for p (supply of rice in India);

(b) $S = \alpha + \beta P$ for P (supply function);

(c) $A = \frac{1}{2}gh$ for g (the area of a triangle);

(d) $V = \frac{4}{3}\pi r^3$ for r (the volume of a ball);

(e) $AK^\alpha L^\beta = Y_0$ for L (production function).

SM **4.** Solve the following equations for the indicated variables:

(a) $\alpha x - a = \beta x - b$ for x

(b) $\sqrt{pq} - 3q = 5$ for p

(c) $Y = 94 + 0.2(Y - (20 + 0.5Y))$ for Y

(d) $K^{1/2}\left(\frac{1}{2}\frac{r}{w}K\right)^{1/4} = Q$ for K

(e) $\dfrac{\frac{1}{2}K^{-1/2}L^{1/4}}{\frac{1}{4}L^{-3/4}K^{1/2}} = \dfrac{r}{w}$ for L

(f) $\frac{1}{2}pK^{-1/4}\left(\frac{1}{2}\frac{r}{w}\right)^{1/4} = r$ for K

5. Solve the following equations for the indicated variables:

(a) $\dfrac{1}{s} + \dfrac{1}{T} = \dfrac{1}{t}$ for s

(b) $\sqrt{KLM} - \alpha L = B$ for M

(c) $\dfrac{x - 2y + xz}{x - z} = 4y$ for z

(d) $V = C\left(1 - \dfrac{T}{N}\right)$ for T

3.3 Quadratic Equations

The general quadratic equation has the form

$$ax^2 + bx + c = 0 \tag{3.3.1}$$

where $a \neq 0$, b, and c are given constants, and variable x is the unknown. If we divide each term by a, we get the equivalent equation $x^2 + (b/a)x + c/a = 0$. If $p = b/a$ and $q = c/a$, the equation is

$$x^2 + px + q = 0 \tag{3.3.2}$$

Two special cases are easy to handle. If $q = 0$, so that there is no "constant term", the equation reduces to $x^2 + px = 0$. This is equivalent to $x(x + p) = 0$, and, since the product of two numbers can be 0 only if at least one of the numbers is 0, we conclude that $x = 0$ or $x = -p$. In short,

$$x^2 + px = 0 \text{ if, and only if, } x = 0 \text{ or } x = -p$$

This means that the equation $x^2 + px = 0$ has the solutions $x = 0$ and $x = -p$, and no others.

If $p = 0$, so that there is no term involving x, Eq. (3.3.2) reduces to $x^2 + q = 0$. Then $x^2 = -q$, and there are two cases to consider. If $q > 0$, the equation has no solutions as no number can be squared to get a negative number. In the alternative case, when $q \leq 0$, both $x = \sqrt{-q}$ and $x = -\sqrt{-q}$ solve the equation. Using the notation $x = \pm\sqrt{-q}$ to express that $\sqrt{-q}$ and $-\sqrt{-q}$ are the values that x can take, we can write, in short, that

$$x^2 + q = 0 \text{ if, and only if, } x = \pm\sqrt{-q} \text{ given that } q \leq 0$$

These results can be applied to solve any instance of the two simple cases.

EXAMPLE 3.3.1 Solve the following equations:

(a) $5x^2 - 8x = 0$ (b) $x^2 - 4 = 0$ (c) $x^2 + 3 = 0$

Solution:

(a) Dividing each term by 5 yields $x^2 - (8/5)x = x(x - 8/5) = 0$, so $x = 0$ or $x = 8/5$.

(b) The equation yields $x^2 = 4$, so $x = \pm\sqrt{4} = \pm 2$. Alternatively, one has $x^2 - 4 = (x + 2)(x - 2)$ so the equation is equivalent to $(x + 2)(x - 2) = 0$. Either way, one concludes that x is either 2 or -2.

(c) Because x^2 is no less than 0, the left-hand side of the equation $x^2 + 3 = 0$ is always strictly positive and, hence, the equation has no solution.

Harder Cases

If (3.3.2) has both coefficients different from 0, solving it becomes harder. Consider, for example,

$$x^2 - (4/3)x - 1/4 = 0$$

We could, of course, try to find the values of x that satisfy the equation by trial and error. However, it is not easy that way to find the only two solutions, which are $x = 3/2$ and $x = -1/6$. Here are two attempts to solve the equation that fail:

(a) A first attempt rearranges $x^2 - (4/3)x - 1/4 = 0$ to give $x^2 - (4/3)x = 1/4$, and so $x(x - 4/3) = 1/4$. Thus, the product of x and $x - 4/3$ must be 1/4. But there are infinitely many pairs of numbers whose product is 1/4, so this is of very little help in finding x.

(b) A second attempt is to divide each term by x to get $x - 4/3 = 1/4x$. Because the equation involves terms in both x and $1/x$, as well as a constant term, we have made no progress whatsoever.

Evidently, we need a completely new idea in order to find the solution of (3.3.2). The following example illustrates the idea that will give us a general method to solve this harder equation.

EXAMPLE 3.3.2 Solve the equation $x^2 + 8x - 9 = 0$.

Solution: It is natural to begin by moving 9 to the right-hand side:

$$x^2 + 8x = 9 \tag{$*$}$$

However, because x occurs in two terms, it is not obvious how to proceed. A method called *completing the square*, one of the oldest tricks in mathematics, turns out to work. In the present case this method involves adding 16 to each side of the equation to get

$$x^2 + 8x + 16 = 9 + 16 \tag{$**$}$$

The point of adding 16 is that the left-hand side is then a complete square: $x^2 + 8x + 16 = (x + 4)^2$. Thus, Eq. $(**)$ takes the form

$$(x + 4)^2 = 25 \tag{***}$$

The equation $z^2 = 25$ has two solutions, $z = \pm\sqrt{25} = \pm 5$. Thus, $(***)$ implies that either $x + 4 = 5$ or $x + 4 = -5$. The required solutions are, therefore, $x = 1$ and $x = -9$.

Alternatively, Eq. $(***)$ can be written as $(x+4)^2 - 5^2 = 0$. Using the difference-of-squares formula yields $(x+4-5)(x+4+5) = 0$, which reduces to $(x - 1)(x + 9) = 0$, so we have the following *factorization*

$$x^2 + 8x - 9 = (x - 1)(x + 9)$$

Note that $(x - 1)(x + 9)$ is 0 precisely when $x = 1$ or $x = -9$.

The General Case

We now apply the method of completing the squares to the quadratic equation (3.3.2). This equation obviously has the same solutions as $x^2 + px = -q$. One half of the coefficient of x is $p/2$. Adding the square of this number to each side of the equation yields

$$x^2 + px + \left(\frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q$$

The left-hand side is now a complete square, so

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} - q \tag{*}$$

Note that if $p^2/4 - q < 0$, then the right-hand side is negative. Because $(x + p/2)^2$ is nonnegative for all choices of x, we conclude that Eq. $(*)$ has no solution in this case. On the other hand, if $p^2/4 - q > 0$, $(*)$ yields two possibilities:

$$x + p/2 = \sqrt{p^2/4 - q} \qquad \text{and} \qquad x + p/2 = -\sqrt{p^2/4 - q}$$

The values of x are then easily found. These formulas are correct even if $p^2/4 - q = 0$, though then they give just the one solution $x = -p/2$. In conclusion:

SIMPLE QUADRATIC FORMULA

$$x^2 + px + q = 0 \;\text{ if, and only if, }\; x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \;\text{ provided that }\; \frac{p^2}{4} \geq q \tag{3.3.3}$$

Faced with an equation of the type (3.3.1), we can always find its solutions by first dividing the equation by a and then using (3.3.3). Sometimes it is convenient to have the formula for the solution of (3.3.1) in terms of the coefficients a, b, and c. Recall that, by dividing Eq. (3.3.1) by a, we get the equivalent version of Eq. (3.3.2), with $p = b/a$ and $q = c/a$. Substituting these particular values in (3.3.3) gives the solutions $x = -b/2a \pm \sqrt{b^2/4a^2 - c/a}$.

GENERAL QUADRATIC FORMULA

If $b^2 - 4ac \geq 0$ and $a \neq 0$, then

$$ax^2 + bx + c = 0 \text{ if, and only if, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.3.4}$$

It is probably a good idea to spend a few minutes of your life memorizing this formula, or formula (3.3.3), thoroughly. Once you have done so, you can immediately write down the solutions of any quadratic equation. Only if $b^2 - 4ac \geq 0$ are the solutions real numbers. If we use the formula when $b^2 - 4ac < 0$, the square root of a negative number appears and no real solution exists. The solutions are often called the *roots* of the equation.[4]

EXAMPLE 3.3.3 Use the quadratic formula to find the solutions of the equation

$$2x^2 - 2x - 40 = 0$$

Solution: Write the equation as $2x^2 + (-2)x + (-40) = 0$. Because $a = 2$, $b = -2$, and $c = -40$, the quadratic formula (3.3.4) yields

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot (-40)}}{2 \cdot 2} = \frac{2 \pm \sqrt{4 + 320}}{4} = \frac{2 \pm 18}{4} = \frac{1}{2} \pm \frac{9}{2}$$

The solutions are, therefore, $x = 1/2 + 9/2 = 5$ and $x = 1/2 - 9/2 = -4$.

If we use formula (3.3.3) instead, we divide each term by 2 and get $x^2 - x - 20 = 0$, so $x = 1/2 \pm \sqrt{1/4 + 20} = 1/2 \pm \sqrt{81/4} = 1/2 \pm 9/2$, namely the same solutions as before.

Suppose $p^2/4 - q \geq 0$ and let x_1 and x_2 be the solutions of Eq. (3.3.2). By using the difference-of-squares formula as we did to obtain the factorization in Example 3.3.2, it follows that (∗) is equivalent to $(x - x_1)(x - x_2) = 0$. It follows also that:

QUADRATIC FACTORIZATION

If x_1 and x_2 are the solutions of $ax^2 + bx + c = 0$, then

$$ax^2 + bx + c = a(x - x_1)(x - x_2) \tag{3.3.5}$$

[4] The quadratic formula, thus, is very useful, but you should not be a "quadratic formula fanatic" and use it always. If $b = 0$ or $c = 0$, we explained at the beginning of this section how the equation can be solved very easily. During a past exam, one extreme fanatic of the formula, when faced with solving the equation $(x - 4)^2 = 0$, expanded the parentheses to obtain $x^2 - 8x + 16 = 0$, and then used the quadratic formula eventually to get the (correct) answer, $x = 4$. What would you have done?

This is a very important result, because it shows how to factor a general quadratic function. If $b^2 - 4ac < 0$, there is no factorization of $ax^2 + bx + c$. If $b^2 - 4ac = 0$, then $x_1 = x_2$ and $ax^2 + bx + c = a(x - x_1)^2 = a(x - x_2)^2$.

EXAMPLE 3.3.4 Factor, if possible, the following quadratic polynomials:

(a) $\frac{1}{3}x^2 + \frac{2}{3}x - \frac{14}{3}$ (b) $-2x^2 + 40x - 600$

Solution:

(a) $\frac{1}{3}x^2 + \frac{2}{3}x - \frac{14}{3} = 0$ has the same solutions as $x^2 + 2x - 14 = 0$. By formula (3.3.2), its solutions are $x = -1 \pm \sqrt{1 + 14} = -1 \pm \sqrt{15}$, and these are the solutions of the given equation also. Then, from (3.3.5),

$$\begin{aligned} \frac{1}{3}x^2 + \frac{2}{3}x - \frac{14}{3} &= \frac{1}{3}\left[x - \left(-1 + \sqrt{15}\right)\right]\left[x - \left(-1 - \sqrt{15}\right)\right] \\ &= \frac{1}{3}\left(x + 1 - \sqrt{15}\right)\left(x + 1 + \sqrt{15}\right) \end{aligned}$$

(b) We apply (3.3.4) with $a = -2$, $b = 40$, and $c = -600$. In fact $b^2 - 4ac = 1600 - 4800 = -3200 < 0$. Therefore, no factoring exists in this case.

Expanding the right-hand side of the identity $x^2 + px + q = (x - x_1)(x - x_2)$ yields $x^2 + px + q = x^2 - (x_1 + x_2)x + x_1x_2$. Equating like powers of x gives $x_1 + x_2 = -p$ and $x_1x_2 = q$. Thus:

RULES FOR QUADRATIC FUNCTIONS

If x_1 and x_2 are the roots of $x^2 + px + q = 0$, then

$$x_1 + x_2 = -p \qquad \text{and} \qquad x_1x_2 = q \tag{3.3.6}$$

In words, the sum of the roots is minus the coefficient of the first-order term and the product is the constant term. The formulas (3.3.6) can also obtained by adding and multiplying the two solutions found in (3.3.2).

EXERCISES FOR SECTION 3.3

1. Solve the following quadratic equations, if they have solutions:

(a) $15x - x^2 = 0$ (b) $p^2 - 16 = 0$ (c) $(q - 3)(q + 4) = 0$

(d) $2x^2 + 9 = 0$ (e) $x(x + 1) = 2x(x - 1)$ (f) $x^2 - 4x + 4 = 0$

2. Solve the following quadratic equations by using the method of completing the square, and factor, if possible, the left-hand side:

(a) $x^2 - 5x + 6 = 0$ (b) $y^2 - y - 12 = 0$ (c) $2x^2 + 60x + 800 = 0$

(d) $-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2} = 0$ (e) $m(m - 5) - 3 = 0$ (f) $0.1p^2 + p - 2.4 = 0$

SM 3. Solve the following equations, by using the quadratic formula:

(a) $r^2 + 11r - 26 = 0$ (b) $3p^2 + 45p = 48$ (c) $20\,000 = 300K - K^2$

(d) $r^2 + (\sqrt{3} - \sqrt{2})r = \sqrt{6}$ (e) $0.3x^2 - 0.09x = 0.12$ (f) $\frac{1}{24} = p^2 - \frac{1}{12}p$

4. Solve the following equations, by using the quadratic formula:

(a) $x^2 - 3x + 2 = 0$ (b) $5t^2 - t = 3$ (c) $6x = 4x^2 - 1$

(d) $9x^2 + 42x + 44 = 0$ (e) $30\,000 = x(x + 200)$ (f) $3x^2 = 5x - 1$

SM 5. (a) Find the lengths of the sides of a rectangle whose perimeter is 40 cm and whose area is 75 cm^2.

(b) Find two successive natural numbers whose sum of squares is 13.

(c) In a right-angled triangle, the hypotenuse is 34 cm. One of the short sides is 14 cm longer than the other. Find the lengths of the two short sides.

(d) A motorist drove 80 km. In order to save 16 minutes, he had to drive 10 km/h faster than usual. What was his usual driving speed?

6. [HARDER] Solve the following equations:

(a) $x^3 - 4x = 0$ (b) $x^4 - 5x^2 + 4 = 0$ (c) $z^{-2} - 2z^{-1} - 15 = 0$

3.4 Nonlinear Equations

We now study a more general form of equation, which encompasses linear and quadratic equations, but for which no general method is readily available. These nonlinear equations are ubiquitous in economics, so we must be able to handle them and obtain as much information from them as possible.

EXAMPLE 3.4.1 Solve each of the following three separate equations:

(a) $x^3\sqrt{x + 2} = 0$ (b) $x(y + 3)(z^2 + 1)\sqrt{w - 3} = 0$ (c) $x^2 - 3x^3 = 0$

Solution:

(a) If $x^3\sqrt{x + 2} = 0$, then either $x^3 = 0$ or $\sqrt{x + 2} = 0$. The equation $x^3 = 0$ has only the solution $x = 0$, while $\sqrt{x + 2} = 0$ gives $x = -2$. The solutions of the equation are therefore $x = 0$ and $x = -2$.

(b) There are four factors in the product. One of the factors, $z^2 + 1$, is never 0. Hence, the solutions are: $x = 0$ or $y = -3$ or $w = 3$.

(c) Start by factoring: $x^2 - 3x^3 = x^2(1 - 3x)$. The product $x^2(1 - 3x)$ is 0 if and only if $x^2 = 0$ or $1 - 3x = 0$. Hence, the solutions are $x = 0$ and $x = 1/3$.[5]

In solving these equations, we have repeatedly used the fact that a product of two or more factors is 0 if and only if at least one of the factors is 0. That is, in general,

$$ab = ac \quad \text{is equivalent to} \quad a = 0 \quad \text{or} \quad b = c \tag{3.4.1}$$

because the equation $ab = ac$ is equivalent to $ab - ac = 0$, or $a(b - c) = 0$. If $ab = ac$ and $a \neq 0$, we conclude from Eq. (3.4.1) that $b = c$.

EXAMPLE 3.4.2 What conclusions about the variables can we draw if

(a) $x(x + a) = x(2x + b)$ (b) $\lambda y = \lambda z^2$ (c) $xy^2(1 - y) - 2\lambda(y - 1) = 0$

Solution:

(a) $x = 0$ or $x + a = 2x + b$. The last equation gives $x = a - b$. The solutions are therefore $x = 0$ and $x = a - b$.

(b) $\lambda = 0$ or $y = z^2$. It is easy to forget the former possibility.

(c) The equation is equivalent to

$$xy^2(1 - y) + 2\lambda(1 - y) = 0$$

which can be written as

$$(1 - y)(xy^2 + 2\lambda) = 0$$

We conclude from the last equation that $1 - y = 0$ or $xy^2 + 2\lambda = 0$, that is $y = 1$ or $\lambda = -\frac{1}{2}xy^2$.

Finally, we consider also some equations involving fractions. Recall that the fraction a/b is not defined if $b = 0$. If $b \neq 0$, then $a/b = 0$ is equivalent to $a = 0$.

EXAMPLE 3.4.3 Solve the following equations:

(a) $\dfrac{1 - K^2}{\sqrt{1 + K^2}} = 0$ (b) $\dfrac{45 + 6r - 3r^2}{(r^4 + 2)^{3/2}} = 0$ (c) $\dfrac{x^2 - 5x}{\sqrt{x^2 - 25}} = 0$

[5] When trying to solve an equation, an easy way to make a serious mistake is to cancel a factor which might be zero. It is important to check that the factor being cancelled really is not zero. For instance, suppose one cancels the common factor x^2 in the equation $x^2 = 3x^3$. The result is $1 = 3x$, implying that $x = 1/3$. The solution $x = 0$ has been lost.

Solution:

(a) The denominator is never 0, so the fraction is 0 when $1 - K^2 = 0$, that is when $K = \pm 1$.

(b) Again the denominator is never 0. The fraction is 0 when $45 + 6r - 3r^2 = 0$, that is $3r^2 - 6r - 45 = 0$. Solving this quadratic equation, we find that $r = -3$ or $r = 5$.

(c) The numerator is equal to $x(x-5)$, which is 0 if $x = 0$ or $x = 5$. At $x = 0$ the denominator is $\sqrt{-25}$, which is not defined, and at $x = 5$ the denominator is 0. We conclude that the equation has no solutions.

EXERCISES FOR SECTION 3.4

1. Solve the following equations:

(a) $x(x+3) = 0$ (b) $x^3(1+x^2)(1-2x) = 0$ (c) $x(x-3) = x-3$

(d) $\sqrt{2x+5} = 0$ (e) $\dfrac{x^2+1}{x(x+1)} = 0$ (f) $\dfrac{x(x+1)}{x^2+1} = 0$

SM 2. Solve the following equations:

(a) $\dfrac{5+x^2}{(x-1)(x+2)} = 0$ (b) $1 + \dfrac{2x}{x^2+1} = 0$

(c) $\dfrac{(x+1)^{1/3} - \frac{1}{3}x(x+1)^{-2/3}}{(x+1)^{2/3}} = 0$ (d) $\dfrac{x}{x-1} + 2x = 0$

SM 3. Examine what conclusions can be drawn about the variables if:

(a) $z^2(z-a) = z^3(a+b),\ a \neq 0$ (b) $(1+\lambda)\mu x = (1+\lambda)y\mu$

(c) $\dfrac{\lambda}{1+\mu} = \dfrac{-\lambda}{1-\mu^2}$ (d) $ab - 2b - \lambda b(2-a) = 0$

3.5 Using Implication Arrows

Implication and equivalence arrows are very useful in helping to avoid mistakes when solving equations. Consider first the following example.

EXAMPLE 3.5.1 Solve the equation $(2x-1)^2 - 3x^2 = 2\left(\frac{1}{2} - 4x\right)$.

Solution: By expanding $(2x-1)^2$ and also multiplying out the right-hand side, we obtain a new equation that obviously has the same solutions as the original one:

$$(2x-1)^2 - 3x^2 = 2\left(\frac{1}{2} - 4x\right) \Leftrightarrow 4x^2 - 4x + 1 - 3x^2 = 1 - 8x$$

Adding $8x - 1$ to each side of the second equation and then gathering terms gives an equivalent equation:

$$4x^2 - 4x + 1 - 3x^2 = 1 - 8x \Leftrightarrow x^2 + 4x = 0$$

Now $x^2 + 4x = x(x+4)$, and the latter expression is zero if, and only if, $x = 0$ or $x = -4$. That is,

$$x^2 + 4x = 0 \Leftrightarrow x(x+4) = 0 \Leftrightarrow [x = 0 \text{ or } x = -4]$$

where we have used brackets, for the sake of clarity, in the last expression. Putting everything together, we have derived a chain of equivalence arrows showing that the given equation is satisfied for the two values $x = 0$ and $x = -4$, and for no other values of x.

EXAMPLE 3.5.2 Solve the equation $x + 2 = \sqrt{4 - x}$ for x.[6]

Solution: Squaring both sides of the given equation yields

$$(x+2)^2 = \left(\sqrt{4-x}\right)^2$$

Consequently, $x^2 + 4x + 4 = 4 - x$, that is, $x^2 + 5x = 0$. From the latter equation it follows that $x(x+5) = 0$ which yields $x = 0$ or $x = -5$. Thus, a necessary condition for x to solve $x + 2 = \sqrt{4-x}$ is that $x = 0$ or $x = -5$. Inserting these two possible values of x into the original equation shows that only $x = 0$ satisfies the equation. The unique solution to the equation is, therefore, $x = 0$.

The method used in solving Example 3.5.2 is the most common. It involves setting up a chain of implications that starts from the given equation and ends with all the possible solutions. By testing each of these trial solutions in turn, we find which of them really do satisfy the equation. Even if the chain of implications is also a chain of equivalences, such a test is always a useful check of both logic and calculations.

EXERCISES FOR SECTION 3.5

1. Using implication arrows, solve the equation

$$\frac{(x+1)^2}{x(x-1)} + \frac{(x-1)^2}{x(x+1)} - 2\frac{3x+1}{x^2-1} = 0$$

2. Using implication arrows, solve the following equations:

   (a) $x + 2 = \sqrt{4x + 13}$ (b) $|x + 2| = \sqrt{4 - x}$ (c) $x^2 - 2|x| - 3 = 0$

SM 3. Using implication arrows, solve the following equations:

   (a) $\sqrt{x - 4} = \sqrt{x + 5} - 9$ (b) $\sqrt{x - 4} = 9 - \sqrt{x + 5}$

[6] Recall Example 1.2.1.

4. Consider the following attempt to solve the equation $x + \sqrt{x+4} = 2$:

"From the given equation, it follows that $\sqrt{x+4} = 2 - x$. Squaring both sides gives $x + 4 = 4 - 4x + x^2$. After rearranging the terms, it is seen that this equation implies $x^2 - 5x = 0$. Cancelling x, we obtain $x - 5 = 0$ and this equation is satisfied when $x = 5$."

(a) Mark with arrows the implications or equivalences expressed in the text. Which ones are correct?

(b) Solve the equation correctly.

3.6 Two Linear Equations in Two Unknowns

In the macroeconomic model of Example 3.2.1, we found that two equations were needed to model the economic phenomena at hand. In that example, we focused on the solution for the value of GDP, but economists are interested in the solution of *all* endogenous variables in their models. In the example, total consumption would be solved, too.

For the case of two variables that relate through two linear equations, a general method is easy to develop. An example allows us to develop the ideas before we address the general case.

EXAMPLE 3.6.1 Find the values of x and y that satisfy both of the equations

$$2x + 3y = 18$$
$$3x - 4y = -7$$

Solution:

Method 1: A first possibility is to deal with one of the variables first, as we did in Section 3.2, and then use that variable's solution to solve for the other. That is, to follow a two step procedure: (i) solve one of the equations for one of the variables in terms of the other; (ii) substitute the result into the other equation. This leaves only one equation in one unknown, which is easily solved.[7]

To apply this method to our system, we can solve the first equation for y in terms of x: the first equation, $2x + 3y = 18$, implies that $3y = 18 - 2x$ and, hence, that

[7] What we did in Example 3.2.1 was to give the first of these steps. The second step would be to substitute $(**)$ into (ii), to find the solution for C, which is

$$C = \frac{a + b\bar{I}}{1 - b}$$

This equation completes the reduced form of the model, together with $(***)$.

$y = 6 - (2/3)x$. Substituting this expression for y into the second equation gives

$$
\begin{aligned}
3x - 4\left(6 - \frac{2}{3}x\right) &= -7 \\
3x - 24 + \frac{8}{3}x &= -7 \\
9x - 72 + 8x &= -21 \\
17x &= 51
\end{aligned}
$$

so $x = 3$.

Then we find y by using $y = 6 - (2/3)x$ once again to obtain $y = 6 - (2/3) \cdot 3 = 4$. The solution of the system is, therefore, $x = 3$ and $y = 4$.[8]

Method 2: This method is based on eliminating one of the variables by adding or subtracting a multiple of one equation from the other. Suppose we want to eliminate y. Suppose we multiply the first equation in the system by 4 and the second by 3. Then the coefficients of y in both equations will be the same except for the sign. If we then add the transformed equations, the term in y disappears and we obtain

$$
\begin{array}{r}
8x + 12y = 72 \\
9x - 12y = -21 \\
\hline
17x \qquad = 51
\end{array}
$$

Hence, $x = 3$. To find the value for y, substitute 3 for x in either of the original equations and solve for y. This gives $y = 4$, which agrees with the earlier result.

Systems of two equations and two unknowns are usually known as 2×2 systems. The general 2×2 linear system is

$$ax + by = c \tag{3.6.1}$$

$$dx + ey = f \tag{3.6.2}$$

where a, b, c, d, e, and f are arbitrary given numbers, whereas x and y are the variables, or "unknowns".[9]

Let us first assume that $ae - bd \neq 0$. Using Method 2, we multiply the first equation by e and the second by $-b$, then add, to obtain

$$
\begin{array}{c}
aex + bey = ce \\
-bdx - bey = -bf \\
\hline
(ae - bd)x = ce - bf
\end{array}
$$

[8] A useful check is to verify such a solution by direct substitution. Indeed, substituting $x = 3$ and $y = 4$ in the system gives $2 \cdot 3 + 3 \cdot 4 = 18$ and $3 \cdot 3 - 4 \cdot 4 = -7$.

[9] If $a = 2$, $b = 3$, $c = 18$, $d = 3$, $e = -4$, and $f = -7$, this reduces to the system in the example.

which gives the value for x. We can substitute back in (3.6.1) to find y, and the result is

$$x = \frac{ce - bf}{ae - bd} \qquad \text{and} \qquad y = \frac{af - cd}{ae - bd}$$

We have found expressions for both x and y.

Evidently, these last formulas break down if the denominator $ae - bd$ in both fractions is equal to 0. In this case the second method fails, and indeed solving the two Eqs (3.6.1) and (3.6.2) requires a special method. We begin by following the first method, assuming that $b \neq 0$.[10] Solving for y in Eq. (3.6.1), we get

$$y = \frac{c}{b} - \frac{a}{b}x \tag{$*$}$$

If we now substitute this solution into (3.6.2), we obtain

$$\begin{aligned} dx + e\left(\frac{c}{b} - \frac{a}{b}x\right) &= f \\ dx + \frac{ce}{b} - \frac{ae}{b}x &= f \\ bd \cdot x + ce - ae \cdot x &= bf \\ (bd - ae)x &= bf - ce \end{aligned}$$

The problem is that, since $ae - bd = 0$, the last equation becomes, simply, the statement that $bf - ac = 0$, or that $ac = bf$. Note that we have lost both variables from our analysis, and are left with a statement, which may or may not be true, about the parameters of the system.

If $ae = bd$ and $bf \neq ce$, the system has no solution: it is impossible to find values of x and y that simultaneously satisfy its two equations. Now, suppose that, besides $ae - bd = 0$, we also have that $bf - ce = 0$. Recalling that

$$\begin{aligned} aex + bey &= ce \\ -bdx - bey &= -bf \end{aligned}$$

we can see that our two equations are, in fact, only one. We are dealing with a system where one of the equations repeats the information given by the other, adding nothing new to it. Now, note that if we pick *any* value of x, and let y be determined by ($*$), the first equation is satisfied immediately, by construction. As for the second equation, using the facts that $ae - bd = 0$, $bf - ce = 0$ and $b \neq 0$, so $ae/b = d$ and $ce/b = f$, we get that

$$dx + ey = dx + e\left(\frac{c}{b} - \frac{a}{b}x\right) = \left(d - \frac{ae}{b}\right)x + \frac{ce}{b} = 0 \cdot x + f = f$$

The second equation is satisfied "for free", and we have found "the" solution for the system—in fact, infinitely many of them: with any value of x, and $y = c/b - (a/b)x$.

[10] We only need one of the coefficients a, b, d or e to differ from zero. If all are zero, the problem is trivial and not interesting.

We can summarize these results as follows:

2 × 2 LINEAR SYSTEMS

The solution for system (3.6.1) is

$$x = \frac{ce - bf}{ae - bd} \quad \text{and} \quad y = \frac{af - cd}{ae - bd}, \quad \text{provided that} \quad ae \neq bd \tag{3.6.3}$$

When $ae = bd$ and $bf \neq ce$, the system has no solution.
When $ae = bd$ and $bf = ce$, the system has infinitely many solutions:

$$x \text{ takes } \textit{any} \text{ value and } \; y = \frac{c}{b} - \frac{a}{b}x, \quad \text{provided that} \quad b \neq 0 \tag{3.6.4}$$

This kind of system, and the issues associated with its solution, will be studied in further detail in Section 15.1.

EXERCISES FOR SECTION 3.6

1. Solve the following systems of equations:

(a) $x - y = 5$ and $x + y = 11$ (b) $4x - 3y = 1$ and $2x + 9y = 4$

(c) $3x + 4y = 2.1$ and $5x - 6y = 7.3$

2. Solve the following systems of equations:

(a) $\begin{aligned} 5x + 2y &= 3 \\ 2x + 3y &= -1 \end{aligned}$ (b) $\begin{aligned} x - 3y &= -25 \\ 4x + 5y &= 19 \end{aligned}$ (c) $\begin{aligned} 2x + 3y &= 3 \\ 6x + 6y &= -1 \end{aligned}$

3. Solve the following systems of equations:

(a) $\begin{aligned} 23p + 45q &= 181 \\ 10p + 15q &= 65 \end{aligned}$ (b) $\begin{aligned} 0.01r + 0.21s &= 0.042 \\ -0.25r + 0.55s &= -0.47 \end{aligned}$

SM **4.** (a) Find two numbers whose sum is 52 and whose difference is 26.

(b) Five tables and 20 chairs cost \$1 800, whereas two tables and three chairs cost \$420. What is the price of each table and each chair?

(c) A firm produces headphones in two qualities, Basic (B) and Premium (P). For the coming year, the estimated output of B is 50% higher than that of P. The profit per unit sold is \$300 for P and \$200 for B. If the profit target is \$180 000 over the next year, how much of each of the two qualities must be produced?

(d) At the beginning of the year a person had a total of \$10 000 in two accounts. The interest rates were 5% and 7.2% per year, respectively. If the person has made no transfers during the year, and has earned a total of \$676 interest, what was the initial balance in each of the two accounts?

REVIEW EXERCISES

1. Solve each of the following equations:

(a) $3x - 20 = 16$ (b) $-5x + 8 + 2x = -(4 - x)$ (c) $-6(x - 5) = 6(2 - 3x)$

(d) $\dfrac{4 - 2x}{3} = -5 - x$ (e) $\dfrac{5}{2x - 1} = \dfrac{1}{2 - x}$ (f) $\sqrt{x - 3} = 6$

SM 2. Solve each of the following equations:

(a) $\dfrac{x - 3}{x - 4} = \dfrac{x + 3}{x + 4}$ (b) $\dfrac{3(x + 3)}{x - 3} - 2 = 9\dfrac{x}{x^2 - 9} + \dfrac{27}{(x + 3)(x - 3)}$

(c) $\dfrac{2x}{3} = \dfrac{2x - 3}{3} + \dfrac{5}{x}$ (d) $\dfrac{x - 5}{x + 5} - 1 = \dfrac{1}{x} - \dfrac{11x + 20}{x^2 - 5x}$

3. Solve the following equations for the variables specified:

(a) $x = \frac{2}{3}(y - 3) + y$, for y (b) $ax - b = cx + d$, for x

(c) $AK\sqrt{L} = Y_0$, for L (d) $px + qy = m$, for y

(e) $\dfrac{\dfrac{1}{1 + r} - a}{\dfrac{1}{1 + r} + b} = c$, for r (f) $Px(Px + Q)^{-1/3} + (Px + Q)^{2/3} = 0$, for x

SM 4. Solve the following equations for the variables indicated:

(a) $3K^{-1/2}L^{1/3} = 1/5$, for K (b) $(1 + r/100)^t = 2$, for r

(c) $p - abx_0^{b-1} = 0$, for x_0 (d) $\left[(1 - \lambda)a^{-\rho} + \lambda b^{-\rho}\right]^{-1/\rho} = c$, for b

5. Solve the following quadratic equations:

(a) $z^2 = 8z$ (b) $x^2 + 2x - 35 = 0$ (c) $p^2 + 5p - 14 = 0$

(d) $12p^2 - 7p + 1 = 0$ (e) $y^2 - 15 = 8y$ (f) $42 = x^2 + x$

6. Solve the following equations:

(a) $(x^2 - 4)\sqrt{5 - x} = 0$ (b) $(x^4 + 1)(4 + x) = 0$ (c) $(1 - \lambda)x = (1 - \lambda)y$

7. Johnson invested \$1 500, part of it at 15% interest and the remainder at 20%. His total yearly income from the two investments was \$275. How much did he invest at each rate?

8. Consider the macro model described by the three equations

$$Y = C + \bar{I} + G, \qquad C = b(Y - T), \qquad T = tY$$

Here the parameters b and t lie in the interval $(0, 1)$, Y is the gross domestic product (GDP), C is consumption, $\bar{I}$ is total investment, T denotes taxes, and G is government expenditure.

(a) Express Y and C in terms of $\bar{I}$, G, and the parameters.

(b) What happens to Y and C as t increases?

9. If $5^{3x} = 25^{y+2}$ and $x - 2y = 8$, what is $x - y$?

10. [HARDER] Solve the following systems of equations:

(a) $$\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 4 \\ \frac{3}{x} - \frac{2}{y} &= 19 \end{aligned}$$

(b) $$\begin{aligned} 3\sqrt{x} + 2\sqrt{y} &= 2 \\ 2\sqrt{x} - 3\sqrt{y} &= \frac{1}{4} \end{aligned}$$

(c) $$\begin{aligned} x^2 + y^2 &= 13 \\ 4x^2 - 3y^2 &= 24 \end{aligned}$$

4 FUNCTIONS OF ONE VARIABLE

. . . —mathematics is not so much a subject as a way of studying any subject, not so much a science as a way of life.
—George F.J. Temple (1981)

Functions are important in practically every area of pure and applied mathematics, including mathematics applied to economics. The language of economic analysis is full of terms like demand and supply functions, cost functions, production functions, consumption functions, etc. In this chapter we present a discussion of functions of one real variable, illustrated by some very important economic examples.

4.1 Introduction

One variable is a function of another if the first variable *depends* upon the second. For instance, the area of a circle is a function of its radius. If the radius r is given, then the area A is determined. In fact $A = \pi r^2$, where π is the numerical constant 3.14159 . . . .

One does not need a mathematical formula to convey the idea that one variable is a function of another: A table can also show the relationship. For instance, Table 4.1 shows the evolution of total final consumption expenditure, measured in current euros, without allowing for inflation, in the European Union (28 countries), from the first quarter of 2013, which we write as 13Q1, to the last quarter of 2014, which we write as 14Q4. This table defines consumption expenditure as a function of the calendar quarter.

In ordinary conversation we sometimes use the word "function" in a similar way. For example, we might say that the infant mortality rate of a country is a function of the quality of its health care, or that a country's national product is a function of the level of investment.

The dependence between two real variables can also be illustrated by means of a graph. In Fig. 4.1.1 we have drawn a curve that allegedly played an important role some years ago in the discussion of "supply side economics". It shows the presumed relationship between a country's income tax rate and its total income tax revenue. Obviously, if the tax rate is 0%, then tax revenue is 0. However, if the tax rate is 100%, then tax revenue will also be (about) 0, since nobody is willing to work if their entire income is going to be confiscated.

Table 4.1 Final consumption expenditure in the EU, 2013Q1–2014Q4 (billions of euros)

Quarter	13Q1	13Q2	13Q3	13Q4	14Q1	14Q2	14Q3	14Q4
Consumption	1 917.5	1 924.9	1 934.3	1 946.0	1 958.6	1 973.4	1 995.1	2 008.2

This curve, which has generated considerable controversy, is supposed to have been drawn on the back of a restaurant napkin by an American economist, Arthur Laffer, who later popularized its message with the public.[1]

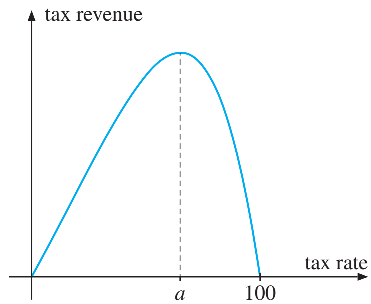


Figure 4.1.1 The "Laffer curve", which relates tax revenue to tax rates

In some instances a graph is preferable to a formula. A case in point is an electrocardiogram (ECG) showing the heartbeat pattern of a patient. Here the doctor studies the pattern of repetitions directly from the graphs; the patient might die before the doctor could understand a formula approximating the ECG picture.

All of the relationships discussed above have one characteristic in common: A definite rule relates each value of one variable to a definite value of another variable. In the ECG example the function is the rule showing electrical activity as a function of time.

In all of our examples it is implicitly assumed that the variables are subject to certain constraints. For instance, in Table 4.1 only the quarters of 2013 and 2014 are relevant.

4.2 Basic Definitions

The examples in the preceding section lead to the following general definition, with D a set of real numbers:

FUNCTION

A (real-valued) *function* of a real variable x with *domain* D is a rule that assigns a unique real number to each real number x in D. As x varies over the whole domain, the set of all possible resulting values $f(x)$ is called the *range* of f.

[1] Actually, there are many economists who previously had the same idea. See, for instance, part (b) in Example 7.2.2.

The word "rule" is used in a very broad sense. *Every* rule with the properties described is called a function, whether that rule is given by a formula, described in words, defined by a table, illustrated by a curve, or expressed by any other means.

Functions are given letter names, such as f, g, F, or φ. If f is a function and x is a number in its domain D, then $f(x)$ denotes the number that the function f assigns to x. The symbol $f(x)$ is pronounced "f of x", or often just "f x". It is important to note the difference between f, which is a symbol for the function (the rule), and $f(x)$, which denotes the value of f at x.

If f is a function, we sometimes let y denote the value of f at x, so $y = f(x)$. Then, we call x the *independent variable*, or the *argument* of f, whereas y is called the *dependent variable*, because the value y (in general) depends on the value of x. The domain of the function f is then the set of all possible values of the independent variable, whereas the range is the set of corresponding values of the dependent variable. In economics, x is often called the *exogenous* variable, which is supposed to be fixed *outside* the economic model, whereas for each given x the equation $y = f(x)$ serves to determine the *endogenous* variable y *inside* the economic model.

A function is often defined by a formula, such as $y = 8x^2 + 3x + 2$. The function is then the rule $x \mapsto 8x^2 + 3x + 2$ that assigns the number $8x^2 + 3x + 2$ to each value of x.

Functional Notation

To become familiar with the relevant notation, it helps to look at some examples of functions that are defined by formulas.

EXAMPLE 4.2.1 A function is defined for all numbers by the following rule:

Assign to any number its third power.

This function will assign $0^3 = 0$ to 0, $3^3 = 27$ to 3, $(-2)^3 = (-2)(-2)(-2) = -8$ to -2, and $(1/4)^3 = 1/64$ to $1/4$. In general, it assigns the number x^3 to the number x. If we denote this third power function by f, then $f(x) = x^3$. So we have $f(0) = 0^3 = 0$, $f(3) = 3^3 = 27$, $f(-2) = (-2)^3 = -8$, and $f(1/4) = (1/4)^3 = 1/64$. In general, substituting a for x in the formula for f gives $f(a) = a^3$, whereas

$$f(a+1) = (a+1)^3 = (a+1)(a+1)(a+1) = a^3 + 3a^2 + 3a + 1$$

A common error is to presume that $f(a) = a^3$ implies $f(a+1) = a^3 + 1$. The error can be illustrated by considering a simple interpretation of f. If a is the edge of a cube measured in metres, then $f(a) = a^3$ is the volume of the cube measured in cubic metres, or m^3. Suppose that each edge of the cube expands by 1 m. Then the volume of the new cube is $f(a+1) = (a+1)^3\ \mathrm{m}^3$. The number $a^3 + 1$ can be interpreted as the number obtained when the volume of a cube with edge a is increased by $1\ \mathrm{m}^3$. In fact, $f(a+1) = (a+1)^3$ is quite a bit more than $a^3 + 1$, as illustrated in Figs 4.2.1 and 4.2.2.

EXAMPLE 4.2.2 The total dollar cost of producing x units of a product is given by

$$C(x) = 100x\sqrt{x} + 500$$

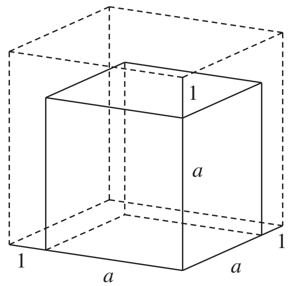


Figure 4.2.1 Volume $f(a+1) = (a+1)^3$

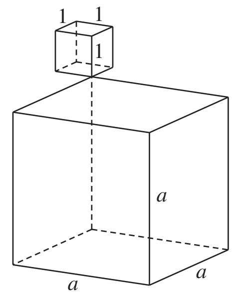


Figure 4.2.2 Volume $a^3 + 1$

for each nonnegative integer x. Find the cost of producing 16 units. Suppose the firm produces a units; find the *increase* in the cost from producing one additional unit.

Solution: The cost of producing 16 units is found by substituting 16 for x in the formula for $C(x)$:

$$C(16) = 100 \cdot 16\sqrt{16} + 500 = 100 \cdot 16 \cdot 4 + 500 = 6900$$

The cost of producing a units is $C(a) = 100a\sqrt{a} + 500$, and the cost of producing $a+1$ units is $C(a+1)$. Thus the increase in cost is

$$\begin{aligned} C(a+1) - C(a) &= 100(a+1)\sqrt{a+1} + 500 - 100a\sqrt{a} - 500 \\ &= 100\left[(a+1)\sqrt{a+1} - a\sqrt{a}\right] \end{aligned}$$

In economic theory, we often study functions that depend on a number of parameters, as well as the independent variable. An obvious generalization of Example 4.2.2 follows.

EXAMPLE 4.2.3 Suppose that the cost of producing x units of a commodity is

$$C(x) = Ax\sqrt{x} + B$$

where A and B are constants. Find the cost of producing 0, 10, and $x+h$ units.

Solution: The cost of producing 0 units is $C(0) = A \cdot 0 \cdot \sqrt{0} + B = 0 + B = B$.[2] Similarly, $C(10) = A10\sqrt{10} + B$. Finally,

$$C(x+h) = A(x+h)\sqrt{x+h} + B$$

which comes from substituting $x+h$ for x in the given formula.

So far we have used x to denote the independent variable, but we could just as well have used almost any other symbol. For example, the following formulas define exactly the same function (and hence we can set $f = g = \varphi$):

$$f(x) = x^4, \qquad g(t) = t^4, \qquad \varphi(\xi) = \xi^4$$

[2] Parameter B simply represents fixed costs. These are the costs that must be paid whether or not anything is actually produced, such as a taxi driver's annual licence fee.

For that matter, we could also express this function as $x \mapsto x^4$, or alternatively as $f(\cdot) = (\cdot)^4$. Here it is understood that the dot between the parentheses can be replaced by an arbitrary number, or an arbitrary letter, or even another function (like $1/y$). Thus,

$$1 \mapsto 1^4 = 1, \quad k \mapsto k^4, \quad \text{and} \quad 1/y \mapsto (1/y)^4$$

or alternatively

$$f(1) = 1^4 = 1, \quad f(k) = k^4, \quad \text{and} \quad f(1/y) = (1/y)^4$$

Specifying the Domain and the Range

The definition of a function is not really complete unless its domain is either obvious or specified explicitly. The natural domain of the function f defined by $f(x) = x^3$ is the set of all real numbers. In Example 4.2.2, where $C(x) = 100x\sqrt{x} + 500$ denotes the cost of producing x units of a product, the domain was specified as the set of nonnegative integers. Actually, a more natural domain is the set of numbers $0, 1, 2, \ldots, x_0$, where x_0 is the maximum number of items the firm can produce. For a producer like an iron mine, however, where output x is a continuous variable, the natural domain is the closed interval $[0, x_0]$.

We shall adopt the convention that *if a function is defined using an algebraic formula, the domain consists of all values of the independent variable for which the formula gives a unique value, unless another domain is explicitly mentioned.*

EXAMPLE 4.2.4 Find the domains of

(a) $f(x) = 1/(x + 3)$ (b) $g(x) = \sqrt{2x + 4}$

Solution:

(a) For $x = -3$, the formula reduces to the meaningless expression "1/0". For all other values of x, the formula makes $f(x)$ a well-defined number. Thus, the domain consists of all numbers $x \neq -3$.

(b) The expression $\sqrt{2x + 4}$ is uniquely defined for all x such that $2x + 4$ is nonnegative. Solving the inequality $2x + 4 \geq 0$ for x gives $x \geq -2$. The domain of g is therefore the interval $[-2, \infty)$.

Let f be a function with domain D. The set of all values $f(x)$ that the function assumes is called the *range* of f. Often, we denote the domain of f by D_f, and the range by R_f. These concepts are illustrated in Fig. 4.2.3, using the idea of the graph of a function which we formally discuss in Section 4.3.

Alternatively, we can think of any function f as an engine operating so that if x in the domain is an input, the output is $f(x)$. The range of f is then all the numbers we get as output using all numbers in the domain as inputs. If we try to use as an input a number not in the domain, the engine does not work, and there is no output.

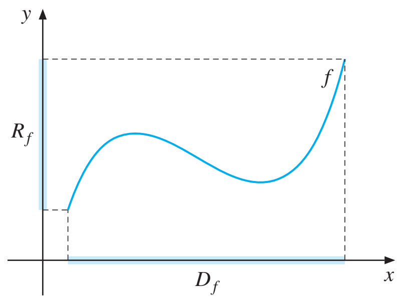


Figure 4.2.3 The domain and range of f

EXAMPLE 4.2.5 Show first that the number 4 belongs to the range of the function defined by $g(x) = \sqrt{2x+4}$. Find the entire range of g. Then, use Example 4.2.4 to show that g has domain $[-2, \infty)$.

Solution: To show that 4 is in the range of g, we must find a number x such that $g(x) = 4$. That is, we must solve the equation $\sqrt{2x+4} = 4$ for x. By squaring both sides of the equation, we get $2x + 4 = 4^2 = 16$, that is, $x = 6$. Because $g(6) = 4$, the number 4 does belong to the range R_g.

In order to determine the whole range of g, we must answer the question: As x runs through the whole of the interval $[-2, \infty)$, what are all the possible values of $\sqrt{2x+4}$? For $x = -2$, one has $\sqrt{2x+4} = 0$, and $\sqrt{2x+4}$ can never be negative. We claim that whatever number $y_0 \geq 0$ is chosen, there exists a number x_0 such that $\sqrt{2x_0+4} = y_0$. Indeed, squaring each side of this last equation gives $2x_0 + 4 = y_0^2$. Hence, $2x_0 = y_0^2 - 4$, which implies that $x_0 = \frac{1}{2}(y_0^2 - 4)$. Because $y_0^2 \geq 0$, we have $x_0 = \frac{1}{2}\left(y_0^2 - 4\right) \geq \frac{1}{2}(-4) = -2$. Hence, for every number $y_0 \geq 0$, we have found a number $x_0 \geq -2$ such that $g(x_0) = y_0$. The range of g is, therefore, $[0, \infty)$.

Even if a function is completely specified by a formula, including a specific domain, it is not always easy to find the range of the function. For example, without using the methods of differential calculus, it is hard to find R_f exactly when $f(x) = 3x^3 - 2x^2 - 12x - 3$ and $D_f = [-2, 3]$.

A function f is called *increasing* if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$, and *strictly increasing* if $x_1 < x_2$ implies $f(x_1) < f(x_2)$. *Decreasing* and *strictly decreasing* functions are defined in the obvious way (see Section 6.3). The function g in Example 4.2.5 is strictly increasing in $[-2, \infty)$.

Calculators (including calculator programs on personal computers or smart phones) often have many special functions built into them. For example, most of them have the key $\boxed{\sqrt{x}}$ which, when given a number x, returns the square root $\sqrt{x}$ of that number. If we enter a nonnegative number such as 25, and then press the square root key, the number 5 appears. If we enter -3, then "Error", or "Not a number" is shown. This is the way the calculator tells us that $\sqrt{-3}$ is not defined within the real number system.

EXERCISES FOR SECTION 4.2

SM **1.** Let $f(x) = x^2 + 1$.

(a) Compute $f(0)$, $f(-1)$, $f(1/2)$, and $f(\sqrt{2})$.

(b) For what values of x is it true that

(i) $f(x) = f(-x)$?

(ii) $f(x+1) = f(x) + f(1)$?

(iii) $f(2x) = 2f(x)$?

2. Suppose $F(x) = 10$, for all x. Find $F(0)$, $F(-3)$, and $F(a+h) - F(a)$.

3. Let $f(t) = a^2 - (t-a)^2$, where a is a constant.

(a) Compute $f(0)$, $f(a)$, $f(-a)$, and $f(2a)$. (b) Compute $3f(a) + f(-2a)$.

4. Let $f(x) = x/(1+x^2)$.

(a) Compute $f(-1/10)$, $f(0)$, $f(1/\sqrt{2})$, $f(\sqrt{\pi})$, and $f(2)$.

(b) Show that $f(-x) = -f(x)$ for all x, and that $f(1/x) = f(x)$ for $x \neq 0$.

5. Let $F(t) = \sqrt{t^2 - 2t + 4}$. Compute $F(0)$, $F(-3)$, and $F(t+1)$.

6. The cost of producing x units of a commodity is given by $C(x) = 1000 + 300x + x^2$.

(a) Compute $C(0)$, $C(100)$, and $C(101) - C(100)$.

(b) Compute $C(x+1) - C(x)$, and explain in words the meaning of the difference.

7. The demand for cotton in the USA, for the period 1915–1919, with appropriate units for the price P and the quantity Q, was estimated to be $Q = D(P) = 6.4 - 0.3P$.

(a) Find the demand quantity in each case if the price is 8, 10, and 10.22.

(b) If the demand quantity is 3.13, what is the price?

8. (a) If $f(x) = 100x^2$, show that for all t, $f(tx) = t^2 f(x)$.

(b) If $P(x) = x^{1/2}$, show that for all $t \geq 0$, $P(tx) = t^{1/2}P(x)$.

9. The cost of removing $p\%$ of the impurities in a lake is given by $b(p) = 10p/(105-p)$.

(a) Find $b(0)$, $b(50)$, and $b(100)$.

(b) What does $b(50+h) - b(50)$ mean (where $h \geq 0$)?

10. Only for very special "additive" functions is it true that $f(a+b) = f(a) + f(b)$ for all a and b. Determine whether $f(2+1) = f(2) + f(1)$ for the following functions:

(a) $f(x) = 2x^2$ (b) $f(x) = -3x$ (c) $f(x) = \sqrt{x}$

11. (a) If $f(x) = Ax$, show that $f(a+b) = f(a) + f(b)$ for all numbers a and b.

(b) If $f(x) = 10^x$, show that $f(a+b) = f(a) \cdot f(b)$ for all natural numbers a and b.

12. A friend of yours claims that $(x+1)^2 = x^2 + 1$. Can you use a geometric argument to show that this is wrong?

SM 13. Find the domains of the functions defined by the following formulas:

(a) $y = \sqrt{5-x}$ (b) $y = \dfrac{2x-1}{x^2-x}$ (c) $y = \sqrt{\dfrac{x-1}{(x-2)(x+3)}}$

14. Let $f(x) = (3x+6)/(x-2)$.

(a) Find the domain of the function.

(b) Show that 5 is in the range of f, by finding an x such that $(3x+6)/(x-2) = 5$.

(c) Show that 3 is not in the range of f.

15. Find the domain and the range of $g(x) = 1 - \sqrt{x+2}$.

4.3 Graphs of Functions

Recall that a *rectangular* (or a *Cartesian*) coordinate system is obtained by first drawing two perpendicular lines, called coordinate axes. The two axes are respectively the *x-axis* (or the *horizontal axis*) and the *y-axis* (or the *vertical axis*). The intersection point O is called the *origin*. We measure the real numbers along each of these lines, as shown in Fig. 4.3.1. The unit distance on the x-axis is not necessarily the same as on the y-axis, although this is the case in Fig. 4.3.1.

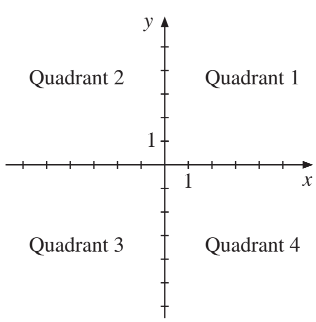


Figure 4.3.1 A coordinate system

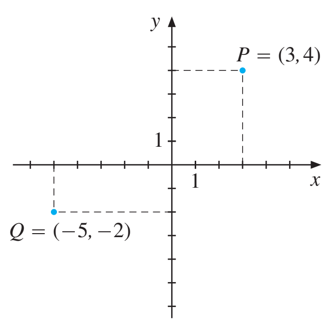


Figure 4.3.2 Points $(3, 4)$ and $(-5, -2)$

The rectangular coordinate system in Fig. 4.3.1 is also called the *xy-plane*. The coordinate axes separate the plane into four quadrants, which traditionally are numbered as in Fig. 4.3.1. Any point P in the plane can be represented by a unique pair (a, b) of real numbers. These can be found by drawing dashed lines, like those in Fig. 4.3.2, which are perpendicular to the two axes. The point represented by (a, b) lies at the intersection of the vertical straight line $x = a$ with the horizontal straight line $y = b$.

Conversely, any pair of real numbers represents a unique point in the plane. For example, in Fig. 4.3.2, if the ordered pair $(3, 4)$ is given, the corresponding point P lies at the intersection of $x = 3$ with $y = 4$. Thus, P lies three units to the right of the y-axis and four units above the x-axis. We call $(3, 4)$ the *coordinates* of P. Similarly, Q lies five units to the left of the y-axis and two units below the x-axis, so the coordinates of Q are $(-5, -2)$.

Note that we call (a, b) an *ordered pair*, because the order of the two numbers in the pair is important. For instance, $(3, 4)$ and $(4, 3)$ represent two different points.

As you surely know, each function of one variable can be represented by a graph in such a rectangular coordinate system. Such a representation helps us visualize the function. This is because the shape of the graph reflects the properties of the function.

GRAPH

The *graph* of a function f is the set of all ordered pairs $(x, f(x))$, where x belongs to the domain of f.

EXAMPLE 4.3.1

Consider the function $f(x) = x^2 - 4x + 3$. The values of $f(x)$ for some special choices of x are given in Table 4.2. If we plot the points $(0, 3)$, $(1, 0)$, $(2, -1)$, $(3, 0)$, and $(4, 3)$, which are obtained from the table, in an xy-plane, and draw a smooth curve through these points, we obtain the graph of the function, as in Fig. 4.3.3.[3]

Table 4.2 Values of $f(x) = x^2 - 4x + 3$

x	0	1	2	3	4
$f(x) = x^2 - 4x + 3$	3	0	-1	0	3

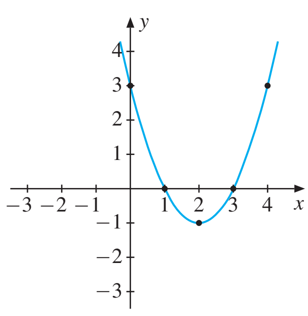


Figure 4.3.3 The graph of $f(x) = x^2 - 4x + 3$

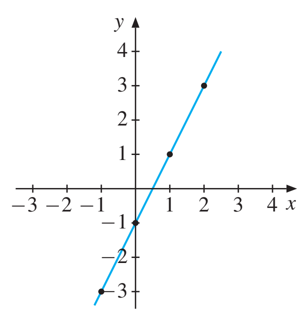


Figure 4.3.4 The graph of $g(x) = 2x - 1$

[3] This graph is called a *parabola*, as you will see in Section 4.6.

EXAMPLE 4.3.2 Find some of the points on the graph of $g(x) = 2x - 1$, and sketch it.

Solution: One has $g(-1) = 2 \cdot (-1) - 1 = -3$, $g(0) = 2 \cdot 0 - 1 = -1$, and $g(1) = 2 \cdot 1 - 1 = 1$. Moreover, $g(2) = 3$. There are infinitely many points on the graph, so we cannot write them all down. In Fig. 4.3.4 the four points $(-1, -3)$, $(0, -1)$, $(1, 1)$, and $(2, 3)$ are marked off, and they seem to lie on a straight line. That line is the graph.

Some Important Graphs

Figures 4.3.5–4.3.10 show some special functions that occur so often in applications that you should learn to recognize their graphs. You should in each case make a table of function values to confirm the form of these graphs.

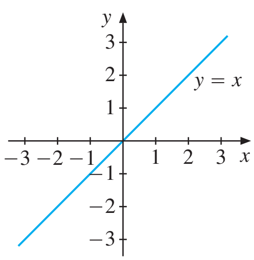


Figure 4.3.5 $y = x$

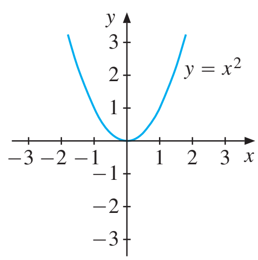


Figure 4.3.6 $y = x^2$

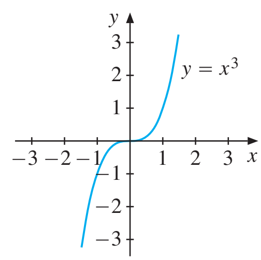


Figure 4.3.7 $y = x^3$

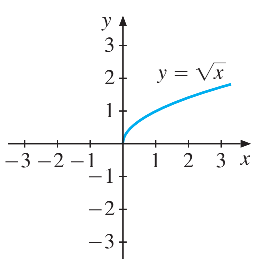


Figure 4.3.8 $y = \sqrt{x}$

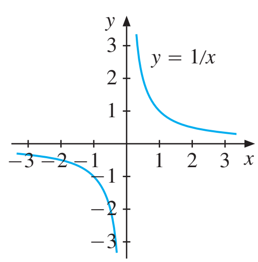


Figure 4.3.9 $y = 1/x$

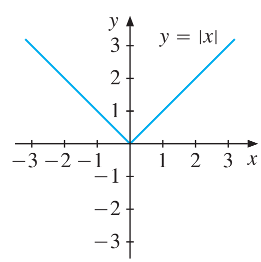


Figure 4.3.10 $y = |x|$

Note that when we try to plot the graph of a function, we must try to include a sufficient number of points, otherwise we might miss some of its important features. Actually, by merely plotting a finite set of points, we can never be entirely sure that there are no wiggles or bumps we have missed. For more complicated functions we have to use differential calculus to decide how many bumps and wiggles there are.

EXERCISES FOR SECTION 4.3

1. Plot all the five points $(2, 3)$, $(-3, 2)$, $(-3/2, -2)$, $(4, 0)$, and $(0, 4)$ in one coordinate system.

2. The graph of function f is given in Fig. 4.3.11.

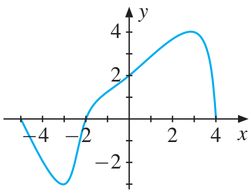


Figure 4.3.11 Exercise 2

(a) Find $f(-5), f(-3), f(-2), f(0), f(3)$, and $f(4)$ by examining the graph.

(b) Determine the domain and the range of f.

3. Fill in the tables and draw the graphs of the following functions:

(a)

x	0	1	2	3	4
$g(x) = -2x + 5$					

(b)

x	−2	−1	0	1	2	3	4
$h(x) = x^2 - 2x - 3$							

(c)

x	−2	−1	0	1	2
$F(x) = 3^x$					

(d)

x	−2	−1	0	1	2	3
$G(x) = 1 - 2^{-x}$						

4.4 Linear Functions

Linear equations occur very often in economics. Recall from Eq. (3.2.1) that they are defined as

$$y = ax + b$$

where a and b are constants. As we saw in Example 4.3.2, the graph of the equation is a straight line. If we let f denote the function that assigns y to x, then $f(x) = ax + b$, and f is called a *linear* function.

Take an arbitrary value of x. Then

$$f(x + 1) - f(x) = a(x + 1) + b - ax - b = a$$

This shows that a measures the change in the value of the function when x increases by one unit. For this reason, the number a is the *slope* of the line (or the function). If the slope a is positive, then the line slants upward to the right, and the larger the value of a, the steeper is the line. On the other hand, if a is negative, then the line slants downward to the right, and the absolute value of a measures the steepness of the line. For example, when $a = -3$, the steepness is 3. In the special case when $a = 0$, the steepness is zero, because the line is horizontal. In this case we have $a = 0$, so the line $y = ax + b$ becomes $y = b$ for all x.

The three different cases are illustrated in Figs 4.4.1 to 4.4.3. If $x = 0$, then $y = ax + b = b$, and b is called the **y**-*intercept*, or often just the intercept.

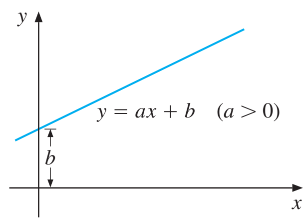


Figure 4.4.1 Increasing

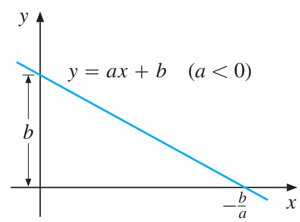


Figure 4.4.2 Decreasing

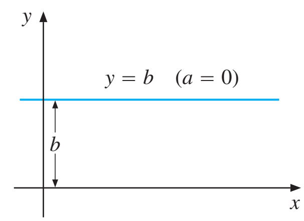


Figure 4.4.3 Constant

EXAMPLE 4.4.1 Find and interpret the slopes of the following straight lines.

(a) The cost function for US Steel Corp. during the period 1917–1938 was estimated to be $C = 55.73x + 182\,100\,000$, where C is the total cost in dollars per year, and x is the production of steel in tons per year.

(b) The demand function for rice in India, for the period 1949–1964, was estimated to be $q = -0.15p + 0.14$, where p is price per kilo in Indian rupees, and q is the annual consumption per person, measured in kilos.

Solution:

(a) The slope is 55.73, which means that if production increases by one ton, then the cost *increases* by \$55.73.

(b) The slope is -0.15, which tells us that if the price increases by one Indian rupee per kilo, then the quantity demanded *decreases* by 0.15 kilos per year.

Computing the slope of a straight line in the plane is easy. Pick two different points on the line $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, as shown in Fig. 4.4.4. The slope of the line is the ratio $(y_2 - y_1)/(x_2 - x_1)$

If we denote the slope by a, then:

SLOPE OF A STRAIGHT LINE

The *slope* of the straight line l is

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_1 \neq x_2$$

where (x_1, y_1) and (x_2, y_2) are any two distinct points on l.

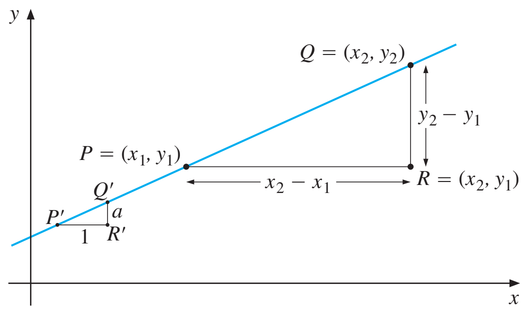


Figure 4.4.4 Slope $a = (y_2 - y_1)/(x_2 - x_1)$

Multiplying both the numerator and the denominator of $(y_2 - y_1)/(x_2 - x_1)$ by -1 gives $(y_1 - y_2)/(x_1 - x_2)$, which does *not* change the ratio. This shows that it does not make any difference which point is P and which is Q. Moreover, the properties of similar triangles imply that the ratios $Q'R'/P'R'$ and QR/PR in Fig. 4.4.4 must be equal. For this reason, the number $a = (y_2 - y_1)/(x_2 - x_1)$ is equal to the change in the value of y when x increases by one unit; this change is the slope.

EXAMPLE 4.4.2 Determine the slopes of the three straight lines l, m, and n in Figs 4.4.5 to 4.4.7.

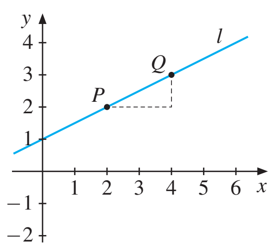


Figure 4.4.5 Line l

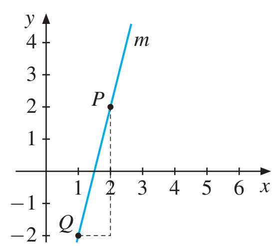


Figure 4.4.6 Line m

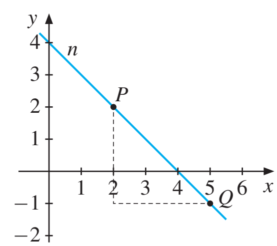


Figure 4.4.7 Line n

Solution: The lines l, m, and n all pass through $P = (2, 2)$. In Fig. 4.4.5, the point Q is $(4, 3)$, whereas in Fig. 4.4.6 it is $(1, -2)$, and in Fig. 4.4.7 it is $(5, -1)$. Therefore, the respective slopes of the lines l, m, and n are

$$a_l = \frac{3-2}{4-2} = \frac{1}{2}, \qquad a_m = \frac{-2-2}{1-2} = 4, \qquad a_n = \frac{-1-2}{5-2} = -1$$

The Point–slope and Point–point Formulas

Let us find the equation of a straight line l passing through the point $P = (x_1, y_1)$ with slope a. If (x, y) is any other point on the line, the slope a is given by the formula:

$$\frac{y - y_1}{x - x_1} = a$$

Multiplying each side by $x - x_1$, we obtain $y - y_1 = a(x - x_1)$. Hence:

POINT—SLOPE FORMULA OF A STRAIGHT LINE

The equation of the straight line passing through (x_1, y_1) with slope a is

$$y - y_1 = a(x - x_1)$$

Note that when using this formula, x_1 and y_1 are fixed numbers giving the coordinates of the given point. On the other hand, x and y are variables denoting the coordinates of an arbitrary point on the line.

EXAMPLE 4.4.3 Find the equation of the line through $(-2, 3)$ with slope -4. Then find the y-intercept and the point at which this line intersects the x-axis.

Solution: The point–slope formula with $(x_1, y_1) = (-2, 3)$ and $a = -4$ gives the equation $y - 3 = (-4)[x - (-2)]$, or $y - 3 = -4(x + 2)$, which reduces to $4x + y = -5$. The y-intercept has $x = 0$, so $y = -5$. The line intersects the x-axis at the point where $y = 0$, that is, where $4x = -5$, so $x = -5/4$. The point of intersection with the x-axis is therefore $(-5/4, 0)$.[4]

Often we need to find the equation of the straight line that passes through two given distinct points. Combining the slope formula and the point–slope formula, we obtain the following:

POINT—POINT FORMULA OF A STRAIGHT LINE

The equation of the straight line passing through (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$, is obtained as follows:

1. Compute the slope of the line,

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

2. Substitute the expression for a into the point–slope formula. The result is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

[4] It is a good exercise for you to draw a graph and verify this solution.

EXAMPLE 4.4.4 Find the equation of the line passing through $(-1, 3)$ and $(5, -2)$.

Solution: Let $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (5, -2)$. Then the point–point formula gives

$$y - 3 = \frac{-2 - 3}{5 - (-1)}[x - (-1)] = -\frac{5}{6}(x + 1)$$

or $5x + 6y = 13$.

Graphical Solutions of Linear Equations

Section 3.6 dealt with algebraic methods for solving a system of two linear equations in two unknowns. The equations are linear, so their graphs are straight lines. The coordinates of any point on a line satisfy the equation of that line. Thus, the coordinates of any point of intersection of these two lines will satisfy both equations. This means that any point where these lines intersect will satisfy the equation system.

EXAMPLE 4.4.5 Solve each of the following three pairs of equations graphically:

(a) $x + y = 5$ and $x - y = -1$ (b) $3x + y = -7$ and $x - 4y = 2$

(c) $3x + 4y = 2$ and $6x + 8y = 24$

Solution:

(a) Figure 4.4.8 shows the graphs of the straight lines $x + y = 5$ and $x - y = -1$. There is only one point of intersection, which is $(2, 3)$. The solution of the system is, therefore, $x = 2$, $y = 3$.

(b) Figure 4.4.9 shows the graphs of the straight lines $3x + y = -7$ and $x - 4y = 2$. There is only one point of intersection, which is $(-2, -1)$. The solution of the system is, therefore, $x = -2$, $y = -1$.

(c) Figure 4.4.10 shows the graphs of the straight lines $3x + 4y = 2$ and $6x + 8y = 24$. These lines are parallel and have no point of intersection. The system has no solutions.

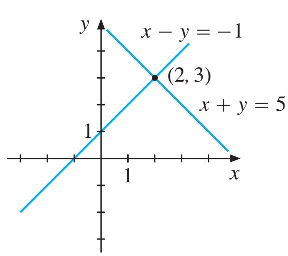


Figure 4.4.8 $x + y = 5$ and $x - y = -1$

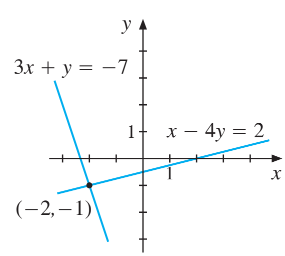


Figure 4.4.9 $3x + y = -7$ and $x - 4y = 2$

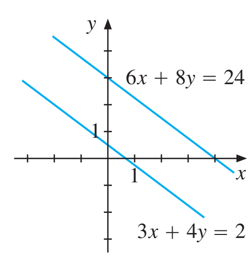


Figure 4.4.10 $3x + 4y = 2$ and $6x + 8y = 24$

Linear Inequalities

This section concludes by discussing how to represent linear inequalities geometrically. We present two examples.

EXAMPLE 4.4.6 Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy the inequality $2x + y \le 4$—that is, using set notation, the set $\{(x, y) : 2x + y \le 4\}$.

Solution: The inequality can be written as $y \le -2x + 4$. The set of points (x, y) that satisfy the equation $y = -2x + 4$ is a straight line. Therefore, the set of points (x, y) that satisfy the inequality $y \le -2x + 4$ must have y-values below those of points on the line $y = -2x + 4$. So it must consist of all points that lie on or below this line. See Fig. 4.4.11.

EXAMPLE 4.4.7 A person has \$$m$ to spend on the purchase of two commodities. The prices of the two commodities are \$$p$ and \$$q$ per unit. Suppose x units of the first commodity and y units of the second commodity are bought. Assuming that negative purchases of either commodity are impossible, one must have both $x \ge 0$ and $y \ge 0$. It follows that the person is restricted to the *budget set* given by

$$B = \{(x, y) : \; px + qy \le m, \; x \ge 0, \; y \ge 0\}$$

as in Eq. (1.1.1). Sketch the budget set B in the xy-plane. Find the slope of the budget line, $px + qy = m$, and its x- and y-intercepts.

Solution: The set of points (x, y) that satisfy $x \ge 0$ and $y \ge 0$ is the first (nonnegative) quadrant. If we impose the additional requirement that $px + qy \le m$, we obtain the triangular domain B shown in Fig. 4.4.12.

If $px + qy = m$, then $qy = -px + m$ and so $y = (-p/q)x + m/q$. This shows that the slope is $-p/q$. The budget line intersects the x-axis when $y = 0$. Then $px = m$, so $x = m/p$. The budget line intersects the y-axis when $x = 0$. Then $qy = m$, so $y = m/q$. So the two points of intersection are $(m/p, 0)$ and $(0, m/q)$, as shown in Fig. 4.4.12.

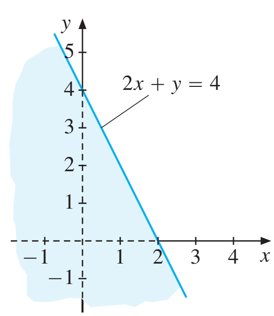


Figure 4.4.11 $\{(x, y) : 2x + y \le 4\}$

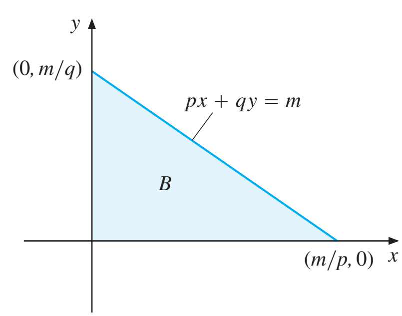


Figure 4.4.12 Budget set: $px + qy \le m$, $x \ge 0$, and $y \ge 0$

EXERCISES FOR SECTION 4.4

1. Find the slopes of the lines passing through the following pairs of points:

(a) $(2, 3)$ and $(5, 8)$ (b) $(-1, -3)$ and $(2, -5)$ (c) $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{1}{3}, -\frac{1}{5}\right)$

2. Draw graphs for the following straight lines:

(a) $3x + 4y = 12$ (b) $\frac{x}{10} - \frac{y}{5} = 1$ (c) $x = 3$

3. Suppose demand D for a good is a linear function of its price per unit, P. When price is \$10, demand is 200 units, and when price is \$15, demand is 150 units. Find the demand function.

4. Decide which of the following relationships are linear:

(a) $5y + 2x = 2$ (b) $P = 10(1 - 0.3t)$

(c) $C = (0.5x + 2)(x - 3)$ (d) $p_1x_1 + p_2x_2 = R$, where p_1, p_2, and R constants.

5. A printing company quotes the price of \$1 400 for producing 100 copies of a report, and \$3 000 for 500 copies. Assuming a linear relation, what would be the price of printing 300 copies?

6. Find the slopes of the five lines L_1 to L_5 shown in Fig. 4.4.13, and give equations describing them.

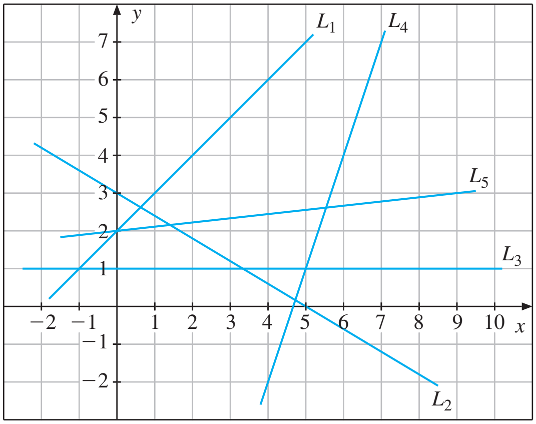


Figure 4.4.13 Lines L_1 to L_5

7. Determine the equations for the following straight lines:

(a) L_1 passes through $(1, 3)$ and has a slope of 2.

(b) L_2 passes through $(-2, 2)$ and $(3, 3)$.

(c) L_3 passes through the origin and has a slope of $-1/2$.

(d) L_4 passes through $(a, 0)$ and $(0, b)$, with $a \neq 0$.

8. Solve the following systems of equations graphically, where possible:

(a) $x - y = 5$ and $x + y = 1$ (b) $x + y = 2$, $x - 2y = 2$ and $x - y = 2$

(c) $3x + 4y = 1$ and $6x + 8y = 6$

9. Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy the following inequalities:

(a) $2x + 4y \geq 5$ (b) $x - 3y + 2 \leq 0$ (c) $100x + 200y \leq 300$

SM 10. Sketch in the xy-plane the set of all pairs of numbers (x, y) that satisfy all the following three inequalities: $3x + 4y \leq 12$, $x - y \leq 1$, and $3x + y \geq 3$.

4.5 Linear Models

Linear relations occur frequently in mathematical models. The relationship between the Celsius and Fahrenheit temperature scales is an example of a linear relation between two variables. (Recall that $F = \frac{9}{5}C + 32$, by definition, from Example 2.6.5.) Most of the linear models in economics are approximations to more complicated models. Two illustrations are those shown in Example 4.4.1. Statistical methods have been devised to construct linear functions that approximate the actual data as closely as possible. Let us consider a very naive attempt to construct a linear model based on some population data.

EXAMPLE 4.5.1 A United Nations report estimated that the European population was 606 million in 1960, and 657 million in 1970. Use these estimates to construct a linear function of t that approximates the population in Europe. Then use the function to estimate the population in 1930, 2000, and 2015.

Solution: Let t be the number of years from 1960, so that $t = 0$ is 1960, $t = 1$ is 1961, and so on. If P denotes the population in millions, we construct an equation of the form $P = at + b$. We know that the graph must pass through the points $(t_1, P_1) = (0, 606)$ and $(t_2, P_2) = (10, 657)$. So we use the point–point formula, replacing x and y with t and P, respectively. This gives

$$P - 606 = \frac{657 - 641}{10 - 0}(t - 0) = \frac{51}{10}t$$

or

$$P = 5.1t + 606 \qquad (*)$$

In Table 4.3, we have compared our estimates with UN forecasts. Note that because $t = 0$ corresponds to 1960, $t = -30$ will correspond to 1930.

Table 4.3 Population estimates for Europe

Year	1930	2000	2015
t	−30	40	55
UN estimates	549	726	738
Formula (∗) gives	555	810	887

Note that the slope of line (∗) is 5.1 This means that if the European population had developed according to (∗), then the annual increase in the population would have been constant and equal to 5.1 million.[5]

EXAMPLE 4.5.2 **(The Consumption Function)** In Keynesian macroeconomic theory, total consumption expenditure on goods and services, C, is assumed to be a function of national income Y, with $C = f(Y)$. Following the work of Keynes's associate Richard F. Kahn, in many models the consumption function is assumed to be linear, so that $C = a + bY$. Then the slope b is called the *marginal propensity to consume*. For example, if C and Y are measured in billions of dollars, the number b tells us by how many billions of dollars consumption would increase if national income were to increase by 1 billion dollars. Following Kahn's insight, the number b is usually thought to lie between 0 and 1.

The Norwegian economist Trygve Haavelmo,[6] in a study of the US economy for the period 1929–1941, estimated the consumption function as $C = 95.05 + 0.712\,Y$. Here, the marginal propensity to consume is equal to 0.712.

EXAMPLE 4.5.3 **(Supply and Demand)** Over a fixed period of time such as a week, the quantity of a specific good that consumers demand (that is, are willing to buy) will depend on the price of that good. Usually, as the price increases the demand will decrease.[7] Also, the number of units that the producers are willing to supply to the market during a certain period depends on the price they are able to obtain. Usually, the supply will increase as the price increases. So typical demand and supply curves are as indicated in Fig. 4.5.1.

The point E in Fig. 4.5.1, at which demand is equal to supply, represents an *equilibrium*. The price P^e at which this occurs is the *equilibrium price* and the corresponding quantity Q^e is the *equilibrium quantity*. The equilibrium price is thus the price at which consumers will buy the same amount of the good as the producers wish to sell at that price.

As a very simple example, consider the following linear demand and supply functions: $D = 100 - P$ and $S = 10 + 2P$; or, in inverse form, $P = 100 - D$ and $P = \frac{1}{2}S - 5$, as in Fig. 4.5.2.[8] The quantity demanded D equals the quantity supplied S provided $100 - P =$

---

5 Actually, Europe's population grew unusually fast during the 1960s. Of course, it grew unusually slowly when millions died during the war years 1939–1945. We see that formula (∗) does not give very good results compared to the UN estimates. For a better way to model population growth see Example 4.9.1.

6 (1911–1999). He was awarded the Nobel prize in 1989.

7 For certain luxury goods like perfume, which are often given as presents, demand might increase as the price increases. For absolutely essential goods, like insulin for diabetics, demand might be almost independent of the price. Occasionally dietary staples could also be "Giffen goods" for which demand rises as price rises. The explanation offered is that these foodstuffs are so essential to a very poor household's survival that a rise in price lowers real income substantially, and so makes alternative sources of nourishment even less affordable.

8 When specifying a linear supply function, the sign of the constant term can be problematic. In the case we just introduced, a negative constant has the unintuitive implication that supply is positive even when the price is zero—in our case, it is 10 units. One possibility for something like this to occur would be when the producer owns a stock of the product and this is highly perishable. But having the supply be positive and increasing at very low prices can be inconsistent with the producer's behaviour. The difficulty is that a positive constant brings about a problem too: that at

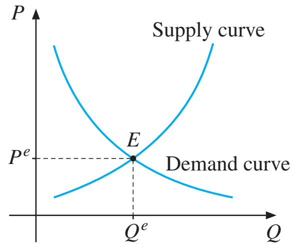


Figure 4.5.1 Demand and supply curves

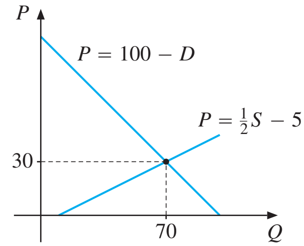


Figure 4.5.2 $D = 100 - P$ and $S = 10 + 2P$

$10 + 2P$, that is, $3P = 90$. So the equilibrium price is $P^e = 30$, with equilibrium quantity $Q^e = 70$.

A peculiarity of Figs 4.5.1 and 4.5.2 is that, although quantity is usually regarded as a function of price, here we measure price on the vertical axis and quantity on the horizontal axis. This has been standard practice in elementary price theory since the fundamental ideas of the French mathematician and economist Antoine-Augustin Cournot (1801–1877) and several other European contemporaries became popularized in the late 19th century by the English economist Alfred Marshall (1842–1924).

EXAMPLE 4.5.4 **(Linear Supply and Demand Functions)** Consider the following general linear demand and supply schedules: $D = a - bP$ and $S = \alpha + \beta P$, where a and b are positive parameters of the demand function D, while α and β are positive parameters of the supply function.[9]

The equilibrium price P^e occurs where demand equals supply. Hence $D = S$ at $P = P^e$ implying that $a - bP^e = \alpha + \beta P^e$, or $a - \alpha = (\beta + b)P^e$. The corresponding equilibrium quantity is $Q^e = a - bP^e$. So equilibrium occurs at

$$P^e = \frac{a - \alpha}{\beta + b} \quad \text{and} \quad Q^e = a - b\frac{a - \alpha}{\beta + b} = \frac{a\beta + \alpha b}{\beta + b}$$

EXERCISES FOR SECTION 4.5

1. The consumption function $C = 4141 + 0.78Y$ was estimated for the UK during the period 1949–1975. What is the marginal propensity to consume?

2. Find the equilibrium price for each of the following linear models of supply and demand:

   (a) $D = 75 - 3P$ and $S = 2P$ (b) $D = 100 - 0.5P$ and $S = -20 + 0.5P$

some low prices, the producer's supply is negative. We will overlook these issues here, but they serve as a warning that overly simplified models can sometimes display undesirable features.

[9] Such linear supply and demand functions play an important role in economics. It is often the case that the market for a particular commodity, such as copper, can be represented approximately by suitably estimated linear demand and supply functions.

3. The total cost C of producing x units of some commodity is a linear function of x. Records show that on one occasion, 100 units were made at a total cost of \$200, and on another occasion, 150 units were made at a total cost of \$275. Express the linear equation for total cost C in terms of the number of units x produced.

4. The expenditure of a household on consumer goods, C, is related to the household's income, y, in the following way: When the household's incomeis \$1 000, the expenditure on consumer goods is \$900, and whenever income increases by \$100, the expenditure on consumer goods increases by \$80. Express the expenditure on consumer goods as a function of income, assuming a linear relationship.

5. For most assets such as cars, electronic goods, and furniture, the value decreases, or *depreciates*, each year. If the value of an asset is assumed to decrease by a fixed percentage of the original value each year, it is referred to as *straight line depreciation*.

   (a) Suppose the value of a car which initially costs \$20 000 depreciates by 10% of its original value each year. Find a formula for its value $P(t)$ after t years.

   (b) If a \$500 washing machine is completely depreciated after ten years (straight line depreciation), find a formula for its value $W(t)$ after t years.

4.6 Quadratic Functions

Economists often find that linear functions are too simple for modelling economic phenomena with acceptable accuracy. Indeed, many economic models involve functions that either decrease down to some minimum value and then increase, or else increase up to some maximum value and then decrease. Some simple functions with this property are the general *quadratic* functions that we first saw in Section 3.3:

$$f(x) = ax^2 + bx + c \tag{4.6.1}$$

where a, b, and c are constants, with $a \neq 0$, otherwise the function would be linear.

In general, the graph of $f(x) = ax^2 + bx + c$ is called a *parabola*. The shape of this parabola roughly resembles $\cap$ when $a < 0$ and $\cup$ when $a > 0$. Three typical cases are illustrated in Figs 4.6.1 to 4.6.3. The graphs are symmetric about the *axis of symmetry*, which is the vertical dashed line in each of the three cases.

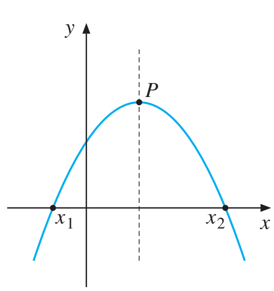


Figure 4.6.1 $a < 0$, $b^2 > 4ac$

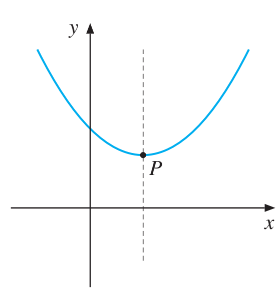


Figure 4.6.2 $a > 0$, $b^2 < 4ac$

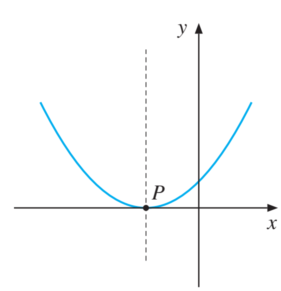


Figure 4.6.3 $a > 0$, $b^2 = 4ac$

In order to investigate the function $f(x) = ax^2 + bx + c$ in more detail, we should find the answers to the following questions:

(a) For which values of x (if any) is $ax^2 + bx + c = 0$?

(b) What are the coordinates of the maximum/minimum point P, also called the *vertex* of the parabola?

The answer to question (a) was given by the quadratic formula (3.3.4) and the subsequent discussion of that formula. The easiest way to handle question (b) is to use derivatives, which is the topic of Chapter 6—see, in particular, Exercise 6.2.7. However, let us briefly consider how the "method of completing the squares" from Section 3.3 can be used to answer question (b).

In fact, this method yields

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \tag{4.6.2}$$

as is easily verified by expanding the right-hand side and gathering terms. Now, when x varies, only the value of $a(x + b/2a)^2$ changes. This term is equal to 0 only when $x = -b/2a$, and if $a > 0$, it is never less than 0. This means that when $a > 0$, then the function $f(x)$ attains its minimum when $x = -b/2a$, and the value of $f(x)$ is then equal to

$$f(-b/2a) = -(b^2 - 4ac)/4a = c - b^2/4a$$

If $a < 0$ on the other hand, then $a(x + b/2a)^2 \leq 0$ for all x, and the squared term is equal to 0 only when $x = -b/2a$. Hence, $f(x)$ attains its maximum when $x = -b/2a$ in this second case.

To summarize, we have shown the following:

EXTREMA OF QUADRATIC FUNCTIONS

$$\text{If } a > 0, \text{ then } f(x) = ax^2 + bx + c \text{ has its } \textit{minimum} \text{ at } x = -b/2a \tag{4.6.3}$$

$$\text{If } a < 0, \text{ then } f(x) = ax^2 + bx + c \text{ has its } \textit{maximum} \text{ at } x = -b/2a \tag{4.6.4}$$

The axis of symmetry for a parabola is the vertical line through its vertex, which is the point P in all three Figs 4.6.1 to 4.6.3.[10] Indeed, formula (4.6.2) implies that, for any number u, one has

$$f\left(-\frac{b}{2a} + u\right) = au^2 - \frac{b^2 - 4ac}{4a} = f\left(-\frac{b}{2a} - u\right)$$

It follows that the quadratic function $f(x) = ax^2 + bx + c$ is symmetric about the vertical line $x = -b/2a$ which passes through P.

[10] The function f is symmetric about $x = x_0$ if $f(x_0 + t) = f(x_0 - t)$ for all x. See Section 5.2.

Quadratic Optimization Problems in Economics

Much of economic analysis is concerned with optimization problems. Economics, after all, is the science of choice, and optimization problems are the form in which economists usually model choice mathematically. A general discussion of such problems must be postponed until we have developed the necessary tools from calculus. Here we show how the simple results from this section on maximizing quadratic functions can be used to illustrate some basic economic ideas.

EXAMPLE 4.6.1 Suppose the price P per unit obtained by a firm in producing and selling Q units is $P = 102 - 2Q$, and the cost of producing and selling Q units is $C = 2Q + \frac{1}{2}Q^2$. Then the profit is[11]

$$\pi(Q) = PQ - C = (102 - 2Q)Q - \left(2Q + \frac{1}{2}Q^2\right) = 100Q - \frac{5}{2}Q^2$$

Find the value of Q which maximizes profit, and the corresponding maximal profit.

Solution: Using formula (4.6.4), we find that profit is maximized at

$$Q = Q^* = -\frac{100}{2 \cdot (-5/2)} = 20$$

with

$$\pi^* = \pi(Q^*) = 100 \cdot 20 - \frac{5}{2} \cdot 400 = 1000$$

This example is a special case of the monopoly problem studied in the next example.

EXAMPLE 4.6.2 **(A Monopoly Problem)** Consider a firm that is the only seller of the commodity it produces, possibly a patented medicine, and so enjoys a monopoly. The total costs of the monopolist are assumed to be given by the quadratic function

$$C = \alpha Q + \beta Q^2$$

of its output level $Q \geq 0$, where α and β are positive constants. For each Q, the price P at which it can sell its output is assumed to be determined from the linear "inverse" demand function

$$P = a - bQ$$

where a and b are constants with $a > 0$ and $b \geq 0$. So for any nonnegative Q, the total revenue R is given by the quadratic function $R = PQ = (a - bQ)Q$, and profit by the quadratic function

$$\pi(Q) = R - C = (a - bQ)Q - \alpha Q - \beta Q^2 = (a - \alpha)Q - (b + \beta)Q^2$$

Assuming that the monopolist's objective is to maximize the profit function $\pi = \pi(Q)$, find the optimal output level Q^M and the corresponding optimal profit π^M.

---

[11] In mathematics the Greek letter π is used to denote the constant ratio 3.1415... between the circumference of a circle and its diameter. In economics, this constant is not used very often. Also, p and P usually denote a price, so π has come to denote profit.

Solution: By using (4.6.4), we see that there is a maximum of π at

$$Q^M = \frac{a-\alpha}{2(b+\beta)} \tag{4.6.5}$$

with

$$\pi^M = \frac{(a-\alpha)^2}{2(b+\beta)} - (b+\beta)\frac{(a-\alpha)^2}{4(b+\beta)^2} = \frac{(a-\alpha)^2}{4(b+\beta)}$$

This result is valid if $a > \alpha$; if $a \leq \alpha$, the firm will not produce, but will have $Q^M = 0$ and $\pi^M = 0$. The two cases are illustrated in Figs 4.6.4 and 4.6.5. In Fig. 4.6.5, the part of the parabola to the left of $Q = 0$ is dashed, because it is not really relevant given the natural requirement that $Q \geq 0$. The price and cost associated with Q^M in (4.6.5) can be found by routine algebra.

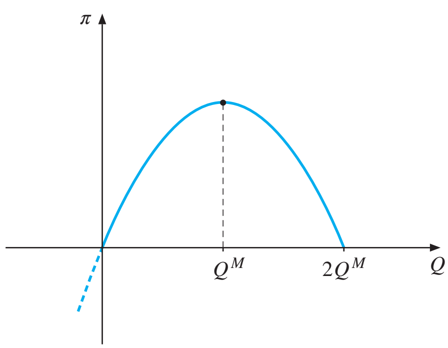


Figure 4.6.4 The profit function, $a > \alpha$

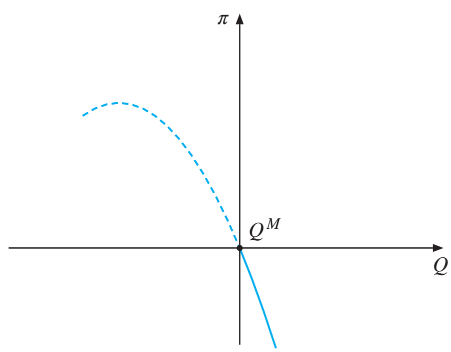


Figure 4.6.5 The profit function, $a \leq \alpha$

If we put $b = 0$ in the price function $P = a - bQ$, then $P = a$ for all Q. In this case, the firm's choice of quantity does not influence the price at all and so the firm is said to be *perfectly competitive*. By replacing a by P in our previous expressions, we see that profit is maximized for a perfectly competitive firm at

$$Q^* = \frac{P-\alpha}{2\beta} \tag{4.6.6}$$

with

$$\pi^* = \frac{(P-\alpha)^2}{4\beta}$$

provided that $P > \alpha$. If $P \leq \alpha$, then $Q^* = 0$ and $\pi^* = 0$.

Solving (4.6.6) for P yields $P = \alpha + 2\beta Q^*$. Thus, the equation

$$P = \alpha + 2\beta Q \tag{4.6.7}$$

represents the *supply curve* of this perfectly competitive firm for $P > \alpha$. For $P \leq \alpha$, the profit-maximizing output Q^* is 0. The supply curve relating the price on the market to the firm's choice of output quantity is shown in Fig. 4.6.6; it includes all the points of the line segment between the origin and $(0, \alpha)$, where the price is too low for the firm to earn any profit by producing a positive output.

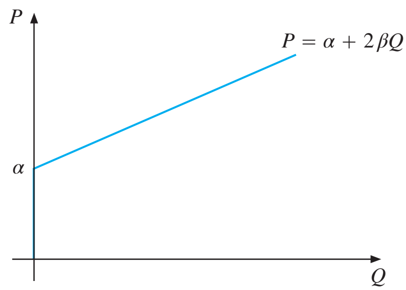


Figure 4.6.6 The supply curve of a perfectly competitive firm

Let us return to the monopoly firm (which has no supply curve). If it could somehow be made to act like a competitive firm, taking price as given, it would be on the supply curve (4.6.7). Given the demand curve $P = a - bQ$, equilibrium between supply and demand occurs when (4.6.7) is also satisfied, and so $P = a - bQ = \alpha + 2\beta Q$. Solving the second equation for Q, and then substituting for P and π in turn, we see that the respective equilibrium levels of output, price, and profit would be

$$Q^e = \frac{a-\alpha}{b+2\beta}, \qquad P^e = \frac{2a\beta + \alpha b}{b+2\beta}, \qquad \pi^e = \frac{\beta(a-\alpha)^2}{(b+2\beta)^2}$$

In order to have the monopolist mimic a competitive firm by choosing to be at (Q^e, P^e), it may be desirable to tax (or subsidize) the output of the monopolist. Suppose that the monopolist is required to pay a specific tax of τ per unit of output. Because the tax payment, $\tau \cdot Q$, is added to the firm's costs, the new total cost function is

$$C = \alpha Q + \beta Q^2 + \tau Q = (\alpha + \tau)Q + \beta Q^2$$

Carrying out the same calculations as before, but with α replaced by $\alpha + \tau$, gives the monopolist's choice of output as

$$Q^M_\tau = \begin{cases} \dfrac{a-\alpha-\tau}{2\,(b+\beta)}, & \text{if } a \geq \alpha + \tau \\ 0, & \text{otherwise} \end{cases}$$

So $Q^M_\tau = Q^e$ when

$$\frac{a-\alpha-\tau}{2(b+\beta)} = \frac{a-\alpha}{b+2\beta}$$

Solving this equation for τ yields

$$\tau = -\frac{(a-\alpha)b}{b+2\beta}$$

Note that τ is actually negative, indicating the desirability of *subsidizing* the output of the monopolist in order to encourage additional production.[12]

[12] Of course, subsidizing monopolists is usually felt to be unjust, and many additional complications need to be considered carefully before formulating a desirable policy for dealing with monopolists. Still the previous analysis suggests that if it is desirable to lower a monopolist's price or its profit, this can be done much better directly than by taxing its output.

EXERCISES FOR SECTION 4.6

1. Let $f(x) = x^2 - 4x$.

   (a) Complete the following table and use it to sketch the graph of f:

x	-1	0	1	2	3	4	5
$f(x)$							

   (b) Using Eq. (4.6.3), determine the minimum point of f.

   (c) Solve the equation $f(x) = 0$.

2. Let $f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$.

   (a) Complete the following table and sketch the graph of f:

x	-4	-3	-2	-1	0	1	2
$f(x)$							

   (b) Using Eq. (4.6.4), determine the maximum point of f.

   (c) Solve the equation $f(x) = 0$.

   (d) Show that $f(x) = -\frac{1}{2}(x - 1)(x + 3)$, and use this to study how the sign of $f(x)$ varies with x. Compare the result with your graph.

3. Determine the maximum/minimum points of the following functions, by using Eqs (4.6.3) or (4.6.4), as appropriate:

   (a) $x^2 + 4x$ (b) $x^2 + 6x + 18$ (c) $-3x^2 + 30x - 30$

   (d) $9x^2 - 6x - 44$ (e) $-x^2 - 200x + 30\,000$ (f) $x^2 + 100x - 20\,000$

4. Find all the zeros of each quadratic function in Exercise 3, and where possible write each function in the form $a(x - x_1)(x - x_2)$.

5. Find solutions to the following equations, where p and q are parameters.

   (a) $x^2 - 3px + 2p^2 = 0$ (b) $x^2 - (p + q)x + pq = 0$ (c) $2x^2 + (4q - p)x = 2pq$

6. A model in the theory of efficient loan markets involves the function

$$U(x) = 72 - (4 + x)^2 - (4 - rx)^2$$

   where r is a constant. Find the value of x for which $U(x)$ attains its largest value.

7. A farmer has 1000 metres of fence wire with which to make a rectangular enclosure, as illustrated in the figure below.

   (a) Find the areas of the three rectangles whose bases are 100, 250, and 350 metres.

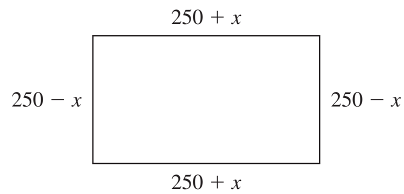


Figure 4.6.7 A plot of land

(b) Let the base have length $250 + x$. Then the height is $250 - x$, as in Fig. 4.6.7. What choice of x gives the maximum area?[13]

8. If a cocoa shipping firm sells Q tons of cocoa in the UK, the price received is given by $P_E = \alpha_1 - \frac{1}{3}Q$. On the other hand, if it buys Q tons from its only source in Ghana, the price it has to pay is given by $P_G = \alpha_2 + \frac{1}{6}Q$. In addition, it costs γ per ton to ship cocoa from its supplier in Ghana to its customers in the UK (its only market). The numbers α_1, α_2, and γ are all positive.

(a) Express the cocoa shipper's profit as a function of Q, the number of tons shipped.

(b) Assuming that $\alpha_1 - \alpha_2 - \gamma > 0$, find the profit-maximizing shipment of cocoa. What happens if $\alpha_1 - \alpha_2 - \gamma \leq 0$?

(c) Suppose the government of Ghana imposes an export tax on cocoa of τ per ton. Find the new expression for the shipper's profits and the new quantity shipped.

(d) Calculate the Ghanaian government's export tax revenue as a function of τ, and compare the graph of this function with the Laffer curve presented in Fig. 4.1.1.

(e) Advise the Ghanaian government on how to obtain as much tax revenue as possible.

SM **9.** [HARDER] Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be arbitrary real numbers. The inequality

$$\left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \tag{4.6.8}$$

is called the *Cauchy–Schwarz inequality*.

(a) Check the inequality for $n = 2$, when $a_1 = -3$, $a_2 = 2$, $b_1 = 5$, and $b_2 = -2$.

(b) Prove (4.6.8) by means of the following trick: first, define f for all x by

$$f(x) = (a_1 x + b_1)^2 + \cdots + (a_n x + b_n)^2$$

It should be obvious that $f(x) \geq 0$ for all x. Write $f(x)$ as $Ax^2 + Bx + C$, where the expressions for A, B, and C are related to the terms in (4.6.8). Because $Ax^2 + Bx + C \geq 0$ for all x, we must have $B^2 - 4AC \leq 0$. Why?

(c) Argue that (4.6.8) then follows.

[13] It is reported that certain surveyors in antiquity, when selling rectangular pieces of land to farmers, would write contracts in which only the perimeter was specified. As a result, the lots they sold were long narrow rectangles.

4.7 Polynomials

After considering linear and quadratic functions, the logical next step is to examine *cubic functions* of the form

$$f(x) = ax^3 + bx^2 + cx + d \tag{4.7.1}$$

where a, b, c, and d are constants and $a \neq 0$. It is relatively easy to examine the behaviour of linear and quadratic functions. Cubic functions are considerably more complicated, because the shape of their graphs changes drastically as the coefficients a, b, c, and d vary. Two examples are given in Figs 4.7.1 and 4.7.2. Cubic functions do occasionally appear in economic models.

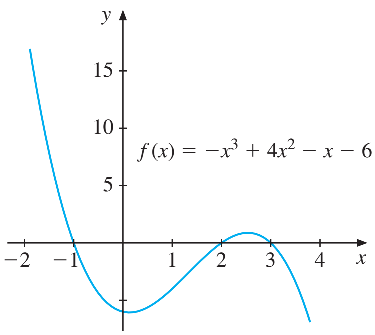


Figure 4.7.1 A cubic function

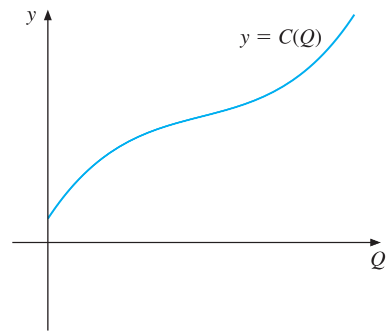


Figure 4.7.2 A cubic cost function

EXAMPLE 4.7.1 Consider a firm producing a single commodity. The total cost of producing Q units of the commodity is $C(Q)$. Cost functions often have the following properties: First, $C(0)$ is positive, because an initial fixed expenditure is involved. When production increases, costs also increase. In the beginning, costs increase rapidly, but the rate of increase slows down as production equipment is used for a higher proportion of each working week. However, at high levels of production, costs again increase at a fast rate, because of technical bottlenecks and overtime payments to workers, for example. It can be shown that the cubic cost function $C(Q) = aQ^3 + bQ^2 + cQ + d$ exhibits this type of behaviour provided that $a > 0$, $b < 0$, $c > 0$, $d > 0$, and $3ac > b^2$. Such a function is sketched in Fig. 4.7.2.

Cubic cost functions whose coefficients have a different sign pattern have also been studied. For instance, a study of a particular electric power generating plant revealed that over a certain period, the cost of fuel y as a function of output Q was given by

$$y = -Q^3 + 214.2Q^2 - 7900Q + 320\,700$$

Note, however, that this cost function cannot be valid for all Q, because it suggests that fuel costs would be negative for large enough Q.

The detailed study of cubic functions is made easier by applying differential calculus, as will be seen later.

General Polynomials

Linear, quadratic, and cubic functions are all examples of *polynomials*.

GENERAL POLYNOMIAL

The function P, defined for all x, by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{4.7.2}$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$, is called the *general polynomial of degree n*, with *coefficients* $a_n, a_{n-1}, \ldots, a_0$.

For instance, when $n = 4$, we obtain $P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$, which is the general *quartic function*, or polynomial of degree 4. Of course, there are many functions like $5 + x^{-2}$ or $1/(x^3 - x + 2)$ that are not polynomials.

Numerous problems in mathematics and its applications involve polynomials. Often, one is particularly interested in finding the number and location of the *zeros* of $P(x)$—that is, the values of x such that $P(x) = 0$. The equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{4.7.3}$$

is called the *general equation of degree n*. It will soon be shown that this equation has *at most n* (real) solutions, also called *roots*, but it need not have any. The corresponding n-th-degree polynomial has a graph which has at most $n - 1$ turning points, but there may be fewer such points. For example, the 100th-degree equation $x^{100} + 1 = 0$ has no solutions because $x^{100} + 1$ is always greater than or equal to 1, and its graph has only one turning point.

According to the *fundamental theorem of algebra*, every polynomial of the form (4.7.2) can be written as a product of polynomials of degree 1 or 2.

Factoring Polynomials

Let $P(x)$ and $Q(x)$ be two polynomials for which the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$. Then, there always exist unique polynomials $q(x)$ and $r(x)$ such that

$$P(x) = q(x)Q(x) + r(x) \tag{4.7.4}$$

where the degree of $r(x)$ is less than the degree of $Q(x)$. This fact is called the *remainder theorem*. When x is such that $Q(x) \neq 0$, then (4.7.4) can be written in the form

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

where $r(x)$ is the remainder. If $r(x) = 0$, we say that $Q(x)$ *is a factor of* $P(x)$, or that $P(x)$ *is divisible by* $Q(x)$. Then $P(x) = q(x)Q(x)$ or $P(x)/Q(x) = q(x)$.

An important special case is when $Q(x) = x - a$. Then $Q(x)$ is of degree 1, so the remainder $r(x)$ must have degree 0, and is therefore a constant. In this special case, for all x,

$$P(x) = q(x)(x - a) + r$$

For $x = a$ in particular, we get $P(a) = r$. Hence, $x - a$ divides $P(x)$ if and only if $P(a) = 0$. This important observation can be formulated as follows:

POLYNOMIAL FACTORIZATION

$$\text{The polynomial } P(x) \text{ has the factor } x - a \Leftrightarrow P(a) = 0. \tag{4.7.5}$$

EXAMPLE 4.7.2 Prove that $x - 5$ is a factor of the polynomial $P(x) = x^3 - 3x^2 - 50$.

Solution: $P(5) = 125 - 75 - 50 = 0$, so according to (4.7.5), $x - 5$ divides $P(x)$. In fact, note that $P(x) = (x - 5)(x^2 + 2x + 10)$.

It follows from (4.7.5) that an n-th-degree polynomial $P(x)$ can have *at most* n different zeros. The reason is that each zero gives rise to a different factor of the form $x - a$, so $P(x)$ can have at most n such factors.

Note that each integer m that satisfies the cubic equation

$$-x^3 + 4x^2 - x - 6 = 0 \tag{$*$}$$

must satisfy the equation $m(-m^2 + 4m - 1) = 6$. Since $-m^2 + 4m - 1$ is also an integer, m must be a factor of the constant term 6. Thus ± 1, ± 2, ± 3, and ± 6 are the only possible integer solutions. Direct substitution into the left-hand side of Eq. ($*$) reveals that of these eight possibilities, -1, 2, and 3 are roots of the equation. A third-degree equation has at most three roots, so we have found them all. In fact,

$$-x^3 + 4x^2 - x - 6 = -(x + 1)(x - 2)(x - 3)$$

In general:

INTEGER ROOTS

Suppose that $a_n, a_{n-1}, \ldots, a_1, a_0$ are all integers. Then, all possible integer roots of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{4.7.6}$$

must be factors of the constant term a_0.

EXAMPLE 4.7.3 Find all possible integer roots of the equation

$$\frac{1}{2}x^3 - x^2 + \frac{1}{2}x - 1 = 0$$

Solution: We multiply both sides of the equation by 2 to obtain an equation whose coefficients are all integers: $x^3 - 2x^2 + x - 2 = 0$. Now, all integer solutions of the equation must be factors of 2, so only ± 1 and ± 2 can be integer solutions. A check shows that $x = 2$ is the only integer solution. In fact, because $x^3 - 2x^2 + x - 2 = (x - 2)(x^2 + 1)$, there is only one real root.

EXAMPLE 4.7.4 Find possible quadratic and cubic functions which have the graphs in Figs 4.7.3 and 4.7.4, respectively.

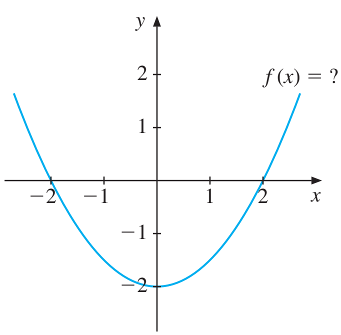


Figure 4.7.3 A quadratic function

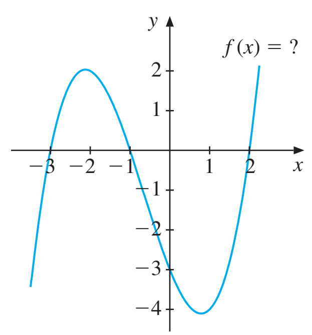


Figure 4.7.4 A cubic function

Solution: For Fig 4.7.3, since the graph intersects the x-axis at the two points $x = -2$ and $x = 2$, we try the quadratic function $f(x) = a(x - 2)(x + 2)$. Then $f(0) = -4a$. According to the graph, $f(0) = -2$, so $a = 1/2$, and hence

$$f(x) = \frac{1}{2}(x - 2)(x + 2) = \frac{1}{2}x^2 - 2$$

For Fig. 4.7.4, because the equation $f(x) = 0$ has roots $x = -3, -1, 2$, we try the cubic function $f(x) = b(x + 3)(x + 1)(x - 2)$. Then $f(0) = -6b$. According to the graph, $f(0) = -3$. So $b = 1/2$, and hence

$$f(x) = \frac{1}{2}(x + 3)(x + 1)(x - 2)$$

Polynomial Division

One can divide polynomials in much the same way as one uses long division to divide numbers. To remind ourselves how long division works, consider a simple numerical example:

$$\begin{array}{rcll}
2735 & \div\ 5 & = & 500+40+7 \\
\underline{2500} & & & \\
235 & & & \\
\underline{200} & & & \\
35 & & & \\
\underline{35} & & & \\
0 & & \leftarrow & \text{the remainder}
\end{array}$$

Hence, $2735 \div 5 = 547$.[14]

Consider next

$$(-x^3 + 4x^2 - x - 6) \div (x - 2)$$

We write the following:

$$\begin{array}{rcccl}
-x^3 + 4x^2 - x - 6 & \div & x - 2 & = & -x^2 + 2x + 3 \\
\underline{-x^3 + 2x^2 \phantom{{}- 4x - 6}} & \leftarrow & -x^2(x-2) & & \\
+\, 2x^2 - x - 6 & & & & \\
\underline{+\, 2x^2 - 4x \phantom{{}- 6}} & \leftarrow & 2x(x-2) & & \\
+\, 3x - 6 & & & & \\
\underline{+\, 3x - 6} & \leftarrow & 3(x-2) & & \\
0 & & & \leftarrow & \text{the remainder}
\end{array}$$

We conclude that $(-x^3 + 4x^2 - x - 6) \div (x - 2) = -x^2 + 2x + 3$. However, it is easy to see that $-x^2 + 2x + 3 = -(x + 1)(x - 3)$. So

$$-x^3 + 4x^2 - x - 6 = -(x + 1)(x - 3)(x - 2)$$

EXAMPLE 4.7.5 Prove that the polynomial $P(x) = -2x^3 + 2x^2 + 10x + 6$ has a zero at $x = 3$, and factor the polynomial.

Solution: Inserting $x = 3$ yields $P(3) = 0$, so $x = 3$ is a zero. According to (4.7.5), the polynomial $P(x)$ has $x - 3$ as a factor. Performing the division $(-2x^3 + 2x^2 + 10x + 6) \div (x - 3)$, we find that the result is $-2(x^2 + 2x + 1) = -2(x + 1)^2$, and so $P(x) = -2(x - 3)(x + 1)^2$.

Polynomial Division with a Remainder

The division $2734 \div 5$ gives 546 and leaves the remainder 4. So $2734/5 = 546 + 4/5$. We consider a similar form of division for polynomials.

[14] Note that the horizontal lines instruct you to subtract the numbers above the lines. You may be more accustomed to a different way of arranging the numbers, but the idea is the same.

EXAMPLE 4.7.6 Perform the division: $(x^4 + 3x^2 - 4) \div (x^2 + 2x)$.

Solution: Proceeding as before,[15]

$$
\begin{array}{rcll}
x^4 \qquad + 3x^2 \qquad - 4 & \div & x^2 + 2x & = x^2 - 2x + 7 \\
x^4 + 2x^3 \qquad\qquad\qquad\quad & \leftarrow & x^2(x^2 + 2x) & \\
\cline{1-1}
-\,2x^3 + 3x^2 \qquad - 4 & & & \\
-\,2x^3 - 4x^2 \qquad\qquad & \leftarrow & -2x(x^2 + 2x) & \\
\cline{1-1}
7x^2 \qquad - 4 & & & \\
7x^2 + 14x \qquad & \leftarrow & 7(x^2 + 2x) & \\
\cline{1-1}
-\,14x - 4 & & & \leftarrow \text{the remainder}
\end{array}
$$

We conclude that

$$x^4 + 3x^2 - 4 = (x^2 - 2x + 7)(x^2 + 2x) + (-14x - 4)$$

Hence,

$$\frac{x^4 + 3x^2 - 4}{x^2 + 2x} = x^2 - 2x + 7 - \frac{14x + 4}{x^2 + 2x}$$

Rational Functions

A *rational function* is a function $R(x) = P(x)/Q(x)$ that can be expressed as the ratio of two polynomials $P(x)$ and $Q(x)$. This function is defined for all x where $Q(x) \neq 0$. The rational function $R(x)$ is called *proper* if the degree of $P(x)$ is less than the degree of $Q(x)$. When the degree of $P(x)$ is greater than or equal to that of $Q(x)$, then $R(x)$ is called an *improper* rational function. By using polynomial division, any improper rational function can be written as a polynomial plus a proper rational function, as in Example 4.7.6.

EXAMPLE 4.7.7 One of the simplest types of rational function is

$$R(x) = \frac{ax + b}{cx + d}$$

where $c \neq 0$ — otherwise, if $c = 0$, then $R(x)$ is either a linear function in case $d \neq 0$, or else is undefined if $d = 0$ as well.

The graph of R is a *hyperbola*. See Fig. 5.1.7 for a typical example where $R(x) = (3x - 5)/(x - 2)$.[16] A very simple case is $R(x) = a/x$, where $a > 0$. Figure 4.7.5 shows the part of the graph of this function where $x > 0$. Note that the shaded area A always equals a, independent of which point P we choose on the curve, since the area is $A = x_0(a/x_0) = a$.

Studying the behaviour of more complicated rational functions becomes easier once we have developed the proper tools from calculus.[17]

[15] The polynomial $x^4 + 3x^2 - 4$ has no terms in x^3 and x, so we inserted some extra space between the powers of x to make room for the terms in x^3 and x that arise in the course of the calculations.

[16] See also the end of Section 5.5.

[17] See, for instance, Exercise 7.9.9.

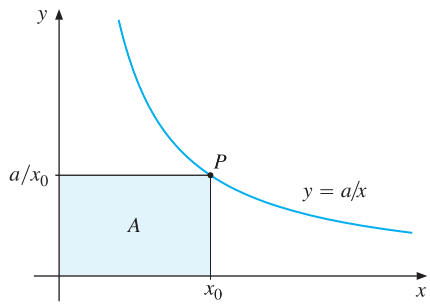


Figure 4.7.5 The area A is independent of P

EXERCISES FOR SECTION 4.7

1. Find all integer roots of the following equations:

   (a) $x^4 - x^3 - 7x^2 + x + 6 = 0$ (b) $2x^3 + 11x^2 - 7x - 6 = 0$

   (c) $x^4 + x^3 + 2x^2 + x + 1 = 0$ (d) $\frac{1}{4}x^3 - \frac{1}{4}x^2 - x + 1 = 0$

2. Find all integer roots of the following equations:

   (a) $x^2 + x - 2 = 0$ (b) $x^3 - x^2 - 25x + 25 = 0$ (c) $x^5 - 4x^3 - 3 = 0$

SM 3. Perform the following divisions:

   (a) $(2x^3 + 2x - 1) \div (x - 1)$ (b) $(x^4 + x^3 + x^2 + x) \div (x^2 + x)$

   (c) $(x^5 - 3x^4 + 1) \div (x^2 + x + 1)$ (d) $(3x^8 + x^2 + 1) \div (x^3 - 2x + 1)$

SM 4. Find possible formulas for each of the three polynomials with graphs shown in Figs 4.7.6 to 4.7.8.

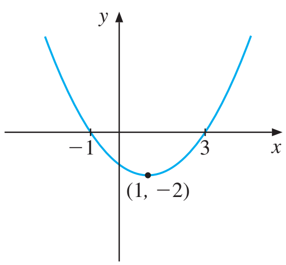


Figure 4.7.6 Exercise 4(a)

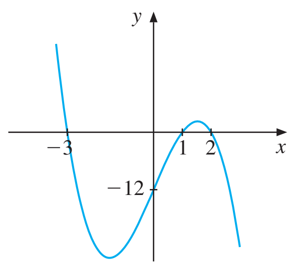


Figure 4.7.7 Exercise 4(b)

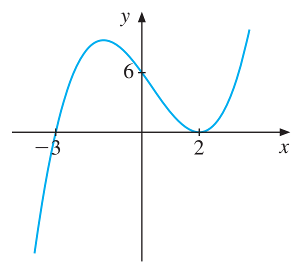


Figure 4.7.8 Exercise 4(c)

5. Perform the following divisions:

   (a) $(x^2 - x - 20) \div (x - 5)$ (b) $(x^3 - 1) \div (x - 1)$ (c) $(-3x^3 + 48x) \div (x - 4)$

6. Show that the division $(x^4 + 3x^2 + 5) \div (x - c)$ leaves a remainder for all values of c.

7. Prove that, provided both $c \neq 0$ and $cx + d \neq 0$, one has $\dfrac{ax + b}{cx + d} = \dfrac{a}{c} + \dfrac{bc - ad}{c(cx + d)}$.

SM **8.** The following function has been used in demand theory:

$$E = \alpha \frac{x^2 - \gamma x}{x + \beta}$$

with α, β, and γ being constants. Perform the division $(x^2 - \gamma x) \div (x + \beta)$, and use the result to express E as a sum of a linear function and a proper fraction.

4.8 Power Functions

We saw in Section 2.5 how the number x^r can be defined for all rational numbers r. We also need to consider x^r when r is irrational in order for x^r to be defined for all real numbers r. How do we define, say, 5 raised to the irrational power π, that is 5^π? Because π is close to 3.1, we should expect that 5^π is approximately

$$5^{3.1} = 5^{31/10} = \sqrt[10]{5^{31}}$$

which *is* defined. An even better approximation is

$$5^\pi \approx 5^{3.14} = 5^{314/100} = 5^{157/50} = \sqrt[50]{5^{157}}$$

We can continue by taking more decimal places in the representation of $\pi = 3.1415926535\ldots$, and our approximation will be better with every additional decimal digit. Then the meaning of 5^π should be reasonably clear. For the moment, however, let us be content with just using a calculator to find that $5^\pi \approx 156.993$. Later Section 7.11 provides a more complete discussion of how to define x^r when r is irrational.

POWER FUNCTION

The general *power function* is defined by the formula

$$f(x) = Ax^r \tag{4.8.1}$$

where r and A are constants, for $x > 0$.

When we consider the power function, we assume that $x > 0$. This is because for many values of r, such as $r = 1/2$, the symbol x^r is not defined for negative values of x. And we exclude $x = 0$ because 0^r is undefined if $r \leq 0$.

EXAMPLE 4.8.1 Here are three illustrations of why powers with rational exponents are needed:

(a) The formula $S \approx 4.84V^{2/3}$ gives the approximate surface area S of a ball as a function of its volume V—see Exercise 6.

(b) The flow of blood, in litres per second, through the heart of an individual is approximately proportional to $x^{0.7}$, where x is the body weight.

(c) Let Y be the net national product, K be capital stock, L be labour, and t be time. The formula $Y = 2.262K^{0.203}L^{0.763}(1.02)^t$ appears in a study of the growth of national product, and shows how powers with fractional exponents can arise in economics.

Graphs of Power Functions

Consider the power function $f(x) = x^r$, defined for all real numbers r provided that $x > 0$. We always have $f(1) = 1^r = 1$, so the graph of the function passes through the point $(1, 1)$ in the xy-plane. The shape of the graph depends crucially on the value of r, as Figs 4.8.1 to 4.8.3 indicate.

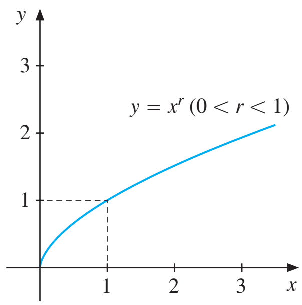


Figure 4.8.1 $0 < r < 1$

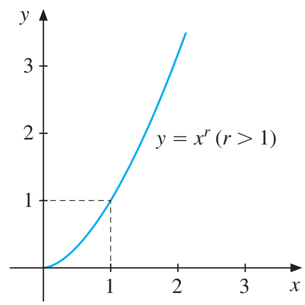


Figure 4.8.2 $r > 1$

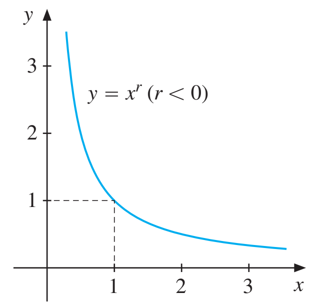


Figure 4.8.3 $r < 0$

If $0 < r < 1$, the graph is like that in Fig. 4.8.1, which resembles the graph of $f(x) = x^{0.5}$ shown in Fig. 4.3.8. For $r > 1$ the graph is like that shown in Fig. 4.8.2; for instance, if $r = 2$ the graph is the right-hand half of the parabola $y = x^2$ shown in Fig. 4.3.6. Finally, for $r < 0$, the graph is shown in Fig. 4.8.3, which, if $r = -1$, is half of the hyperbola $y = 1/x$ shown in Fig. 4.3.9. Fig. 4.8.4 further illustrates how the graph of $y = x^r$ changes with changing positive values of the exponent.

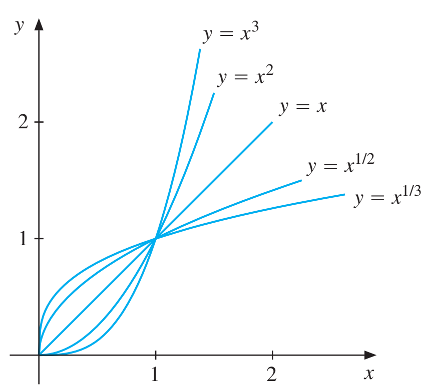


Figure 4.8.4 $y = x^r$

EXERCISES FOR SECTION 4.8

1. Sketch the graphs of $y = x^{-3}$, $y = x^{-1}$, $y = x^{-1/2}$, and $y = x^{-1/3}$, defined for $x > 0$.

2. Use a calculator to find approximate values for $\sqrt{2}^{\sqrt{2}}$ and π^{π}.

3. Solve the following equations for x:

(a) $2^{2x} = 8$ (b) $3^{3x+1} = 1/81$ (c) $10^{x^2-2x+2} = 100$

SM **4.** Match five of the graphs A–F in Figs 4.8.5 to 4.8.10 with each of the functions (a)–(e) below. Then specify a suitable function in (f) that matches the sixth graph.

(a) $y = \frac{1}{2}x^2 - x - \frac{3}{2}$ has graph ___ (b) $y = 2\sqrt{2-x}$ has graph ___

(c) $y = -\frac{1}{2}x^2 + x + \frac{3}{2}$ has graph ___ (d) $y = \left(\frac{1}{2}\right)^x - 2$ has graph ___

(e) $y = 2\sqrt{x-2}$ has graph ___ (f) $y =$ has graph ___

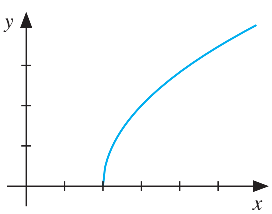


Figure 4.8.5 Graph A

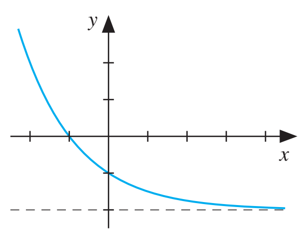


Figure 4.8.6 Graph B

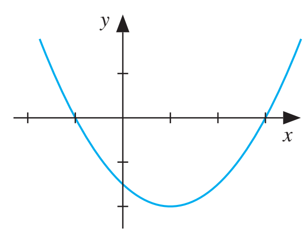


Figure 4.8.7 Graph C

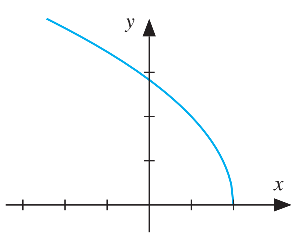


Figure 4.8.8 Graph D

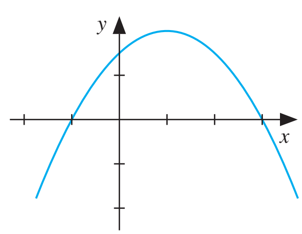


Figure 4.8.9 Graph E

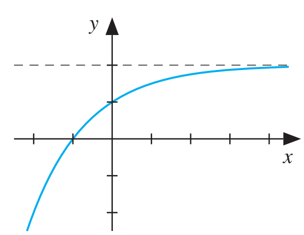


Figure 4.8.10 Graph F

5. Find t when: (a) $3^{5t}9^t = 27$, and (b) $9^t = (27)^{1/5}/3$.

6. The formulas for the surface area S and the volume V of a ball with radius r are $S = 4\pi r^2$ and $V = (4/3)\pi r^3$. Express S as a power function of V.

4.9 Exponential Functions

A quantity that increases (or decreases) by a fixed factor per unit of time is said to *increase* (or *decrease*) *exponentially*. If the fixed factor is a, this leads to the exponential function:

$$f(t) = Aa^t \tag{4.9.1}$$

where $a > 0$ and A are constants. In what follows, we shall consider the case where A is positive, but it will be obvious how to modify the subsequent discussion for the case when A is negative.

Note that if $f(t) = Aa^t$, then $f(t+1) = Aa^{t+1} = Aa^t \cdot a^1 = af(t)$, so the value of f at time $t+1$ is a times the value of f at time t. If $a > 1$, then f is increasing; if $0 < a < 1$, then f is decreasing—see Figs 4.9.1 and 4.9.2. Because $f(0) = Aa^0 = A$, we can always write $f(t) = f(0)a^t$.

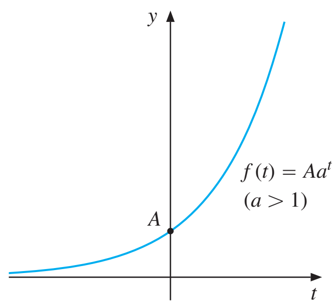


Figure 4.9.1 $f(t) = Aa^t$, $a > 1$

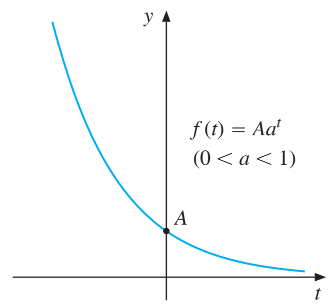


Figure 4.9.2 $f(t) = Aa^t$, $0 < a < 1$

It is important to recognize the fundamental difference between the exponential functions $f(x) = a^x$ and the typical power function $g(x) = x^a$ that was discussed in Section 4.8. Indeed, for the exponential function a^x, it is the exponent x that varies, while the base a is constant; for the power function x^a, on the other hand, the exponent a is constant, while the base x varies.

Exponential functions appear in many important economic, social, and physical models. For instance, economic growth, population growth, continuously accumulated interest, radioactive decay, and decreasing illiteracy have all been described by exponential functions. In addition, the exponential function is one of the most important functions in statistics. Here is one application:

EXAMPLE 4.9.1 **(Population Growth)** Consider a growing population like that of Europe during the 20th century. In Example 4.5.1, we constructed a linear function $P = 5.1t + 606$, where P denotes the population in millions, $t = 0$ corresponds to the year 1960 when the population was 606 million, and $t = 10$ corresponds to the year 1970 when the population estimate was 657 million. According to this formula, the annual increase in population would be constant and equal to 5.1 million. This is a very unreasonable assumption. After all, the linear function implies that, for $t \leq -119$ (i.e., for years before 1841), the population of Europe was negative!

In fact, according to UN estimates, the European population was to grow by approximately 0.45% annually during the period 1960 to 2000. With a population of 606 million in

1960, the population in 1961 would then be $606 \cdot 1.0045$ (see Section 2.2), which is approximately 609 million. Next year, in 1962, it would have grown to $606 \cdot 1.0045^2$, which is approximately 611 million. If growth were to continue at 0.45% annually, the population figure would grow by the factor 1.0045 each year. Then, t years after 1960, the population would be given by

$$P(t) = 606 \cdot 1.0045^t$$

Thus, $P(t)$ is an exponential function of the form (4.9.1). For the year 2015, corresponding to $t = 55$, the formula yields the estimate $P(55) \approx 776$ million.[18]

Many countries, particularly in Africa, have recently had far faster population growth than Europe. For instance, during the 1970s and 1980s, the growth rate of Zimbabwe's population was close to 3.5% annually. If we let $t = 0$ correspond to the census year 1969 when the population was 5.1 million, the population t years after 1969 is $P(t) = 5.1 \cdot 1.035^t$. If we calculate $P(20)$, $P(40)$, and $P(60)$ using this formula, we get roughly 10, 20, and 40. Thus, the population of Zimbabwe roughly doubles after 20 years; during the next 20 years, it doubles again, and so on. We say that the *doubling time* of the population is approximately 20 years. Of course, this kind of extrapolation is quite dubious, because exponential growth of population cannot go on forever: if the growth rate were to continue at 3.5% annually and there was no emigration, then by year 2296 each Zimbabwean would have only one square metre of land on average—see Exercise 6.

If $a > 1$ and $A > 0$, the exponential function $f(t) = Aa^t$ is increasing. Its *doubling time* is the time it takes for it to double. Its value at $t = 0$ is A, so the doubling time t^* is given by the equation $f(t^*) = Aa^{t^*} = 2A$, or after cancelling A, by $a^{t^*} = 2$. Thus the doubling time of the exponential function $f(t) = Aa^t$ is the power to which a must be raised in order to get 2. In Exercise 7 you will show that the doubling time is independent of which year you take as the base. Ultimately, in Example 4.10.4, we will use the *natural logarithm function*, which is denoted by ln, to determine that $t^* = \ln 2/ \ln a$.

EXAMPLE 4.9.2 Use your calculator to find the doubling time of:

(a) a population, like that of Zimbabwe, increasing at 3.5% annually (thus confirming the earlier calculations);

(b) the population of Kenya in the 1980s, which then had the world's highest annual growth rate of population, 4.2%.

Solution:

(a) The doubling time t^* is given by the equation $1.035^{t^*} = 2$. Using a calculator shows that $1.035^{15} \approx 1.68$, whereas $1.035^{25} \approx 2.36$. Thus, t^* must lie between 15 and 25. Because $1.035^{20} \approx 1.99$, t^* is close to 20. In fact, $t^* \approx 20.15$.

(b) The doubling time t^* is given by the equation $1.042^{t^*} = 2$. Using a calculator, we find that $t^* \approx 16.85$. Thus, with a growth rate of 4.2%, Kenya's population would double in less than 17 years.

[18] The actual figure turned out to be about 738 million, which shows the limitations of naive projections.

EXAMPLE 4.9.3 **(Compound Interest)** A savings account of K that increases by $p\%$ interest each year will have increased after t years to $K(1 + p/100)^t$, as seen in Section 2.2. According to this formula with $K = 1$, a deposit of \$1 earning interest at 8% per year (so $p = 8$) will have increased after t years to $(1 + 8/100)^t = 1.08^t$ dollars.

Table 4.4 How \$1 of savings increases with time at 8% annual interest

t	1	2	5	10	20	30	50	100	200
$(1.08)^t$	1.08	1.17	1.47	2.16	4.66	10.06	46.90	2199.76	4 838 949.60

As shown in Table 4.4, after 50 years, \$1 of savings will have increased to more than \$10, and after 200 years, to more than \$4.8 million!

Observe that the expression 1.08^t defines an exponential function of the type (4.9.1), with $a = 1.08$. Even if a is only slightly larger than 1, $f(t)$ will increase very quickly as t becomes large.

EXAMPLE 4.9.4 **(Continuous Depreciation)** Each year the value of most assets such as cars, electronic goods, or furniture decreases, or *depreciates*. If the value of an asset is assumed to decrease by a fixed percentage each year, then the depreciation is called *continuous*.[19]

Assume that a car, which at time $t = 0$ has the value P_0, depreciates at the rate of 20% each year over a five-year period. What is its value $A(t)$ at time t, for $t = 1, 2, 3, 4, 5$?

Solution: After one year, its value is $P_0 - (20P_0/100) = P_0(1 - 20/100) = P_0(0.8)^1$. Thereafter, it depreciates each subsequent year by the factor 0.8. Thus, after t years, its value is $A(t) = P_0(0.8)^t$. In particular, $A(5) = P_0(0.8)^5 \approx 0.32P_0$, so after five years its value has decreased to about 32% of its original value.

The most important properties of the exponential function are summed up by the following:

EXPONENTIAL FUNCTION

The *general exponential function* with base $a > 0$ is

$$f(x) = Aa^x$$

where a is the factor by which $f(x)$ changes when x increases by 1.

If $a = 1 + p/100$, where $p > 0$ and $A > 0$, then $f(x)$ will increase by $p\%$ for each unit increase in x. If $a = 1 - p/100$, where $0 < p < 100$ and $A > 0$, then $f(x)$ will decrease by $p\%$ for each unit increase in x.

[19] Recall the case of linear depreciation discussed in Exercise 4.5.5.

The Natural Exponential Function

Each base a of $f(x) = Aa^x$ gives a different exponential function. In mathematics, one particular value of a gives an exponential function that is far more important than all others. One might guess that $a = 2$ or $a = 10$ would be this special base. Certainly, powers to the base of 2 are important in computing, and powers to the base 10 occur in our usual decimal number system. Nevertheless, once we have studied some calculus, it will turn out that the most important base for an exponential function is an irrational number a little larger than 2.7. In fact, it is so distinguished that it is denoted by the single letter e, possibly because it is the first letter of the word "exponential". Its value to 15 decimal places is[20]

$$e = 2.718281828459045\ldots$$

Many formulas in calculus become much simpler when e is used as the base for exponential functions. Given this base e, the corresponding exponential function

$$f(x) = e^x \tag{4.9.2}$$

is called the *natural exponential function*. In Examples 7.5.4 and 7.6.2 we shall give an explicit way of approximating e^x to an arbitrary degree of accuracy. Of course, all the usual rules for powers apply also to the natural exponential function. In particular,

(a) $e^s e^t = e^{s+t}$ (b) $e^s/e^t = e^{s-t}$ (c) $(e^s)^t = e^{st}$

The graphs of $f(x) = e^x$ and $f(x) = e^{-x}$ are given in Fig. 4.9.3.

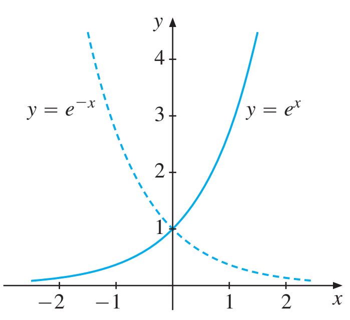


Figure 4.9.3 The graphs of $y = e^x$ and $y = e^{-x}$

Powers with e as their base, even e^1, are difficult to compute by hand. A pocket calculator with an $\boxed{e^x}$ function key can do this immediately, however. For instance, one finds that $e^{1.0} \approx 2.7183$, $e^{0.5} \approx 1.6487$, and $e^{-\pi} \approx 0.0432$.

Sometimes the notation $\exp(u)$, or even $\exp u$, is used in place of e^u. If u is a complicated expression like $x^3 + x\sqrt{x - 1/x} + 5$, it is easier to read and write $\exp(x^3 + x\sqrt{x - 1/x} + 5)$ instead of $e^{x^3 + x\sqrt{x-1/x}+5}$.

[20] Though this number had been defined implicitly over 100 years earlier, the Swiss scientist and mathematician Leonhard Euler (1707–1783) was the first to denote it by the letter e. He subsequently proved that it was irrational and calculated it to 23 decimal places.

EXERCISES FOR SECTION 4.9

1. If the population of Europe grew at the rate of 0.72% annually, what would be its doubling time?

2. The population of Botswana was estimated to be 1.22 million in 1989, and to be growing at the rate of 3.4% annually. If $t = 0$ denotes 1989, find a formula for the population $P(t)$ at date t. What is the doubling time?

3. A savings account with an initial deposit of \$100 earns 12% interest per year. What is the amount of savings after t years? Make a table similar to Table 4.4, stopping at 50 years.

4. Fill in the following table and sketch the graphs of $y = 2^x$ and $y = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
$2x$							
2^{-x}							

5. The *normal density function*

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

is one of the most important functions in statistics. Its graph is often called the "bell curve" because of its shape. Use your calculator to fill in the following table:

x	-2	-1	0	1	2
$y = \varphi(x)$					

6. The area of Zimbabwe is approximately $3.91 \cdot 10^{11}$ m^2. Referring to Example 4.9.1 and using a calculator, solve the equation $5.1 \cdot 10^6 \cdot 1.035^t = 3.91 \cdot 10^{11}$ for t, and interpret the solution.

7. With $f(t) = Aa^t$, if $f(t + t^*) = 2f(t)$, prove that $a^{t^*} = 2$. This shows that the doubling time t^* of the general exponential function is independent of the initial time t.

8. Which of the following equations do *not* define exponential functions of x?

(a) $y = 3^x$ (b) $y = x^{\sqrt{2}}$ (c) $y = (\sqrt{2})^x$

(d) $y = x^x$ (e) $y = (2.7)^x$ (f) $y = 1/2^x$

9. Suppose that all prices rise at the same proportional (inflation) rate of 19% per year. For an item that currently costs P_0, use the implied formula for the price after t years in order to predict the prices of:

(a) A 20 kg bag of corn, presently costing \$16, after five years.

(b) A \$4.40 can of coffee after ten years.

(c) A \$250 000 house after four years.

10. Find possible exponential functions for the graphs A to C, in Figs 4.9.4 to 4.9.6.

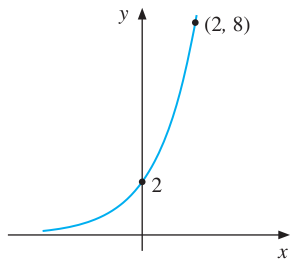


Figure 4.9.4 Graph A

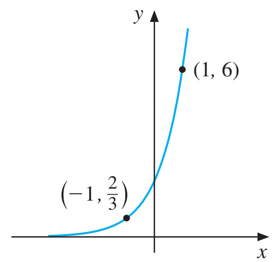


Figure 4.9.5 Graph B

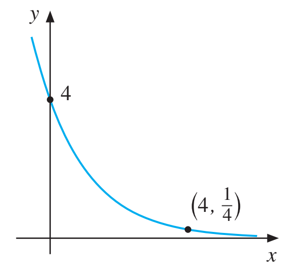


Figure 4.9.6 Graph C

4.10 Logarithmic Functions

The doubling time of an exponential function $f(t) = Aa^t$ was defined as the time it takes for $f(t)$ to become twice as large. In order to find the doubling time t^*, we must solve the equation $a^{t^*} = 2$ for t^*. In economics, we often need to solve similar problems:

(a) At the present rate of inflation, how long will it take the price level to triple?

(b) If the world's population grows at 2% per year, how long does it take to double its size?

(c) If \$1 000 is invested in a savings account bearing interest at the annual rate of 8%, how long does it take for the account to reach \$10 000?

All these questions involve solving equations of the form $a^x = b$ for x. For instance, problem (c) reduces to finding which x solves the equation $1000(1.08)^x = 10\,000$.

We begin with equations in which the base of the exponential is e, which was, as you recall, the irrational number $2.718\ldots$. Here are some examples: $e^x = 4$; $5e^{-3x} = 16$; and $A\alpha e^{-\alpha x} = k$. In all these equations, the unknown x occurs as an exponent. We therefore introduce the following useful definition. If $e^u = a$, we call u the *natural logarithm* of a, and we write $u = \ln a$. Hence, we have the following definition of the symbol $\ln a$:

NATURAL LOGARITHM

For any positive number a,

$$e^{\ln a} = a \tag{4.10.1}$$

Thus, $\ln a$ is the power of e you need to get a.

Because e^u is a strictly increasing function of u, it follows that $\ln a$ *is* uniquely determined by the definition (4.10.1). You should memorize this definition. It is the foundation for everything in this section, and for a good part of what comes later. The following example illustrates how to use this definition.

EXAMPLE 4.10.1 Find the following numbers:

(a) $\ln 1$ (b) $\ln e$ (c) $\ln(1/e)$ (d) $\ln 4$ (e) $\ln(-6)$

Solution:

(a) $\ln 1 = 0$, because $e^0 = 1$ and so 0 is the power of e that you need to get 1.

(b) $\ln e = 1$, because $e^1 = e$ and so 1 is the power of e that you need to get e.

(c) $\ln(1/e) = \ln e^{-1} = -1$, because -1 is the power of e that you need to get $1/e$.

(d) $\ln 4$ is the power of e you need to get 4. Because $e^1 \approx 2.7$ and $e^2 = e^1 \cdot e^1 \approx 7.3$, the number $\ln 4$ must lie between 1 and 2. By using the a calculator, you should be able to find a good approximation to $\ln 4$ by trial and error. Of course, it is easier to press 4 and the $\ln x$ key. Then you find that $\ln 4 \approx 1.386$. Thus, $e^{1.386} \approx 4$.

(e) $\ln(-6)$ would be the power of e you need to get -6. Because e^x is positive for all x, it is obvious that $\ln(-6)$ must be undefined. (The same is true for $\ln x$ whenever $x \leq 0$.)

The following box collects some useful rules for natural logarithms.

RULES FOR THE NATURAL LOGARITHMIC FUNCTION

(a) The logarithm of a *product* is the *sum* of the logarithms of the factors: if x and y are positive, then $\ln(xy) = \ln x + \ln y$.

(b) The logarithm of a *quotient* is the *difference* between the logarithms of its numerator and denominator: if x and y are positive, then

$$\ln \frac{x}{y} = \ln x - \ln y$$

(c) The logarithm of a *power* is the exponent multiplied by the logarithm of the base: if x is positive, then $\ln x^p = p \ln x$.

(d) $\ln 1 = 0$, $\ln e = 1$, and, for general x,

$$\ln e^x = x \quad \text{and} \quad x = e^{\ln x} \tag{4.10.2}$$

where the latter expression assumes that $x > 0$.

To show (a), observe first that the definition of $\ln(xy)$ implies that $e^{\ln(xy)} = xy$. Furthermore, $x = e^{\ln x}$ and $y = e^{\ln y}$, so

$$e^{\ln(xy)} = xy = e^{\ln x}e^{\ln y} = e^{\ln x + \ln y} \tag{$*$}$$

where the last step uses the rule $e^s e^t = e^{s+t}$. In general, $e^u = e^v$ implies $u = v$, so we conclude from $(*)$ that $\ln(xy) = \ln x + \ln y$.

The proofs of (b) and (c) are based on the rules $e^s/e^t = e^{s-t}$ and $(e^s)^t = e^{st}$, respectively, and are left to the reader. Finally, (d) displays some important properties for convenient reference.

It *is* tempting to replace $\ln(x + y)$ by $\ln x + \ln y$, but this is quite wrong. In fact, $\ln x + \ln y$ is equal to $\ln(xy)$, not to $\ln(x + y)$.

LOG OF A SUM

There are *no* simple formulas for $\ln(x+y)$ and $\ln(x-y)$.

Here are some examples that apply the previous rules.

EXAMPLE 4.10.2 Express in terms of $\ln 2$: (a) $\ln 4$; (b) $\ln \sqrt[3]{2^5}$; and (c) $\ln(1/16)$.

Solution:

(a) $\ln 4 = \ln(2 \cdot 2) = \ln 2 + \ln 2 = 2\ln 2$, or, alternatively $\ln 4 = \ln 2^2 = 2\ln 2$.

(b) We have $\sqrt[3]{2^5} = 2^{5/3}$. Therefore, $\ln \sqrt[3]{2^5} = \ln 2^{5/3} = (5/3)\ln 2$.

(c) $\ln(1/16) = \ln 1 - \ln 16 = 0 - \ln 2^4 = -4\ln 2$. Or, $\ln(1/16) = \ln 2^{-4} = -4\ln 2$.

EXAMPLE 4.10.3 Solve the following equations for x:

(a) $5e^{-3x} = 16$ (b) $A\alpha e^{-\alpha x} = k$ (c) $(1.08)^x = 10$ (d) $e^x + 4e^{-x} = 4$

Solution:

(a) Take ln of each side of the equation to obtain $\ln(5e^{-3x}) = \ln 16$. The product rule gives $\ln(5e^{-3x}) = \ln 5 + \ln e^{-3x}$. Here, $\ln e^{-3x} = -3x$, by rule (d). Hence, $\ln 5 - 3x = \ln 16$, which gives

$$x = \frac{1}{3}(\ln 5 - \ln 16) = \frac{1}{3}\ln\frac{5}{16}$$

(b) We argue as in (a) and obtain $\ln(A\alpha e^{-\alpha x}) = \ln k$, or $\ln(A\alpha) + \ln e^{-\alpha x} = \ln k$, so $\ln(A\alpha) - \alpha x = \ln k$. Finally, therefore,

$$x = \frac{1}{\alpha}[\ln(A\alpha) - \ln k] = \frac{1}{\alpha}\ln\frac{A\alpha}{k}$$

(c) Again we take the ln of each side of the equation and obtain $x \ln 1.08 = \ln 10$. So the solution is $x = \ln 10/\ln 1.08$, which is ≈ 29.9. Thus, it takes just short of 30 years for \$1 to increase to \$10 when the interest rate is 8%. (See Table 4.4, in Example 4.9.3.)

(d) It is very tempting to begin with $\ln(e^x + 4e^{-x}) = \ln 4$, but this leads nowhere, because $\ln(e^x + 4e^{-x})$ cannot be further evaluated. Instead, we argue like this: Putting $u = e^x$ gives $e^{-x} = 1/e^x = 1/u$, so the equation is $u + 4/u = 4$, or $u^2 + 4 = 4u$. Solving this quadratic equation for u yields $u = 2$ as the only solution. Hence, $e^x = 2$, and so $x = \ln 2$.

The Function $g(x) = \ln x$

For each positive number x, the number $\ln x$ is defined by $e^{\ln x} = x$. In other words, $u = \ln x$ is the solution of the equation $e^u = x$. This definition is illustrated in Fig. 4.10.1. We call the resulting function

$$g(x) = \ln x \tag{4.10.3}$$

where $x > 0$, the *natural logarithm* of x. Think of x as a point moving upwards on the vertical axis from the origin. As x increases from values less than 1 to values greater than 1, so $g(x)$ increases from negative to positive values. Because e^u tends to 0 as u becomes large and negative, so $g(x)$ becomes large and negative as x tends to 0. Repeating the definition of $\ln x$, then inserting $y = \ln x$ and taking the ln of each side, yields Eq. (4.10.2): $e^{\ln x} = x$ for all $x > 0$; and $\ln e^y = y$ for all y.

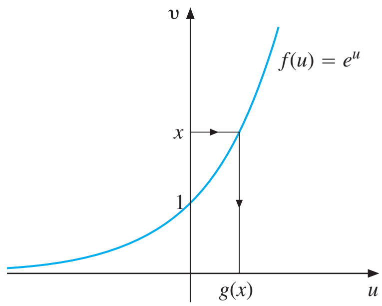


Figure 4.10.1 Construction of $g(x) = \ln x$

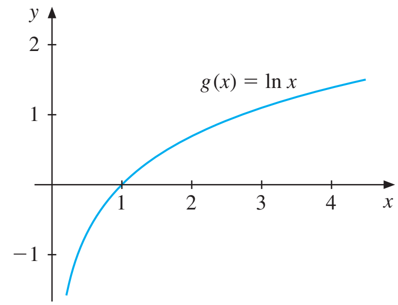


Figure 4.10.2 $g(x) = \ln x$

In Fig. 4.10.2 we have drawn the graph of $g(x) = \ln x$. The shape of this graph ought to be remembered. It can be obtained by reflecting the graph of Fig. 4.10.1 about the 45° line, so that the u- and v-axes are interchanged and become the y- and x-axes of Fig. 4.10.2, respectively. According to Example 4.10.1, we have $g(1/e) = -1$, $g(1) = 0$, and $g(e) = 1$. Observe that this corresponds well with the graph.

Logarithms with Bases other than e

Recall that we defined $\ln x$ as the exponent to which we must raise the base e in order to obtain x. From time to time, it is useful to have logarithms based on numbers other than e. For many years, until the use of mechanical and then electronic calculators became widespread, tables of logarithms to the base 10 were frequently used to simplify complicated calculations involving multiplication, division, square roots, and so on.

Suppose that a is a fixed positive number (usually chosen larger than 1). If $a^u = x$, then we call u the *logarithm of x to base a* and write $u = \log_a x$. The symbol $\log_a x$ is then defined for every positive number x by the following:

LOGARITHM OF x TO BASE a

$$a^{\log_a x} = x \tag{4.10.4}$$

For instance, $\log_2 32 = 5$, because $2^5 = 32$, whereas $\log_{10}(1/100) = -2$, because $10^{-2} = 1/100$. Note that $\ln x$ is $\log_e x$.

By taking the ln on each side of (4.10.4), we obtain $\log_a x \cdot \ln a = \ln x$, so that

$$\log_a x = \frac{1}{\ln a} \ln x \tag{4.10.5}$$

This reveals that the logarithm of x in the system with base a is proportional to $\ln x$*, with a proportionality factor* $1/\ln a$. It follows immediately that $\log_a$ obeys the same rules as ln:

(a) $\log_a(xy) = \log_a x + \log_a y$ (b) $\log_a(x/y) = \log_a x - \log_a y$

(c) $\log_a x^p = p \log_a x$ (d) $\log_a 1 = 0$ and $\log_a a = 1$

Rule (a), for example, follows directly from the corresponding rule for ln, because

$$\log_a(xy) = \frac{1}{\ln a} \ln(xy) = \frac{1}{\ln a}(\ln x + \ln y) = \frac{1}{\ln a} \ln x + \frac{1}{\ln a} \ln y = \log_a x + \log_a y$$

EXAMPLE 4.10.4 Recall from Section 4.9 that the doubling time, t^*, of an exponential function $f(t) = Aa^t$ is given by the formula $a^{t^*} = 2$. Solve this equation for t^*.

Solution: Taking the logarithm to base 2 of both sides of the equation yields $\log_2 a^{t^*} = \log_2 2 = 1$. By rule (c), this implies that $t^* \log_2 a = 1$, and so $t^* = 1/\log_2 a$.

EXERCISES FOR SECTION 4.10

1. Express as multiples of ln 3: (a) $\ln 9$; (b) $\ln \sqrt{3}$; (c) $\ln \sqrt[5]{3^2}$; (d) $\ln(1/81)$.

2. Solve the following equations for x:

   (a) $3^x = 8$ (b) $\ln x = 3$ (c) $\ln(x^2 - 4x + 5) = 0$

   (d) $\ln[x(x-2)] = 0$ (e) $\dfrac{x \ln(x+3)}{x^2+1} = 0$ (f) $\ln(\sqrt{x} - 5) = 0$

SM 3. Solve the following equations for x:

   (a) $3^x 4^{x+2} = 8$ (b) $3 \ln x + 2 \ln x^2 = 6$ (c) $4^x - 4^{x-1} = 3^{x+1} - 3^x$

   (d) $\log_2 x = 2$ (e) $\log_x e^2 = 2$ (f) $\log_3 x = -3$

SM 4. Suppose that $f(t) = Ae^{rt}$ and $g(t) = Be^{st}$, where $A > 0$, $B > 0$, and $r \neq s$. Solve the equation $f(t) = g(t)$ for t.

SM 5. In 1990 the GDP of China was estimated to be $1.2 \cdot 10^{12}$ US dollars, whereas that of the USA was reported to be $5.6 \cdot 10^{12}$ US dollars. The two countries' annual rates of growth were estimated to be 9% and 2% respectively, implying that t years after 1990, their GDP should be Ae^{rt} and Be^{st} respectively, where $r = 0.09$, $s = 0.02$, and A, B are suitable constants. Use the answer to Exercise 4 to determine the date when the two nations' GDP should be the same.

6. Assume that all the variables in the formulas below are positive. Which of them are always true, and which are sometimes false?

(a) $(\ln A)^4 = 4\ln A$

(b) $\ln B = 2\ln\sqrt{B}$

(c) $\ln A^{10} - \ln A^4 = 3\ln A^2$

(d) $\ln \dfrac{A+B}{C} = \ln A + \ln B - \ln C$

(e) $\ln \dfrac{A+B}{C} = \ln(A+B) - \ln C$

(f) $\ln \dfrac{A}{B} + \ln \dfrac{B}{A} = 0$

(g) $p\ln(\ln A) = \ln(\ln A^p)$

(h) $p\ln(\ln A) = \ln(\ln A)^p$

(i) $\dfrac{\ln A}{\ln B + \ln C} = \ln A(BC)^{-1}$

7. Simplify the following expressions:

(a) $\exp[\ln(x)] - \ln\left[\exp(x)\right]$

(b) $\ln\left[x^4\exp(-x)\right]$

(c) $\exp\left[\ln(x^2) - 2\ln y\right]$

REVIEW EXERCISES

1. Let $f(x) = 3 - 27x^3$.

(a) Compute $f(0)$, $f(-1)$, $f(1/3)$, and $f(\sqrt[3]{2})$.

(b) Show that $f(x) + f(-x) = 6$ for all x.

2. Let

$$F(x) = 1 + \frac{4x}{x^2+4}$$

(a) Compute $F(0)$, $F(-2)$, $F(2)$, and $F(3)$.

(b) What happens to $F(x)$ when x becomes large positive or negative?

(c) Give a rough sketch of the graph of F.

3. Figure 4.R.1 combines the graphs of a quadratic function f and a linear function g. Use the two graphs to find those x where: (a) $f(x) \le g(x)$; (b) $f(x) \le 0$; and (c) $g(x) \ge 0$.

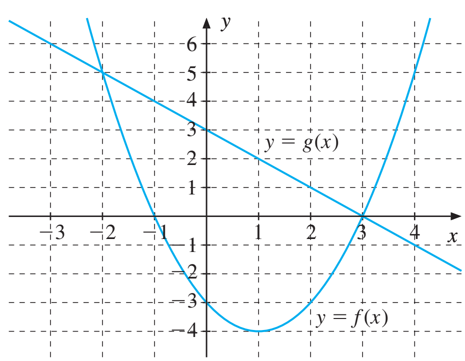


Figure 4.R.1 Two functions

4. Find the domains of the following functions:

(a) $f(x) = \sqrt{x^2 - 1}$

(b) $g(x) = \dfrac{1}{\sqrt{x-4}}$

(c) $h(x) = \sqrt{(x-3)(5-x)}$

5. The cost of producing x units of a commodity is given by $C(x) = 100 + 40x + 2x^2$.

   (a) Find $C(0)$, $C(100)$, and $C(101) - C(100)$.

   (b) Find $C(x+1) - C(x)$, and explain in words the meaning of the difference.

6. Find the slopes of the following straight lines:

   (a) $y = -4x + 8$ (b) $3x + 4y = 12$ (c) $\dfrac{x}{a} + \dfrac{y}{b} = 1$

7. Find equations for the following straight lines:

   (a) L_1 passes through $(-2, 3)$ and has a slope of -3.

   (b) L_2 passes through $(-3, 5)$ and $(2, 7)$.

   (c) L_3 passes through (a, b) and $(2a, 3b)$, where $a \neq 0$.

8. If $f(x) = ax + b$, $f(2) = 3$, and $f(-1) = -3$, then $f(-3) =$?

9. Fill in the following table, then make a rough sketch of the graph of $y = x^2 e^x$.

x	-5	-4	-3	-2	-1	0	1
$y = x^2 e^x$							

10. Find the equation for the parabola $y = ax^2 + bx + c$ that passes through the three points $(1, -3)$, $(0, -6)$, and $(3, 15)$—that is, determine a, b, and c.

11. If a firm sells Q tons of a product, the price P received per ton is $P = 1000 - \frac{1}{3}Q$. The price it has to pay per ton is $P = 800 + \frac{1}{5}Q$. In addition, it has transportation costs of 100 per ton.

    (a) Express the firm's profit π as a function of Q, the number of tons sold, and find the profit-maximizing quantity.

    (b) Suppose the government imposes a tax on the firm's product of 10 per ton. Find the new expression for the firm's profits $\hat{\pi}$ and the new profit-maximizing quantity.

12. In Example 4.6.1, suppose a tax of τ per unit produced is imposed. If $\tau < 100$, what production level now maximizes profits?

13. A firm produces a commodity and receives \$100 for each unit sold. The cost of producing and selling x units is $20x + 0.25x^2$ dollars.

    (a) Find the production level that maximizes profits.

    (b) If a tax of \$10 per unit is imposed, what is the new optimal production level?

    (c) Answer the question in (b) if the sales price per unit is p, the total cost of producing and selling x units is $\alpha x + \beta x^2$, and the tax per unit is τ where $\tau \leq p - \alpha$.

SM 14. Write the following polynomials as products of linear factors:

    (a) $p(x) = x^3 + x^2 - 12x$ (b) $q(x) = 2x^3 + 3x^2 - 18x + 8$

15. Suppose that a and b are constants, while n is a natural number. Which of the following divisions leave no remainder?

(a) $(x^3 - x - 1)/(x - 1)$ (b) $(2x^3 - x - 1)/(x - 1)$

(c) $(x^3 - ax^2 + bx - ab)/(x - a)$ (d) $(x^{2n} - 1)/(x + 1)$

16. Find the values of k that make the polynomial $q(x)$ divide the polynomial $p(x)$ in:

(a) $p(x) = x^2 - kx + 4,\ q(x) = x - 2$ (b) $p(x) = k^2x^2 - kx - 6,\ q(x) = x + 2$

(c) $p(x) = x^3 - 4x^2 + x + k,\ q(x) = x + 2$ (d) $p(x) = k^2x^4 - 3kx^2 - 4,\ q(x) = x - 1$

SM **17.** The cubic function $p(x) = \frac{1}{4}x^3 - x^2 - \frac{11}{4}x + \frac{15}{2}$ has three real zeros. Verify that $x = 2$ is one of them, and find the other two.

18. In 1964 a five-year plan was introduced in Tanzania. One objective was to double the real income per capita over the next 15 years. What is the average annual rate of growth of real income per capita required to achieve this objective?

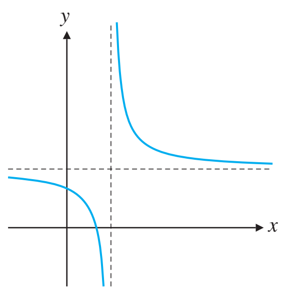


Figure 4.R.2 Graph of $y = \dfrac{ax + b}{x + c}$

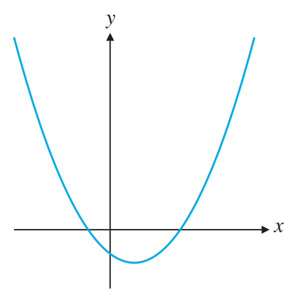


Figure 4.R.3 Graph of $y = px^2 + qx + r$

SM **19.** Figure 4.R.2 shows the graph of the function $y = f(x) = (ax + b)/(x + c)$. Check which of the constants a, b, and c are positive, zero, or negative.

20. Figure 4.R.3 shows the graph of the function $y = g(x) = px^2 + qx + r$. Check which of the constants p, q, and r are positive, zero, or negative.

21. Recall that: (i) the relationship between the Celsius (C) and Fahrenheit (F) temperature scales is linear; (ii) water freezes at 0° C and 32° F; and (iii) water boils at 100° C and 212° F.

(a) Determine the equation that converts C to F.

(b) Which temperature is represented by the same number in both scales?

22. Solve for t in the following equations:

(a) $x = e^{at+b}$ (b) $e^{-at} = 1/2$ (c) $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = \dfrac{1}{8}$

SM 23. Prove the following equalities, with appropriate restrictions on the variables:

(a) $\ln x - 2 = \ln(x/e^2)$

(b) $\ln x - \ln y + \ln z = \ln \dfrac{xz}{y}$

(c) $3 + 2\ln x = \ln(e^3 x^2)$

(d) $\frac{1}{2}\ln x - \frac{3}{2}\ln \dfrac{1}{x} - \ln(x+1) = \ln \dfrac{x^2}{x+1}$

5 PROPERTIES OF FUNCTIONS

The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete facts.
—Alfred North Whitehead (1925)

This chapter begins by examining more closely functions of one variable and their graphs. In particular, we shall consider how changes in a function relate to shifts in its graph, and how to construct new functions from old ones. Next we discuss when a function has an inverse, and explain how an inverse function reverses the effect of the original function, and vice versa.

Any equation in two variables can be represented by a curve (or a set of points) in the *xy*-plane. Some examples illustrate this. The chapter ends with a discussion of the general concept of a function, which is one of the most fundamental in mathematics, of great importance also in economics.

5.1 Shifting Graphs

Bringing a significant new oil field into production will affect the supply curve for oil, with consequences for its equilibrium price. Adopting an improved technology in the production of a commodity will imply an upward shift in its production function, and a downward shift in its cost function.

This section studies in general how the graph of a function $f(x)$ relates to the graphs of the associated functions

$$f(x)+c, \quad f(x+c), \quad cf(x), \quad \text{and} \quad f(-x)$$

where c is a positive or negative constant. Before formulating any general rules, consider the following example.

EXAMPLE 5.1.1 The graph of $y=\sqrt{x}$ is one of the three drawn in Fig. 5.1.1. Sketch the graphs of $y=\sqrt{x}+2$, $y=\sqrt{x}-2$, $y=\sqrt{x+2}$, $y=\sqrt{x-2}$, $y=2\sqrt{x}$, $y=-\sqrt{x}$, and $y=\sqrt{-x}$.

Solution: The graphs of $y=\sqrt{x}+2$ and $y=\sqrt{x}-2$, shown in Fig. 5.1.1, are obviously obtained by moving the graph of $y=\sqrt{x}$ upwards or downwards by two units, respectively.

The function $y = \sqrt{x+2}$ is defined for $x + 2 \geq 0$, that is, for $x \geq -2$. Its graph, which is shown in Fig. 5.1.2, is obtained by moving the graph of $y = \sqrt{x}$ two units to the left. In the same way the graph of $y = \sqrt{x-2}$ is obtained by moving the graph of $y = \sqrt{x}$ two units to the right, as shown in Fig. 5.1.2.

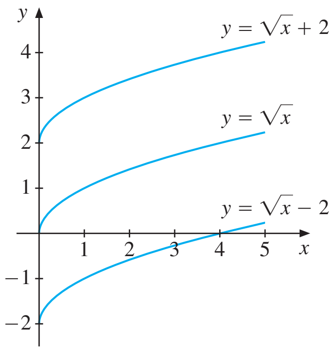


Figure 5.1.1 $y = \sqrt{x} \pm 2$

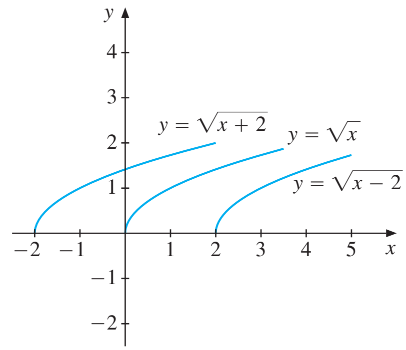


Figure 5.1.2 $y = \sqrt{x \pm 2}$

The graph of $y = 2\sqrt{x}$ is obtained by stretching the graph of f vertically upwards by a factor of two, as shown in Fig. 5.1.3. The graph of $y = -\sqrt{x}$ is obtained by reflecting the graph of $y = \sqrt{x}$ about the x-axis, as shown in Fig. 5.1.4.

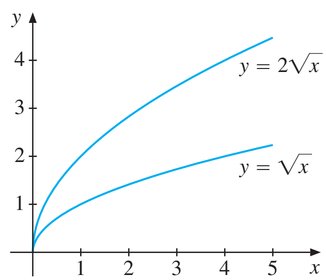


Figure 5.1.3 $y = \sqrt{x}$ and $y = 2\sqrt{x}$

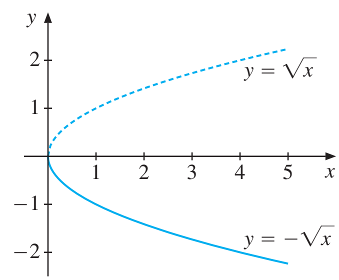


Figure 5.1.4 $y = \sqrt{x}$ and $y = -\sqrt{x}$

Finally, the function $y = \sqrt{-x}$ is defined for $-x \geq 0$, that is, for $x \leq 0$, and its graph is shown in Fig. 5.1.5. It is obtained by reflecting the graph of $y = \sqrt{x}$ about the y-axis.

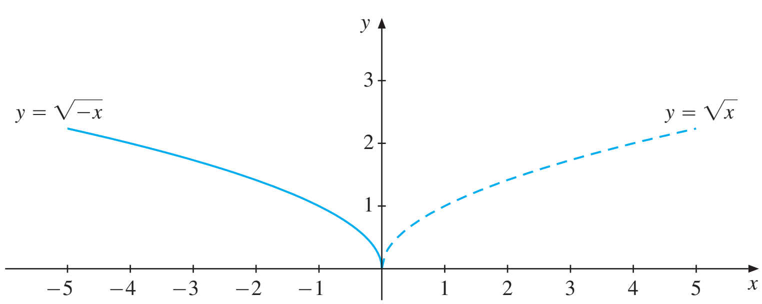


Figure 5.1.5 $y = \sqrt{x}$ and $y = \sqrt{\pm x}$

Here are some general rules for shifting the graph of a function.

GENERAL RULES FOR SHIFTING THE GRAPH OF $y = f(x)$

(i) If $y = f(x)$ is replaced by $y = f(x) + c$, the graph is moved upwards by c units if $c > 0$; it is moved downwards if $c < 0$.

(ii) If $y = f(x)$ is replaced by $y = f(x + c)$, the graph is moved c units to the left if $c > 0$; it is moved to the right if $c < 0$.

(iii) If $y = f(x)$ is replaced by $y = cf(x)$, the graph is stretched vertically if $c > 0$; it is stretched vertically and reflected about the x-axis if $c < 0$.

(iv) If $y = f(x)$ is replaced by $y = f(-x)$, the graph is reflected about the y-axis.

In the case where the independent variable is y and $x = g(y)$, then in the first two rules you should interchange the words "upwards" with "to the right", and "downwards" with "to the left". In the third rule, the word "vertically" would become "horizontally", and in this and the last rule the term "y-axis" would become "x-axis".

Combining these rules with Figs 4.3.5 to 4.3.10, a large number of useful graphs can be sketched with ease, as the following example illustrates.

EXAMPLE 5.1.2 Sketch the graphs of

(a) $y = 2 - (x + 2)^2$ (b) $y = \dfrac{1}{x - 2} + 3$

Solution:

(a) First, $y = x^2$ is reflected about the x-axis to obtain the graph of $y = -x^2$. This graph is then moved two units to the left, which results in the graph of $y = -(x + 2)^2$. Finally, this new graph is raised by two units, and we obtain the graph shown in Fig. 5.1.6.

(b) The graph of $y = 1/(x - 2)$ is obtained by moving two units to the right the graph of $y = 1/x$ in Fig. 4.3.9. By moving the new graph three units up, we get the graph in Fig. 5.1.7.

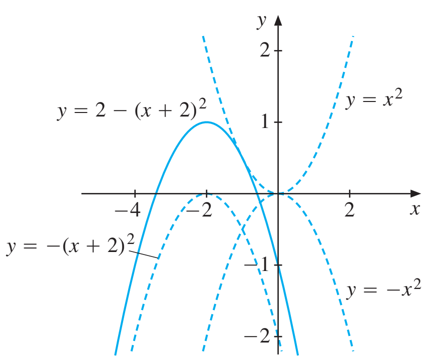


Figure 5.1.6 $y = 2 - (x + 2)^2$

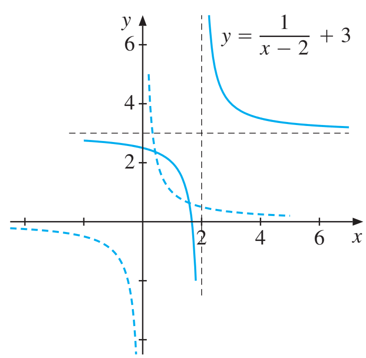


Figure 5.1.7 $y = 1/(x - 2) + 3$

EXAMPLE 5.1.3 In Example 4.5.3 we studied the simple demand and supply functions $D = 100 - P$ and $S = 10 + 2P$, which gave the equilibrium price $P^e = 30$ with corresponding quantity $Q^e = 70$. Suppose that there is a shift to the right in the supply curve, so that the new supply at price P is $\tilde{S} = 16 + 2P$. Then the new equilibrium price $\tilde{P}^e$ is determined by the equation $100 - \tilde{P}^e = 16 + 2\tilde{P}^e$, which gives $\tilde{P}^e = 28$, with corresponding quantity $\tilde{Q}^e = 100 - 28 = 72$. Hence the new equilibrium price is lower than the old one, while the quantity is higher. The outward shift in the supply curve from S to $\tilde{S}$ implies that the equilibrium point moves down to the right along the unchanged demand curve. This is shown in Fig. 5.1.8.

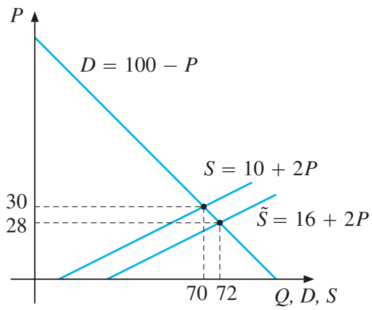


Figure 5.1.8 A shift in supply

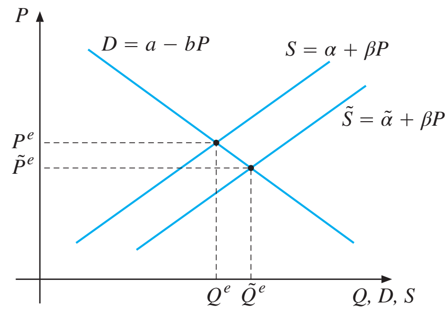


Figure 5.1.9 A shift in supply

In Example 4.5.4 we studied the general linear demand and supply functions $D = a - bP$ and $S = \alpha + \beta P$. The equilibrium price P^e and corresponding equilibrium quantity Q^e were given by

$$P^e = \frac{a - \alpha}{\beta + b} \quad \text{and} \quad Q^e = \frac{a\beta + \alpha b}{\beta + b}$$

Suppose that there is a shift in the supply curve so that the new supply at each price P is $\tilde{S} = \tilde{\alpha} + \beta P$, where $\tilde{\alpha} > \alpha$. Then the new equilibrium price $\tilde{P}^e$ is determined by the equation $a - b\tilde{P}^e = \tilde{\alpha} + \beta\tilde{P}^e$, implying that

$$\tilde{P}^e = \frac{a - \tilde{\alpha}}{\beta + b}, \quad \text{with} \quad \tilde{Q}^e = a - b\tilde{P}^e = \frac{a\beta + \tilde{\alpha} b}{\beta + b}$$

The differences between the new and the old equilibrium prices and quantities are

$$\tilde{P}^e - P^e = \frac{\alpha - \tilde{\alpha}}{\beta + b} \quad \text{and} \quad \tilde{Q}^e - Q^e = -b(\tilde{P}^e - P^e) = \frac{(\tilde{\alpha} - \alpha)b}{\beta + b}$$

We see that $\tilde{P}^e$ is less than P^e (because $\tilde{\alpha} > \alpha$), while $\tilde{Q}^e$ is larger than Q^e. This is shown in Fig. 5.1.9. The rightward shift in the supply curve from S to $\tilde{S}$ implies that the equilibrium point moves down and to the right along the unchanged demand curve. Upward shifts in the supply curve resulting from, for example, taxation or increased cost, can be analysed in the same way, as can shifts in the demand curve.

EXAMPLE 5.1.4 Suppose a person earning y dollars in a given year pays $f(y)$ dollars that year in income tax. The government decides to reduce taxes. One proposal is to allow every individual to deduct d dollars from their taxable income before the tax is calculated. An alternative proposal involves calculating income tax on the full amount of taxable income, and then allowing each person a "tax credit" that deducts c dollars from the total tax due. Illustrate graphically the two proposals for a "normal" tax function f, and mark off the income y^* where the two proposals yield the same tax revenue.

Solution: Figure 5.1.10 illustrates the situation for a so-called "progressive" tax schedule in which the average tax rate, $T/y = f(y)/y$, is an increasing function of y.[1] First draw the graph of the original tax function, $T = f(y)$. If taxable income is y and the deduction is d, then $y - d$ is the reduced taxable income, and so the tax liability is $f(y - d)$. By shifting the graph of the original tax function d units to the right, we obtain the graph of $T_1 = f(y - d)$.

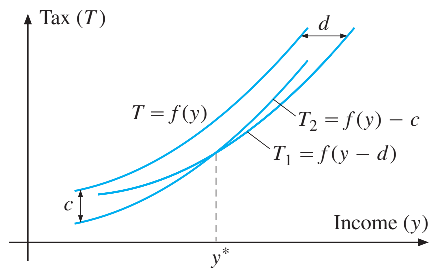


Figure 5.1.10 The graphs of $T_1 = f(y - d)$ and $T_2 = f(y) - c$

The graph of $T_2 = f(y) - c$ is obtained by lowering the graph of $T = f(y)$ by c units. The income y^* which gives the same tax under the two different schemes is given by the equation

$$f(y^* - d) = f(y^*) - c$$

Note that $T_1 > T_2$ when $y < y^*$, but that $T_1 < T_2$ when $y > y^*$. Thus, the tax credit is worth more to those with low incomes; the deduction is worth more to those with high incomes (as one might expect).

EXERCISES FOR SECTION 5.1

1. Use Fig. 4.3.6 and the rules for shifting graphs to sketch the graphs of the following functions:

(a) $y = x^2 + 1$ (b) $y = (x + 3)^2$ (c) $y = 3 - (x + 1)^2$

2. If $y = f(x)$ has the graph drawn in Fig. 5.1.11, sketch the graphs of:

(a) $y = f(x - 2)$ (b) $y = f(x) - 2$ (c) $y = f(-x)$

3. Suppose that in the model of Example 5.1.3 there is a positive shift in demand so that the new demand quantity at price P is $\tilde{D} = 106 - P$. Find the new equilibrium point and illustrate.

4. Use Fig. 4.3.10 and the rules for shifting graphs to sketch the graph of $y = 2 - |x + 2|$.

[1] Example 5.4.4 considers the US Federal Income Tax, which is an example having this property.

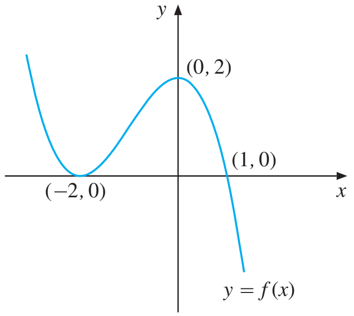


Figure 5.1.11 Function f for Exercise 2

5. Starting with the graph of $f(x) = 1/x^2$, sketch the graph of $g(x) = 2 - (x + 2)^{-2}$.

6. Suppose in Example 5.1.4 that $f(y) = Ay + By^2$ where A and B are positive parameters. Find y^* in this case.

5.2 New Functions from Old

Figure 5.2.1 gives a graphical representation of the number of male and female students registered at a certain university in the period 1986 to 1997.

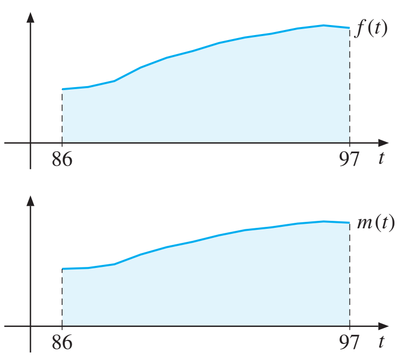


Figure 5.2.1 Male and female students

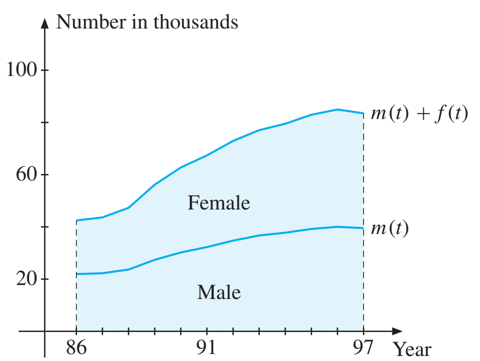


Figure 5.2.2 Total students

Let $f(t)$ and $m(t)$ denote the number of female and male students in year t, while $n(t)$ denotes the total number of students. Of course, $n(t) = f(t) + m(t)$. The graph of the total number $n(t)$ is obtained by piling the graph of $f(t)$ on top of that of $m(t)$, as in Fig. 5.2.2.

Suppose in general that f and g are functions which are both defined in a set A of real numbers. The function F defined by the formula $F(x) = f(x) + g(x)$ is called the *sum* of f and g, and we write $F = f + g$. The function G defined by $G(x) = f(x) - g(x)$ is called the *difference* between f and g, and we write $G = f - g$.

Sums and differences of functions are often seen in economic models. Consider the following typical examples.

EXAMPLE 5.2.1 The cost of producing $Q > 0$ units of a commodity is $C(Q)$. The cost per unit of output, $A(Q) = C(Q)/Q$, is called the *average cost*. If, in particular,

$$C(Q) = aQ^3 + bQ^2 + cQ + d$$

is a cost function of the type shown in Fig. 4.7.2, the average cost is

$$A(Q) = aQ^2 + bQ + c + \frac{d}{Q}$$

Thus $A(Q)$ is a sum of a quadratic function $y = aQ^2 + bQ + c$ and the hyperbola $y = d/Q$. Figure 5.2.5 shows how the graph of the average cost function $A(Q)$ is obtained by piling the graph of the hyperbola $y = d/Q$, which appears in Fig. 5.2.4, on top of the graph of the parabola $y = aQ^2 + bQ + c$, from Fig. 5.2.3.

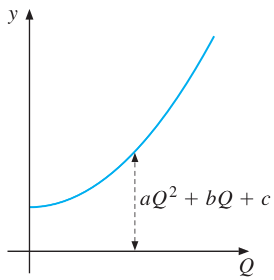


Figure 5.2.3 $aQ^2 + bQ + c$

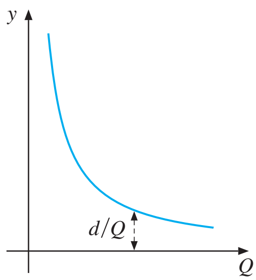


Figure 5.2.4 d/Q

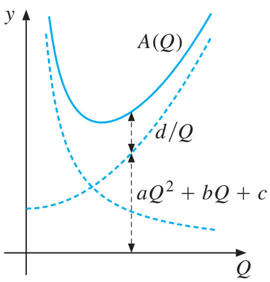


Figure 5.2.5 $A(Q)$

Note that for small values of Q the graph of $A(Q)$ is close to the graph of $y = d/Q$, while for large values of Q, the graph is close to the parabola, since d/Q is small when Q is large.

Now, let $R(Q)$ denote the *revenue* obtained by selling Q units. Then, the *profit* $\pi(Q)$ is given by $\pi(Q) = R(Q) - C(Q)$, An example showing how to construct the graph of the profit function $\pi(Q)$ is given in Fig. 5.2.6. In this case the firm gets a fixed price p per unit, so that the graph of $R(Q)$ is a straight line through the origin. The graph of $-C(Q)$ must be added to that of $R(Q)$. The production level that maximizes profit is Q^*.

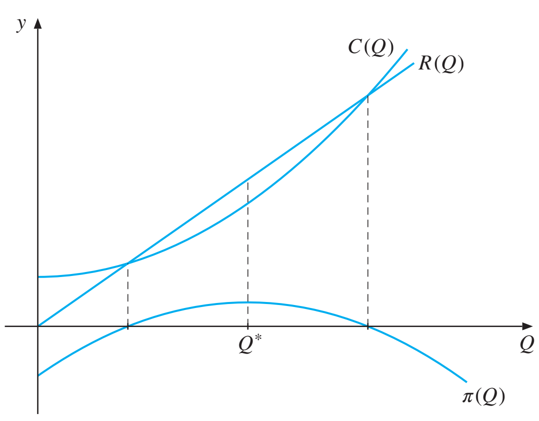


Figure 5.2.6 $\pi(Q) = R(Q) - C(Q)$

Products and Quotients

If f and g are defined in a set A, the function F defined by $F(x) = f(x) \cdot g(x)$ is called the *product* of f and g, and we put $F = f \cdot g$ (or fg). The function F defined where $g(x) \neq 0$ by $F(x) = f(x)/g(x)$ is called the *quotient* of f and g, and we write $F = f/g$. We have already seen examples of these operations. It is difficult to give useful general rules about the behaviour of the graphs of fg and f/g given the graphs of f and g.

Composite Functions

Suppose the demand for a commodity is a function x of its price p. Suppose that price p is not constant, but depends on time t. Then it is natural to regard x as a function of t.

In general, if y is a function of u, and u is a function of x, then y can be regarded as a function of x. We call y a *composite function* of x. If we denote the two functions involved by f and g, with $y = f(u)$ and $u = g(x)$, then we can replace u by $g(x)$ and so write y in the form

$$y = f(g(x))$$

Note that when computing y, we first apply g to x to obtain $g(x)$, and then we apply f to $g(x)$. Here $g(x)$ is called the *kernel*, or *interior function*, while f is called the *exterior function*.

The function that maps x to $f(g(x))$ is often denoted by $f \circ g$. This is read as "f of g" or "f after g", and is called the *composition* of f with g. Correspondingly, $g \circ f$ denotes the function that maps x to $g(f(x))$. Thus, we have

$$(f \circ g)(x) = f(g(x)) \quad \text{and} \quad (g \circ f)(x) = g(f(x))$$

Usually, $f \circ g$ and $g \circ f$ are quite different functions. For instance, if $g(x) = 2 - x^2$ and $f(u) = u^3$, then $(f \circ g)(x) = (2 - x^2)^3$, whereas $(g \circ f)(x) = 2 - (x^3)^2 = 2 - x^6$; the two resulting polynomials are not the same.

It is easy to confuse $f \circ g$ with $f \cdot g$, especially typographically. But these two functions are defined in entirely different ways. When we evaluate $f \circ g$ at x, we first compute $g(x)$ and then evaluate f at $g(x)$. On the other hand, the product $f \cdot g$ of f and g is the function whose value at a particular number x is simply the product of $f(x)$ and $g(x)$, so $(f \cdot g)(x) = f(x) \cdot g(x)$.

Many calculators have built-in functions. When we enter a number x_0 and press the key for the function f, we obtain $f(x_0)$. When we compute a composite function given f and g, and try to obtain the value of $f(g(x))$, we proceed in a similar manner: enter the number x_0, then press the g key to get $g(x_0)$, and again press the f key to get $f(g(x_0))$. Suppose the calculator has the functions $\boxed{1/x}$ and $\boxed{\sqrt{x}}$. If we enter the number 9, then press $\boxed{1/x}$ and $\boxed{\sqrt{x}}$, we get $1/3 = 0.33\ldots$.

The computation we have performed can be illustrated as follows:

$$9 \xrightarrow{1/x} 1/9 \xrightarrow{\sqrt{x}} 1/3$$

Using function notation, $f(x) = \sqrt{x}$ and $g(x) = 1/x$, so $f(g(x)) = f(1/x) = \sqrt{1/x} = 1/\sqrt{x}$. In particular, $f(g(9)) = 1/\sqrt{9} = 1/3$.

EXAMPLE 5.2.2 Write the following as composite functions:

(a) $y = (x^3 + x^2)^{50}$ (b) $y = e^{-(x-\mu)^2}$, where μ is a constant

Solution:

(a) Given a value of x, you first compute $x^3 + x^2$, which gives the interior function, $g(x) = x^3 + x^2$. Then take the 50th power of the result, so the exterior function is $f(u) = u^{50}$. Hence,

$$f(g(x)) = f(x^3 + x^2) = (x^3 + x^2)^{50}$$

(b) We can choose the interior function as $g(x) = -(x - \mu)^2$ and the exterior function as $f(u) = e^u$. Then $f(g(x)) = f(-(x - \mu)^2) = e^{-(x-\mu)^2}$. Alternatively, we could choose $g(x) = (x - \mu)^2$ and $f(u) = e^{-u}$.

Symmetry

The function $f(x) = x^2$ satisfies $f(-x) = f(x)$, as indeed does any even power x^{2n}, with n an integer, positive or negative. So if $f(-x) = f(x)$ for all x in the domain of f, then f is called an *even* function. This condition implies that the graph of f is *symmetric about the y-axis* as shown in Fig. 5.2.7.

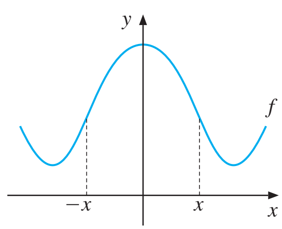


Figure 5.2.7 Even function

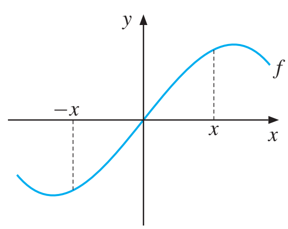


Figure 5.2.8 Odd function

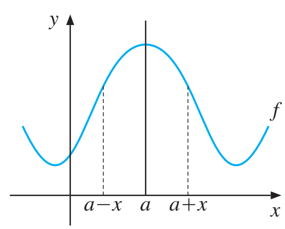


Figure 5.2.9 Symmetry

On the other hand, any odd power x^{2n+1} such as $f(x) = x^3$ satisfies $f(-x) = -f(x)$. So if $f(-x) = -f(x)$ for all x in the domain of f, implying that the graph of f is *symmetric about the origin*, as shown in Fig. 5.2.8, then f is called an *odd* function.

Finally, f is *symmetric about a* if $f(a + x) = f(a - x)$ for all x. The *graph* of f is then *symmetric about the line $x = a$* as in Fig. 5.2.9. In Section 4.6 we showed that the quadratic function $f(x) = ax^2 + bx + c$ is symmetric about $x = -b/2a$. The function $y = e^{-(x-\mu)^2}$ from Example 5.2.2(b) is symmetric about $x = \mu$.

EXERCISES FOR SECTION 5.2

1. Assuming $x > 0$, show graphically how to find the graph of $y = \frac{1}{4}x^2 + 1/x$, by adding the graph of $1/x$ to the graph of $y = \frac{1}{4}x^2$.

2. Sketch the graphs of the following functions:

(a) $y = \sqrt{x} - x$ (b) $y = e^x + e^{-x}$ (c) $y = e^{-x^2} + x$

3. If $f(x) = 3x - x^3$ and $g(x) = x^3$, compute: $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, $(f/g)(x)$, $f(g(1))$, and $g(f(1))$.

4. Let $f(x) = 3x + 7$. Compute $f(f(x))$, and find the value x^* when $f(f(x^*)) = 100$.

5. Compute $\ln(\ln e)$ and $(\ln e)^2$. What do you notice?[2]

5.3 Inverse Functions

Suppose that the demand quantity D for a commodity depends on the price per unit P according to $D = 30/P^{1/3}$. This formula tells us directly the demand D corresponding to a given price P. If, for example, $P = 27$, then $D = 30/27^{1/3} = 10$. So D is a function of P. That is, $D = f(P)$ with $f(P) = 30/P^{1/3}$. Note that demand decreases as the price increases.

If we look at the matter from a producer's point of view, however, it may be more natural to treat output as something it can choose and consider the resulting price. The producer wants to know the *inverse* function, in which price depends on the quantity sold. This functional relationship is obtained by solving $D = 30/P^{1/3}$ for P. First we obtain $P^{1/3} = 30/D$ and then $(P^{1/3})^3 = (30/D)^3$, so that $P = 27\,000/D^3$. This equation gives us directly the price P corresponding to a given output D. For example, if $D = 10$, then $P = 27\,000/10^3 = 27$. In this case, P is a function $g(D)$ of D, with $g(D) = 27\,000/D^3$.

The two variables D and P in this example are related in a way that allows each to be regarded as a function of the other. In fact, the two functions

$$f(P) = 30p^{-1/3} \quad \text{and} \quad g(D) = 27000D^{-3}$$

are *inverses* of each other. We say that f is the inverse of g, and that g is the inverse of f.

Note that the two functions f and g convey exactly the same information. For example, the fact that demand is 10 at price 27 can be expressed using either f or g: $f(27) = 10$ or $g(10) = 27$. In Example 4.5.3 we considered an even simpler demand function $D = 100 - P$. Solving for P we get $P = 100 - D$, which was referred to as the inverse demand function.

Suppose in general that f is a function with domain $D_f = A$, meaning that to each x in A there corresponds a unique number $f(x)$. Recall that if f has domain A, then the range of f is the set $B = R_f = \{f(x) : x \in A\}$, which is also denoted by $f(A)$. The range B consists of all numbers $f(x)$ obtained by letting x vary in A. Furthermore, f is said to be *one-to-one* in A if f never has the same value at any two different points in A. In other words, for each one y in B, there is exactly one x in A such that $y = f(x)$. Equivalently, f is one-to-one in

[2] This illustrates how, if we define the function f^2 by $f^2(x) = (f(x))^2$, then, in general, $f^2(x) \neq f(f(x))$.

A provided that it has the property that, whenever x_1 and x_2 both lie in A and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

It is evident that if a function is strictly increasing in all of A, or strictly decreasing in all of A, then it is one-to-one. A particular one-to-one function f is illustrated in Fig. 5.3.1. The function g shown in Fig. 5.3.2 is not one-to-one because, for example, the two x-values x_1 and x_2 are both associated with y_1.

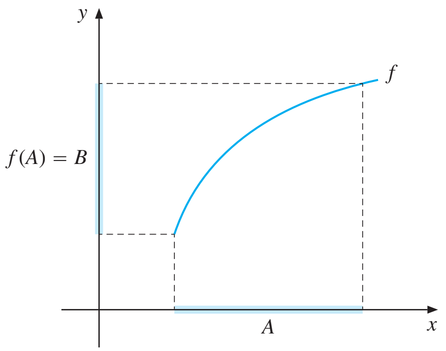


Figure 5.3.1 f is one-to-one with domain A and range B. f has an inverse

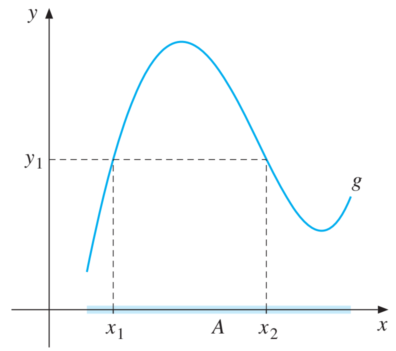


Figure 5.3.2 g is *not* one-to-one and hence has no inverse over A

INVERSE FUNCTION

Let f be a function with domain A and range B. If and only if f is one-to-one, it has an *inverse function* g with domain B and range A. The function g is given by the following rule: For each y in B, the value $g(y)$ is the unique number x in A such that $f(x) = y$. Then

$$g(y) = x \Leftrightarrow y = f(x) \quad (x \in A,\ y \in B) \tag{5.3.1}$$

A direct implication of (5.3.1) is that

$$g(f(x)) = x \text{ for all } x \text{ in } A \text{ and } f(g(y)) = y \text{ for all } y \text{ in } B \tag{5.3.2}$$

The equation $g(f(x)) = x$ shows what happens if we first apply f to x and then apply g to $f(x)$: we get x back because *g undoes what f did to x*. Note that if g is the inverse of a function f, then f is also the inverse of g. If g is the inverse of f, it is standard to use the notation f^{-1} for g.[3]

In simple cases, we can use the same method as in the introductory example to find the inverse of a given function (and hence automatically verify that the inverse exists). Some more examples follow.

[3] This sometimes leads to confusion. If a is a number such that $a \neq 0$, then a^{-1} means $1/a$. But $f^{-1}(x)$ does *not* mean $1/f(x)$ — which equals, instead, $f(x)^{-1}$. For example: the functions defined by $y = 1/(x^2 + 2x + 3)$ and $y = x^2 + 2x + 3$ are *not* inverses of each other, but reciprocals.

EXAMPLE 5.3.1 Solve the following equations for x and find the corresponding inverse functions:

(a) $y = 4x - 3$ (b) $y = \sqrt[5]{x+1}$ (c) $y = \dfrac{3x-1}{x+4}$

Solution:

(a) Solving the equation for x, we have the following equivalences:

$$y = 4x - 3 \Leftrightarrow 4x = y + 3 \Leftrightarrow x = \frac{1}{4}y + \frac{3}{4}$$

for all x and y. We conclude that $f(x) = 4x - 3$ and $g(y) = \frac{1}{4}y + \frac{3}{4}$ are inverses of each other.

(b) We begin by raising each side to the fifth power and so obtain the equivalences

$$y = \sqrt[5]{x+1} \Leftrightarrow y^5 = x + 1 \Leftrightarrow x = y^5 - 1$$

These are valid for all x and all y. Hence, we have shown that $f(x) = \sqrt[5]{x+1}$ and $g(y) = y^5 - 1$ are inverses of each other.

(c) Here we begin by multiplying both sides of the equation by $x + 4$ to obtain $y(x + 4) = 3x - 1$. From this equation, we obtain $yx + 4y = 3x - 1$ or $x(3 - y) = 4y + 1$. Hence,

$$x = \frac{4y+1}{3-y}$$

We conclude that $f(x) = (3x - 1)/(x + 4)$ and $g(y) = (4y + 1)/(3 - y)$ are inverses of each other. Observe that f is only defined for $x \neq -4$, and g is only defined for $y \neq 3$. So the equivalence in (5.3.1) is valid only with these restrictions.

A Geometric Characterization of Inverse Functions

In our introductory example, we saw that $f(P) = 30p^{-1/3}$ and $g(D) = 27\,000D^{-3}$ were inverse functions. Because of the concrete interpretation of the symbols P and D, it was natural to describe the functions the way we did. In other circumstances, it may be convenient to use the same variable as argument in both f and g. In Example 5.3.1(a), we saw that $f(x) = 4x - 3$ and $g(y) = \frac{1}{4}y + \frac{3}{4}$ were inverses of each other. If also we use x instead of y as the variable of the function g, we find that

$$f(x) = 4x - 3 \quad \text{and} \quad g(x) = \tfrac{1}{4}x + \tfrac{3}{4} \quad \text{are inverses of each other} \tag{$*$}$$

In the same way, on the basis of part (b) of the same example, we can say that

$$f(x) = (x+1)^{1/5} \quad \text{and} \quad g(x) = x^5 - 1 \text{ are inverses of each other} \tag{$**$}$$

There is an interesting geometric property of the graphs of inverse functions. For the pairs of inverse functions in ($*$) and ($**$), the graphs of f and g are mirror images of each other with respect to the line $y = x$. This is illustrated in Figs 5.3.3 and 5.3.4.

Suppose in general that f and g are inverses of each other. The fact that (a, b) lies on the graph f means that $b = f(a)$. According to (5.3.1), this implies that $g(b) = a$, so that (b, a)

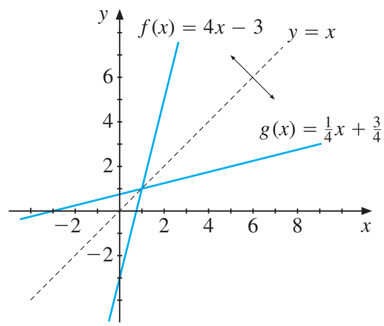


Figure 5.3.3 f and g are inverses of each other

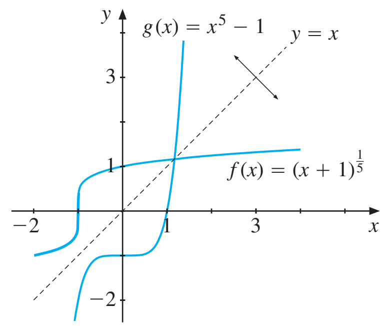


Figure 5.3.4 f and g are inverses of each other

lies on the graph of g. Because (a, b) and (b, a) lie symmetrically about the line $y = x$ (see Exercise 8), we have the following conclusion:

SYMMETRY OF INVERSE FUNCTIONS

Suppose the two functions f and g are inverses of each other. Provided that the scales of the coordinate axes are the same, the graphs of $y = f(x)$ and $y = g(x)$ are symmetric about the line $y = x$.

When the functions f and g are inverses of each other, then by definition (5.3.1), the equations $y = f(x)$ and $x = g(y)$ are equivalent. The two functions actually have exactly the same graph, though in the second case we should think of x depending on y, instead of the other way around. On the other hand, the graphs of $y = f(x)$ and $y = g(x)$ are symmetric about the line $y = x$.

For instance, Examples 4.5.3 and 5.1.3 discuss demand and supply curves. These can be thought of as the graphs of a function where quantity Q depends on price P, or equivalently of the inverse function where price P depends on quantity Q.

In all the examples examined so far, the inverse could be expressed in terms of known formulas. It turns out that even if a function has an inverse, it may be impossible to express it in terms of a function we know. *Inverse functions are actually an important source of new functions*. A typical case arises in connection with the exponential function. In Section 4.9 we showed that $y = e^x$ is strictly increasing and that it tends to 0 as x tends to $-\infty$ and to ∞ as x tends to ∞. For each positive y there exists a uniquely determined x such that $e^x = y$. In Section 4.10 we called the new function the natural logarithm function, ln, and we have the equivalence $y = e^x \Leftrightarrow x = \ln y$. The *functions $f(x) = e^x$ and $g(y) = \ln y$ are therefore inverses of each other*. Because the ln function appears in so many connections, it is tabulated, and moreover represented by a separate key on many calculators.

If a calculator has a certain function f represented by a key, then it will usually have another which represents its inverse function f^{-1}. If, for example, it has an $\boxed{e^x}$-key, it also has an $\boxed{\ln x}$-key. Since $f^{-1}(f(x)) = x$, if we enter a number x, press the $\boxed{f}$-key and then press the $\boxed{f^{-1}}$-key, then we should get x back again. Try entering 5, then using the $\boxed{e^x}$-key followed by the $\boxed{\ln x}$-key. You should then get 5 back again.[4]

If f and g are inverses of each other, the domain of f is equal to the range of g, and vice versa. Consider the following examples.

EXAMPLE 5.3.2 The function $f(x) = \sqrt{3x+9}$, defined in the interval $[-3, \infty)$, *is* strictly increasing and hence has an inverse. Find a formula for the inverse. Use x as the free variable for both functions.

Solution: When x increases from -3 to ∞, $f(x)$ increases from 0 to ∞, so the range of f is $[0, \infty)$. Hence f has an inverse g defined on $[0, \infty)$. To find a formula for the inverse, we solve the equation $y = \sqrt{3x+9}$ for x. Squaring gives $y^2 = 3x + 9$, which solved for x gives $x = \frac{1}{3}y^2 - 3$. Interchanging x and y in this expression to make x the free variable, we find that the inverse function of f is $y = g(x) = \frac{1}{3}x^2 - 3$, defined on $[0, \infty)$. The graphs of the two functions $f(x)$ and $gf(x)$ are shown in Fig. 5.3.5.

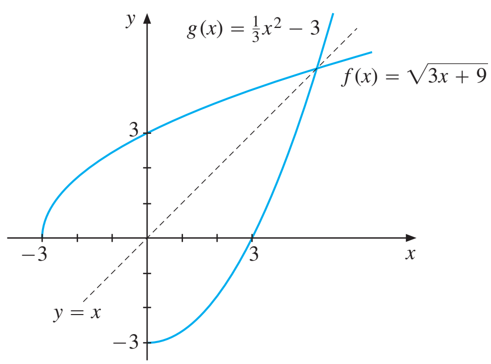


Figure 5.3.5 $f(x) = \sqrt{3x+9}$ and $g(x) = \frac{1}{3}x^2 - 3$

EXAMPLE 5.3.3 Consider the function f defined by the formula $f(x) = 4\ln(\sqrt{x+4} - 2)$.

(a) For which values of x is $f(x)$ defined? Determine the range of f.

(b) Find a formula for its inverse. Use x as the free variable.

Solution:

(a) In order for $\sqrt{x+4}$ to be defined, x must be ≥ -4. But we also have to make sure that $\sqrt{x+4} - 2 > 0$, otherwise the logarithm is not defined. Now, $\sqrt{x+4} - 2 > 0$ means that $\sqrt{x+4} > 2$, or $x + 4 > 4$, that is, $x > 0$. The domain of f is therefore $(0, \infty)$. As x varies from near 0 to ∞, $f(x)$ increases from $-\infty$ to ∞. The range of f is therefore $(-\infty, \infty)$.

(b) If $y = 4\ln(\sqrt{x+4} - 2)$, then $\ln(\sqrt{x+4} - 2) = y/4$, so that $\sqrt{x+4} - 2 = e^{y/4}$ and then $\sqrt{x+4} = 2 + e^{y/4}$. By squaring each side we obtain $x + 4 = (2 + e^{y/4})^2 = 4 + 4e^{y/4} + e^{y/2}$, so that $x = 4e^{y/4} + e^{y/2}$. The inverse function, with x as the free variable, is therefore $y = e^{x/2} + 4e^{x/4}$. It is defined in $(-\infty, \infty)$ with range $(0, \infty)$.

[4] One reason why you might not get exactly 5 is a rounding error.

EXERCISES FOR SECTION 5.3

1. Demand D as a function of price P is given by $D = \frac{32}{5} - \frac{3}{10}P$. Solve the equation for P and find the inverse function.

2. The demand D for sugar in the USA in the period 1915–1929, as a function of the price P, was estimated to be $D = f(P) = 157.8/P^{0.3}$. Solve the equation for P and so find the inverse of f.

3. Find the domains, ranges, and inverses of the functions given by the following formulas:

   (a) $y = -3x$ (b) $y = 1/x$ (c) $y = x^3$ (d) $y = \sqrt{\sqrt{x} - 2}$

SM 4. The function f is defined by the following table:

x	-4	-3	-2	-1	0	1	2
$f(x)$	-4	-2	0	2	4	6	8

   (a) Denote the inverse of f by f^{-1}. What is its domain? What is the value of $f^{-1}(2)$?

   (b) Find a formula for a function $f(x)$, defined for all real x, which agrees with this table. What is the formula for its inverse?

5. Why does $f(x) = x^2$, for x in $(-\infty, \infty)$, have no inverse function? Show that f restricted to $[0, \infty)$ has an inverse, and find that inverse.

6. Formalize the following statements:

   (a) Halving and doubling are inverse operations.

   (b) The operation of multiplying a number by 3 and then subtracting 2 is the inverse of the operation of adding 2 to the number and then dividing by 3.

   (c) The operation of subtracting 32 from a number and then multiplying the result by 5/9 is the inverse of the operation of multiplying a number by 9/5 and then adding 32. "Fahrenheit to Celsius, and Celsius to Fahrenheit".[5]

7. If f is the function that tells you how many kilograms of carrots you can buy for a specified amount of money, then what does f^{-1} tell you?

8. On a coordinate system in the plane:

   (a) Show that points $(3, 1)$ and $(1, 3)$ are symmetric about the line $y = x$, and the same for $(5, 3)$ and $(3, 5)$.

   (b) Use properties of congruent triangles to prove that points (a, b) and (b, a) in the plane are symmetric about the line $y = x$. What is the point half-way between them?

[5] Recall Example 2.6.5 and Review Exercise 4.21.

SM **9.** Find inverses of the following functions, where x is the independent variable:

(a) $f(x) = (x^3 - 1)^{1/3}$ (b) $f(x) = \dfrac{x+1}{x-2}$ (c) $f(x) = (1 - x^3)^{1/5} + 2$

SM **10.** The functions defined by the following formulas are strictly increasing in their domains. Find the domain of each inverse function, and a formula for the corresponding inverse.

(a) $y = e^{x+4}$ (b) $y = \ln x - 4,\ x > 0$ (c) $y = \ln\left(2 + e^{x-3}\right)$

11. [HARDER] Find the inverse of $f(x) = \frac{1}{2}(e^x - e^{-x})$. (*Hint*: Solve a quadratic equation in $z = e^x$.)

5.4 Graphs of Equations

The equations $x\sqrt{y} = 2$, $x^2 + y^2 = 16$, and $y^3 + 3x^2 y = 13$ are three examples of equations in two variables x and y. A *solution* of such an equation is an ordered pair (a, b) such that the equation is satisfied when we replace x by a and y by b. The *solution set* of the equation is the set of all solutions. Representing all pairs in the solution set in a Cartesian coordinate system gives a set called the *graph* of the equation.

EXAMPLE 5.4.1 Find some solutions of each of the equations $x\sqrt{y} = 2$ and $x^2 + y^2 = 16$, and try to sketch their graphs.

Solution: From $x\sqrt{y} = 2$ we obtain $y = 4/x^2$. Hence it is easy to find corresponding values for x and y as given in Table 5.1. The graph is drawn in Fig. 5.4.1, along with the four points in the table.

Table 5.1 Solutions of $x\sqrt{y} = 2$

x	1	2	4	6
y	4	1	1/4	1/9

For $x^2 + y^2 = 16$, if $y = 0$, $x^2 = 16$, so $x = \pm 4$. Thus $(4, 0)$ and $(-4, 0)$ are two solutions. Table 5.2 combines these with some other solutions.

Table 5.2 Solutions of $x^2 + y^2 = 16$

x	-4	-3	-1	0	1	3	4
y	0	$\pm\sqrt{7}$	$\pm\sqrt{15}$	± 4	$\pm\sqrt{15}$	$\pm\sqrt{7}$	0

In Fig. 5.4.2 we have plotted the points given in the table, and the graph seems to be a circle, as will be confirmed in Section 5.5.

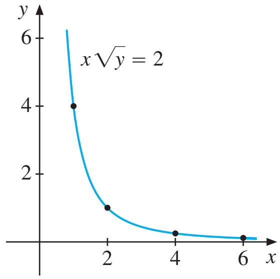


Figure 5.4.1 $x\sqrt{y} = 2$

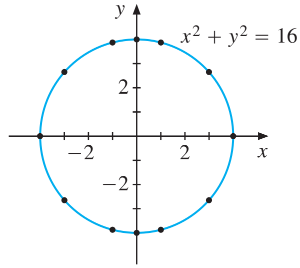


Figure 5.4.2 $x^2 + y^2 = 16$

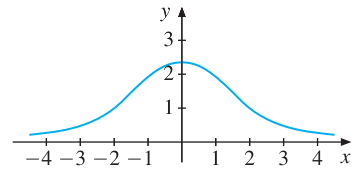


Figure 5.4.3 $y^3 + 3x^2y = 13$

EXAMPLE 5.4.2 What can you say about the graph of the equation $y^3 + 3x^2y = 13$?

Solution: If $x = 0$, then $y^3 = 13$, so that $y = \sqrt[3]{13} \approx 2.35$. Hence $(0, \sqrt[3]{13})$ lies on the graph. Note that if (x_0, y_0) lies on the graph, so does $(-x_0, y_0)$, since x is raised to the second power. Hence the graph is symmetric about the y-axis. You may notice that $(2, 1)$, and hence $(-2, 1)$, are solutions.

If we write the equation in the form

$$y = \frac{13}{y^2 + 3x^2} \tag{$*$}$$

we see that no point (x, y) on the graph can have $y \leq 0$, so that the whole graph lies above the x-axis. From $(*)$ it also follows that if x is large positive or negative, then y must be small.

Figure 5.4.3 displays the graph, which accords with these findings. It consists of all points (x, y) satisfying $x = \pm\sqrt{(13 - y^3)/3y}$.

Vertical-line Test

Graphs of different functions can have innumerable different shapes. However, not all curves in the plane are graphs of functions. By definition, a function assigns to each point x in the domain only one y-value. *The graph of a function therefore has the property that a vertical line through any point on the x-axis has at most one point of intersection with the graph.* This simple *vertical-line test* is illustrated in Figs 5.4.4 and 5.4.5.

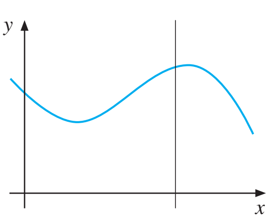


Figure 5.4.4 A function

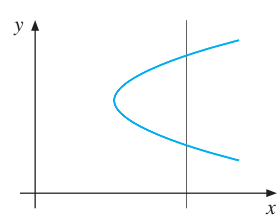


Figure 5.4.5 *Not* a function

The graph of the circle $x^2 + y^2 = 16$, shown in Fig. 5.4.2, is a typical example of a graph that does *not* represent a function, since it does not pass the vertical-line test. A vertical line

$x = a$ for any a with $-4 < a < 4$ intersects the circle at *two* points. Solving the equation $x^2 + y^2 = 16$ for y, we obtain $y = \pm\sqrt{16 - x^2}$. Note that the upper semicircle alone is the graph of the function $y = \sqrt{16 - x^2}$, and the lower semicircle is the graph of the function $y = -\sqrt{16 - x^2}$. Both these functions are defined on the interval $[-4, 4]$.

Choosing Units

A function of one variable is a rule assigning numbers in its range to numbers in its domain. When we describe an empirical relationship by means of a function, we must first choose the units of measurement. For instance we might measure time in years, days, or weeks. We might measure money in dollars, yen, or euros. The choice we make will influence the visual impression conveyed by the graph of the function.

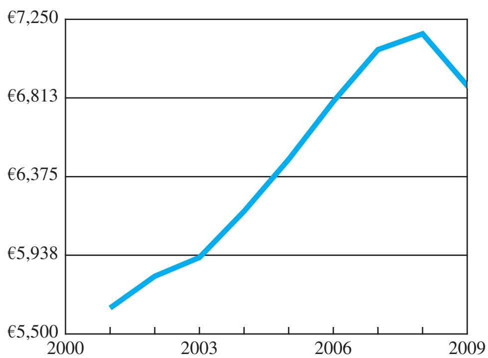


Figure 5.4.6 An optimistic view

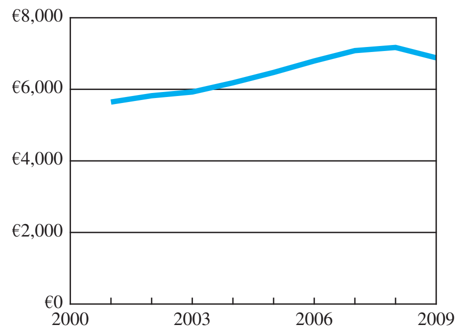


Figure 5.4.7 A pessimistic view

Figures 5.4.6 and 5.4.7 display the time-series of per-capita expenditure during the period 2001–2009 for the 27 countries that belonged to EU in 2009. The data are in *current* euros, meaning that there has been no correction to allow for the effect of inflation. These graphs illustrate a standard trick which is often used to influence people's impressions of empirical relationships. In both diagrams time is measured in years and consumption in billions of euros. They both graph the same function. But if you were trying to impress an audience with the performance of the European economy, which one would you choose?

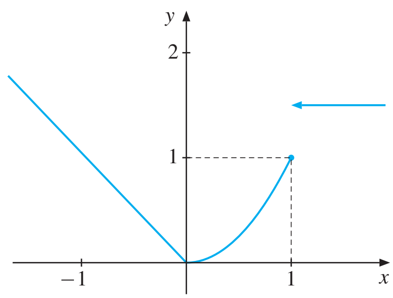
Compound Functions

Sometimes a function is defined in several pieces, by giving a separate formula for each of a number of disjoint parts of the domain. Two examples of such *compound functions* are presented next.

EXAMPLE 5.4.3 Draw the graph of the function f defined by

$$f(x) = \begin{cases} -x & \text{for } x \leq 0 \\ x^2 & \text{for } 0 < x \leq 1 \\ 1.5 & \text{for } x > 1 \end{cases}$$

Solution: The graph is drawn in Fig. 5.4.8. The arrow at $(1, 1.5)$ indicates that this point is not part of the graph of the function. As we shall explain in Section 7.8, the function has a *discontinuity* at $x = 1$. ▮



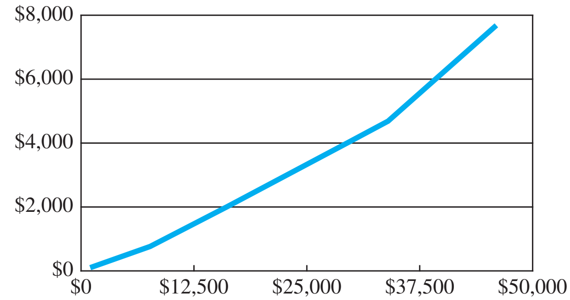


Figure 5.4.8 The function in Example 5.4.3 **Figure 5.4.9** US Federal Income Tax in 2009

EXAMPLE 5.4.4 **(US Federal Income Tax, 2009)** In Fig. 5.4.9 we show a part of the graph of the income tax a single person had to pay, as a function of that person's net income.[6] For income below \$8 350, the tax rate was 10%, so a person with income x this low paid $y = 0.1x$ in taxes. For incomes between \$8 351 and \$33 950, the tax was \$835 plus 15% of the income above \$8 350: a person with income x in this bracket paid $y = 835 + 0.15(x - 8350)$ in taxes. This latter tax rate, 15%, is known as *marginal rate* for incomes in the bracket from \$8 351 to \$33 950. The marginal tax rates for higher income brackets are higher, which explains the fact that the graph becomes steeper as we move to the right. For instance, for incomes between \$33 951 and \$82 250, the marginal tax was 25%; it reached a maximum of 35% for incomes above \$372 952, which are not shown in the graph. In public economics, tax functions whose marginal rates increase with the tax-payer's income are known as *progressive*. ▮

EXERCISES FOR SECTION 5.4

SM 1. Find some particular solutions of the following two equations, then sketch their graphs:

(a) $x^2 + 2y^2 = 6$ (b) $y^2 - x^2 = 1$

2. Try to sketch the graph of $\sqrt{x} + \sqrt{y} = 5$ by finding some particular solutions.

3. The function F is defined for all $r \geq 0$ by the following formulas:

$$F(r) = \begin{cases} 0 & \text{for } r \leq 7500 \\ 0.044(r - 7500) & \text{for } r > 7500 \end{cases}$$

Compute $F(100\,000)$, and sketch the graph of F.

[6] Of course, Fig. 5.4.9 is an idealization. The true income tax function is defined only for integer numbers of dollars — or, more precisely, it is a discontinuous "step function" which jumps up slightly whenever income rises by another dollar.

5.5 Distance in the Plane

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the xy-plane, as shown in Fig. 5.5.1. By Pythagoras's theorem, stated in the appendix, the distance d between P_1 and P_2 satisfies the equation $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. This gives the following important formula:

DISTANCE FORMULA

The distance between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{5.5.1}$$

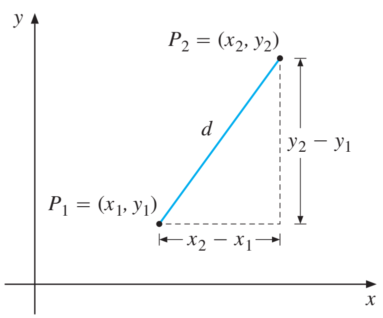


Figure 5.5.1 Distance between two points

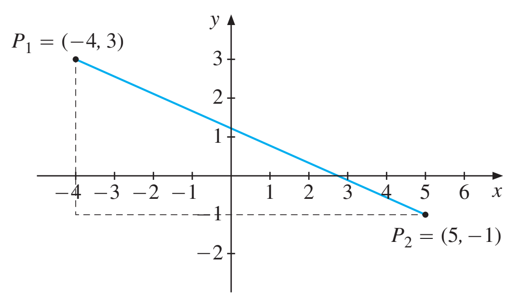


Figure 5.5.2 Between $(-4, 3)$ and $(5, -1)$

We considered two points in the first quadrant to prove the distance formula. It turns out that the same formula is valid irrespective of where the two points P_1 and P_2 lie. Note also that since $(x_1 - x_2)^2 = (x_2 - x_1)^2$ and $(y_1 - y_2)^2 = (y_2 - y_1)^2$, it makes no difference which point is P_1 and which is P_2.

Some find formula (5.5.1) hard to grasp. In words it tells us that we can find the distance between two points in the plane as follows: *Take the difference between the x-coordinates and square what you get. Do the same with the y-coordinates. Add the results and then take the square root.*

EXAMPLE 5.5.1 Find the distance d between $P_1 = (-4, 3)$ and $P_2 = (5, -1)$.

Solution: See Fig. 5.5.2 for an illustration. Using (5.5.1) with $x_1 = -4$, $y_1 = 3$ and $x_2 = 5$, $y_2 = -1$, we have

$$d = \sqrt{(5 - (-4))^2 + (-1 - 3)^2} = \sqrt{9^2 + (-4)^2} = \sqrt{81 + 16} = \sqrt{97} \approx 9.85$$

Circles

Let (a, b) be a point in the plane. *The circle with radius r and centre at (a, b) is the set of all points (x, y) whose distance from (a, b) is equal to r.* Applying the distance formula to the typical point (x, y) on the circle shown in Fig. 5.5.3 gives

$$\sqrt{(x-a)^2 + (y-b)^2} = r$$

Squaring each side yields:

EQUATION OF A CIRCLE

The equation of a circle with centre at (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2 \tag{5.5.2}$$

A graph of (5.5.2) is shown in Fig. 5.5.3. Note that if we let $a = b = 0$ and $r = 4$, then (5.5.2) reduces to $x^2 + y^2 = 16$. This is the equation of a circle with centre at $(0, 0)$ and radius 4, as shown in Fig. 5.4.2.

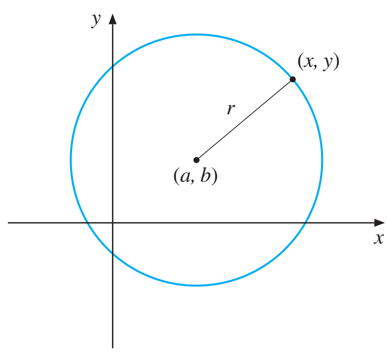


Figure 5.5.3 Circle with centre at (a, b) and radius r

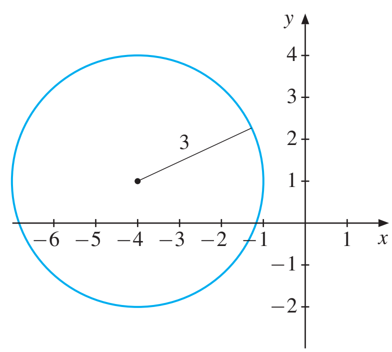


Figure 5.5.4 Circle with centre at $(-4, 1)$ and radius 3

EXAMPLE 5.5.2 Find the equation of the circle with centre at $(-4, 1)$ and radius 3.

Solution: See Fig. 5.5.4. Here $a = -4$, $b = 1$, and $r = 3$. So according to (5.5.2), the equation for the circle is

$$(x+4)^2 + (y-1)^2 = 9 \tag{$*$}$$

Expanding the squares to obtain $x^2 + 8x + 16 + y^2 - 2y + 1 = 9$, and then collecting terms, we have

$$x^2 + y^2 + 8x - 2y + 8 = 0 \tag{$**$}$$

The equation of the circle given in $(**)$ has the disadvantage that we cannot immediately read off its centre and radius. If we are given Eq. $(**)$, however, we can follow the method of "completing the squares" used in Exercise 5 to deduce $(*)$ from $(**)$.

Ellipses and Hyperbolas

All the planets, including the Earth, move around the Sun in orbits that are approximately elliptical. This makes ellipses a very important type of curve in physics and astronomy. Occasionally, ellipses also appear in economics and statistics. The simplest type of ellipse has the equation

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1 \tag{5.5.3}$$

This ellipse has centre at (x_0, y_0) and its graph is shown in Fig. 5.5.5. Note that when $a = b$, the ellipse degenerates into a circle.

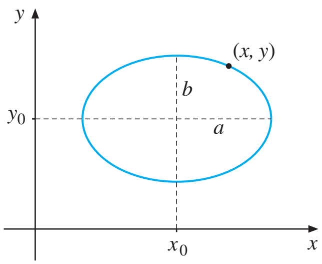


Figure 5.5.5 Ellipse

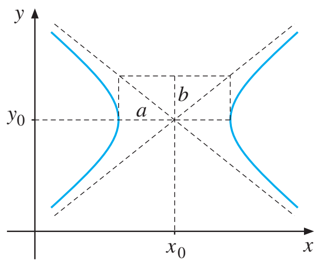


Figure 5.5.6 Hyperbola

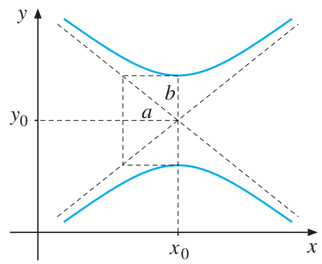


Figure 5.5.7 Hyperbola

Figures 5.5.6 and 5.5.7 show the graphs of the two *hyperbolas*

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=+1 \quad\text{and}\quad \frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=-1 \tag{5.5.4}$$

respectively. These have *asymptotes* which are like 'tangents at infinity', represented by the same two dashed lines in both figures. Their equations are $y - y_0 = \pm(b/a)(x - x_0)$.

We end this section by considering the graph of the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{5.5.5}$$

where A, B, and C are not all 0. This will have one of the following shapes:

(i) If $4AC > B^2$, either an ellipse (possibly a circle), or a single point, or empty.

(ii) If $4AC = B^2$, either a parabola, or one line or two parallel lines, or empty.

(iii) If $4AC < B^2$, either a hyperbola, or two intersecting lines.

EXERCISES FOR SECTION 5.5

1. Determine the distances between the following pairs of points:

(a) $(1, 3)$ and $(2, 4)$ (b) $(-1, 2)$ and $(-3, 3)$ (c) $(3/2, -2)$ and $(-5, 1)$

(d) (x, y) and $(2x, y + 3)$ (e) (a, b) and $(-a, b)$ (f) $(a, 3)$ and $(2 + a, 5)$

2. The distance between $(2, 4)$ and $(5, y)$ is $\sqrt{13}$. Find y, and explain geometrically why there must be two values of y.

3. Find the distances between each pair of points:

(a) $(3.998, 2.114)$ and $(1.130, -2.416)$; (b) $(\pi, 2\pi)$ and $(-\pi, 1)$.

4. Find the equations of: (a) The circle with centre at $(2, 3)$ and radius 4. (b) The circle with centre at $(2, 5)$ and one point at $(-1, 3)$.

5. To show that the graph of $x^2 + y^2 - 10x + 14y + 58 = 0$ is a circle, we can argue like this: First rearrange the equation to read $(x^2 - 10x) + (y^2 + 14y) = -58$. Completing the two squares gives: $(x^2 - 10x + 5^2) + (y^2 + 14y + 7^2) = -58 + 5^2 + 7^2 = 16$. Thus the equation becomes

$$(x - 5)^2 + (y + 7)^2 = 16$$

whose graph is a circle with centre $(5, -7)$ and radius $\sqrt{16} = 4$. Use this method to find the centre and the radius of the two circles with equations:

(a) $x^2 + y^2 + 10x - 6y + 30 = 0$ (b) $3x^2 + 3y^2 + 18x - 24y = -39$

6. Prove that if the distance from a point (x, y) to the point $(-2, 0)$ is twice the distance from (x, y) to $(4, 0)$, then (x, y) must lie on the circle with centre $(6, 0)$ and radius 4.

7. In Example 4.7.7 we considered the function $y = (ax + b)/(cx + d)$, and we claimed that for $c \neq 0$ the graph was a hyperbola. See how this accords with the classification (i) to (iii) given after Eq. (5.5.5).

SM 8. [HARDER] Consider the equation $x^2 + y^2 + Ax + By + C = 0$, where A, B, and C are constants. Show that its graph is a circle if $A^2 + B^2 > 4C$. Use the method of Exercise 5 to find its centre and radius. What happens if $A^2 + B^2 \leq 4C$?

5.6 General Functions

So far we have studied functions of one variable. These are functions whose domain is a set of real numbers, and whose range is also a set of real numbers. Yet a realistic description of many economic phenomena requires considering a large number of variables simultaneously. For example, the demand for a good like butter is a function of several variables such as the price of the good, the prices of complements like bread, substitutes like olive oil or margarine, as well as consumers' incomes, their doctors' advice, and so on.

Actually, you have probably already seen many special functions of several variables. For instance, the formula $V = \pi r^2 h$ for the volume V of a cylinder with base radius r and height h involves a function of two variables.[7] A change in one of these variables will not affect the value of the other variable. For each pair of positive numbers (r, h), there is a definite value for the volume V. To emphasize that V depends on the values of both r and h, we write

$$V(r, h) = \pi r^2 h$$

[7] Of course, in this case π denotes the mathematical constant that satisfies $\pi \approx 3.14159$.

For $r = 2$ and $h = 3$, we obtain $V(2, 3) = 12\pi$, whereas $r = 3$ and $h = 2$ give $V(3, 2) = 18\pi$. Also, $r = 1$ and $h = 1/\pi$ give $V(1, 1/\pi) = 1$. Note in particular that $V(2, 3) \neq V(3, 2)$.

In some abstract economic models, it may be enough to know that there is some functional relationship between variables, without specifying the dependence more closely. For instance, suppose a market sells three commodities whose prices per unit are respectively p, q, and r. Then economists generally assume that the demand for one of the commodities by an individual with income m is given by a function $f(p, q, r, m)$ of four variables, without necessarily specifying the precise form of that function.

An extensive discussion of functions of several variables begins in Chapter 11. This section introduces an even more general type of function. In fact, general functions of the kind presented here are of fundamental importance in practically every area of pure and applied mathematics, including mathematics applied to economics. Here is the general definition:

FUNCTION

> A *function* is a rule which to each element in a set A associates one, and only one, element in a set B.

The following example indicates how very wide is the concept of a function.

EXAMPLE 5.6.1

(a) The function that assigns to each triangle in a plane the area of that triangle, measured, say, in cm^2.

(b) The function that determines the social security number, or other identification number, of each taxpayer.

(c) The function that for each point P in a horizontal plane determines the point lying three units above P.

(d) Let A be the set of possible actions that a person can choose in a certain situation. Suppose that every action a in A produces a certain result (say, a certain profit) $\varphi(a)$. In this way, we have defined a function φ with domain A.

If we denote the function by f, the set A is called the *domain* of f, and B is called its *target set* or its *codomain*. This generalizes the definitions given in Section 4.2: the two sets A and B need not consist of numbers, but can be sets of arbitrary elements. The definition of a function requires three objects to be specified: (i) a domain, A; (ii) a target set, B; and (iii) a rule that assigns a *unique* element in B to *each* element in A.[8]

An important requirement in the definition of a function is that to each element in the domain A, there corresponds a *unique* element in the target B. While it is meaningful to talk

[8] Nevertheless, in many cases, we refrain from specifying the sets A and/or B explicitly when it is obvious from the context what these sets are.

about the function that assigns the natural mother to every child, the rule that assigns the aunt to any child does not, in general, define a function, because many children have more than one aunt. To test your understanding of these ideas, explain why the following rule, as opposed to the one in Example 5.6.1(c), does not define a function: "to a point P in the plane assign a point that lies three units from P".

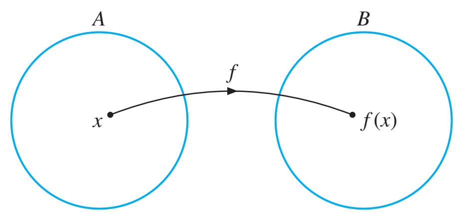


Figure 5.6.1 A function from A to B

If f is a function with domain A and target B, we often say that f is a *function from A to B*, and write $f : A \to B$. The functional relationship is often represented as in Fig. 5.6.1. Other words that are sometimes used instead of "function" include *transformation* and *map* or *mapping*.

The particular value $f(x)$ is often called the *image* of the element x by the function f. The set of elements in B that are images of at least one element in A is called the *range* of the function. Thus, the range is a subset of the target. If we denote the range of f by R_f, then $R_f = \{f(x) : x \in A\}$. This is also written as $f(A)$. The range of the function in Example 5.6.1(a) is the set of all positive numbers. In Example (c), the range is the (whole) horizontal plane that results from shifting the original plane upwards, by three units.

The definition of a function requires that only *one* element in B be assigned to each element in A. However, different elements in A might be mapped to the same element in B. In Example 5.6.1(a), for instance, many different triangles have the same area.

If each element of B is the image of at most one element in A, the function f is called *one-to-one*. Otherwise, if one or more elements of B are the images of more than one element in A, the function f is many-to-one.[9]

The social security function in Example 5.6.1(b) is one-to-one, because two different taxpayers should always have different social security numbers. Can you explain why the function defined in Example 5.6.1(c) is also one-to-one, whereas the function that assigns to each child his or her mother is not?

Inverse Functions

The definition of inverse function in Section 5.3 can easily be extended to general functions. Suppose f is a one-to-one function from a set A to a set B, and assume that the range of f is all of B. We can then define a function g from B to A by the following obvious rule: Assign to each element v of B the one and only element $u = g(v)$ of A that f maps to v—that is,

[9] If a relation is one-to-many, it is not even a function.

the u satisfying $v = f(u)$. Because f is one-to-one, there can be only one u in A such that $v = f(u)$, so g is a function, its domain is B and its target and range are both equal to A. The function g is called the *inverse function* of f. For instance, the inverse of the social security function mentioned in Example 5.6.1(b) is the function that, to each social security number in its range, assigns the person carrying that number.

EXERCISES FOR SECTION 5.6

SM **1.** Which of the following rules define functions?

(a) The rule that assigns to each person in a classroom his or her height.

(b) The rule that assigns to each mother her youngest child alive today.

(c) The rule that assigns the perimeter of a rectangle to its area.

(d) The rule that assigns the surface area of a spherical ball to its volume.

(e) The rule that assigns the pair of numbers $(x + 3, y)$ to the pair of numbers (x, y).

2. Determine which of the functions defined in Exercise 1 are one-to-one, and which then have an inverse. Determine each inverse when it exists.

REVIEW EXERCISES

1. Use Figs 4.3.5 to 4.3.10 and the rules for shifting graphs to sketch the graphs of the following functions:

(a) $y = |x| + 1$ (b) $y = |x + 3|$ (c) $y = 3 - |x + 1|$

2. If $f(x) = x^3 - 2$ and $g(x) = (1 - x)x^2$, compute the six functions:

$$(f + g)(x); \quad (f - g)(x); \quad (fg)(x); \quad (f/g)(x); \quad f(g(1)); \quad \text{and} \quad g(f(1)).$$

3. Consider the demand and supply curves $D = 150 - \frac{1}{2}P$ and $S = 20 + 2P$.

(a) Find the equilibrium price P^*, and the corresponding quantity Q^*.

(b) Suppose a tax of \$2 per unit is imposed on the producer's output. How will this influence the equilibrium price?

(c) Compute the total revenue obtained by the producer before the tax is imposed (R^*) and after ($\hat{R}$).

4. As a function of the price P per unit, the demand quantity D is given by $D = \dfrac{32}{5} - \dfrac{3}{10}P$. Solve the equation for P and find the inverse demand function.[10]

5. As a function of the price P per unit, the demand quantity D for a product is given by $D = 120 - 5P$. Solve the equation for P and so find the inverse demand function.

---

[10] See Exercise 4.2.7 for an economic interpretation.

6. Find the inverses of the functions given by the formulas:

(a) $y = 100 - 2x$ (b) $y = 2x^5$ (c) $y = 5e^{3x-2}$

SM 7. The following functions are strictly increasing in their domains. Find the domains of their inverses and formulas for the inverses, using x as the free variable.

(a) $f(x) = 3 + \ln(e^x - 2)$, for $x > \ln 2$

(b) $f(x) = \dfrac{a}{e^{-\lambda x} + a}$, where a and λ are positive, for $x \in (-\infty, \infty)$

8. Determine the distances between the following pairs of points:

(a) $(2, 3)$ and $(5, 5)$ (b) $(-4, 4)$ and $(-3, 8)$ (c) $(2a, 3b)$ and $(2 - a, 3b)$

9. Find the equations of the circles with:

(a) centre at $(2, -3)$ and radius 5 (b) centre at $(-2, 2)$ and passing through $(-10, 1)$

10. A point P moves in the plane so that it is always equidistant between the two points $A = (3, 2)$ and $B = (5, -4)$. Find a simple equation that the coordinates (x, y) of P must satisfy. (*Hint:* Compute the square of the distance from P to the points A and B, respectively.)

11. Each person in a team is known to have red blood cells that belong to one and only one of four blood groups denoted A, B, AB, and O. Consider the function that assigns each person in the team to his or her blood group. Can this function be one-to-one if the team consists of five people?

6

DIFFERENTIATION

To think of it [differential calculus] merely as a more advanced technique is to miss its real content. In it, mathematics becomes a dynamic mode of thought, and that is a major mental step in the ascent of man.

—Jacob Bronowski (1973)

An important topic in many scientific disciplines, including economics, is the study of how quickly quantities change over time. In order to compute the future position of a planet, to predict the growth in population of a biological species, or to estimate the future demand for a commodity, we need information about rates of change.

The concept used to describe the rate of change of a function is the derivative, which is *the* central concept in mathematical analysis. This chapter defines the derivative of a function and presents some of the important rules for calculating it.

Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) discovered most of these general rules independently of each other. This initiated differential calculus, which has been the foundation for the development of modern science. It has also been of central importance to the theoretical development of modern economics.

6.1 Slopes of Curves

We begin this chapter with a geometrical motivation for the concept. When we study the graph of a function, we would like to have a precise measure of its steepness at a point. We know that for the line $y = px + q$, the number p denotes its slope. If p is large and positive, then the line rises steeply from left to right; if p is large and negative, the line falls steeply. But for an arbitrary function f, what is the steepness of its graph? A natural answer is to define the steepness or *slope* of a curve *at a particular point* as the slope of the tangent to the curve at that point—that is, as the slope of the straight line which just touches the curve at that point. For the curve in Fig. 6.1.1 the steepness at point P is seen to be $1/2$, because the slope of the tangent line L is 1/2.

In Fig. 6.1.1, point P has coordinates $(a, f(a))$. The slope of the tangent to the graph at P is called the *derivative* of $f(x)$ at $x = a$, and we denote this number by $f'(a)$, read as "*f prime a*". In Fig. 6.1.1, we have $f'(a) = 1/2$. In general,

$$f'(a) \text{ is the slope of the tangent to the curve } y = f(x) \text{ at the point } (a, f(a))$$

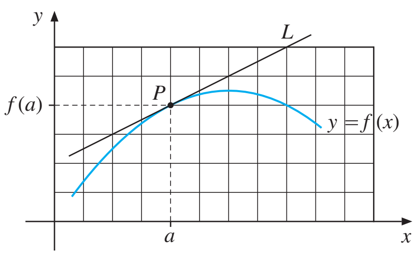


Figure 6.1.1 $f'(a) = 1/2$

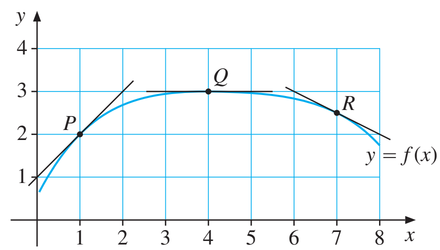


Figure 6.1.2 Example 6.1.1

EXAMPLE 6.1.1 Find $f'(1)$, $f'(4)$, and $f'(7)$ for the function whose graph is shown in Fig. 6.1.2.

Solution: At the point $P = (1, 2)$, the tangent goes through the point $(0, 1)$, and so has slope 1. At the point $Q = (4, 3)$ the tangent is horizontal, and so has slope 0. At the point $R = (7, 2\frac{1}{2})$, the tangent goes through $(8, 2)$, and so has slope $-1/2$. Hence, $f'(1) = 1$, $f'(4) = 0$, and $f'(7) = -1/2$.

EXERCISES FOR SECTION 6.1

1. Figure 6.1.3 shows the graph of a function f. Find the values of $f(3)$ and $f'(3)$.

2. Figure 6.1.4 shows the graph of a function g. Find the values of $g(5)$ and $g'(5)$.

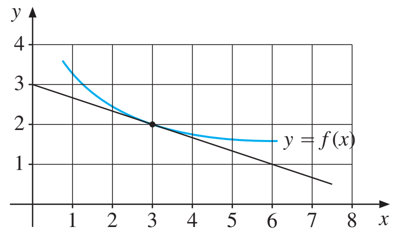


Figure 6.1.3 Exercise 1

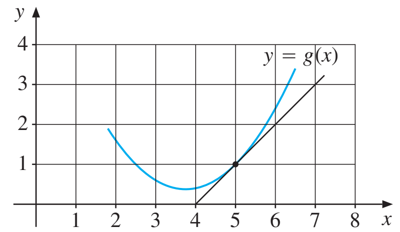


Figure 6.1.4 Exercise 2

6.2 Tangents and Derivatives

The previous section gave a rather vague definition of the tangent to a curve at a point. All we said is that it is a straight line which just touches the curve at that point. We now give a more formal definition of the same concept.

The geometric idea behind the definition is easy to understand. Consider a point P on a curve in the xy-plane — see Fig. 6.2.1. Take another point Q on the curve. The entire straight line through P and Q is called a *secant*. If we keep P fixed, but let Q move along the curve toward P, then the secant will rotate around P, as indicated in Fig. 6.2.2. The limiting straight line PT toward which the secant tends is called the *tangent (line)* to the curve at P. Suppose that the curve in Figs 6.2.1 and 6.2.2 is the graph of a function f. The approach illustrated in Fig. 6.2.2 allows us to find the slope of the tangent PT to the graph of f at the point P.

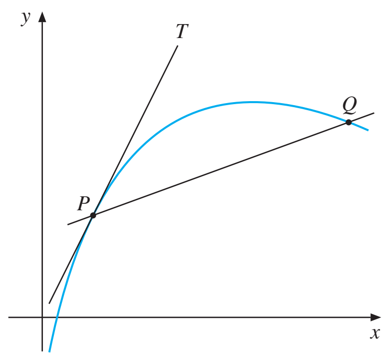


Figure 6.2.1 A secant

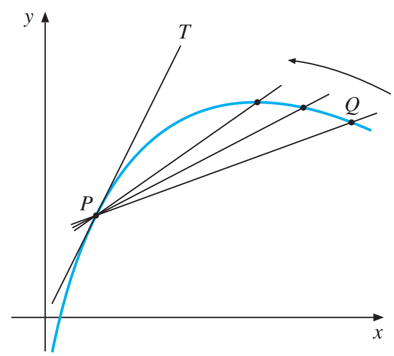


Figure 6.2.2 Secants and the tangent

Figure 6.2.3 reproduces the curve, the points P and Q, and the tangent PT in Fig. 6.2.2. Point P in Fig. 6.2.3 has coordinates $(a, f(a))$. Point Q lies close to P and is also on the graph of f. Suppose that the x-coordinate of Q is $a + h$, where h is a small number different from 0. Then the x-coordinate of Q is not a (because $Q \neq P$), but a number close to a. Because Q lies on the graph of f, the y-coordinate of Q is equal to $f(a + h)$. Hence, the point Q has coordinates $(a + h, f(a + h))$.

The slope of the secant PQ is, therefore,

$$\frac{f(a+h) - f(a)}{h} \tag{6.2.1}$$

This fraction is called a *Newton quotient* of f. Note that when $h = 0$, the fraction becomes 0/0 and so is undefined. But choosing $h = 0$ corresponds to letting $Q = P$. When Q moves toward P along the graph of f, the x-coordinate of Q, which is $a + h$, must tend to a, and so h tends to 0. Simultaneously, the secant PQ tends to the tangent to the graph at P. This suggests that we ought to *define* the slope of the tangent at P as the number that the slope of the secant approaches as h tends to 0. In the previous section we called this number $f'(a)$, so we propose the following definition of $f'(a)$: *$f'(a)$ is the limit, as h tends to 0, of the Newton quotient.*

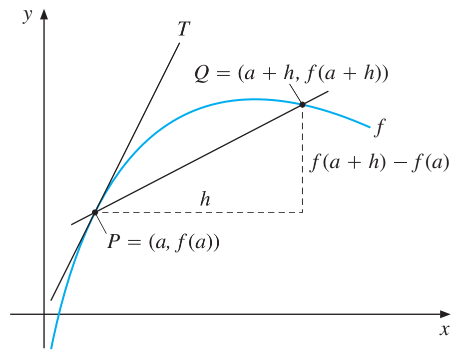


Figure 6.2.3 Newton quotient

It is common to use the abbreviated notation $\lim_{h\to 0}$ for "the limit as h tends to zero" of an expression involving h. We therefore have the following definition:

DEFINITION OF DERIVATIVE

The derivative of function f at point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{6.2.2}$$

The number $f'(a)$ gives the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$. The equation for a straight line passing through (x_1, y_1) and having a slope b is given by $y - y_1 = b(x - x_1)$. Hence, we obtain:

DEFINITION OF TANGENT

The equation for the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a) \tag{6.2.3}$$

So far the concept of a limit in the definition of $f'(a)$ is somewhat imprecise. Section 6.5 discusses the concept of limit in more detail. Because it is relatively complicated, we rely on intuition for the time being. Consider a simple example.

EXAMPLE 6.2.1 Use (6.2.2) to compute $f'(a)$ when $f(x) = x^2$. Find in particular $f'(1/2)$ and $f'(-1)$. Give geometric interpretations, and find the equation for the tangent at each of the points $(1/2, 1/4)$ and $(-1, 1)$.

Solution: For $f(x) = x^2$, we have $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2$, and so

$$f(a+h) - f(a) = (a^2 + 2ah + h^2) - a^2 = 2ah + h^2$$

Hence, for all $h \neq 0$, we obtain

$$\frac{f(a+h)-f(a)}{h} = \frac{2ah+h^2}{h} = \frac{h(2a+h)}{h} = 2a+h \tag{$*$}$$

because we can cancel h whenever $h \neq 0$. But as h tends to 0, so $2a+h$ obviously tends to $2a$. Thus, we can write

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}(2a+h) = 2a$$

This shows that when $f(x) = x^2$, then $f'(a) = 2a$. For the special case when $a = 1/2$, we obtain $f'(1/2) = 2 \cdot 1/2 = 1$. Similarly, $f'(-1) = 2 \cdot (-1) = -2$.

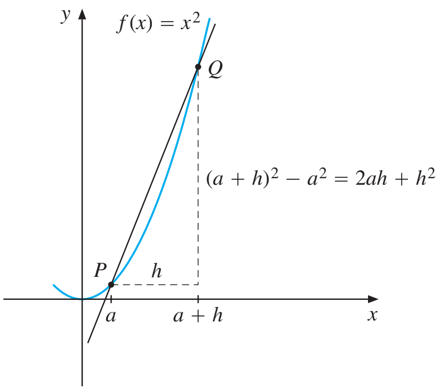


Figure 6.2.4 A secant of $f(x) = x^2$

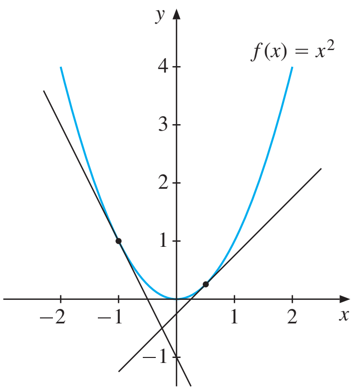


Figure 6.2.5 Tangents of $f(x) = x^2$

Figure 6.2.4 provides a geometric interpretation of ($*$). In Fig. 6.2.5, we have drawn the tangents to the curve $y = x^2$ corresponding to the values $a = 1/2$ and $a = -1$ of x. At $a = 1/2$, we have $f(a) = (1/2)^2 = 1/4$ and $f'(1/2) = 1$. According to (6.2.3), the equation of the tangent is $y - 1/4 = 1 \cdot (x - 1/2)$ or $y = x - 1/4$.[1] Note that the formula $f'(a) = 2a$ shows that $f'(a) < 0$ when $a < 0$, and $f'(a) > 0$ when $a > 0$. Does this agree with the graph?

If f is a relatively simple function, we can find $f'(a)$ by using the following recipe:

COMPUTING THE DERIVATIVE

(i) Add h to a and compute $f(a+h)$.

(ii) Compute the corresponding change in the function value: $f(a+h) - f(a)$.

(iii) For $h \neq 0$, form the Newton quotient (6.2.1).

(iv) Simplify the fraction in step (iii) as much as possible. Wherever possible, cancel h from the numerator and denominator.

(v) Then $f'(a)$ is the limit, as h tends to 0, of the Newton quotient.

[1] Can you show that the other tangent drawn in Fig. 6.2.5 has the equation $y = -2x - 1$?

Let us apply this recipe to another example.

EXAMPLE 6.2.2 Compute $f'(a)$ when $f(x) = x^3$.

Solution: We follow the recipe step by step:

(i) $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$

(ii) $f(a+h) - f(a) = (a^3 + 3a^2h + 3ah^2 + h^3) - a^3 = 3a^2h + 3ah^2 + h^3$

(iii)–(iv) $\dfrac{f(a+h) - f(a)}{h} = \dfrac{3a^2h + 3ah^2 + h^3}{h} = 3a^2 + 3ah + h^2$

(v) As h tends to 0, so $3ah + h^2$ also tends to 0; hence, the entire expression $3a^2 + 3ah + h^2$ tends to $3a^2$. Therefore, $f'(a) = 3a^2$.

We have thus shown that the graph of the function $f(x) = x^3$ at the point $x = a$ has a tangent with slope $3a^2$. Note that $f'(a) = 3a^2 > 0$ when $a \neq 0$, and $f'(0) = 0$. The tangent points upwards to the right for all $a \neq 0$, and is horizontal at the origin. You should look at the graph of $f(x) = x^3$ in Fig. 4.3.7 to confirm this behaviour.

The recipe works well for simple functions like those in Examples 6.2.1 and 6.2.2. But for more complicated functions such as $f(x) = \sqrt{3x^2 + x + 1}$ it is unnecessarily cumbersome. The powerful rules explained in Section 6.6 allow the derivatives of even quite complicated functions to be found quite easily. Understanding these rules, however, relies on the more precise concept of limits that we will provide in Section 6.5.

On Notation

We showed in Example 6.2.1 that if $f(x) = x^2$, then for every a we have $f'(a) = 2a$. We frequently use x as the symbol for a quantity that can take any value, so we write $f'(x) = 2x$. Using this notation, our results from the two last examples are as follows:

$$f(x) = x^2 \Rightarrow f'(x) = 2x \tag{6.2.4}$$

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \tag{6.2.5}$$

The result in (6.2.4) is a special case of the following rule, which you are asked to show in Exercise 7: given constants a, b and c,

$$f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b \tag{6.2.6}$$

Here are some applications of (6.2.6):

$$f(x) = 3x^2 + 2x + 5 \Rightarrow f'(x) = 2 \cdot 3x + 2 = 6x + 2$$

$$f(x) = -16 + \frac{1}{2}x - \frac{1}{16}x^2 \Rightarrow f'(x) = \frac{1}{2} - \frac{1}{8}x$$

$$f(x) = (x-p)^2 = x^2 - 2px + p^2 \Rightarrow f'(x) = 2x - 2p$$

where p is a constant. If we use y to denote the typical value of the function $y = f(x)$, we often denote the derivative simply by y'. We can then write

$$y = x^3 \Rightarrow y' = 3x^2$$

Several other forms of notation for the derivative are often used in mathematics and its applications. One of them, originally due to Leibniz, is called the *differential notation*. If $y = f(x)$, then in place of $f'(x)$, we write

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \quad \frac{\mathrm{d}f(x)}{\mathrm{d}x} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

For instance, if $y = x^2$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$$

We can think of the symbol "$\mathrm{d}/\mathrm{d}x$" as an instruction to differentiate what follows with respect to x. Differentiation occurs so often in mathematics that it has become standard to use *w.r.t.* as an abbreviation for *with respect to*.[2]

When we use letters other than f, x, and y, the notation for the derivative changes accordingly. For example:

$$P(t) = t^2 \Rightarrow P'(t) = 2t; \;\; Y = K^3 \Rightarrow Y' = 3K^2; \;\; \text{and } A = r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = 2r$$

EXERCISES FOR SECTION 6.2

1. Let $f(x) = 4x^2$. Show that $f(5+h) - f(5) = 40h + 4h^2$, implying that $\dfrac{f(5+h) - f(5)}{h} = 40 + 4h$ for $h \neq 0$. Use this result to find $f'(5)$. Compare the answer with (6.2.6).

2. Let $f(x) = 3x^2 + 2x - 1$.

   (a) Show that $\dfrac{f(x+h) - f(x)}{h} = 6x + 2 + 3h$ for $h \neq 0$, and use this result to find $f'(x)$.

   (b) Find in particular $f'(0)$, $f'(-2)$, and $f'(3)$. Find also the equation of the tangent to the graph at the point $(0, -1)$.

3. The demand function for a commodity with price P is given by the formula $D(P) = a - bP$. Use rule (6.2.6) to find $\mathrm{d}D(P)/\mathrm{d}P$.

4. The cost of producing x units of a commodity is given by the formula $C(x) = p + qx^2$. Use rule (6.2.6) to find $C'(x)$.[3]

5. Show that $[f(x+h) - f(x)]/h = -1/x(x+h)$, and use this to show that

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -\frac{1}{x^2} = -x^{-2}$$

[2] At this point, we will only think of the symbol "$\mathrm{d}y/\mathrm{d}x$" as meaning $f'(x)$ and will not consider how it might relate to $\mathrm{d}y$ divided by $\mathrm{d}x$. Later chapters discuss this notation in greater detail.

[3] In Section 6.4, this is interpreted as the *marginal cost*.

SM **6.** In each case below, find the slope of the tangent to the graph of f at the specified point:

(a) $f(x) = 3x + 2$, at $(0, 2)$ (b) $f(x) = x^2 - 1$, at $(1, 0)$ (c) $f(x) = 2 + 3/x$, at $(3, 3)$

(d) $f(x) = x^3 - 2x$, at $(0, 0)$ (e) $f(x) = x + 1/x$, at $(-1, -2)$ (f) $f(x) = x^4$, at $(1, 1)$

7. Let $f(x) = ax^2 + bx + c$.

(a) Show that $[f(x + h) - f(x)]/h = 2ax + b + ah$. Use this to show that $f'(x) = 2ax + b$.

(b) For what value of x is $f'(x) = 0$? Explain this result in the light of (4.6.3) and (4.6.4).

8. Figure 6.2.6 shows the graph of a function f. Determine the sign of the derivative $f'(x)$ at each of the four points a, b, c, and d.

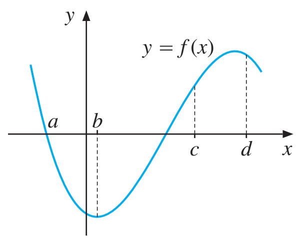


Figure 6.2.6 Exercise 8

SM **9.** Let $f(x) = \sqrt{x}$.

(a) Show that $(\sqrt{x + h} - \sqrt{x})(\sqrt{x + h} + \sqrt{x}) = h$.

(b) Use the result in part (a) to show that the Newton quotient of $f(x)$ is $1/(\sqrt{x + h} + \sqrt{x})$.

(c) Use the result in part (b) to show that for $x > 0$ one has $f'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}x^{-1/2}$.

10. Let $f(x) = ax^3 + bx^2 + cx + d$.

(a) Show that $\dfrac{f(x + h) - f(x)}{h} = 3ax^2 + 2bx + c + 3axh + ah^2 + bh$ for $h \neq 0$, and find $f'(x)$.

(b) Show that the result in part (a) generalizes Example 6.2.2 and Exercise 7.

11. [HARDER] Apply the results of Exercise 8 to prove first that

$$\left[(x + h)^{1/3} - x^{1/3}\right]\left[(x + h)^{2/3} + (x + h)^{1/3}x^{1/3} + x^{2/3}\right] = h$$

Then follow the argument used to solve Exercise 9 to show that $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$.

6.3 Increasing and Decreasing Functions

The terms *increasing* and *decreasing* functions have been used previously to describe the behaviour of a function as we travel from *left to right* along its graph. In order to establish a definite terminology, we introduce the following definitions. We assume that f is defined in an interval I and that x_1 and x_2 are numbers from that interval.

INCREASING AND DECREASING FUNCTIONS

(i) If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, then f is *increasing* in I.
(ii) If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is *strictly increasing* in I.
(iii) If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, then f is *decreasing* in I.
(iv) If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is *strictly decreasing* in I

Figures 6.3.1–6.3.4 illustrate these definitions. Note that we allow an increasing, or decreasing, function to have sections where the graph is horizontal. This does not quite agree with common language: few people would say that their salary increases when it stays constant! For this reason, therefore, sometimes an increasing function is called non-decreasing, and a decreasing function is called nonincreasing.

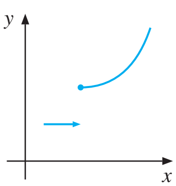


Figure 6.3.1
Increasing

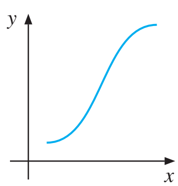


Figure 6.3.2
Strictly increasing

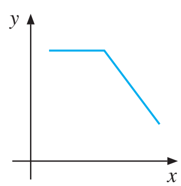


Figure 6.3.3
Decreasing

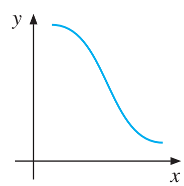


Figure 6.3.4
Strictly decreasing

To find out on which intervals a function is (strictly) increasing or (strictly) decreasing using the definitions, we have to consider the sign of $f(x_2) - f(x_1)$ whenever $x_2 > x_1$. This is usually quite difficult to do directly by checking the values of $f(x)$ at different points x, but we already know a good test of whether a function is increasing or decreasing, in terms of the sign of its derivative:

$$f'(x) \geq 0 \text{ for all } x \text{ in the interval } I \iff f \text{ is increasing in } I \tag{6.3.1}$$

$$f'(x) \leq 0 \text{ for all } x \text{ in the interval } I \iff f \text{ is increasing in } I \tag{6.3.2}$$

Using the fact that the derivative of a function is the slope of the tangent to its graph, the equivalences in (6.3.1) and (6.3.2) seem almost obvious. An observation which is equally correct is the following:

$$f'(x) = 0 \text{ for all } x \text{ in the interval } I \iff f \text{ is constant in } I \tag{6.3.3}$$

A precise proof of (6.3.1)– (6.3.3) relies on the mean-value theorem, which we will study in Section 8.4.

EXAMPLE 6.3.1 Use result (6.2.6) to find the derivative of $f(x) = \frac{1}{2}x^2 - 2$. Then examine where f is increasing/decreasing.

Solution: We find that $f'(x) = x$, which is nonnegative for $x \geq 0$, and nonpositive if $x \leq 0$, and thus $f'(0) = 0$. We conclude that f is increasing in $[0, \infty)$ and decreasing in $(-\infty, 0]$. Draw the graph of f to confirm this.

EXAMPLE 6.3.2 Use the result in Exercise 6.2.10 in order to find the derivative of the cubic function $f(x) = -\frac{1}{3}x^3 + 2x^2 - 3x + 1$. Then examine where f is increasing/decreasing.

Solution: The formula in the exercise can be used with $a = -1/3$, $b = 2$, $c = -3$, and $d = 1$. Thus $f'(x) = -x^2 + 4x - 3$. Solving the equation $f'(x) = -x^2 + 4x - 3 = 0$ yields $x = 1$ and $x = 3$, and thus $f'(x) = -(x-1)(x-3) = (x-1)(3-x)$. A sign diagram for $(x-1)(3-x)$ reveals that $f'(x) = (x-1)(3-x)$ is nonnegative in the interval $[1, 3]$, and nonpositive in $(-\infty, 1]$ and in $[3, \infty)$.[4] We conclude that $f(x)$ is increasing in $[1, 3]$, but decreasing in $(-\infty, 1]$ and in $[3, \infty)$. See Fig. 6.3.5.

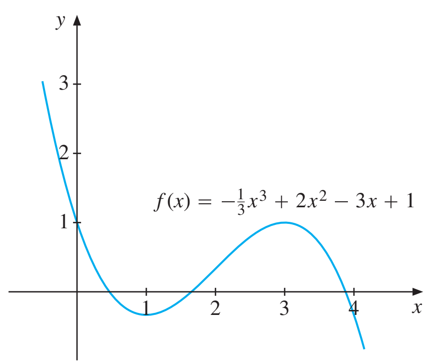


Figure 6.3.5 $f(x) = -\frac{1}{3}x^3 + 2x^2 - 3x + 1$

If $f'(x)$ is strictly positive in an interval, the function should be strictly increasing. Indeed,

$$f'(x) > 0 \text{ for all } x \text{ in the interval } I \implies f \text{ is strictly increasing in } I \tag{6.3.4}$$

$$f'(x) < 0 \text{ for all } x \text{ in the interval } I \implies f \text{ is strictly decreasing in } I \tag{6.3.5}$$

The implications in (6.3.4) and (6.3.4) give sufficient conditions for f to be strictly increasing or decreasing. They cannot be reversed to give necessary conditions. For example, if $f(x) = x^3$, then $f'(0) = 0$. Yet f is strictly increasing — see Exercise 3.[5]

---

[4] See Example 2.6.2.

[5] The following statement is often seen: "Suppose that f is strictly increasing — that is, $f'(x) > 0$." The example $f(x) = x^3$ shows that the statement is wrong. A function can be strictly increasing even though the derivative is 0 at certain points. In fact, suppose that $f'(x) \geq 0$ for all x in I and $f'(x) = 0$ at only a finite number of points in I. Then $f'(x) > 0$ in the subinterval between any two adjacent zeros of $f'(x)$, and so f is strictly increasing on each such subinterval. It follows that f is strictly increasing on the whole interval.

EXERCISES FOR SECTION 6.3

1. Use (6.2.6), (6.3.1), and (6.3.2) to examine where $f(x) = x^2 - 4x + 3$ is increasing/decreasing. Compare with Fig. 4.3.3.

2. Use the result in Exercise 6.2.10 to examine where $f(x) = -x^3 + 4x^2 - x - 6$ is increasing/decreasing. Compare with Fig. 4.7.1.

3. Show algebraically that $f(x) = x^3$ is strictly increasing by studying the sign of

$$x_2^3 - x_1^3 = (x_2 - x_1)(x_1^2 + x_1x_2 + x_2^2) = (x_2 - x_1)\left[\left(x_1 + \frac{1}{2}x_2\right)^2 + \frac{3}{4}x_2^2\right]$$

6.4 Rates of Change

The derivative of a function at a particular point was defined as the slope of the tangent to its graph at that point. Economists interpret the derivative in many important ways, starting with the rate of change of an economic variable.

Suppose that a quantity y is related to a quantity x by $y = f(x)$. If x has the value a, then the value of the function is $f(a)$. Suppose that a is changed to $a + h$. The new value of y is $f(a + h)$, and the change in the value of the function when x is changed from a to $a + h$ is $f(a + h) - f(a)$. The change in y per unit change in x has a particular name; it is called the *average rate of change of f over the interval from a to $a + h$*. It is equal to

$$\frac{f(a+h) - f(a)}{h}$$

Note that this fraction is precisely the Newton quotient of f at a. Taking the limit as h tends to 0 gives the derivative of f at a, which we interpret as follows:

> The *instantaneous rate of change of f at a* is $f'(a)$.

This very important concept appears whenever we study quantities that change. When time is the independent variable, we often use the "dot notation" for differentiation with respect to time. For example, if $x(t) = t^2$, we write $\dot{x}(t) = 2t$.

Sometimes we are interested in studying the proportion $f'(a)/f(a)$. This proportion can be interpreted as follows:

> The *relative rate of change of f at a* is $\dfrac{f'(a)}{f(a)}$.

In economics, such relative rates of change are often seen. Sometimes they are called *proportional rates of change*. They are usually quoted in percentages per unit of time—for example, percentages per year.[6] Often we will describe a variable as increasing at, say, 3% a year if there is a relative rate of change of $3/100 = 0.03$ each year.

EXAMPLE 6.4.1 Let $N(t)$ be the number of individuals in a population of animals at time t. If t increases to $t + h$, then the change in population is equal to $N(t + h) - N(t)$ individuals. Hence,

$$\frac{N(t + h) - N(t)}{h}$$

is *the average rate of change*. Taking the limit as h tends to 0 gives $\dot{N}(t) = \mathrm{d}N/\mathrm{d}t$ for *the rate of change of population at time* t.

In Example 4.5.1, the formula $P = 5.1t + 606$ was used as an (inaccurate) estimate of Europe's population, in millions, at a date which comes t years after 1960. In this case, the rate of change is $\mathrm{d}P/\mathrm{d}t = 5.1$ million per year, the same for all t.

EXAMPLE 6.4.2 Let $K(t)$ be the capital stock in an economy at time t. The rate of change $\dot{K}(t)$ of $K(t)$ is called the *net rate of investment* at time t,[7] and is usually denoted by $I(t)$:

$$\dot{K}(t) = I(t) \tag{6.4.1}$$

EXAMPLE 6.4.3 Consider a firm producing some commodity in a given period, and let $C(x)$ denote its cost of producing x units. The derivative $C'(x)$ at x is called the *marginal cost* at x. According to the definition, it is equal to

$$C'(x) = \lim_{h \to 0} \frac{C(x + h) - C(x)}{h} \tag{6.4.2}$$

When h is small in absolute value, we obtain the approximation

$$C'(x) \approx \frac{C(x + h) - C(x)}{h} \tag{6.4.3}$$

The difference $C(x + h) - C(x)$ is called the *incremental cost* of producing h units of extra output. For h small, a linear approximation to this incremental cost is $hC'(x)$, the product of the marginal cost and the change in output. This is true even when $h < 0$, signifying a decrease in output and, provided that $C'(x) > 0$, a lower cost.

Note that putting $h = 1$ in (6.4.3) makes marginal cost *approximately* equal to

$$C'(x) \approx C(x + 1) - C(x) \tag{6.4.4}$$

[6] Or per annum, for those who think Latin is still a useful language.

[7] This differs from gross investment because some investment is needed to replace depreciated capital.

Marginal cost is then approximately equal to the *incremental cost* $C(x+1) - C(x)$, that is, the *additional cost of producing one more unit than* x. In elementary economics books marginal cost is often defined as the difference $C(x+1) - C(x)$ because more appropriate concepts from differential calculus cannot be used.

In this book, we will sometimes offer comparable economic interpretations that consider the change in a function when a variable x is increased by one unit; it would be more accurate to consider the change in the function per unit increase, for small increases. Here is an example.

EXAMPLE 6.4.4 Let $C(x)$ denote the cost in millions of dollars for removing $x\%$ of the pollution in a lake. Give an economic interpretation of the equality $C'(50) = 3$.

Solution: Because of the linear approximation $C(50+h) - C(50) \approx hC'(50)$, the precise interpretation of $C'(50) = 3$ is that, starting at 50%, for each extra 1% of pollution that is removed, the extra cost is about 3 million dollars. Much less precisely, $C'(50) = 3$ means that it costs about 3 million dollars extra to remove 51% instead of 50% of the pollution.

Following these examples, economists often use the word "marginal" to indicate a derivative. To mention just two of many examples we shall encounter, the *marginal propensity to consume* is the derivative of the consumption function with respect to income; similarly, the *marginal product*, or *productivity*, of labour is the derivative of the production function with respect to labour input.

The concept is so important that it underlies most of our understanding of economics. For example, Adam Smith, seen by many as the founder of the science, struggled to understand why a non-essential commodity such as a diamond could be worth so much more than an essential one, such as water. Using marginal analysis, Carl Menger (1840–1921), Leon Walras (1834–1910) and Stanley Jevons (1835–1882) explained this seeming paradox: if offered a choice between *only* water or *only* diamonds, people would surely choose water, as it is essential; but, given the water and the diamonds a person already owns, they may value one *extra* glass of water less than one *extra* diamond. This fundamental understanding of optimal decisions led to the three economists being considered founders of the "Marginalist" school of economic thought.

EXERCISES FOR SECTION 6.4

1. Let $C(x) = x^2 + 3x + 100$ be the cost function of a firm. Show that when x is changed from 100 to $100 + h$, where $h \neq 0$, the average rate of change per unit of output is

$$\frac{C(100+h) - C(100)}{h} = 203 + h$$

What is the marginal cost $C'(100)$? Then use (6.2.6) to find $C'(x)$ and, in particular, $C'(100)$.

2. If the cost function of a firm is $C(x) = \bar{C} + cx$, give economic interpretations of the parameters c and $\bar{C}$.

3. If the total saving of a country is a function $S(Y)$ of the national product Y, then $S'(Y)$ is called the *marginal propensity to save*, or MPS. Find the MPS for the following functions:

   (a) $S(Y) = \bar{S} + sY$ (b) $S(Y) = 100 + 0.1Y + 0.0002Y^2$

4. Let $T(y)$ denote the income tax a person is liable to pay, as a function of its income y. Then $T'(y)$ is called the *marginal tax rate*. Consider the case when $T(y) = ty$, where t is a constant number in the interval $(0, 1)$. Characterize this tax function by determining its marginal rate.

5. Let $x(t)$ denote the number of barrels of oil left in a well at time t, where time is measured in minutes. What is the interpretation of the equation $x'(0) = -3$?

6. The total cost of producing $x \geq 0$ units of a commodity is $C(x) = x^3 - 90x^2 + 7500x$.

   (a) Use the result in Exercise 6.2.10 to compute the marginal cost function $C'(x)$.

   (b) For what value of x is the marginal cost the least?

7. (a) A firm's profit function is $\pi(Q) = 24Q - Q^2 - 5$. Find the marginal profit, and the value Q^* of Q which maximizes profits.

   (b) A firm's revenue function is $R(Q) = 500Q - \frac{1}{3}Q^3$. Find the marginal revenue.

   (c) For the particular cost function $C(Q) = -Q^3 + 214.2Q^2 - 7900Q + 320\,700$ which was considered in Example 4.7.1, find the marginal cost.

8. Referring to the definition given in Example 6.4.3, compute the marginal cost in the following cases:

   (a) $C(x) = a_1x^2 + b_1x + c_1$ (b) $C(x) = a_1x^3 + b_1$

6.5 A Dash of Limits

In Section 6.2, we defined the derivative of a function based on the concept of a limit. The same concept has many other uses in mathematics, as well as in economic analysis, so now we should take a closer look. Here we give a preliminary definition and formulate some important rules for limits. In Section 7.9, we will discuss the limit concept more closely.

EXAMPLE 6.5.1 Consider the function F defined by the formula

$$F(x) = \frac{e^x - 1}{x}$$

where the number $e \approx 2.718$ is the base for the natural exponential function that was introduced in Section 4.9. Note that if $x = 0$, then $e^0 = 1$, and the fraction collapses to the absurd expression "0/0". Thus, the function F is not defined for $x = 0$. Yet one can still ask what happens to $F(x)$ when x is close to 0. Using a calculator, we find the values shown in Table 6.1.

From the table, it appears that as x gets closer and closer to 0, so the fraction $F(x)$ gets closer and closer to 1. It therefore seems reasonable to assume that $F(x)$ tends to 1 in

Table 6.1 Values of $F(x) = (e^x - 1)/x$ when x is close to 0

x	-1	-0.1	-0.001	-0.0001	0	0.0001	0.001	0.1	1
$F(x)$	0.632	0.956	0.999	1.000	Not defined	1.000	1.001	1.052	1.718

the limit as x tends to 0. Indeed, as we argue later, our definition of e is motivated by the requirement that this limit equal 1. So we write:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \quad \text{or} \quad \frac{e^x - 1}{x} \to 1 \quad \text{as} \quad x \to 0$$

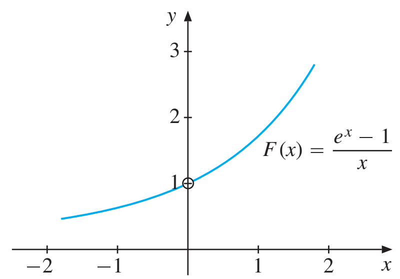


Figure 6.5.1 $y = \dfrac{e^x - 1}{x}$

Figure 6.5.1 shows a portion of the graph of F. The function F is defined for all x, except at $x = 0$, and $\lim_{x\to 0} F(x) = 1$. A small circle is used to indicate that the corresponding point $(0, 1)$ is not in the graph of F.

Suppose, in general, that a function f is defined for all x near a, but not necessarily at $x = a$. *Then we say that the number A is the limit of $f(x)$ as x tends to a if $f(x)$ tends to A as x tends to (but is not equal to) a*. We write:

$$\lim_{x\to a} f(x) = A, \quad \text{or} \quad f(x) \to A \text{ as } x \to a$$

It is possible, however, that the value of $f(x)$ does not tend to any fixed number as x tends to a. Then we say that $\lim_{x\to a} f(x)$ *does not exist*, or that $f(x)$ *does not have a limit as x tends to a*.

EXAMPLE 6.5.2 Using a calculator, examine the limit

$$\lim_{h\to 0} \frac{\sqrt{h+1} - 1}{h}$$

Solution: By choosing numbers h close to 0, we construct Table 6.2.

Table 6.2 Values of $F(h) = (\sqrt{h+1} - 1)/h$ when h is close to 0

h	-0.5	-0.2	-0.1	-0.01	0	0.01	0.1	0.2	0.5
$F(h)$	0.586	0.528	0.513	0.501	Not defined	0.499	0.488	0.477	0.449

This suggests that the desired limit is 0.5.

The limits that we claimed to have found in Examples 6.5.1 and 6.5.2 are both based on a rather shaky numerical procedure. For instance, in Example 6.5.2, how can we really be certain that our guess is correct? Could it be that if we chose h values even closer to 0, the fraction would tend to a limit other than 0.5, or maybe not have any limit at all? Further numerical computations will support our belief that the initial guess is correct, but we can never make a table that has *all* the values of h close to 0, so numerical computation alone can never establish with certainty what the limit is.

This illustrates the need to have a rigorous procedure for finding limits, based on a precise mathematical definition of the limit concept. This precise definition is given in Section 7.9; here we merely give a preliminary definition which will convey the right idea.

LIMIT

The expression

$$\lim_{x \to a} f(x) = A \tag{6.5.1}$$

means that we can make $f(x)$ as close to A as we want for all x sufficiently close to, but not equal to, a.

We emphasize:

(a) The number $\lim_{x \to a} f(x)$ depends on how $f(x)$ behaves for values of x close to a, but not on what happens to f at the precise value $x = a$. Indeed, when finding $\lim_{x \to a} f(x)$, we are simply not interested in the value $f(a)$, or even in whether f is defined at a.

(b) When computing $\lim_{x \to a} f(x)$, we must consider values of x on both sides of a.

Rules for Limits

Since limits cannot really be determined merely by numerical computations, we use simple rules instead. Their validity can be shown later once we have a precise definition of the limit concept. These rules are very straightforward; we have even used a few of them already in the previous section.

Suppose that f and g are defined as functions of x in a neighbourhood of a (but not necessarily at a). Then we have the following rules, written down in a way that makes them easy to refer to later:[8]

[8] Because of the identities $f(x) - g(x) = f(x) + (-1)g(x)$, and $f(x)/g(x) = f(x)(g(x))^{-1}$, it is clear that some of these rules follow from others.

RULES FOR LIMITS

If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then

$$\lim_{x\to a} [f(x) \pm g(x)] = A \pm B \tag{6.5.2}$$

$$\lim_{x\to a} [f(x) \cdot g(x)] = A \cdot B \tag{6.5.3}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}, \quad \text{if } B \neq 0 \tag{6.5.4}$$

$$\lim_{x\to a} [f(x)]^r = A^r, \quad \text{if } A^r \text{ is defined and } r \text{ is a real number} \tag{6.5.5}$$

It is easy to give intuitive explanations for these rules. Suppose that $\lim_{x\to a} f(x) = A$ and that $\lim_{x\to a} g(x) = B$. These equations imply that, when x is close to a, then $f(x)$ is close to A and $g(x)$ is close to B. So intuitively the sum $f(x) + g(x)$ is close to $A + B$, the difference $f(x) - g(x)$ is close to $A - B$, the product $f(x)g(x)$ is close to $A \cdot B$, and so on.

These rules can be used repeatedly to obtain new extended rules such as

$$\lim_{x\to a} \big[f_1(x) + f_2(x) + \cdots + f_n(x)\big] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x) + \cdots + \lim_{x\to a} f_n(x)$$

$$\lim_{x\to a} \big[f_1(x) \cdot f_2(x) \cdots f_n(x)\big] = \lim_{x\to a} f_1(x) \cdot \lim_{x\to a} f_2(x) \cdots \lim_{x\to a} f_n(x)$$

In words: *the limit of a sum is the sum of the limits, and the limit of a product is equal to the product of the limits.*

There are two special cases when the limit is obvious. First, suppose the function $f(x)$ is equal to the same constant value c for every x. Then, at every point a, $\lim_{x\to a} c = c$. Second, it is also evident that if $f(x) = x$, then, again at every point a, $\lim_{x\to a} f(x) = \lim_{x\to a} x = a$. Combining these two simple limits with the general rules allows easy computation of the limits for certain combinations of functions.

EXAMPLE 6.5.3 Use the rules specified in (6.5.2) to (6.5.5) to compute the following limits:

(a) $\lim_{x\to -2} (x^2 + 5x)$ (b) $\lim_{x\to 4} \dfrac{2x^{3/2} - \sqrt{x}}{x^2 - 15}$ (c) $\lim_{x\to a} Ax^n$

Solution:

(a) By the first rule, $\lim_{x\to -2}(x^2 + 5x)$ equals $\lim_{x\to -2}(x \cdot x) + \lim_{x\to -2}(5 \cdot x)$. Using the second rule twice, the latter can be computed as

$$\lim_{x\to -2} x \cdot \lim_{x\to -2} x + \lim_{x\to -2} 5 \cdot \lim_{x\to -2} x$$

so it follows that

$$\lim_{x\to -2} (x^2 + 5x) = (-2)(-2) + 5 \cdot (-2) = -6$$

(b) $\lim_{x\to 4} \dfrac{2x^{3/2}-\sqrt{x}}{x^2-15} = \dfrac{2\lim_{x\to 4} x^{3/2} - \lim_{x\to 4}\sqrt{x}}{\lim_{x\to 4} x^2 - 15} = \dfrac{2\cdot 4^{3/2}-\sqrt{4}}{4^2-15} = \dfrac{2\cdot 8-2}{16-15} = 14$

(c) $\lim_{x\to a} Ax^n = \lim_{x\to a} A \cdot \lim_{x\to a} x^n = A\cdot \left(\lim_{x\to a} x\right)^n = A\cdot a^n$, where n is a natural number.

This last example was straightforward. Examples 6.5.1 and 6.5.2 were more difficult, as they involved a fraction whose numerator and denominator both tend to 0. A simple observation can sometimes help us find such limits, provided that they exist. Because $\lim_{x\to a} f(x)$ can only depend on the values of f when x is close to, but not equal to a, we have the following:

If the functions f and g are equal for all x close to a, but not necessarily at $x = a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) \tag{6.5.6}$$

whenever either limit exists.

Here are some examples of how this rule works.

EXAMPLE 6.5.4 Compute the limit $\lim_{x\to 2} \dfrac{3x^2+3x-18}{x-2}$.

Solution: We see that both numerator and denominator tend to 0 as x tends to 2. Because the numerator $3x^2+3x-18$ is equal to 0 for $x = 2$, it has $x-2$ as a factor. In fact, $3x^2+3x-18 = 3(x-2)(x+3)$. Hence,

$$f(x) = \frac{3x^2+3x-18}{x-2} = \frac{3(x-2)(x+3)}{x-2}$$

For $x \neq 2$, we can cancel $x-2$ from both numerator and denominator to obtain $3(x+3)$. So the functions $f(x)$ and $g(x) = 3(x+3)$ are equal for all $x \neq 2$. By (6.5.6), it follows that

$$\lim_{x\to 2} \frac{3x^2+3x-18}{x-2} = \lim_{x\to 2} 3(x+3) = 3(2+3) = 15$$

EXAMPLE 6.5.5 Compute the limits: (a) $\lim_{h\to 0} \dfrac{\sqrt{h+1}-1}{h}$; (b) $\lim_{x\to 4} \dfrac{x^2-16}{4\sqrt{x}-8}$.

Solution:

(a) The numerator and the denominator both tend to 0 as h tends to 0. Here we must use a little trick. We multiply both numerator and denominator by $\sqrt{h+1}+1$ to get

$$\frac{\sqrt{h+1}-1}{h} = \frac{\left(\sqrt{h+1}-1\right)\left(\sqrt{h+1}+1\right)}{h\left(\sqrt{h+1}+1\right)} = \frac{h+1-1}{h\left(\sqrt{h+1}+1\right)} = \frac{1}{\sqrt{h+1}+1}$$

for all $h \neq 0$, after cancelling the common factor h. For all $h \neq 0$ (and $h \geq -1$), the given function is equal to $1/(\sqrt{h+1}+1)$, which tends to $1/2$ as h tends to 0. We conclude that the limit of our function is equal to $1/2$, which confirms the result in Example 6.5.2.

(b) We must try to simplify the fraction, because $x = 4$ gives $0/0$. Again we can use a trick to factorize the fraction as follows:

$$\frac{x^2-16}{4\sqrt{x}-8} = \frac{(x+4)(x-4)}{4\left(\sqrt{x}-2\right)} = \frac{(x+4)\left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)}{4\left(\sqrt{x}-2\right)} \tag{$*$}$$

Here we have used the factorization $x-4 = \left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)$, which is correct for $x \geq 0$. In the last fraction of $(*)$, we can cancel $\sqrt{x}-2$ when $\sqrt{x}-2 \neq 0$ — that is, when $x \neq 4$. Using (6.5.6) again gives

$$\lim_{x\to 4}\frac{x^2-16}{4\sqrt{x}-8} = \lim_{x\to 4}\frac{1}{4}(x+4)(\sqrt{x}+2) = \frac{1}{4}(4+4)(\sqrt{4}+2) = 8$$

EXERCISES FOR SECTION 6.5

1. Determine the following by using the rules for limits:

(a) $\lim\limits_{x\to 0}(3+2x^2)$ (b) $\lim\limits_{x\to -1}\dfrac{3+2x}{x-1}$ (c) $\lim\limits_{x\to 2}(2x^2+5)^3$

(d) $\lim\limits_{t\to 8}\left(5t+t^2-\frac{1}{8}t^3\right)$ (e) $\lim\limits_{y\to 0}\dfrac{(y+1)^5-y^5}{y+1}$ (f) $\lim\limits_{z\to -2}\dfrac{1/z+2}{z}$

2. Examine the following limits numerically by using a calculator:

(a) $\lim\limits_{h\to 0}\dfrac{1}{h}(2^h-1)$ (b) $\lim\limits_{h\to 0}\dfrac{1}{h}(3^h-1)$ (c) $\lim\limits_{\lambda\to 0}\dfrac{1}{\lambda}(3^\lambda-2^\lambda)$

3. Consider the limit $\lim\limits_{x\to 1}\dfrac{x^2+7x-8}{x-1}$.

(a) Examine the limit numerically by making a table of values of the fraction when x is close to 1.

(b) Find the limit precisely by using the method in Example 6.5.4.

4. Compute the following limits, where $h \neq 0$ in (f):

(a) $\lim\limits_{x\to 2}(x^2+3x-5)$ (b) $\lim\limits_{y\to -3}\dfrac{1}{y+8}$ (c) $\lim\limits_{x\to 0}\dfrac{x^3-2x-1}{x^5-x^2-1}$

(d) $\lim\limits_{x\to 0}\dfrac{x^3+3x^2-2x}{x}$ (e) $\lim\limits_{h\to 0}\dfrac{(x+h)^3-x^3}{h}$ (f) $\lim\limits_{x\to 0}\dfrac{(x+h)^3-x^3}{h}$

SM **5.** Compute the following limits:

(a) $\lim\limits_{h\to 2}\dfrac{\frac{1}{3}-\frac{2}{3h}}{h-2}$ (b) $\lim\limits_{x\to 0}\dfrac{x^2-1}{x^2}$ (c) $\lim\limits_{t\to 3}\dfrac{\sqrt[3]{32t-96}}{t^2-2t-3}$

(d) $\lim\limits_{h\to 0}\dfrac{\sqrt{h+3}-\sqrt{3}}{h}$ (e) $\lim\limits_{t\to -2}\dfrac{t^2-4}{t^2+10t+16}$ (f) $\lim\limits_{x\to 4}\dfrac{2-\sqrt{x}}{4-x}$

SM **6.** If $f(x) = x^2 + 2x$, compute the following limits:

(a) $\displaystyle\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$ (b) $\displaystyle\lim_{x\to 2} \frac{f(x) - f(1)}{x - 1}$ (c) $\displaystyle\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$

(d) $\displaystyle\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ (e) $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ (f) $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{h}$

7. [HARDER] Compute the following limits, where in part (c) n denotes any natural number:

(a) $\displaystyle\lim_{x\to 2} \frac{x^2 - 2x}{x^3 - 8}$ (b) $\displaystyle\lim_{h\to 0} \frac{\sqrt[3]{27+h} - 3}{h}$ (*Hint*: Put $u = \sqrt[3]{27+h}$.) (c) $\displaystyle\lim_{x\to 1} \frac{x^n - 1}{x - 1}$

6.6 Simple Rules for Differentiation

Recall that the derivative of a function f was defined by the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{$*$}$$

If this limit exists, we say that f is *differentiable* at x. The process of finding the derivative of a function is called *differentiation*. It is useful to think of this as an operation that transforms one function f into a new function f'. The function f' is then defined for the values of x where the limit in $(*)$ exists. If $y = f(x)$, we can use the symbols y' and $\mathrm{d}y/\mathrm{d}x$ as alternatives to $f'(x)$.

In Section 6.2 we used formula $(*)$ to find the derivatives of some simple functions. However, it is difficult and time consuming to apply the definition directly in each separate case. We now embark on a systematic programme to find general rules which ultimately will give mechanical and efficient procedures for finding the derivative of very many differentiable functions specified by a formula, even one that is complicated. We start with some simple rules.

If f is a constant function, then its derivative is 0:

$$f(x) = A \Rightarrow f'(x) = 0 \tag{6.6.1}$$

The result is easy to see geometrically. The graph of $f(x) = A$ is a straight line parallel to the x-axis. The tangent to the graph is the line itself, which has slope 0 at each point—see Fig. 6.6.1. You should now use the definition of $f'(x)$ to get the same answer.

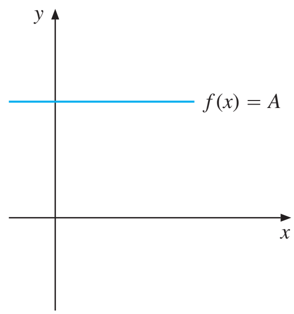


Figure 6.6.1 $f(x) = A$

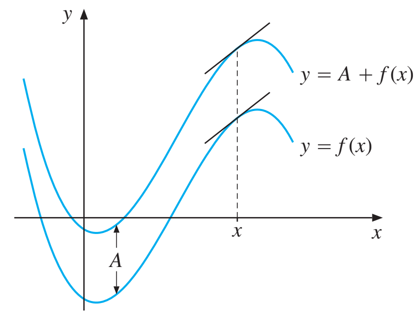


Figure 6.6.2 $f(x)$ and $A + f(x)$

The next two rules are also very useful.

SIMPLE RULES

When taking derivatives, additive constants disappear while multiplicative constants are preserved:

$$y = A + f(x) \Rightarrow y' = f'(x) \tag{6.6.2}$$

$$y = Af(x) \Rightarrow y' = Af'(x) \tag{6.6.3}$$

Rule (6.6.2) is illustrated in Fig. 6.6.2, where A is positive. Recall, from Section 5.1, that the graph of $A + f(x)$ is that of $f(x)$ shifted upwards by A units in the direction of the y-axis. So the tangents to the two curves $y = f(x)$ and $y = f(x) + A$ at the same value of x must be parallel. In particular, they must have the same slope. Again you should try to use the definition of $f'(x)$ to give a formal proof of this assertion.

Let us prove rule (6.6.3) by using the definition of a derivative. If $g(x) = Af(x)$, then

$$g(x + h) - g(x) = Af(x + h) - Af(x) = A\,[f(x + h) - f(x)]$$

and so

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = A \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = Af'(x)$$

For an economic illustration of rule (6.6.3), suppose that $R(t)$ denotes the revenue at time t of firm A, and firm B's revenue $S(t)$ at each time is three times larger than that of A. Then the absolute growth rate of B's revenue is three times larger than that of A. In mathematical notation: $S(t) = 3R(t) \Rightarrow S'(t) = 3R'(t)$. Nevertheless, the firms' *relative* growth rates $R'(t)/R(t)$ and $S'(t)/S(t)$ will be equal.

In Leibniz's notation, the results (6.6.1), (6.6.2), and (6.6.3) are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x} A = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}\,[A + f(x)] = \frac{\mathrm{d}}{\mathrm{d}x} f(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}\,[Af(x)] = A \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

EXAMPLE 6.6.1 Suppose we know $f'(x)$. Find the derivatives of:

(a) $5 + f(x)$ (b) $f(x) - \frac{1}{2}$ (c) $4f(x)$

(d) $-\frac{1}{5}f(x)$ (e) $\dfrac{Af(x)+B}{C}$, where $C \neq 0$

Solution: Applying rules (6.6.2) and (6.6.3), we obtain:

(a) $\dfrac{d}{dx}(5 + f(x)) = f'(x)$

(b) $\dfrac{d}{dx}\left(f(x) - \dfrac{1}{2}\right) = \dfrac{d}{dx}\left(-\dfrac{1}{2} + f(x)\right) = f'(x)$

(c) $\dfrac{d}{dx}(4f(x)) = 4f'(x)$

(d) $\dfrac{d}{dx}\left(-\dfrac{1}{5}f(x)\right) = \dfrac{d}{dx}\left(-\dfrac{1}{5}f(x)\right) = -\dfrac{1}{5}f'(x)$

(e) $\dfrac{d}{dx}\left(\dfrac{Af(x)+B}{C}\right) = \dfrac{d}{dx}\left(\dfrac{A}{C}f(x) + \dfrac{B}{C}\right) = \dfrac{A}{C}f'(x)$

Few rules of differentiation are more useful than the following:

THE POWER RULE

Given any constant a,

$$f(x) = x^a \Rightarrow f'(x) = ax^{a-1} \tag{6.6.4}$$

For $a = 2$ and $a = 3$ this rule was confirmed in Section 6.2. The method used in these two examples can be generalized to the case where a is an arbitrary natural number. Later we shall see that the rule is valid for all real numbers a.

EXAMPLE 6.6.2 Use rule (6.6.4) to compute the derivatives of:

(a) $y = x^5$ (b) $y = 3x^8$ (c) $y = \dfrac{x^{100}}{100}$

Solution:

(a) $y = x^5 \Rightarrow y' = 5x^{5-1} = 5x^4$

(b) $y = 3x^8 \Rightarrow y' = 3 \cdot 8x^{8-1} = 24x^7$

(c) $y = \dfrac{x^{100}}{100} = \dfrac{1}{100}x^{100} \Rightarrow y' = \dfrac{1}{100}100x^{100-1} = x^{99}$

EXAMPLE 6.6.3 Use rule (6.6.4) to compute:

(a) $\dfrac{d}{dx}(x^{-0.33})$ (b) $\dfrac{d}{dr}(-5r^{-3})$ (c) $\dfrac{d}{dp}(Ap^{\alpha} + B)$ (d) $\dfrac{d}{dx}\left(\dfrac{A}{\sqrt{x}}\right)$

Solution:

(a) $\dfrac{d}{dx}\left(x^{-0.33}\right) = -0.33x^{-0.33-1} = -0.33x^{-1.33}$

(b) $\dfrac{d}{dr}(-5r^{-3}) = (-5)(-3)r^{-3-1} = 15r^{-4}$

(c) $\dfrac{d}{dp}(Ap^{\alpha} + B) = A\alpha p^{\alpha-1}$

(d) $\dfrac{d}{dx}\left(\dfrac{A}{\sqrt{x}}\right) = \dfrac{d}{dx}(Ax^{-1/2}) = A\left(-\dfrac{1}{2}\right)x^{-1/2-1} = -\dfrac{1}{2}Ax^{-3/2} = \dfrac{-A}{2x\sqrt{x}}$

EXAMPLE 6.6.4 Let $r > 0$ denote a household's income measured in, say, dollars per year. The *Pareto income distribution* is described by the formula

$$f(r) = \frac{B}{r^{\beta}} = Br^{-\beta} \tag{6.6.5}$$

where B and β are positive constants. As explained more fully in Section 9.4, $f(r)\Delta r$ is approximately the proportion of the population whose income is between r and $r + \Delta r$. The distribution function gives a good approximation for incomes above a certain threshold level. For these, empirical estimates of β have usually been in the interval $2.4 < \beta < 2.6$.

Using (6.6.4), we find that $f'(r) = -\beta Br^{-\beta-1} = -\beta B/r^{\beta+1}$ so $f'(r) < 0$, and $f(r)$ is strictly decreasing.

EXERCISES FOR SECTION 6.6

1. Compute the derivatives of the following functions:

(a) $y = 5$ (b) $y = x^4$ (c) $y = 9x^{10}$ (d) $y = \pi^7$

2. Suppose we know $g'(x)$. Find expressions for the derivatives of the following:

(a) $2g(x) + 3$ (b) $-\frac{1}{6}g(x) + 8$ (c) $\dfrac{g(x) - 5}{3}$

3. Find the derivatives of the following:

(a) x^6 (b) $3x^{11}$ (c) x^{50} (d) $-4x^{-7}$

(e) $\dfrac{x^{12}}{12}$ (f) $\dfrac{-2}{x^2}$ (g) $\dfrac{3}{\sqrt[3]{x}}$ (h) $\dfrac{-2}{x\sqrt{x}}$

4. Compute the following:

(a) $\dfrac{d}{dr}(4\pi r^2)$ (b) $\dfrac{d}{dy}\left(Ay^{b+1}\right)$ (c) $\dfrac{d}{dA}\left(\dfrac{1}{A^2\sqrt{A}}\right)$

5. Explain why $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$. Then use this formula to find $f'(a)$ when $f(x) = x^2$.

6. For each of the following functions, find a function $F(x)$ having $f(x)$ as its derivative—that is, a function that satisfies $F'(x) = f(x)$.[9]

(a) $f(x) = x^2$ (b) $f(x) = 2x + 3$ (c) $f(x) = x^a$, for $a \neq -1$

[9] Note that you are not asked to find $f'(x)$.

7. [HARDER] The following limits all take the form $\lim_{h\to 0}[f(a+h)-f(a)]/h$. Use your knowledge of derivatives to find the limits.

(a) $\lim_{h\to 0}\dfrac{(5+h)^2-5^2}{h}$ (b) $\lim_{s\to 0}\dfrac{(s+1)^5-1}{s}$ (c) $\lim_{h\to 0}\dfrac{5(x+h)^2+10-5x^2-10}{h}$

6.7 Sums, Products, and Quotients

If we know $f'(x)$ and $g'(x)$, then what are the derivatives of the four functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\cdot g(x)$, and $f(x)/g(x)$? You will probably guess the first two correctly, but are less likely to be right about the last two, unless you have already learned the answers.

Sums and Differences

Suppose f and g are both defined on a set A of real numbers.

DERIVATIVES OF SUMS AND DIFFERENCES

If both f and g are differentiable at x, then the sum $f+g$ and the difference $f-g$ are both differentiable at x, with

$$F(x)=f(x)\pm g(x)\Rightarrow F'(x)=f'(x)\pm g'(x) \tag{6.7.1}$$

In Leibniz's notation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)\pm g(x))=\frac{\mathrm{d}}{\mathrm{d}x}f(x)\pm\frac{\mathrm{d}}{\mathrm{d}x}g(x)$$

We can give a formal proof of (6.7.1).

Consider the case when $F(x)=f(x)+g(x)$, The Newton quotient of F is

$$\begin{aligned}\frac{F(x+h)-F(x)}{h}&=\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}\\&=\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\end{aligned}$$

When $h\to 0$, the last two fractions tend to $f'(x)$ and $g'(x)$, respectively, and thus the sum of the fractions tends to $f'(x)+g'(x)$. Hence,

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f'(x)+g'(x)$$

This proves (6.7.1) for the sum. The proof of the result for the difference is similar—only some of the signs change in an obvious way.

EXAMPLE 6.7.1 Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^8 + x^{100}/100\right)$.

Solution: Using (6.7.1) and the results from Example 6.6.2,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^8 + x^{100}/100\right) = \frac{\mathrm{d}}{\mathrm{d}x}(3x^8) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{100}/100\right) = 24x^7 + x^{99}$$

EXAMPLE 6.7.2 In Example 6.4.3, $C(x)$ denoted the cost of producing x units of some commodity in a given period. If $R(x)$ is the revenue from selling those x units, then the profit function $\pi(x) = R(x) - C(x)$ is the difference between the revenue and the cost. According to (6.7.1), $\pi'(x) = R'(x) - C'(x)$. In particular, $\pi'(x) = 0$ when $R'(x) = C'(x)$. In words: *Marginal profit is 0 when marginal revenue is equal to marginal cost.*

Rule (6.7.1) can be extended to sums of an arbitrary number of terms. For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) - g(x) + h(x)) = f'(x) - g'(x) + h'(x)$$

which we see by writing $f(x) - g(x) + h(x)$ as $(f(x) - g(x)) + h(x)$, and then using (6.7.1) twice. Using the rules (6.6.2)–(6.6.4) and (6.7.1) makes it easy to differentiate any polynomial.

Products

Suppose $f(x) = x$ and $g(x) = x^2$, then $(f \cdot g)(x) = x^3$. Here $f'(x) = 1$, $g'(x) = 2x$, and $(f \cdot g)'(x) = 3x^2$. Hence, the derivative of $(f \cdot g)(x)$ is *not* equal to $f'(x) \cdot g'(x) = 2x$. The correct rule for differentiating a product is a little more complicated.

DERIVATIVE OF A PRODUCT

If both f and g are differentiable at the point x, then so is $F = f \cdot g$, and

$$F(x) = f(x) \cdot g(x) \Rightarrow F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \tag{6.7.2}$$

Formulated in words: *The derivative of the product of two functions is equal to the derivative of the first times the second, plus the first times the derivative of the second.* The formula, however, is much easier to digest than these words.

In Leibniz's notation, the product rule is expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \cdot g(x)] = \left[\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right] \cdot g(x) + f(x) \cdot \left[\frac{\mathrm{d}}{\mathrm{d}x}g(x)\right]$$

Before demonstrating why (6.7.2) is valid, here are two examples.

EXAMPLE 6.7.3 Find $h'(x)$ when $h(x) = (x^3 - x) \cdot (5x^4 + x^2)$. Confirm the answer by expanding $h(x)$ as a single polynomial, then differentiating the result.

Solution: We see that $h(x) = f(x) \cdot g(x)$ with $f(x) = x^3 - x$ and $g(x) = 5x^4 + x^2$. Here $f'(x) = 3x^2 - 1$ and $g'(x) = 20x^3 + 2x$. Thus, from (6.7.2)

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) = (3x^2 - 1) \cdot (5x^4 + x^2) + (x^3 - x) \cdot (20x^3 + 2x)$$

Usually we simplify the answer by expanding to obtain just one polynomial. Simple computations give $h'(x) = 35x^6 - 20x^4 - 3x^2$.

Alternatively, expanding $h(x)$ as a single polynomial gives $h(x) = 5x^7 - 4x^5 - x^3$, whose derivative, from rules (6.6.4) and (6.7.1), is $h'(x) = 35x^6 - 20x^4 - 3x^2$.

EXAMPLE 6.7.4 Let $D(P)$ denote the demand function for a product. By selling $D(P)$ units at price P per unit, the producer earns revenue $R(P)$ given by $R(P) = PD(P)$. Usually $D'(P)$ is negative because demand goes down when the price increases. According to the product rule for differentiation,

$$R'(P) = D(P) + PD'(P) \tag{$*$}$$

For an economic interpretation, suppose P increases by one dollar. The revenue $R(P)$ changes for two reasons. First, $R(P)$ increases by $1 \cdot D(P)$, because each of the $D(P)$ units brings in one dollar more. But a one dollar increase in the price per unit causes demand to change by $D(P+1) - D(P)$ units, which is approximately $D'(P)$. The (positive) loss due to a one dollar increase in the price per unit is then $-PD'(P)$, which must be subtracted from $D(P)$ to obtain $R'(P)$, as in Eq. ($*$). The resulting expression merely expresses the simple fact that $R'(P)$, the total rate of change of $R(P)$, is what you gain minus what you lose.

Now we proceed with the proof of (6.7.2):

Suppose f and g are differentiable at x, so that the two Newton quotients

$$\frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \frac{g(x+h) - g(x)}{h}$$

tend to the limits $f'(x)$ and $g'(x)$, respectively, as h tends to 0. We must show that the Newton quotient of F also tends to a limit, which is given by $f'(x)g(x) + f(x)g'(x)$. The Newton quotient of F is

$$\frac{F(x+h) - F(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \tag{$*$}$$

To proceed further we must somehow transform the right-hand side (RHS) to involve the Newton quotients of f and g. We use a trick: The numerator of the RHS is unchanged if we both subtract and add the number $f(x)g(x+h)$. Hence, with a suitable regrouping of terms, we have

$$\begin{aligned}\frac{F(x+h) - F(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h}\end{aligned}$$

As h tends to 0, the two Newton quotients tend to $f'(x)$ and $g'(x)$, respectively. Now we can write $g(x+h)$ for $h \neq 0$ as

$$g(x+h) = \left[\frac{g(x+h) - g(x)}{h}\right] h + g(x)$$

By the product rule for limits and the definition of $g'(x)$, this tends to $g'(x) \cdot 0 + g(x) = g(x)$ as h tends to 0. It follows that the Newton quotient of F tends to $f'(x)g(x) + f(x)g'(x)$ as h tends to 0.

To conclude, now that we have seen how to differentiate products of two functions, let us consider products of more than two functions. For example, suppose that $y = f(x)g(x)h(x)$. What is y'? We extend the same technique shown earlier and put $y = [f(x)g(x)]h(x)$. Then the product rule gives

$$\begin{aligned} y' &= [f(x)g(x)]'\,h(x) + [f(x)g(x)]\,h'(x) \\ &= \left[f'(x)g(x) + f(x)g'(x)\right]h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \end{aligned}$$

If none of the three functions is equal to 0, we can write the result as follows:[10]

$$\frac{(fgh)'}{fgh} = \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}$$

By analogy, it is easy to write down the corresponding result for a product of n functions. In words, the relative rate of growth of the product is the sum of the relative rates at which each factor is changing.

Quotients

Suppose $F(x) = f(x)/g(x)$, where f and g are differentiable in x with $g(x) \neq 0$. Bearing in mind the complications in the formula for the derivative of a product, one should be somewhat reluctant to make a quick guess as to the correct formula for $F'(x)$.

In fact, it is quite easy to find the formula for $F'(x)$ if we *assume* that $F(x)$ *is* differentiable. From $F(x) = f(x)/g(x)$ it follows that $f(x) = F(x)g(x)$. Thus, the product rule gives $f'(x) = F'(x) \cdot g(x) + F(x) \cdot g'(x)$. Solving for $F'(x)$ yields $F'(x) \cdot g(x) = f'(x) - F(x) \cdot g'(x)$, and so

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)} = \frac{f'(x) - [f(x)/g(x)]\,g'(x)}{g(x)}$$

Multiplying both numerator and denominator of the last fraction by $g(x)$ gives the following important formula.

DERIVATIVE OF A QUOTIENT

If f and g are differentiable at x and $g(x) \neq 0$, then $F = f/g$ is differentiable at x, and

$$F(x) = \frac{f(x)}{g(x)} \Rightarrow F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2} \tag{6.7.3}$$

[10] If all the functions are positive, this result is easier to show using logarithmic differentiation. See Section 6.11.

In words: *The derivative of a quotient is equal to the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, this difference then being divided by the square of the denominator*. In simpler notation, we have $(f/g)' = (f'g - fg')/g^2$.

Note that in the product rule formula, the two functions appear symmetrically, so that it is easy to remember. In the formula for the derivative of a quotient, the expressions in the numerator must be in the right order. Here is one way to check that you have the order right. Write down the formula you believe is correct. Put $g \equiv 1$. Then $g' \equiv 0$, and your formula ought to reduce to f'. If you get $-f'$, then your signs are wrong.

EXAMPLE 6.7.5 Compute $F'(x)$ and $F'(4)$ when

$$F(x) = \frac{3x-5}{x-2}$$

Solution: We apply (6.7.3), with $f(x) = 3x - 5$, $g(x) = x - 2$. Then $f'(x) = 3$ and $g'(x) = 1$. So we obtain, for $x \neq 2$:

$$F'(x) = \frac{3 \cdot (x-2) - (3x-5) \cdot 1}{(x-2)^2} = \frac{3x - 6 - 3x + 5}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

To find $F'(4)$, we put $x = 4$ in the formula for $F'(x)$ to get $F'(4) = -1/(4-2)^2 = -1/4$. Note that $F'(x) < 0$ for all $x \neq 2$. Hence F is strictly decreasing both for $x < 2$ and for $x > 2$. Note that $(3x-5)/(x-2) = 3 + 1/(x-2)$. The graph is shown in Fig. 5.1.7.

EXAMPLE 6.7.6 Let $C(x)$ be the total cost of producing x units of a commodity. Then $C(x)/x$ is the *average cost* of producing x units. Find an expression for $\frac{d}{dx}[C(x)/x]$.

Solution:

$$\frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{C'(x)x - C(x)}{x^2} = \frac{1}{x}\left(C'(x) - \frac{C(x)}{x}\right) \tag{6.7.4}$$

Note that the marginal cost $C'(x)$ exceeds the average cost $C(x)/x$ if, and only if, average cost increases as output increases.[11]

The formula for the derivative of a quotient becomes more symmetric if we consider relative rates of change. By using (6.7.3), simple computation shows that

$$F(x) = \frac{f(x)}{g(x)} \Rightarrow \frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \tag{6.7.5}$$

That is, *the relative rate of change of a quotient is equal to the relative rate of change of the numerator minus the relative rate of change of the denominator*.

Let $W(t)$ be the nominal wage rate and $P(t)$ the price index at time t. Then $w(t) = W(t)/P(t)$ is called the *real wage rate*. According to (6.7.5),

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{W}(t)}{W(t)} - \frac{\dot{P}(t)}{P(t)}$$

[11] In a similar way, if a basketball team recruits a new player, the average height of the team increases if and only if the new player's height exceeds the old average height.

The relative rate of change of the real wage rate is equal to the difference between the relative rates of change of the nominal wage rate and the price index. Thus, if nominal wages increase at the rate of 5% per year but prices rise by 6% per year, then real wages fall by 1%. Also, if inflation leads to wages and prices increasing at the same relative rate, then the real wage rate is constant.

EXERCISES FOR SECTION 6.7

1. Differentiate w.r.t x the following functions:

   (a) $x + 1$ (b) $x + x^2$ (c) $3x^5 + 2x^4 + 5$

   (d) $8x^4 + 2\sqrt{x}$ (e) $\frac{1}{2}x - \frac{3}{2}x^2 + 5x^3$ (f) $1 - 3x^7$

2. Differentiate w.r.t x the following functions:

   (a) $\frac{3}{5}x^2 - 2x^7 + \frac{1}{8} - \sqrt{x}$ (b) $(2x^2 - 1)(x^4 - 1)$ (c) $\left(x^5 + \dfrac{1}{x}\right)(x^5 + 1)$

SM 3. Differentiate w.r.t x the following functions:

   (a) $\dfrac{1}{x^6}$ (b) $x^{-1}(x^2 + 1)\sqrt{x}$ (c) $\dfrac{1}{\sqrt{x^3}}$ (d) $\dfrac{x + 1}{x - 1}$

   (e) $\dfrac{x + 1}{x^5}$ (f) $\dfrac{3x - 5}{2x + 8}$ (g) $3x^{-11}$ (h) $\dfrac{3x - 1}{x^2 + x + 1}$

4. Differentiate w.r.t x the following functions:

   (a) $\dfrac{\sqrt{x} - 2}{\sqrt{x} + 1}$ (b) $\dfrac{x^2 - 1}{x^2 + 1}$ (c) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$

5. Let $x = f(L)$ be the output when L units of labour are used as input. Assume that $f(0) = 0$ and that $f'(L) > 0, f''(L) < 0$ for all $L > 0$. Average productivity is defined by the formula $g(L) = f(L)/L$.

   (a) Let $L^* > 0$. Indicate on a figure the values of $f'(L^*)$ and $g(L^*)$. Which is larger?

   (b) How does the average productivity change when labour input increases?

SM 6. For each of the following functions, determine the intervals where it is increasing.

   (a) $y = 3x^2 - 12x + 13$ (b) $y = \frac{1}{4}(x^4 - 6x^2)$ (c) $y = \dfrac{2x}{x^2 + 2}$ (d) $y = \dfrac{x^2 - x^3}{2(x + 1)}$

SM 7. Find the equations for the tangents to the graphs of the following functions at the specified points:

   (a) $y = 3 - x - x^2$ at $x = 1$ (b) $y = \dfrac{x^2 - 1}{x^2 + 1}$ at $x = 1$

   (c) $y = \left(\dfrac{1}{x^2} + 1\right)(x^2 - 1)$ at $x = 2$ (d) $y = \dfrac{x^4 + 1}{(x^2 + 1)(x + 3)}$ at $x = 0$

8. Consider an oil well where $x(t)$ denotes the rate of extraction in barrels per day and $p(t)$ denotes the price in dollars per barrel, both at time t. Then $R(t) = p(t)x(t)$ is the revenue in dollars per day. Find an expression for $\dot{R}(t)$, and give it an economic interpretation in the case when $p(t)$ and $x(t)$ are both increasing. (*Hint*: $R(t)$ increases for two reasons.)

SM 9. Differentiate the following functions w.r.t. t:

(a) $\dfrac{at+b}{ct+d}$ (b) $t^n\left(a\sqrt{t}+b\right)$ (c) $\dfrac{1}{at^2+bt+c}$

10. If $f(x) = \sqrt{x}$, then $f(x) \cdot f(x) = x$. Differentiate this equation using the product rule in order to find a formula for $f'(x)$. Compare this with the result in Exercise 6.2.9.

11. Suppose that $a = -n$ where n is any natural number. By using the relation $x^{-n} = 1/x^n$ and the quotient rule (6.7.3), prove the power rule stating that $y = x^a \Rightarrow y' = ax^{a-1}$.

6.8 The Chain Rule

Suppose that y is a function of u, and that u is a function of x. Then y is a composite function of x. Suppose that x changes. This gives rise to a two-stage "chain reaction": first, u reacts directly to the change in x; second, y reacts to this change in u. If we know the rates of change $\mathrm{d}u/\mathrm{d}x$ and $\mathrm{d}y/\mathrm{d}u$, then what is the rate of change $\mathrm{d}y/\mathrm{d}x$? It turns out that the relationship between these rates of change is simple.

THE CHAIN RULE

If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x, and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \tag{6.8.1}$$

It is easier to remember the chain rule when using Leibniz's notation, as the left-hand side of 6.8.1 is exactly what results if we "cancel" the $\mathrm{d}u$ on the right-hand side. Of course this is just a mnemonic and one must be careful: because $\mathrm{d}y/\mathrm{d}u$ and $\mathrm{d}u/\mathrm{d}x$ are not fractions, but merely symbols for derivatives, and $\mathrm{d}u$ is not a number, cancelling is *not* defined![12]

[12] It has been suggested that proving (6.8.1) by cancelling $\mathrm{d}u$ is not much better than proving that $64/16 = 4$ by cancelling the two sixes: $\cancel{6}4/1\cancel{6} = 4$.

An important special case is when y is a power function.

THE GENERALIZED POWER RULE

If $y = u^a$ and u is a differentiable function of x, then

$$y' = au^{a-1}u' \tag{6.8.2}$$

The chain rule is very powerful. Facility in applying it comes from a lot of practice.

EXAMPLE 6.8.1 Find $\mathrm{d}y/\mathrm{d}x$ when:

(a) $y = u^5$ and $u = 1 - x^3$ (b) $y = \dfrac{10}{(x^2 + 4x + 5)^7}$

Solution:

(a) Here we can use (6.8.1) directly. Since $\mathrm{d}y/\mathrm{d}u = 5u^4$ and $\mathrm{d}u/\mathrm{d}x = -3x^2$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = 5u^4(-3x^2) = -15x^2u^4 = -15x^2(1 - x^3)^4$$

(b) If we write $u = x^2 + 4x + 5$, then $y = 10u^{-7}$. By the generalized power rule (6.8.2), one has

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 10(-7)u^{-8}u' = -70u^{-8}(2x + 4) = \frac{-140(x + 2)}{(x^2 + 4x + 5)^8}$$

After a little training, the intermediate steps become unnecessary. For example, to differentiate the compound function

$$y = (\underbrace{1 - x^3}_{u})^5$$

suggested by part (a) of Example 6.8.1, we can *think* of y as $y = u^5$, where $u = 1 - x^3$. Now we can differentiate both u^5 and $1 - x^3$ in our heads, and so write down $y' = 5(1 - x^3)^4(-3x^2)$ immediately.

Note that if you differentiate $y = x^5/5$ using the quotient rule, you obtain the right answer, but commit a small "mathematical crime". This is because it is much easier to write y as $y = (1/5)x^5$ to get $y' = (1/5)5x^4 = x^4$. In the same way, it is unnecessarily cumbersome to apply the quotient rule to the function given in part (b) of Example 6.8.1. The generalized power rule is much more effective.

EXAMPLE 6.8.2 Differentiate the functions:

(a) $y = (x^3 + x^2)^{50}$ (b) $y = \left(\dfrac{x - 1}{x + 3}\right)^{1/3}$ (c) $y = \sqrt{x^2 + 1}$

Solution:

(a) $y = (x^3 + x^2)^{50} = u^{50}$ where $u = x^3 + x^2$, so $u' = 3x^2 + 2x$. Then (6.8.2) gives

$$y' = 50u^{50-1} \cdot u' = 50(x^3 + x^2)^{49}(3x^2 + 2x)$$

(b) Again, we use (6.8.2):

$$y = \left(\frac{x-1}{x+3}\right)^{1/3} = u^{1/3}$$

where $u = (x-1)/(x+3)$. The quotient rule gives

$$u' = \frac{1 \cdot (x+3) - (x-1) \cdot 1}{(x+3)^2} = \frac{4}{(x+3)^2}$$

and hence

$$y' = \frac{1}{3}u^{(1/3)-1} \cdot u' = \frac{1}{3}\left(\frac{x-1}{x+3}\right)^{-2/3} \cdot \frac{4}{(x+3)^2} = \frac{4}{3}(x+1)^{-2/3}(x+3)^{-4/3}$$

(c) Note first that $y = \sqrt{x^2+1} = (x^2+1)^{1/2}$, so $y = u^{1/2}$ where $u = x^2 + 1$. Hence,

$$y' = \frac{1}{2}u^{(1/2)-1} \cdot u' = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

The formulation of the chain rule may appear abstract and difficult. However, when we interpret the derivatives involved in (6.8.1) as rates of change, the chain rule becomes rather intuitive, as the next example from economics will indicate.

EXAMPLE 6.8.3 The demand quantity x for a commodity depends on price p. Suppose that price p is not constant, but depends on time t. Then x is a composite function of t, and according to the chain rule,

$$\frac{dx}{dt} = \frac{dx}{dp} \cdot \frac{dp}{dt}$$

Suppose, for instance, that the demand for butter decreases by 5000 pounds if the price goes up by one dollar per pound. So $dx/dp \approx -5000$. Suppose further that the price per pound increases by five cents per month, so $dp/dt \approx 0.05$. What is the decrease in demand in pounds per month?

Solution: Because the price per pound increases by \$0.05 per month, and the demand decreases by 5000 pounds for every dollar increase in the price, the demand *decreases* by approximately $5000 \cdot 0.05 = 250$ pounds per month. This means that $dx/dt \approx -250$, measured in pounds per month.

The next example uses the chain rule several times.

EXAMPLE 6.8.4 Find $x'(t)$ when $x(t) = 5\left(1 + \sqrt{t^3+1}\right)^{25}$.

Solution: The initial step is easy: let $x(t) = 5u^{25}$, where $u = 1 + \sqrt{t^3+1}$, to obtain

$$x'(t) = 5 \cdot 25u^{24}\frac{du}{dt} = 125u^{24}\frac{du}{dt} \qquad (*)$$

The new feature in this example is that we cannot write down $\mathrm{d}u/\mathrm{d}t$ at once. Finding $\mathrm{d}u/\mathrm{d}t$ requires using the chain rule a second time. Let $u = 1 + \sqrt{v} = 1 + v^{1/2}$, where $v = t^3 + 1$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{2}v^{(1/2)-1} \cdot \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{2}v^{-1/2} \cdot 3t^2 = \frac{1}{2}(t^3+1)^{-1/2} \cdot 3t^2 \tag{$**$}$$

From $(*)$ and $(**)$, we get

$$x'(t) = 125\left(1 + \sqrt{t^3+1}\right)^{24} \cdot \tfrac{1}{2}(t^3+1)^{-1/2} \cdot 3t^2$$

Suppose, as in the last example, that x is a function of u, u is a function of v, and v is in turn a function of t. Then x is a composite function of t, and the chain rule can be used twice to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}t}$$

This is precisely the formula used in the last example. Again the notation is suggestive because the left-hand side is exactly what results if we “cancel” both $\mathrm{d}u$ and $\mathrm{d}v$ on the right-hand side.

An Alternative Formulation of the Chain Rule

Although Leibniz’s notation makes it very easy to remember the chain rule, it suffers from the defect of not specifying where each derivative is evaluated. We remedy this by introducing names for the functions involved. So let $y = f(u)$ and $u = g(x)$. Then y can be written as

$$y = f(g(x))$$

Here y is a *composite function* of x, as considered in Section 5.2, with $g(x)$ as the *kernel*, and f as the *exterior function*.

THE CHAIN RULE

If g is differentiable at x_0 and f is differentiable at $u_0 = g(x_0)$, then the composite function $F(x) = f(g(x))$ is differentiable at x_0, and

$$F'(x_0) = f'(u_0)g'(x_0) = f'(g(x_0))g'(x_0) \tag{6.8.3}$$

In words: *to differentiate a composite function, first differentiate the exterior function w.r.t. the kernel, and then multiply by the derivative of the kernel.*

EXAMPLE 6.8.5 Find the derivative of the compound function $F(x) = f(g(x))$ at $x_0 = -3$ in case $f(u) = u^3$ and $g(x) = 2 - x^2$.

Solution: In this case one has $f'(u) = 3u^2$ and $g'(x) = -2x$. So according to (6.8.3), one has $F'(-3) = f'(g(-3))\, g'(-3)$. Now $g(-3) = 2 - (-3)^2 = 2 - 9 = -7$; $g'(-3) = 6$; and $f'(g(-3)) = f'(-7) = 3(-7)^2 = 3 \cdot 49 = 147$. So $F'(-3) = f'(g(-3))\, g'(-3) = 147 \cdot 6 = 882$.

Finally, we prove that the Chain Rule is correct. Using this alternative formulation, it is tempting to try to argue (6.8.3) as follows:

In simplified notation, with $y = F(x) = f(u)$ and $u = g(x)$, as above, it is tempting to argue as follows: Since $u = g(x)$ is continuous, $\Delta u = g(x) - g(x_0) \to 0$ as $x \to x_0$, and so

$$F'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right) = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = f'(u_0) g'(x_0)$$

There is a catch, however, because Δu may be equal to 0 for values of x arbitrarily close to x_0, and then $\Delta y/\Delta u$ will be undefined. A correct argument goes as follows:

Define functions φ and γ as:

$$\varphi(u) = \begin{cases} \dfrac{f(u) - f(u_0)}{u - u_0} & \text{if } u \neq u_0 \\ f'(u_0) & \text{if } u = u_0 \end{cases} \quad \text{and} \quad \gamma(x) = \begin{cases} \dfrac{g(x) - g(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ g'(x_0) & \text{if } x = x_0 \end{cases}$$

Then, by definition of the derivatives $f'(u_0)$ and $g'(x_0)$, one has $\lim_{u \to u_0} \varphi(u) = \varphi(u_0)$ and $\lim_{x \to x_0} \gamma(x) = \gamma(x_0)$. Moreover,

$$f(u) - f(u_0) = \varphi(u)(u - u_0) \text{ and } g(x) - g(x_0) = \gamma(x)(x - x_0)$$

for all u in an interval around u_0 and all x in an interval around x_0. So for h close to 0, it follows that

$$\begin{aligned} F(x_0 + h) - F(x_0) &= f(g(x_0 + h)) - f(g(x_0)) \\ &= \varphi(g(x_0 + h)) \cdot [g(x_0 + h) - g(x_0)] \\ &= \varphi(g(x_0 + h)) \cdot \gamma(x_0 + h) \cdot h \end{aligned}$$

Hence

$$F'(x_0) = \lim_{h \to 0} \varphi(g(x_0 + h))\gamma(x_0 + h) = \varphi(g(x_0))\gamma(x_0) = f'(g(x_0))g'(x_0)$$

EXERCISES FOR SECTION 6.8

1. Use the chain rule (6.8.1) to find $\mathrm{d}y/\mathrm{d}x$ for the following:

(a) $y = 5u^4$, where $u = 1 + x^2$ (b) $y = u - u^6$, where $u = 1 + 1/x$

2. Compute the following:

(a) $\mathrm{d}Y/\mathrm{d}t$, when $Y = -3(V + 1)^5$ and $V = \frac{1}{3}t^3$.

(b) $\mathrm{d}K/\mathrm{d}t$, when $K = AL^a$ and $L = bt + c$, where A, a, b, and c are positive constants.

SM **3.** Find the derivatives of the following functions, where a, p, q, and b are constants:

(a) $y = \dfrac{1}{(x^2+x+1)^5}$ (b) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$ (c) $y = x^a(px+q)^b$

4. If Y is a function of K, and K is a function of t, find the formula for the derivative of Y with respect to t at $t = t_0$.

5. If $Y = F(K)$ and $K = h(t)$, find the formula for $\mathrm{d}Y/\mathrm{d}t$.

6. Consider the demand function $x = b - \sqrt{ap - c}$, where a, b, and c are positive constants, x is the quantity demanded, and p is the price, for $p > c/a$. Compute $\mathrm{d}x/\mathrm{d}p$.

7. Find a formula for $h'(x)$ when: (a) $h(x) = f(x^2)$; and (b) $h(x) = f\left(x^n g(x)\right)$.

8. Let $s(t)$ be the distance in kilometres a car goes in t hours. Let $B(s)$ be the number of litres of fuel the car uses to go s kilometres. Provide an interpretation of the function $b(t) = B\left(s(t)\right)$, and find a formula for $b'(t)$.

9. Suppose that $C = 20q - 4q\left(25 - \frac{1}{2}x\right)^{1/2}$, where q is a constant and $x < 50$. Find $\mathrm{d}C/\mathrm{d}x$.

10. Differentiate each of the following in two different ways:

(a) $y = (x^4)^5 = x^{20}$ (b) $y = (1-x)^3 = 1 - 3x + 3x^2 - x^3$

11. Suppose you invest €1 000 at $p\%$ interest per year. Let $g(p)$ denote how many euros you will have after ten years.

(a) Give economic interpretations of: (i) $g(5) \approx 1629$; and (ii) $g'(5) \approx 155$.

(b) To check the numbers in (a), find a formula for $g(p)$, then compute $g(5)$ and $g'(5)$.

12. If f is differentiable at x, find expressions for the derivatives of the following functions:

(a) $x + f(x)$ (b) $[f(x)]^2 - x$ (c) $[f(x)]^4$ (d) $x^2 f(x) + [f(x)]^3$

(e) $xf(x)$ (f) $\sqrt{f(x)}$ (g) $\dfrac{x^2}{f(x)}$ (h) $\dfrac{[f(x)]^2}{x^3}$

6.9 Higher-Order Derivatives

The derivative f' of a function f is often called the *first derivative* of f. If f' is also differentiable, then we can differentiate f' in turn. The result $(f')'$ is called the *second derivative*, written more concisely as f''. We use $f''(x)$ to denote the second derivative of f evaluated at the particular point x.

EXAMPLE 6.9.1 Find $f'(x)$ and $f''(x)$ when $f(x) = 2x^5 - 3x^3 + 2x$.

Solution: The rules for differentiating polynomials imply that $f'(x) = 10x^4 - 9x^2 + 2$. Then we differentiate each side of this equality to get $f''(x) = 40x^3 - 18x$.

The different forms of notation for the second derivative are analogous to those for the first derivative. For example, we write $y'' = f''(x)$ in order to denote the second derivative of $y = f(x)$. The Leibniz notation for the second derivative is also used. In the notation $\mathrm{d}y/\mathrm{d}x$ or $\mathrm{d}f(x)/\mathrm{d}x$ for the first derivative, we interpreted the symbol $\mathrm{d}/\mathrm{d}x$ as an operator indicating that what follows is to be differentiated with respect to x. The second derivative is obtained by using the operator $\mathrm{d}/\mathrm{d}x$ twice: $f''(x) = (\mathrm{d}/\mathrm{d}x)(\mathrm{d}/\mathrm{d}x)f(x)$. We usually think of this as $f''(x) = (\mathrm{d}/\mathrm{d}x)^2 f(x)$, and so write

$$f''(x) = \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} \quad \text{or} \quad y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

Pay special attention to where the superscripts are placed! Of course, the notation for the second derivative must change if the variables have other names.

EXAMPLE 6.9.2 Find:

(a) Y'' when $Y = AK^a$ is a function of $K > 0$, with A and a as constants.

(b) $\mathrm{d}^2L/\mathrm{d}t^2$ when $L = \dfrac{t}{t+1}$, and $t \geq 0$.

Solution:

(a) Differentiating $Y = AK^a$ once with respect to K gives $Y' = AaK^{a-1}$. Differentiating a second time with respect to K yields $Y'' = Aa(a-1)K^{a-2}$.

(b) First, use the quotient rule to find that $\dfrac{\mathrm{d}L}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{t}{t+1}\right) = \dfrac{1 \cdot (t+1) - t \cdot 1}{(t+1)^2}$ $= (t+1)^{-2}$. Then,

$$\frac{\mathrm{d}^2 L}{\mathrm{d}t^2} = -2(t+1)^{-3} = \frac{-2}{(t+1)^3}$$

Convex and Concave Functions

Recall from Section 6.3 how the sign of the first derivative determines whether a function is increasing or decreasing on an interval I. If $f'(x) \geq 0$ ($f'(x) \leq 0$) on I, then f is increasing (decreasing) on I, and conversely. The second derivative $f''(x)$ is the derivative of $f'(x)$. Hence:

$$f''(x) \geq 0 \text{ on } I \iff f' \text{ is increasing on } I \tag{6.9.1}$$

$$f''(x) \leq 0 \text{ on } I \iff f' \text{ is decreasing on } I \tag{6.9.2}$$

The equivalence in (6.9.1) is illustrated in Fig. 6.9.1. The slope of the tangent, $f'(x)$, is increasing as x increases. On the other hand, the slope of the tangent to the graph in Fig. 6.9.2 is decreasing as x increases, which illustrates (6.9.2).

To help visualize this, imagine sliding a ruler along the curve and keeping it aligned with the tangent to the curve at each point. As the ruler moves along the curve from left to right, the tangent rotates anti-clockwise in Fig. 6.9.1, clockwise in Fig. 6.9.2.

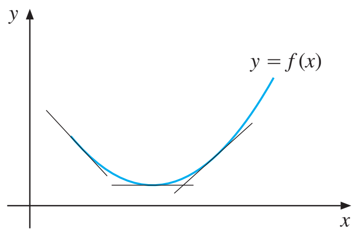


Figure 6.9.1 The slope of the tangent line increases as x increases

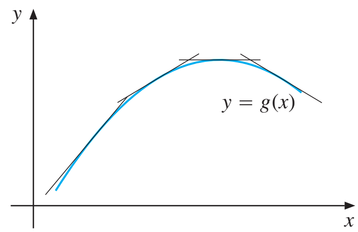


Figure 6.9.2 The slope of the tangent line decreases as x increases

Suppose that f is continuous in the interval I and twice differentiable in the interior of I. Then we can introduce the following definitions:

CONVEX AND CONCAVE FUNCTIONS

$$f \text{ is } \textit{convex} \text{ on } I \iff f''(x) \geq 0 \text{ for all } x \text{ in } I \tag{6.9.3}$$

$$f \text{ is } \textit{concave} \text{ on } I \iff f''(x) \leq 0 \text{ for all } x \text{ in } I \tag{6.9.4}$$

These properties are illustrated in Figs 6.9.3–6.9.6, which should be studied carefully. Whether a function is concave or convex is crucial to many results in economic analysis, especially the many that involve maximization or minimization problems. We note that often I is the whole real line, in which case the interval is not mentioned explicitly.

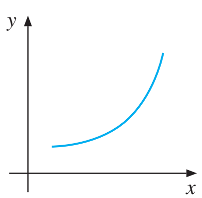


Figure 6.9.3 Increasing convex

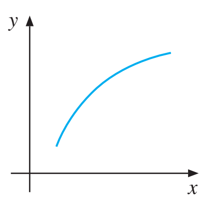


Figure 6.9.4 Increasing concave

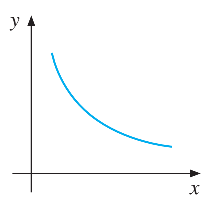


Figure 6.9.5 Decreasing convex

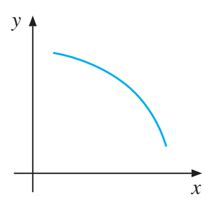


Figure 6.9.6 Decreasing concave

EXAMPLE 6.9.3 Check the convexity/concavity of the following functions:

(a) $f(x) = x^2 - 2x + 2$ (b) $f(x) = ax^2 + bx + c$

Solution:

(a) Here $f'(x) = 2x - 2$ so $f''(x) = 2$. Because $f''(x) > 0$ for all x, the function f is convex.

(b) Here $f'(x) = 2ax + b$, so $f''(x) = 2a$. If $a = 0$, then f is linear. In this case, the function f meets both the definitions in (6.9.3) and (6.9.4), so it is both concave and convex. If $a > 0$, then $f''(x) > 0$, so f is convex. If $a < 0$, then $f''(x) < 0$, so f is concave. The last two cases are illustrated by the graphs in Figs 4.6.1 and 4.6.2.

We consider two typical examples of convex and concave functions. Fig. 6.9.7 shows a rough graph of the function P, for dates between 1500 and 2000, where $P(t)$ measures the world population (in billions) in year t. It appears from the figure that not only is $P(t)$ increasing, but the rate of increase increases: each year the *increase* becomes larger. So, for the last five centuries, $P(t)$ has been convex.

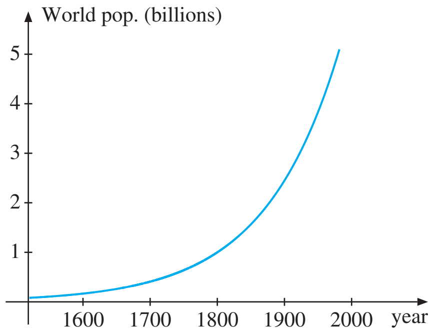


Figure 6.9.7 World population

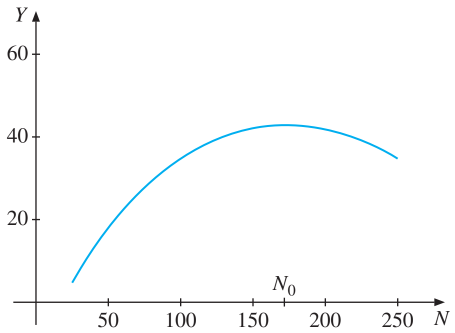


Figure 6.9.8 Wheat production

The graph in Fig. 6.9.8 shows the crop of wheat $Y(N)$ when N pounds of fertilizer per acre are used. The curve is based on fertilizer experiments in Iowa during 1952. The function has a maximum at $N = N_0 \approx 172$. Increasing the amount of fertilizer beyond N_0 will cause wheat production to decline. Moreover, $Y(N)$ is concave. If $N < N_0$, increasing N by one unit will *increase* $Y(N)$ by less, the larger is N. On the other hand, if $N > N_0$, increasing N by one unit will *decrease* $Y(N)$ by more, the larger is N.

EXAMPLE 6.9.4 Examine the concavity/convexity of the production function $Y = AK^a$, defined for all $K \geq 0$, where $A > 0$ and $a > 0$.

Solution: From Example 6.9.2, one has $Y'' = Aa(a-1)K^{a-2}$.

If $a \in (0, 1)$, then the coefficient $Aa(a-1) < 0$, so that $Y'' < 0$ for all $K > 0$. Hence, the function is concave. The graph of $Y = AK^a$ for $0 < a < 1$, is shown in Fig. 6.9.9.

On the other hand, if $a > 1$, then $Y'' > 0$ and Y is a convex function of K, as shown in Fig. 6.9.10.

Finally, if $a = 1$, then Y is linear, so it is both concave and convex.

EXAMPLE 6.9.5 Suppose that the two functions U and g are both increasing and concave, with $U' \geq 0$, $U'' \leq 0$, $g' \geq 0$, and $g'' \leq 0$. Prove that the composite function $f(x) = g(U(x))$ is also increasing and concave.

Solution: Using the chain rule yields

$$f'(x) = g'(U(x)) \cdot U'(x) \tag{$*$}$$

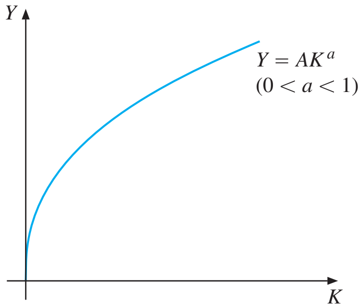


Figure 6.9.9 Concave production function

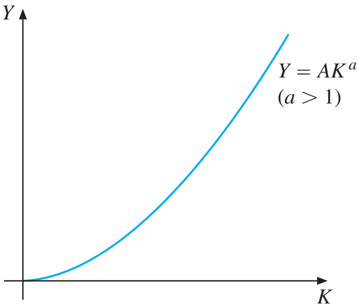


Figure 6.9.10 Convex production function

Because g' and U' are both ≥ 0, so $f'(x) \geq 0$. Hence, f is increasing. In words: *an increasing transformation of an increasing function is increasing.*

In order to compute $f''(x)$, we must differentiate the product of the two functions $g'(U(x))$ and $U'(x)$. According to the chain rule, the derivative of $g'(U(x))$ is equal to $g''(U(x)) \cdot U'(x)$. Hence,

$$f''(x) = g''(U(x)) \cdot [U'(x)]^2 + g'(U(x)) \cdot U''(x) \qquad (**)$$

Because $g'' \leq 0$, $g' \geq 0$, and $U'' \leq 0$, it follows that $f''(x) \leq 0$. Again, in words: *an increasing concave transformation of a concave function is concave.*

Third and Higher Derivatives

If $y = f(x)$, the derivative of $y'' = f''(x)$ is called the *third derivative*, customarily denoted by $y''' = f'''(x)$. It is notationally cumbersome to continue using more and more primes to indicate repeated differentiation, so the *fourth derivative* is usually denoted by $y^{(4)} = f^{(4)}(x)$.[13] The same derivative can be expressed as $\mathrm{d}^4y/\mathrm{d}x^4$. In general, let

$$y^{(n)} = f^{(n)}(x) \quad \text{or} \quad \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

denote the *n-th derivative* of f at x. The number n is called the *order* of the derivative. For example, $f^{(6)}(x_0)$ denotes the sixth derivative of f calculated at x_0, found by differentiating six times.

EXAMPLE 6.9.6 Compute all the derivatives up to and including order 4 of

$$f(x) = 3x^{-1} + 6x^3 - x^2$$

where $x \neq 0$.

Solution: Repeated differentiation gives

$$f'(x) = -3x^{-2} + 18x^2 - 2x, \qquad f''(x) = 6x^{-3} + 36x - 2$$
$$f'''(x) = -18x^{-4} + 36, \qquad f^{(4)}(x) = 72x^{-5}$$

[13] We put the number 4 in parentheses in order to avoid confusion with y^4, the fourth power of y.

EXERCISES FOR SECTION 6.9

1. Compute the second derivatives of:

   (a) $y = x^5 - 3x^4 + 2$ (b) $y = \sqrt{x}$ (c) $y = (1 + x^2)^{10}$

2. Find $\mathrm{d}^2y/\mathrm{d}x^2$ when $y = \sqrt{1 + x^2} = \left(1 + x^2\right)^{1/2}$.

3. Compute:

   (a) y'' for $y = 3x^3 + 2x - 1$ (b) Y''' for $Y = 1 - 2x^2 + 6x^3$

   (c) $d^3z/\mathrm{d}t^3$ for $z = 120t - (1/3)t^3$ (d) $f^{(4)}(1)$ for $f(z) = 100z^{-4}$

4. Find $g''(2)$ when $g(t) = \dfrac{t^2}{t - 1}$.

5. Find formulas for y'' and y''' when $y = f(x)g(x)$.

6. Find $\mathrm{d}^2L/\mathrm{d}t^2$ when $L = 1/\sqrt{2t - 1}$.

7. If $u(y)$ denotes an individual's utility of having income (or consumption) y, then $R = -yu''(y)/u'(y)$ is the coefficient of *relative risk aversion*.[14] Compute R for the following utility functions, where A_1, A_2, and ρ are positive constants with $\rho \neq 1$, and we assume that $y > 0$:

   (a) $u(y) = A_1 y$ (b) $u(y) = \sqrt{y}$ (c) $u(y) = A_1 - A_2 y^{-2}$ (d) $u(y) = A_1 + A_2 \dfrac{y^{1-\rho}}{1 - \rho}$

8. Let $U(x) = \sqrt{x}$ and $g(u) = u^3$. Then $f(x) = g(U(x)) = x^{3/2}$, which is not a concave function. Why does this not contradict the conclusion in Example 6.9.5?

9. The US defence secretary claimed in 1985 that Congress had reduced the defence budget. Representative Gray pointed out that the budget had not been reduced; Congress had only reduced the rate of increase. If P denotes the size of the defence budget, translate the statements into statements about the signs of P' and P''.

10. Sentence in a newspaper: "The rate of increase of bank loans is increasing at an increasing rate". If $L(t)$ denotes total bank loans at time t, represent the sentence by a mathematical statement about the sign of an appropriate derivative of L.

6.10 Exponential Functions

Exponential functions were introduced in Section 4.9. They were shown to be well suited to describing certain economic phenomena such as growth and compound interest. In particular we introduced the *natural* exponential function $f(x) = e^x$, where $e \approx 2.71828$, as well as the alternative notation $\exp x$.

[14] By contrast, $R_A = -u''(y)/u'(y)$ is the coefficient of *absolute risk aversion*.

Now we explain why this particular exponential function deserves to be called "natural". Consider the Newton quotient of $f(x) = e^x$, which is

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h} \tag{$*$}$$

If this fraction tends to a limit as h tends to 0, then $f(x) = e^x$ is differentiable and $f'(x)$ is precisely equal to this limit.

To simplify the right-hand side of $(*)$, we make use of the rule $e^{x+h} = e^x \cdot e^h$ to write $e^{x+h} - e^x$ as $e^x(e^h - 1)$. So $(*)$ can be rewritten as

$$\frac{f(x+h) - f(x)}{h} = e^x \cdot \frac{e^h - 1}{h}$$

We now evaluate the limit of the right-hand side as $h \to 0$. Note that e^x is a constant when we vary only h. As for $(e^h - 1)/h$, in Example 6.5.1 we argued that this fraction tends to 1 as h tends to 0, although in that example the variable was x and not h. It follows that

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$f(x) = e^x \quad \Longrightarrow \quad f'(x) = e^x \tag{6.10.1}$$

Thus the *natural exponential function $f(x) = e^x$ has the remarkable property that its derivative is equal to the function itself*. This is the main reason why the function appears so often in mathematics and applications. An implication of (6.10.1) is that $f''(x) = e^x$. Because $e^x > 0$ for all x, both $f'(x)$ and $f''(x)$ are positive. Hence, both f and f' are strictly increasing. This confirms the increasing convex shape indicated in Fig. 4.9.3.

Combining (6.10.1) with the other rules of differentiation, we can differentiate many expressions involving the exponential function e^x.

EXAMPLE 6.10.1 Find the first and second derivatives of:

(a) $y = x^3 + e^x$ (b) $y = x^5 e^x$ (c) $y = e^x/x$

Solution:

(a) We find that $y' = 3x^2 + e^x$ and $y'' = 6x + e^x$.

(b) By the product rule, $y' = 5x^4 e^x + x^5 e^x = x^4 e^x (5 + x)$. To find y'', differentiate $y' = 5x^4 e^x + x^5 e^x$ once more to obtain $y'' = 20x^3 e^x + 5x^4 e^x + 5x^4 e^x + x^5 e^x = x^3 e^x (x^2 + 10x + 20)$.

(c) The quotient rule yields

$$y = \frac{e^x}{x} \Rightarrow y' = \frac{e^x x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

Differentiating $y' = \dfrac{e^x x - e^x}{x^2}$ once more w.r.t. x gives

$$y'' = \frac{(e^x x + e^x - e^x)x^2 - (e^x x - e^x)2x}{(x^2)^2} = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

Combining (6.10.1) with the chain rule (6.8.1) allows some rather complicated functions to be differentiated. First, note that $y = e^{g(x)}$ can be rewritten as $y = e^u$, where $u = g(x)$. Then $y' = e^u \cdot u'$ and $u' = g'(x)$, so that:

$$y = e^{g(x)} \quad \Longrightarrow \quad y' = e^{g(x)} g'(x) \tag{6.10.2}$$

EXAMPLE 6.10.2 Differentiate the functions:

(a) $y = e^{-x}$ (b) $y = x^p e^{ax}$, where p and a are constants (c) $y = \sqrt{e^{2x} + x}$

Solution:

(a) Direct use of (6.10.2) gives $y = e^{-x} \Rightarrow y' = e^{-x} \cdot (-1) = -e^{-x}$. This derivative is always negative, so the function is strictly decreasing. Furthermore, $y'' = e^{-x} > 0$, so the function is convex. This, again, agrees with the graph shown in Fig. 4.9.3.

(b) By the chain rule, the derivative of e^{ax} is ae^{ax}. Hence, using the product rule,

$$y' = px^{p-1}e^{ax} + x^p a e^{ax} = x^{p-1}e^{ax}(p + ax)$$

(c) Let $y = \sqrt{e^{2x} + x} = \sqrt{u}$, with $u = e^{2x} + x$. Then $u' = 2e^{2x} + 1$, where we used the chain rule. Using the chain rule again with $v = e^{2x} + x$, we obtain

$$y = \sqrt{e^{2x} + x} = \sqrt{v} \Rightarrow y' = \frac{1}{2\sqrt{v}} \cdot v' = \frac{2e^{2x} + 1}{2\sqrt{e^{2x} + x}}$$

EXAMPLE 6.10.3 For each of the following functions, find the intervals where they are increasing:

(a) $y = \dfrac{e^x}{x}$ (b) $y = x^4 e^{-2x}$ (c) $y = xe^{-\sqrt{x}}$

Solution:

(a) According to part (c) of Example 6.10.1, $y' = e^x(x - 1)/x^2$, so $y' \geq 0$ if and only if $x \geq 1$. Thus y is increasing in $[1, \infty)$.

(b) According to part (b) of Example 6.10.2, with $p = 4$ and $a = -2$, we have $y' = x^3 e^{-2x}(4 - 2x)$. A sign diagram reveals that y is increasing in $[0, 2]$.

(c) The function is only defined for $x \geq 0$. Using the chain rule, for $x > 0$ the derivative of $e^{-\sqrt{x}}$ is $-e^{-\sqrt{x}}/2\sqrt{x}$, so by the product rule, the derivative of $y = xe^{-\sqrt{x}}$ is

$$y' = 1 \cdot e^{-\sqrt{x}} - \frac{xe^{-\sqrt{x}}}{2\sqrt{x}} = e^{-\sqrt{x}}\left(1 - \frac{1}{2}\sqrt{x}\right)$$

where we have used the fact that $x/\sqrt{x} = \sqrt{x}$. It follows that y is increasing when $x > 0$ and $1 - \frac{1}{2}\sqrt{x} \geq 0$. Because $y = 0$ when $x = 0$ and $y > 0$ when $x > 0$, it follows that y is increasing in $[0, 4]$.

A common error when differentiating exponential functions is to believe that the derivative of e^x w.r.t x is "xe^{x-1}". Actually, this is the derivative of e^x w.r.t e. The exponential function e^x of x has been confused with the power function e^x of e!

SURVEY OF THE PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION

The natural exponential function

$$f(x) = \exp(x) = e^x \quad (e = 2.71828\ldots)$$

is differentiable, strictly increasing and convex. In fact,

$$f(x) = e^x \Rightarrow f'(x) = f(x) = e^x$$

The following properties hold for all exponents s and t:

$$e^s e^t = e^{s+t}, \quad e^s/e^t = e^{s-t} \quad \text{and} \quad (e^s)^t = e^{st}$$

Moreover,

$$\lim_{x\to-\infty} e^x = 0 \quad \text{and} \quad \lim_{x\to\infty} e^x = \infty$$

Differentiating Other Exponential Functions

So far we have considered only the derivative of e^x, where $e = 2.71828\ldots$. How can we differentiate $y = a^x$, where a is any other positive number? According to definition (4.10.1), we have $a = e^{\ln a}$. So, using the general property $(e^r)^s = e^{rs}$, we have the formula

$$a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$$

This shows that in functions involving the expression a^x, we can just as easily work with the special exponential function e^{bx}, where b is a constant equal to $\ln a$. In particular, we can differentiate a^x by differentiating $e^{x \ln a}$. According to (6.10.2), with $g(x) = (\ln a)x$, we have

$$y = a^x \Rightarrow y' = a^x \ln a \tag{6.10.3}$$

EXAMPLE 6.10.4 Find the derivatives of: (a) $f(x) = 10^{-x}$; and (b) $g(x) = x2^{3x}$.

Solution:

(a) Rewrite $f(x) = 10^{-x} = 10^u$, where $u = -x$. Using (6.10.3) and the chain rule gives $f'(x) = -10^{-x} \ln 10$.

(b) Rewrite $y = 2^{3x} = 2^u$, where $u = 3x$. By (6.10.3) and the chain rule,

$$y' = (2^u \ln 2)u' = (2^{3x} \ln 2) \cdot 3 = 3 \cdot 2^{3x} \ln 2$$

Finally, using the product rule we obtain

$$g'(x) = 1 \cdot 2^{3x} + x \cdot 3 \cdot 2^{3x} \ln 2 = 2^{3x}(1 + 3x \ln 2)$$

EXERCISES FOR SECTION 6.10

1. Find the first-order derivatives of:

(a) $y = e^x + x^2$ (b) $y = 5e^x - 3x^3 + 8$ (c) $y = \dfrac{x}{e^x}$ (d) $y = \dfrac{x + x^2}{e^x + 1}$

(e) $y = -x - 5 - e^x$ (f) $y = x^3 e^x$ (g) $y = e^x x^{-2}$ (h) $y = (x + e^x)^2$

2. Find the first derivatives w.r.t. t of the following functions, where a, b, c, p, and q are constants:

(a) $x = (a + bt + ct^2)e^t$ (b) $x = \dfrac{p + qt^3}{te^t}$ (c) $x = \dfrac{(at + bt^2)^2}{e^t}$

3. Find the first and second derivatives of:

(a) $y = e^{-3x}$ (b) $y = 2e^{x^3}$ (c) $y = e^{1/x}$ (d) $y = 5e^{2x^2 - 3x + 1}$

SM 4. Find the intervals where the following functions are increasing:

(a) $y = x^3 + e^{2x}$ (b) $y = 5x^2 e^{-4x}$ (c) $y = x^2 e^{-x^2}$

5. Find the intervals where the following functions are increasing:

(a) $y = x^2/e^{2x}$ (b) $y = e^x - e^{3x}$ (c) $y = \dfrac{e^{2x}}{x + 2}$

6. Find:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{(e^x)}\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(e^{t/2} + e^{-t/2}\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{e^t + e^{-t}}\right)$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(e^{z^3} - 1\right)^{1/3}$

7. Differentiate:

(a) $y = 5^x$ (b) $y = x2^x$ (c) $y = x^2 2^{x^2}$ (d) $y = e^x 10^x$

6.11 Logarithmic Functions

In Section 4.10 we introduced the natural logarithmic function, $g(x) = \ln x$. It is defined for all $x > 0$ and has the graph shown in Fig. 4.10.2.

According to Section 5.3, this function has $f(x) = e^x$ as its *inverse*. If we *assume* that $g(x) = \ln x$ has a derivative for all $x > 0$, we can easily find that derivative. To do so, we differentiate w.r.t x the equation defining $g(x) = \ln x$, which is

$$e^{g(x)} = x \tag{$*$}$$

Using (6.10.2) to differentiate each side of ($*$), we obtain $e^{g(x)}g'(x) = 1$. Since $e^{g(x)} = x$, this implies $xg'(x) = 1$, and thus the derivative of $\ln x$ at x is simply the number $1/x$.

DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

$$g(x) = \ln x \Rightarrow g'(x) = \frac{1}{x} \tag{6.11.1}$$

For $x > 0$, we have $g'(x) > 0$, so that $g(x)$ is *strictly* increasing. Note moreover that $g''(x) = -1/x^2$, which is less than 0 for all $x > 0$, so that $g(x)$ is concave. This confirms the shape of the graph in Fig. 4.10.2. In fact, the growth of $\ln x$ is quite slow: for example, $\ln x$ does not attain the value 10 until $x > 22\,026$, because $\ln x = 10$ gives $x = e^{10} \approx 22\,026.5$.

EXAMPLE 6.11.1 Compute y' and y'' when:

(a) $y = x^3 + \ln x$ (b) $y = x^2 \ln x$ (c) $y = \ln x/x$

Solution:

(a) We find easily that $y' = 3x^2 + 1/x$. Furthermore, $y'' = 6x - 1/x^2$.

(b) The product rule gives $y' = 2x \ln x + x^2(1/x) = 2x \ln x + x$. Differentiating the last expression w.r.t. x gives $y'' = 2 \ln x + 2x(1/x) + 1 = 2 \ln x + 3$.

(c) Here we use the quotient rule:

$$y' = \frac{(1/x)x - (\ln x) \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Differentiating again yields

$$y'' = \frac{-(1/x)x^2 - (1 - \ln x)2x}{(x^2)^2} = \frac{2 \ln x - 3}{x^3}$$

Often, we need to consider composite functions involving natural logarithms. Because $\ln u$ is defined only when $u > 0$, a composite function of the form $y = \ln h(x)$ will only be defined for values of x satisfying $h(x) > 0$.

Combining the rule for differentiating $\ln x$ with the chain rule allows us to differentiate many different types of function. Suppose, for instance, that $y = \ln h(x)$, where $h(x)$ is differentiable and positive. By the chain rule, $y = \ln u$ with $u = h(x)$ implies that $y' = (1/u)u' = (1/h(x))\, h'(x)$, so:

$$y = \ln h(x) \Rightarrow y' = \frac{h'(x)}{h(x)} \tag{6.11.2}$$

Note that if $N(t)$ is a function of t, then the derivative of its natural logarithm

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln N(t) = \frac{1}{N(t)} \frac{\mathrm{d}N(t)}{\mathrm{d}t} = \frac{\dot{N}(t)}{N(t)}$$

is the relative rate of growth of $N(t)$.

EXAMPLE 6.11.2 Find the domains of the following functions and compute their derivatives:

(a) $y = \ln(1 - x)$ (b) $y = \ln(4 - x^2)$ (c) $y = \ln\left(\dfrac{x-1}{x+1}\right) - \dfrac{1}{4}x$

Solution:

(a) $\ln(1 - x)$ is defined if $1 - x > 0$, that is if $x < 1$. To find its derivative, we use (6.11.2), with $h(x) = 1 - x$. Then $h'(x) = -1$, and

$$y' = \frac{-1}{1-x} = \frac{1}{x-1}$$

(b) $\ln(4 - x^2)$ is defined if $4 - x^2 > 0$, that is if $(2 - x)(2 + x) > 0$. This is satisfied if and only if $-2 < x < 2$. Formula (6.11.2) gives

$$y' = \frac{-2x}{4-x^2} = \frac{2x}{x^2-4}$$

(c) We can write $y = \ln u - \frac{1}{4}x$, where $u = (x - 1)/(x + 1)$. For the function to be defined, we require that $u > 0$. A sign diagram shows that this is satisfied if $x < -1$ or $x > 1$. Using (6.11.2), we obtain

$$y' = \frac{u'}{u} - \frac{1}{4}$$

where

$$u' = \frac{1 \cdot (x+1) - 1 \cdot (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

So

$$y' = \frac{2(x+1)}{(x+1)^2(x-1)} - \frac{1}{4} = \frac{9-x^2}{4(x^2-1)} = \frac{(3-x)(3+x)}{4(x-1)(x+1)}$$

EXAMPLE 6.11.3 Find the intervals where the following functions are increasing:

(a) $y = x^2 \ln x$ (b) $y = 4x - 5\ln(x^2 + 1)$ (c) $y = 3\ln(1 + x) + x - \frac{1}{2}x^2$

Solution:

(a) The function is defined for $x > 0$, and

$$y' = 2x\ln x + x^2(1/x) = x(2\ln x + 1)$$

Hence, $y' \geq 0$ when $\ln x \geq -1/2$, that is, when $x \geq e^{-1/2}$. That is, y is increasing in the interval $[e^{-1/2}, \infty)$.

(b) We find that

$$y' = 4 - \frac{10x}{x^2+1} = 4(x-2)\left(x - \frac{1}{2}\right)x^2 + 1$$

A sign diagram reveals that y is increasing in each of the intervals $(-\infty, \frac{1}{2}]$ and $[2, \infty)$.

(c) The function is defined for $x > -1$, and

$$y' = \frac{3}{1+x} + 1 - x = \frac{(2-x)(2+x)}{x+1}$$

A sign diagram reveals that y is increasing in $(-1, 2]$.

SURVEY OF THE PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION

The natural logarithmic function

$$g(x) = \ln x$$

is differentiable, strictly increasing and concave in $(0, \infty)$. In fact,

$$g'(x) = 1/x, \quad g''(x) = -1/x^2$$

By definition, $e^{\ln x} = x$ for all $x > 0$, and $\ln e^x = x$ for all x. The following properties hold for all $x > 0$, $y > 0$:

$$\ln(xy) = \ln x + \ln y, \ \ln(x/y) = \ln x - \ln y, \quad \text{and} \quad \ln x^p = p \ln x$$

Moreover,

$$\ln x \to -\infty \text{ as } x \to 0 \text{ from the right}$$

while

$$\ln x \to +\infty \text{ as } x \to +\infty$$

Logarithmic Differentiation

When differentiating an expression containing products, quotients, roots, powers, and combinations of these, it is often an advantage to use *logarithmic differentiation*. The method is illustrated by two examples:

EXAMPLE 6.11.4 Find the derivative of $y = x^x$ defined for all $x > 0$.

Solution: The power rule of differentiation, $y = x^a \Rightarrow y' = ax^{a-1}$, requires the exponent a to be a constant, while the rule $y = a^x \Rightarrow y' = a^x \ln a$ requires that the base a is constant. In the expression x^x both the exponent and the base vary with x, so neither of the two rules can be used.

Begin by taking the natural logarithm of each side, $\ln y = x \ln x$. Differentiating w.r.t x gives $y'/y = 1 \cdot \ln x + x(1/x) = \ln x + 1$. Multiplying by $y = x^x$ gives us the result:

$$y = x^x \Rightarrow y' = x^x(\ln x + 1)$$

EXAMPLE 6.11.5 Find the derivative of $y = [A(x)]^\alpha [B(x)]^\beta [C(x)]^\gamma$, where α, β, and γ are constants and A, B, and C are positive functions.

Solution: First, take the natural logarithm of each side to obtain

$$\ln y = \alpha \ln(A(x)) + \beta \ln(B(x)) + \gamma \ln(C(x))$$

Differentiation w.r.t. x yields

$$\frac{y'}{y} = \alpha \frac{A'(x)}{A(x)} + \beta \frac{B'(x)}{B(x)} + \gamma \frac{C'(x)}{C(x)}$$

Multiplying by y, we have

$$y' = \left[\alpha \frac{A'(x)}{A(x)} + \beta \frac{B'(x)}{B(x)} + \gamma \frac{C'(x)}{C(x)}\right] [A(x)]^{\alpha} [B(x)]^{\beta} [C(x)]^{\gamma}$$

In Eq. (4.10.5), we showed that the logarithm of x in the system with base a, denoted by $\log_a x$, satisfies $\log_a x = (1/\ln a)\ln x$. Differentiating each side w.r.t x, it follows immediately that:

$$y = \log_a x \Rightarrow y' = \frac{1}{\ln a}\frac{1}{x} \tag{6.11.3}$$

Approximating the Number e

If $g(x) = \ln x$, then $g'(x) = 1/x$, and, in particular, $g'(1) = 1$. We use in turn: (i) the definition of $g'(1)$; (ii) the fact that $\ln 1 = 0$; (iii) the rule $\ln x^p = p \ln x$. The result is

$$1 = g'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{1}{h} \ln(1+h) = \lim_{h\to 0} \ln(1+h)^{1/h}$$

Because $\ln(1+h)^{1/h}$ tends to 1 as h tends to 0 and the exponential mapping is continuous, it follows that $(1+h)^{1/h} = \exp\left[\ln(1+h)^{1/h}\right]$ itself must tend to $\exp 1 = e$. That is,

$$\boxed{e = \lim_{h\to 0}(1+h)^{1/h}} \tag{6.11.4}$$

To illustrate this limit, Table 6.3 gives some function values that were computed using a calculator. These numbers seem to confirm that the decimal expansion 2.718281828 ... we gave for e starts out correctly. Of course, this by no means proves that the limit exists, but it does suggest that closer and closer approximations to e can be obtained by choosing h smaller and smaller.[15]

Table 6.3 Values of $(1+h)^{1/h}$ when h gets smaller and smaller

h	1	1/2	1/10	1/1000	1/100 000	1/1 000 000
$(1+h)^{1/h}$	2	2.25	2.5937 ...	2.7169 ...	2.71825 ...	2.718281828 ...

[15] A better way to approximate e^x, for general real x, is suggested in Examples 7.5.4 and 7.6.2.

Power Functions

In Section 6.6 we claimed that, for all real numbers a,

$$f(x) = x^a \Rightarrow f'(x) = ax^{a-1} \tag{$*$}$$

This important rule has only been established for certain special values of a, particularly the rational numbers. Because $x = e^{\ln x}$, we have $x^a = (e^{\ln x})^a = e^{a \ln x}$. Using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^a) = \frac{\mathrm{d}}{\mathrm{d}x}(e^{a \ln x}) = e^{a \ln x} \cdot \frac{a}{x} = x^a \frac{a}{x} = ax^{a-1}$$

This justifies using the same power rule even when a is an irrational number.

EXERCISES FOR SECTION 6.11

1. Compute the first and second derivatives of:

   (a) $y = \ln x + 3x - 2$ (b) $y = x^2 - 2\ln x$ (c) $y = x^3 \ln x$ (d) $y = \dfrac{\ln x}{x}$

2. Find the derivatives of:

   (a) $y = x^3(\ln x)^2$ (b) $y = \dfrac{x^2}{\ln x}$ (c) $y = (\ln x)^{10}$ (d) $y = (\ln x + 3x)^2$

SM 3. Find the derivatives of:

   (a) $\ln(\ln x)$ (b) $\ln\sqrt{1 - x^2}$ (c) $e^x \ln x$ (d) $e^{x^3} \ln x^2$

   (e) $\ln(e^x + 1)$ (f) $\ln(x^2 + 3x - 1)$ (g) $2(e^x - 1)^{-1}$ (h) $e^{2x^2 - x}$

4. Determine the domains of the functions defined by:

   (a) $y = \ln(x + 1)$ (b) $y = \ln\left(\dfrac{3x - 1}{1 - x}\right)$ (c) $y = \ln|x|$

SM 5. Determine the domains of the functions defined by:

   (a) $y = \ln(x^2 - 1)$ (b) $y = \ln(\ln x)$ (c) $y = \dfrac{1}{\ln(\ln x) - 1}$

SM 6. Find the intervals where the following functions are increasing:

   (a) $y = \ln(4 - x^2)$ (b) $y = x^3 \ln x$ (c) $y = \dfrac{(1 - \ln x)^2}{2x}$

7. Find the equation for the tangent to the graph of

   (a) $y = \ln x$ at the three points with x-coordinates: 1, $\frac{1}{2}$, and e;

   (b) $y = xe^x$ at the three points with x-coordinates: 0, 1, and -2.

8. Use logarithmic differentiation to find $f'(x)/f(x)$ when:

   (a) $f(x) = x^{2x}$ (b) $f(x) = \sqrt{x - 2}(x^2 + 1)(x^4 + 6)$ (c) $f(x) = \left(\dfrac{x + 1}{x - 1}\right)^{1/3}$

SM **9.** Differentiate the following functions using logarithmic differentiation:

(a) $y = (2x)^x$ (b) $y = x^{\sqrt{x}}$ (c) $y = \left(\sqrt{x}\right)^x$

10. Prove that if u and v are differentiable functions of x, and $u > 0$, then

$$y = u^v \Rightarrow y' = u^v \left(v' \ln u + \frac{vu'}{u} \right)$$

SM **11.** [HARDER] If $f(x) = e^x - 1 - x$, then $f'(x) = e^x - 1 > 0$ for all $x > 0$. The function $f(x)$ is therefore strictly increasing in the interval $[0, \infty)$. Since $f(0) = 0$, it follows that $f(x) > 0$ for all $x > 0$, and so $e^x > 1 + x$ for all $x > 0$. Use the same method to prove the following inequalities:

(a) $e^x > 1 + x + x^2/2$ for $x > 0$ (b) $\frac{1}{2}x < \ln(1 + x) < x$ for $0 < x < 1$

(c) $\ln x < 2(\sqrt{x} - 1)$ for $x > 1$

REVIEW EXERCISES

1. Let $f(x) = x^2 - x + 2$. Show that $[f(x + h) - f(x)]/h = 2x - 1 + h$, and use this result to find $f'(x)$.

2. Let $f(x) = -2x^3 + x^2$. Compute $[f(x + h) - f(x)]/h$, and find $f'(x)$.

3. Compute the first- and second-order derivatives of the following functions:

(a) $y = 2x - 5$ (b) $y = \frac{1}{3}x^9$ (c) $y = 1 - \frac{1}{10}x^{10}$ (d) $y = 3x^7 + 8$

(e) $y = \dfrac{x - 5}{10}$ (f) $y = x^5 - x^{-5}$ (g) $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}5^2$ (h) $y = \dfrac{1}{x} + \dfrac{1}{x^3}$

4. Let $C(Q)$ denote the cost of producing Q units per month of a commodity. What is the interpretation of $C'(1000) = 25$? Suppose the price obtained per unit is fixed at 30 and that the current output per month is 1000. Is it profitable to increase production?

5. For each of the following functions, find the equation for the tangent to the graph at the specified point:

(a) $y = -3x^2$ at $x = 1$ (b) $y = \sqrt{x} - x^2$ at $x = 4$ (c) $y = \dfrac{x^2 - x^3}{x + 3}$ at $x = 1$

6. Let $A(x)$ denote the dollar cost of building a house with a floor area of x square metres. What is the interpretation of $A'(100) = 250$?

7. Differentiate the following functions:

(a) $f(x) = x(x^2 + 1)$ (b) $g(w) = w^{-5}$ (c) $h(y) = y(y - 1)(y + 1)$

(d) $G(t) = \dfrac{2t + 1}{t^2 + 3}$ (e) $\varphi(\xi) = \dfrac{2\xi}{\xi^2 + 2}$ (f) $F(s) = \dfrac{s}{s^2 + s - 2}$

8. Find the derivatives:

(a) $\dfrac{d}{da}(a^2t - t^2)$ (b) $\dfrac{d}{dt}(a^2t - t^2)$ (c) $\dfrac{d}{d\varphi}\left(x\varphi^2 - \sqrt{\varphi}\right)$

9. Use the chain rule to find dy/dx for the following:

(a) $y = 10u^2$ where $u = 5 - x^2$ (b) $y = \sqrt{u}$ where $u = \dfrac{1}{x} - 1$

10. Compute the following:

(a) dZ/dt when $Z = (u^2 - 1)^3$ and $u = t^3$ (b) dK/dt when $K = \sqrt{L}$ and $L = 1 + 1/t$

11. If $a(t)$ and $b(t)$ are positive valued differentiable functions of t, and if A, α, and β are constants, find expressions for $\dot{x}/x$ where:

(a) $x = a(t)^2 \cdot b(t)$ (b) $x = A \cdot a(t)^\alpha \cdot b(t)^\beta$ (c) $x = A \cdot [a(t)^\alpha + b(t)^\beta]^{\alpha+\beta}$

12. If $R = S^\alpha$, $S = 1 + \beta K^\gamma$, and $K = At^p + B$, find an expression for dR/dt.

13. Find the derivatives of the following functions, where a, b, p, and q are constants:

(a) $h(L) = (L^a + b)^p$ (b) $C(Q) = aQ + bQ^2$ (c) $P(x) = \left(ax^{1/q} + b\right)^q$

14. Find the first derivatives of:

(a) $y = -7e^x$ (b) $y = e^{-3x^2}$ (c) $y = \dfrac{x^2}{e^x}$ (d) $y = e^x \ln(x^2 + 2)$

(e) $y = e^{5x^3}$ (f) $y = 2 - x^4e^{-x}$ (g) $y = (e^x + x^2)^{10}$ (h) $y = \ln\left(\sqrt{x} + 1\right)$

SM **15.** Find the intervals where the following functions are increasing:

(a) $y = (\ln x)^2 - 4$ (b) $y = \ln(e^x + e^{-x})$ (c) $y = x - \frac{3}{2}\ln(x^2 + 2)$

16. (a) Suppose $\pi(Q) = QP(Q) - cQ$, where P is a differentiable function and c is a constant. Find an expression for $d\pi/dQ$.

(b) Suppose $\pi(L) = PF(L) - wL$, where F is a differentiable function and P and w are constants. Find an expression for $d\pi/dL$.

7

DERIVATIVES IN USE

Although this may seem a paradox, all science is dominated by the idea of approximation.
—Bertrand Russell

Many economic models involve functions that are defined implicitly by one or more equations. In some simple but economically relevant cases, we begin this chapter by showing how to compute derivatives of such functions, including how to differentiate the inverse. It is very important for economists to master the technique of implicit differentiation.

Next we consider linear approximations and differentials, followed by a discussion of quadratic and higher-order polynomial approximations. Section 7.6 studies Taylor's formula, which makes it possible to analyse the resulting error when a function is approximated by a polynomial. A discussion of the important economic concept of elasticity follows in Section 7.7.

The word *continuous* is common even in everyday language. We use it, in particular, to characterize changes that are gradual rather than sudden. This usage is closely related to the idea of a continuous function. In Section 7.8 we discuss this concept and explain its close relationship with the limit concept. Limits and continuity are key ideas in mathematics, and also very important in the application of mathematics to economic problems. The preliminary discussion of limits in Section 6.5 was, by necessity, very sketchy. In Section 7.9 we take a closer look at this concept and extend it in several directions.

Next we present the intermediate value theorem, which makes precise the idea that a continuous function has a "connected" graph. This makes it possible to prove that certain equations have solutions. A brief discussion of Newton's method for finding approximate solutions to equations is given. A short section on infinite sequences follows. Finally, Section 7.12 presents l'Hôpital's rule for indeterminate forms, which is sometimes useful for evaluating limits.

7.1 Implicit Differentiation

We know how to differentiate functions given by explicit formulas like $y = f(x)$. Now we consider how to differentiate functions defined implicitly by an equation such as $g(x, y) = c$, where c is a constant. We begin with a very simple case.

EXAMPLE 7.1.1 Consider the following equation in x and y,

$$xy = 5 \tag{$*$}$$

If $x = 1$, then $y = 5$. Also, $x = 3$ gives $y = 5/3$. And $x = 5$ gives $y = 1$. In general, for each number $x \neq 0$, there is a unique number y such that the pair (x, y) satisfies the equation. We say that Eq. $(*)$ *defines y implicitly as a function of x*. The graph of Eq. $(*)$ for $x > 0$ is shown in Fig. 7.1.1.

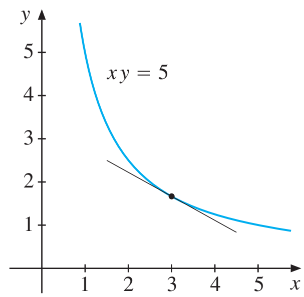


Figure 7.1.1 $xy = 5$, with $x > 0$

Economists often need to know the slope of the tangent at an arbitrary point on such a graph—that is, to know the derivative of y as a function of x. The answer can be found by implicit differentiation of Eq. $(*)$, which defines y as a function of x. If we denote this function by f, then replacing y by $f(x)$ gives

$$xf(x) = 5 \text{ for all } x > 0 \tag{$**$}$$

Because the left and right sides of the equation are equal for all $x > 0$, the derivative of the left-hand side w.r.t. x must be equal to the derivative of the right-hand side w.r.t. x. The derivative of the constant 5 is 0. When we differentiate $xf(x)$, we must use the product rule. Therefore, by differentiating $(**)$ w.r.t. x, we obtain

$$1 \cdot f(x) + xf'(x) = 0$$

It follows that for $x > 0$,

$$f'(x) = -\frac{f(x)}{x}$$

If $x = 3$, then $f(3) = 5/3$, and thus $f'(3) = -(5/3)/3 = -5/9$, which agrees with Fig. 7.1.1.

Usually, we do not introduce a name like f for y as a function of x. Instead, we differentiate $(*)$ w.r.t. x, while recalling that y is a differentiable function of x. Again, because of the product rule, this leads to $y + xy' = 0$. Solving for y' gives

$$y' = -\frac{y}{x} \tag{$***$}$$

For this particular example, there is another way to find the answer. Solving Eq. $(*)$ for y gives $y = 5/x = 5x^{-1}$, and hence direct differentiation gives $y' = 5(-1)x^{-2} = -5/x^2$. Note that substituting $5/x$ for y in $(***)$ yields $y' = -5/x^2$ again.

EXAMPLE 7.1.2 The graph of

$$y^3 + 3x^2y = 13 \tag{$*$}$$

was studied in Example 5.4.2 and is drawn larger in Fig. 7.1.2. It passes through the point $(2, 1)$. Find the slope of the graph at that point.

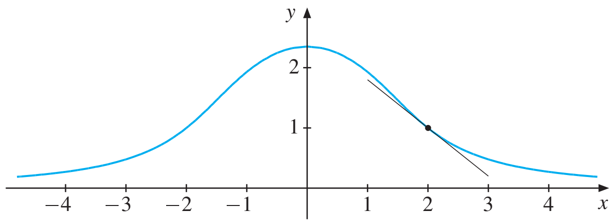


Figure 7.1.2 The graph of $y^3 + 3x^2y = 13$

Solution: Since in this case there is no simple way of expressing y as an explicit function of x, we use implicit differentiation. We think of replacing y with an unspecified function of x wherever y occurs. Then $y^3 + 3x^2y$ becomes a function of x which is equal to the constant 13 for all x. So the derivative of $y^3 + 3x^2y$ w.r.t. x must be equal to zero for all x. According to the chain rule, the derivative of y^3 w.r.t. x is equal to $3y^2y'$. Using the product rule, the derivative of $3x^2y$ is equal to $6xy + 3x^2y'$. Hence, differentiating $(*)$ gives

$$3y^2y' + 6xy + 3x^2y' = 0 \tag{$**$}$$

Solving this equation for y' yields

$$y' = \frac{-6xy}{3x^2 + 3y^2} = \frac{-2xy}{x^2 + y^2} \tag{7.1.1}$$

For $x = 2$, $y = 1$ we find $y' = -4/5$, which agrees with Fig. 7.1.2.[1]

Examples 7.1.1 and 7.1.2 illustrate the following general method.

IMPLICIT DIFFERENTIATION

To find y' when an equation relates two variables x and y:

(i) Differentiate each side of the equation w.r.t. x, considering y as a function of x.

(ii) Solve the resulting equation for y'.

We note that usually the chain rule will be needed at the first of these two steps.

The next section shows several economic examples of this procedure. A particularly important application of this method occurs in the next chapter where we consider what happens to the solution of an optimization problem whose parameters change.

[1] Recall Fig. 5.4.3.

EXAMPLE 7.1.3 The equation $x^2y^3 + (y+1)e^{-x} = x + 2$ defines y as a differentiable function of x in a neighbourhood of $(x, y) = (0, 1)$. Compute y' at this point.

Solution: Implicit differentiation w.r.t. x gives

$$2xy^3 + x^2 3y^2 y' + y'e^{-x} + (y+1)(-e^{-x}) = 1$$

Inserting $x = 0$ and $y = 1$ yields $y' + 2(-1) = 1$, implying that $y' = 3$.

EXAMPLE 7.1.4 Suppose y is defined implicitly as a function of x by the equation

$$g(xy^2) = xy + 1 \tag{$*$}$$

where g is a given differentiable function of one variable. Find an expression for y'.

Solution: We differentiate each side of the equation w.r.t. x, considering y as a function of x. The derivative of $g(xy^2)$ w.r.t. x is $g'(xy^2)(y^2 + x2yy')$. So differentiating $(*)$ yields $g'(xy^2)(y^2 + x2yy') = y + xy'$. Solving for y' gives us

$$y' = \frac{y\left[yg'(xy^2) - 1\right]}{x\left[1 - 2yg'(xy^2)\right]}$$

EXAMPLE 7.1.5 Suppose that a person has to decide how much of her current income she will save for future consumption.[2] It is common to assume that there exists a so-called utility function, $u(c)$, defined over the positive real numbers, that measures the value to the consumer of consuming c at a given period. If she consumes c_t in year t, her "instantaneous utility" is $u(c_t)$. One normally assumes that the individual is impatient and values present consumption more than future consumption. Assuming for simplicity that the individual lives for only two periods, her "intertemporal utility" is modelled as

$$u(c_1) + \beta u(c_2)$$

where $0 < \beta < 0$ is a constant that measures the individual impatience—her "discount factor".

Consider now a savings policy that will reduce the individual's present consumption, c_1. By how much will her future consumption have to change, if she is to be compensated so that her intertemporal utility remains constant? Suppose that without the policy her utility level is $\bar{U}$. Denoting $x = c_1$ and $y = c_2$, so as to keep with our previous notation, it follows that y is implicitly defined by

$$u(x) + \beta u(y) = \bar{U} \tag{$*$}$$

as a function of x. The question we want to answer is by how much does y change when we change x by a small amount, which is measured by y'. Using implicit differentiation on $(*)$, we have

$$u'(x) + \beta u'(y)y' = 0$$

[2] Example 8.5.4 will study this problem further.

implying that

$$y' = -\frac{u'(x)}{u'(y)} \tag{7.1.2}$$

It is normal to assume that $u'(c) > 0$ for all c, so that the individual prefers to consume more. Under that assumption, $y' < 0$, which is as it should be: if the individual's present consumption *decreases*, her future consumption *must increase* if she is to remain indifferent. The ratio $u'(x)/u'(y)$ is an example of what is known in economics as the "marginal rate of substitution", explored further in Section 12.5.

The Second Derivative of Functions Defined Implicitly

The following examples suggest how to compute the second derivative of a function that is defined implicitly by an equation.

EXAMPLE 7.1.6 Compute y'' when y is given implicitly as a function of x by

$$xy = 5$$

Solution: In Example 7.1.1 we used implicit differentiation to find that $y + xy' = 0$. Differentiating this equation implicitly w.r.t. x once more, while recognizing that both y and y' depend on x, gives us

$$y' + y' + xy'' = 0$$

Inserting the expression $-y/x$ we already have for y' gives $-2y/x + xy'' = 0$. Solving for y'' finally yields

$$y'' = \frac{2y}{x^2}$$

We see that if $y > 0$, then $y'' > 0$, which accords with Fig. 7.1.1 since the graph is convex. Because $y = 5/x$, we also get $y'' = 10/x^3$.

In this simple case, we can check the answer directly. Since $y = 5x^{-1}$ and $y' = -5x^{-2}$, we have $y'' = 10x^{-3}$.

In order to find y'' we can also use formula $(***)$ in Example 7.1.1 and differentiate the fraction w.r.t. x, again taking into account that y depends on x. This gives:

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{y}{x}\right) = -\frac{y'x - y}{x^2} = -\frac{(-y/x)x - y}{x^2} = \frac{2y}{x^2}$$

EXAMPLE 7.1.7 For the function defined by the equation $y^3 + 3x^2y = 13$ in Example 7.1.2, find y'' at the point $(2, 1)$.

Solution: The easiest approach is to differentiate Eq. $(**)$ in Example 7.1.2 w.r.t. x. The derivative of $3y^2y'$ w.r.t. x is $(6yy')y' + 3y^2y'' = 6y(y')^2 + 3y^2y''$. The two other terms are differentiated in the same way, and we obtain

$$6y(y')^2 + 3y^2y'' + 6y + 6xy' + 6xy' + 3x^2y'' = 0$$

Now insert $x = 2$, $y = 1$, and the value $y' = -4/5$ that was already found in Example 7.1.2. Solving the resulting equation gives $y'' = 78/125$.

An obvious alternative approach starts by taking the fraction on the right-hand side of (7.1.1), and then differentiates it w.r.t. x.

EXAMPLE 7.1.8 Recall the intertemporal decision problem studied in Example 7.1.5. The pairs (x, y) that satisfy $(*)$ are the combinations of present and future consumption that leave the individual's intertemporal utility constant. The set of all such pairs is known as the consumer's *indifference curve*, which is shown in Fig. 7.1.3. The result of Example 7.1.5 tells us that the graph of the function defined implicitly by $(*)$ is downward sloping, as shown in Fig. 7.1.3. This graph is precisely the indifference curve.

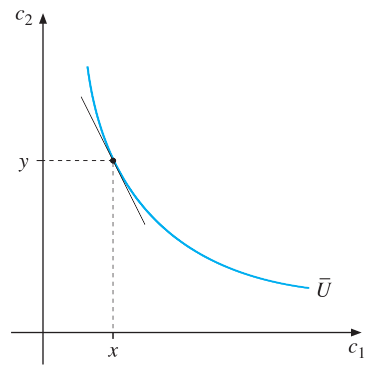


Figure 7.1.3 An indifference curve

Using the results of Section 6.9, we may want to understand the sign of y'', in order to know more about the shape of the indifference curve. Now, differentiating (7.1.2) gives

$$y'' = -\frac{u'(y)u''(x) - u'(x)u''(y)y'}{[u'(y)]^2} = -\frac{1}{[u'(y)]^2}\left(u'(y)u''(x) + \frac{[u'(x)]^2}{u'(y)}u''(y)\right) \qquad (*)$$

where we have used (7.1.2) a second time in order to substitute for y'. Economists normally assume that the first derivative of u, while positive, is decreasing. The idea is that each additional unit of consumption gives the individual a smaller increase in utility than the previous one. This means that $u''(c) < 0$, which is to say that u is a concave function. Now, that concavity assumption implies that $u''(x) < 0$ and $u''(y) < 0$ in the previous expression, and, therefore, that $y'' > 0$. So the assumption that the utility function is concave implies that the indifference curve, seen as a function $y(x)$, is convex.

Figure 7.1.3 presents a typical indifference curve. The straight line is the tangent to the indifference curve at the point (x, y). Its slope is y', given by Eq. (7.1.2). The absolute value of that slope is the marginal rate of substitution; it is called the "intertemporal" marginal rate of substitution in the context of this example.

Now, Eq. $(*)$ tells us that $y'' > 0$, so that y' is increasing. The absolute value of y' is therefore decreasing, which is reflected by the fact that the indifference curve becomes flatter as one moves down and to the right. The intuition is that the increase in future consumption required to compensate a sacrifice in present consumption is lower, the higher the person's present consumption is.

EXERCISES FOR SECTION 7.1

1. Find y' by implicit differentiation if $3x^2 + 2y = 5$. Check by solving the equation for y and then differentiating.

2. For the equation $x^2y = 1$, find dy/dx and d^2y/dx^2 by implicit differentiation. Check by solving the equation for y and then differentiating.

SM 3. Find dy/dx and d^2y/dx^2 by implicit differentiation when: (a) $x - y + 3xy = 2$; and (b) $y^5 = x^6$.

4. A curve in the uv-plane is given by $u^2 + uv - v^3 = 0$. Compute dv/du by implicit differentiation. Find the point (u, v) on the curve where $dv/du = 0$ and $u \neq 0$.

5. Suppose that y is a differentiable function of x that satisfies the equation $2x^2 + 6xy + y^2 = 18$. Find y' and y'' at the point $(x, y) = (1, 2)$.

6. For each of the following equations, answer the question: If $y = f(x)$ is a differentiable function that satisfies the equation, what is y'? Here, a is a positive constant.

   (a) $x^2 + y^2 = a^2$ (b) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (c) $x^4 - y^4 = x^2y^3$ (d) $e^{xy} - x^2y = 1$

7. Consider the curve $2xy - 3y^2 = 9$.

   (a) Find the slope of the tangent line to the curve at $(x, y) = (6, 1)$.

   (b) Compute also the second derivative at the point.

SM 8. In each of the following equations, suppose g is a given differentiable function of one variable. Suppose the equation defines y implicitly as a function of x. Find an expression for y' in each case.

   (a) $xy = g(x) + y^3$ (b) $g(x + y) = x^2 + y^2$ (c) $(xy + 1)^2 = g(x^2y)$

9. Suppose F is a differentiable function of one variable, with $F(0) = 0$ and $F'(0) \neq -1$, and that y is defined implicitly as a differentiable function of x by the equation

$$x^3F(xy) + e^{xy} = x$$

   Find an expression for y' at the point $(x, y) = (1, 0)$.

SM 10. The elegant curve shown in Fig. 7.1.4 is known as a *lemniscate*. In the late 1600s, the Swiss mathematician Johann Bernoulli (1667–1748) discovered that it is the graph of the equation

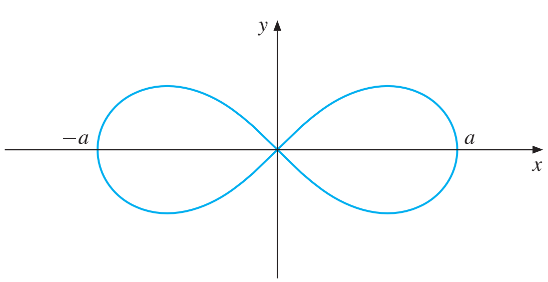


Figure 7.1.4 A lemniscate

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

where a is a positive constant.

(a) Find the slope of the tangent to this curve at any point (x, y) where $y \neq 0$.

(b) Determine those points on the curve where the tangent is parallel to the x-axis.

7.2 Economic Examples

Few mathematical techniques are more important in economics than implicit differentiation. This is because so many functions in economic models are defined implicitly by an equation or by a system of equations. Often the variables have names other than x and y, so one needs to practise differentiating equations with other names for the variables.

EXAMPLE 7.2.1 A more general version of the standard macroeconomic model for determining national income that we saw in Example 3.2.1 is that: (i) $Y = C + \bar{I}$; and (ii) $C = f(Y)$. Here, (ii) is the consumption function discussed in Example 4.5.2, whereas (i) states that GDP, Y, is divided up between consumption, C, and investment, $\bar{I}$, which is assumed to be exogenous. Assume that $f'(Y)$, the *marginal propensity to consume*, exists and lies between 0 and 1.

(a) Suppose first that $C = f(Y) = 95.05 + 0.712Y$, as in Example 4.5.2, and use equations (i) and (ii) to find Y in terms of $\bar{I}$.

(b) Inserting the expression for C from (ii) into (i) gives $Y = f(Y) + \bar{I}$. Suppose that this equation defines Y as a differentiable function of $\bar{I}$. Find an expression for $\mathrm{d}Y/\mathrm{d}\bar{I}$.

(c) Assuming that $f''(Y)$ also exists, find $Y'' = \mathrm{d}^2Y/\mathrm{d}\bar{I}^2$.

Solution:

(a) In this case, we find that $Y = 95.05 + 0.712Y + \bar{I}$. Solving for Y yields

$$Y = (95.05 + \bar{I})/(1 - 0.712) \approx 3.47\bar{I} + 330.03$$

In particular, $\mathrm{d}Y/\mathrm{d}\bar{I} \approx 3.47$, so if $\bar{I}$ is increased by \$1 billion, then the corresponding increase in GDP is approximately \$3.47 billion.

(b) Differentiating $Y = f(Y) + \bar{I}$ w.r.t. $\bar{I}$, and using the chain rule, we have

$$\frac{\mathrm{d}Y}{\mathrm{d}\bar{I}} = f'(Y)\frac{\mathrm{d}Y}{\mathrm{d}\bar{I}} + 1 \quad \text{or} \quad \frac{\mathrm{d}Y}{\mathrm{d}\bar{I}}[1 - f'(Y)] = 1 \tag{$*$}$$

Solving for $\mathrm{d}y/\mathrm{d}\bar{I}$ yields

$$\frac{\mathrm{d}Y}{\mathrm{d}\bar{I}} = \frac{1}{1 - f'(Y)} \tag{$**$}$$

For example, if $f'(Y) = 1/2$, then $\mathrm{d}Y/\mathrm{d}\bar{I} = 2$. If $f'(Y) = 0.712$, then $\mathrm{d}Y/\mathrm{d}\bar{I} \approx 3.47$. In general, we see that because of the assumption that $f'(Y)$ lies between 0 and 1, so $1 - f'(Y)$ also lies between 0 and 1. Hence $1/[1 - f'(Y)]$ is always greater than 1. In this model, therefore, a \$1 billion increase in investment will always lead to a more than \$1 billion increase in GDP. Also, the greater is $f'(Y)$, the marginal propensity to consume, the smaller is $1 - f'(Y)$, and so the greater is $\mathrm{d}Y/\mathrm{d}\bar{I}$.

(c) We differentiate the first equation in $(*)$ implicitly w.r.t. $\bar{I}$. The derivative of $f'(Y)$ w.r.t. $\bar{I}$ is $f''(Y)(\mathrm{d}Y/\mathrm{d}\bar{I})$. According to the product rule, the derivative of the product $f'(Y)(\mathrm{d}Y/\mathrm{d}\bar{I})$ w.r.t. $\bar{I}$ is

$$\frac{\mathrm{d}}{\mathrm{d}\bar{I}}\left(f'(Y)\frac{\mathrm{d}Y}{\mathrm{d}\bar{I}}\right) = f''(Y)\frac{\mathrm{d}Y}{\mathrm{d}\bar{I}}\frac{\mathrm{d}Y}{\mathrm{d}\bar{I}} + f'(Y)\frac{\mathrm{d}^2Y}{\mathrm{d}\bar{I}^2}$$

Hence,

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\bar{I}^2} = f''(Y)\left(\frac{\mathrm{d}Y}{\mathrm{d}\bar{I}}\right)^2 + f'(Y)\frac{\mathrm{d}^2Y}{\mathrm{d}\bar{I}^2}$$

Since $\mathrm{d}Y/\mathrm{d}\bar{I} = 1/(1 - f'(Y))$, easy algebra yields

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\bar{I}^2} = \frac{f''(Y)}{[1 - f'(Y)]^3}$$

EXAMPLE 7.2.2 In the linear supply and demand model of Example 4.5.4, suppose that consumers are required to pay a tax of τ per unit, thus raising the price they face from P to $P + \tau$. Then

$$D = a - b(P + \tau), \quad S = \alpha + \beta P \tag{7.2.1}$$

Here a, b, α, and β are positive constants. The equilibrium price is determined by equating supply and demand, so that

$$a - b(P + \tau) = \alpha + \beta P \tag{7.2.2}$$

(a) Equation (7.2.2) implicitly defines the price P as a function of the unit tax τ. Compute $\mathrm{d}P/\mathrm{d}\tau$ by implicit differentiation. What is its sign? What is the sign of $(\mathrm{d}/\mathrm{d}\tau)(P + \tau)$? Check the result by first solving Eq. (7.2.2) for P and then finding $\mathrm{d}P/\mathrm{d}\tau$ explicitly.

(b) Compute tax revenue T as a function of τ. For what value of τ does the quadratic function T reach its maximum?

(c) Generalize the foregoing model by assuming that $D = f(P + \tau)$ and $S = g(P)$, where f and g are differentiable functions with $f' < 0$ and $g' > 0$. The equilibrium condition $f(P + \tau) = g(P)$ defines P implicitly as a differentiable function of τ. Find an expression for $\mathrm{d}P/\mathrm{d}\tau$ by implicit differentiation. Illustrate geometrically.

Solution:

(a) Differentiating (7.2.2) w.r.t. τ yields $-b\left(\frac{\mathrm{d}P}{\mathrm{d}\tau} + 1\right) = \beta\,\frac{\mathrm{d}P}{\mathrm{d}\tau}$. Solving for $\frac{\mathrm{d}P}{\mathrm{d}\tau}$ gives

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{-b}{b + \beta}$$

We see that $\mathrm{d}P/\mathrm{d}\tau$ is negative. Because P is the price received by the producer, this price will go down if the tax rate τ increases. But $P + \tau$ is the price paid by the consumer. Because

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(P+\tau) = \frac{\mathrm{d}P}{\mathrm{d}\tau} + 1 = \frac{-b}{b+\beta} + 1 = \frac{-b+b+\beta}{b+\beta} = \frac{\beta}{b+\beta}$$

we see that $0 < \mathrm{d}(P+t)/\mathrm{d}\tau < 1$. Thus, the consumer price $P + \tau$ increases, but by less than the increase in the tax.

If we solve (7.2.2) for P, we obtain

$$P = \frac{a-\alpha-b\tau}{b+\beta} = \frac{a-\alpha}{b+\beta} - \frac{b}{b+\beta}\tau$$

This equation shows that the equilibrium producer price P is a linear function of τ, the tax per unit, with slope $-b/(b+\beta)$.

(b) The total tax revenue is $T = S\tau = (\alpha + \beta P)\tau$, where P is the equilibrium price. Thus,

$$T = \left[\alpha + \beta\left(\frac{a-\alpha}{b+\beta} - \frac{b}{b+\beta}\tau\right)\right]\tau = \frac{-b\beta}{b+\beta}\tau^2 + \frac{\alpha b + \beta a}{b+\beta}\tau$$

This quadratic function has its maximum at $\tau = (\alpha b + \beta a)/2b\beta$.

(c) Differentiating the equation $f(P+\tau) = g(P)$ w.r.t. τ yields

$$f'(P+\tau)\left(\frac{\mathrm{d}P}{\mathrm{d}\tau} + 1\right) = g'(P)\frac{\mathrm{d}P}{\mathrm{d}\tau} \tag{7.2.3}$$

Solving for $\mathrm{d}P/\mathrm{d}\tau$ gives

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{f'(P+\tau)}{g'(P) - f'(P+\tau)}$$

Because $f' < 0$ and $g' > 0$, we see that $\mathrm{d}P/\mathrm{d}\tau$ is negative in this case as well. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(P+\tau) = \frac{\mathrm{d}P}{\mathrm{d}\tau} + 1 = \frac{f'(P+\tau)}{g'(P) - f'(P+\tau)} + 1 = \frac{g'(P)}{g'(P) - f'(P+\tau)}$$

Again, because $f' < 0$ and $g' > 0$, this implies that $0 < \mathrm{d}(P+\tau)/\mathrm{d}\tau < 1$.

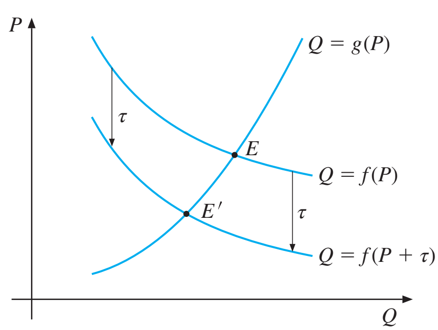


Figure 7.2.1 Shift in the demand curve

Figure 7.2.1 has a graph which illustrates this answer. As usual in economics, we have quantity on the horizontal axis, and price on the vertical axis. The demand curve with the tax is represented by the curve $Q = f(P + \tau)$. Its graph is obtained by shifting the graph of $Q = f(P)$—or equivalently, the graph of the inverse demand curve $P = f^{-1}(Q)$—down by τ units, so it becomes $P = f^{-1}(Q) - \tau$, or $Q = f(P + \tau)$. The figure confirms that, when τ increases, the new equilibrium E' corresponds to a decreased price P. Nevertheless, $P + \tau$ increases because the decrease in P is smaller than the increase in τ.

EXERCISES FOR SECTION 7.2

1. According to a study, the demand quantity Q for butter in Stockholm during the period 1925–1937 was related to the price P by the equation $Q \cdot P^{1/2} = 38$. Find $\mathrm{d}Q/\mathrm{d}P$ by implicit differentiation. Check the answer by using a different method to compute the derivative.

2. Consider a profit-maximizing firm producing a single commodity. If the firm gets a fixed price P per unit sold, its profit from selling Q units is $\pi(Q) = PQ - C(Q)$, where $C(Q)$ is the cost function. Assume that $C'(Q) > 0$ and $C''(Q) > 0$. In Example 8.5.1, it will be shown that $Q = Q^* > 0$ maximizes profits w.r.t. Q provided that

$$P = C'(Q^*) \tag{$*$}$$

Thus, at the optimum, marginal cost must equal the price per unit.

(a) By implicitly differentiating ($*$) w.r.t. P, find an expression for $\mathrm{d}Q^*/\mathrm{d}P$.

(b) Comment on the sign of $\mathrm{d}Q^*/\mathrm{d}P$.

3. Consider the equation $AP^{-\alpha}r^{-\beta} = S$ where A, α, β, and S are positive constants. The left-hand side of the equation expresses the demand for a commodity as a decreasing function of both its price P and the interest rate r. In equilibrium, this demand must equal a fixed supply quantity S.

(a) Take natural logarithms of both sides and find $\mathrm{d}P/\mathrm{d}r$ by implicit differentiation.

(b) How does the equilibrium price react to an increase in the interest rate?

SM 4. Extending the standard macroeconomic model of Example 7.2.1 for an economy open to international trade gives:

(i) $Y = C + \bar{I} + \bar{X} - M$; (ii) $C = f(Y)$, with $0 < f'(Y) < 1$; (iii) $M = g(Y)$.

Here $\bar{X}$ is an exogenous constant that denotes exports, whereas M denotes the volume of imports. The function g in (iii) is called an *import function*, which is assumed to satisfy $0 < g'(Y) < f'(Y)$.

(a) By inserting (ii) and (iii) into (i), obtain an equation that defines Y as a function of exogenous investment $\bar{I}$.

(b) Find an expression for $\mathrm{d}Y/\mathrm{d}\bar{I}$ by implicit differentiation. Discuss the sign of $\mathrm{d}Y/\mathrm{d}\bar{I}$.

(c) Find an expression for $\mathrm{d}^2Y/\mathrm{d}\bar{I}^2$.

5. In part (c) of Example 7.2.2, find an expression for $\mathrm{d}^2P/\mathrm{d}t^2$ by differentiating (7.2.3) w.r.t. t.

6. In Example 7.2.2 we studied a model of supply and demand where a tax is imposed on the consumers. Instead, suppose that the producers have to pay a tax per unit sold that is equal to

a fraction t of the sales price P they receive, where $0 < t < 1$. This implies that the equilibrium condition with the tax is

$$f(P) = g(P - tP) \tag{$*$}$$

We assume that $f' < 0$ and $g' > 0$.

(a) Differentiate ($*$) w.r.t. t and find an expression for $\mathrm{d}P/\mathrm{d}t$.

(b) Find the sign of $\mathrm{d}P/\mathrm{d}t$ and give an economic interpretation.

7.3 Differentiating the Inverse

Section 5.3 dealt with inverse functions. As explained there, if f is a one-to-one function defined on an interval I, it has an inverse function g defined on the range $f(I)$ of f. What relationship is there between the derivatives of f and g?

EXAMPLE 7.3.1 Provided that $a \neq 0$, the two linear functions $f(x) = ax + b$ and $g(x) = (x - b)/a$ are inverses of each other, as you should verify. The graphs are straight lines which are symmetric about the line $y = x$. The slopes are respectively a and $1/a$. Look back at Fig. 5.3.3, and notice that this result is confirmed, since the slope of f is 4 and the slope of g is $1/4$.

Recall that, if f and g are inverses of each other, then, for all x in I,

$$g\left(f(x)\right) = x \tag{7.3.1}$$

By implicit differentiation, *provided that* both f and g are differentiable, it is easy to find the relationship between the derivatives of f and g. Indeed, differentiating (7.3.1) w.r.t. x gives $g'\left(f(x)\right)f'(x) = 1$. Hence, at any x where $f'(x) \neq 0$, one has $g'\left(f(x)\right) = 1/f'(x)$.

The most important facts about inverse functions are summed up in this theorem:

THEOREM 7.3.1 (INVERSE FUNCTION THEOREM)

If f is differentiable and strictly increasing (strictly decreasing) in an interval I, then f has an inverse function g, which is strictly increasing (strictly decreasing) in the interval $f(I)$. If x_0 is an interior point of I and $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$, and

$$g'(y_0) = \frac{1}{f'(x_0)} \tag{7.3.2}$$

Formula (7.3.2) is used as follows to find the derivative of g at a point y_0. First find, if possible, the point x_0 in I at which $f(x_0) = y_0$. Thereafter, compute $f'(x)$, and then find $f'(x_0)$. If $f'(x_0) \neq 0$, then g has a derivative at y_0 given by $g'(y_0) = 1/f'(x_0)$. An implication of (7.3.2) is that f' and g' must have the same sign. So if f is strictly increasing (decreasing), then g is strictly increasing (decreasing), and vice versa.

The geometric interpretation of formula (7.3.2) is shown in Fig. 7.3.1, where f and g are inverses of each other. Let the slope of the tangent at P be $a = f'(x_0)$; in the figure $a \approx 1/3$. At the point Q on the graph of the inverse function is $f(x_0)$, and the slope of the tangent at that point is $g'(f(x_0))$. This number is equal to $1/a$.

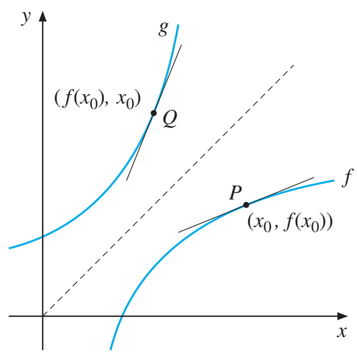


Figure 7.3.1 If the slope at P is a, then the slope at Q is $1/a$

EXAMPLE 7.3.2 Suppose the function f is defined for all real x by the following formula: $f(x) = x^5 + 3x^3 + 6x - 3$. Show that f has an inverse function g, and then, given that $f(1) = 7$, use formula (7.3.2) to find $g'(7)$.

Solution: Differentiating $f(x)$ yields $f'(x) = 5x^4 + 9x^2 + 6$. Clearly, $f'(x) > 0$ for all x, so f is strictly increasing and consequently it is one-to-one. It therefore has an inverse function g. To find $g'(7)$, we use formula (7.3.2) with $x_0 = 1$ and $y_0 = 7$. Since $f'(1) = 20$, we obtain $g'(7) = 1/f'(1) = 1/20$. Note that we have found $g'(7)$ exactly even though it is impossible to find any algebraic formula for the inverse function g.

EXAMPLE 7.3.3 Suppose that f and g are twice differentiable functions which are inverses of each other. By differentiating $g'(f(x)) = 1/f'(x)$ w.r.t. x, find an expression for $g''(f(x))$ where $f'(x) \neq 0$. Do f'' and g'' have the same, or opposite signs?

Solution: Differentiating $g'(f(x)) = 1/f'(x)$ w.r.t. x yields

$$g''(f(x))f'(x) = (-1)\left(f'(x)\right)^{-2} f''(x)$$

It follows that, if $f'(x) \neq 0$, then

$$g''(f(x)) = -\frac{f''(x)}{(f'(x))^3} \tag{7.3.3}$$

If $f' > 0$, then $f''(x)$ and $g''(f(x))$ have opposite signs, but they have the same sign if $f' < 0$. In particular, if f is increasing and concave, the inverse g is increasing and convex, as shown in Fig. 7.3.1.

It is common to present the formula in (7.3.2) in the deceptively simple way:

$$\frac{dx}{dy} = \frac{1}{dy/dx} \tag{7.3.4}$$

as if dx and dy could be manipulated like ordinary numbers. Formula (7.3.4) shows that similar use of the differential notation for second derivatives fails drastically. The formula "$\mathrm{d}^2x/\mathrm{d}y^2 = 1/(\mathrm{d}y^2/\mathrm{d}^2x)$", for instance, makes no sense at all.

EXAMPLE 7.3.4 Suppose that, instead of the linear demand function of Example 4.5.4, one has the *log-linear* function $\ln Q = a - b \ln P$.

(a) Express Q as a function of P, and show that $\mathrm{d}Q/\mathrm{d}P = -bQ/P$.

(b) Express P as a function of Q, and find $\mathrm{d}P/\mathrm{d}Q$.

(c) Check that your answer satisfies the version $\mathrm{d}P/\mathrm{d}Q = 1/(\mathrm{d}Q/\mathrm{d}P)$ of (7.3.4).

Solution:

(a) Taking exponentials gives $Q = e^{a-b\ln P} = e^a(e^{\ln P})^{-b} = e^aP^{-b}$, from which it follows that $\mathrm{d}Q/\mathrm{d}P = -be^aP^{-b-1} = -bQ/P$.

(b) Solving $Q = e^aP^{-b}$ for P gives $P = e^{a/b}Q^{-1/b}$, so $\mathrm{d}P/\mathrm{d}Q = (-1/b)e^{a/b}Q^{-1-1/b}$.

(c) From part (b) one has $\mathrm{d}P/\mathrm{d}Q = (-1/b)P/Q = 1/(\mathrm{d}Q/\mathrm{d}P)$.

Note that in this example the inverse demand function $P = e^{a/b}Q^{-1/b}$ is also log-linear.

EXERCISES FOR SECTION 7.3

1. The function defined for all x by $f(x) = e^{2x-2}$ has an inverse g. Find x such that $f(x) = 1$. Then use (7.3.2) to find $g'(1)$. Check your result by finding a formula for g.

2. The function f is defined, for $-2 \le x \le 2$, by the formula $f(x) = \frac{1}{3}x^3\sqrt{4-x^2}$.

   (a) Find the intervals where f increases, and the intervals where f decreases, then sketch its graph.

   (b) Explain why f has an inverse g on $[0, \sqrt{3}]$, and find $g'(\frac{1}{3}\sqrt{3})$. (*Hint*: $f(1) = \frac{1}{3}\sqrt{3}$.)

3. Let f be defined by $f(x) = \ln(2 + e^{x-3})$, for all x.

   (a) Show that f is strictly increasing and find the range of f.

   (b) Find an expression for the inverse function, g, of f. Where is g defined?

   (c) Verify that $f'(3) = 1/g'(f(3))$.

4. According to Exercise 5.3.2, during the period 1915–1929 the demand for sugar in the USA, as a function of the price P, was given by $D = 157.8/P^{0.3}$. Use (7.3.4) to find $\mathrm{d}P/\mathrm{d}D$.

SM 5. Use (7.3.4) to find $\mathrm{d}x/\mathrm{d}y$ when:

   (a) $y = e^{-x-5}$ (b) $y = \ln(e^{-x} + 3)$ (c) $xy^3 - x^3y = 2x$

7.4 Linear Approximations

Much of modern economic analysis relies on numerical calculations, nearly always only approximate. Often, therefore, rather than work with a complicated function, we approximate it by one that is much simpler. Since linear functions are especially simple, it seems natural to try using a "linear approximation" first.

Consider a function $f(x)$ that is differentiable at $x = a$. Suppose we approximate the graph of f by its tangent line at $(a, f(a))$, as shown in Fig. 7.4.1. This tangent line is the graph of the function $y = p(x) = f(a) + f'(a)(x - a)$, as we saw in formula (6.2.3).

THE LINEAR APPROXIMATION TO f ABOUT $x = a$

For x close to a,

$$f(x) \approx f(a) + f'(a)(x - a) \tag{7.4.1}$$

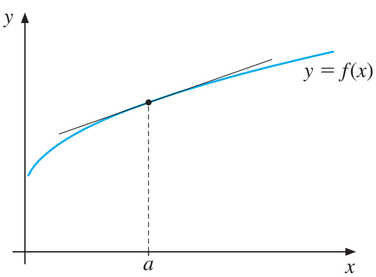


Figure 7.4.1 Approximation of a function by its tangent

Note that $f(x)$ and its linear approximation $p(x) = f(a) + f'(a)(x - a)$ have the same value and also the same derivative at $x = a$.[3]

EXAMPLE 7.4.1 Find the linear approximation to $f(x) = \sqrt[3]{x}$ about $x = 1$.

Solution: We have $f(x) = \sqrt[3]{x} = x^{1/3}$, so $f(1) = 1$, and $f'(x) = \frac{1}{3}x^{-2/3}$, implying that $f'(1) = \frac{1}{3}$. Inserting these values into (7.4.1), when $a = 1$, yields

$$\sqrt[3]{x} \approx f(1) + f'(1)(x - 1) = 1 + \tfrac{1}{3}(x - 1) \qquad (x \text{ close to } 1)$$

For example, $\sqrt[3]{1.03} \approx 1 + \frac{1}{3}(1.03 - 1) = 1.01$. The correct value to four decimals is 1.0099.

EXAMPLE 7.4.2 Use (7.4.1) to show that $\ln(1 + x) \approx x$ for x close to 0.

Solution: With $f(x) = \ln(1 + x)$, we get $f(0) = 0$ and $f'(x) = 1/(1 + x)$, implying that $f'(0) = 1$. Then (1) yields $\ln(1 + x) \approx x$.

---

[3] One can prove that if f is differentiable, then $f(x) - f(a) = [f'(a) + \varepsilon(x)](x - a)$ where $\varepsilon(x) \to 0$ as $x \to a$. That is, if $x - a$ is very small, then $\varepsilon(x)$ is very small, and $\varepsilon(x)(x - a)$ is "very very small".

EXAMPLE 7.4.3 **(Rule of 70)** If an amount K earns interest at the rate of $p\%$ a year, the doubling time of money in the account is $t^* = \ln 2 / \ln(1 + p/100)$ — see Example 4.10.4. The approximations $\ln 2 \approx 0.7$ and $\ln(1 + x) \approx x$ imply that

$$t^* = \frac{\ln 2}{\ln(1 + p/100)} \approx \frac{0.7}{p/100} = \frac{70}{p}$$

This yields the "rule of 70" according to which, if the interest rate is $p\%$ per year, then the doubling time is approximately 70 divided by p. For instance, if $p = 3.5$, then t^* is 20, which is close to the exact value $t^* = \ln 2 / \ln 1.035 \approx 20.1$.[4]

EXAMPLE 7.4.4 Use (7.4.1) to find an approximate value for $(1.001)^{50}$.

Solution: We put $f(x) = x^{50}$. Then $f(1) = 1$ and $f'(x) = 50x^{49}$, implying that $f'(1) = 50 \cdot 1^{49} = 50$. Hence, by formula (7.4.1), with $x = 1.001$ and $a = 1$,

$$(1.001)^{50} \approx 1 + 50 \cdot 0.001 = 1.05$$

(Using a calculator, we find $(1.001)^{50} \approx 1.0512$.)

The Differential of a Function

Consider a differentiable function $f(x)$, and let $\mathrm{d}x$ denote an arbitrary change in the variable x. In this notation, "$\mathrm{d}x$" is not a product of d and x. Rather, $\mathrm{d}x$ is a single symbol representing the change in the value of x. The expression $f'(x)\,\mathrm{d}x$ is called the *differential* of $y = f(x)$, and it is denoted by $\mathrm{d}y$ (or $\mathrm{d}f$), so that

$$\mathrm{d}y = f'(x)\,\mathrm{d}x \tag{7.4.2}$$

Note that $\mathrm{d}y$ is proportional to $\mathrm{d}x$, with $f'(x)$ as the factor of proportionality.

Now, if x changes by $\mathrm{d}x$, then the corresponding change in $y = f(x)$ is

$$\Delta y = f(x + \mathrm{d}x) - f(x) \tag{7.4.3}$$

In approximation (7.4.1), suppose we replace x by $x + \mathrm{d}x$ and a by x. The result is $f(x + \mathrm{d}x) \approx f(x) + f'(x)\,\mathrm{d}x$. Using the definitions of $\mathrm{d}y$ and Δy in (7.4.2) and (7.4.3) above, we get $\Delta y \approx \mathrm{d}y = f'(x)\,\mathrm{d}x$.

The differential $\mathrm{d}y$ is not the actual increment in y as x is changed to $x + \mathrm{d}x$, but rather the change in y that would occur if y continued to change at the fixed rate $f'(x)$ as x changes to $x + \mathrm{d}x$. The error resulting from using the approximation $\mathrm{d}y$ rather than the exact change Δy is illustrated graphically in Fig. 7.4.2.

Consider, first, the movement from P to Q along the curve $y = f(x)$: as x changes by $\mathrm{d}x$, the actual change in the vertical height of the point is Δy. Suppose instead that we are only allowed to move along the tangent to the graph at P. Thus, as we move from P to R along the

[4] Luca Pacioli, considered by many as the father of modern accounting, writing in Venice in the year 1494, proposed an equivalent "rule of 72" whereby $t^* = 72/p$. This is more convenient because 72 is divisible by more integers than 70. A more accurate approximation is the "rule of 69.3", because $\ln 2 \approx 0.693$ to three decimal places.

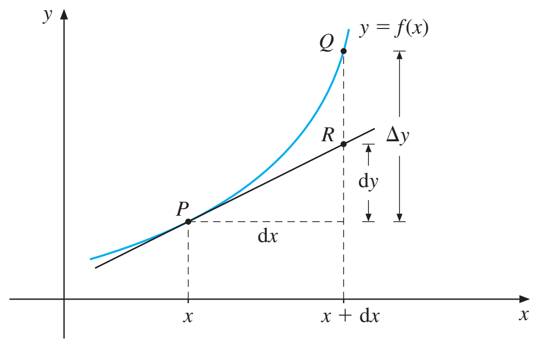


Figure 7.4.2 The differential $\mathrm{d}y$ and $\Delta y = f(x + \mathrm{d}x) - f(x)$

tangent, the change in height that corresponds to $\mathrm{d}x$ is $\mathrm{d}y$. As suggested by Fig. 7.4.2, the approximation $\Delta y \approx \mathrm{d}y$ is usually better if $\mathrm{d}x$ is smaller in absolute value. This is because the length $|RQ| = |\Delta y - \mathrm{d}y|$ of the line segment RQ, representing the difference between Δy and $\mathrm{d}y$, tends to 0 as $\mathrm{d}x$ tends to 0. In fact, $|RQ|$ becomes small so fast that the ratio $|RQ|/\mathrm{d}x$ tends to 0 as $\mathrm{d}x \to 0$.

Rules for Differentials

The notation $(\mathrm{d}/\mathrm{d}x)(\cdot)$ calls for the expression in parentheses to be differentiated with respect to x. For example, $(\mathrm{d}/\mathrm{d}x)(x^3) = 3x^2$. In the same way, we let $\mathrm{d}(\cdot)$ denote the differential of whatever is inside the parentheses.

EXAMPLE 7.4.5 Compute the following differentials:

(a) $\mathrm{d}(Ax^a + B)$, where A, B, and a are constants

(b) $\mathrm{d}(f(K))$, where f is a differentiable function of K

Solution:

(a) Putting $f(x) = Ax^a + B$, we get $f'(x) = Aax^{a-1}$, so $\mathrm{d}(Ax^a + B) = Aax^{a-1}\mathrm{d}x$.

(b) $\mathrm{d}(f(K)) = f'(K)\,\mathrm{d}K$.

RULES FOR DIFFERENTIALS

Let f and g be differentiable functions of x, and let a and b be constants. Then the following rules hold:

$$\mathrm{d}(af + bg) = a\,\mathrm{d}f + b\,\mathrm{d}g \tag{7.4.4}$$

$$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g \tag{7.4.5}$$

and, if $g \neq 0$,

$$\mathrm{d}\left(\frac{f}{g}\right) = \frac{g\,\mathrm{d}f - f\,\mathrm{d}g}{g^2} \tag{7.4.6}$$

Here is a proof of the second of these formulas:

$$d(fg) = (fg)'\,dx = (f'g + fg')\,dx = gf'\,dx + fg'\,dx = g\,df + f\,dg$$

The others are proved in a similar way.

Suppose that $y = f(x)$ and that $x = g(t)$ is a function of t. Then $y = h(t) = f(g(t))$ is a function of t. The differential of $y = h(t)$ is $dy = h'(t)\,dt$.According to the chain rule, $h'(t) = f'(g(t))g'(t)$, so that $dy = f'(g(t))g'(t)\,dt$. Because $x = g(t)$, however, the differential of x is equal to $dx = g'(t)\,dt$. It follows that $dy = f'(x)\,dx$. This shows that *if $y = f(x)$, then the differential of y is equal to $dy = f'(x)\,dx$, whether x depends on another variable or not.*

Economists often use differentials in their models. A typical example follows.

EXAMPLE 7.4.6 Consider again the model in Example 7.2.1. Find the differential dY, expressed in terms of $d\bar{I}$. If employment is also a function $N = g(Y)$ of Y, find the differential dN expressed in terms of $d\bar{I}$.

Solution: Taking the differential of (i) in Example 7.2.1, we obtain $dY = dC + d\bar{I}$. Doing the same for (ii) gives $dC = f'(Y)\,dY$. Substituting dC from the latter into the former, and solving for dY yields

$$dY = \frac{1}{1 - f'(Y)}\,d\bar{I}$$

which is the same formula found previously. From $N = g(Y)$ we get $dN = g'(Y)dY$, so

$$dN = \frac{g'(Y)}{1 - f'(Y)}\,d\bar{I}$$

Economists usually claim that employment increases as GDP increases ($g'(Y) > 0$), and that the marginal propensity to consume, $f'(Y)$, is between 0 and 1. From the above formula for dN, these claims imply that if investment increases, then employment increases as well.

EXERCISES FOR SECTION 7.4

1. Prove that $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for x close to 0, and illustrate this approximation by drawing the two graphs of $y = 1 + \frac{1}{2}x$ and $y = \sqrt{1+x}$ in the same coordinate system.

2. Use (7.4.1) to find the linear approximation to $f(x) = (5x+3)^{-2}$ about $x = 0$.

SM 3. Find the linear approximations to the following functions about $x = 0$:

   (a) $f(x) = (1+x)^{-1}$ (b) $f(x) = (1+x)^5$ (c) $f(x) = (1-x)^{1/4}$

4. Find the linear approximation to $F(K) = AK^{\alpha}$ about $K = 1$.

5. Let p, q, and r be constants. Find the following differentials:

   (a) $d(10x^3)$ (b) $d(5x^3 - 5x^2 + 5x + 5)$ (c) $d(1/x^3)$ (d) $d(\ln x)$

   (e) $d(x^p + x^q)$ (f) $d(x^p x^q)$ (g) $d(px+q)^r$ (h) $d(e^{px} + e^{qx})$

6. (a) Prove that $(1+x)^m \approx 1 + mx$ for x close to 0.

   (b) Use this to find approximations to the following numbers:

   (i) $\sqrt[3]{1.1} = \left(1+\frac{1}{10}\right)^{1/3}$ (ii) $\sqrt[5]{33} = 2\left(1+\frac{1}{32}\right)^{1/5}$ (iii) $\sqrt[3]{9} = \sqrt[3]{8+1}$ (iv) $(0.98)^{25}$

7. Compute $\Delta y = f(x+\mathrm{d}x) - f(x)$ and the differential $\mathrm{d}y = f'(x)\,\mathrm{d}x$ for the following cases:

   (a) $f(x) = x^2 + 2x - 3$ when $x = 2$ and: (i) $\mathrm{d}x = 1/10$; or (ii) $\mathrm{d}x = 1/100$

   (b) $f(x) = 1/x$ when $x = 3$ and: (i) $\mathrm{d}x = -1/10$; or (ii) $\mathrm{d}x = -1/100$

   (c) $f(x) = \sqrt{x}$ when $x = 4$ and: (i) $\mathrm{d}x = 1/20$; (ii) $\mathrm{d}x = 1/100$

SM 8. The equation $3xe^{xy^2} - 2y = 3x^2 + y^2$ defines y as a differentiable function of x about the point $(x, y) = (1, 0)$.

   (a) Find the slope of the graph at this point by implicit differentiation.

   (b) What is the linear approximation to y about $x = 1$?

9. A circle with radius r has area $A(r) = \pi r^2$. Then $A'(r) = 2\pi r$, the circumference of the circle.

   (a) Explain geometrically the approximation $A(r+\mathrm{d}r) - A(r) \approx 2\pi r\,\mathrm{d}r$.

   (b) Explain geometrically the approximation $V(r+\mathrm{d}r) - V(r) \approx 4\pi r^2\,\mathrm{d}r$, where $V(r) = \frac{4}{3}\pi r^3$ is the volume of a ball with radius r, and $V'(r) = 4\pi r^2$ is the surface area of a sphere with radius r.

10. If an amount K is charged to a credit card on which interest is $p\%$ per year, then unless some payments are made beforehand, after t years the balance will have grown to $K_t = K(1 + p/100)^t$ (even without any penalty charges). Using the approximation $\ln(1 + p/100) \approx p/100$, derived in Example 7.4.2, prove that $\ln K_t \approx \ln K + pt/100$. Find the percentage interest rate p at which the balance doubles after t years.

11. Consider the function $g(\mu) = A(1+\mu)^{a/(1+b)} - 1$ where A, a, and b are positive constants. Find the linear approximation to the function about the point $\mu = 0$.

7.5 Polynomial Approximations

The previous section discussed approximations of functions of one variable by linear functions. In particular, Example 7.4.1 established the approximation

$$\sqrt[3]{x} \approx 1 + \tfrac{1}{3}(x-1)$$

for x close to 1. In this case, at $x = 1$, the two functions $y = \sqrt[3]{x}$ and $y = 1 + \frac{1}{3}(x-1)$ both have the same value 1, and the same derivative $1/3$. Approximation by linear functions, however, may well be insufficiently accurate. So it is natural to try quadratic approximations, or approximations by polynomials of a higher order.

Quadratic Approximations

We begin by showing how a twice differentiable function $y = f(x)$ can be approximated near $x = a$ by a quadratic polynomial

$$f(x) \approx p(x) = A + B(x - a) + C(x - a)^2$$

With a fixed, there are three coefficients A, B, and C, to determine. We use three conditions to do so. Specifically, at $x = a$, we arrange that $f(x)$ and $p(x) = A + B(x - a) + C(x - a)^2$ should have: (i) the same value; (ii) the same derivative; and (iii) the same second derivative. In symbols, we require $f(a) = p(a)$, $f'(a) = p'(a)$, and $f''(a) = p''(a)$. Now $p'(x) = B + 2C(x - a)$ and $p''(x) = 2C$. So, after inserting $x = a$ into our expressions for $p(x)$, $p'(x)$, and $p''(x)$, it follows that $A = p(a)$, $B = p'(a)$, and $C = \frac{1}{2}p''(a)$. This justifies the following:

THE QUADRATIC APPROXIMATION TO $f(x)$ ABOUT $x = a$

For x close to a,

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 \tag{7.5.1}$$

Note that, compared with (7.4.1), we have simply added one extra term. For $a = 0$, in particular, we obtain the following: for x close to 0,

$$f(x) \approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \tag{7.5.2}$$

EXAMPLE 7.5.1 Find the quadratic approximation to $f(x) = \sqrt[3]{x}$ about $x = 1$.

Solution: Here $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-5/3}$. It follows that $f'(1) = \frac{1}{3}$ and $f''(1) = -\frac{2}{9}$. Because $f(1) = 1$, using (7.5.1) yields

$$\sqrt[3]{x} \approx 1 + \tfrac{1}{3}(x - 1) - \tfrac{1}{9}(x - 1)^2 \ (x \text{ close to } 1)$$

For example, $\sqrt[3]{1.03} \approx 1 + \frac{1}{3} \cdot 0.03 - \frac{1}{9}(0.03)^2 = 1 + 0.01 - 0.0001 = 1.0099$. This is correct to four decimals, and so it is better than the linear approximation derived in Example 7.4.1.

EXAMPLE 7.5.2 Find the quadratic approximation to $y = y(x)$ about $x = 0$ when y is defined implicitly as a function of x near $(x, y) = (0, 1)$ by $xy^3 + 1 = y$.

Solution: Implicit differentiation w.r.t. x yields

$$y^3 + 3xy^2y' = y' \tag{$*$}$$

Substituting $x = 0$ and $y = 1$ into $(*)$ gives $y' = 1$. Differentiating $(*)$ w.r.t x now yields

$$3y^2y' + (3y^2 + 6xyy')y' + 3xy^2y'' = y''$$

Substituting $x = 0$, $y = 1$, and $y' = 1$, we obtain $y'' = 6$. Hence, according to (7.5.2),

$$y(x) \approx y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 = 1 + x + 3x^2$$

Higher-Order Approximations

So far, we have considered linear and quadratic approximations. For functions with third- and higher-order derivatives, we can find even better approximations near one point by using polynomials of a higher degree. Suppose we want to approximate a function $f(x)$ over an interval centred at $x = a$ with an n-th-degree polynomial of the form

$$p(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \cdots + A_n(x-a)^n \tag{7.5.3}$$

Because $p(x)$ has $n+1$ coefficients, we can impose the following $n+1$ conditions on this polynomial:

$$f(a) = p(a),\ f'(a) = p'(a),\ \ldots,\ f^{(n)}(a) = p^{(n)}(a)$$

These conditions require that $p(x)$ and its first n derivatives agree with the value of $f(x)$ and its first n derivatives at $x = a$. Let us see what these conditions become when $n = 3$. In this case,

$$p(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3$$

Repeated differentiation gives

$$\begin{aligned} p'(x) &= A_1 + 2A_2(x-a) + 3A_3(x-a)^2 \\ p''(x) &= 2A_2 + 2 \cdot 3A_3(x-a) \\ p'''(x) &= 2 \cdot 3A_3 \end{aligned}$$

Thus, when $x = a$, we have $p(a) = A_0$, $p'(a) = 1!A_1$, $p''(a) = 2!A_2$, $p'''(a) = 3!A_3$. This implies the approximation:

$$f(x) \approx f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

Thus, we have added an extra term to the quadratic approximation (7.5.1).

The general case follows the same pattern, and we obtain the following approximation to $f(x)$ by an n-th-degree polynomial:

Taylor Approximation to $f(x)$ about $x = a$: for x close to a,

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{7.5.4}$$

The polynomial on the right-hand side of (7.5.4) is called the n-th-order *Taylor polynomial*, or *Taylor approximation* for f about $x = a$. The function f and its n-th-order Taylor polynomial have such a high degree of contact at $x = a$ that it is reasonable to expect the approximation in (7.5.4) to be good over some (possibly small) interval centred about $x = a$. The next section analyses the error that results from using such polynomial approximations. In the case when f is itself a polynomial whose degree does not exceed n, the formula becomes exact, without any approximation error at any point.

EXAMPLE 7.5.3 Find the third-order Taylor approximation to $f(x) = \sqrt{1+x}$ about $x = 0$.

Solution: We write $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Then

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-5/2}$$

Putting $x = 0$ gives $f(0) = 1$, $f'(0) = 1/2$, $f''(0) = (1/2)(-1/2) = -1/4$, and finally $f'''(0) = (1/2)(-1/2)(-3/2) = 3/8$. Hence, by (7.5.4) for the case $n = 3$, we have

$$f(x) \approx 1 + \frac{1}{1!}\frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{4}\right)x^2 + \frac{1}{3!}\frac{3}{8}x^3 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

EXAMPLE 7.5.4 For any natural number n, write the n-th-order Taylor approximation to $f(x) = e^x$ about $x = 0$.

Solution: This case is particularly simple, because all derivatives of f are equal to e^x, and thus $f^{(k)}(0) = 1$ for all $k = 1, 2, \ldots, n$. Hence, (7.5.4) yields

$$e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \tag{7.5.5}$$

which is an important result.

EXERCISES FOR SECTION 7.5

1. Find quadratic approximations to the following functions about the specified points:

   (a) $f(x) = (1+x)^5$ about $x = 0$

   (b) $F(K) = AK^\alpha$ about $K = 1$

   (c) $f(\varepsilon) = \left(1 + \frac{3}{2}\varepsilon + \frac{1}{2}\varepsilon^2\right)^{1/2}$ about $\varepsilon = 0$

   (d) $H(x) = (1-x)^{-1}$ about $x = 0$

SM 2. Find the fifth-order Taylor approximation to $f(x) = \ln(1+x)$ about $x = 0$.

SM 3. Find the second-order Taylor approximation to $f(x) = 5\left(\ln(1+x) - \sqrt{1+x}\right)$ about $x = 0$.

4. A study of attitudes to risk is based on the following approximation

$$U(y+m) \approx U(y) + U'(y)\,m + \frac{1}{2}U''(y)\,m^2$$

to a consumer's utility function, where y represents the consumer's initial income, and m is a random prize she may receive. Explain how to derive this approximation.

5. Find the quadratic approximation for y about $(x, y) = (0, 1)$ when y is defined implicitly, as a function of x, by the equation $1 + x^3y + x = y^{1/2}$.

6. Let the function $x(t)$ be given by the conditions $x(0) = 1$ and $\dot{x}(t) = tx(t) + 2[x(t)]^2$. Determine the second-order Taylor polynomial for $x(t)$ about $t = 0$.

7. Establish the approximation $e^{\sigma\sqrt{t/n}} \approx 1 + \sigma\sqrt{t/n} + \sigma^2 t/2n$.

8. Establish the approximation

$$\left(1 + \frac{p}{100}\right)^n \approx 1 + n\,\frac{p}{100} + \frac{n(n-1)}{2}\left(\frac{p}{100}\right)^2$$

9. The function h is defined, for all $x > 0$, by

$$h(x) = \frac{x^p - x^q}{x^p + x^q}$$

where $p > q > 0$. Find its first-order Taylor approximation about $x = 1$.

7.6 Taylor's Formula

Any approximation like (7.5.4) is of limited use unless something is known about the error it implies. Taylor's formula remedies this deficiency. This formula is often used by economists, and is regarded as one of the main results in mathematical analysis. Consider the Taylor approximation about $x = 0$ in (7.5.4):

$$f(x) \approx f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n \qquad (*)$$

Except at $x = 0$, function $f(x)$ and the Taylor polynomial on the right-hand side of $(*)$ are usually different. The difference between the two will depend on x as well as on n. It is called the *remainder* after n terms, which we denote by $R_{n+1}(x)$. Hence,

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + R_{n+1}(x) \qquad (7.6.1)$$

The following theorem gives an important explicit formula for the remainder.[5] Its proof is deferred to Section 8.4, after we have introduced the Mean Value Theorem.

[5] The English mathematician Brook Taylor (1685–1731) had already found polynomial approximations of the general form $(*)$ in 1715. Italian Joseph-Louis Lagrange (1736–1813) proved (7.6.2) approximately 50 years later.

LAGRANGE'S FORM OF THE REMAINDER

Suppose f is differentiable $n+1$ times in an interval that includes 0 and x. Then the remainder $R_{n+1}(x)$ given in (7.6.1) can be written as

$$R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(z) x^{n+1} \tag{7.6.2}$$

for some number z between 0 and x.

Using this formula for $R_{n+1}(x)$ in (7.6.1), we obtain

TAYLOR'S FORMULA

Suppose f is differentiable $n+1$ times in an interval that includes 0 and x. Then

$$f(x) = f(0) + \frac{1}{1!} f'(0)x + \cdots + \frac{1}{n!} f^{(n)}(0)x^n + \frac{1}{(n+1)!} f^{(n+1)}(z) x^{n+1} \tag{7.6.3}$$

for some number z between 0 and x.

Note that the remainder resembles the preceding terms in the sum. The only difference is that in the formula for the remainder, the $(n+1)$-th derivative $f^{(n+1)}$ is evaluated at a point z, where z is some unspecified number between 0 and x. This is in contrast to all the other terms, where the derivative is evaluated at 0. The number z is not fixed because it depends, in general, on x as well as on n.

If we put $n = 1$ in formula (7.6.3), we obtain that, for some z between 0 and x,

$$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(z)x^2 \tag{7.6.4}$$

This formula tells us that $\frac{1}{2} f''(z)x^2$ is the error that results if we replace $f(x)$ by its linear approximation about $x = 0$.

How do we use the remainder formula? It suggests an upper limit for the error that results if we replace f with its nth-order Taylor polynomial. Suppose, for instance, that for all x in an interval I, the absolute value of $f^{(n+1)}(x)$ is at most M. Then we can conclude that in this interval

$$\left|R_{n+1}(x)\right| \leq \frac{M}{(n+1)!} |x|^{n+1} \tag{7.6.5}$$

Note that if n is a large number and if x is close to 0, then $|R_{n+1}(x)|$ is small for two reasons: first, if n is large, the number $(n+1)!$ in the denominator in (7.6.5) is large; second, if $|x|$ is less than 1, then $|x|^{n+1}$ is also small when n is large.

EXAMPLE 7.6.1 Use formula (7.6.4) to approximate the function

$$f(x) = \sqrt{25 + x} = (25 + x)^{1/2}$$

Use the result to estimate $\sqrt{25.01}$, with a bound on the absolute value of the remainder.

Solution: To apply (7.6.4), we differentiate to obtain

$$f'(x) = \frac{1}{2}(25+x)^{-1/2}, \quad \text{and} \quad f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(25+x)^{-3/2}$$

Then $f(0) = 5$, whereas $f'(0) = 1/2 \cdot 1/5 = 1/10$ and $f''(z) = -(1/4)\ (25+z)^{-3/2}$. So by (7.6.4), there exists z between 0 and x such that

$$\sqrt{25+x} = 5 + \frac{1}{10}x + \frac{1}{2}\left(-\frac{1}{4}\right)(25+z)^{-3/2}x^2 = 5 + \frac{1}{10}x - \frac{1}{8}(25+z)^{-3/2}x^2 \qquad (*)$$

In order to estimate $\sqrt{25.01}$, we write $25.01 = 25 + 0.01$ and use $(*)$. If $x = 0.01$, then z lies between 0 and 0.01, so $25 + z > 25$. Then $(25+z)^{-3/2} < (25)^{-3/2} = 1/125$, so the absolute value of the remainder is

$$|R_2(0.01)| = \left|-\frac{1}{8}(25+z)^{-3/2}\left(\frac{1}{100}\right)^2\right| < \frac{1}{80\,000} \cdot \frac{1}{125} = 10^{-7}$$

We conclude that $\sqrt{25.01} \approx 5 + 1/10 \cdot 1/100 = 5.001$, with an error less than 10^{-7}.

EXAMPLE 7.6.2 Find Taylor's formula for $f(x) = e^x$, and estimate the error term for $n = 3$ and $x = 0.1$.

Solution: From Example 7.5.4, it follows that there exists a number z between 0 and x such that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^z \qquad (7.6.6)$$

One can prove that for each fixed number x the remainder term in (7.6.6) approaches 0 as n approaches infinity. For any x one can therefore use (7.6.6) to find the value of e^x to an arbitrary degree of accuracy. However, if $|x|$ is large, a large number of terms have to be used in order to obtain a good degree of accuracy, because the remainder approaches 0 very slowly as n approaches infinity.

For $n = 3$ and $x = 0.1$, Eq. (7.6.6) implies that, for some z in the interval $(0, 0.1)$, one has

$$e^{0.1} = 1 + \frac{1}{10} + \frac{1}{200} + \frac{1}{6000} + \frac{(0.1)^4}{24}e^z \qquad (*)$$

For $z < 0.1$, we have $e^z < e^{0.1}$. We claim that $e^{0.1} < 1.2$. To prove this note that $(1.2)^{10} \approx 6.2 > e$, so $e < (1.2)^{10}$, and thus $e^z < e^{0.1} < \left[(1.2)^{10}\right]^{0.1} = 1.2$, implying that

$$\left|R_4\left(\frac{1}{10}\right)\right| = \frac{(0.1)^4}{24}e^z < \frac{1}{240\,000}1.2 = 0.000\,005 = 5 \cdot 10^{-6}$$

The error that results from dropping the remainder from $(*)$ is therefore smaller than $5 \cdot 10^{-6}$.

Suppose we consider the Taylor formula on an interval about $x = a$ instead of $x = 0$. The first $n + 1$ terms on the right-hand side of Eq. (7.6.3) become replaced by those of Eq. (7.5.4), and the new remainder is

$$R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(z)(x-a)^{n+1} \tag{7.6.7}$$

for some z between x and a. It is easy to show that (7.6.7) follows from Eqs (7.6.2) and (7.6.3) by considering the function g defined by $g(t) = f(a + t)$ when t is close to 0.

EXERCISES FOR SECTION 7.6

1. Write Taylor's formula (7.6.3) with $n = 2$ for $f(x) = \ln(1 + x)$.

2. Use the approximation $(1+x)^m \approx 1 + mx + \frac{1}{2}m(m-1)x^2$ and the fact that $\sqrt[3]{25} = 3(1 - 2/27)^{1/3}$ in order to find values of: (a) $\sqrt[3]{25}$; and (b) $\sqrt[5]{33}$. Then check these approximations by using a calculator.

3. Show that $\sqrt[3]{9} = 2\,(1 + 1/8)^{1/3}$. Use formula (7.6.3), with $n = 2$, to compute $\sqrt[3]{9}$ to three decimal places.

SM 4. Let $g(x) = \sqrt[3]{1 + x}$.

   (a) Find the Taylor polynomial of $g(x)$ of order 2 about the origin.

   (b) For $x \geq 0$ show that $|R_3(x)| \leq 5x^3/81$.

   (c) Find $\sqrt[3]{1003}$ to seven decimal places.

7.7 Elasticities

Economists often study how demand for a certain commodity such as coffee reacts to price changes. We can ask by how many units such as kilograms the quantity demanded will change per dollar increase in price. In this way, we obtain a concrete number, measured in units of the commodity per unit of money. There are, however, several unsatisfactory aspects to this way of measuring the sensitivity of demand to price changes. For instance, a \$1 increase in the price of a kilo of coffee may be considerable, whereas a \$1 increase in the price of a car is insignificant.

This problem arises because the sensitivity of demand to price changes is being measured in the same arbitrary units as those used to measure both quantity demanded and price. The difficulties are eliminated if we use relative changes instead. We ask by what percentage the quantity demanded changes when the price increases by 1%. The number we obtain in this way will be independent of the units in which both quantities and prices are measured. This number is called the *price elasticity of the demand*, measured at a given price.

In 1960, the price elasticity of butter in a certain country was estimated to be -1. This means that an increase of 1% in the price would lead to a decrease of 1% in the demand, if all the other factors that influence the demand remained constant. The price elasticity for potatoes was estimated to be -0.2. What is the interpretation? Why do you think the absolute value of this elasticity is so much less than that for butter?

Assume now that the demand for a commodity can be described by the function $x = D(p)$ of the price p. When the price changes from p to $p + \Delta p$, the quantity demanded, x, also changes. The absolute change in x is $\Delta x = D(p + \Delta p) - D(p)$, and the *relative*, or *proportional*, change is

$$\frac{\Delta x}{x} = \frac{D(p + \Delta p) - D(p)}{D(p)}$$

The ratio between the relative change in the quantity demanded and the relative change in the price is

$$\frac{\Delta x}{x} \div \frac{\Delta p}{p} = \frac{p}{x}\frac{\Delta x}{\Delta p} = \frac{p}{D(p)}\frac{D(p + \Delta p) - D(p)}{\Delta p} \tag{$*$}$$

When $\Delta p = p/100$ so that p increases by 1%, then $(*)$ becomes $(\Delta x/x) \cdot 100$, which is the percentage change in the quantity demanded. We call the proportion in $(*)$ the *average elasticity of x in the interval* $[p, p + \Delta p]$. Observe that the number defined in $(*)$ depends both on the price change Δp and on the price p, but is unit-free. Thus, it makes no difference whether the quantity is measured in tons, kilograms, or pounds, or whether the price is measured in dollars, pounds, or euros.

We would like to define the elasticity of D at p so that it does not depend on the size of the increase in p. We can do this if D is a differentiable function of p. For then it is natural to define the elasticity of D w.r.t. p as the limit of the ratio in $(*)$ as Δp tends to 0. Because the Newton quotient $[D(p + \Delta p) - D(p)]/\Delta p$ tends to $D'(p)$ as Δp tends to 0, we obtain:

PRICE ELASTICITY OF DEMAND

The elasticity of the demand function $D(p)$ (with respect to the price p) is

$$\frac{p}{D(p)}\frac{\mathrm{d}D(p)}{\mathrm{d}p} \tag{7.7.1}$$

Usually, we get a good approximation to the elasticity by letting $\Delta p/p = 1/100 = 1\%$ and computing $p\Delta x/(x\Delta p)$.

The General Definition of Elasticity

The above definition of elasticity concerned a function determining quantity demanded as a function of price. Economists, however, also consider income elasticities of demand, when demand is regarded as a function of income. They also consider elasticities of supply, elasticities of substitution, and several other kinds of elasticity. It is therefore helpful to see how elasticity can be defined for a general differentiable function.

ELASTICITY

If f is differentiable at x and $f(x) \neq 0$, the *elasticity of f w.r.t. x* is

$$\mathrm{El}_x\, f(x) = \frac{x}{f(x)} f'(x) \tag{7.7.2}$$

EXAMPLE 7.7.1 Find the elasticity of $f(x) = Ax^b$, where A and b are constants, with $A \neq 0$.

Solution: In this case, $f'(x) = Abx^{b-1}$. Hence, $\mathrm{El}_x(Ax^b) = (x/Ax^b)Abx^{b-1} = b$, so

$$f(x) = Ax^b \;\Rightarrow\; \mathrm{El}_x\, f(x) = b \tag{7.7.3}$$

The elasticity of the power function Ax^b w.r.t. x is simply the exponent b. So this function has constant elasticity. In fact, it is the only type of function which has constant elasticity. This is shown in Exercise 9.9.6.

EXAMPLE 7.7.2 Assume that the quantity demanded of a particular commodity is given by $D(p) = 8000p^{-1.5}$. Compute the elasticity of $D(p)$ and find the exact percentage change in quantity demanded when the price increases by 1% from $p = 4$.

Solution: Using 7.7.3, we find that $\mathrm{El}_p\, D(p) = -1.5$, so that an increase in the price of 1% causes the quantity demanded to decrease by about 1.5%. In this case, we can also compute the decrease in demand exactly. When the price is 4, the quantity demanded is $D(4) = 8000 \cdot 4^{-1.5} = 1000$. If the price $p = 4$ is increased by 1%, the new price will be $4 + 4/100 = 4.04$, so that the change in demand is

$$\Delta D = D(4.04) - D(4) = 8000 \cdot 4.04^{-1.5} - 1000 = -14.81$$

The percentage change in demand from $D(4)$ is $-(14.81/1000) \cdot 100 = -1.481\%$.

EXAMPLE 7.7.3 Let $D(P)$ denote the demand function for a product. By selling $D(P)$ units at price P, the producer earns revenue $R(P) = P \cdot D(P)$. The elasticity of $R(P)$ w.r.t. P is

$$\mathrm{El}_P\, R(P) = \frac{P}{R(P)} \frac{\mathrm{d}}{\mathrm{d}P}[P \cdot D(P)] = \frac{1}{D(P)}[D(P) + P \cdot D'(P)] = 1 + \mathrm{El}_P\, D(P)$$

Observe that if $\mathrm{El}_P\, D(P) = -1$, then $\mathrm{El}_P\, R(P) = 0$. Thus, when the price elasticity of the demand at a point is equal to -1, a small price change will have (almost) no influence on the revenue. More generally, the marginal revenue $\mathrm{d}R/\mathrm{d}P$ generated by a price change is positive if the price elasticity of demand is greater than -1, and negative if the elasticity is less than -1.

Economists sometimes use the following terminology:

1. If $|\mathrm{El}_x\, f(x)| > 1$, then f is elastic at x.
2. If $|\mathrm{El}_x\, f(x)| = 1$, then f is unit elastic at x.
3. If $|\mathrm{El}_x\, f(x)| < 1$, then f is inelastic at x.

4. If $\left|\mathrm{El}_x f(x)\right| = 0$, then f is perfectly inelastic at x.
5. If $\left|\mathrm{El}_x f(x)\right| = \infty$, then f is perfectly elastic at x.

If $y = f(x)$ has an inverse function $x = g(y)$, then Theorem 7.3.1 implies that

$$\mathrm{El}_y(g(y)) = \frac{y}{g(y)} g'(y) = \frac{f(x)}{x} \frac{1}{f'(x)} = \frac{1}{\mathrm{El}_x f(x)} \tag{7.7.4}$$

A formulation that corresponds nicely to (7.3.2) follows:

$$\mathrm{El}_y x = \frac{1}{\mathrm{El}_x y} \tag{7.7.5}$$

There are some rules for elasticities of sums, products, quotients, and composite functions that are occasionally useful. You might like to derive these rules by solving Exercise 9.

Elasticities as Logarithmic Derivatives

Suppose that two variables x and y are related by the equation $y = Ax^b$, where x, y, and A are positive, as in Example 7.7.1. Taking the natural logarithm of each side of the equation and applying the rules for logarithms learned in Section 4.10, we find that

$$\ln y = \ln A + b \ln x \tag{7.7.6}$$

From here, we see that $\ln y$ is a linear function of $\ln x$, and so we say that (7.7.6) is a *log-linear* relation between x and y.

For the function $y = Ax^b$, we know from Example 7.7.1 that $\mathrm{El}_x y = b$. So from (7.7.6) we see that $\mathrm{El}_x y$ is equal to the (double) logarithmic derivative $\mathrm{d}(\ln y)/\mathrm{d}(\ln x)$, which is the constant slope of this log-linear relationship. This example illustrates the general rule that elasticities are equal to such logarithmic derivatives. In fact, whenever x and y are both positive variables, with y a differentiable function of x, a proof based on repeatedly applying the chain rule shows that

$$\mathrm{El}_x y = \frac{x}{y} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(\ln y)}{\mathrm{d}(\ln x)} \tag{7.7.7}$$

The transformation from the original equation $y = Ax^b$ to Eq. (7.7.6) is often seen in economic models, sometimes using logarithms to a base other than e.

EXERCISES FOR SECTION 7.7

1. Find the elasticities of the functions given by the following formulas:

(a) $3x^{-3}$ (b) $-100x^{100}$ (c) $\sqrt{x}$ (d) $A/x\sqrt{x}$, where A is a constant

2. A study of transport economics uses the relation $T = 0.4K^{1.06}$, where K is expenditure on building roads, and T is a measure of traffic volume. Find the elasticity of T w.r.t. K. In this model, if expenditure increases by 1%, by what percentage (approximately) does traffic volume increase?

3. A study of Norway's State Railways revealed that, for rides up to 60 km, the price elasticity of the volume of traffic was approximately -0.4.

(a) According to this study, what is the consequence of a 10% increase in fares?

(b) The corresponding elasticity for journeys over 300 km was calculated to be approximately -0.9. Can you think of a reason why this elasticity is larger in absolute value than the previous one?

4. Use definition (7.7.2) to find $\mathrm{El}_x\, y$ for the following functions, where a and p are constants:

(a) $y = e^{ax}$ (b) $y = \ln x$ (c) $y = x^p e^{ax}$ (d) $y = x^p \ln x$

5. Prove that $\mathrm{El}_x(f(x)^p) = p\,\mathrm{El}_x\, f(x)$, where p is a constant.

6. The demand D for apples in the USA during the period 1927 to 1941, as a function of income r, was estimated as $D = Ar^{1.23}$, where A is a constant. Find and interpret the income elasticity of demand, or *Engel elasticity*, defined as the elasticity of D w.r.t. r.

7. A study of the transportation systems in 37 American cities estimated the average travel time to work, m (in minutes), as a function of the number of inhabitants, N. They found that $m = e^{-0.02}N^{0.19}$. Write the relation in log-linear form. What is the value of m when $N = 480\,000$?

8. Show that, when finding elasticities:

(a) Multiplicative constants disappear: $\mathrm{El}_x\,(Af(x)) = \mathrm{El}_x\, f(x)$.

(b) Additive constants do *not* disappear: $\mathrm{El}_x\,(A + f(x)) = \dfrac{f(x)\,\mathrm{El}_x\, f(x)}{A + f(x)}$.

SM 9. [HARDER] Prove that if f and g are positive-valued differentiable functions of x and A is a constant, then the following rules hold. Here we write, for instance, $\mathrm{El}_x\, f$ instead of $\mathrm{El}_x\, f(x)$.

(a) $\mathrm{El}_x\, A = 0$ (b) $\mathrm{El}_x(fg) = \mathrm{El}_x\, f + \mathrm{El}_x\, g$

(c) $\mathrm{El}_x(f/g) = \mathrm{El}_x\, f - \mathrm{El}_x\, g$ (d) $\mathrm{El}_x(f + g) = \dfrac{f\,\mathrm{El}_x\, f + g\,\mathrm{El}_x\, g}{f + g}$

(e) $\mathrm{El}_x(f - g) = \dfrac{f\,\mathrm{El}_x\, f - g\,\mathrm{El}_x\, g}{f - g}$ (f) $\mathrm{El}_x\, f(g(x)) = \mathrm{El}_u f(u)\,\mathrm{El}_x\, g \quad (u = g(x))$

10. [HARDER] Use the rules of Exercise 9 to evaluate the following:

(a) $\mathrm{El}_x(-10x^{-5})$ (b) $\mathrm{El}_x(x + x^2)$ (c) $\mathrm{El}_x(x^3 + 1)^{10}$

(d) $\mathrm{El}_x(\mathrm{El}_x\, 5x^2)$ (e) $\mathrm{El}_x(1 + x^2)$ (f) $\mathrm{El}_x\left(\dfrac{x - 1}{x^5 + 1}\right)$

7.8 Continuity

Roughly speaking, a function $y = f(x)$ is continuous if small changes in the independent variable x lead to small changes in the function value y. Geometrically, *a function is continuous on an interval if its graph is connected—that is, if it has no breaks*. An example is presented in Fig. 7.8.1.

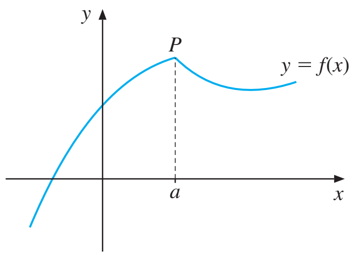


Figure 7.8.1 A continuous function

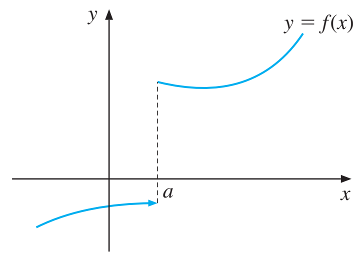


Figure 7.8.2 A discontinuous function

It is often said that a function is continuous if its graph can be drawn without lifting one's pencil off the paper. On the other hand, if the graph makes one or more jumps, we say that f is *discontinuous*. Thus, the function whose graph is shown in Fig. 7.8.2 is discontinuous at $x = a$, but continuous at all other points of its domain. The graph indicates that $f(x) < 0$ for all $x < a$, but $f(x) > 0$ for all $x \geq a$, so there is a jump at $x = a$.

Why are we interested in distinguishing between continuous and discontinuous functions? One important reason is that we must usually work with numerical approximations. For instance, if a function f is given by some formula and we wish to compute $f(\sqrt{2})$, we usually take it for granted that we can compute $f(1.4142)$ and obtain a good approximation to $f(\sqrt{2})$. In fact, this implicitly assumes that f is continuous. Then, because 1.4142 is close to $\sqrt{2}$, the value $f(1.4142)$ must be close to $f(\sqrt{2})$.

In applications of mathematics to natural sciences and economics, a function will often represent how some phenomenon changes over time. Continuity of the function will then reflect continuity of the phenomenon, in the sense of gradual rather than sudden changes. For example, a person's body temperature is a function of time which changes from one value to another only after passing through all the intermediate values.

On the other hand, the market price of Brent crude oil is actually a discontinuous function of time when examined closely enough. One reason is that the price (measured in dollars or some other currency) must always be a rational number. A second, more interesting, reason for occasional large jumps in the price is the sudden arrival of news or a rumour that significantly affects either the demand or supply function—for example, a sudden change in the government of a major oil-exporting country.

Before we can use the concept of continuity just discussed in mathematical arguments, it must obviously be made more precise. In fact, we need a definition of continuity that is not based solely on geometric intuition.

Continuity in Terms of Limits

As discussed above, a function $y = f(x)$ is continuous at $x = a$ if small changes in x lead to small changes in $f(x)$. Stated differently, if x is close to a, then $f(x)$ must be close to $f(a)$. This motivates the following definition:

CONTINUITY

$$\text{The function } f \text{ is } \textit{continuous at } x = a \text{ iff } \lim_{x \to a} f(x) = f(a) \tag{7.8.1}$$

Hence, we see that in order for f to be continuous at $x = a$, the following three conditions must all be fulfilled:

(i) The function f must be defined at $x = a$.

(ii) The limit of $f(x)$ as x tends to a must exist.

(iii) This limit must be equal to $f(a)$.

Unless all three of these conditions are satisfied, we say that f is *discontinuous* at a.

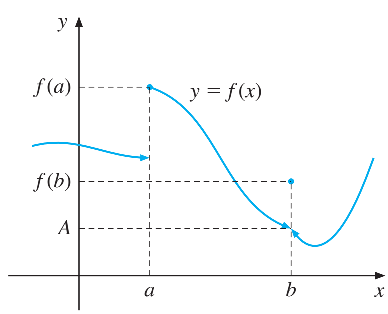


Figure 7.8.3 A discontinuous function

Figure 7.8.3 distinguishes between two important different types of discontinuity that can occur. At $x = a$, the function is discontinuous because $f(x)$ clearly has no limit as x tends to a. Hence, condition (ii) is not satisfied. This is an “irremovable” discontinuity. On the other hand, the limit of $f(x)$ as x tends to b exists and is equal to A. Because $A \neq f(b)$, however, condition (iii) is not satisfied, so f is discontinuous at b. This is a “removable” discontinuity that would disappear if the function were redefined at $x = b$ to make $f(b)$ equal to A.

Properties of Continuous Functions

Mathematicians have discovered many important results that are true only for continuous functions. It is therefore important to be able to determine whether or not a given function is continuous. The rules for limits given in Section 6.5 make it is easy to establish continuity of many types of function. Note that because $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$, at each point a, the two functions

$$f(x) = c \text{ and } f(x) = x \text{ are continuous everywhere} \tag{7.8.2}$$

This is as it should be, because the graphs of these functions are straight lines. Now, using definition (7.8.1) and the limit rules (6.5.2)–(6.5.5), the following result is immediate.

PROPERTIES OF CONTINUOUS FUNCTIONS

If f and g are continuous at a, then:

(a) $f + g$ and $f - g$ are continuous at a.

(b) fg and, in case $g(a) \neq 0$, the quotient f/g are continuous at a.

(c) $[f(x)]^r$ is continuous at a, if $[f(a)]^r$ is defined, where r is a real number.

(d) If f has an inverse on the interval I, then its inverse f^{-1} is continuous on $f(I)$.

For instance, to prove the first half of property (b), if both f and g are continuous at a, then $f(x) \to f(a)$ and $g(x) \to g(a)$ as $x \to a$. But then, according to rules for limits, $f(x)g(x) \to f(a)g(a)$ as $x \to a$, which means precisely that fg is continuous at $x = a$. The result in (d) is a little trickier to prove, but it is easy to believe because the graphs of f and its inverse f^{-1} are symmetric about the line $y = x$.

By combining these properties and (7.8.2), it follows that functions like $h(x) = x + 8$ and $k(x) = 3x^3 + x + 8$ are continuous. In general, because a polynomial is a sum of continuous functions, it is continuous everywhere. Moreover, any rational function

$$R(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P(x) \text{ and } Q(x) \text{ are polynomials}$$

is continuous at all x where $Q(x) \neq 0$.

Consider a composite function $f(g(x))$ where f and g are assumed to be continuous. If x is close to a, then continuity of g at a implies that $g(x)$ is close to $g(a)$. In turn, $f(g(x))$ becomes close to $f(g(a))$ because f is continuous at $g(a)$, and thus $f(g(x))$ is continuous at a. In short, *composites of continuous functions are continuous*: If g is continuous at $x = a$, and f is continuous at $g(a)$, then $f(g(x))$ is continuous at $x = a$. In general:

PRESERVATION OF CONTINUITY

Any function that can be constructed from continuous functions by combining one or more operations of addition, subtraction, multiplication, division (except by zero), and composition is continuous at all points where it is defined.

By using the results just discussed, a mere glance at the formula defining a function will often suffice to determine the points at which it is continuous.

EXAMPLE 7.8.1 Determine for which values of x the functions f and g are continuous:

(a) $f(x) = \dfrac{x^4 + 3x^2 - 1}{(x-1)(x+2)}$ (b) $g(x) = (x^2 + 2)\left(x^3 + \dfrac{1}{x}\right)^4 + \dfrac{1}{\sqrt{x+1}}$

Solution:

(a) This is a rational function that is continuous at all x, except where the denominator $(x-1)(x+2)$ vanishes. Hence, f is continuous at all x different from 1 and -2.

(b) This function is defined when $x \neq 0$ and $x + 1 > 0$. By properties (a), (b), and (c) on the preceding page, it follows that g is continuous in the domain $(-1, 0) \cup (0, \infty)$.

Knowing where a function is continuous simplifies the computation of many limits. If f is continuous at $x = a$, then the limit of $f(x)$ as x tends to a is found simply by evaluating $f(a)$. For instance, since the function $f(x) = x^2 + 5x$ studied in Example 6.5.3 is a continuous function of x, one has

$$\lim_{x \to -2} \left(x^2 + 5x\right) = f(-2) = (-2)^2 + 5(-2) = 4 - 10 = -6$$

Of course, simply finding $f(-2)$ like this is much easier than using the rules for limits.

Compound functions, such as Examples 5.4.3 and 5.4.4, are defined "piecewise" by different formulas which apply to disjoint intervals. Such functions are frequently discontinuous at the junction points. As another example, the amount of postage you pay for a letter is a discontinuous function of its weight. (As long as we use preprinted postage stamps, it would be extremely inconvenient to have the "postage function" be even approximately continuous.) On the other hand, given any tax schedule that looks like the one in Example 5.4.4, the tax you pay is (essentially) a continuous function of your net income (although many people seem to believe that it is not).

EXAMPLE 7.8.2 An economically significant example of a discontinuous function is the system for taxing house purchases in the UK that existed prior to the reform of 3rd December 2014. Any house buyer had to pay a tax that was called "stamp duty"—or officially, "Stamp Duty Land Tax", usually abbreviated to SDLT. Prior to 3rd December 2014, SDLT was levied at increasing average rates under a "slab system". From 24th March 2012 to 3rd December 2014, these rates were as shown in Table 7.1.

An important implication of this slab system was that the amount of tax to be paid undergoes a discontinuous jump whenever the rate increased. Specifically, the tax on a house bought for £125 000 was zero. But if the house were bought for £125 001 instead, the tax payable would rise to 2% of the purchase price, which is £2 500.02. Similarly, the tax on a house bought for £250 000 was 2% of the purchase price, which is £5 000. But if the house were bought for £250 001 instead, the tax payable would rise to 5% of the purchase price, which is £12 500.05.

Table 7.1 Rates of Stamp Duty on English Houses Prior to 3rd December 2014 (http://webarchive.nationalarchives.gov.uk/20141124143249/http:/www.hmrc.gov.uk/sdlt/rates-tables.htm#3)

Purchase price of property	Rate of SDLT
Up to £125 000	Zero
Over £125 000 to £250 000	2%
Over £250 000 to £925 000	5%
Over £925 000 to £1.5 million	10%
Over £1.5 million	12%

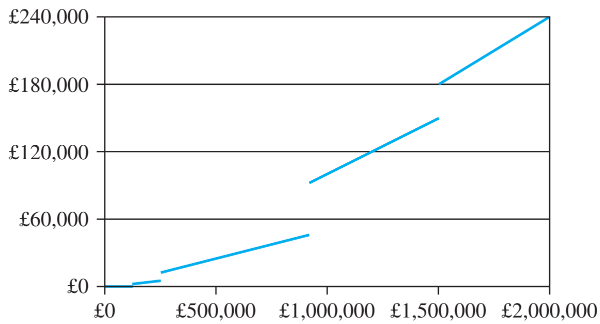


Figure 7.8.4 SDLT revenue function

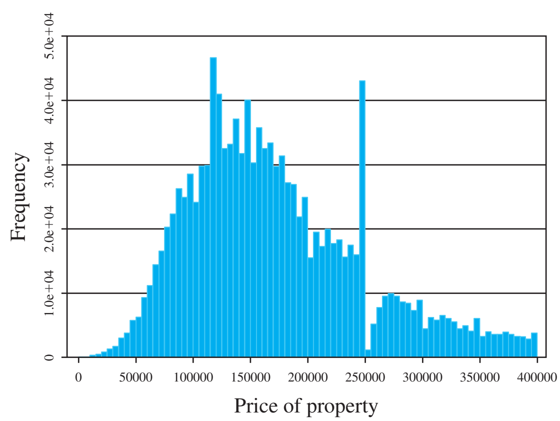


Figure 7.8.5 Frequency distribution of house prices

Figure 7.8.4 has a graph of this old SDLT revenue function, with discontinuous jumps at each of the four prices where there is an increase in the rate. Figure 7.8.5 is a bar chart showing the frequency distribution of house purchases at different prices for the year 2006.[6]

[6] This bar chart is adapted, with the authors' kind permission, from the paper by Teemu Lyytikäinen and Christian Hilber entitled "Housing transfer taxes and household mobility: Distortion on the housing or labour market?" available at http://econpapers.repec.org/paper/ferwpaper/47.htm

Not surprisingly, there are huge troughs in the distribution at a price just above one where the rate increases. In particular, notice the huge increase in frequency in the bar just to the left of £250 000, and the huge drop almost to zero in the bar just to the right of £250 000. An economist finds this easy to explain. After all, if both buyer and seller agree that a house is really worth £251 000, on which the tax payable is £12 550, they could instead agree to record the purchase price at £249 000, on which the tax payable is only £4 950. This saves the purchaser £7 600 in tax, some of which could be used to pay the seller at least £2 000 extra for "fittings" like carpets and curtains which, in the UK, are often not included in the price of the house itself. [7]

Eventually, the UK Treasury became aware that this was a serious anomaly in the tax system.[8] So on 3 December 2014, it was announced that the SDLT tax schedule would be reformed immediately. It became more like the US Federal income tax system described in Examples 5.4.4 and 7.9.7, with several bands. Between these bands, the marginal rate would increase, but the average tax rate and total tax payable are both continuous functions of the price. It remains to be seen whether this move to continuity leads to a more regular frequency distribution of prices paid for English houses, and the land on which they are built.

By the way, the *Economist*, in its discussion of this tax reform in the issue dated December 6th 2014, described the new revenue function as "less kinky". This is actually mathematical nonsense. Kinks are corners where the slope of the tangent to the graphs changes discontinuously, as discussed in (7.9.4), so kinks are different from discontinuous jumps. The old schedule had jumps but otherwise no kinks. The new schedule has kinks but, because it is continuous, it has no jumps.

EXERCISES FOR SECTION 7.8

1. Which of the following functions are likely to be continuous functions of time?

(a) The price of an ounce of gold in the Zurich gold market.

(b) The height of a growing child.

(c) The height of an aeroplane above sea level.

(d) The distance travelled by a car.

2. Let f and g be defined for all x by

$$f(x) = \begin{cases} x^2 - 1, & \text{for } x \le 0 \\ -x^2, & \text{for } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3x - 2, & \text{for } x \le 2 \\ -x + 6, & \text{for } x > 2 \end{cases}$$

Draw a graph of each function. Is f continuous at $x = 0$? Is g continuous at $x = 2$?

---

[7] Lyytikäinen and Hilber point out that when SDLT replaced an older system in 2003, it "was designed to crack down on tax evasion. In the old system it was possible to evade taxes by selling 'fixtures and fittings' separately at excessive prices. In the current system [in 2006], the sale of fixtures and fittings is declared together with the property and the Land Registry compares purchase prices with typical prices paid in the area to detect evasion." The anomalies in the frequency distribution of house prices suggest that such evasion was imperfectly deterred, to say the least.

[8] Actually, these taxes applied only in England and Wales. Scotland and Northern Ireland had different systems.

SM **3.** Determine the values of x at which each of the functions defined by the following formulas is continuous:

(a) $x^5 + 4x$ (b) $\dfrac{x}{1-x}$ (c) $\dfrac{1}{\sqrt{2-x}}$

(d) $\dfrac{x}{x^2+1}$ (e) $\dfrac{x^8 - 3x^2 + 1}{x^2 + 2x - 2}$ (f) $\dfrac{1}{\sqrt{x}} + \dfrac{x^7}{(x+2)^{3/2}}$

4. Draw the graph of y as a function of x if y depends on x as indicated in Fig. 7.8.6—that is, y is the height of the aeroplane above the point on the ground vertically below. Is y a continuous function of x?

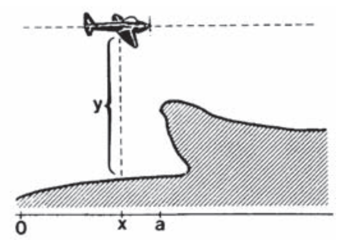


Figure 7.8.6 Exercise 4

5. For what value of a is the following function continuous for all x?

$$f(x) = \begin{cases} ax - 1 & \text{for } x \le 1 \\ 3x^2 + 1 & \text{for } x > 1 \end{cases}$$

6. Sketch the graph of a function f that is one-to-one on an interval, but neither strictly increasing nor strictly decreasing. (*Hint*: f cannot be continuous.)

7.9 More on Limits

Section 6.5 gave a preliminary discussion of limits. We now supplement this with some additional concepts and results, still keeping the discussion at an intuitive level. The reason for this gradual approach is that it is important and quite easy to acquire a working knowledge of limits. Experience suggests, however, that the precise definition is rather difficult to understand, as are proofs based on this definition.

Suppose f is defined for all x close to a, but not necessarily at a. According to Section 6.5, as x tends to a, the function $f(x)$ has A as its limit provided that the number $f(x)$ can be made as close to A as one pleases by making x sufficiently close to, but not equal to, a. Then we say that the limit exists. Now consider a case in which the limit does not exist.

EXAMPLE 7.9.1 Examine $\lim_{x \to -2} \dfrac{1}{(x+2)^2}$ using a calculator.

Solution: Choosing x-values close to -2, we obtain the values in Table 7.2.

Table 7.2 Values of $1/(x+2)^2$ when x is close to -2

x	-1.8	-1.9	-1.99	-1.999	-2.0	-2.001	-2.01	-2.1	-2.2
$\frac{1}{(x+2)^2}$	25	100	10 000	1 000 000	Not defined	1 000 000	10 000	100	25

As x gets closer and closer to -2, we see that the value of the fraction becomes larger and larger. By extending the values in the table, we see, for example, that for $x = -2.0001$ and $x = -1.9999$, the value of the fraction is 100 million. Figure 7.9.1 shows the graph of $f(x) = 1/(x+2)^2$. The line $x = -2$ is called a *vertical asymptote* for the graph of f.

We can obviously make the fraction as large as we like by choosing x sufficiently close to -2, so it does not tend to any limit as x tends to -2. Instead, we say that it tends to infinity, and we write

$$\frac{1}{(x+2)^2} \to \infty \quad \text{as} \quad x \to -2$$

Note that ∞ is *not* a number, so ∞ is not a limit.

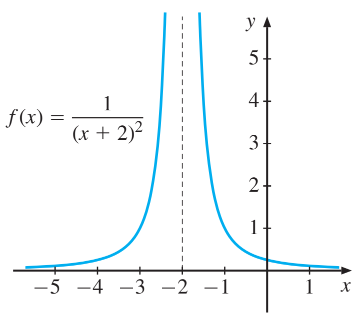


Figure 7.9.1 $f(x) \to \infty$ as $x \to -2$

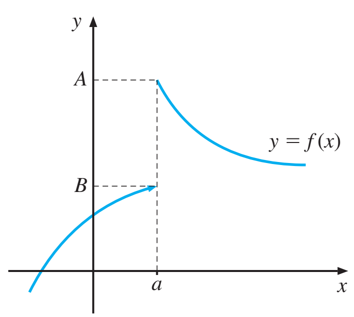


Figure 7.9.2 $\lim_{x \to a} f(x)$ does not exist

One-Sided Limits

The function whose graph is shown in Fig. 7.9.2 also fails to have a limit as x tends to a. However, it seems from the figure that if x tends to a with values less than a, then $f(x)$ tends to the number B. We say, therefore, that the *limit of $f(x)$ as x tends to a from below is B*, and we write

$$\lim_{x \to a^-} f(x) = B \quad \text{or} \quad f(x) \to B \quad \text{as} \quad x \to a^-$$

Analogously, also referring to Fig. 7.9.2, we say that the *limit of $f(x)$ as x tends to a from above is A*, and we write

$$\lim_{x\to a^+} f(x) = A \text{ or } f(x) \to A \text{ as } x \to a^+$$

We call these *one-sided limits*, the first *from below* and the second *from above*. They can also be called the *left limit* and *right limit*, respectively.

A necessary and sufficient condition for the (ordinary) limit to exist is that the two one-sided limits of f at a exist and are equal:

$$\lim_{x\to a} f(x) = A \iff \left[\lim_{x\to a^-} f(x) = A \text{ and } \lim_{x\to a^+} f(x) = A\right] \tag{7.9.1}$$

It should now also be clear what is meant by

$$f(x) \to \pm\infty \text{ as } x \to a^- \text{ and } f(x) \to \pm\infty \text{ as } x \to a^+$$

EXAMPLE 7.9.2 Figure 7.9.3 shows the graph of a function f defined on $[0, 9]$. Use the figure to check that the following limits seem correct:

$$\lim_{x\to 2} f(x) = 3, \ \lim_{x\to 4^-} f(x) = 1/2, \ \lim_{x\to 4^+} f(x) = 3, \ \text{and} \ \lim_{x\to 6^-} f(x) = \lim_{x\to 6^+} f(x) = -\infty$$

In a situation like this, we further write $\lim_{x\to 6} f(x) = -\infty$.

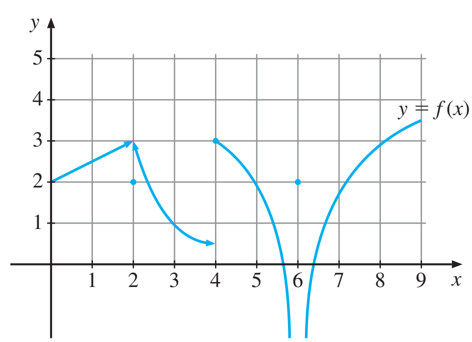


Figure 7.9.3 A function defined on $[0, 9]$

EXAMPLE 7.9.3 Explain the following limits:

$$\frac{1}{\sqrt{2-x}} \to \infty \text{ as } x \to 2^-, \text{ and } \frac{-1}{\sqrt{x-2}} \to -\infty \text{ as } x \to 2^+$$

Solution: If x is slightly smaller than 2, then $2 - x$ is small and positive. Hence, $\sqrt{2-x}$ is close to 0, and $1/\sqrt{2-x}$ is a large positive number. For example, $1/\sqrt{2-1.9999} = 1/\sqrt{0.0001} = 100$. As x tends to 2^-, so $1/\sqrt{2-x}$ tends to ∞.

The other limit is similar, because if x is slightly larger than 2, then $\sqrt{x-2}$ is positive and close to 0, and $-1/\sqrt{x-2}$ is a large negative number.

One-Sided Continuity

The introduction of one-sided limits allows us to define one-sided continuity. Suppose f is defined on the half-open interval $(c, a]$. If $f(x)$ tends to $f(a)$ as x tends to a^-, we say that f is *left continuous* at a. Similarly, if f is defined on $[a, d)$, we say that f is *right continuous* at a if $f(x)$ tends to $f(a)$ as x tends to a^+. Because of (7.9.1), we see that a function f is continuous at a if and only if f is both left and right continuous at a.

EXAMPLE 7.9.4 Figure 7.9.3 shows that f is right continuous at $x = 4$ since $\lim_{x\to 4^+} f(x)$ exists and is equal to $f(4) = 3$. At $x = 2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3$, but $f(2) = 2$, so f is neither right nor left continuous at $x = 2$.

If a function f is defined on a closed bounded interval $[a, b]$, we usually say that f is continuous in $[a, b]$ if it is continuous at each point of the open interval (a, b), and is in addition right continuous at a and left continuous at b. It should be obvious how to define continuity on half-open intervals. The continuity of a function at all points of an interval (including one-sided continuity at the end points) is often a minimum requirement we impose when speaking about "well-behaved" functions.

Limits at Infinity

We can also use the language of limits to describe the behaviour of a function as its argument becomes infinitely large through positive or negative values. Let f be defined for arbitrarily large positive numbers x. We say that $f(x)$ *has the limit A as x tends to infinity* if $f(x)$ can be made arbitrarily close to A by making x sufficiently large. We write

$$\lim_{x\to\infty} f(x) = A \quad \text{or} \quad f(x) \to A \quad \text{as} \quad x \to \infty$$

In the same way,

$$\lim_{x\to-\infty} f(x) = B \quad \text{or} \quad f(x) \to B \quad \text{as} \quad x \to -\infty$$

indicates that $f(x)$ can be made arbitrarily close to B by making x a sufficiently large negative number. The two limits are illustrated in Fig. 7.9.4. The horizontal line $y = A$ is a *horizontal asymptote* for the graph of f as x tends to ∞, whereas $y = B$ is a horizontal asymptote for the graph as x tends to $-\infty$.

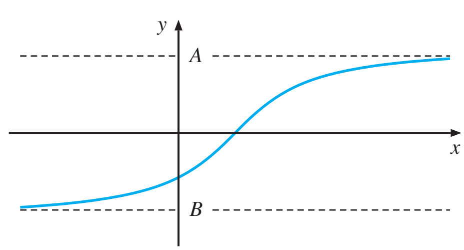


Figure 7.9.4 $y = A$ and $y = B$ are horizontal asymptotes

EXAMPLE 7.9.5 Examine the following functions as $x \to \infty$ and as $x \to -\infty$:

(a) $f(x) = \dfrac{3x^2 + x - 1}{x^2 + 1}$ (b) $g(x) = \dfrac{1 - x^5}{x^4 + x + 1}$

Solution:

(a) Away from $x = 0$ we can divide each term in the numerator and the denominator by the highest power of x, which is x^2, to obtain

$$f(x) = \frac{3x^2 + x - 1}{x^2 + 1} = \frac{3 + (1/x) - (1/x^2)}{1 + (1/x^2)}$$

If x is large in absolute value, then both $1/x$ and $1/x^2$ are close to 0. Thus, $f(x)$ is arbitrarily close to 3 if $|x|$ is sufficiently large, and $f(x) \to 3$ both as $x \to -\infty$ and $x \to \infty$.

(b) Noting that

$$g(x) = \frac{1 - x^5}{x^4 + x + 1} = \frac{(1/x^4) - x}{1 + (1/x^3) + (1/x^4)}$$

you should be able to finish the argument yourself, along the lines given in part (a).

Warnings

We have extended the original definition of a limit in several different directions. For these extended limit concepts, the previous limit rules set out in Section 6.5 still apply. For example, all the usual limit rules are valid if we consider left-hand limits or right-hand limits. Also, if we replace $x \to a$ by $x \to \infty$ or $x \to -\infty$, then again the corresponding limit properties hold. Provided at least one of the two limits A and B is nonzero, the four rules in (6.5.2)–(6.5.5) remain valid if at most one of A and B is infinite.

When $f(x)$ and $g(x)$ both tend to ∞ as x tends to a, however, much more care is needed. Because $f(x)$ and $g(x)$ can each be made arbitrarily large if x is sufficiently close to a, both $f(x) + g(x)$ and $f(x)g(x)$ can also be made arbitrarily large. But, in general, we cannot say what are the limits of $f(x) - g(x)$ and $f(x)/g(x)$. The limits of these expressions will depend on how "fast" $f(x)$ and $g(x)$, respectively, tend to ∞ as x tends to a. Briefly formulated:

PROPERTIES OF INFINITE LIMITS

If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then,

$$f(x) + g(x) \to \infty \text{ and } f(x)g(x) \to \infty \text{ as } x \to a \tag{7.9.2}$$

However, there is no rule for the limits of $f(x) - g(x)$ and $f(x)/g(x)$ as $x \to a$.

It is important to note that we cannot determine the limits of $f(x) - g(x)$ and $f(x)/g(x)$ without more information about f and g. We do not even know if these limits exist or not. The following example illustrates some of the possibilities.

EXAMPLE 7.9.6 Let $f(x) = 1/x^2$ and $g(x) = 1/x^4$. As $x \to 0$, so $f(x) \to \infty$ and $g(x) \to \infty$. Examine the limits as $x \to 0$ of the following expressions:

$$f(x) - g(x),\ g(x) - f(x),\ \frac{f(x)}{g(x)},\ \text{and}\ \frac{g(x)}{f(x)}$$

Solution: $f(x) - g(x) = (x^2 - 1)/x^4$. As $x \to 0$, the numerator tends to -1 and the denominator to 0, so the fraction tends to $-\infty$. For the other three limits we have:

$$g(x) - f(x) = \frac{1 - x^2}{x^4} \to \infty,\ \frac{f(x)}{g(x)} = x^2 \to 0,\ \text{and}\ \frac{g(x)}{f(x)} = \frac{1}{x^2} \to \infty$$

These examples serve to illustrate that infinite limits require extreme care. Let us consider some other tricky examples. Suppose we study the product $f(x)g(x)$ of two functions, where $g(x)$ tends to 0 as x tends to a. Will the product $f(x)g(x)$ also tend to 0 as x tends to a? Not necessarily. If $f(x)$ tends to a limit A, then we know that $f(x)g(x)$ tends to $A \cdot 0 = 0$. On the other hand, if $f(x)$ tends to $\pm\infty$, then it is easy to construct examples in which the product $f(x)g(x)$ does not tend to 0 at all. You should try to construct some examples of your own before turning to Exercise 4.

Continuity and Differentiability

Consider the function f graphed in Fig. 7.9.5. At point $(a, f(a))$ the graph does not have a unique tangent. Thus f has no derivative at $x = a$, but f is continuous at $x = a$. So a function can be continuous at a point without being differentiable at that point. A standard example is the absolute value function whose graph is shown in Fig. 4.3.10: that function is continuous everywhere, but not differentiable at the origin.

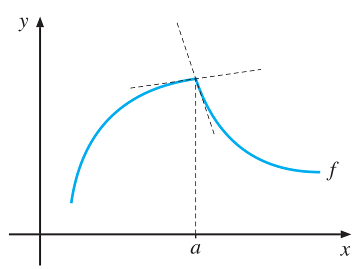


Figure 7.9.5 f is continuous, but not differentiable at $x = a$

On the other hand, differentiability implies continuity:

$$\boxed{\text{If function } f \text{ is differentiable at } x = a, \text{ then it is continuous at } x = a.} \qquad (7.9.3)$$

The proof of this result is, in fact, not difficult:

The function f is continuous at $x = a$ provided $f(a+h) - f(a)$ tends to 0 as $h \to 0$. Now, for $h \neq 0$,

$$f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \cdot h \tag{$*$}$$

If f is differentiable at $x = a$, the Newton quotient $[f(a+h) - f(a)]/h$ tends to the number $f'(a)$ as $h \to 0$. So the right-hand side of $(*)$ tends to $f'(a) \cdot 0 = 0$ as $h \to 0$. Thus, f is continuous at $x = a$.

Suppose that f is some function whose Newton quotient tends to a limit as h tends to 0 through positive values. Then the limit is called the *right derivative* of f at a. The *left derivative* of f at a is defined similarly. If the one-sided limits of the Newton quotient exist, then we use the notation

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \quad \text{and } f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \tag{7.9.4}$$

If f is continuous at a and has different left and right derivatives, $f'(a^+) \neq f'(a^-)$, then the graph of f is said to *have a kink* at $(a, f(a))$.

EXAMPLE 7.9.7 **(US Federal Income Tax, 2009)** This income tax function was discussed in Example 5.4.4. Figure 5.4.9 gives the graph of the tax function. If $\tau(x)$ denotes the tax paid at income x, its graph has kinks, for instance, at $x = 8375$ and at $x = 34\,000$. Indeed, for incomes below \$8 375, the tax rate was 10%, while a tax-payer with an income between \$8 375 and \$34 000 paid in taxes 10% of the "first" \$8 375 and 15% of the part of his income above \$8 375. Thus, $\tau'(8375^-) = 0.1$ while $\tau'(8375^+) = 0.15$. Similarly, $\tau'(34\,000^-) = 0.15$ while $\tau'(34\,000^+) = 0.25$. The highest tax rate was $\tau'(373\ 680^+) = 0.35$.

A Rigorous Definition of Limits

Our preliminary definition of the limit concept in Section 6.5 was as follows: $\lim_{x \to a} f(x) = A$ means that we can make $f(x)$ as close to A as we want, for all x sufficiently close to (but not equal to) a. The closeness, or, more generally, the distance, between two numbers can be measured by the absolute value of the difference between them. Using absolute values, the definition can be reformulated in this way:

$\lim_{x \to a} f(x) = A$ *means that we can make* $|f(x) - A|$ *as small as we want for all* $x \neq a$ *with* $|x - a|$ *sufficiently small.*

Towards the end of the 19th century some of the world's best mathematicians gradually realized that this definition can be made precise in the following way:[9]

[9] This specific idea is often attributed to German mathematicians Eduard Heine (1821–1881) and Karl Weierstrass (1815–1897), although there really is no consensus about this.

THE ϵ—δ DEFINITION OF LIMIT

We say that $f(x)$ has limit A as x tends to a if, for each number $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x with $0 < |x - a| < \delta$.

When this holds, we write

$$\lim_{x \to a} f(x) = A$$

and also say that $f(x)$ tends to A as x tends to a.

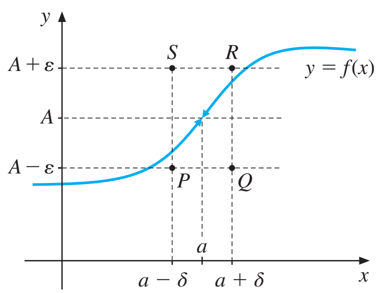


Figure 7.9.6 Definition of limit

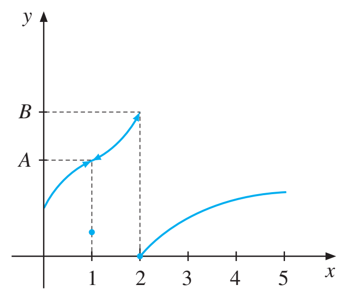


Figure 7.9.7 Exercise 1

This definition is the basis of all mathematically rigorous work on limits. It is illustrated in Fig. 7.9.6. The definition implies that the graph of f must remain within the rectangular box $PQRS$, for all $x \neq a$ in $(a - \delta, a + \delta)$. *Seeing* this formal ε–δ definition of the limit concept should be regarded as a part of anybody's general mathematical education. In this book, however, we rely only on an intuitive understanding of limits.

EXERCISES FOR SECTION 7.9

1. Function f, defined for $0 < x < 5$, has the graph that appears in Fig. 7.9.7. Determine the following limits:

(a) $\lim_{x \to 1^-} f(x)$ (b) $\lim_{x \to 1^+} f(x)$ (c) $\lim_{x \to 2^-} f(x)$ (d) $\lim_{x \to 2^+} f(x)$

SM **2.** Evaluate the following limits:

(a) $\lim_{x \to 0^+} (x^2 + 3x - 4)$ (b) $\lim_{x \to 0^-} \dfrac{x + |x|}{x}$ (c) $\lim_{x \to 0^+} \dfrac{x + |x|}{x}$

(d) $\lim_{x \to 0^+} \dfrac{-1}{\sqrt{x}}$ (e) $\lim_{x \to 3^+} \dfrac{x}{x - 3}$ (f) $\lim_{x \to 3^-} \dfrac{x}{x - 3}$

3. Evaluate

(a) $\lim_{x\to\infty} \dfrac{x-3}{x^2+1}$ (b) $\lim_{x\to-\infty} \sqrt{\dfrac{2+3x}{x-1}}$ (c) $\lim_{x\to\infty} \dfrac{(ax-b)^2}{(a-x)(b-x)}$

4. Let $f_1(x) = x$, $f_2(x) = x$, $f_3(x) = x^2$, and $f_4(x) = 1/x$. Determine $\lim_{x\to\infty} f_i(x)$ for $i = 1, 2, 3, 4$. Then examine whether the rules for limits in Section 6.5 apply to the following limits as $x \to \infty$.

(a) $f_1(x) + f_2(x)$ (b) $f_1(x) - f_2(x)$ (c) $f_1(x) - f_3(x)$ (d) $f_1(x)/f_2(x)$

(e) $f_1(x)/f_3(x)$ (f) $f_1(x)f_2(x)$ (g) $f_1(x)f_4(x)$ (h) $f_3(x)f_4(x)$

SM **5.** The non-vertical line $y = ax + b$ is said to be an *asymptote* as $x \to \infty$ (or $x \to -\infty$) to the curve $y = f(x)$ if

$$f(x) - (ax + b) \to 0 \text{ as } x \to \infty \text{ (or } x \to -\infty)$$

This condition means that the vertical distance between point $(x, f(x))$ on the curve and the point $(x, ax + b)$ on the line tends to 0 as $x \to \pm\infty$, as in Fig. 7.9.8.

Suppose that $f(x) = P(x)/Q(x)$ is a rational function where the degree of the polynomial $P(x)$ is *one greater* than that of the polynomial $Q(x)$. In this case, $f(x)$ will have an asymptote that can be found by performing the method of polynomial division $P(x) \div Q(x)$ that was explained in Section 4.7 in order to obtain a polynomial of degree 1, plus a remainder term that tends to 0 as $x \to \pm\infty$. Use this method to find asymptotes for the graph of each of the following functions of x:

(a) $\dfrac{x^2}{x+1}$ (b) $\dfrac{2x^3 - 3x^2 + 3x - 6}{x^2+1}$ (c) $\dfrac{3x^2 + 2x}{x-1}$ (d) $\dfrac{5x^4 - 3x^2 + 1}{x^3 - 1}$

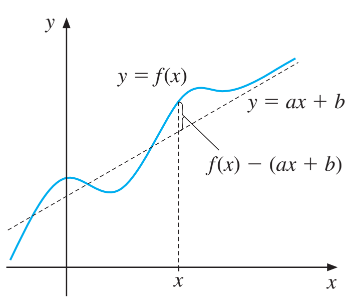


Figure 7.9.8 An asymptote

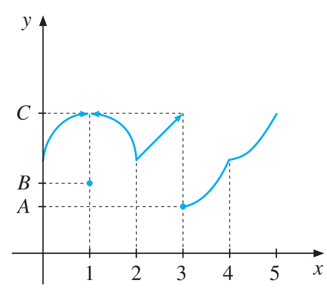


Figure 7.9.9 Exercise 7

6. Consider the cost function defined for $x \geq 0$ by

$$C(x) = A\frac{x(x+b)}{x+c} + d$$

where A, b, c, and d are positive constants. Find its asymptotes.

7. Let f be the function, defined for $0 < x < 5$, whose graph appears in Fig. 7.9.9. Study the continuity and differentiability of the function at: (a) $x = 1$; (b) $x = 2$; (c) $x = 3$; and (d) $x = 4$.

8. Graph the function f defined by $f(x) = 0$ for $x \leq 0$, and $f(x) = x$ for $x > 0$. Compute $f'(0^+)$ and $f'(0^-)$.

SM 9. Consider the function f defined by the formula

$$f(x) = \frac{3x}{-x^2 + 4x - 1}$$

Compute $f'(x)$ and use a sign diagram to determine where the function increases. (The function is not defined for $x = 2 \pm \sqrt{3}$.)

7.10 The Intermediate Value Theorem and Newton's Method

An important reason for introducing the concept of a continuous function was to make precise the idea of a function whose graph is connected—that is, it lacks any breaks. The following result, which can be proved by using the ε–δ definition of limit, expresses this property in mathematical language.

THEOREM 7.10.1 (THE INTERMEDIATE VALUE THEOREM)

Let f be a function which is continuous in the interval $[a, b]$.

(i) If $f(a)$ and $f(b)$ have different signs, then there is at least one c in (a, b) such that $f(c) = 0$.

(ii) If $f(a) \neq f(b)$, then for every *intermediate value* y in the open interval between $f(a)$ and $f(b)$ there is at least one c in (a, b) such that $f(c) = y$.

The conclusion in part (ii) follows from applying part (i) to the function $g(x) = f(x) - y$. You should draw a figure to help convince yourself that a function for which there is no such c must have at least one discontinuity.

Theorem 7.10.1 is important in assuring the existence of solutions to some equations that cannot be solved explicitly. We defer its proof until Section 7.11.

EXAMPLE 7.10.1 Prove that the following equation has at least one solution c between 0 and 1:

$$x^6 + 3x^2 - 2x - 1 = 0$$

Solution: The polynomial $f(x) = x^6 + 3x^2 - 2x - 1$ is continuous for all x—in particular for x in $[0, 1]$. Moreover, $f(0) = -1$ and $f(1) = 1$. So Theorem 7.10.1 implies that there exists at least one number c in $(0, 1)$ such that $f(c) = 0$.

Sometimes it is important to prove that a particular equation has a unique solutions. Consider the following example.

EXAMPLE 7.10.2 Prove that the equation

$$2x - 5e^{-x}(1 + x^2) = 0$$

has a unique solution, which lies in the interval $(0, 2)$.

Solution: Define $g(x) = 2x - 5e^{-x}(1 + x^2)$. Then $g(0) = -5$ and $g(2) = 4 - 25/e^2$. In fact $g(2) > 0$ because $e > 5/2$. According to the intermediate value theorem, therefore, the continuous function g must have at least one zero in $(0, 2)$. Moreover, note that $g'(x) = 2 + 5e^{-x}(1 + x^2) - 10xe^{-x} = 2 + 5e^{-x}(1 - 2x + x^2) = 2 + 5e^{-x}(x - 1)^2$. But then $g'(x) > 0$ for all x, so g is strictly increasing. It follows that g can have only one zero.

Newton's Method

The intermediate value theorem can often be used to show that an equation $f(x) = 0$ has a solution in a given interval, but it says nothing more about where to find this zero. In this subsection we shall explain an effective method for finding a good approximate solution. The method was first suggested by Isaac Newton. It has an easy geometric explanation.

Consider the graph of the function $y = f(x)$ shown in Fig. 7.10.1. It has a zero at $x = a$, but this zero is not known. To find it, start with x_0 as an initial estimate of a. It is usually better to start with x_0 not too far from a, if possible.

In order to improve the estimate, construct the tangent line to the graph at the point $(x_0, f(x_0))$, then find the point x_1 at which the tangent crosses the x-axis, as shown in Fig. 7.10.1.

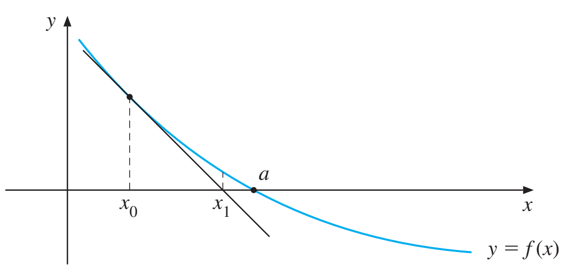


Figure 7.10.1 Newton's method

Usually x_1 is a significantly better estimate of a than x_0 was. After having found x_1, repeat the procedure by constructing the tangent line to the curve at the point $(x_1, f(x_1))$. Denote by x_2 the point where this new tangent line crosses the x-axis. Repeating this procedure, we obtain a sequence of points which usually converges very quickly to a.

It is easy to find formulas for $x_1, x_2, \ldots$. The slope of the tangent at x_0 is $f'(x_0)$. According to the point–slope formula, the equation for the tangent line through the point $(x_0, f(x_0))$ with slope $f'(x_0)$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

At the point where this tangent line crosses the x-axis, we have $y = 0$ and $x = x_1$. Hence $-f(x_0) = f'(x_0)(x_1 - x_0)$. Solving this equation for x_1, we get the first new approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, given x_1, the formula for the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, one has the following formula for the $(n+1)$-th approximation x_{n+1}, expressed in terms of the n-th approximation x_n:

NEWTON'S METHOD

As long as $f'(x_n) \neq 0$, Newton's method generates the sequence of points given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots \tag{7.10.1}$$

Usually, the sequence $\{x_n\}$ converges quickly to a zero of f.

EXAMPLE 7.10.3 In Example 7.10.1, we considered the function

$$f(x) = x^6 + 3x^2 - 2x - 1$$

Use Newton's method once to find an approximate value for the zero of f in the interval $[0, 1]$.

Solution: Choose $x_0 = 1$. Then $f(x_0) = f(1) = 1$. Because $f'(x) = 6x^5 + 6x - 2$, we have $f'(1) = 10$. Hence, Eq. (7.10.1) for $n = 0$ yields

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{10} = \frac{9}{10} = 0.9$$

EXAMPLE 7.10.4 Use Newton's method twice to find an approximate value for $\sqrt[15]{2}$.

Solution: We need an equation of the form $f(x) = 0$ which has $x = \sqrt[15]{2} = 2^{1/15}$ as a root. The equation $x^{15} = 2$ has this root, so we let $f(x) = x^{15} - 2$. Choose $x_0 = 1$. Then $f(x_0) = f(1) = -1$, and because $f'(x) = 15x^{14}$, we have $f'(1) = 15$. Thus, for $n = 0$, (7.10.1) gives

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{15} = \frac{16}{15} \approx 1.0667$$

Moreover,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{16}{15} - \frac{f(16/15)}{f'(16/15)} = \frac{16}{15} - \frac{(16/15)^{15} - 2}{15(16/15)^{14}} \approx 1.04729412$$

This is actually correct to eight decimal places.

A frequently used rule of thumb says that, to obtain an approximation that is correct to n decimal places, use Newton's method until it gives the same n decimal places twice in a row.

How Fast Does Newton's Method Converge, When it Does?

THEOREM 7.10.2 (CONVERGENCE OF NEWTON'S METHOD)

Suppose that f is twice differentiable, with $f(x^*) = 0$ and $f'(x^*) \neq 0$, and that there exist numbers $K > 0$ and $\delta > 0$, with $K\delta < 1$, such that

$$\frac{|f(x)f''(x)|}{f'(x)^2} \leq K|x - x^*|$$

for all x in the open interval $I = (x^* - \delta, x^* + \delta)$.

Then, provided that the sequence $\{x_n\}$ in Eq. (7.10.1) starts at an x_0 in I, it will converge to x^*, with an error $\|x_n - x^*\|$ that satisfies

$$|x_n - x^*| \leq \frac{(\delta K)^{2^n}}{K}$$

In most cases, Newton's method is very efficient, but it can happen that the sequence $\{x_n\}$ defined in (7.10.1) does not converge. Figure 7.10.2 shows an example where x_1 is a much worse approximation to a than x_0 was. Usually, Newton's method fails only if the absolute value of $f'(x_n)$ becomes too small, for some n. Of course, formula (7.10.1) breaks down entirely if $f'(x_n) = 0$.

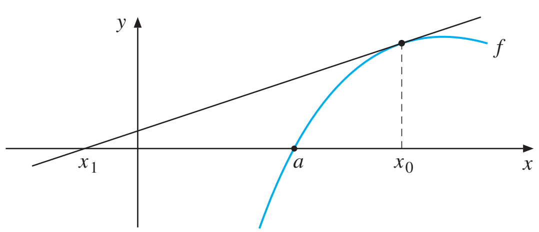


Figure 7.10.2 Newton's method

EXERCISES FOR SECTION 7.10

1. Show that each of the following equations has at least one root in the given interval.

(a) $x^7 - 5x^5 + x^3 - 1 = 0$, in $(-1, 1)$

(b) $x^3 + 3x - 8 = 0$, in $(1, 3)$

(c) $\sqrt{x^2 + 1} = 3x$, in $(0, 1)$

(d) $e^{x-1} = 2x$, in $(0, 1)$

2. Explain why anybody who is taller than 1 metre today was once exactly 1 metre tall.

3. Find a better approximation to $\sqrt[3]{17} \approx 2.5$ by using Newton's method once.

SM 4. The equation $x^4 + 3x^3 - 3x^2 - 8x + 3 = 0$ has an integer root. Find it. The three additional roots are close to -1.9, 0.4, and 1.5. Find better approximations by using Newton's method once for each root that is not an integer.

5. The equation $(2x)^x = 15$ has a solution which is approximately an integer. Find a better approximation by using Newton's method once.

6. In Fig. 7.10.1, $f(x_0) > 0$ and $f'(x_0) < 0$. Moreover, x_1 is to the right of x_0. Verify that this agrees with the formula (7.10.1) for $n = 0$. Check the other combinations of signs for $f(x_0)$ and $f'(x_0)$ to see both geometrically and analytically on which side of x_0 the point x_1 lies.

7.11 Infinite Sequences

We often encounter functions like those in Newton's method which associate a number $s(n)$ to each natural number n. Such a function is called an *infinite sequence*, or just a sequence. Its terms $s(1), s(2), s(3), \ldots, s(n), \ldots$ are usually denoted by using subscripts: thus, they become $s_1, s_2, s_3, \ldots, s_n, \ldots$. We often use the notation $\{s_n\}_{n=1}^{\infty}$, or simply $\{s_n\}$, for an arbitrary infinite sequence. For example, if $s(n) = 1/n$ for $n = 1, 2, 3, \ldots$, then the terms of the sequence are

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$$

If we choose n large enough, the terms of this sequence can be made as small as we like. We say that the sequence *converges* to 0. In general, we introduce the following definition: *A sequence $\{s_n\}$ is said to* converge *to a number s if s_n can be made arbitrarily close to s by choosing n sufficiently large.* We write

$$\lim_{n\to\infty} s_n = s \quad \text{or} \quad s_n \to s \text{ as } n \to \infty$$

This definition is just a special case of the previous definition that $f(x) \to A$ as $x \to \infty$. All the ordinary limit rules in Section 6.5 apply to limits of sequences.

A sequence that does not converge to any real number is said to *diverge*. Consider the following two sequences

$$\left\{2^n\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{(-1)^n\right\}_{n=1}^{\infty}$$

Explain why they both diverge.[10]

[10] Occasionally, as in the first of these sequences, the starting index is not 1, but another integer, which is 0 in this case.

EXAMPLE 7.11.1 For $n \geq 3$ let A_n be the area of a regular polygon with n equal sides and n equal angles, or n-gon, inscribed in a circle with radius 1. For $n = 1$ or $n = 2$, the polygon collapses to a single point or a line interval, respectively, Both have zero area, so we take $A_1 = A_2 = 0$.

Thereafter, for $n = 3$, A_3 is the area of a triangle; for $n = 4$, A_4 is the area of a square; for $n = 5$, A_5 is the area of a pentagon; and so on—see Fig. 7.11.1.

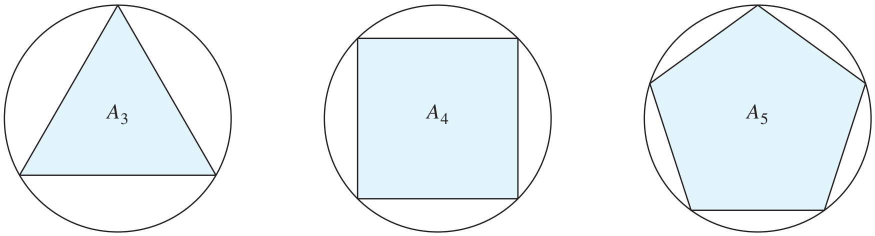


Figure 7.11.1 Three n-gons

The area A_n increases with n, but is always less than π, the area of a circle with radius 1. It seems intuitively evident that we can make the difference between A_n and π as small as we wish provided that n becomes sufficiently large, so that $A_n \to \pi$ as $n \to \infty$.

EXAMPLE 7.11.2 In Section 6.11 we argued that $\lim_{h\to 0}(1 + h)^{1/h}$ is $e = 2.718\ldots$. If we let $h = 1/n$, where the natural number $n \to \infty$ as $h \to 0$, we obtain the following important limit:

$$e = \lim_{n\to\infty} (1 + 1/n)^n \tag{7.11.1}$$

Proof of the Intermediate Value Theorem

A fundamental property of the real line states that, provided the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ together satisfy the two conditions:

$$\text{(i) for all } n \geq 1 \text{ one has } a_n \leq a_{n+1} \leq b_{n+1} \leq b_n; \quad \text{(ii)} \lim_{n\to\infty} |b_n - a_n| = 0 \tag{7.11.2}$$

there exists a common limit c^* of the two sequences.

This property allows us to prove Theorem 7.10.1 as follows:

Consider the following construction of a shrinking sequence of intervals $[a_n, b_n]$. Start with $a_0 = a$ and $b_0 = b$. Then $f(a_0)$ and $f(b_0)$ have opposite signs, by hypothesis. Given a_0 and b_0, construct the point $c_0 = \frac{1}{2}(a_0 + b_0)$—that is, the mid-point of the interval $[a_0, b_0]$. If it happens that $f(c_0) = 0$, then we can take $c = c_0$, and the construction is complete.

Otherwise, if $f(c_0) \neq 0$, then either $f(c_0)$ and $f(a_0)$ have opposite signs, or $f(c_0)$ and $f(b_0)$ have opposite signs. In the first case, choose $a_1 = a_0$ and $b_1 = c_0$; in the second case, choose $a_1 = c_0$ and $b_1 = b_0$. In this way, we have constructed a new interval $[a_1, b_1]$ such that $f(a_1)$ and $f(b_1)$ have opposite signs. Moreover, our construction implies that either $|b_1 - a_1| = |c_0 - a_0| = \frac{1}{2}|b_0 - a_0|$ or $|b_1 - a_1| = |b_0 - c_0| = \frac{1}{2}|b_0 - a_0|$. In either case, the new interval is half as long as the old. Finally, note that $a_0 \leq a_1 \leq b_1 \leq b_0$.

This construction can be repeated as often as necessary to yield a sequence of intervals $[a_n, b_n]$ with $|b_{n+1} - a_{n+1}| = \frac{1}{2}|b_n - a_n|$ at whose end points the function values $f(a_n)$ and $f(b_n)$ have opposite signs. The construction will stop after n steps if we happen to reach a point c_n at which $f(c_n) = 0$. Otherwise, one obtains an infinite sequence of intervals $[a_n, b_n]$ whose lengths satisfy $|b_n - a_n| = 2^{-n}|b_0 - a_0|$, and so converge to zero. Also, the sequence a_n of lower bounds is non-decreasing, whereas the sequence b_n of upper bounds is non-increasing.

Since sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy (7.11.2), there exists a common limit c^* such that a_n converges to c^* from below, and b_n converges to c^* from above. But we have assumed that the function f is continuous on the interval $[a, b]$, so definition (7.8.1) implies that both $f(a_n) \to f(c^*)$ and $f(b_n) \to f(c^*)$ as $n \to \infty$.

Now, note that because $f(a_n)$ and $f(b_n)$ always have opposite signs, one has $f(a_n)f(b_n) \leq 0$ for all $n = 0, 1, 2, \ldots$. Then the above limit properties imply that

$$[f(c^*)]^2 = \lim_{n\to\infty} f(a_n)f(b_n) \leq 0$$

But this is only possible if $f(c^*) = 0$, so we can take $c = c^*$.

We note that the fundamental property (7.11.2) of the real line used in this proof does not hold if we restrict ourselves to the set of rational numbers. Indeed, consider the function $f(x) = x^2 - 2$ on the interval $[1, 2]$. All the other conditions of the intermediate value theorem hold. One can even construct an infinite sequence of intervals $[a_n, b_n]$ with all the above properties, except that there is no limit point among the rational numbers. Indeed, there is no rational number such that $f(x) = 0$ because $\sqrt{2}$ is irrational.

Irrational Numbers as Limits of Sequences

The sequence $\{A_n\}$ in Example 7.11.1 converges to the irrational number $\pi = 3.14159265\ldots$. Another sequence that converges to π starts this way: $s_1 = 3.1$, $s_2 = 3.14$, $s_3 = 3.141$, $s_4 = 3.1415$, etc. Each new number is obtained by including an additional digit in the decimal expansion for π. This sequence is constructed so that $s_n \to \pi$ as $n \to \infty$.

Section 2.5 defined the power a^x when x is rational, and Section 4.8 suggested how to define a^x when x is irrational, by considering the special case of 5^{π}. Now, let r be an arbitrary irrational number. Then, just as for π, there exists a sequence r_n of rational numbers such that $r_n \to r$ as $n \to \infty$. The power a^{r_n} is well-defined for all n. Since r_n converges to r, it is reasonable to define a^r as the limit of a^{r_n} as n approaches infinity:

$$a^r = \lim_{n\to\infty} a^{r_n} \qquad (*)$$

Actually, there are infinitely many sequences $\{r_n\}$ of rational numbers that converge to any given irrational number r. Nevertheless, one can show that the limit in $(*)$ exists and is independent of which sequence we choose.

EXERCISES FOR SECTION 7.11

1. Let $\alpha_n = \dfrac{3-n}{2n-1}$ and $\beta_n = \dfrac{n^2+2n-1}{3n^2-2}$, for $n = 1, 2, \ldots$. Find the following limits:

(a) $\lim_{n\to\infty} \alpha_n$ (b) $\lim_{n\to\infty} \beta_n$ (c) $\lim_{n\to\infty} (3\alpha_n + 4\beta_n)$

(d) $\lim_{n\to\infty} \alpha_n\beta_n$ (e) $\lim_{n\to\infty} \alpha_n/\beta_n$ (f) $\lim_{n\to\infty} \sqrt{\beta_n - \alpha_n}$

2. Examine the convergence of the sequences whose general terms are as follows:

(a) $s_n = 5 - \dfrac{2}{n}$ (b) $s_n = \dfrac{n^2-1}{n}$ (c) $s_n = \dfrac{3n}{\sqrt{2n^2-1}}$

3. Prove that $e^x = \lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n$ for $x > 0$.[11]

7.12 L'Hôpital's Rule

We often need to examine the limit as x tends to a of a quotient in which both numerator and denominator tend to 0. Then we write

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \text{“}0/0\text{”}$$

We call such a limit an *indeterminate form of type* $0/0$. Here a may be replaced by a^+, a^-, ∞, or $-\infty$. The words "indeterminate form" indicate that the limit—or one-sided limit—cannot be found without further examination.

We start with the simple case of an indeterminate form $f(x)/g(x)$, where f and g are differentiable and $f(a) = g(a) = 0$. When $x \neq a$ and $g(x) \neq g(a)$, then some routine algebra allows us to write

$$\frac{f(x)}{g(x)} = \frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)}$$

The right-hand side is the ratio of two Newton quotients. Taking the limit as $x \to a$, we see that provided $g'(a) \neq 0$, this ratio tends to $f'(a)/g'(a)$. This gives the following result:

L'HÔPITAL'S RULE

If $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \tag{7.12.1}$$

[11] The same limit is valid also for $x < 0$.

According to (7.12.1), provided that $g'(a) \neq 0$, we can find the limit of an indeterminate form of type "0/0" by differentiating the numerator and the denominator separately.

EXAMPLE 7.12.1 Use (7.12.1) to confirm the limit found in Example 6.5.1—namely,

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

Solution: In this case, put $f(x) = e^x - 1$ and $g(x) = x$ in (7.12.1). Note that $f(0) = e^0 - 1 = 0$ and $g(0) = 0$. Also $f'(x) = e^x$ and $g'(x) = 1$, so $f'(0) = g'(0) = 1$. Thus (7.12.1) implies that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \frac{f'(0)}{g'(0)} = \frac{1}{1} = 1$$

EXAMPLE 7.12.2 Assuming that $x > 0$ and $y > 0$, compute

$$\lim_{\lambda\to 0} \frac{x^\lambda - y^\lambda}{\lambda}$$

Solution: In this limit x and y are kept fixed. Define $f(\lambda) = x^\lambda - y^\lambda$ and $g(\lambda) = \lambda$. Then $f(0) = g(0) = 0$. Using the rule $(\mathrm{d}/\mathrm{d}x)a^x = a^x \ln a$, we obtain $f'(\lambda) = x^\lambda \ln x - y^\lambda \ln y$, so that $f'(0) = \ln x - \ln y$. Moreover, $g'(\lambda) = 1$, so $g'(0) = 1$. Using l'Hôpital's rule,

$$\lim_{\lambda\to 0} \frac{x^\lambda - y^\lambda}{\lambda} = \frac{\ln x - \ln y}{1} = \ln \frac{x}{y}$$

In particular, if $y = 1$, then

$$\lim_{\lambda\to 0} \frac{x^\lambda - 1}{\lambda} = \ln x \tag{7.12.2}$$

which is a useful result.

Suppose we have a "0/0" form as in (7.12.1), but that $f'(a)/g'(a)$ is also of the type "0/0". Because $g'(a) = 0$, the argument for (7.12.1) breaks down. What do we do then? The answer is to differentiate once more both numerator and denominator separately. If we still obtain an expression of the type "0/0", we go on differentiating numerator and denominator repeatedly until the limit is determined, if possible. Here is an example from statistics.

EXAMPLE 7.12.3 Find

$$\lim_{x\to 0} \frac{e^{xt} - 1 - xt}{x^2}$$

Solution: The numerator and denominator are both 0 at $x = 0$. Applying l'Hôpital's rule twice, we have

$$\lim_{x\to 0} \frac{e^{xt} - 1 - xt}{x^2} = \text{“}0/0\text{”} = \lim_{x\to 0} \frac{te^{xt} - t}{2x} = \text{“}0/0\text{”} = \lim_{x\to 0} \frac{t^2 e^{xt}}{2} = \frac{1}{2}t^2$$

Here are two important warnings concerning the most common errors in attempting to apply l'Hôpital's rule:

1. Check that you really do have an indeterminate form; otherwise, as Exercise 5 shows, the method usually gives an erroneous result.
2. Do not differentiate f/g as a fraction, but compute f'/g' instead.

The method explained here and used to solve Example 7.12.3 is built on the following theorem. Note that the requirements on f and g are weaker than one might have supposed based on the examples presented so far. For instance, f and g need not even be differentiable at $x = a$. Thus the theorem actually gives a more general version of l'Hôpital's rule.

THEOREM 7.12.1 (L'HÔPITAL'S RULE FOR "0/0" FORMS)

Suppose that f and g are differentiable in an interval (α, β) that contains a, except possibly at a, and suppose that $f(x)$ and $g(x)$ both tend to 0 as x tends to a. If $g'(x) \neq 0$ for all $x \neq a$ in (α, β), and if $\lim_{x\to a} f'(x)/g'(x) = L$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

This is true whether L is finite, ∞, or $-\infty$.

Extensions of L'Hôpital's Rule

L'Hôpital's rule can be extended to some other cases. For instance, a can be an end points of the interval (α, β). Thus, $x \to a$ can be replaced by $x \to a^+$ or $x \to a^-$. Also it is easy to see that a may be replaced by ∞ or $-\infty$, as Exercise 6 shows. The rule also applies to other indeterminate forms such as "$\pm\infty/\pm\infty$", although the proof is more complicated—see Exercise 8 for more details. Here is an example:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \text{“}\infty/\infty\text{”} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \tag{7.12.3}$$

Indeed, a variety of other indeterminate forms can sometimes be transformed into expressions of the type we have already mentioned by means of algebraic manipulations or substitutions.

EXAMPLE 7.12.4 Find $L = \lim_{x\to\infty} \left(\sqrt[5]{x^5 - x^4} - x\right)$.

Solution: We reduce this "$\infty - \infty$" case to a "0/0" case by some algebraic manipulation. Note first that for $x \neq 0$,

$$\sqrt[5]{x^5 - x^4} - x = \left[x^5(1 - 1/x)\right]^{1/5} - x = x(1 - 1/x)^{1/5} - x$$

Thus,

$$\lim_{x\to\infty} \left(\sqrt[5]{x^5 - x^4} - x\right) = \lim_{x\to\infty} \frac{(1 - 1/x)^{1/5} - 1}{1/x} = \text{“}0/0\text{”}$$

Using l'Hôpital's rule, we have

$$L = \lim_{x\to\infty} \frac{(1/5)\,(1-1/x)^{-4/5}\,\left(1/x^2\right)}{-1/x^2} = \lim_{x\to\infty}\left[-\frac{1}{5}\left(1-\frac{1}{x}\right)^{-4/5}\right] = -\frac{1}{5}$$

Functions of two variables are not studied systematically until Chapter 11. Nevertheless, here is an example where applying l'Hôpital's rule to a function of two variables yields an economically significant result.

EXAMPLE 7.12.5 Consider the "constant elasticity of substitution", or CES, function

$$F(K,L) = A\left(aK^{-\rho} + (1-a)L^{-\rho}\right)^{-1/\rho} \tag{$*$}$$

where $A > 0$, $K > 0$, $L > 0$, $a \in (0,1)$, and $\rho \neq 0$. Keeping A, K, L, and a fixed, apply l'Hôpital's rule to $z = \ln[F(K,L)/A]$ as $\rho \to 0$ in order to show that $F(K,L)$ converges to the Cobb–Douglas function AK^aL^{1-a}.

Solution: We get

$$z = \ln\left(aK^{-\rho} + (1-a)L^{-\rho}\right)^{-1/\rho} = -\ln\left(aK^{-\rho} + (1-a)L^{-\rho}\right)/\rho \to \text{“0/0” as } \rho \to 0$$

Because $(\mathrm{d}/\mathrm{d}\rho)K^{-\rho} = -K^{-\rho}\ln K$ and $(\mathrm{d}/\mathrm{d}\rho)L^{-\rho} = -L^{-\rho}\ln L$, applying l'Hôpital's rule gives

$$\begin{aligned}\lim_{\rho\to 0} z &= \lim_{\rho\to 0}\left[\frac{aK^{-\rho}\ln K + (1-a)L^{-\rho}\ln L}{aK^{-\rho} + (1-a)L^{-\rho}}\right] \div 1 \\ &= a\ln K + (1-a)\ln L \\ &= \ln K^aL^{1-a}\end{aligned}$$

Hence $e^z \to K^aL^{1-a}$. By definition of z, it follows that $F(K,L) \to AK^aL^{1-a}$ as $\rho \to 0$.

An Important Limit

If a is an arbitrary number greater than 1, then $a^x \to \infty$ as $x \to \infty$. For example, $(1.0001)^x \to \infty$ as $x \to \infty$. Furthermore, if p is an arbitrary positive number, then $x^p \to \infty$ as $x \to \infty$. If we compare $(1.0001)^x$ and x^{1000}, it is clear that the former increases quite slowly at first, whereas the latter increases very quickly. Nevertheless, $(1.0001)^x$ eventually "overwhelms" x^{1000}. In general, given $a > 1$, for any fixed positive number p,

$$\lim_{x\to\infty} \frac{x^p}{a^x} = 0 \tag{7.12.4}$$

For example, x^2/e^x and $x^{10}/(1.1)^x$ both tend to 0 as x tends to ∞. This result is actually quite remarkable. It can be expressed briefly by saying that, for an arbitrary base $a > 1$, *the exponential function a^x increases faster than any power x^p of x*. Even more succinctly, one may say that *"Exponentials overwhelm powers"*. (If $p \leq 0$, the limit is obviously 0.)

To prove (7.12.4), consider the logarithm of the left-hand side, which is

$$\ln \frac{x^p}{a^x} = p \ln x - x \ln a = x\left(p \frac{\ln x}{x} - \ln a\right) \tag{$*$}$$

Now, as $x \to \infty$, we have $\ln x / x \to 0$ because of (7.12.3). So the term in parentheses in ($*$) converges to $-\ln a$, which is negative because $a > 1$. It follows from ($*$) that $\ln(x^p/a^x) \to -\infty$, and so $x^p/a^x = \exp[\ln(x^p/a^x)] \to 0$ because $e^z \to 0$ as $z \to -\infty$.

EXERCISES FOR SECTION 7.12

1. Use l'Hôpital's rule to find:

(a) $\displaystyle\lim_{x\to 3} \frac{3x^2 - 27}{x - 3}$ (b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{3x^3}$ (c) $\lim_{x\to 0} \dfrac{e^{-3x} - e^{-2x} + x}{x^2}$

2. Find the limits:

(a) $\displaystyle\lim_{x\to a} \frac{x^2 - a^2}{x - a}$ (b) $\displaystyle\lim_{x\to 0} \frac{2\sqrt{1+x} - 2 - x}{2\sqrt{1+x+x^2} - 2 - x}$

SM **3.** Use l'Hôpital's rule to find the following limits:

(a) $\displaystyle\lim_{x\to 1} \frac{x - 1}{x^2 - 1}$ (b) $\displaystyle\lim_{x\to -2} \frac{x^3 + 3x^2 - 4}{x^3 + 5x^2 + 8x + 4}$ (c) $\displaystyle\lim_{x\to 2} \frac{x^4 - 4x^3 + 6x^2 - 8x + 8}{x^3 - 3x^2 + 4}$

(d) $\displaystyle\lim_{x\to 1} \frac{\ln x - x + 1}{(x-1)^2}$ (e) $\displaystyle\lim_{x\to 1} \frac{1}{x-1} \ln\left(\frac{7x+1}{4x+4}\right)$ (f) $\displaystyle\lim_{x\to 1} \frac{x^x - x}{1 - x + \ln x}$

4. Find the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ (b) $\displaystyle\lim_{x\to 0^+} x \ln x$ (c) $\displaystyle\lim_{x\to 0^+} (xe^{1/x} - x)$

5. Find the error in the following line of reasoning:

$$\lim_{x\to 1} \frac{x^2 + 3x - 4}{2x^2 - 2x} = \lim_{x\to 1} \frac{2x + 3}{4x - 2} = \lim_{x\to 1} \frac{2}{4} = \frac{1}{2}$$

What is the correct value of the first limit?

6. With $\beta > 0$ and $\gamma > 0$, find $\lim_{v\to 0^+} \dfrac{1 - (1 + v^\beta)^{-\gamma}}{v}$. (*Hint:* Consider first the case $\beta = 1$.)

7. In the context of Examples 7.1.5 and 7.1.8, the family of CES utility functions is given by[12]

$$u(c) = \begin{cases} \dfrac{c^{1-\rho} - 1}{1 - \rho}, & \text{if } \rho \neq 1 \\ \ln c, & \text{if } \rho = 1 \end{cases}$$

for all $c > 0$. Use l'Hôpital's rule to show that $\lim_{\rho\to 1} \dfrac{c^{1-\rho} - 1}{1 - \rho} = \ln c$. In this sense, the family is "continuous in ρ".

[12] See Example 7.12.5; in the context of Examples 7.1.5 and 7.1.8, these functions are also known as constant relative risk aversion, or CRRA, preferences.

8. [HARDER] Suppose that f and g are both differentiable for all large x and that $f(x)$ and $g(x)$ both tend to 0 as $x \to \infty$. If, in addition, $\lim_{x\to\infty} g'(x) \neq 0$, show that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \text{“}0/0\text{”} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

by introducing $x = 1/t$ in the first fraction and then using l'Hôpital's rule as $t \to 0^+$.

SM 9. [HARDER] Suppose that $\lim_{x\to a} f(x)/g(x) = \pm\text{“}\infty/\infty\text{”} = L \neq 0$ where f and g are differentiable functions whose derivatives $f'(x)$ and $g'(x)$ converge to non-zero limits as x tends to a. By applying l'Hôpital's rule to the equivalent limit, $\lim_{x\to a} [1/g(x)] / [1/f(x)] = \text{“}0/0\text{”}$, show that one has $L = \lim_{x\to a}[f'(x)/g'(x)]$ provided this limit exists.

REVIEW EXERCISES

1. Use implicit differentiation to find dy/dx and d^2y/dx^2 for each of the following equations:

   (a) $5x + y = 10$ (b) $xy^3 = 125$ (c) $e^{2y} = x^3$

   Check by solving each equation for y as a function of x, then differentiating.

2. Compute y' when y is defined implicitly by the equation $y^5 - xy^2 = 24$. Is y' ever 0?

3. The graph of the equation $x^3 + y^3 = 3xy$ passes through the point $(3/2, 3/2)$. Find the slope of the tangent line to the curve at this point. This equation has a nice graph, called *Descartes's folium*, which appears in Fig. 7.R.1.

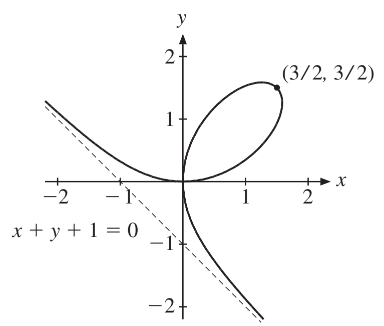


Figure 7.R.1 Descartes' folium

4. (a) Find the slope of the tangent to the curve $x^2y + 3y^3 = 7$ at $(x, y) = (2, 1)$.

   (b) Prove that $y'' = -210/13^3$ at $(2, 1)$.

5. If $K^{1/3}L^{1/3} = 24$, compute dL/dK by implicit differentiation.

6. The equation

$$\ln y + y = 1 - 2\ln x - 0.2(\ln x)^2$$

defines y as a function of x for $x > 0$, $y > 0$. Compute y' and show that $y' = 0$ for $x = e^{-5}$.

7. Consider the following macroeconomic model:

(i) $Y = C + I$ (ii) $C = f(Y - T)$ (iii) $T = \alpha + \beta Y$

where Y is GDP, C is consumption, T denotes taxes, and α and β are constants. Assume that $f' \in (0, 1)$ and $\beta \in (0, 1)$.

(a) From equations (i)–(iii) derive the equation $Y = f((1 - \beta)Y - \alpha) + I$.

(b) Differentiate the equation in (a) implicitly w.r.t. I and find an expression for $\mathrm{d}y/\mathrm{d}I$.

(c) Examine the sign of $\mathrm{d}y/\mathrm{d}I$.

8. (a) Find y' when y is given implicitly by the equation

$$x^2 - xy + 2y^2 = 7$$

(b) Find the points where the graph has horizontal tangent and the points where it has vertical tangent. Do your results accord with Fig. 7.R.2, which shows the graph of the equation?

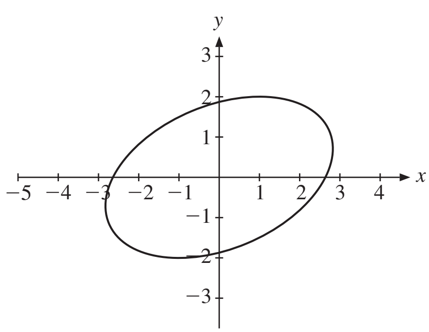


Figure 7.R.2 Exercise 8

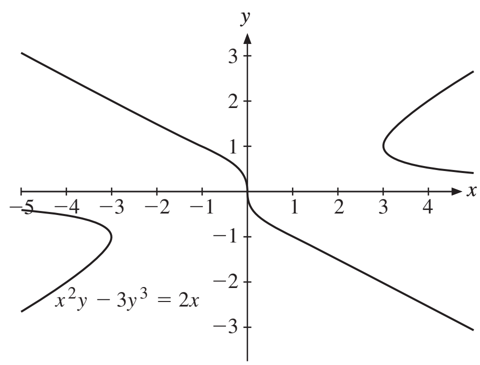


Figure 7.R.3 Exercise 9

9. The graph of the equation $x^2y - 3y^3 = 2x$ passes through the point $(x, y) = (-1, 1)$.

(a) Find the slope of the graph at this point.

(b) Find the points at which the curve has vertical tangent. Show that no point on the curve has a horizontal tangent. Do your results accord with Fig. 7.R.3, which shows the graph of the equation?

SM 10. Let function f be defined by the formula $f(x) = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$.

(a) Determine the domain and range of f.

(b) Prove that f has an inverse g, and find a formula for the inverse. Note that $f\left(\frac{1}{2}\right) = \frac{1}{2} \ln 3$. Find $g'\left(\frac{1}{2} \ln 3\right)$ in two different ways.

11. Let $f(x)$ be defined for all $x > 0$ by $f(x) = (\ln x)^3 - 2(\ln x)^2 + \ln x$.

(a) Compute $f(e^2)$ and find the zeros of $f(x)$.

(b) Prove that $f(x)$ defined on $[e, \infty)$ has an inverse function h, then determine $h'(2)$.

SM 12. Find the quadratic approximations about $x = 0$ to the following functions:

(a) $f(x) = \ln(2x + 4)$ (b) $g(x) = (1 + x)^{-1/2}$ (c) $h(x) = xe^{2x}$

13. Find the differentials:

(a) $d(\sqrt{1 + x^2})$ (b) $d(4\pi r^2)$ (c) $d(100K^4 + 200)$ (d) $d[\ln(1 - x^3)]$

14. Compute the differential of $f(x) = \sqrt{1 + x^3}$. What is the approximate change in $f(x)$ when x changes from $x = 2$ to $x = 2 + dx$, where $dx = 0.2$?

SM 15. Use formula (7.6.6) with $n = 5$ to find an approximate value of $\sqrt{e}$. Show that the answer is correct to three decimal places. (*Hint*: For $0 < c < 1/2$, note that $e^c < e^{1/2} < 2$.)

16. Find the quadratic approximation to $y = y(x)$ about $(x, y) = (0, 1)$ when y is defined implicitly as a function of x by the equation $y + \ln y = 1 + x$.

17. Determine the values of x at which each of the functions defined by the following formulas is continuous:

(a) $e^x + e^{1/x}$ (b) $\dfrac{\sqrt{x} + 1/x}{x^2 + 2x + 2}$ (c) $\dfrac{1}{\sqrt{x + 2}} + \dfrac{1}{\sqrt{2 - x}}$

18. Let f be a given differentiable function of one variable. Suppose that each of the following equations defines y implicitly as a function of x. Find an expression for y' in each case.

(a) $x = f(y^2)$ (b) $xy^2 = f(x) - y^3$ (c) $f(2x + y) = x + y^2$

19. The demands for margarine (marg) and for meals away from home (mah) in the UK during the period 1920–1937, as functions of personal income r, were estimated to be $D_{\text{marg}} = Ar^{-0.165}$ and $D_{\text{mah}} = Br^{2.39}$, respectively, for suitable constants A and B. Find and interpret the (Engel) elasticities of D_{marg} and D_{mah} w.r.t. r.

20. Find the elasticities of the functions given by the following formulas:

(a) $50x^5$ (b) $\sqrt[3]{x}$ (c) $x^3 + x^5$ (d) $\dfrac{x - 1}{x + 1}$

21. The equation $x^3 - x - 5 = 0$ has a root close to 2. Find an approximation to this root by using Newton's method once, with $x_0 = 2$.

22. Prove that $f(x) = e^{\sqrt{x}} - 3$ has a unique zero in the interval $(1, 4)$. Find an approximate value for this zero by using Newton's method once, with $x_0 = 1$.

SM 23. Evaluate the limits:

(a) $\lim\limits_{x \to 3^-} (x^2 - 3x + 2)$ (b) $\lim\limits_{x \to -2^+} \dfrac{x^2 - 3x + 14}{x + 2}$ (c) $\lim\limits_{x \to -1} \dfrac{3 - \sqrt{x + 17}}{x + 1}$

(d) $\lim\limits_{x \to 0} \dfrac{(2 - x)e^x - x - 2}{x^3}$ (e) $\lim\limits_{x \to 3} \left(\dfrac{1}{x - 3} - \dfrac{5}{x^2 - x - 6} \right)$ (f) $\lim\limits_{x \to 4} \dfrac{x - 4}{2x^2 - 32}$

(g) $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2}$ (h) $\lim\limits_{x \to -1} \dfrac{4 - \sqrt{x + 17}}{2x + 2}$ (i) $\lim\limits_{x \to \infty} \dfrac{(\ln x)^2}{3x^2}$

SM 24. Examine the following limit for different values of the constants a, b, c, and d, assuming that b and d are positive:

$$\lim_{x \to 0} \frac{\sqrt{ax+b} - \sqrt{cx+d}}{x}$$

25. Evaluate $\lim_{x \to 0} \dfrac{a^x - b^x}{e^{ax} - e^{bx}}$, where $a \neq b$, with a and b both positive.

26. The equation $x^{21} - 11x + 10 = 0$ has a root at $x = 1$, and another root in the interval $(0, 1)$. Starting from $x_0 = 0.9$, use Newton's method as many times as necessary to find the latter root to three decimal places.

8

SINGLE-VARIABLE OPTIMIZATION

If you want literal realism, look at the world around you; if you want understanding, look at theories.
—Robert Dorfman (1964)

Finding the best way to do a specific task involves what is called an *optimization problem*. Examples abound in almost all areas of human activity. A manager seeks those combinations of inputs, such as capital and labour, that maximize profit or minimize cost. A doctor might want to know when is the best time of day to inject a drug, so as to avoid the concentration in the bloodstream becoming dangerously high. A farmer might want to know what amount of fertilizer per square yard will maximize profits. An oil company may wish to find the optimal rate of extraction from its wells.

Studying an optimization problem of this sort systematically requires a mathematical model. Constructing one is usually not easy, and only in simple cases will the model lead to the problem of maximizing or minimizing a function of a single variable—the main topic of this chapter.

In general, no mathematical methods are more important in economics than those designed to solve optimization problems. Though economic optimization problems usually involve several variables, the examples of quadratic optimization in Section 4.6 indicate how useful economic insights can be gained even from simple one-variable optimization.

8.1 Extreme Points

Those points in the domain of a function where it reaches its largest and its smallest values are usually referred to as maximum and minimum points. If we do not need to bother about the distinction between maxima and minima, we call them *extreme points*, or *extrema*. Thus, if $f(x)$ has domain D, then

$$c \in D \text{ is a } \textit{maximum point} \text{ for } f \Leftrightarrow f(x) \le f(c) \text{ for all } x \in D \tag{8.1.1}$$

$$d \in D \text{ is a } \textit{minimum point} \text{ for } f \Leftrightarrow f(x) \ge f(d) \text{ for all } x \in D \tag{8.1.2}$$

In (8.1.1), we call $f(c)$ the *maximum value*, and in (8.1.2), we call $f(d)$ the *minimum value*.[1] If the value of f at c is strictly larger than at any other point in D, then c is a *strict maximum* point. Similarly, d is a *strict minimum* point if $f(x) > f(d)$ for all $x \in D$, $x \neq d$. As collective names, we use the terms *optimal points* and *values*, or *extreme points* and *values*.

If f is a function with domain D, then the function $-f$ is defined in D by $(-f)(x) = -f(x)$. Note that $f(x) \le f(c)$ for all x in D if and only if $-f(x) \ge -f(c)$ for all x in D. Thus, point c maximizes f in D if and only if it minimizes $-f$ in D. This simple observation, which is illustrated in Fig. 8.1.1, can be used to convert maximization problems to minimization problems and vice versa.

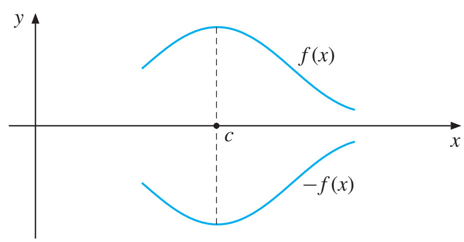


Figure 8.1.1 Point c is a maximum point for $f(x)$, and a minimum point for $-f(x)$

Sometimes we can find the maximum and minimum points of a function simply by studying the formula that defines it.

EXAMPLE 8.1.1 Find possible maximum and minimum points for:

(a) $f(x) = 3 - (x-2)^2$ (b) $g(x) = \sqrt{x-5} - 100$, for $x \ge 5$

Solution:

(a) Because $(x-2)^2 \ge 0$ for all x, it follows that $f(x) \le 3$ for all x. But $f(x) = 3$ when $(x-2)^2 = 0$ at $x = 2$. Therefore, $x = 2$ is a maximum point for f. Because $f(x) \to -\infty$ as $x \to \infty$, f has no minimum.

(b) Since $\sqrt{x-5}$ is ≥ 0 for all $x \ge 5$, it follows that $f(x) \ge -100$ for all $x \ge 5$. Since $f(5) = -100$, we conclude that $x = 5$ is a minimum point. Since $f(x) \to \infty$ as $x \to \infty$, f has no maximum.

[1] Some authors use different terminology, referring to the extreme values as the maximum or minimum, and to the points where these values are reached as *maximizers* or *minimizers*.

Rarely can we find extreme points as simply as in Example 8.1.1. The main task of this chapter is to explain how to locate possible extreme points in more complicated cases.

Suppose that c is a point in some interval I. If c is not an end point of I, it is possible to construct another interval, perhaps very small, that contains points on both sides of c, but which is completely included in I. Suppose, for instance, that c lies in $I = (a, b]$. If $c < b$, then the smaller of the two numbers $c - a$ and $b - c$ will be larger than zero. Let us denote that smaller number by δ and define the interval $J = (c - \delta, c + \delta)$. Then we will have numbers in J on both sides of c, while $J \subseteq I$. The same is not true if $c = b$, as any interval that includes numbers to the right of c will have elements outside of I. In order to distinguish these two situations, we say that any $c < b$ that lies in I is *interior* to I, while b is on the *boundary* of I, as is a. To make this idea explicit:

INTERIOR OF AN INTERVAL

Let a and b be real numbers. All the points in the open interval (a, b) are *interior* to the intervals $[a, b]$, $[a, b)$, $(a, b]$ and (a, b). For the intervals $[a, b)$ and (a, b), the end point b can be ∞; for the intervals $(a, b]$ and (a, b), the end point a can be $-\infty$.

An essential observation is that if f is a differentiable function that has a maximum or minimum at an interior point c of its domain, then the tangent line to its graph must be horizontal (parallel to the x-axis) at that point. Hence, $f'(c) = 0$. Points c at which f is differentiable and $f'(c) = 0$ are called *critical*, or *stationary*, *points* for f. Precisely formulated, one has the following theorem:

THEOREM 8.1.1 (NECESSARY FIRST-ORDER CONDITION)

Suppose that a function f is differentiable in an interval I and that c is an interior point of I. For $x = c$ to be a maximum or minimum point for f in I, a necessary condition is that it is a critical point for f—i.e., $x = c$ is a solution of

$$f'(x) = 0 \tag{8.1.3}$$

A proof of the theorem is as follows:

Suppose that f has a maximum at c. If the absolute value of h is sufficiently small, then $c + h \in I$ because c is an interior point of I. Because c is a maximum point, $f(c + h) - f(c) \leq 0$. If h is sufficiently small and positive, the Newton quotient satisfies $[f(c + h) - f(c)]/h \leq 0$. The limit of this quotient as $h \to 0^+$ is therefore ≤ 0 as well. But because $f'(c)$ exists, this limit is equal to $f'(c)$, so $f'(c) \leq 0$. For small negative values of h, on the other hand, we get $[f(c + h) - f(c)]/h \geq 0$. The limit of this expression as $h \to 0^-$ is therefore ≥ 0. So $f'(c) \geq 0$. We have now proved that $f'(c) \leq 0$ and $f'(c) \geq 0$, so $f'(c) = 0$.

The proof in the case when c is a minimum point is similar.

Before starting to explore systematically other properties of maxima and minima, we provide some geometric examples. They will indicate for us the role played by the critical points of a function in the theory of optimization. Figure 8.1.2 shows the graph of a function f defined in an interval $[a, b]$ and having two critical points, c and d. At c, there is a maximum; at d, there is a minimum.

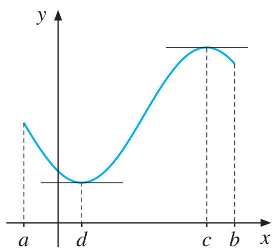


Figure 8.1.2 Two critical points

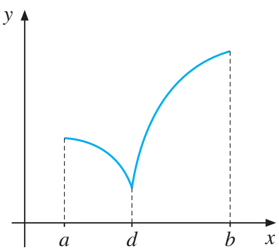


Figure 8.1.3 No critical points

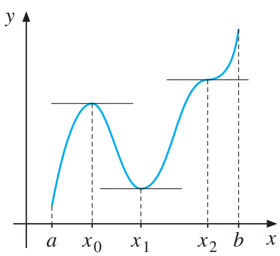


Figure 8.1.4 No interior extrema

In Fig. 8.1.3, the function has no critical points. There is a maximum at the end point b and a minimum at d. At d, the function is not differentiable. At b, the derivative (the left-hand derivative) is not 0.

Condition (8.1.3) is known as a first-order condition, or FOC, as it refers to the function's first derivative. Theorem 8.1.1 implies that (8.1.3) is a *necessary* condition for a differentiable function f to have a maximum or minimum at an interior point x in its domain. The condition is far from sufficient. This is illustrated in Fig. 8.1.4, where f has three critical points, x_0, x_1, and x_2, but none of them are extrema. At the end point a there is a minimum, whereas at end point b there is a maximum.[2] At the critical point x_0 the function f has a "local maximum", in the sense that its value at that point is higher than at all neighbouring points. Similarly, at x_1 it has a local "minimum", whereas at x_2 there is a critical point that is neither a local minimum nor a local maximum—in fact, x_2 is a special case of an *inflection point*.

The situation suggested in Fig. 8.1.2 is the most typical for problems arising in applications, since maximum and minimum points usually will be attained at critical points. But Figs 8.1.3 and 8.1.4 illustrate situations that *can* occur, also in economic problems. Actually, the three figures represent important aspects of single-variable optimization problems. Because the theory is so important in economics, we must not simply rely on vague geometric insights. Instead, we must develop a firmer analytical foundation by formulating precise mathematical results.

[2] Or, if you prefer, suppose that b is not in the domain of the function, and that $f(x)$ approaches ∞ as x tends to b.

EXERCISES FOR SECTION 8.1

1. Use arguments similar to those in Example 8.1.1 to find the maximum or minimum points for the following functions:

(a) $f(x) = \dfrac{8}{3x^2 + 4}$ (b) $g(x) = 5(x+2)^4 - 3$ (c) $h(x) = \dfrac{1}{1+x^4}$ for $x \in [-1, 1]$

(d) $F(x) = \dfrac{-2}{2+x^2}$ (e) $G(x) = 2 - \sqrt{1-x}$ (f) $H(x) = 100 - e^{-x^2}$

8.2 Simple Tests for Extreme Points

In many cases we can find maximum or minimum values for a function just by studying the sign of its first derivative. Suppose $f(x)$ is differentiable in an interval I and that it has only one critical point, $x = c$. Suppose $f'(x) \geq 0$ for all x in I such that $x \leq c$, whereas $f'(x) \leq 0$ for all x in I such that $x \geq c$. Then $f(x)$ is increasing to the left of c and decreasing to the right of c. It follows that $f(x) \leq f(c)$ for all $x \leq c$, and $f(c) \geq f(x)$ for all $x \geq c$. Hence, $x = c$ is a maximum point for f in I, as illustrated in Fig. 8.2.1.

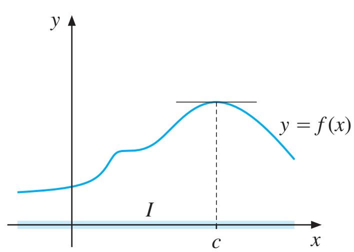


Figure 8.2.1 $x = c$ is a maximum point

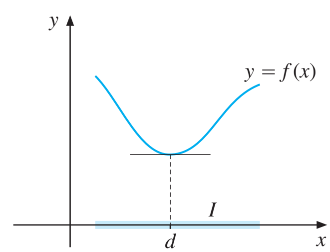


Figure 8.2.2 $x = d$ is a minimum point

With obvious modifications, a similar result holds for minimum points, as illustrated in Fig. 8.2.2. Briefly stated:[3]

THEOREM 8.2.1 (FIRST-DERIVATIVE TEST FOR EXTREMA)

Suppose the function $f(x)$ is differentiable in an interval I that includes c.

(i) If $f'(x) \geq 0$ for $x \leq c$ and $f'(x) \leq 0$ for $x \geq c$, then $x = c$ is a maximum point for f in I.

(ii) If $f'(x) \leq 0$ for $x \leq c$ and $f'(x) \geq 0$ for $x \geq c$, then $x = c$ is a minimum point for f in I.

[3] Many books in mathematics for economists instruct students always to check so-called second-order conditions, even when this first-derivative test is much easier to use.

EXAMPLE 8.2.1 Measured in milligrams per litre, the concentration of a drug in the bloodstream, t hours after injection, is given by the formula

$$c(t) = \frac{t}{t^2 + 4}, \quad t \geq 0$$

Find the time of maximum concentration.

Solution: Differentiating with respect to t yields

$$c'(t) = \frac{1 \cdot (t^2 + 4) - t \cdot 2t}{(t^2 + 4)^2} = \frac{4 - t^2}{(t^2 + 4)^2} = \frac{(2 + t)(2 - t)}{(t^2 + 4)^2}$$

For $t \geq 0$, the term $2 - t$ alone determines the sign of the fraction, because the other terms are positive: if $t \leq 2$, then $c'(t) \geq 0$; whereas if $t \geq 2$, then $c'(t) \leq 0$. We conclude that $t = 2$ maximizes $c(t)$. Thus, the concentration of the drug is highest two hours after injection. Because $c(2) = 0.25$, the maximum concentration is 0.25 mg.

EXAMPLE 8.2.2 Consider the function f that is defined for all x by

$$f(x) = e^{2x} - 5e^x + 4 = (e^x - 1)(e^x - 4)$$

(a) Find the zeros of $f(x)$ and compute $f'(x)$.

(b) Find the intervals where f increases and decreases, and determine its possible extreme points and values.

(c) Examine $\lim_{x \to -\infty} f(x)$, and sketch the graph of f.

Solution:

(a) $f(x) = (e^x - 1)(e^x - 4) = 0$ when $e^x = 1$ and when $e^x = 4$. Hence $f(x) = 0$ for $x = 0$ and for $x = \ln 4$. By differentiating $f(x)$, we obtain $f'(x) = 2e^{2x} - 5e^x$.

(b) $f'(x) = 2e^{2x} - 5e^x = e^x(2e^x - 5)$. Thus $f'(x) = 0$ for $e^x = 5/2 = 2.5$; that is, $x = \ln 2.5$. Furthermore, $f'(x) \leq 0$ for $x \leq \ln 2.5$, and $f'(x) \geq 0$ for $x \geq \ln 2.5$. So $f(x)$ is decreasing in the interval $(-\infty, \ln 2.5]$ and increasing in $[\ln 2.5, \infty)$. Hence $f(x)$ has a minimum at $x = \ln 2.5$, and $f(\ln 2.5) = (2.5 - 1)(2.5 - 4) = -2.25$. Since $f(x) \to \infty$ as $x \to \infty$, $f(x)$ has no maximum.

(c) When $x \to -\infty$, then e^x tends to 0, and $f(x)$ tends to 4. The graph is drawn in Fig. 8.2.3. Note that $y = 4$ is an asymptote as $x \to -\infty$.

Extreme Points for Concave and Convex Functions

Recall the definitions of concave and convex functions in Section 6.9. Suppose that f is concave with $f''(x) \leq 0$ for all x in an interval I. Then $f'(x)$ is decreasing in I. If $f'(c) = 0$ at an interior point c of I, then $f'(x)$ must be nonnegative to the left of c, and nonpositive to the right of c. This implies that the function itself is increasing to the left of c and decreasing to the right of c. We conclude that $x = c$ is a maximum point for f in I. We obviously get a corresponding result for a minimum of a convex function.

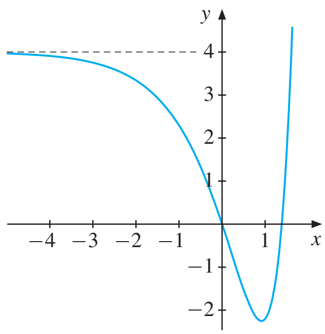


Figure 8.2.3 $f(x) = e^{2x} - 5e^x + 4$

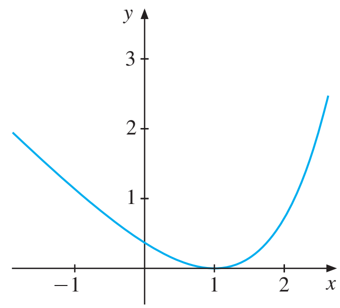


Figure 8.2.4 $f(x) = e^{x-1} - x$

THEOREM 8.2.2 (EXTREMA FOR CONCAVE AND CONVEX FUNCTIONS)

Suppose that f is a function defined in an interval I and that c is a critical point for f in the interior of I.

(i) If f is concave, then c is a maximum point for f in I.

(ii) If f is convex, then c is a minimum point for f in I.

EXAMPLE 8.2.3 Consider the function f defined for all x by $f(x) = e^{x-1} - x$. Show that f is convex and find its minimum point. Sketch the graph.

Solution: $f'(x) = e^{x-1} - 1$ and $f''(x) = e^{x-1} > 0$, so f is convex. Note that $f'(x) = e^{x-1} - 1 = 0$ for $x = 1$. From Theorem 8.2.2 it follows that $x = 1$ minimizes f. See Fig. 8.2.4 for the graph of f, which confirms the result.

EXERCISES FOR SECTION 8.2

1. Let y denote the weekly average quantity of pork produced in Chicago during 1948, in millions of pounds, and let x be the total weekly work effort, in thousands of hours. A study estimated the relation $y = -2.05 + 1.06x - 0.04x^2$. Determine the value of x that maximizes y by studying the sign variation of y'.

SM 2. Find the derivative of the function h, defined for all x by the formula $h(x) = 8x/(3x^2 + 4)$. Note that $h(x) \to 0$ as $x \to \pm\infty$. Use the sign variation of $h'(x)$ to find the extreme points of $h(x)$.

3. The height of a flowering plant after t months is given by $h(t) = \sqrt{t} - \frac{1}{2}t$, for $t \in [0, 3]$. At what time is the plant at its tallest?

4. Show that

$$f(x) = \frac{2x^2}{x^4 + 1} \quad \Rightarrow \quad f'(x) = \frac{4x(1 + x^2)(1 + x)(1 - x)}{(x^4 + 1)^2}$$

and find the maximum value of f on $[0, \infty)$.

5. Find possible extreme points for $g(x) = x^3 \ln x$, for $x \in (0, \infty)$.

6. Find possible extreme points for $f(x) = e^{3x} - 6e^x$, for $x \in (-\infty, \infty)$.

7. Find the maximum of $y = x^2 e^{-x}$ on $[0, 4]$.

SM 8. Use Theorem 8.2.2 to find the values of x that maximize/minimize the functions given by the following formulas:

(a) $y = e^x + e^{-2x}$ (b) $y = 9 - (x - a)^2 - 2(x - b)^2$ (c) $y = \ln x - 5x$, for $x > 0$

9. Consider n numbers $a_1, a_2, \dots, a_n$. Find the number $\bar{x}$ which gives the best approximation to these numbers, in the sense of minimizing the distance function

$$d(x) = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2$$

SM 10. [HARDER] After the North Sea flood catastrophe in 1953, the Dutch government initiated a project to determine the optimal height of the dykes. One of the models involved finding the value of x minimizing $f(x) = I_0 + kx + Ae^{-\alpha x}$, for $x \geq 0$. Here x denotes the extra height in metres that should be added to the dykes, $I_0 + kx$ is the construction cost, and $Ae^{-\alpha x}$ is an estimate of the expected loss caused by flooding. The parameters I_0, k, A, and α are all positive constants.

(a) Suppose that $A\alpha > k$ and find $x_0 > 0$ that minimizes $f(x)$.

(b) The constant A is defined as $A = p_0 V(1 + 100/\delta)$, where p_0 is the probability that the dykes will be flooded if they are not rebuilt, V is an estimate of the cost of flood damage, and δ is an interest rate. Show that

$$x_0 = \frac{1}{\alpha} \ln \left[\frac{\alpha p_0 V}{k} \left(1 + \frac{100}{\delta} \right) \right]$$

Examine what happens to x_0 when one of the variables p_0, V, δ, or k increases. Comment on the reasonableness of the results.[4]

8.3 Economic Examples

This section presents some interesting instances of economic optimization problems.

EXAMPLE 8.3.1 Suppose $Y(N)$ bushels of wheat are harvested per acre of land when N pounds of fertilizer per acre are used. If P is the dollar price per bushel of wheat and q is the dollar price per pound of fertilizer, then profits in dollars per acre are

$$\pi(N) = PY(N) - qN$$

for $N \geq 0$. Suppose there exists N^* such that $\pi'(N) \geq 0$ for $N \leq N^*$, whereas $\pi'(N) \leq 0$ for $N \geq N^*$. Then N^* maximizes profits, and $\pi'(N^*) = 0$. That is, $PY'(N^*) - q = 0$, so

$$PY'(N^*) = q \tag{$*$}$$

[4] This problem is discussed in D. van Dantzig, "Economic Decision Problems for Flood Prevention". *Econometrica*, 24 (1956): 276–287.

Let us give an economic interpretation of this condition. Suppose N^* units of fertilizer are used and we contemplate increasing N^* by one unit. What do we gain? If N^* increases by one unit, then $Y(N^*+1) - Y(N^*)$ more bushels are produced. Now $Y(N^*+1) - Y(N^*) \approx Y'(N^*)$. For each of these bushels, we get P dollars, so *by increasing N^* by one unit, we gain approximately $PY'(N^*)$ dollars*. On the other hand, *by increasing N^* by one unit, we lose q dollars*, because this is the cost of one unit of fertilizer. Hence, we can interpret (∗) as follows: In order to maximize profits, you should increase the amount of fertilizer to the level N^* at which an additional pound of fertilizer equates the changes in your gains and losses from the extra pound.

(a) In an (unrealistic) example, suppose that $Y(N) = \sqrt{N}$, $P = 10$, and $q = 0.5$. Find the amount of fertilizer which maximizes profits in this case.

(b) An agricultural study in Iowa estimated the yield function $Y(N)$ for the year 1952 as

$$Y(N) = -13.62 + 0.984N - 0.05N^{1.5}$$

If the price of wheat is \$1.40 per bushel and the price of fertilizer is \$0.18 per pound, find the amount of fertilizer that maximizes profits.

Solution:

(a) The profit function is

$$\pi(N) = PY(N) - qN = 10N^{1/2} - 0.5N, \; N \geq 0$$

Then $\pi'(N) = 5N^{-1/2} - 0.5$. We see that $\pi'(N^*) = 0$ when $(N^*)^{-1/2} = 0.1$, hence $N^* = 100$. Moreover, it follows that $\pi'(N) \geq 0$ when $N \leq 100$ and $\pi'(N) \leq 0$ when $N \geq 100$. We conclude that $N^* = 100$ maximizes profits. See Fig. 8.3.1.

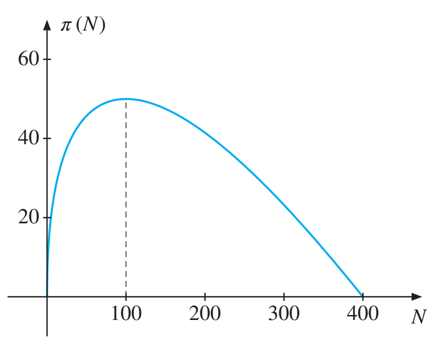


Figure 8.3.1 Example 8.3.1(a)

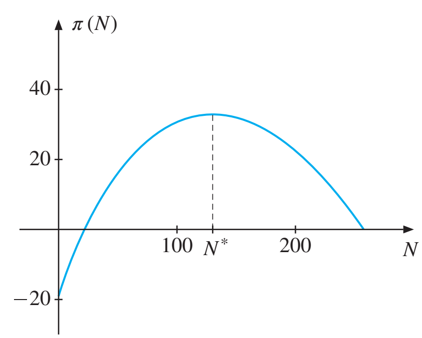


Figure 8.3.2 Example 8.3.1(b)

(b) In this case,

$$\begin{aligned}\pi(N) &= 1.4(-13.62 + 0.984N - 0.05N^{1.5}) - 0.18N \\ &= -19.068 + 1.1976N - 0.07N^{1.5}\end{aligned}$$

so that

$$\pi'(N) = 1.1976 - 0.07 \cdot 1.5N^{0.5} = 1.1976 - 0.105\sqrt{N}$$

Hence $\pi'(N^*) = 0$ when $0.105\sqrt{N^*} = 1.1976$. This implies that

$$\sqrt{N^*} = 1.1976/0.105 \approx 11.4 \text{ and so } N^* \approx (11.4)^2 \approx 130$$

By studying the expression for $\pi'(N)$, we see that $\pi'(N)$ is positive to the left of N^* and negative to the right of N^*. Hence, $N^* \approx 130$ maximizes profits. The graph of $\pi(N)$ is shown in Fig. 8.3.2.

EXAMPLE 8.3.2 Suppose that the total cost of producing $Q > 0$ units of a commodity is $C(Q) = aQ^2 + bQ + c$, where a, b, and c are positive constants.

(a) Find the value of Q that minimizes the average cost defined by $A(Q) = C(Q)/Q$ in the special case when $C(Q) = 2Q^2 + 10Q + 32$.

(b) Show that in the general case, the average cost function has a minimum at $Q^* = \sqrt{c/a}$. In the same coordinate system, draw the graphs of the average cost, the marginal cost, and the straight line $P = aQ + b$.

Solution:

(a) We find that here $A(Q) = 2Q + 10 + 32/Q$, so $A'(Q) = 2 - 32/Q^2$ and $A''(Q) = 64/Q^3$. Since $A''(Q) > 0$ for all $Q > 0$, the function A is convex, and since $A'(Q) = 0$ for $Q = 4$, this is a minimum point.

(b) We find that here $A(Q) = aQ + b + c/Q$, $A'(Q) = a - c/Q^2$ and $A''(Q) = 2c/Q^3$. Since $A''(Q) > 0$ for all $Q > 0$, the function A is convex, and since $A'(Q) = 0$ for $Q^* = \sqrt{c/a}$, this is a minimum point. The graphs are drawn in Fig. 8.3.3. Note that at the minimum point Q^*, marginal cost is equal to average cost. This is no coincidence, because it is true in general that $A'(Q) = 0$ if and only if $C'(Q) = A(Q)$.[5]

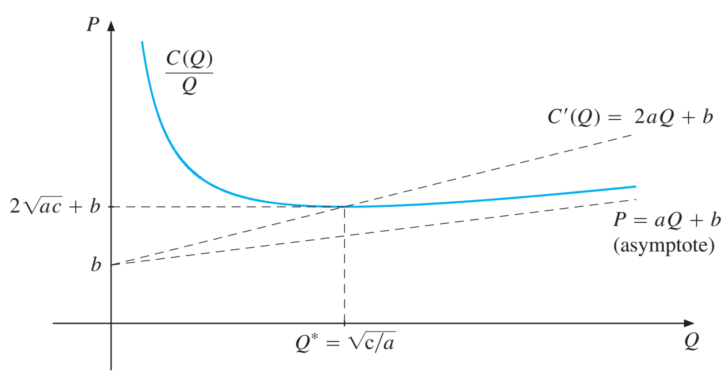


Figure 8.3.3 Average cost function

The following example is typical of how economists use implicit differentiation in connection with optimization problems.

[5] See Example 6.7.6. The minimum average cost is $A(Q^*) = a\sqrt{c/a} + b + c/\sqrt{c/a} = \sqrt{ac} + b + \sqrt{ac} = 2\sqrt{ac} + b$.

EXAMPLE 8.3.3 A monopolist is faced with the inverse demand function $P(Q)$ denoting the price when output is Q. The monopolist has a constant average cost k per unit produced.

(a) Find the profit function $\pi(Q)$, and prove that the first-order condition for maximal profit at $Q^* > 0$ is

$$P(Q^*) + Q^*P'(Q^*) = k \tag{$*$}$$

(b) By implicit differentiation of $(*)$ find how the monopolist's choice of optimal production is affected by changes in k.

(c) How does the optimal profit react to a change in k?

Solution:

(a) The profit function is $\pi(Q) = QP(Q) - kQ$, so $\pi'(Q) = P(Q) + QP'(Q) - k$. In order for $Q^* > 0$ to maximize $\pi(Q)$, one must have $\pi'(Q^*) = 0$, or equivalently $(*)$.

(b) Assuming that Eq. $(*)$ defines Q^* as a differentiable function of k, we obtain

$$P'(Q^*)\frac{\mathrm{d}Q^*}{\mathrm{d}k} + \frac{\mathrm{d}Q^*}{\mathrm{d}k}P'(Q^*) + Q^*P''(Q^*)\frac{\mathrm{d}Q^*}{\mathrm{d}k} = 1$$

Solving for $\mathrm{d}Q^*/\mathrm{d}k$ gives

$$\frac{\mathrm{d}Q^*}{\mathrm{d}k} = \frac{1}{Q^*P''(Q^*) + 2P'(Q^*)}$$

(c) Because $\pi(Q^*) = Q^*P(Q^*) - kQ^*$, differentiating w.r.t. k gives

$$\frac{\mathrm{d}\pi(Q^*)}{\mathrm{d}k} = \frac{\mathrm{d}Q^*}{\mathrm{d}k}P(Q^*) + Q^*P'(Q^*)\frac{\mathrm{d}Q^*}{\mathrm{d}k} - Q^* - k\frac{\mathrm{d}Q^*}{\mathrm{d}k}$$

But the three terms containing $\mathrm{d}Q^*/\mathrm{d}k$ all cancel because of the first-order condition $(*)$. So $\mathrm{d}\pi^*/\mathrm{d}k = -Q^*$. Thus, if the cost increases by one unit, the optimal profit will decrease by approximately Q^*, the optimal output level.

EXERCISES FOR SECTION 8.3

1. (a) A firm produces $Q = 2\sqrt{L}$ units of a commodity when L units of labour are employed. If the price obtained per unit is €160, and the price per unit of labour is €40, what value of L maximizes profits $\pi(L)$?

(b) A firm produces $Q = f(L)$ units of a commodity when L units of labour are employed. Assume that $f'(L) > 0$ and $f''(L) < 0$. If the price obtained per unit is 1 and price per unit of labour is w, what is the first-order condition for maximizing profits at $L = L^*$?

(c) By implicitly differentiating the first-order condition in (b) w.r.t. w, find how L^* changes when w changes.

SM 2. In Example 8.3.3, suppose that $P(Q) = a - Q$, and assume that $0 < k < a$.

(a) Find the profit maximizing output Q^* and the associated monopoly profit $\pi(Q^*)$.

(b) How does the monopoly profit react to changes in k? Find $\mathrm{d}\pi(Q^*)/\mathrm{d}k$.

(c) The government argues that the monopoly produces too little. It wants to induce the monopolist to produce $\hat{Q} = a - k$ units by granting a subsidy s per unit of output. Calculate the subsidy s required to reach the target.

3. A square tin plate whose edges are 18 cm long is to be made into an open square box of depth x cm by cutting out equally sized squares of width x in each corner and then folding over the edges. Draw a figure, and show that the volume of the box is, for $x \in [0, 9]$:

$$V(x) = x(18 - 2x)^2 = 4x^3 - 72x^2 + 324x$$

Also find the maximum point of V in $[0, 9]$.

4. In an economic model, the proportion of families whose income is no more than x, and who have a home computer, is given by

$$p(x) = a + k(1 - e^{-cx})$$

where a, k, and c are positive constants. Determine $p'(x)$ and $p''(x)$. Does $p(x)$ have a maximum? Sketch the graph of p.

5. Suppose the tax T a person pays on income w is given by $T = a(bw + c)^p + kw$, where a, b, c, and k are positive constants, and $p > 1$. Then the average tax rate is

$$\bar{T}(w) = \frac{T}{w} = a\frac{(bw + c)^p}{w} + k$$

Find the value of income that minimizes the average tax rate.

8.4 The Extreme Value Theorem

The main theorems used so far in this chapter to locate extreme points require the function to be steadily increasing on one side of the point and steadily decreasing on the other side. Many functions with a derivative whose sign varies in a more complicated way may still have a maximum or minimum. This section shows how to locate possible extreme points for an important class of such functions.

It is not difficult to think of functions that have no extreme points. Example 8.2.3 illustrates one such function, but there are even simpler cases such as, for instance, the function $f(x) = x$, defined over the whole real line. The following theorem gives important sufficient conditions for extreme points to exist.

THEOREM 8.4.1 (THE EXTREME VALUE THEOREM)

Suppose that f is a continuous function over a closed and bounded interval $[a, b]$. Then there exists a point d in $[a, b]$ where f has a minimum, and a point c in $[a, b]$ where f has a maximum—that is, one has $f(d) \leq f(x) \leq f(c)$ for all x in $[a, b]$.

One of the most common misunderstandings of the extreme value theorem is illustrated by the following statement from a student's exam paper: "The function is continuous, but since it is not defined on a closed, bounded interval, the extreme value theorem shows that there is no maximum." The misunderstanding here is that, although the conditions of the theorem are sufficient, they certainly are not *necessary* for the existence of an extreme point. In Exercise 9, you will study a function defined in an interval that is neither closed nor bounded, and moreover the function is not even continuous. Even so, it has both a maximum and a minimum.

This observation does not mean, however, that we can dispense with the assumptions of the theorem. Figures 8.4.1 to 8.4.3 display cases where two of the assumptions of Theorem 8.4.1 are satisfied, but the remaining one is not. In each case, the function fails to attain a maximum—even though it possesses a minimum.

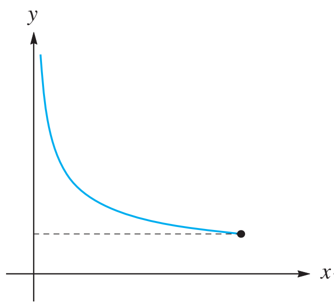


Figure 8.4.1 The domain is not closed

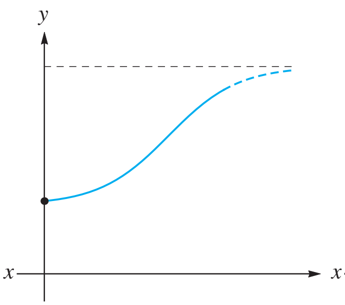


Figure 8.4.2 The domain is unbounded

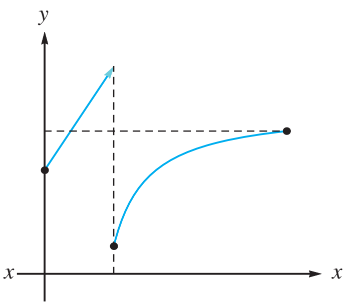


Figure 8.4.3 A discontinuous function

The proof of the extreme value theorem is surprisingly difficult.[6] Yet the result is not hard to believe. Imagine, for example, a mountainous stage of a cycle race like the Tour de France. Since roads avoid going over cliffs, the height of the road above sea level is a continuous function of the distance travelled, as illustrated in Fig. 8.4.4. As the figure also shows, the stage must take the cyclist over some highest point P, as well as through a lowest point Q. Of course, these points could also be at the start or finish of the ride, and the ride must finish eventually!

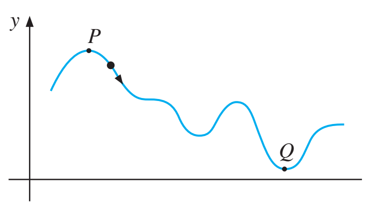


Figure 8.4.4 Altitude as a function of distance

[6] The original proof was given by German mathematician Karl Weierstrass (1815–1897). Essentially, the argument is that the range of a continuous function defined over a closed and bounded interval is, itself, closed and bounded. The end points of the range, which are the extreme values of the function, therefore exist.

How to Search for Maxima and Minima

Suppose we know that a function f has a maximum and/or a minimum in some bounded interval I. The optimum must occur either at an interior point of I or else at one of the end points. If it occurs inside the interval I—namely at an interior point—and if f is differentiable, then the derivative f' is zero at that point. In addition, there is the possibility that the optimum occurs at a point where f is not differentiable. Hence, every extreme point must belong to one of the following three different sets:

(a) interior points in I where $f'(x) = 0$;

(b) end points of I, if included in I; and

(c) interior points in I where f' does not exist.

Points satisfying any one of these three conditions will be called *candidate extreme points*. Whether they are actual extreme points depends on a careful comparison of function values, as explained below. A typical example showing that a minimum can occur at a point of type (c) is shown in Fig. 8.1.3. However, most functions that economists study are differentiable everywhere, so the following recipe covers most problems of interest.

FINDING THE EXTREMA OF FUNCTIONS

> In order to find the maximum and minimum values of a differentiable function f defined on a closed, bounded interval $[a, b]$.
>
> (i) Find all critical points of f in (a, b) — that is, find all points x in (a, b) that satisfy the FOC, Eq. (8.1.3).
>
> (ii) Evaluate f at the end points a and b of the interval and also at all critical points.
>
> (iii) The largest function value found in (ii) is the maximum value, and the smallest function value is the minimum value of f in $[a, b]$.

A differentiable function is continuous, so the extreme value theorem assures us that maximum and minimum points do exist, provided that its domain is closed and bounded. Following the procedure just given, we can, in principle, find these extreme points.

EXAMPLE 8.4.1 Find the maximum and minimum values, for $x \in [0, 3]$, of:

$$f(x) = 3x^2 - 6x + 5$$

Solution: The function is differentiable everywhere, and $f'(x) = 6x - 6 = 6(x - 1)$. Hence $x = 1$ is the only critical point. The candidate extreme points are the end points 0 and 3, as well as $x = 1$. We calculate the value of f at these three points. The results are $f(0) = 5$, $f(3) = 14$, and $f(1) = 2$. We conclude that the maximum value is 14, obtained at $x = 3$, and the minimum value is 2 at $x = 1$.

EXAMPLE 8.4.2 Find the maximum and minimum values, for $x \in [-1, 3]$, of

$$f(x) = \frac{1}{4}x^4 - \frac{5}{6}x^3 + \frac{1}{2}x^2 - 1$$

Solution: The function is differentiable everywhere, and

$$f'(x) = x^3 - \frac{5}{2}x^2 + x = x\left(x^2 - \frac{5}{2}x + 1\right)$$

Solving the quadratic equation $x^2 - \frac{5}{2}x + 1 = 0$, we get the roots $x = 1/2$ and $x = 2$. Thus $f'(x) = 0$ for $x = 0, x = 1/2$, and $x = 2$. These three points, together with the two end points $x = -1$ and $x = 3$ of the interval, constitute the five candidate extreme points. We find that $f(-1) = 7/12, f(0) = -1, f(1/2) = -185/192, f(2) = -5/3$ and $f(3) = 5/4$. Thus, the maximum value of f is $5/4$, at $x = 3$; the minimum value is $-5/3$, at $x = 2$.

Note that it was unnecessary to study the sign variation of $f'(x)$ or to use other tests, such as second-order conditions, in order to verify that we have found the maximum and minimum values. In the two previous examples, we had no trouble in finding the solutions to the equation $f'(x) = 0$. However, in some cases, finding all the solutions to $f'(x) = 0$ might constitute a formidable, or even insuperable, problem. For instance, the function,

$$f(x) = x^{26} - 32x^{23} - 11x^5 - 2x^3 - x + 28$$

defined for x in $[-1, 5]$ is continuous, so it does have a maximum and a minimum in $[-1, 5]$. Yet it is impossible to find any exact solution to the equation $f'(x) = 0$.

Difficulties of this kind are often encountered in practical optimization problems. In fact, only in very special cases can the equation $f'(x) = 0$ be solved exactly. Fortunately, there are standard numerical methods for use on a computer that in most cases will find points arbitrarily close to the actual solutions of such equations—see, for example, Newton's method discussed in Section 7.10.

The Mean Value Theorem

This section deals with the mean value theorem, which is a principal tool for the precise demonstration of results in calculus. The section is a bit more advanced than the rest of the book, and hence may be considered optional.

Consider a function f defined on an interval $[a, b]$, and suppose that the graph of f is connected and lacks kinks, as illustrated in Fig. 8.4.5. Because the graph of f joins A to B by a connected curve having a tangent at each point, it is geometrically plausible that for at least one value of x between a and b, the tangent to the graph at x should be parallel to the line AB. In Fig. 8.4.5, x^* appears to be such a value of x. The line AB has slope $[f(b) - f(a)]/(b - a)$. So the condition for the tangent line at $(x^*, f(x^*))$ to be parallel to the line AB is that $f'(x^*) = [f(b) - f(a)]/(b - a)$. In fact, x^* can be chosen so that the vertical distance between the graph of f and AB is as large as possible. The proof that follows is based on this fact.

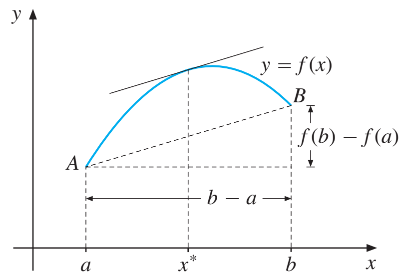


Figure 8.4.5 The mean value theorem

THEOREM 8.4.2 (THE MEAN VALUE THEOREM)

If f is continuous in the closed and bounded interval $[a, b]$, and differentiable in the open interval (a, b), then there exists at least one point x^* in (a, b) such that

$$f'(x^*) = \frac{f(b) - f(a)}{b - a} \tag{8.4.1}$$

We can prove this theorem as follows:

According to the point–point formula, the straight line through A and B in Fig. 8.4.5 has the equation

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

The function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

therefore measures the vertical distance between the graph of f and that line segment. Note that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \tag{$*$}$$

Obviously, $g(a) = g(b) = 0$. The function $g(x)$ inherits from f the properties of being continuous in $[a, b]$ and differentiable in (a, b). By the extreme value theorem, $g(x)$ has a maximum and a minimum over $[a, b]$. Because $g(a) = g(b)$, at least one of these extreme points x^* must lie in (a, b). Theorem 8.1.1 tells us that $g'(x^*) = 0$, and the conclusion follows from $(*)$.

EXAMPLE 8.4.3 Test the mean value theorem on the function $f(x) = x^3 - x$, defined over $[0, 2]$.

Solution: We find that $[f(2) - f(0)]/(2 - 0) = 3$ and $f'(x) = 3x^2 - 1$. The equation $f'(x) = 3$ has two solutions, $x = \pm 2\sqrt{3}/3$. The positive root $x^* = 2\sqrt{3}/3 \in (0, 2)$, and

$$f'(x^*) = \frac{f(2) - f(0)}{2 - 0}$$

This confirms the mean value theorem in this case. ▮

Recall from Section 6.3 that a function f is *increasing* in I if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$ with x_1 and x_2 in I. Using the definition of the derivative, we see easily that if $f(x)$ is increasing and differentiable, then $f'(x) \geq 0$. The mean value theorem can be used to make this statement precise, and to prove the converse. Let f be a function which is continuous in the interval I and differentiable in the interior of I. Suppose $f'(x) \geq 0$ for all x in the interior of I. Let $x_2 > x_1$ be two arbitrary numbers in I. According to the mean value theorem, there exists a number x^* in (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(x^*)(x_2 - x_1) \tag{8.4.2}$$

Because $x_2 > x_1$ and $f'(x^*) \geq 0$, it follows that $f(x_2) \geq f(x_1)$, so $f(x)$ is increasing. This proves statement (6.3.1). The equivalence in (6.3.2) can be proved by considering the condition for $-f$ to be increasing. Finally, (6.3.3) involves both f and $-f$ being increasing.[7]

We can also use the mean value theorem to prove Lagrange's remainder formula (7.6.2):

We start by proving that the formula is correct for $n = 1$. This means that we want to prove formula (7.6.4). For $x \neq 0$, define the function $S(x)$ implicitly by the equation

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}S(x)x^2 \tag{$*$}$$

If we can prove that there exists a number c between 0 and x such that $S(x) = f''(c)$, then (7.6.4) is established. Keep x fixed and define the function g, for all t between 0 and x, by

$$g(t) = f(x) - [f(t) + f'(t)(x - t) + \tfrac{1}{2}S(x)(x - t)^2] \tag{$**$}$$

Then $(*)$ and $(**)$ imply that $g(0) = f(x) - [f(0) + f'(0)x + \frac{1}{2}S(x)x^2] = 0$ and that $g(x) = f(x) - [f(x) + 0 + 0] = 0$. So, by the mean value theorem, there exists a number c strictly between 0 and x such that $g'(c) = 0$. Differentiating $(**)$ with respect to t, with x fixed, we get

$$g'(t) = -f'(t) + f'(t) - f''(t)(x - t) + S(x)(x - t)$$

Thus, $g'(c) = -f''(c)(x - c) + S(x)(x - c)$. Because $g'(c) = 0$ and $c \neq x$, it follows that $S(x) = f''(c)$. Hence, we have proved (7.6.4).

The proof for the case when $n > 1$ is based on the same idea, generalizing $(*)$ and $(**)$ in the obvious way.

EXERCISES FOR SECTION 8.4

1. Find the maximum and minimum and draw the graph of $f(x) = 4x^2 - 40x + 80$, for $x \in [0, 8]$.

SM **2.** Find the maximum and minimum of each function over the indicated interval:

(a) $f(x) = -2x - 1$ over $[0, 3]$

(b) $f(x) = x^3 - 3x + 8$ over $[-1, 2]$

(c) $f(x) = (x^2 + 1)/x$ over $[1/2, 2]$

(d) $f(x) = x^5 - 5x^3$ over $[-1, \sqrt{5}\,]$

(e) $f(x) = x^3 - 4500x^2 + 6 \cdot 10^6 x$ over $[0, 3000]$

3. Suppose the function g is defined for all $x \in [-1, 2]$ by $g(x) = \frac{1}{5}(e^{x^2} + e^{2-x^2})$. Calculate $g'(x)$ and find the extreme points of g.

[7] Alternatively it follows easily by using Eq. (8.4.2).

4. A sports club plans to charter a plane, and charge its members 10% commission on the price they pay to buy seats. That price is arranged by the charter company. The standard fare for each passenger is $800. For each additional person above 60, all travellers (including the first 60) get a discount of $10. The plane can take at most 80 passengers.

   (a) How much commission is earned when there are 61, 70, 80, and $60 + x$ passengers?

   (b) Find the number of passengers that maximizes the total commission earned by the sports club.

5. Let the function f be defined for $x \in [1, e^3]$ by $f(x) = (\ln x)^3 - 2(\ln x)^2 + \ln x$.

   (a) Compute $f(e^{1/3})$, $f(e^2)$, and $f(e^3)$. Find the zeros of $f(x)$.

   (b) Find the extreme points of f.

   (c) Show that f defined over $[e, e^3]$ has an inverse function g and determine $g'(2)$.

SM 6. [HARDER] For the following functions determine all numbers x^* in the specified intervals such that $f'(x^*) = [f(b) - f(a)]/(b - a)$:

   (a) $f(x) = x^2$, in $[1, 2]$

   (b) $f(x) = \sqrt{1 - x^2}$, in $[0, 1]$

   (c) $f(x) = 2/x$, in $[2, 6]$

   (d) $f(x) = \sqrt{9 + x^2}$, in $[0, 4]$

7. [HARDER] You are supposed to sail from point A in a lake to point B. What does the mean value theorem have to say about your trip?

8. [HARDER] Consider the function f defined, for all $x \in [-1, 1]$, by

$$f(x) = \begin{cases} x, & \text{for } x \in (-1, 1) \\ 0, & \text{for } x = -1 \text{ and for } x = 1 \end{cases}$$

   Is this function continuous? Does it attain a maximum or minimum?

9. [HARDER] Let f be defined for all x in $(0, \infty)$ by

$$f(x) = \begin{cases} x + 1, & \text{for } x \in (0, 1] \\ 1, & \text{for } x \in (1, \infty) \end{cases}$$

   Prove that f attains maximum and minimum values. Verify that, nevertheless, *none* of the conditions in the extreme value theorem is satisfied.

8.5 Further Economic Examples

EXAMPLE 8.5.1 A firm that produces a single commodity wants to maximize its profits. The total revenue generated in a certain period by producing and selling Q units is $R(Q)$ dollars, whereas $C(Q)$ denotes the associated total dollar cost. The profit obtained as a result of producing and selling Q units is, then,

$$\pi(Q) = R(Q) - C(Q) \qquad (*)$$

Suppose that because of technical limitations, there is a maximum quantity $\bar{Q}$ that can be produced by the firm in a given period. Assume that R and C are differentiable functions in the interval $(0, \bar{Q})$. The profit function is then differentiable, so it is also continuous. Consequently π does have a maximum value. In special cases, that maximum might occur at $Q = 0$ or at $Q = \bar{Q}$. If not, it has an "interior maximum" where the production level Q^* satisfies $\pi'(Q^*) = 0$, and so

$$R'(Q^*) = C'(Q^*) \tag{$**$}$$

Hence, *production should be adjusted to a point where the marginal revenue is equal to the marginal cost*.

Let us assume that the firm gets a fixed price P per unit sold. Then $R(Q) = PQ$, and $(**)$ takes the form

$$P = C'(Q^*) \tag{8.5.1}$$

Thus, in the case when the firm has no control over the price, *production should be adjusted to a level at which the marginal cost is equal to the price per unit of the commodity*—assuming an interior maximum.

It is quite possible that the firm has functions $R(Q)$ and $C(Q)$ for which Eq. $(**)$ has several solutions. If so, the maximum profit occurs at that point Q^* among the solutions of $(**)$ which gives the highest value of $\pi(Q^*)$.

Equation $(**)$ has an economic interpretation rather like that for the corresponding optimality condition in Example 8.3.1. Indeed, suppose we contemplate increasing production from the level Q^* by one unit. We would increase revenue by the amount $R(Q^* + 1) - R(Q^*) \approx R'(Q^*)$. We would increase cost by the amount $C(Q^* + 1) - C(Q^*) \approx C'(Q^*)$. Equation $(**)$ equates $R'(Q^*)$ and $C'(Q^*)$, so that the approximate extra revenue earned by selling an extra unit is offset by the approximate extra cost of producing that unit.

EXAMPLE 8.5.2 Suppose that the firm in the preceding example obtains a fixed price $P = 121$ per unit, and that the cost function is $C(Q) = 0.02Q^3 - 3Q^2 + 175Q + 500$. The firm can produce at most $\bar{Q} = 110$ units.

(a) Make a table of the values of the functions $R(Q) = 121Q$, $C(Q)$, and $\pi(Q) = R(Q) - C(Q)$, for Q taking the values 0, 10, 30, 50, 70, 90, and 110. Draw the graphs of $R(Q)$ and $C(Q)$ in the same coordinate system.

(b) Answer the following questions, approximately, by using the graphs in (a):
   (i) How many units must be produced in order for the firm to make a profit?
   (ii) How many units must be produced for the profit to be \$2 000?
   (iii) Which production level maximizes profits?

(c) Answer the question in (b.iii) by computation.

(d) What is the smallest price per unit the firm must charge in order not to lose money, if capacity is fully utilized—that is, if it produces 110 units?

Solution:

(a) We form the following table:

Q	0	10	30	50	70	90	110
$R(Q) = 121Q$	0	1 210	3 630	6 050	8 470	10 890	13 310
$C(Q)$	500	1 970	3 590	4 250	4 910	6 530	10 070
$\pi(Q) = R(Q) - C(Q)$	−500	−760	40	1 800	3 560	4 360	3 240

The graphs of $R(Q)$ and $C(Q)$ are shown in Fig. 8.5.1.

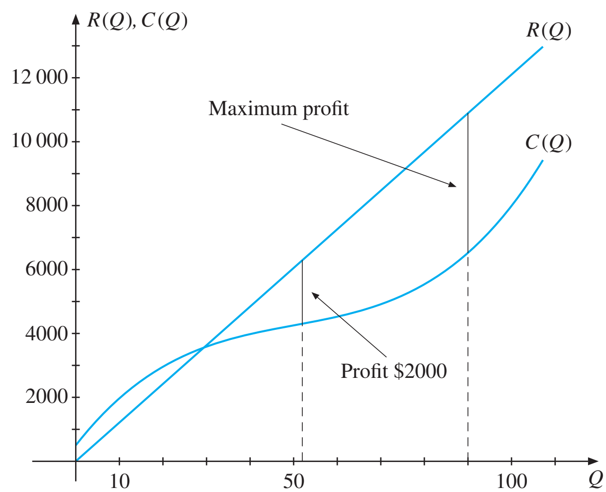


Figure 8.5.1 Revenue, cost, and profit

(b) (i) The firm earns a profit if $\pi(Q) > 0$, that is when $R(Q) > C(Q)$. On the figure we see that $R(Q) > C(Q)$ when Q is larger than, approximately, 30.

(ii) We must find where the "gap" between $R(Q)$ and $C(Q)$ is \$2 000. This occurs when $Q \approx 52$.

(iii) The profit is largest when the gap between $R(Q)$ and $C(Q)$ is largest. This seems to occur when $Q \approx 90$.

(c) When the formula for $C'(Q)$ is inserted into Eq. (8.5.1) with $P = 121$, the result is $121 = 0.06Q^2 - 6Q + 175$. Solving this quadratic equation yields $Q = 10$ and $Q = 90$. We know that $\pi(Q)$ must have a maximum point in $[0, 110]$, and there are four candidates: $Q = 0$, $Q = 10$, $Q = 90$, and $Q = 110$. Using the table from part (a), we see that

$$\pi(0) = -500, \ \pi(10) = -760, \ \pi(90) = 4360, \ \pi(110) = 3240$$

The firm therefore attains maximum profit by producing 90 units.

(d) If the price per unit is P, the profit from producing 110 units is

$$\pi(110) = P \cdot 110 - C(110) = 110P - 10\,070$$

The smallest price P which ensures that the firm does not lose money when producing 110 units, satisfies $\pi(110) = 0$, that is $110P = \$10\,070$ with solution $P \approx \$91.55$. This is the average cost of producing 110 units. The price must be at least \$91.55 if revenue is going to be enough to cover the cost of producing at full capacity.

EXAMPLE 8.5.3 In the model of the previous example, the firm took the price as given. Consider an example at the other extreme, where the firm has a monopoly in the sale of the commodity. Assume that the price $P(Q)$ per unit varies with Q according to the formula $P(Q) = 100 - \frac{1}{3}Q$ for $Q \in [0, 300]$. Suppose now the cost function is

$$C(Q) = \frac{1}{600}Q^3 - \frac{1}{3}Q^2 + 50Q + \frac{1000}{3}$$

The profit is, then,

$$\pi(Q) = QP(Q) - C(Q) = -\frac{1}{600}Q^3 + 50Q - \frac{1000}{3}$$

Find the production level that maximizes profit, and compute the maximum profit.

Solution: The derivative of $\pi(Q)$ is $\pi'(Q) = -\frac{1}{200}Q^2 + 50$. Hence, $\pi'(Q) = 0$ for $Q^2 = 10\,000$. Because $Q < 0$ is not permissible, the maximum is at $Q = 100$.

The values of $\pi(Q)$ at the end points of $[0, 300]$ are $\pi(0) = -1000/3$ and $\pi(300) = -91\,000/3$. Since $\pi(100) = 3000$, we conclude that $Q = 100$ maximizes profit, and the maximum profit is 3000.

EXAMPLE 8.5.4 **(Either a borrower or a lender be)**[8] Recall Example 7.1.5, and suppose that a student has current income y_1 and expects future income y_2. She plans current consumption, $c_1 > 0$, and future consumption, $c_2 > 0$, in order to maximize the utility function

$$U = \ln c_1 + \frac{1}{1+\delta} \ln c_2$$

where δ is her discount rate.[9] If she borrows now, so that $c_1 > y_1$, then future consumption, after repaying the loan amount $c_1 - y_1$ with interest charged at rate r, will be

$$c_2 = y_2 - (1+r)(c_1 - y_1)$$

Alternatively, if she saves now, so that $c_1 < y_1$, then future consumption will be

$$c_2 = y_2 + (1+r)(y_1 - c_1)$$

after receiving interest at rate r on her savings. Find the optimal borrowing or saving plan.

[8] According to Shakespeare, Polonius's advice to Hamlet was: "Neither a borrower nor a lender be".

[9] In terms of Example 7.1.5, $\beta = 1/(1+\delta)$.

Solution: Whether the student borrows or saves, second period consumption is

$$c_2 = y_2 - (1+r)(c_1 - y_1)$$

in either case. So the student will want to maximize, by choosing c_1,

$$U = \ln c_1 + \frac{1}{1+\delta}\ln[y_2 - (1+r)(c_1 - y_1)] \qquad (*)$$

We can obviously restrict attention to the interval $0 < c_1 < y_1 + (1+r)^{-1}y_2$, where both c_1 and c_2 are positive. Differentiating $(*)$ w.r.t. the choice variable c_1 gives

$$\frac{dU}{dc_1} = \frac{1}{c_1} - \frac{1+r}{1+\delta}\cdot\frac{1}{y_2 - (1+r)(c_1 - y_1)}$$

Rewriting the fractions so that they have a common denominator yields

$$\frac{dU}{dc_1} = \frac{(1+\delta)[y_2 - (1+r)(c_1 - y_1)] - (1+r)c_1}{c_1(1+\delta)[y_2 - (1+r)(c_1 - y_1)]}$$

Rearranging the numerator and equating the derivative to 0, we have

$$\frac{dU}{dc_1} = \frac{(1+\delta)[(1+r)y_1 + y_2] - (2+\delta)(1+r)c_1}{c_1(1+\delta)[y_2 - (1+r)(c_1 - y_1)]} = 0 \qquad (**)$$

The unique solution of this equation is

$$c_1^* = \frac{(1+\delta)[(1+r)y_1 + y_2]}{(2+\delta)(1+r)} = y_1 + \frac{(1+\delta)y_2 - (1+r)y_1}{(2+\delta)(1+r)}$$

From $(**)$, we see that for $c_1 < c_1^*$ one has $dU/dc_1 > 0$, whereas for $c_1 > c_1^*$ one has $dU/dc_1 < 0$. We conclude that c_1^* indeed maximizes U. Moreover, the student lends if and only if $(1+\delta)y_2 < (1+r)y_1$. In the more likely case when $(1+\delta)y_2 > (1+r)y_1$ because future income is considerably higher than present income, she will borrow. Only if by some chance $(1+\delta)y_2$ is exactly equal to $(1+r)y_1$ will she be neither a borrower nor a lender. However, this discussion has neglected the difference between borrowing and lending rates of interest that one always observes in reality.

EXERCISES FOR SECTION 8.5

1. With reference to Example 8.5.1, suppose that $R(Q) = 10Q - Q^2/1000$ for $Q \in [0, 10\,000]$ and $C(Q) = 5000 + 2Q$ for all $Q \geq 0$. Find the value of Q that maximizes profit.

2. With reference to Example 8.5.1, let $R(Q) = 80Q$ and $C(Q) = Q^2 + 10Q + 900$. The firm can produce at most 50 units.

   (a) Draw the graphs of R and C in the same coordinate system.

   (b) Answer the following questions both graphically and by computation:

      (i) How many units must be produced for the firm to make a profit?

      (ii) How many units must be produced for the firm to maximize profit?

3. A pharmaceutical firm produces penicillin. The sales price per unit is 200, while the cost of producing x units is given by $C(x) = 500\,000 + 80x + 0.003x^2$. The firm can produce at most 30 000 units. What value of x maximizes profits?

SM 4. Consider Example 8.5.1 and find the production level which maximizes profits when

(a) $R(Q) = 1840Q$ and $C(Q) = 2Q^2 + 40Q + 5000$

(b) $R(Q) = 2240Q$ and $C(Q) = 2Q^2 + 40Q + 5000$

(c) $R(Q) = 1840Q$ and $C(Q) = 2Q^2 + 1940Q + 5000$

5. The price a firm obtains for a commodity varies with demand Q according to the formula $P(Q) = 18 - 0.006Q$. Total cost is $C(Q) = 0.004Q^2 + 4Q + 4500$.

(a) Find the firm's profit $\pi(Q)$ and the value of Q which maximizes profit.

(b) Find a formula for the elasticity of $P(Q)$ w.r.t. Q, and find the particular value Q^* of Q at which the elasticity is equal to -1.

(c) Show that the marginal revenue is 0 at Q^*.

6. With reference to Example 8.5.1, let $R(Q) = PQ$ and $C(Q) = aQ^b + c$, where P, a, b, and c are positive constants, and $b > 1$. Find the value of Q which maximizes the profit $\pi(Q) = PQ - (aQ^b + c)$. Make use of Theorem 8.2.2.

8.6 Local Extreme Points

So far this chapter has discussed what are often referred to as *global* optimization problems. The reason for this terminology is that we have been seeking the largest or smallest values of a function when we compare the function values at *all* points in the domain, without exception. In applied optimization problems, especially those arising in economics, it is usually these global extrema that are of interest. However, sometimes one is interested in the local maxima and minima of a function. In this case, we compare the function value at the point in question only with alternative function values at nearby points.

Consider Fig. 8.6.1 and think of the graph as representing the profile of a landscape. Then the mountain tops P_1 and P_2 represent local maxima, whereas the valley bottoms Q_1 and Q_2 represent local minima. The precise definitions are as follows:

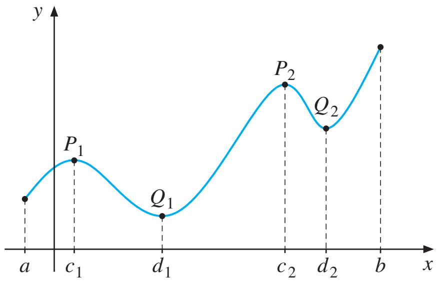


Figure 8.6.1 c_1, c_2, and b are local maximum points; a, d_1, and d_2 are local minimum points

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The function f has a *local maximum* at c if there exists an interval (α, β) about c such that

$$f(x) \leq f(c) \text{ for all } x \text{ in } (\alpha, \beta) \text{ which are in the domain of } f \tag{8.6.1}$$

It has a *local minimum* at c if there exists an interval (α, β) about c such that

$$f(x) \geq f(c) \text{ for all } x \text{ in } (\alpha, \beta) \text{ which are in the domain of } f \tag{8.6.2}$$

These definitions imply that point a in Fig. 8.6.1 is a local minimum point, while b is a local (and global) maximum point.[10] Function values corresponding to local maximum (minimum) points are called *local maximum (minimum) values*. As collective names we use *local extreme points* and *local extreme values*.

In searching for (global) maximum and minimum points, Theorem 8.1.1 was very useful. Actually, the same result is valid for local extreme points:

At a local extreme point in the interior of the domain of a differentiable function, the derivative must be zero.

This is clear if we recall that the proof of Theorem 8.1.1 needed to consider the behaviour of the function in only a small interval about the optimal point. Consequently, in order to find possible local maxima and minima for a function f defined in an interval I, we can again search among the following types of point:

(i) interior points in I where $f'(x) = 0$;

(ii) end points of I, if included in I; and

(iii) interior points in I where f' does not exist.

We have thus established *necessary* conditions for a function f defined in an interval I to have a local extreme point. But how do we decide whether a point satisfying the necessary conditions is a local maximum, a local minimum, or neither? In contrast to global extreme points, it does not help to calculate the function value at the different points satisfying these necessary conditions. To see why, consider again the function whose graph is given in Fig. 8.6.1. Point P_1 is a local maximum point and Q_2 is a local minimum point, but the function value at P_1 is *smaller* than the function value at Q_2.

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[10] Some authors restrict the definition of local maximum/minimum points only to *interior* points of the domain of the function. According to this definition, a global maximum point that is not an interior point of the domain is not a local maximum point. It seems desirable that a global maximum/minimum point should always be a local maximum/minimum point as well, so we stick to definitions (8.6.1) and (8.6.2).

The First-Derivative Test

There are two main ways of determining whether a given critical point is a local maximum, a local minimum, or neither. One of them is based on studying the sign of the first derivative about the critical point, and is an easy modification of Theorem 8.2.1.

THEOREM 8.6.1 (FIRST-DERIVATIVE TEST FOR LOCAL EXTREMA)

Suppose c is a critical point for $y = f(x)$.

(i) If $f'(x) \geq 0$ throughout some interval (a, c) to the left of c and $f'(x) \leq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local maximum point for f.

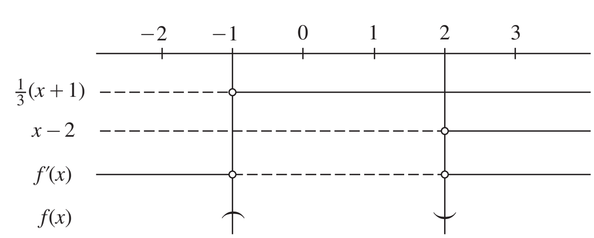
(ii) If $f'(x) \leq 0$ throughout some interval (a, c) to the left of c and $f'(x) \geq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local minimum point for f.

(iii) If $f'(x) > 0$ both throughout some interval (a, c) to the left of c and throughout some interval (c, b) to the right of c, then $x = c$ is not a local extreme point for f. The same conclusion holds if $f'(x) < 0$ on both sides of c.

Only case (iii) is not already covered by Theorem 8.2.1. In fact, if $f'(x) > 0$ in (a, c) and also in (c, b), then $f(x)$ is strictly increasing in $(a, c]$ as well as in $[c, b)$. Then $x = c$ cannot be a local extreme point.

EXAMPLE 8.6.1 Classify the critical points of $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$.

Solution: We get $f'(x) = \frac{1}{3}(x + 1)(x - 2)$, so $x = -1$ and $x = 2$ are the critical points. The sign diagram for $f'(x)$ is:



We conclude from this sign diagram that $x = -1$ is a local maximum point whereas $x = 2$ is a local minimum point.

EXAMPLE 8.6.2 Classify the critical points of $f(x) = x^2 e^x$.

Solution: Differentiating, we get $f'(x) = 2xe^x + x^2e^x = xe^x(2 + x)$. Then $f'(x) = 0$ for $x = 0$ and for $x = -2$. A sign diagram shows that f has a local maximum point at $x = -2$ and a local, as well as global, minimum point at $x = 0$. The graph of f is given in Fig. 8.6.2.

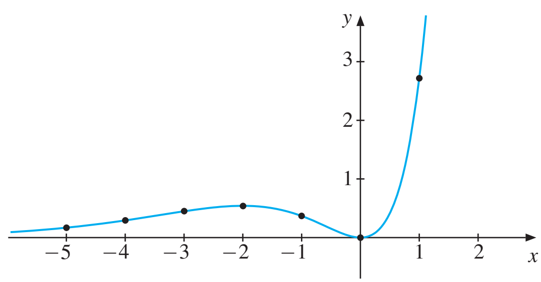


Figure 8.6.2 $f(x) = x^2 e^x$

The Second-Derivative Test

For most problems of practical interest in which an explicit function is specified, the first-derivative test on its own will determine whether a critical point is a local maximum, a local minimum, or neither. Note that the theorem requires knowing the sign of $f'(x)$ at points both to the left and to the right of the given critical point. The next test requires knowing the first two derivatives of the function, but only at the critical point itself.

THEOREM 8.6.2 (SECOND-DERIVATIVE TEST FOR LOCAL EXTREMA)

Let f be a twice differentiable function in an interval I, and let c be an interior point of I.

(i) If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a strict local maximum point.

(ii) If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a strict local minimum point.

(iii) If $f'(c) = 0$ and $f''(c) = 0$, the character of $x = c$ remains undetermined.

The proof is as follows:

To prove part (i), assume $f'(c) = 0$ and $f''(c) < 0$. By definition of $f''(c)$ as the derivative of $f'(x)$ at c,

$$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h}$$

Because $f''(c) < 0$, it follows that $f'(c+h)/h < 0$ if $|h|$ is sufficiently small. In particular, if h is a small positive number, then $f'(c+h) < 0$, so f' is negative in an interval to the right of c. In the same way, we see that f' is positive in some interval to the left of c. But then c is a strict local maximum point for f.

Part (ii) can be proved in the same way.

For the inconclusive part (iii), where $f'(c) = f''(c) = 0$, "anything" can happen. Each of three functions, $f(x) = x^4$, $f(x) = -x^4$, and $f(x) = x^3$, satisfies $f'(0) = f''(0) = 0$. At $x = 0$, they have, as shown in Figs 8.6.3 to 8.6.5, respectively, a minimum, a maximum, and what will be called a point of inflection in Section 8.7. Usually, as here, the first-derivative test can be used to classify critical points at which $f'(c) = f''(c) = 0$.

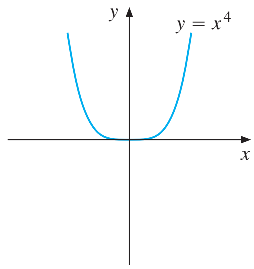


Figure 8.6.3 Minimum point

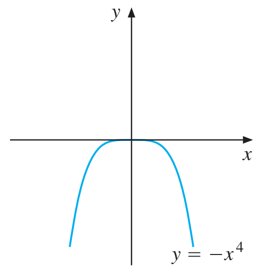


Figure 8.6.4 Maximum point

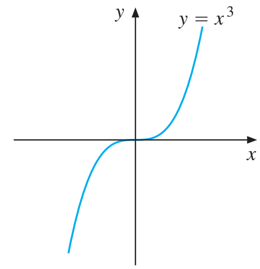


Figure 8.6.5 Inflection point

EXAMPLE 8.6.3 Classify the critical points of $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$, by using the second-derivative test.

Solution: We saw in Example 8.6.1 that

$$f'(x) = \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = \frac{1}{3}(x+1)(x-2)$$

with two critical points $x = -1$ and $x = 2$. Furthermore, $f''(x) = \frac{2}{3}x - \frac{1}{3}$, so that $f''(-1) = -1$ and $f''(2) = 1$. From Theorem 8.6.2 it follows that $x = -1$ is a local maximum point and $x = 2$ is a local minimum point. This confirms the results in Example 8.6.1.

EXAMPLE 8.6.4 Classify the critical points of $f(x) = x^2 e^x$, using the second-derivative test.

Solution: From Example 8.6.2, $f'(x) = 2xe^x + x^2 e^x$, with $x = 0$ and $x = -2$ as the two critical points. The second derivative of f is

$$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2 e^x = e^x(2 + 4x + x^2)$$

We find that $f''(0) = 2 > 0$ and $f''(-2) = -2e^{-2} < 0$. From Theorem 8.6.2, it follows that $x = 0$ is a local minimum point and $x = -2$ is a local maximum point. This confirms the results in Example 8.6.2.

Theorem 8.6.2 can be used to obtain a useful necessary condition for local extrema. Suppose that f is twice differentiable in the interval I and that c is an interior point of I where there is a local maximum. Then $f'(c) = 0$. Moreover, $f''(c) > 0$ is impossible, because by part (ii) in Theorem 8.6.2, this inequality would imply that c is a strict local minimum. Hence, $f''(c)$ has to be ≤ 0. In the same way, we see that $f''(c) \geq 0$ is a necessary condition for local minimum. Briefly formulated:

NECESSARY SECOND-ORDER CONDITIONS

$$\text{Point } c \text{ is a local maximum for } f \;\Rightarrow\; f''(c) \le 0 \tag{8.6.3}$$

$$\text{Point } c \text{ is a local minimum for } f \;\Rightarrow\; f''(c) \ge 0 \tag{8.6.4}$$

Many results in economic analysis rely on postulating an appropriate sign for the second derivative rather than suitable variations in the sign of the first derivative.

EXAMPLE 8.6.5 Suppose that the firm in Example 8.5.1 faces a sales tax of τ dollars per unit. The firm's profit from producing and selling Q units is, then, $\pi(Q) = R(Q) - C(Q) - \tau Q$. In order to maximize profits at some quantity Q^* satisfying $0 < Q^* < \bar{Q}$, one must have $\pi'(Q^*) = 0$. Hence,

$$R'(Q^*) - C'(Q^*) - \tau = 0 \tag{$*$}$$

Suppose $R''(Q^*) < 0$ and $C''(Q^*) > 0$. Equation $(*)$ implicitly defines Q^* as a differentiable function of τ. Find an expression for $\mathrm{d}Q^*/\mathrm{d}\tau$ and discuss its sign. Also compute the derivative w.r.t. τ of the optimal value $\pi(Q^*)$ of the profit function, and show that $\mathrm{d}\pi(Q^*)/\mathrm{d}\tau = -Q^*$.

Solution: Differentiating $(*)$ with respect to τ yields

$$R''(Q^*)\frac{\mathrm{d}Q^*}{\mathrm{d}\tau} - C''(Q^*)\frac{\mathrm{d}Q^*}{\mathrm{d}\tau} - 1 = 0$$

Solving for $\mathrm{d}Q^*/\mathrm{d}\tau$ gives

$$\frac{\mathrm{d}Q^*}{\mathrm{d}\tau} = \frac{1}{R''(Q^*) - C''(Q^*)} \tag{$**$}$$

The sign assumptions on R'' and C'' imply that $\mathrm{d}Q^*/\mathrm{d}\tau < 0$. Thus, the optimal number of units produced will decline if the tax rate τ increases.

The optimal value of the profit function is $\pi(Q^*) = R(Q^*) - C(Q^*) - \tau Q^*$. Taking into account the dependence of Q^* on τ, we get

$$\begin{aligned}
\frac{\mathrm{d}\pi(Q^*)}{\mathrm{d}\tau} &= R'(Q^*)\frac{\mathrm{d}Q^*}{\mathrm{d}\tau} - C'(Q^*)\frac{\mathrm{d}Q^*}{\mathrm{d}\tau} - Q^* - \tau\frac{\mathrm{d}Q^*}{\mathrm{d}\tau} \\
&= \left[R'(Q^*) - C'(Q^*) - \tau\right]\frac{\mathrm{d}Q^*}{\mathrm{d}\tau} - Q^* \\
&= -Q^*
\end{aligned}$$

Note how the square bracket disappears from this last expression because of the FOC $(*)$. This is an instance of the "envelope theorem", which will be discussed in Section 14.7. For each 1¢ increase in the sales tax, profit decreases by approximately Q^* cents, where Q^* is the number of units produced at the optimum.

EXERCISES FOR SECTION 8.6

1. Consider the function f defined for all x by $f(x) = x^3 - 12x$. Find the critical points of f, and classify them by using both the first- and second-derivative tests.

SM 2. Determine possible local extreme points and values for the following functions:

(a) $f(x) = -2x - 1$ (b) $f(x) = x^3 - 3x + 8$ (c) $f(x) = x + \dfrac{1}{x}$

(d) $f(x) = x^5 - 5x^3$ (e) $f(x) = \frac{1}{2}x^2 - 3x + 5$ (f) $f(x) = x^3 + 3x^2 - 2$

SM 3. Let function f be given by the formula $f(x) = (1 + 2/x)\sqrt{x + 6}$.

(a) Find the domain of f and the intervals where $f(x)$ is positive.

(b) Find possible local extreme points.

(c) Examine $f(x)$ as $x \to 0^-$, $x \to 0^+$, and $x \to \infty$. Also determine the limit of $f'(x)$ as $x \to \infty$. Does f have a maximum or a minimum in the domain?

4. Figure 8.6.6 graphs the *derivative* of a function f. Which of points a, b, c, d, and e are local maximum points for f, local minimum points for f, or neither?

5. Let $f(x) = x^3 + ax^2 + bx + c$. What requirements must be imposed on the constants a, b, and c in order that f will have: (a) a local minimum at $x = 0$? (b) critical points at $x = 1$ and $x = 3$?

6. Find the local extreme points for: (a) $f(x) = x^3 e^x$; and (b) $g(x) = x^2 2^x$.

SM 7. [HARDER] Find the local extreme points of $f(x) = x^3 + ax + b$. Use the answer to show that the equation $f(x) = 0$ has three different real roots if, and only if, $4a^3 + 27b^2 < 0$.

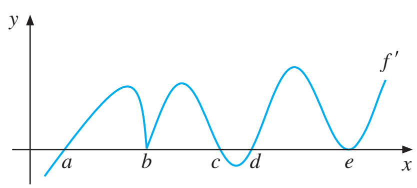


Figure 8.6.6 Exercise 4

8.7 Inflection Points, Concavity, and Convexity

Recall that in Section 6.9 we defined a twice differentiable function $f(x)$ to be concave in an interval I if $f''(x) \le 0$ for all x in I, and convex if $f''(x) \ge 0$ for all x in I. Points at which a function changes from being convex to being concave, or *vice versa*, are called *inflection points*.[11] For twice differentiable functions they can be defined this way:

[11] We note that what a mathematician would call a turning point of a function f, which is a point at which the sign of $f'(x)$ changes, is often erroneously called an inflection point in popular parlance. Perhaps it is too much to expect popular parlance to take account of changes in the sign of the second derivative!

INFLECTION POINTS

If the function f is twice differentiable, the point c is called an *inflection point* for f if there exists an interval (a, b) about c such that:

(a) $f''(x) \geq 0$ in (a, c) and $f''(x) \leq 0$ in (c, b); *or*

(b) $f''(x) \leq 0$ in (a, c) and $f''(x) \geq 0$ in (c, b).

Briefly, $x = c$ is an inflection point if $f''(x)$ *changes sign* at $x = c$, and we refer to the point $(c, f(c))$ as an inflection point on the graph. Figure 8.7.1 gives an abstract example from mathematics, while Fig. 8.7.2 gives a sporting example: it shows the profile of a ski jump. The point P, where the slope is steepest, is an inflection point.

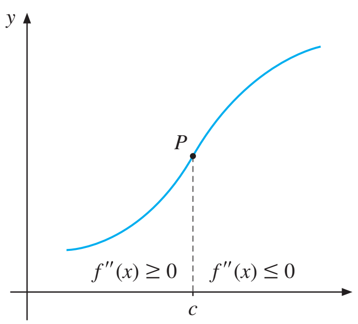


Figure 8.7.1 Point P is an inflection point on the graph; $x = c$ is an inflection point for the function

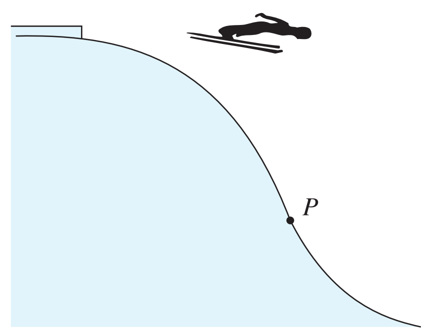


Figure 8.7.2 The point P, where the slope is steepest, is an inflection point

When looking for possible inflection points of a function, we usually use part (ii) in the following theorem:

THEOREM 8.7.1 (TEST FOR INFLECTION POINTS)

Let f be a function with a continuous second derivative in an interval I, and let c be an interior point of I.

(i) If c is an inflection point for f, then $f''(c) = 0$.

(ii) If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f.

The proof of this theorem is rather simple:

(i) Because $f''(x) \leq 0$ on one side of c and $f''(x) \geq 0$ on the other, and because f'' is continuous, it must be true that $f''(c) = 0$.

(ii) If f'' changes sign at c, then c is an inflection point for f, by definition.

This theorem implies that $f''(c) = 0$ is a *necessary* condition for c to be an inflection point. It is not a sufficient condition, however, because $f''(c) = 0$ does not imply that f'' changes sign at $x = c$. A typical case is given in the next example.

EXAMPLE 8.7.1 Show that the function $f(x) = x^4$ does not have an inflection point at $x = 0$, even though $f''(0) = 0$.

Solution: Here $f'(x) = 4x^3$ and $f''(x) = 12x^2$, so that $f''(0) = 0$. But $f''(x) > 0$ for all $x \neq 0$, and so f'' does not change sign at $x = 0$. Hence, $x = 0$ is not an inflection point—in fact, it is a global minimum, as shown in Fig. 8.6.3.

EXAMPLE 8.7.2 Find possible inflection points for $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$.

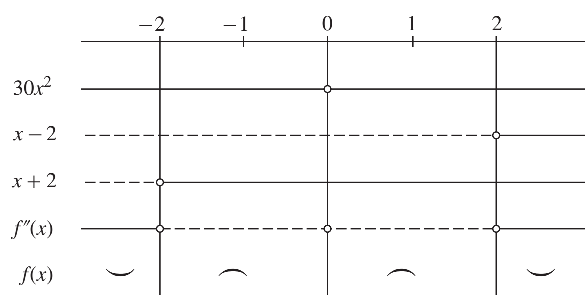
Solution: From Example 8.6.1, we have $f''(x) = \frac{2}{3}x - \frac{1}{3} = \frac{2}{3}\left(x - \frac{1}{2}\right)$. Hence, $f''(x) \leq 0$ for $x \leq 1/2$, whereas $f''(1/2) = 0$ and $f''(x) \geq 0$ for $x > 1/2$. According to part (ii) in Theorem 8.7.1, $x = 1/2$ is an inflection point for f.

EXAMPLE 8.7.3 Find possible inflection points for $f(x) = x^6 - 10x^4$.

Solution: In this case $f'(x) = 6x^5 - 40x^3$ and

$$f''(x) = 30x^4 - 120x^2 = 30x^2(x^2 - 4) = 30x^2(x - 2)(x + 2)$$

A sign diagram for f'' is as follows:



From the sign diagram we see that f'' changes sign at $x = -2$ and at $x = 2$, so these are inflection points. Since f'' does not change sign at $x = 0$, it is not an inflection point, even though $f''(0) = 0$.

Economic models often involve functions having inflection points. The cost function in Fig. 4.7.2 is a typical example. Here is another.

EXAMPLE 8.7.4 A firm produces a commodity using only one input. Let $y = f(x)$, for $x \geq 0$, be the output obtained when x units of the input are used. Then f is called a *production function*. Its first derivative measures the increase in output that is obtained by increasing the input used infinitesimally; this derivative is called the firm's *marginal product*. It is often assumed that the graph of a production function is "S-shaped". That is, the marginal product, $f'(x)$, is increasing up to a certain production level c, and then decreasing. Such a production function is indicated in Fig. 8.7.1. If f is twice differentiable, then $f''(x) \geq 0$ in $[0, c]$, and $f'(x) \leq 0$ in $[c, \infty)$. Hence, f is first convex and then concave, with c as an inflection point. Note that at $x = c$ a unit increase in input gives the maximum increase in output.

More General Definitions of Concave and Convex Functions

So far the convexity and concavity properties of functions have been defined by looking at the sign of the second derivative. An alternative geometric characterization of convexity and concavity suggests a more general definition that is valid even for functions that are not differentiable.

CONCAVE AND CONVEX FUNCTIONS

A function f is called *concave* if the line segment joining any two points on the graph is below the graph, or on it. It is called *convex* if any such line segment lies above, or on the graph.

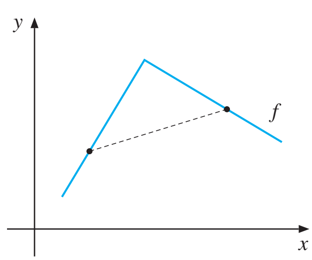


Figure 8.7.3 f is concave

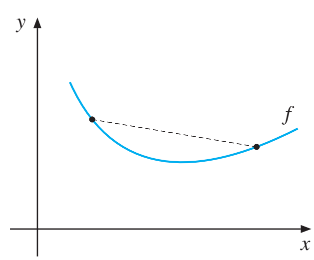


Figure 8.7.4 f is convex

These definitions are illustrated in Figs 8.7.3 and 8.7.4. Because the graph has a "kink" in Fig. 8.7.3, this function is not even differentiable, let alone twice differentiable. For twice differentiable functions, one can prove that this general definition is equivalent to the definitions in (6.9.3) and (6.9.4). Now, in order to use the definition to examine the convexity or concavity of a given function, we need an algebraic formulation. This will be discussed in FMEA.

Strictly Concave and Strictly Convex Functions

A function f is called *strictly concave* if the line segment joining any two points on the graph is strictly below the graph, except at the end points of the segment; it is called *strictly convex* if any such segment lies strictly above the graph, again except at the end points of the segment. For instance, the function whose graph is shown in Fig. 8.7.3 has two linear pieces, on which line segments joining two points coincide with part of the graph. Thus this function is concave, but not strictly concave. By contrast, the function graphed in Fig. 8.7.4 is strictly convex.

Fairly obvious sufficient conditions for strict concavity/convexity are the following, which will be further discussed in FMEA:

$$f''(x) < 0 \text{ for all } x \in (a, b) \implies f \text{ is strictly concave in } (a, b) \tag{8.7.1}$$

$$f''(x) > 0 \text{ for all } x \in (a, b) \implies f \text{ is strictly convex in } (a, b) \tag{8.7.2}$$

The reverse implications are not correct. For instance, one can prove that $f(x) = x^4$ is strictly convex in the interval $(-\infty, \infty)$, but $f''(x)$ is not > 0 everywhere, because $f''(0) = 0$—see Fig. 8.6.3.

For twice differentiable functions, it is usually much easier to check concavity/convexity by considering the sign of the second derivative than by using the definitions of the properties. However, in theoretical arguments the definitions are often very useful, especially because they generalize easily to functions of several variables. (See FMEA.)

EXERCISES FOR SECTION 8.7

1. Let f be defined for all x by $f(x) = x^3 + \frac{3}{2}x^2 - 6x + 10$.

(a) Find the critical points of f and determine the intervals where f increases.

(b) Find the inflection point for f.

2. Decide where the following functions are convex and determine possible inflection points:

(a) $f(x) = \dfrac{x}{1+x^2}$ (b) $g(x) = \dfrac{1-x}{1+x}$ (c) $h(x) = xe^x$

SM **3.** Find local extreme points and inflection points for the functions defined by the following formulas:

(a) $y = (x+2)e^{-x}$ (b) $y = \ln x + 1/x$ (c) $y = x^3 e^{-x}$

(d) $y = \dfrac{\ln x}{x^2}$ (e) $y = e^{2x} - 2e^x$ (f) $y = (x^2 + 2x)e^{-x}$

4. A competitive firm receives a price p for each unit of its output, and pays a price w for each unit of its only variable input. It also incurs set up costs of F. Its output from using x units of variable input is $f(x) = \sqrt{x}$.

(a) Determine the firm's revenue, cost, and profit functions.

(b) Write the first-order condition for profit maximization, and give it an economic interpretation. Check whether profit really is maximized at a point satisfying the first-order condition.

5. Find the extreme points and the inflection points of the function f whose graph is shown in Fig. 8.7.5.

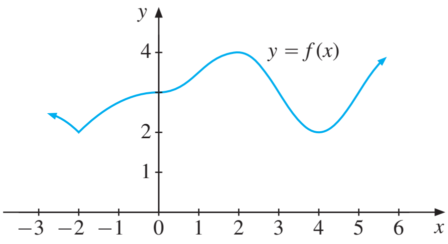


Figure 8.7.5 Exercise 5

6. Find numbers a and b such that the graph of $f(x) = ax^3 + bx^2$ passes through $(-1, 1)$ and has an inflection point at $x = 1/2$.

7. Consider the following cubic cost function, defined for $x \geq 0$: $C(x) = ax^3 + bx^2 + cx + d$, where $a > 0$, $b < 0$, $c > 0$, and $d > 0$. Find the intervals where the function is convex and where it is concave. Find also the unique inflection point.

8. Use the same coordinate system to draw the graphs of two concave functions f and g, both defined for all x. Let the function h be defined by $h(x) = \min\{f(x), g(x)\}$—that is, for each given x, the number $h(x)$ is the smaller of $f(x)$ and $g(x)$. Draw the graph of h and explain why it is also concave.

REVIEW EXERCISES

1. Let $f(x) = \dfrac{x^2}{x^2 + 2}$.

   (a) Compute $f'(x)$ and determine where $f(x)$ is increasing/decreasing.

   (b) Find possible inflection points.

   (c) Determine the limit of $f(x)$ as $x \to \pm\infty$, and sketch the graph of $f(x)$.

2. A firm's production function is $Q(L) = 12L^2 - \frac{1}{20}L^3$, where L denotes the number of workers, with $L \in [0, 200]$.

   (a) What size of the work force maximizes output $Q(L)$?

   (b) What size of the work force maximizes output per worker, $Q(L)/L$? Letting L^* denote such size, note that $Q'(L^*) = Q(L^*)/L^*$. Is this a coincidence?

3. A farmer has one thousand metres of fence wire to make a rectangular enclosure, as in Exercise 4.6.7, but this time no fencing is needed on one side of the enclosure that is a straight canal bank. What should be the dimensions of the enclosure in order to maximize area?

4. By producing and selling Q units of some commodity a firm earns a total revenue $R(Q) = -0.0016Q^2 + 44Q$ and incurs a cost of $C(Q) = 0.0004Q^2 + 8Q + 64\,000$.

   (a) What production level maximizes profits?

   (b) The elasticity $\mathrm{El}_Q\, C(Q) \approx 0.12$ for $Q = 1000$. Interpret this result.

5. The unit price P obtained by a firm in producing and selling $Q \geq 0$ units is $P(Q) = a - bQ^2$, and the cost of producing and selling Q units is $C(Q) = \alpha - \beta Q$. All constants are positive. Find the level of production that maximizes profits.

6. Let $g(x) = x - 2\ln(x + 1)$.

(a) Where is g defined?

(b) Find $g'(x)$ and $g''(x)$.

(c) Find possible extreme points and inflection points, and sketch the graph.

7. Let $f(x) = \ln(x + 1) - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$.

(a) Find the domain of the function and prove that, for x in the domain:

$$f'(x) = \frac{x^2 - x^3}{2(x + 1)}$$

(b) Find possible extreme points and inflection points.

(c) Check $f(x)$ as $x \to (-1)^+$, and sketch the graph on the interval $(-1, 2]$.

SM 8. Consider the function defined, for all x, by $h(x) = e^x/(2 + e^{2x})$.

(a) Where is h increasing/decreasing? Find possible maximum and minimum points for h.

(b) If one restricts the domain of h to $(-\infty, 0]$, it has an inverse. Why? Find an expression for the inverse function.

9. Let $f(x) = \left(e^{2x} + 4e^{-x}\right)^2$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine where f is increasing/decreasing, and show that f is convex.

(c) Find possible global extreme points for f.

SM 10. [HARDER] Letting $a > 0$, consider the function

$$f(x) = \frac{x}{\sqrt[3]{x^2 - a}}$$

(a) Find the domain D_f of f and the intervals where $f(x)$ is positive. Show that the graph of f is symmetric about the origin.

(b) Where is f increasing and where is it decreasing? Find possible local extreme points.

(c) Find possible inflection points for f.

SM 11. [HARDER] Classify the critical points of

$$f(x) = \frac{6x^3}{x^4 + x^2 + 2}$$

by using the first-derivative test. Sketch the graph of f.

9

INTEGRATION

Is it right I ask; is it even prudence; to bore thyself and bore the students?
—Mephistopheles to Faust, in Johann Wolfgang von Goethe's *Faust*

The main topic of the preceding three chapters was differentiation, which can be directly applied to many interesting economic problems. Economists, however, especially when doing statistics, often face the mathematical problem of finding a function from information about its derivative. This process of reconstructing a function from its derivative can be regarded as the "inverse" of differentiation. Mathematicians call this process *integration*.

There are simple formulas that have been known since ancient times for calculating the area of any triangle, and so of any polygon that, by definition, is entirely bounded by straight lines. Over 4000 years ago, however, the Babylonians were concerned with accurately measuring the area of plane surfaces, like circles, that are not bounded by straight lines. Solving this kind of area problem is intimately related to integration, as will be explained in Section 9.2.

Apart from providing an introduction to integration, this chapter will also discuss some important applications of integrals that economists are expected to know. A brief introduction to some simple differential equations concludes the chapter.

9.1 Indefinite Integrals

Suppose we do not know the function F, but we do know that its derivative is x^2, so that $F'(x) = x^2$. What is F? Since the derivative of x^3 is $3x^2$, we see that $\frac{1}{3}x^3$ has x^2 as its derivative. But so does $\frac{1}{3}x^3 + C$ where C is an arbitrary constant, since additive constants disappear with differentiation.

In fact, let $G(x)$ denote an arbitrary function having x^2 as its derivative. Then the derivative of $G(x) - \frac{1}{3}x^3$ is equal to 0 for all x. But, by (6.3.3), a function that has derivative equal to 0 for all x must be constant. This shows that

$$F'(x) = x^2 \iff F(x) = \tfrac{1}{3}x^3 + C$$

with C as an arbitrary constant.

EXAMPLE 9.1.1 Assume that the marginal cost function of a firm is $C'(Q) = 2Q^2 + 2Q + 5$, and that the fixed costs are 100. Find the cost function $C(Q)$.

Solution: Considering separately each of the three terms in the expression for $C'(Q)$, we realize that the cost function must have the form $C(Q) = \frac{2}{3}Q^3 + Q^2 + 5Q + c$, because if we differentiate this function we obtain precisely $2Q^2 + 2Q + 5$. But the fixed costs are 100, which means that $C(0) = 100$. Inserting $Q = 0$ into the proposed formula for $C(Q)$ yields $c = 100$. Hence, the required cost function must be $C(Q) = \frac{2}{3}Q^3 + Q^2 + 5x + 100$.

Suppose $f(x)$ and $F(x)$ are two functions of x having the property that $f(x) = F'(x)$ for all x in some interval I. We pass from F to f by taking the derivative, so the reverse process of passing from f to F could appropriately be called taking the *antiderivative*. But following usual mathematical practice, we call F an *indefinite integral* of f over the interval I, and denote it by $\int f(x)\,\mathrm{d}x$. Two functions having the same derivative throughout an interval must differ by a constant, so:

THE INDEFINITE INTEGRAL

If $F'(x) = f(x)$, then

$$\int f(x)\,\mathrm{d}x = F(x) + C \tag{9.1.1}$$

where C is an arbitrary constant.

For instance, the solution to Example 9.1.1 implies that

$$\int (2x^2 + 2x + 5)\,\mathrm{d}x = \frac{2}{3}x^3 + x^2 + 5x + C$$

The symbol $\int$ is the *integral sign*, and the function $f(x)$ appearing in (9.1.1) is the *integrand*. Then we write $\mathrm{d}x$ to indicate that x is the *variable of integration*. Finally, C is a *constant of integration*. We read (9.1.1) this way: The indefinite integral of $f(x)$ w.r.t. x is $F(x)$ plus a constant. We call it an *indefinite* integral because $F(x) + C$ is not to be regarded as one definite function, but as a whole class of functions, all having the same derivative f.

Differentiating each side of (9.1.1) shows directly that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\,\mathrm{d}x = f(x) \tag{9.1.2}$$

namely, that the derivative of an indefinite integral equals the integrand. Also, (9.1.1) can obviously be rewritten as

$$\int F'(x)\,\mathrm{d}x = F(x) + C \tag{9.1.3}$$

Thus, *integration and differentiation cancel each other out*.

Some Important Integrals

There are some important integration formulas which follow immediately from the corresponding rules for differentiation. Let a be a fixed number, different from -1. Because the derivative of $x^{a+1}/(a+1)$ is x^a, one has

If $a \neq -1$, then

$$\int x^a \, \mathrm{d}x = \frac{1}{a+1} x^{a+1} + C \tag{9.1.4}$$

This very important result states that the indefinite integral of any power of x, except x^{-1}, is obtained by increasing the exponent of x by 1, then dividing by the new exponent, and finally adding a constant of integration. Here are three prominent examples.

EXAMPLE 9.1.2

(a) $\int x \, \mathrm{d}x = \int x^1 \, \mathrm{d}x = \dfrac{1}{1+1} x^{1+1} + C = \dfrac{1}{2} x^2 + C$

(b) $\int \frac{1}{x^3} \, \mathrm{d}x = \int x^{-3} \, \mathrm{d}x = \dfrac{1}{-3+1} x^{-3+1} + C = -\dfrac{1}{2x^2} + C$

(c) $\int \sqrt{x} \, \mathrm{d}x = \int x^{1/2} \, \mathrm{d}x = \dfrac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C = \dfrac{2}{3} x^{3/2} + C$

When $a = -1$, the formula in (9.1.4) is not valid, because the right-hand side involves division by zero and so becomes meaningless. The integrand is then $1/x$, and the problem is thus to find a function having $1/x$ as its derivative. Now $F(x) = \ln x$ has this property, but it is only defined for $x > 0$. Note, however, that $\ln(-x)$ is defined for $x < 0$, and according to the chain rule, its derivative is $[1/(-x)](-1) = 1/x$. Recall too that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. Thus, whether $x > 0$ or $x < 0$, we have:

$$\int \frac{1}{x} \, \mathrm{d}x = \ln |x| + C \tag{9.1.5}$$

Consider next the exponential function. The derivative of e^x is e^x. Thus $\int e^x \, \mathrm{d}x = e^x + C$. More generally, since the derivative of $(1/a)e^{ax}$ is e^{ax}, we have:

If $a \neq 0$, then

$$\int e^{ax} \, \mathrm{d}x = \frac{1}{a} e^{ax} + C \tag{9.1.6}$$

For $a > 0$ we can write $a^x = e^{(\ln a)x}$. As a special case of (9.1.6), when $\ln a \neq 0$ because $a \neq 1$, we obtain:

When $a > 0$ and $a \neq 1$,

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C \tag{9.1.7}$$

The above were examples of how knowing the derivative of a function given by a formula automatically gives us a corresponding indefinite integral. Indeed, suppose it were possible to construct a complete table with every formula that we knew how to differentiate in the first column, and the corresponding derivative in the second column. For example, corresponding to the entry $y = x^2 e^x$ in the first column, there would be $y' = 2xe^x + x^2 e^x$ in the second column. Because integration is the reverse of differentiation, we infer the corresponding integration result that $\int (2xe^x + x^2 e^x) \, dx = x^2 e^x + C$ for a constant C.

Even after this superhuman effort, you would look in vain for e^{-x^2} in the second column of this table. The reason is that there is no "elementary" function that has e^{-x^2} as its derivative. Indeed, the integral of e^{-x^2} is used in the definition of a new very special "error function" that plays a prominent role in statistics because of its relationship to the "normal distribution" — see Exercises 4.9.5 and 9.7.12. A list of a few such impossible "integrals" is given in (9.3.9).

Using the proper rules systematically allows us to *differentiate* very complicated functions. On the other hand, finding the indefinite integral of even quite simple functions can be very difficult, or even impossible. Where it is possible, mathematicians have developed a number of *integration methods* to help in the task. Some of these methods will be explained in the rest of this chapter.

It is usually quite easy, however, to check whether a proposed indefinite integral is correct. We simply differentiate the proposed function to see if its derivative really is equal to the integrand.

EXAMPLE 9.1.3 Verify that, for $x > 0$, $\int \ln x \, dx = x \ln x - x + C$.

Solution: We put $F(x) = x \ln x - x + C$. Then

$$F'(x) = 1 \cdot \ln x + x \cdot (1/x) - 1 = \ln x + 1 - 1 = \ln x$$

which shows that the integral formula is correct.

Some General Rules

The two differentiation rules (6.7.1) and (6.7.2) immediately imply that $(aF(x))' = aF'(x)$ and $(F(x) + G(x))' = F'(x) + G'(x)$. These equalities then imply the following:

BASIC INTEGRATION RULES

$$\int af(x)\,dx = a\int f(x)\,dx, \text{ where } a \neq 0 \text{ is a constant} \tag{9.1.8}$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{9.1.9}$$

Rule (9.1.8) says that a constant factor can be moved outside the integral, while rule (9.1.9) shows that the integral of a sum is the sum of the integrals. Repeated use of these two rules yields:

$$\int \left[a_1 f_1(x) + \cdots + a_n f_n(x)\right] dx = a_1 \int f_1(x)\,dx + \cdots + a_n \int f_n(x)\,dx \tag{9.1.10}$$

EXAMPLE 9.1.4 Use (9.1.10) to evaluate: (a) $\int (3x^4 + 5x^2 + 2)\,dx$; (b) $\int \left(\frac{3}{x} - 8e^{-4x}\right) dx$.

Solution:

(a)
$$\begin{aligned}\int (3x^4 + 5x^2 + 2)\,dx &= 3\int x^4\,dx + 5\int x^2\,dx + 2\int 1\,dx \\ &= 3\left(\frac{1}{5}x^5 + C_1\right) + 5\left(\frac{1}{3}x^3 + C_2\right) + 2(x + C_3) \\ &= \frac{3}{5}x^5 + \frac{5}{3}x^3 + 2x + 3C_1 + 5C_2 + 2C_3 \\ &= \frac{3}{5}x^5 + \frac{5}{3}x^3 + 2x + C\end{aligned}$$

Because C_1, C_2, and C_3 are arbitrary constants, $3C_1 + 5C_2 + 2C_3$ is also an arbitrary constant. So in the last line we have replaced it by just one constant C. In future examples of this kind, we will usually drop the two middle lines of the equations.

(b) $\int \left(3/x - 8e^{-4x}\right) dx = 3\int (1/x)\,dx + (-8)\int e^{-4x}\,dx = 3\ln|x| + 2e^{-4x} + C$

So far, we have always used x as the variable of integration. In applications, the variables often have other labels, but this makes no difference to the rules of integration.

EXAMPLE 9.1.5 Evaluate:

(a) $\int (B/r^{2.5})\,dr$ (b) $\int (a + bq + cq^2)\,dq$ (c) $\int (1+t)^5\,dt$

Solution:

(a) Writing $B/r^{2.5}$ as $Br^{-2.5}$, formula (9.1.4) can be used with r replacing x, and so

$$\int \frac{B}{r^{2.5}}\,dr = B\int r^{-2.5}\,dr = B\frac{1}{-2.5+1}r^{-2.5+1} + C = -\frac{B}{1.5r^{1.5}} + C$$

(b) $\int (a + bq + cq^2)\,dq = aq + \frac{1}{2}bq^2 + \frac{1}{3}cq^3 + C$
(c) $\int (1+t)^5\,dt = \frac{1}{6}(1+t)^6 + C$

EXERCISES FOR SECTION 9.1

1. Find the following integrals by using formula (9.1.4):

(a) $\int x^{13}\,dx$ (b) $\int x\sqrt{x}\,dx$ (c) $\int \frac{1}{\sqrt{x}}\,dx$ (d) $\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx$

(e) $\int e^{-x}\,dx$ (f) $\int e^{x/4}\,dx$ (g) $\int 3e^{-2x}\,dx$ (h) $\int 2^x\,dx$

2. In the manufacture of a product, the marginal cost of producing x units is $C'(x)$ and fixed costs are $C(0)$. Find the total cost function $C(x)$ when:

(a) $C'(x) = 3x + 4$ and $C(0) = 40$ (b) $C'(x) = ax + b$ and $C(0) = C_0$

SM 3. Find the following integrals:

(a) $\int (t^3 + 2t - 3)\,dt$ (b) $\int (x-1)^2\,dx$ (c) $\int (x-1)(x+2)\,dx$

(d) $\int (x+2)^3\,dx$ (e) $\int \left(e^{3x} - e^{2x} + e^x\right)\,dx$ (f) $\int \frac{x^3 - 3x + 4}{x}\,dx$

SM 4. Find the following integrals:

(a) $\int \frac{(y-2)^2}{\sqrt{y}}\,dy$ (b) $\int \frac{x^3}{x+1}\,dx$ (c) $\int x(1+x^2)^{15}\,dx$

(*Hints*: In part (a), first expand $(y-2)^2$, and then divide each term by $\sqrt{y}$. In part (b), use polynomial division as in Section 4.7. In part (c), what is the derivative of $(1+x^2)^{16}$?)

5. Show that

(a) $\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

(b) $\int \sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left(x + \sqrt{x^2+1}\right) + C$

6. Suppose that $f(0) = 2$ and that the *derivative* of f has the graph given in Fig. 9.1.1. First suggest a formula for $f'(x)$, then sketch the graph of $f(x)$, and finally find an explicit function $f(x)$ which has this graph.

7. Suppose that $f(0) = 0$ and that the *derivative* of f has the graph given in Fig. 9.1.2. Sketch the graph of $f(x)$ and find an explicit function $f(x)$ which has this graph.

8. Prove that $\int 2x \ln(x^2 + a^2)\,dx = (x^2 + a^2)\ln(x^2 + a^2) - x^2 + C$.

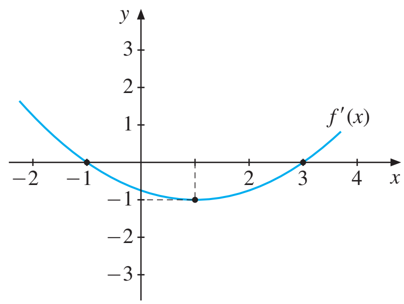


Figure 9.1.1 Problem 6

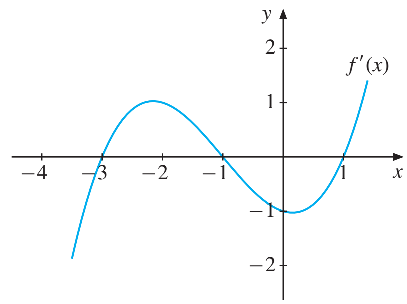


Figure 9.1.2 Problem 7

9. (a) Show that if $a \neq 0$ and $p \neq -1$ then $\displaystyle\int (ax+b)^p\,dx = \frac{1}{a(p+1)}(ax+b)^{p+1} + C$.

(b) Use part (a) to evaluate: (i) $\displaystyle\int (2x+1)^4\,dx$; (ii) $\displaystyle\int \sqrt{x+2}\,dx$; and (iii) $\displaystyle\int \frac{1}{\sqrt{4-x}}\,dx$.

(c) Find $F(x)$ if: (i) $F'(x) = \frac{1}{2}e^x - 2x$ and $F(0) = \frac{1}{2}$; (ii) $F'(x) = x(1-x^2)$ and $F(1) = \frac{5}{12}$.

10. Find the general form of a function f whose second derivative is x^2. If we require in addition that $f(0) = 1$ and $f'(0) = -1$, what is $f(x)$?

SM **11.** Suppose that $f''(x) = x^{-2} + x^3 + 2$ for $x > 0$, and $f(1) = 0$, $f'(1) = 1/4$. Find $f(x)$.

9.2 Area and Definite Integrals

This section will show how the concept of the integral can be used to calculate the area of many plane regions. This problem has been important in economics for over 4000 years. Like all major rivers, the Tigris and Euphrates in Mesopotamia (now part of Iraq) and the Nile in Egypt would occasionally change course as a result of severe floods. Some farmers would gain new land from the river, while others would lose land. Since taxes were often assessed on land area, it became necessary to re-calculate the area of a parcel of land whose boundary might be an irregularly shaped river bank.

Rather later, but still around 360 *b.c.*, the Greek mathematician Eudoxos developed a general *method of exhaustion* for determining the areas of irregularly shaped plane regions. The idea was to exhaust the area by inscribing within it an expanding sequence of polygonal regions, whose area can be calculated exactly by summing the areas of a finite collection of triangles. Provided this sequence does indeed "exhaust" the area by including every point in the limit, we can define the *area* of the region as the limit of the increasing sequence of areas of the inscribed polygonal regions. Moreover, one can bound the error of any finite approximation by circumscribing the region within a decreasing sequence of polygonal regions, whose intersection is the region itself.

Eudoxos and Archimedes, amongst others, used the method of exhaustion in order to determine quite accurate approximations to the areas of a number of specific plane regions, especially for a circular disk — see Example 7.11.1 for an illustration of how this might work. The method was able to provide exact answers, however, only for a limited number of special cases, largely because of the algebraic problems encountered. Nearly 1900 years passed after Eudoxos before an exact method could be devised, combining what we now call integration with the new differential calculus due to Newton and Leibniz. Besides allowing areas to be measured with complete accuracy, their ideas have many other applications. Demonstrating the precise logical relationship between differentiation and integration is one of the main achievements of mathematical analysis. It has even been argued that this discovery is the single most important in all of science.

The problem to be considered and solved in this section is illustrated in Fig. 9.2.1: *How do we compute the area A under the graph of a continuous and nonnegative function f over the interval $[a, b]$?*

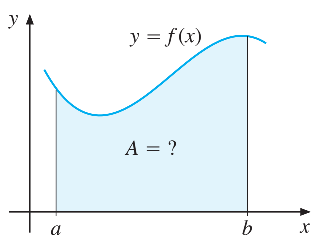


Figure 9.2.1 Area A under the graph

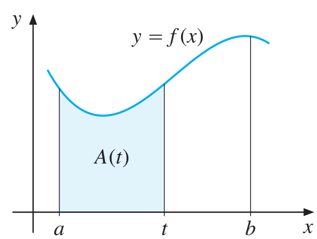


Figure 9.2.2 Area $A(t)$ under the graph up to t

Let t be an arbitrary point in $[a, b]$, and let $A(t)$ denote the area under the curve $y = f(x)$ over the interval $[a, t]$, as shown in Fig. 9.2.2. Clearly, $A(a) = 0$, because there is no area from a to a. On the other hand, the area in Fig. 9.2.1 is $A = A(b)$. It is obvious from Fig. 9.2.2 that, because f is always positive, $A(t)$ increases as t increases. Suppose we increase t by a positive amount Δt. Then $A(t + \Delta t)$ is the area under the curve $y = f(x)$ over the interval $[a, t + \Delta t]$. Hence, $A(t + \Delta t) - A(t)$ is the area ΔA under the curve over the interval $[t, t + \Delta t]$, as shown in Fig. 9.2.3.

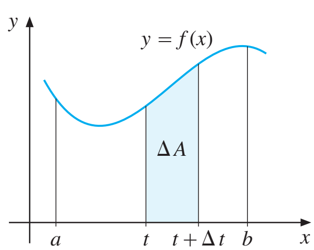


Figure 9.2.3 Change in area, ΔA

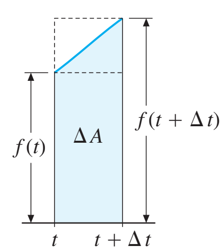


Figure 9.2.4 Approximating ΔA

In Fig. 9.2.4, the area ΔA has been magnified. It cannot be larger than the area of the rectangle with base Δt and height $f(t + \Delta t)$, nor smaller than the area of the rectangle with base Δt and height $f(t)$. Hence, for all $\Delta t > 0$, one has

$$f(t)\Delta t \le A(t + \Delta t) - A(t) \le f(t + \Delta t)\Delta t \tag{$*$}$$

Because $\Delta t > 0$, this implies

$$f(t) \leq \frac{A(t+\Delta t) - A(t)}{\Delta t} \leq f(t+\Delta t) \tag{**}$$

Let us consider what happens to (**) as $\Delta t \to 0$. The interval $[t, t+\Delta t]$ shrinks to the single point t, and by continuity of f, the value $f(t+\Delta t)$ approaches $f(t)$. The Newton quotient $[A(t+\Delta t) - A(t)]/\Delta t$ is squeezed between $f(t)$ and a quantity $f(t+\Delta t)$ that tends to $f(t)$. This quotient must therefore tend to $f(t)$ in the limit as $\Delta t \to 0$.

So we arrive at the remarkable conclusion that the function $A(t)$, which measures the area under the graph of f over the interval $[a, t]$, is differentiable, with derivative given by

$$A'(t) = f(t), \quad \text{for all } t \text{ in } (a, b) \tag{***}$$

This proves that:

The derivative of the area function $A(t)$ is the curve's "height" function $f(t)$, and the area function is therefore one of the indefinite integrals of $f(t)$.[1]

Let us now use x as the free variable, and suppose that $F(x)$ is an arbitrary indefinite integral of $f(x)$. Then $A(x) = F(x) + C$ for some constant C. Recall that $A(a) = 0$. Hence, $0 = A(a) = F(a) + C$, so $C = -F(a)$. Therefore,

$$A(x) = F(x) - F(a), \quad \text{where } F(x) = \int f(x)\,\mathrm{d}x \tag{9.2.1}$$

Suppose $G(x)$ is another function with $G'(x) = f(x)$. Then $G(x) = F(x) + C_1$ for some other constant C_1, and so $G(x) - G(a) = F(x) + C_1 - (F(a) + C_1) = F(x) - F(a)$. This argument tells us that the area we compute using (9.2.1) is independent of which indefinite integral of f we choose.

EXAMPLE 9.2.1 Calculate the area under the parabola $f(x) = x^2$ over the interval $[0, 1]$.

Solution: The area in question is the shaded region in Fig. 9.2.5. The area is equal to $A = F(1) - F(0)$ where $F(x)$ is an indefinite integral of x^2. Now, $\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + C$, so we choose $F(x) = \frac{1}{3}x^3$. Thus the required area is

$$A = F(1) - F(0) = \frac{1}{3} \cdot 1^3 - \frac{1}{3} \cdot 0^3 = \frac{1}{3}$$

Figure 9.2.5 suggests that this answer is reasonable, because the shaded region appears to have roughly 1/3 the area of a square whose side is of length 1.

[1] The function f in the figures is increasing in the interval $[t, t+\Delta t]$. It is easy to see that the same conclusion is obtained whenever the function f is continuous on the closed interval $[t, t+\Delta t]$. On the left-hand side of (*), just replace $f(t)$ by $f(c)$, where c is a minimum point of the continuous function f in the interval; and on the right-hand side, replace $f(t+\Delta t)$ by $f(d)$, where d is a maximum point of f in $[t, t+\Delta t]$. By continuity, both $f(c)$ and $f(d)$ must tend to $f(t)$ as $\Delta t \to 0$. So (***) holds also for general continuous functions f.

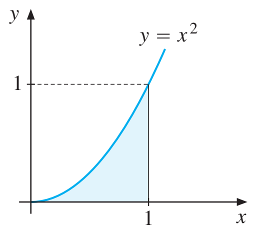


Figure 9.2.5 $y = x^2$

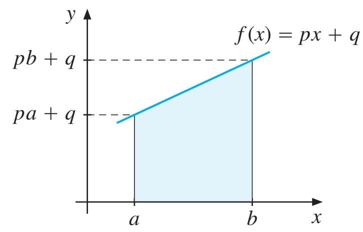


Figure 9.2.6 $y = px + q$

The argument leading to (9.2.1) is based on rather intuitive considerations. Formally, mathematicians choose to *define* the area under the graph of a continuous and nonnegative function f over the interval $[a, b]$ as the number $F(b) - F(a)$, where $F'(x) = f(x)$. The concept of area that emerges agrees with the usual concept for regions bounded by straight lines. The next example verifies this in a special case.

EXAMPLE 9.2.2 Find the area A under the straight line $f(x) = px + q$ over the interval $[a, b]$, where a, b, p, and q are all positive, with $b > a$.

Solution: The area is shown shaded in Fig. 9.2.6. It is equal to $F(b) - F(a)$ where $F(x)$ is an indefinite integral of $px + q$. Now, $\int (px + q)\,\mathrm{d}x = \frac{1}{2}px^2 + qx + C$. The obvious choice of an indefinite integral is $F(x) = \frac{1}{2}px^2 + qx$, and so

$$A = F(b) - F(a) = \left(\frac{1}{2}pb^2 + qb\right) - \left(\frac{1}{2}pa^2 + qa\right) = \frac{1}{2}p(b^2 - a^2) + q(b - a)$$

As Fig. 9.2.6 shows, the area A is the sum of a rectangle whose area is $(b - a)(pa + q)$, and a triangle whose area is $\frac{1}{2}p(b - a)^2$, which you should check gives the same answer.

The Definite Integral

Let f be a continuous function defined in the interval $[a, b]$. Suppose that the function F is continuous in $[a, b]$ and has a derivative with $F'(x) = f(x)$ for every x in (a, b). Then the difference $F(b) - F(a)$ is called the *definite integral* of f over $[a, b]$. We observed above that this difference does not depend on which of the indefinite integrals of f we choose as F. The definite integral of f over $[a, b]$ is therefore a *number* that depends only on the function f and the numbers a and b. We denote this number by

$$\int_a^b f(x)\,\mathrm{d}x \tag{9.2.2}$$

This notation makes explicit the function $f(x)$ we integrate and the interval of integration $[a, b]$. The numbers a and b are called, respectively, the *lower* and *upper limit of integration*.

The variable x in Eq. (9.2.2) is a *dummy variable* in the sense that it could be replaced by any other variable that does not occur elsewhere in the expression. For instance,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(y)\,\mathrm{d}y = \int_a^b f(\xi)\,\mathrm{d}\xi$$

are all equal to $F(b) - F(a)$. But do not write anything like $\int_a^y f(y)\,\mathrm{d}y$, with the same variable as both the upper limit and the dummy variable of integration, because that is meaningless.

The difference $F(b) - F(a)$ is denoted by $\Big|_a^b F(x)$, or by $[F(x)]_a^b$. Thus:

THE DEFINITE INTEGRAL

$$\int_a^b f(x)\,\mathrm{d}x = \Big|_a^b F(x) = F(b) - F(a) \tag{9.2.3}$$

where F is any indefinite integral of f over an interval containing both a and b.

EXAMPLE 9.2.3 Evaluate the definite integrals: (a) $\int_2^5 e^{2x}\,\mathrm{d}x$; (b) $\int_{-2}^2 \left(x - x^3 - x^5\right)\,\mathrm{d}x$.

Solution:

(a) Here $\int e^{2x}\,\mathrm{d}x = \frac{1}{2}e^{2x} + C$, so $\int_2^5 e^{2x}\,\mathrm{d}x = \Big|_2^5 \frac{1}{2}e^{2x} = \frac{1}{2}e^{10} - \frac{1}{2}e^4 = \frac{1}{2}e^4(e^6 - 1)$.

(b) $\int_{-2}^2 (x - x^3 - x^5)\,\mathrm{d}x = \Big|_{-2}^2 \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{6}x^6\right) = \left(2 - 4 - \frac{64}{6}\right) - \left(2 - 4 - \frac{64}{6}\right) = 0.$

After reading the next subsection and realizing that the graph of $f(x) = x - x^3 - x^5$ is symmetric about the origin, you should understand better why the answer to part (b) must be 0.

Definition (9.2.1) does not necessarily require $a < b$. However, if $a > b$ and $f(x)$ is positive throughout the interval $[b, a]$, then $\int_a^b f(x)\,\mathrm{d}x$ is a negative number. Note, also, that in (9.2.1) we have defined the definite integral without necessarily giving it a geometric interpretation as the area under a curve. In fact, depending on the context, it can have different interpretations. For instance, if $f(r)$ is an income density function, as in Section 9.4 below, then $\int_a^b f(r)\mathrm{d}r$ is the proportion of people with income between a and b.

Although the notation for definite and indefinite integrals is similar, the two integrals are entirely different. In fact, $\int_a^b f(x)\,\mathrm{d}x$ denotes a single number, whereas $\int f(x)\,\mathrm{d}x$ represents any one of the infinite set of functions that all have $f(x)$ as their derivative.

The Area when $f(x)$ is Negative

If $f(x) \geq 0$ over $[a, b]$, then $\int_a^b f(x)\,\mathrm{d}x$ is the area below the graph of f over $[a, b]$. If f is defined in $[a, b]$ and $f(x) \leq 0$ for all x in $[a, b]$, then the graph of f, the x-axis, and the lines $x = a$ and $x = b$ still enclose an area, shown as area A in Fig. 9.2.7. Defining $g(x) = -f(x)$,

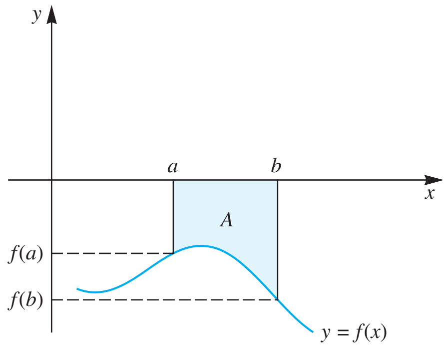


Figure 9.2.7 $f(x) \leq 0$

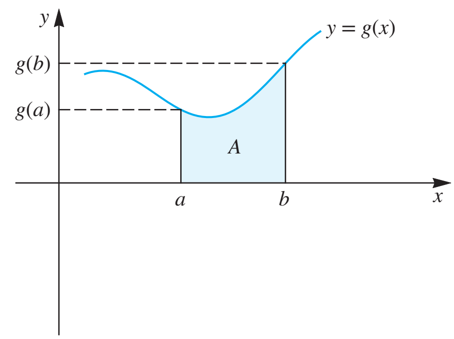


Figure 9.2.8 $g(x) = -f(x) \geq 0$

we have $g(x) \geq 0$, so that $\int_a^b g(x)\,\mathrm{d}x$ measures the area below the graph of g over $[a, b]$. But it follows, by construction, that this is the same as area A, as depicted in Fig. 9.2.8. It follows, then, that the area over f and under $[a, b]$ is $\int_a^b (-f)(x)\,\mathrm{d}x$, with a minus sign before the integrand because the area of a region must be positive (or zero), whereas the definite integral of f is negative. Shortly, in Section 9.3, we will see that we can equivalently put the minus sign in front of the integral: see rule (9.3.3).

EXAMPLE 9.2.4 Figure 9.2.9 shows the graph of $f(x) = e^{x/3} - 3$. Evaluate the shaded area A between the x-axis and this graph over the interval $[0, b]$, where $b = 3 \ln 3$ is chosen because there $f(b) = 0$.

Solution: Because $f(x) \leq 0$ in the interval $[0, 3 \ln 3]$, we obtain

$$A = -\int_0^{3\ln 3} \left(e^{x/3} - 3\right) \mathrm{d}x = -\Big|_0^{3\ln 3} (3e^{x/3} - 3x) = -(3e^{\ln 3} - 3 \cdot 3 \ln 3) + 3e^0$$

$$= -9 + 9 \ln 3 + 3 = 9 \ln 3 - 6 \approx 3.89$$

Is the answer reasonable? Yes, because the shaded set in Fig. 9.2.9 seems to have an area somewhat less than that of the triangle enclosed by the points $(0, 0)$, $(0, -2)$, and $(4, 0)$, whose area is 4, and a little more than the area of the inscribed triangle with vertices $(0, 0)$, $(0, -2)$, and $(3 \ln 3, 0)$, whose area is $3 \ln 3 \approx 3.30$.

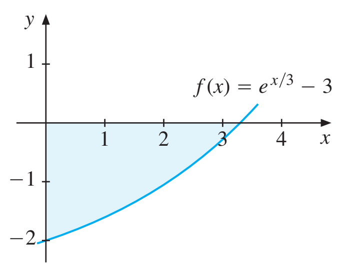


Figure 9.2.9 $e^{x/3} - 3$

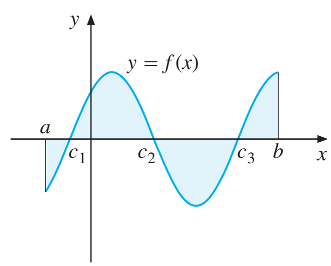


Figure 9.2.10 A function that takes positive and negative values

Suppose the function f is defined and continuous in $[a, b]$, and that it is positive in some subintervals, negative in others, as shown in Fig. 9.2.10. Let c_1, c_2, c_3 denote three roots of the equation $f(x) = 0$—that is, three points where the graph crosses the x-axis. The definite integral $\int_a^b f(x)\,dx$ is the sum of the two shaded areas above the x-axis, minus the sum of the two shaded areas below the x-axis. The total area bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$, on the other hand, is calculated by computing the positive areas in each subinterval $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$, and $[c_3, b]$ in turn according to the previous definitions, and then adding these areas. Specifically, the total shaded area is

$$-\int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx - \int_{c_2}^{c_3} f(x)\,dx + \int_{c_3}^{b} f(x)\,dx$$

In fact, this illustrates a general result: the area between the graph of a function f and the x-axis is given by the definite integral $\int_a^b |f(x)|\,dx$ of the absolute value of the integrand $f(x)$, which equals the area under the graph of the nonnegative-valued function $|f(x)|$.

EXERCISES FOR SECTION 9.2

1. Compute the areas under the graphs, over $[0, 1]$, of: (a) $f(x) = x^2$; (b) $f(x) = x^{10}$.

2. Compute the area bounded by the graph of the function over the indicated interval. In (c), sketch the graph and indicate by shading the area in question.

   (a) $f(x) = 3x^2$, in $[0, 2]$ (b) $f(x) = x^6$, in $[0, 1]$

   (c) $f(x) = e^x$, in $[-1, 1]$ (d) $f(x) = 1/x^2$, in $[1, 10]$

3. Compute the area bounded by the graph of $f(x) = 1/x^3$, the x-axis, and the two lines $x = -2$ and $x = -1$. Make a drawing. (*Hint*: $f(x) < 0$ in $[-2, -1]$.)

4. Compute the area bounded by the graph of $f(x) = \frac{1}{2}(e^x + e^{-x})$, the x-axis, and the lines $x = -1$ and $x = 1$.

SM 5. Evaluate the following integrals:

   (a) $\displaystyle\int_0^1 x\,dx$ (b) $\displaystyle\int_1^2 (2x + x^2)\,dx$ (c) $\displaystyle\int_{-2}^3 \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right) dx$

   (d) $\displaystyle\int_0^2 (t^3 - t^4)\,dt$ (e) $\displaystyle\int_1^2 \left(2t^5 - \frac{1}{t^2}\right) dt$ (f) $\displaystyle\int_2^3 \left(\frac{1}{t-1} + t\right) dt$

SM 6. Let $f(x) = x(x - 1)(x - 2)$.

   (a) Calculate $f'(x)$. Where is $f(x)$ increasing?

   (b) Sketch the graph and calculate $\int_0^1 f(x)\,dx$.

7. The profit of a firm as a function of its output $x > 0$ is given by

$$f(x) = 4000 - x - \frac{3\,000\,000}{x}$$

(a) Find the level of output that maximizes profit. Sketch the graph of f.

(b) The actual output varies between 1000 and 3000 units. Compute the average profit

$$I = \frac{1}{2000}\int_{1000}^{3000} f(x)\,\mathrm{d}x$$

8. Evaluate the integrals

(a) $\displaystyle\int_1^3 \frac{3x}{10}\,\mathrm{d}x$ (b) $\displaystyle\int_{-3}^{-1} \xi^2\,\mathrm{d}\xi$ (c) $\displaystyle\int_0^1 \alpha e^{\beta\tau}\,\mathrm{d}\tau$, with $\beta \neq 0$ (d) $\displaystyle\int_{-2}^{-1} \frac{1}{y}\,\mathrm{d}y$

9.3 Properties of Definite Integrals

From the definition of the definite integral, a number of properties can be derived.

PROPERTIES OF DEFINITE INTEGRALS

If f is a continuous function in an interval that contains the points a, b, and c, then

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x \tag{9.3.1}$$

$$\int_a^a f(x)\,\mathrm{d}x = 0 \tag{9.3.2}$$

$$\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x, \text{ where } \alpha \text{ is an arbitrary number} \tag{9.3.3}$$

and

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \tag{9.3.4}$$

When the definite integral is interpreted as an area, (9.3.4) is the additivity property of areas, as illustrated in Fig. 9.3.1. Of course, rule (9.3.4) easily generalizes to the case in which we partition the interval $[a, b]$ into an arbitrary finite number of subintervals.

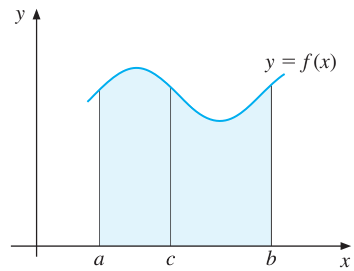


Figure 9.3.1 $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$

Equations (9.3.3) and (9.3.4) are counterparts for definite integrals of, respectively, the constant multiple property (9.1.8) and the summation property (9.1.9) for indefinite integrals. In fact, if f and g are continuous in $[a, b]$, and if α and β are real numbers, then it is easy to prove that

$$\int_a^b [\alpha f(x) + \beta g(x)]\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x \tag{9.3.5}$$

Again, this rule can obviously be extended to more than two functions.

Differentiation with Respect to the Limits of Integration

Suppose that $F'(x) = f(x)$ for all x in an open interval (a, b). Suppose too that $a < t < b$. It follows that $\int_a^t f(x)\,\mathrm{d}x = \Big|_a^t F(x) = F(t) - F(a)$, so

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(x)\,\mathrm{d}x = F'(t) = f(t) \tag{9.3.6}$$

In words: *The derivative of the definite integral with respect to the upper limit of integration is equal to the integrand evaluated at that limit.*

Correspondingly, $\int_t^b f(x)\,\mathrm{d}x = \Big|_t^b F(x) = F(b) - F(t)$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_t^b f(x)\,\mathrm{d}x = -F'(t) = -f(t) \tag{9.3.7}$$

In words: *The derivative of the definite integral with respect to the lower limit of integration is equal to minus the integrand evaluated at that limit.*

These results are not surprising: Suppose that $f(x) \geq 0$ and $t < b$. We can interpret $\int_t^b f(x)\,\mathrm{d}x$ as the area below the graph of f over the interval $[t, b]$. Then the interval shrinks as t increases, and the area will decrease by the value of the integrand at the lower limit.

The results in (9.3.6) and (9.3.7) can be generalized. In fact, if $a(t)$ and $b(t)$ are differentiable and $f(x)$ is continuous, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x = f(b(t))b'(t) - f(a(t))a'(t) \tag{9.3.8}$$

To prove this formula, suppose F is an indefinite integral of f, so that $F'(x) = f(x)$. Then $\int_u^v f(x)\,\mathrm{d}x = F(v) - F(u)$, so in particular,

$$\int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x = F(b(t)) - F(a(t))$$

Using the chain rule to differentiate the right-hand side of this equation w.r.t. t, we obtain $F'(b(t))b'(t) - F'(a(t))a'(t)$. But $F'(b(t)) = f(b(t))$ and $F'(a(t)) = f(a(t))$, so Eq. (9.3.8) holds.[2]

Continuous Functions are Integrable

Suppose $f(x)$ is a continuous function in $[a, b]$. Then we defined $\int_a^b f(x)\,\mathrm{d}x$ as the number $F(b) - F(a)$, provided that $F(x)$ is some function whose derivative is $f(x)$. In some cases, we are able to find an explicit expression for $F(x)$. But this is not always the case. For example, it is impossible to find an explicit standard function of x whose derivative is the positive valued function $f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$, the standard normal density function in statistics, whose graph is shown in the answer to Exercise 4.9.5. Yet $f(x)$ is continuous on any interval $[a, b]$ of the real line, so the area under the graph of f over this interval definitely exists and is equal to $\int_a^b f(x)\,\mathrm{d}x$.

In fact, one can prove that any continuous function has an antiderivative. Here are some integrals that really are impossible to "solve", except by introducing special new functions:

$$\int e^{x^2}\,\mathrm{d}x, \quad \int e^{-x^2}\,\mathrm{d}x, \quad \int \frac{e^x}{x}\,\mathrm{d}x, \quad \int \frac{1}{\ln x}\,\mathrm{d}x, \quad \text{and} \quad \int \frac{1}{\sqrt{x^4+1}}\,\mathrm{d}x \tag{9.3.9}$$

The Riemann Integral

The kind of integral discussed so far, which is based on the antiderivative, is called the *Newton–Leibniz, or N–L, integral*. Several other kinds of integral are considered by mathematicians. For continuous functions, they all give the same result as the N–L integral. We briefly sketch the so-called *Riemann integral*. The idea behind the definition is closely related to the exhaustion method that was described in Section 9.2.

Let f be a *bounded* function in the interval $[a, b]$, and let n be a natural number. Subdivide $[a, b]$ into n parts by choosing points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Let $\Delta x_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, n-1$, and choose an arbitrary number ξ_i in each interval $[x_i, x_{i+1}]$. Then the sum

$$f(\xi_0)\Delta x_0 + f(\xi_1)\Delta x_1 + \cdots + f(\xi_{n-1})\Delta x_{n-1} \tag{9.3.10}$$

[2] Formula (9.3.8) is a special case of Leibniz's formula discussed in FMEA, Section 4.2.

is called *a Riemann sum* associated with the function f. You should draw a figure to help understand this construction.

The sum (9.3.10) depends on f as well as on the subdivision and on the choice of the different points ξ_i. Suppose however that, when n approaches infinity and simultaneously the largest of the numbers $\Delta x_0, \Delta x_1, \dots, \Delta x_{n-1}$ approaches 0, the limit of the sum exists. Then f is called *Riemann, or R, integrable* in the interval $[a, b]$, and we put

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$$

Textbooks on mathematical analysis show that the value of the integral is independent of how the ξ_i are chosen. They also show that every continuous function is R integrable, and that the R integral in this case satisfies (9.2.3). The N–L integral and the R integral thus coincide for continuous functions. But the R integral is defined for some (discontinuous) functions whose N–L integral does not exist.

EXERCISES FOR SECTION 9.3

1. Evaluate the following integrals:

(a) $\displaystyle\int_0^5 (x + x^2)\,\mathrm{d}x$ (b) $\displaystyle\int_{-2}^2 (e^x - e^{-x})\,\mathrm{d}x$ (c) $\displaystyle\int_2^{10} \frac{1}{x-1}\,\mathrm{d}x$ (d) $\displaystyle\int_0^1 2xe^{x^2}\,\mathrm{d}x$

(e) $\displaystyle\int_{-4}^4 (x-1)^3\,\mathrm{d}x$ (f) $\displaystyle\int_1^2 (x^5 + x^{-5})\,\mathrm{d}x$ (g) $\displaystyle\int_0^4 \frac{1}{2}\sqrt{x}\,\mathrm{d}x$ (h) $\displaystyle\int_1^2 \frac{1+x^3}{x^2}\,\mathrm{d}x$

2. If $\int_a^b f(x)\,\mathrm{d}x = 8$ and $\int_a^c f(x)\,\mathrm{d}x = 4$, what is $\int_c^b f(x)\,\mathrm{d}x$?

3. If $\int_0^1 (f(x) - 2g(x))\,\mathrm{d}x = 6$ and $\int_0^1 (2f(x) + 2g(x))\,\mathrm{d}x = 9$, find $I = \int_0^1 (f(x) - g(x))\,\mathrm{d}x$.

SM **4.** Let p, q, and r be positive constants. Evaluate the integral $\int_0^1 x^p(x^q + x^r)\,\mathrm{d}x$.

SM **5.** Find the function $f(x)$ if $f'(x) = ax^2 + bx$, and the following three equations all hold:

(i) $f'(1) = 6$ (ii) $f''(1) = 18$ (iii) $\int_0^2 f(x)\,\mathrm{d}x = 18$

SM **6.** Evaluate the following integrals, assuming, in (d), that all constants are positive:

(a) $\displaystyle\int_0^3 \left(\frac{1}{3}e^{3t-2} + (t+2)^{-1}\right)\mathrm{d}t$ (b) $\displaystyle\int_0^1 (x^2+2)^2\,\mathrm{d}x$

(c) $\displaystyle\int_0^1 \frac{x^2 + x + \sqrt{x+1}}{x+1}\,\mathrm{d}x$ (d) $\displaystyle\int_1^b \left(A\frac{x+b}{x+c} + \frac{d}{x}\right)\mathrm{d}x$

7. Let $F(x) = \int_0^x (t^2 + 2)\,\mathrm{d}t$ and $G(x) = \int_0^{x^2} (t^2 + 2)\,\mathrm{d}t$. Find $F'(x)$ and $G'(x)$.

8. Define $H(t) = \int_0^{t^2} K(\tau)e^{-\rho\tau}\,d\tau$, where $K(\tau)$ is a given continuous function and ρ is a constant. Find $H'(t)$.

9. Find:

(a) $\dfrac{d}{dt}\displaystyle\int_0^t x^2\,dx$ (b) $\dfrac{d}{dt}\displaystyle\int_t^3 e^{-x^2}\,dx$

(c) $\dfrac{d}{dt}\displaystyle\int_{-t}^t \frac{1}{\sqrt{x^4+1}}\,dx$ (d) $\dfrac{d}{d\lambda}\displaystyle\int_{-\lambda}^2 (f(t)-g(t))\,dt$

10. Find the area between the two parabolas defined by the equations $y+1=(x-1)^2$ and $3x=y^2$. (*Hint:* The points of intersection have integer coordinates.)

SM 11. [HARDER] A theory of investment has used a function W defined for all $T>0$ by

$$W(T) = \frac{K}{T}\int_0^T e^{-\rho t}\,dt$$

where K and ρ are positive constants. Evaluate the integral, then prove that $W(T)$ takes values in the interval $(0, K)$ and is strictly decreasing. (*Hint*: See Exercise 6.11.11.)

SM 12. [HARDER] Consider the function f defined, for $x>0$, by $f(x) = 4\ln(\sqrt{x+4}-2)$.

(a) Show that f has an inverse function g, and find a formula for g.

(b) Draw the graphs of f and g in the same coordinate system.

(c) Give a geometric interpretation of $A = \int_5^{10} 4\ln(\sqrt{x+4}-2)\,dx$, and explain why

$$A = 10a - \int_0^a (e^{x/2}+4e^{x/4})\,dx$$

where $a = f(10)$. Use this equality to express A in terms of a.

9.4 Economic Applications

We motivated the definite integral as a tool for computing the area under a curve. However, the integral has many other important interpretations. In statistics, many important probability distributions are expressed as integrals of continuous probability density functions. This section presents some other examples showing why integrals are important in economics.

Extraction from an Oil Well

Assume that at time $t=0$ an oil producer starts extracting oil from a well that contains K barrels at that time. Let us define $x(t)$ as the number of barrels of oil that is left at time t.

In particular, $x(0) = K$. Assuming it is impractical to put oil back into the well, $x(t)$ is a decreasing function of t. The amount of oil that is extracted in a time interval $[t, t + \Delta t]$, where $\Delta t > 0$, is $x(t) - x(t + \Delta t)$. Extraction per unit of time is, therefore,

$$\frac{x(t) - x(t + \Delta t)}{\Delta t} = -\frac{x(t + \Delta t) - x(t)}{\Delta t} \tag{$*$}$$

If we assume that $x(t)$ is differentiable, then as $\Delta t \to 0$ the fraction $(*)$ tends to $-\dot{x}(t)$. Letting $u(t)$ denote the *rate of extraction* at time t, we have $\dot{x}(t) = -u(t)$, with $x(0) = K$. The solution to this equation is

$$x(t) = K - \int_0^t u(\tau)\, \mathrm{d}\tau \tag{$**$}$$

Indeed, we check $(**)$ as follows. First, setting $t = 0$ gives $x(0) = K$. Moreover, differentiating $(**)$ w.r.t. t according to rule (9.3.6) yields $\dot{x}(t) = -u(t)$.

The result $(**)$ may be interpreted as follows: The amount of oil left at time t is equal to the initial amount K, minus the total amount that has been extracted during the time span $[0, t]$, namely $\int_0^t u(\tau)\, \mathrm{d}\tau$.

If the rate of extraction is constant, with $u(t) = \bar{u}$, then $(**)$ yields

$$x(t) = K - \int_0^t \bar{u}\, \mathrm{d}\tau = K - \Big|_0^t \bar{u}\tau = K - \bar{u}t$$

In particular, the well will be empty when $x(t) = 0$, or when $K - \bar{u}t = 0$, that is when $t = K/\bar{u}$. (Of course, this particular answer could have been found more directly, without recourse to integration.)

The example illustrates two concepts that it is important to distinguish in many economic arguments. The quantity $x(t)$ is a *stock*, measured in barrels. On the other hand, $u(t)$ is a *flow*, measured in barrels *per unit of time*.

Income Distribution

In many countries, data collected by income tax authorities can be used to reveal some properties of the income distribution within a given year, as well as how the distribution changes from year to year. Suppose we measure annual income in dollars and let $F(r)$ denote the proportion of individuals that receive no more than r dollars in a particular year. Thus, if there are n individuals in the population, $nF(r)$ is the number of individuals with income no greater than r. If r_0 is the lowest and r_1 is the highest (registered) income in the group, we are interested in the function F defined on the interval $[r_0, r_1]$. By definition, F is not continuous and therefore also not differentiable in $[r_0, r_1]$, because r has to be a multiple of \$0.01 and $F(r)$ has to be a multiple of $1/n$. However, if the population consists of a large number of individuals, then it is usually possible to find a "smooth" function that gives a good approximation to the true income distribution. Assume, therefore, that F is a function with a continuous derivative denoted by f, so that $f(r) = F'(r)$ for all r in (r_0, r_1).

According to the definition of the derivative, we have

$$f(r)\Delta r \approx F(r+\Delta r) - F(r)$$

for all small Δr. Thus, $f(r)\Delta r$ is approximately equal to the proportion of individuals who have incomes between r and $r+\Delta r$. The function f is called an *income density function*, and F is the associated *cumulative distribution function*.[3]

Suppose that f is a continuous income distribution for a certain population with incomes in the interval $[r_0, r_1]$. If $r_0 \le a \le b \le r_1$, then the previous discussion and the definition of the definite integral imply that $\int_a^b f(r)\mathrm{d}r$ is the proportion of individuals with incomes in $[a, b]$. Thus,

$$N = n\int_a^b f(r)\,\mathrm{d}r \tag{9.4.1}$$

is the *number of individuals* with incomes in $[a, b]$.

We will now find an expression for the combined income of those who earn between a and b dollars. Let $M(r)$ denote the total income of those who earn no more than r dollars during the year, and consider the income interval $[r, r+\Delta r]$. There are approximately $nf(r)\Delta r$ individuals with incomes in this interval. Each of them has an income approximately equal to r, so that the total income of these individuals, $M(r+\Delta r) - M(r)$, is approximately equal to $nrf(r)\Delta r$. So we have

$$\frac{M(r+\Delta r) - M(r)}{\Delta r} \approx nrf(r)$$

The approximation improves (in general) as Δr decreases. By taking the limit as $\Delta r \to 0$, we obtain $M'(r) = nrf(r)$. Integrating over the interval from a to b gives $M(b) - M(a) = n\int_a^b rf(r)\mathrm{d}r$. Hence,

$$M = n\int_a^b rf(r)\,\mathrm{d}r \tag{9.4.2}$$

is the *total income* of individuals with income in $[a, b]$.

The argument that leads to (9.4.2) can be made more exact: $M(r+\Delta r) - M(r)$ is the total income of those who have income in the interval $[r, r+\Delta r]$, when $\Delta r > 0$. In this income interval, there are $n[F(r+\Delta r) - F(r)]$ individuals each of whom earns at least r and at most $r+\Delta r$. Thus,

$$nr\,[F(r+\Delta r) - F(r)] \le M(r+\Delta r) - M(r) \le n(r+\Delta r)\,[F(r+\Delta r) - F(r)] \tag{$*$}$$

If $\Delta r > 0$, division by Δr yields

$$nr\frac{F(r+\Delta r) - F(r)}{\Delta r} \le \frac{M(r+\Delta r) - M(r)}{\Delta r} \le n(r+\Delta r)\frac{F(r+\Delta r) - F(r)}{\Delta r} \tag{$**$}$$

[3] Readers who know some statistics may see the analogy with probability density functions and with cumulative (probability) distribution functions.

(If $\Delta r < 0$, then the inequalities in (∗) are left unchanged, whereas those in (∗∗) are reversed.) Letting $\Delta r \to 0$ gives $nrF'(r) \le M'(r) \le nrF'(r)$, so that $M'(r) = nrF'(r) = nrf(r)$.

The ratio between the total income and the number of individuals belonging to a certain income interval $[a, b]$ is called the mean income for the individuals in this income interval. Therefore, the *mean income* of individuals with incomes in the interval $[a, b]$ is:

$$m = \frac{\int_a^b rf(r)\,\mathrm{d}r}{\int_a^b f(r)\,\mathrm{d}r} \tag{9.4.3}$$

A function that approximates actual income distributions quite well, particularly for large incomes, is the *Pareto distribution*. In this case, the proportion of individuals who earn at most r dollars is given by

$$f(r) = \frac{B}{r^\beta} \tag{9.4.4}$$

Here B and β are positive constants. Empirical estimates of β are usually in the range $2.4 < \beta < 2.6$. For values of r close to 0, the formula is of no use. In fact, the integral $\int_0^a f(r)\,\mathrm{d}r$ diverges to ∞, as will be seen using the arguments of Section 9.7.

EXAMPLE 9.4.1 Consider a population of n individuals in which the income density function for those with incomes between a and b is given by $f(r) = B/r^{2.5}$. Here $b > a > 0$, and B is positive. Determine the mean income of this group.

Solution: According to (9.4.1), the total number of individuals in this group is

$$N = n\int_a^b Br^{-2.5}\,\mathrm{d}r = nB\Big|_a^b \left(-\tfrac{2}{3}r^{-1.5}\right) = \tfrac{2}{3}nB\left(a^{-1.5} - b^{-1.5}\right)$$

According to (9.4.2), the total income of these individuals is

$$M = n\int_a^b rBr^{-2.5}\,\mathrm{d}r = nB\int_a^b r^{-1.5}\,\mathrm{d}r = -2nB\Big|_a^b r^{-0.5} = 2nB\left(a^{-0.5} - b^{-0.5}\right)$$

So the mean income of the group is

$$m = \frac{M}{N} = 3\frac{a^{-0.5} - b^{-0.5}}{a^{-1.5} - b^{-1.5}}$$

by (9.4.3). Suppose that b is very large. Then $b^{-0.5}$ and $b^{-1.5}$ are both close to 0, and so $m \approx 3a$. The mean income of those who earn at least a is therefore approximately $3a$.

The Influence of Income Distribution on Demand

Obviously each consumer's demand for a particular commodity depends on its price p. In addition, economists soon learn that it depends on the consumer's income r as well. Here, we consider the total demand quantity for a group of consumers whose individual demands are given by the same continuous function $D(p, r)$ of the single price p, as well as of individual income r whose distribution is given by a continuous density function $f(r)$ on the interval $[a, b]$.

Given a particular price p, let $T(r)$ denote the total demand for the commodity by all individuals whose income does not exceed r. Consider the income interval $[r, r + \Delta r]$. There are approximately $nf(r)\Delta r$ individuals with incomes in this interval. Because each of them demands approximately $D(p, r)$ units of the commodity, the total demand of these individuals will be approximately $nD(p, r)f(r)\Delta r$. However, the actual total demand of individuals with incomes in the interval $[r, r + \Delta r]$ is $T(r + \Delta r) - T(r)$, by definition. So we must have $T(r + \Delta r) - T(r) \approx nD(p, r)f(r)\Delta r$, and thus

$$\frac{T(r + \Delta r) - T(r)}{\Delta r} \approx nD(p, r)f(r)$$

The approximation improves, in general, as Δr decreases. Taking the limit as $\Delta r \to 0$, we obtain $T'(r) = nD(p, r)f(r)$. By definition of the definite integral,

$$T(b) - T(a) = n \int_a^b D(p, r)f(r)\,dr$$

But $T(b) - T(a)$ is the desired measure of total demand for the commodity by all the individuals in the group. In fact, this total demand will depend on the price p. So we denote it by $x(p)$, and thus we have that total demand is

$$x(p) = \int_a^b nD(p, r)f(r)\,dr \tag{9.4.5}$$

EXAMPLE 9.4.2 Let the income distribution function be that of Example 9.4.1, and let $D(p, r) = Ap^{-1.5}r^{2.08}$. Compute the total demand.

Solution: Using (9.4.5) gives

$$x(p) = \int_a^b nAp^{-1.5}r^{2.08}Br^{-2.5}\,dr = nABp^{-1.5}\int_a^b r^{-0.42}\,dr$$

Hence,

$$x(p) = nABp^{-1.5} \times \Big|_a^b \frac{1}{0.58}r^{0.58} = \frac{nAB}{0.58}p^{-1.5}(b^{0.58} - a^{0.58})$$

Consumer and Producer Surplus

Economists are interested in studying how much consumers and producers as a whole benefit (or lose) when market conditions change. A common (but theoretically questionable) measure of these benefits used by many applied economists is the total amount of consumer and producer surplus defined below.[4] At the equilibrium point E in Fig. 9.4.1, demand

[4] See, for example, H. Varian: *Intermediate Microeconomics: A Modern Approach*, 8th ed., Norton, 2009 for a more detailed treatment.

quantity is equal to supply. The corresponding equilibrium price P^* is the one which induces consumers to purchase (demand) precisely the same aggregate amount that producers are willing to offer (supply) at that price, as in Example 4.5.3. According to the demand curve in Fig. 9.4.1, there are consumers who are willing to pay more than P^* per unit. In fact, even if the price is almost as high as P_1, some consumers still wish to buy some units at that price. The total amount "saved" by all such consumers is called the *consumer surplus*, denoted by CS.

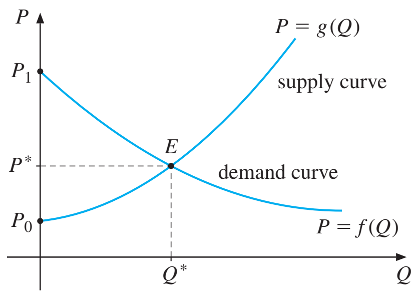


Figure 9.4.1 Market equilibrium

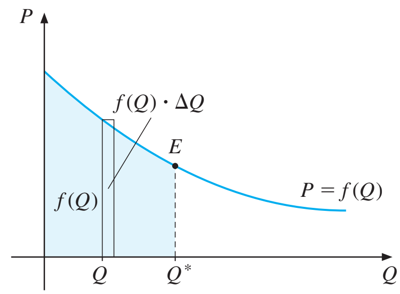


Figure 9.4.2 Consumer surplus, CS

Consider the small rectangle indicated in Fig. 9.4.2. It has base ΔQ and height $f(Q)$, so its area is $f(Q) \cdot \Delta Q$. It is approximately the maximum additional amount that consumers as a whole are willing to pay for an extra ΔQ units at price $f(Q)$, after they have already bought Q units. For those willing to buy the commodity at price P^* or higher, the total amount they are willing to pay is the total area below the demand curve over the interval $[0, Q^*]$, that is $\int_0^{Q^*} f(Q)\, dQ$. This area is shaded in Fig. 9.4.2. If all consumers together buy Q^* units of the commodity, the total cost is P^*Q^*. This represents the area of the rectangle with base Q^* and height P^*. It can therefore be expressed as the integral $\int_0^{Q^*} P^*\, dQ$. The consumer surplus is defined as the integral

$$\text{CS} = \int_0^{Q^*} [f(Q) - P^*]\, dQ \tag{9.4.6}$$

which equals the total amount consumers are willing to pay for Q^*, minus what they actually pay. In Fig. 9.4.3, $\int_0^{Q^*} f(Q)\, dQ$ is the area OP_1EQ^*, whereas OP^*EQ^* is P^*Q^*. So CS is equal to the area P^*P_1E between the demand curve and the horizontal line $P = P^*$. This is also the area to the left of the demand curve—that is between the demand curve and the P-axis. So the consumer surplus CS is the lighter-shaded area in Fig. 9.4.3.

Most producers also derive positive benefit or "surplus" from selling at the equilibrium price P^* because they would be willing to supply the commodity for less than P^*. In Fig. 9.4.3, even if the price is almost as low as P_0, some producers are still willing to supply the commodity. Consider the total surplus of all the producers who receive more than the price at which they are willing to sell. We call this the *producer surplus*, denoted by PS. Geometrically it is represented by the darker-shaded area in Fig. 9.4.3. Analytically,

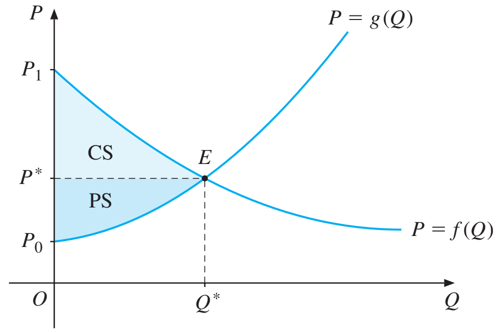


Figure 9.4.3 Consumer and producer surplus, CS and PS

it is defined by

$$\mathrm{PS} = \int_0^{Q^*} [P^* - g(Q)]\, \mathrm{d}Q \tag{9.4.7}$$

since this is the total revenue producers actually receive, minus what would make them willing to supply Q^*. In Fig. 9.4.3, the area OP^*EQ^* is again P^*Q^*, and $\int_0^{Q^*} g(Q)\, \mathrm{d}Q$ is the area OP_0EQ^*. So PS is equal to the area P^*P_0E between the supply curve and the line $P = P^*$. This is also the area to the left of the supply curve—that is between the supply curve and the P-axis.

EXAMPLE 9.4.3 Suppose that the demand curve is $P = f(Q) = 50 - 0.1Q$ and the supply curve is $P = g(Q) = 0.2Q + 20$. Find the equilibrium price and compute the consumer and producer surplus.

Solution: The equilibrium quantity is determined by the equation $50 - 0.1Q^* = 0.2Q^* + 20$, which gives $Q^* = 100$. Then $P^* = 0.2Q^* + 20 = 40$. Hence,

$$\mathrm{CS} = \int_0^{100} [50 - 0.1Q - 40]\, \mathrm{d}Q = \int_0^{100} [10 - 0.1Q]\, \mathrm{d}Q = \Big|_0^{100} (10Q - 0.05Q^2) = 500$$

and

$$\mathrm{PS} = \int_0^{100} [40 - (0.2Q + 20)]\, \mathrm{d}Q = \int_0^{100} [20 - 0.2Q]\, \mathrm{d}Q = \Big|_0^{100} (20Q - 0.1Q^2) = 1000$$

EXERCISES FOR SECTION 9.4

1. Assume that the rate of extraction $u(t)$ from an oil well decreases exponentially over time, with $u(t) = \bar{u}e^{-at}$, where $\bar{u}$ and a are positive constants. Given the initial stock $x(0) = K$, find an expression $x(t)$ for the remaining amount of oil at time t. Under what condition will the well never be exhausted?

SM 2. Follow the pattern in Examples 9.4.1 and 9.4.2, and:

(a) Find the mean income m over the interval $[b, 2b]$ when $f(r) = Br^{-2}$, assuming that there are n individuals in the population.

(b) Assume that the individuals' demand function is $D(p, r) = Ap^{\gamma} r^{\delta}$ with $A > 0$ $\gamma < 0$, $\delta > 0$, $\delta \neq 1$. Compute the total demand $x(p)$ by using formula (9.4.5).

3. Solve the equation $S = \int_0^T e^{rt}\, dt$ for T.

4. Let $K(t)$ denote the capital stock of an economy at time t. Then *net investment* at time t, denoted by $I(t)$, is given by the rate of increase $\dot{K}(t)$ of $K(t)$.

(a) If $I(t) = 3t^2 + 2t + 5$ $(t \geq 0)$, what is the total increase in the capital stock during the interval from $t = 0$ to $t = 5$?

(b) If $K(t_0) = K_0$, find an expression for the total increase in the capital stock from time $t = t_0$ to $t = T$ when the investment function $I(t)$ is as in part (a).

5. An oil company is planning to extract oil from one of its fields, starting today at $t = 0$, where t is time measured in years. It has a choice between two extraction profiles f and g giving the rates of flow of oil, measured in barrels per year. Both extraction profiles last for 10 years, with $f(t) = 10t^2 - t^3$ and $g(t) = t^3 - 20t^2 + 100t$ for t in $[0, 10]$.

(a) Sketch the two profiles in the same coordinate system.

(b) Show that $\int_0^t g(\tau)\, d\tau \geq \int_0^t f(\tau)\, d\tau$ for all t in $[0, 10]$.

(c) The company sells its oil at a price per unit given by $p(t) = 1 + 1/(t + 1)$. Total revenues from the two profiles are then given by $\int_0^{10} p(t)f(t)\, dt$ and $\int_0^{10} p(t)g(t)\, dt$ respectively. Compute these integrals. Which of the two extraction profiles earns the higher revenue?

6. Suppose that the inverse demand and supply curves for a particular commodity are, respectively, $P = f(Q) = 200 - 0.2Q$ and $P = g(Q) = 20 + 0.1Q$. Find the equilibrium price and quantity, and compute the consumer and producer surplus.

7. Suppose the inverse demand and supply curves for a particular commodity are, respectively, $P = f(Q) = 6000/(Q + 50)$ and $P = g(Q) = Q + 10$. Find the equilibrium price and quantity, and compute the consumer and producer surplus.

9.5 Integration by Parts

Mathematicians, statisticians and economists often need to evaluate integrals like $\int x^3 e^{2x}\, dx$, whose integrand is a product of two functions. We know that $\frac{1}{4}x^4$ has x^3 as its derivative and that $\frac{1}{2}e^{2x}$ has e^{2x} as its derivative, but $(\frac{1}{4}x^4)(\frac{1}{2}e^{2x})$ certainly does not have $x^3 e^{2x}$ as its derivative. In general, because the derivative of a product is *not* the product of the derivatives, the integral of a product is not the product of the integrals.

The correct rule for differentiating a product allows us to derive an important and useful rule for integrating products. The product rule for differentiation, Eq. (6.7.2), states that

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \tag{$*$}$$

Now take the indefinite integral of each side in (∗), and then use the rule for integrating a sum. The result is

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

where the constants of integration have been left implicit in the indefinite integrals on the right-hand side of this equation. Rearranging this last equation yields the following formula:

FORMULA FOR INTEGRATION BY PARTS

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{9.5.1}$$

At first sight, this formula does not look at all helpful. Yet the examples that follow show how this impression is quite wrong, once one has learned to use the formula properly.

Indeed, suppose we are asked to integrate a function $H(x)$ that can be written in the form $f(x)g'(x)$. By using (9.5.1), the problem can be transformed into that of integrating $f'(x)g(x)$. Usually, a function $H(x)$ can be written as $f(x)g'(x)$ in several different ways. The point is, therefore, to choose f and g so that it is easier to find $\int f'(x)g(x)\,dx$ than it is to find $\int f(x)g'(x)\,dx$.

EXAMPLE 9.5.1 Use integration by parts to evaluate $\int xe^x\,dx$.

Solution: In order to use (9.5.1), we must write the integrand in the form $f(x)g'(x)$. Let $f(x) = x$ and $g'(x) = e^x$, implying that $g(x) = e^x$. Then $f(x)g'(x) = xe^x$, so (9.5.1) gives

$$\int \underbrace{x \cdot e^x}_{f(x)g'(x)}\,dx = \underbrace{x \cdot e^x}_{f(x)g(x)} - \int \underbrace{1 \cdot e^x}_{f'(x)g(x)}\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

The derivative of $xe^x - e^x + C$ is indeed $e^x + xe^x - e^x = xe^x$, so the integration has been carried out correctly.

An appropriate choice of f and g enabled us to evaluate the integral. Let us see what happens if we interchange the roles of f and g, and try $f(x) = e^x$ and $g'(x) = x$ instead. Then $g(x) = \frac{1}{2}x^2$. Again $f(x)g'(x) = e^x x = xe^x$, and by (9.5.1):

$$\int \underbrace{e^x \cdot x}_{f(x)g'(x)}\,dx = \underbrace{e^x \cdot \frac{1}{2}x^2}_{f(x)g(x)} - \int \underbrace{e^x \cdot \frac{1}{2}x^2}_{f'(x)g(x)}\,dx$$

In this case, the integral on the right-hand side is more complicated than the original one. Thus, this second choice of f and g does not simplify the integral.

The example illustrates that we must be careful how we split the integrand. Insights into making a good choice, if there is one, come only with practice.

Sometimes integration by parts works not by producing a simpler integral, but one that is similar, as in part (a) of the next example.

EXAMPLE 9.5.2 Evaluate the following: (a) $I = \int (1/x) \ln x \, dx$; and (b) $J = \int x^3 e^{2x} \, dx$.

Solution:

(a) Choosing $f(x) = 1/x$ and $g'(x) = \ln x$ leads nowhere. Choosing $f(x) = \ln x$ and $g'(x) = 1/x$ works better:

$$I = \int \frac{1}{x} \ln x \, dx = \int \underbrace{\ln x \cdot \frac{1}{x}}_{f(x)g'(x)} \, dx = \underbrace{\ln x \cdot \ln x}_{f(x)g(x)} - \int \underbrace{\frac{1}{x} \cdot \ln x}_{f'(x)g(x)} \, dx$$

In this case, the last integral is exactly the one we started with, namely I. So it must be true that $I = (\ln x)^2 - I + C_1$ for some constant C_1. Solving for I yields $I = \frac{1}{2}(\ln x)^2 + \frac{1}{2}C_1$. Putting $C = \frac{1}{2}C_1$, we conclude that

$$\int \frac{1}{x} \ln x \, dx = \frac{1}{2}(\ln x)^2 + C$$

(b) We begin by arguing rather loosely as follows. Differentiation makes x^3 simpler by reducing the power in the derivative $3x^2$ from 3 to 2. On the other hand, e^{2x} becomes about equally simple whether we differentiate or integrate it. Therefore, we choose $f(x) = x^3$ and $g'(x) = e^{2x}$, so that integration by parts tells us to differentiate f and integrate g'. This yields $f'(x) = 3x^2$ and we can choose $g(x) = \frac{1}{2}e^{2x}$. Therefore,

$$J = \int x^3 e^{2x} \, dx = x^3 \left(\frac{1}{2}e^{2x}\right) - \int (3x^2)\left(\frac{1}{2}e^{2x}\right) dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x} \, dx \quad (*)$$

The last integral *is* somewhat simpler than the one we started with, because the power of x has been reduced. Integrating by parts once more yields

$$\int x^2 e^{2x} \, dx = x^2 \left(\frac{1}{2}e^{2x}\right) - \int (2x)\left(\frac{1}{2}e^{2x}\right) dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x} \, dx \qquad (**)$$

Using integration by parts a third and final time gives

$$\int x e^{2x} \, dx = x\left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x} \, dx = \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + C_1 \qquad (***)$$

Successively inserting the results of $(***)$ and $(**)$ into $(*)$ yields, with $3C_1/2 = C$:

$$J = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + C$$

It is a good idea to double-check your work by verifying that $dJ/dx = x^3 e^{2x}$.

There is a corresponding result for definite integrals. From the definition of the definite integral and the product rule for differentiation, we have

$$\int_a^b \left[f'(x)g(x) + f(x)g'(x)\right] dx = \int_a^b \frac{d}{dx}[f(x)g(x)] \, dx = \Big|_a^b f(x)g(x)$$

implying that

$$\int_a^b f(x)g'(x)\,dx = \Big|_a^b f(x)g(x) - \int_a^b f'(x)g(x)\,dx \tag{9.5.2}$$

EXAMPLE 9.5.3 Evaluate $\int_0^{10}(1+0.4t)e^{-0.05t}\,dt$.

Solution: Put $f(t) = 1 + 0.4t$ and $g'(t) = e^{-0.05t}$. Then we can choose $g(t) = -20e^{-0.05t}$, and (9.5.2) yields

$$\int_0^{10}(1+0.4t)e^{-0.05t}\,dt = \Big|_0^{10}(1+0.4t)(-20)e^{-0.05t} - \int_0^{10}(0.4)(-20)e^{-0.05t}\,dt$$

$$= -100e^{-0.5} + 20 + 8\int_0^{10} e^{-0.05t}\,dt$$

$$= -100e^{-0.5} + 20 - 160(e^{-0.5} - 1) \approx 22.3$$

EXERCISES FOR SECTION 9.5

SM 1. Use integration by parts to evaluate the following:

(a) $\int xe^{-x}\,dx$ (b) $\int 3xe^{4x}\,dx$ (c) $\int(1+x^2)e^{-x}\,dx$ (d) $\int x\ln x\,dx$

SM 2. Use integration by parts to evaluate the following:

(a) $\int_{-1}^{1} x\ln(x+2)\,dx$ (b) $\int_0^2 x2^x\,dx$ (c) $\int_0^1 x^2e^x\,dx$ (d) $\int_0^3 x\sqrt{1+x}\,dx$

In part (d) you should graph the integrand and decide if your answer is reasonable.

3. Use integration by parts to evaluate the following:

(a) $\int_1^4 \sqrt{t}\ln t\,dt$ (b) $\int_0^2 (x-2)e^{-x/2}\,dx$ (c) $\int_0^3 (3-x)3^x\,dx$

4. Of course, $f(x) = 1 \cdot f(x)$ for any function $f(x)$. Use this fact and integration by parts to prove that $\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$. Apply this formula to $f(x) = \ln x$. Compare with Example 9.1.3.

5. Given $\rho \neq -1$, show that $\displaystyle\int x^\rho \ln x\,dx = \frac{x^{\rho+1}}{\rho+1}\ln x - \frac{x^{\rho+1}}{(\rho+1)^2} + C$.

SM 6. Evaluate the following integrals, for $r \neq 0$:

(a) $\displaystyle\int_0^T bte^{-rt}\,dt$ (b) $\displaystyle\int_0^T (a+bt)e^{-rt}\,dt$ (c) $\displaystyle\int_0^T (a-bt+ct^2)e^{-rt}\,dt$

9.6 Integration by Substitution

In this section we shall see how the chain rule for differentiation leads to an important method for evaluating many complicated integrals. We start with some simple examples.

EXAMPLE 9.6.1 Evaluate the integrals:

(a) $\int (x^2 + 10)^{50}\, 2x\, dx$ (b) $\int_0^a xe^{-cx^2}\, dx$, where $c \neq 0$

Solution:

(a) Attempts to use integration by parts fail. Expanding $(x^2 + 10)^{50}$ to get a polynomial of 51 terms, and then integrating term by term, would work in principle, but would be extremely cumbersome. Instead, let us introduce $u = x^2 + 10$ as a new variable. Using differential notation, we see that $du = 2x\, dx$. Inserting these into the integral in (a) yields $\int u^{50}\, du$. This integral is easy, $\int u^{50}\, du = \frac{1}{51}u^{51} + C$. Because $u = x^2 + 10$, it appears that

$$\int (x^2 + 10)^{50}\, 2x\, dx = \frac{1}{51}(x^2 + 10)^{51} + C$$

By the chain rule, the derivative of $\frac{1}{51}(x^2 + 10)^{51} + C$ is precisely $(x^2 + 10)^{50}\, 2x$, so the result *is* confirmed.

(b) First, we consider the indefinite integral $\int xe^{-cx^2}\, dx$ and substitute $u = -cx^2$. Then $du = -2cx\, dx$, and thus $x\, dx = -du/2c$. Therefore

$$\int xe^{-cx^2}\, dx = \int -\frac{1}{2c}e^u\, du = -\frac{1}{2c}e^u + C = -\frac{1}{2c}e^{-cx^2} + C$$

The definite integral is

$$\int_0^a xe^{-cx^2}\, dx = -\frac{1}{2c} \times \Big|_0^a e^{-cx^2} = \frac{1}{2c}(1 - e^{-ca^2})$$

In both of these examples, the integrand could be written in the form $f(u)u'$, where $u = g(x)$. In part (a) of Example 9.6.1 we put $f(u) = u^{50}$ with $u = g(x) = x^2 + 10$. In part (b), we put $f(u) = e^u$ with $u = g(x) = -cx^2$. Then the integrand is a constant, $-1/(2c)$, multiplied by $f(g(x))g'(x)$. Let us try the same method on the more general integral

$$\int f(g(x))g'(x)\, dx$$

If we put $u = g(x)$, then $du = g'(x)\, dx$, and so the integral reduces to $\int f(u)\, du$. Suppose we could find an antiderivative function $F(u)$ such that $F'(u) = f(u)$. Then, we would have $\int f(u)\, du = F(u) + C$, which implies that $\int f(g(x))g'(x)\, dx = F(g(x)) + C$. Does this purely formal method always give the right result? To convince you that it does, we use the chain rule to differentiate $F(g(x)) + C$ w.r.t. x. The derivative is $F'(g(x))g'(x)$, which is precisely equal to $f(g(x))g'(x)$, thus confirming the following rule:

CHANGE OF VARIABLE

Suppose that g is continuously differentiable, and $f(u)$ is continuous at all points u belonging to the relevant range of g. Then,

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u \tag{9.6.1}$$

where $u = g(x)$.

EXAMPLE 9.6.2 Evaluate $\int 8x^2(3x^3-1)^{16}\,\mathrm{d}x$.

Solution: Substitute $u = 3x^3 - 1$. Then $\mathrm{d}u = 9x^2\,\mathrm{d}x$, so that $8x^2\,\mathrm{d}x = \frac{8}{9}\,\mathrm{d}u$. Hence

$$\int 8x^2(3x^3-1)^{16}\,\mathrm{d}x = \frac{8}{9}\int u^{16}\,\mathrm{d}u = \frac{8}{9}\cdot\frac{1}{17}u^{17} + C = \frac{8}{153}(3x^3-1)^{17} + C$$

The definite integral in part (b) of Example 9.6.1 can be evaluated more simply by "carrying over" the limits of integration. We substituted $u = -cx^2$. As x varies from 0 to a, so u varies from 0 to $-ca^2$. This allows us to write:

$$\int_0^a xe^{-cx^2}\,\mathrm{d}x = \int_0^{-ca^2} -\frac{1}{2c}e^u\,\mathrm{d}u = -\frac{1}{2c}\times\Big|_0^{-ca^2} e^u = \frac{1}{2c}(1-e^{-ca^2})$$

This method of carrying over the limits of integration can be used in general. In fact, for $u = g(x)$,

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u \tag{9.6.2}$$

The argument is simple: Provided that $F'(u) = f(u)$, we obtain

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \Big|_a^b F(g(x)) = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

EXAMPLE 9.6.3 Evaluate the integral $\displaystyle\int_1^e \frac{1+\ln x}{x}\,\mathrm{d}x$.

Solution: We suggest the substitution $u = 1 + \ln x$. Then $\mathrm{d}u = (1/x)\,\mathrm{d}x$. Also, if $x = 1$ then $u = 1$; and if $x = e$, then $u = 2$. So, we have

$$\int_1^e \frac{1+\ln x}{x}\,\mathrm{d}x = \int_1^2 u\,\mathrm{d}u = \frac{1}{2}\Big|_1^2 u^2 = \frac{1}{2}(4-1) = \frac{3}{2}$$

More Complicated Cases

The examples of integration by substitution considered so far were rather simple. More challenging applications of this integration method are studied in this subsection.

EXAMPLE 9.6.4 Find a substitution that allows $\int \frac{x-\sqrt{x}}{x+\sqrt{x}}\,dx$ to be evaluated, assuming $x > 0$.

Solution: Because $\sqrt{x}$ occurs in both the numerator and the denominator, we try to simplify the integral by substituting $u = \sqrt{x}$. Then $x = u^2$ and $dx = 2u\,du$, so we get

$$\int \frac{x-\sqrt{x}}{x+\sqrt{x}}\,dx = \int \frac{u^2-u}{u^2+u}2u\,du = 2\int \frac{u^2-u}{u+1}\,du = 2\int \left(u-2+\frac{2}{u+1}\right)du$$
$$= u^2 - 4u + 4\ln|u+1| + C$$

where we have performed the polynomial division $(u^2-u) \div (u+1)$ with a remainder, as in Section 4.7, in order to derive the third equality. Replacing u by $\sqrt{x}$ in the last expression yields the answer

$$\int \frac{x-\sqrt{x}}{x+\sqrt{x}}\,dx = x - 4\sqrt{x} + 4\ln\left(\sqrt{x}+1\right) + C$$

where we use the fact that $\sqrt{x}+1 > 0$ for all x.

The last example shows the method that is used most frequently. We can summarize it as follows:

A GENERAL METHOD

In order to find $\int G(x)\,dx$

1. Pick out a "part" of $G(x)$ and introduce this "part" as a new variable, $u = g(x)$.
2. Compute $du = g'(x)\,dx$.
3. Using the substitution $u = g(x)$, $du = g'(x)\,dx$, transform, if possible, $\int G(x)\,dx$ to an integral of the form $\int f(u)\,du$.
4. Find, if possible, $\int f(u)\,du = F(u) + C$.
5. Replace u by $g(x)$.

Then the final answer is $\int G(x)\,dx = F(g(x)) + C$.

At the third step of this procedure, it is crucial that the substitution results in an integrand $f(u)$ that only contains u (and du), without any x's. Probably the most common error when integrating by substitution is to replace dx by du, rather than use the correct formula $du = g'(x)\,dx$.

Note that if one particular substitution does not work, one can try another. But as explained in Section 9.3, there is always the possibility that no substitution at all will work.

EXAMPLE 9.6.5 Find the following: (a) $\int x^3\sqrt{1+x^2}\,dx$; and (b) $\int_0^1 x^3\sqrt{1+x^2}\,dx$.

Solution:

(a) We follow steps 1 to 5:

1. We pick a "part" of $x^3\sqrt{1+x^2}$ as a new variable. Let us try $u = \sqrt{1+x^2}$.
2. When $u = \sqrt{1+x^2}$, then $u^2 = 1 + x^2$ and so $2u\,du = 2x\,dx$, implying that $u\,du = x\,dx$. Note that this is easier than differentiating u directly.
3. $\int x^3\sqrt{1+x^2}\,dx = \int x^2\sqrt{1+x^2}\,x\,dx = \int (u^2-1)uu\,du = \int (u^4-u^2)\,du$
4. $\int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$
5. $\int x^3\sqrt{1+x^2}\,dx = \frac{1}{5}(\sqrt{1+x^2}\,)^5 - \frac{1}{3}(\sqrt{1+x^2}\,)^3 + C$

(b) We combine the results in steps 3 and 4 of part (a), while noting that $u = 1$ when $x = 0$ and $u = \sqrt{2}$ when $x = 1$. The implication is

$$\int_0^1 x^3\sqrt{1+x^2}\,dx = \Big|_1^{\sqrt{2}} \left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) = \frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3} = \frac{2}{15}(\sqrt{2}+1)$$

In this example the substitution $u = 1 + x^2$ also works.

Integrating Rational Functions and Partial Fractions

In Section 4.7 we defined a rational function as the ratio $P(x)/Q(x)$ of two polynomials. Just occasionally economists need to integrate such functions. So we will merely give two examples that illustrate a procedure one can use more generally. One example has already appeared in part (b) of Exercise 9.1.4., where the integrand was the rational function $x^3/(x+1)$. As explained in Section 4.7, this fraction can be simplified by polynomial division with a remainder into a form that can be integrated directly.

That first example was particularly simple because the denominator is a polynomial of degree 1 in x. When degree of the denominator exceeds 1, however, it is generally necessary to combine polynomial division with a *partial fraction expansion* of the remainder. Here is an example:

EXAMPLE 9.6.6 Calculate the integral $\displaystyle\int \frac{x^4+3x^2-4}{x^2+2x}\,dx$.

Solution: We apply polynomial division to the integrand, which yields

$$\frac{x^4+3x^2-4}{x^2+2x} = x^2 - 2x + 7 - \frac{14x+4}{x^2+2x}$$

We can easily integrate the first three terms of the right-hand side, to obtain $\int (x^2 - 2x + 7)\,dx = \frac{1}{3}x^3 - x^2 + 7x + C_0$. The fourth term, however, has a denominator equal to the product of the two degree-one factors x and $x+2$. To obtain an integrand we can integrate, we expand this term as

$$\frac{14x+4}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

— i.e., the sum of two partial fractions, where A and B are constants to be determined. Multiplying each side of the equation by the common denominator $x(x+2)$ gives $14x+4 = A(x+2)+Bx$, or equivalently $(14-A-B)x+4-2A=0$. To make this true for all $x \neq 0$ and all $x \neq -2$ (points where the fraction is undefined), we require that both the coefficient $14-A-B$ of x and the constant $4-2A$ are 0. Solving these two simultaneous equations gives $A=2$ and $B=12$. Finally, therefore, we can integrate the fourth remainder term of the integrand to obtain

$$\int \frac{14x+4}{x^2+2x}\,dx = \int \frac{2}{x}\,dx + \int \frac{12}{x+2}\,dx = 2\ln|x| + 12\ln|x+2| + C$$

Hence, the overall answer is

$$\int \frac{x^4+3x^2-4}{x^2+2x}\,dx = \frac{1}{3}x^3 - x^2 + 7x + 2\ln|x| + 12\ln|x+2| + C$$

This answer, of course, can be verified by differentiation.

EXERCISES FOR SECTION 9.6

1. Find the following integrals by using (9.6.1):

(a) $\int (x^2+1)^8 2x\,dx$ (b) $\int (x+2)^{10}\,dx$ (c) $\int \frac{2x-1}{x^2-x+8}\,dx$

SM **2.** Find the following integrals by means of an appropriate substitution:

(a) $\int x(2x^2+3)^5\,dx$ (b) $\int x^2 e^{x^3+2}\,dx$ (c) $\int \frac{\ln(x+2)}{2x+4}\,dx$

(d) $\int x\sqrt{1+x}\,dx$ (e) $\int \frac{x^3}{(1+x^2)^3}\,dx$ (f) $\int x^5\sqrt{4-x^3}\,dx$

3. Find the following integrals:

(a) $\int_0^1 x\sqrt{1+x^2}\,dx$ (b) $\int_1^e \frac{\ln y}{y}\,dy$ (c) $\int_1^3 \frac{1}{x^2}e^{2/x}\,dx$ (d) $\int_5^8 \frac{x}{x-4}\,dx$

Hint: In (d), use both integration by substitution and expansion of partial fractions, as alternative methods to find the integral.

4. Solve, for $x > 2$, the equation: $\int_3^x \frac{2t-2}{t^2-2t}\,dt = \ln\left(\frac{2}{3}x - 1\right)$.

5. Show that $\int_{t_0}^{t_1} S'(x(t))x(t)\,\mathrm{d}t = S(x(t_1)) - S(x(t_0))$.

SM **6.** [HARDER] Calculate the following integrals:

(a) $\int_0^1 (x^4 - x^9)(x^5 - 1)^{12}\,\mathrm{d}x$ (b) $\int (\ln x/\sqrt{x})\,\mathrm{d}x$ (c) $\int_0^4 \frac{1}{\sqrt{1+\sqrt{x}}}\,\mathrm{d}x$

SM **7.** [HARDER] Calculate the following integrals:

(a) $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}(1+e^{\sqrt{x}})}\,\mathrm{d}x$ (b) $\int_0^{1/3} \frac{1}{e^x+1}\,\mathrm{d}x$ (c) $\int_{8.5}^{41} \frac{1}{\sqrt{2x-1} - \sqrt[4]{2x-1}}\,\mathrm{d}x$

Hints: For (b), substitute $t = e^{-x}$; for (c), substitute $z^4 = 2x - 1$.

8. [HARDER] Use one substitution that eliminates both fractional exponents in $x^{1/2}$ and $x^{1/3}$ in order to find the integral $I = \int \frac{x^{1/2}}{1 - x^{1/3}}\,\mathrm{d}x$.

9. [HARDER] Use the method of partial fractions suggested in Example 9.6.6 in order to write $f(x) = \frac{cx+d}{(x-a)(x-b)}$ as a sum of two fractions. Then use the result to integrate:

(a) $\int \frac{x}{(x+1)(x+2)}\,\mathrm{d}x$ (b) $\int \frac{1-2x}{x^2-2x-15}\,\mathrm{d}x$

9.7 Infinite Intervals of Integration

In part (b) of Example 9.6.1, we proved that

$$\int_0^a xe^{-cx^2}\,\mathrm{d}x = \frac{1}{2c}(1 - e^{-ca^2})$$

Suppose c is a positive number and let a tend to infinity. Then the right-hand expression tends to $1/(2c)$. This makes it seem natural to write

$$\int_0^\infty xe^{-cx^2}\,\mathrm{d}x = \frac{1}{2c}$$

In statistics and economics it is common to encounter such integrals over an infinite interval.

In general, suppose f is a function that is continuous for all $x \geq a$. Then $\int_a^b f(x)\,\mathrm{d}x$ is defined for each $b \geq a$. If the limit of this integral as $b \to \infty$ exists (and is finite), then we say that f is *integrable over* $[a, \infty)$, and define

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x \tag{9.7.1}$$

The *improper integral* $\int_a^\infty f(x)\,\mathrm{d}x$ is then said to *converge*. If the limit does *not* exist, however, the improper integral is said to *diverge*. If $f(x) \geq 0$ in $[a, \infty)$, we interpret the integral (9.7.1) as the *area* below the graph of f over the infinite interval $[a, \infty)$.

Analogously, we define

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^b f(x)\,\mathrm{d}x \tag{9.7.2}$$

when f is continuous in $(-\infty, b]$. If this limit exists, the improper integral is said to converge. Otherwise, it diverges.

EXAMPLE 9.7.1 The *exponential distribution* in statistics is defined by the density function $f(x) = \lambda e^{-\lambda x}$, where $x \geq 0$ and λ is a positive constant. The area below the graph of f over $[0, \infty)$ is illustrated in Fig. 9.7.1. Show that this area is equal to 1.

Solution: For $b > 0$, the area below the graph of f over $[0, b]$ is equal to

$$\int_0^b \lambda e^{-\lambda x}\,\mathrm{d}x = \Big|_0^b \left(-e^{-\lambda x}\right) = -e^{-\lambda b} + 1$$

As $b \to \infty$, so $-e^{-\lambda b} + 1$ approaches 1. Therefore,

$$\int_0^\infty \lambda e^{-\lambda x}\,\mathrm{d}x = \lim_{b\to\infty} \int_0^b \lambda e^{-\lambda x}\,\mathrm{d}x = \lim_{b\to\infty} \left(-e^{-\lambda b} + 1\right) = 1$$

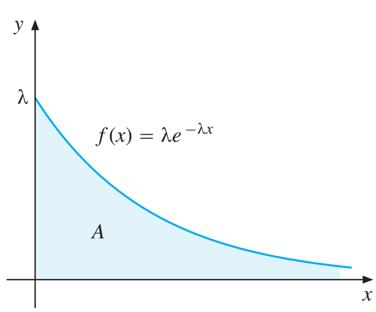


Figure 9.7.1 Area A has an unbounded base but the height decreases to 0 so rapidly that the total area is 1

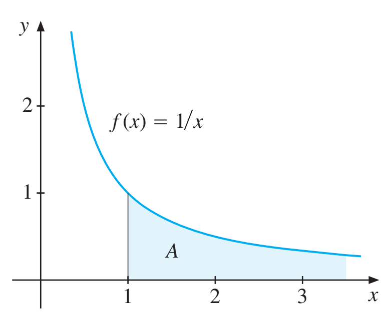


Figure 9.7.2 "$A = \int_1^\infty (1/x)\,\mathrm{d}x = \infty$." $1/x$ does not approach 0 sufficiently fast, so the improper integral diverges

EXAMPLE 9.7.2 For $a > 1$, show that

$$\int_1^\infty \frac{1}{x^a}\,\mathrm{d}x = \frac{1}{a-1} \tag{$*$}$$

Then study the case $a \leq 1$.

Solution: For $a \neq 1$ and $b > 1$,

$$\int_1^b \frac{1}{x^a}\,\mathrm{d}x = \int_1^b x^{-a}\,\mathrm{d}x = \Big|_1^b \frac{1}{1-a}x^{1-a} = \frac{1}{1-a}(b^{1-a}-1) \tag{$**$}$$

For $a > 1$, one has $b^{1-a} = 1/b^{a-1} \to 0$ as $b \to \infty$. Hence, $(*)$ follows from $(**)$ by letting $b \to \infty$. For $a = 1$, the right-hand side of $(**)$ is undefined. Nevertheless, $\int_1^b (1/x)\,\mathrm{d}x = \ln b - \ln 1 = \ln b$, which tends to ∞ as b tends to ∞, so $\int_1^\infty (1/x)\,\mathrm{d}x$ diverges—see Fig. 9.7.2. For $a < 1$, the last expression in $(**)$ tends to ∞ as b tends to ∞. Hence, the integral diverges in this case also.

If both limits of integration are infinite, the improper integral of a continuous function f on $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^0 f(x)\,\mathrm{d}x + \int_0^\infty f(x)\,\mathrm{d}x \tag{9.7.3}$$

If *both* integrals on the right-hand side converge, the improper integral $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ is said to *converge*; otherwise, it *diverges*. Instead of using 0 as the point of subdivision, one could use an arbitrary fixed real number c. The value assigned to the integral will always be the same, provided that the integral does converge.

It is important to note that definition (9.7.3) requires both integrals on the right-hand side to converge. Note in particular that

$$\lim_{b\to\infty} \int_{-b}^b f(x)\,\mathrm{d}x \tag{9.7.4}$$

is *not* the definition of $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$. Exercise 4 provides an example in which (9.7.3) exists, yet the integral in (9.7.4) diverges because $\int_{-b}^0 f(x)\,\mathrm{d}x \to -\infty$ as $b \to \infty$, and $\int_0^b f(x)\,\mathrm{d}x \to \infty$ as $b \to \infty$. So (9.7.4) is not an acceptable definition, whereas (9.7.3) is.

The following result is very important in statistics. It is also related to Exercise 12.

EXAMPLE 9.7.3 For $c > 0$, prove that the following integral converges, and find its value:

$$\int_{-\infty}^{+\infty} xe^{-cx^2}\,\mathrm{d}x$$

Solution: In the introduction to this section we proved that $\int_0^\infty xe^{-cx^2}\,dx = 1/2c$. In the same way we see that

$$\int_{-\infty}^{0} xe^{-cx^2}\,dx = \lim_{a\to-\infty}\int_a^0 xe^{-cx^2}\,dx = \lim_{a\to-\infty}\Big|_a^0 -\frac{1}{2c}e^{-cx^2} = -\frac{1}{2c}$$

It follows that

$$\int_{-\infty}^{\infty} xe^{-cx^2}\,dx = -\frac{1}{2c}+\frac{1}{2c} = 0 \tag{9.7.5}$$

In fact, the function $f(x) = xe^{-cx^2}$ satisfies $f(-x) = -f(x)$ for all x, and so its graph is symmetric about the origin. Therefore the integral $\int_{-\infty}^{0} xe^{-cx^2}\,dx$ must also exist and be equal to $-1/2c$.

Integrals of Unbounded Functions

We turn next to improper integrals where the *integrand* is not bounded. Consider first the function $f(x) = 1/\sqrt{x}$, with $x \in (0, 2]$—see Fig. 9.7.3. Note that $f(x) \to \infty$ as $x \to 0^+$. The function f is continuous in the interval $[h, 2]$ for any fixed number h in $(0, 2)$. Therefore, the definite integral of f over the interval $[h, 2]$ exists, and in fact

$$\int_h^2 \frac{1}{\sqrt{x}}\,dx = \Big|_h^2 2\sqrt{x} = 2\sqrt{2} - 2\sqrt{h}$$

The limit of this expression as $h \to 0^+$ is $2\sqrt{2}$. Then, by definition,

$$\int_0^2 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{2}$$

The improper integral is said to converge in this case, and the area below the graph of f over the interval $(0, 2]$ is $2\sqrt{2}$. The area over the interval $(h, 2]$ is shown in Fig. 9.7.3. As $h \to 0$ the shaded area becomes unbounded, but the graph of f approaches the y-axis so quickly that the total area is finite.

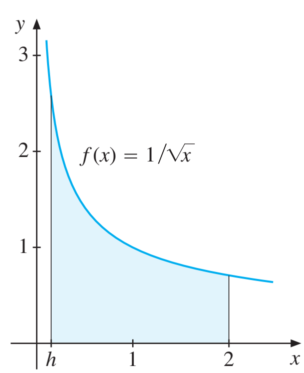


Figure 9.7.3 $f(x) = 1/\sqrt{x}$

More generally, suppose that f is a continuous function in the interval $(a, b]$, but $f(x)$ is not defined at $x = a$. Then we can define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{h \to 0^+} \int_{a+h}^b f(x)\,\mathrm{d}x \tag{9.7.6}$$

if the limit exists, and the improper integral of f is said to *converge* in this case. If $f(x) \geq 0$ in $(a, b]$, we identify the integral as the *area under the graph* of f over the interval $(a, b]$.

In the same way, if f is not defined at b, we can define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{h \to 0^+} \int_a^{b-h} f(x)\,\mathrm{d}x \tag{9.7.7}$$

if the limit exists, in which case the improper integral of f is said to *converge*.

Suppose f is continuous in (a, b). We may not even have f defined at a or b. For instance, suppose $f(x) \to -\infty$ as $x \to a^+$ and $f(x) \to +\infty$ as $x \to b^-$. In this case, f is said to be *integrable* in (a, b), and we can define

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \tag{9.7.8}$$

provided that both integrals on the right-hand side of (9.7.8) converge. Here c is an arbitrary fixed number in (a, b), and neither the convergence of the integral nor its value depends on the choice of c. If either of the integrals on the right-hand side of (9.7.8) does not converge, the left-hand side is not well defined.

A Test for Convergence

The following convergence test for integrals is occasionally useful because it does not require that the integral be evaluated.

THEOREM 9.7.1 (A COMPARISON TEST FOR CONVERGENCE)

Suppose that for all $x \geq a$, f and g are continuous, and $|f(x)| \leq g(x)$. If $\int_a^\infty g(x)\,\mathrm{d}x$ converges, then $\int_a^\infty f(x)\,\mathrm{d}x$ also converges, and

$$\left| \int_a^\infty f(x)\,\mathrm{d}x \right| \leq \int_a^\infty g(x)\,\mathrm{d}x$$

Considering the case in which $f(x) \geq 0$, Theorem 9.7.1 can be interpreted as follows: If the area below the graph of g is finite, then the area below the graph of f is finite as well, because at no point in $[a, \infty)$ does the graph of f lie above the graph of g. This result seems quite plausible, especially after drawing a suitable figure, so we shall not give an analytical proof. A corresponding theorem holds for the case where the lower limit of integration is $-\infty$. Also, similar comparison tests can be proved for unbounded functions defined on bounded intervals.

EXAMPLE 9.7.4 Integrals of the form

$$\int_{t_0}^{\infty} U(c(t))e^{-\alpha t}\,dt \tag{$*$}$$

often appear in economic growth theory. Here, $c(t)$ denotes consumption at time t, whereas U is an instantaneous utility function, and α is a positive discount rate. Suppose that there exist numbers M and β, with $\beta < \alpha$, such that

$$|U(c(t))| \le Me^{\beta t} \tag{$**$}$$

for all $t \ge t_0$ and for each possible consumption level $c(t)$ at time t. Thus, the absolute value of the utility of consumption is growing at a rate less than the discount rate α. Prove that then $(*)$ converges.

Solution: From $(**)$, we have

$$\left|U(c(t))e^{-\alpha t}\right| \le Me^{-(\alpha-\beta)t}$$

for all $t \ge t_0$. Moreover,

$$\int_{t_0}^{T} Me^{-(\alpha-\beta)t}\,dt = \Big|_{t_0}^{T} -\frac{M}{\alpha-\beta}e^{-(\alpha-\beta)t} = \frac{M}{\alpha-\beta}\left[e^{-(\alpha-\beta)t_0} - e^{-(\alpha-\beta)T}\right]$$

Because $\alpha - \beta > 0$, the last expression tends to

$$[M/(\alpha-\beta)]\,e^{-(\alpha-\beta)t_0}$$

as $T \to \infty$. From Theorem 9.7.1 it follows that $(*)$ converges. ▮

EXAMPLE 9.7.5 The function $f(x) = e^{-x^2}$ is extremely important in statistics. When multiplied by a suitable constant, $1/\sqrt{\pi}$, it is the density function associated with a *Gaussian*, or *normal*, distribution. We want to show that the improper integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx \tag{$*$}$$

converges. Recall from Section 9.1 that the indefinite integral of $f(x) = e^{-x^2}$ cannot be expressed in terms of "elementary" functions. Because $f(x) = e^{-x^2}$ is symmetric about the y-axis, one has $\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx$, so it suffices to prove that $\int_0^{\infty} e^{-x^2}\,dx$ converges. To show this, subdivide the interval of integration so that

$$\int_0^{\infty} e^{-x^2}\,dx = \int_0^{1} e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx \tag{$**$}$$

Of course, $\int_0^1 e^{-x^2}\,\mathrm{d}x$ presents no problem because it is the integral of a continuous function over a bounded interval. For $x \geq 1$, one has $x^2 \geq x$ and so $0 \leq e^{-x^2} \leq e^{-x}$. Now $\int_1^{\infty} e^{-x}\,\mathrm{d}x$ converges (to $1/e$), so according to Theorem 9.7.1, the integral $\int_1^{\infty} e^{-x^2}\,\mathrm{d}x$ must also converge. From (∗∗), it follows that $\int_0^{\infty} e^{-x^2}\,\mathrm{d}x$ converges. Thus, the integral (∗) does converge, but we have not found its value. In fact, one can use a more advanced technique of integration to prove that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi} \tag{9.7.9}$$

as discussed in FMEA.

EXERCISES FOR SECTION 9.7

1. Determine the following integrals, if they converge. Indicate those that diverge.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,\mathrm{d}x$ (b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,\mathrm{d}x$ (c) $\displaystyle\int_{-\infty}^{0} e^x\,\mathrm{d}x$ (d) $\displaystyle\int_0^{a} \frac{x}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$, where $a > 0$

2. In statistics, the *uniform*, or *rectangular*, *distribution* on the interval $[a, b]$ is described by the density function f defined for all x by $f(x) = 1/(b - a)$ for $x \in [a, b]$, and $f(x) = 0$ for $x \notin [a, b]$. Find the following:

(a) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ (b) $\displaystyle\int_{-\infty}^{+\infty} x f(x)\,\mathrm{d}x$ (c) $\displaystyle\int_{-\infty}^{+\infty} x^2 f(x)\,\mathrm{d}x$

SM 3. In connection with Example 9.7.1, find the following:

(a) $\displaystyle\int_0^{\infty} x\lambda e^{-\lambda x}\,\mathrm{d}x$ (b) $\displaystyle\int_0^{\infty} (x - 1/\lambda)^2\,\lambda e^{-\lambda x}\,\mathrm{d}x$ (c) $\displaystyle\int_0^{\infty} (x - 1/\lambda)^3\,\lambda e^{-\lambda x}\,\mathrm{d}x$

The three numbers you obtain are called respectively the *expectation*, the *variance*, and the *third central moment* of the exponential distribution.

4. Prove that $\int_{-\infty}^{+\infty} x/(1 + x^2)\,\mathrm{d}x$ diverges, but that $\lim_{b\to\infty} \int_{-b}^{b} x/(1 + x^2)\,\mathrm{d}x$ converges.

SM 5. The function f is defined for $x > 0$ by $f(x) = (\ln x)/x^3$.

(a) Find the maximum and minimum points of f, if there are any.

(b) Examine the convergence of $\int_0^1 f(x)\,\mathrm{d}x$ and $\int_1^{\infty} f(x)\,\mathrm{d}x$.

6. Use Theorem 9.7.1 to prove the convergence of $\displaystyle\int_1^{\infty} \frac{1}{1 + x^2}\,\mathrm{d}x$.

SM 7. Show that $\displaystyle\int_{-2}^{3} \left(\frac{1}{\sqrt{x + 2}} + \frac{1}{\sqrt{3 - x}}\right)\mathrm{d}x = 4\sqrt{5}$.

8. The integral

$$z = \int_0^\infty e^{-rs} D(s)\, \mathrm{d}s$$

represents the present discounted value, at interest rate r, of the time-dependent stream of depreciation allowances $D(s)$, for $0 \le s < \infty$. Find z as a function of τ in the following cases:[5]

(a) $D(s) = 1/\tau$ for $0 \le s \le \tau$; $D(s) = 0$ for $s > \tau$.

(b) $D(s) = 2(\tau - s)/\tau^2$ for $0 \le s \le \tau$; $D(s) = 0$ for $s > \tau$.

9. Suppose you evaluate $\int_{-1}^{+1} (1/x^2)\, \mathrm{d}x$ by using the definition of the definite integral without thinking carefully. Show that you get a negative answer even though the integrand is never negative. What has gone wrong?

10. Prove that the integral $\int_0^1 (\ln x/\sqrt{x})\, \mathrm{d}x$ converges and find its value. (*Hint*: See part (b) of Exercise 9.6.6.)

11. Find the integral

$$I_k = \int_1^\infty \left(\frac{k}{x} - \frac{k^2}{1 + kx} \right) \mathrm{d}x$$

where k is a positive constant. Find the limit of I_k as $k \to \infty$, if it exists.

SM **12.** [HARDER] In statistics, the normal, or Gaussian, density function with mean μ and variance σ^2 is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-(x-\mu)^2/2\sigma^2\right]$$

in the interval $(-\infty, \infty)$.[6] Prove that:

(a) $\displaystyle\int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = 1$ (b) $\displaystyle\int_{-\infty}^{+\infty} x f(x)\, \mathrm{d}x = \mu$ (c) $\displaystyle\int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, \mathrm{d}x = \sigma^2$

(*Hint*: Use the substitution $u = (x - \mu)/\sqrt{2}\sigma$, together with Eqs (9.7.9) and (9.7.5).)

9.8 A Glimpse at Differential Equations

In economic growth theory, in studies of the extraction of natural resources, in many models in environmental economics, and in several other areas of economics, one encounters equations where the unknowns are functions, and where the derivatives of these functions

[5] The first case models constant depreciation over τ years; the second is known as straight-line depreciation.

[6] The formula for this function, along with its bell-shaped graph and a portrait of its inventor Carl Friedrich Gauss (1777–1855), all appeared on the German 10 Deutsche mark banknote that was used between 1991 and 2001, in the decade before the euro currency started to circulate instead.

also appear. Equations of this general type are called *differential equations*, and their study is one of the most fascinating fields of mathematics. Here we shall consider only a few simple types of such equations. We denote the independent variable by t, because most of the differential equations in economics have time as the independent variable.

We have already solved the simplest type of differential equation: Let $f(t)$ be a given function. Find all functions that have $f(t)$ as their derivative—that is, find all functions that solve $\dot{x}(t) = f(t)$ for $x(t)$, where $\dot{x}$ denotes the derivative of x w.r.t. time t. We already know that the answer is an indefinite integral:

$$\dot{x}(t) = f(t) \Leftrightarrow x(t) = \int f(t)\,\mathrm{d}t + C$$

We call $x(t) = \int f(t)\,\mathrm{d}t + C$ the *general* solution of the equation $\dot{x}(t) = f(t)$.

Consider next some more challenging types of differential equation.

The Exponential Growth Law

Let $x(t)$ denote an economic quantity such as the GDP of China. The ratio $\dot{x}(t)/x(t)$ has previously been called the *relative rate of change* of this quantity. Several economic models postulate that the relative rate of change is approximately a constant, r. Thus, for all t

$$\dot{x}(t) = rx(t) \tag{9.8.1}$$

Which functions have a constant relative rate of change? For $r = 1$ the differential equation is $\dot{x} = x$, and we know that the derivative of $x = e^t$ is again $\dot{x} = e^t$. More generally, the function $x = Ae^t$ satisfies the equation $\dot{x} = x$ for all values of the constant A. By trial and error you will probably be able to come up with $x(t) = Ae^{rt}$ as a solution of (9.8.1). In any case, it is easy to verify: If $x = Ae^{rt}$, then $\dot{x}(t) = Are^{rt} = rx(t)$. Moreover, we can prove that no other function satisfies (9.8.1): Indeed, multiply Eq. (9.8.1) by the positive function e^{-rt} and collect all terms on the left-hand side. This gives

$$\dot{x}(t)e^{-rt} - rx(t)e^{-rt} = 0 \tag{9.8.2}$$

Equation (9.8.2) must have precisely the same solutions as 9.8.1. But the left-hand side of this equation is the derivative of the product $x(t)e^{-rt}$. So Eq. (9.8.2) can be rewritten as $\frac{\mathrm{d}}{\mathrm{d}t}[x(t)e^{-rt}] = 0$. It follows that $x(t)e^{-rt}$ must equal a constant A. Hence, $x(t) = Ae^{rt}$. If the value of $x(t)$ at $t = 0$ is x_0, then $x_0 = Ae^0 = A$. We conclude that:

$$\boxed{\dot{x}(t) = rx(t) \ \text{ with } \ x(0) = x_0 \ \Leftrightarrow \ x(t) = x_0 e^{rt}} \tag{9.8.3}$$

EXAMPLE 9.8.1 Let $S(t)$ denote the sales volume of a particular commodity, per unit of time, evaluated at time t. In a stable market where no sales promotion is carried out, the decrease in $S(t)$ per unit of time is proportional to $S(t)$. Thus sales decelerate at the constant proportional rate $a > 0$, implying that $\dot{S}(t) = -aS(t)$.

(a) Find an expression for $S(t)$ when sales at time 0 are S_0.

(b) Solve the equation $S_0 e^{-at} = \frac{1}{2} S_0$ for t. Interpret the answer.

Solution:

(a) This is an equation of type (9.8.1) with $x = S$ and $r = -a$. According to (9.8.3), the solution is $S(t) = S_0 e^{-at}$.

(b) From $S_0 e^{-at} = \frac{1}{2} S_0$, we obtain $e^{-at} = \frac{1}{2}$. Taking the natural logarithm of each side yields $-at = \ln(1/2) = -\ln 2$. Hence $t = \ln 2/a$. This is the time it takes before sales fall to half their initial level.

Equation (9.8.1) has often been called the *law for natural growth*. Whatever it may be called, this law is probably the most important differential equation that economists have to know.

Suppose that $x(t)$ denotes the number of individuals in a population at time t. The population could be, for instance, a particular colony of bacteria, or polar bears in the Arctic. We call $\dot{x}(t)/x(t)$ the *per capita growth rate* of the population. If there is neither immigration nor emigration, then the per capita rate of increase will be equal to the difference between the per capita birth and death rates. These rates will depend on many factors such as food supply, age distribution, available living space, predators, disease, and parasites, among other things.

Equation (9.8.1) specifies a simple model of population growth, following what is often called *Malthus's law*. According to (9.8.3), if the per capita growth rate is constant, then the population must grow exponentially. In reality, of course, exponential growth can go on only for a limited time. Let us consider some alternative models for population growth.

Another way to solve (9.8.1) is to take logarithms. Note that $\mathrm{d} \ln x/\mathrm{d}t = \dot{x}/x = r$, so $\ln x(t) = \int r \,\mathrm{d}t = rt + C$. This implies that $x(t) = e^{rt+C} = e^{rt}e^{C} = Ae^{rt}$, where $A = e^{C}$. In fact, a generalized version of Eq. (9.8.1) allows for the growth rate to be a function of time:

$$\dot{x}(t) = r(t)x(t) \tag{9.8.4}$$

Provided that $x(t) \neq 0$, this can be rearranged to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln x(t) = \frac{\dot{x}(t)}{x(t)} = r(t)$$

whose solution is $\ln x(t) - \ln x(0) = R(t)$ or, taking exponentials, $x(t) = x(0)e^{R(t)}$ where $R(t) = \int_0^t r(s)\,\mathrm{d}s$.

In applications, it is sometimes useful to have an "initial value" for a differential equation at a period t other than $t = 0$. This is easily done, as $t = 0$ is essentially no more than a convention. That is, if the evolution of x is given by Eq. (9.8.1), we know from (9.8.3) that for any t_0 and any t,

$$x(t_0) = x_0 e^{rt_0} \quad \text{and} \quad x(t) = x_0 e^{rt}$$

Now, the latter is equivalent to

$$x(t) = x_0 e^{r(t-t_0+t_0)} = x_0 e^{r(t-t_0)} e^{t_0} = (x_0 e^{t_0}) e^{r(t-t_0)} = x(t_0) e^{r(t-t_0)}$$

where one uses t_0 as the initial reference point of the equation.

Growth Towards an Upper Limit

Suppose the population size $x(t)$ cannot exceed some carrying capacity K, and that the rate of change of population is proportional to its deviation from this carrying capacity:

$$\dot{x}(t) = a(K - x(t)) \qquad (*)$$

With a little trick, it is easy to find all the solutions to this equation. Define a new function $u(t) = K - x(t)$, which at each time t measures the deviation of the population size from the carrying capacity K. Then $\dot{u}(t) = -\dot{x}(t)$. Inserting this into $(*)$ gives $-\dot{u}(t) = au(t)$, or $\dot{u}(t) = -au(t)$. This is an equation like (9.8.1). The solution is $u(t) = Ae^{-at}$, so that $K - x(t) = Ae^{-at}$, hence $x(t) = K - Ae^{-at}$. If $x(0) = x_0$, then $x_0 = K - A$, and so $A = K - x_0$. It follows that:

$$\dot{x}(t) = a(K - x(t)) \text{ with } x(0) = x_0 \iff x(t) = K - (K - x_0)e^{-at} \qquad (9.8.5)$$

In Exercise 3 we shall see that the same equation describes the population in countries where the indigenous population has a fixed relative rate of growth, but where there is immigration each year. The same equation can also represent several other phenomena, some of which are discussed in the problems for this section.

EXAMPLE 9.8.2 Suppose that a population has a carrying capacity of $K = 200$ (million) and that at time $t = 0$ there are 50 (million). Let $x(t)$ denote the population in millions at time t. Suppose that $a = 0.05$ and solve Eq. $(*)$ in this case. Sketch a graph of the solution.

Solution: Using Eq. (9.8.5) we find that

$$x(t) = 200 - (200 - 50)e^{-0.05t} = 200 - 150e^{-0.05t}$$

The graph is drawn in Fig. 9.8.1.

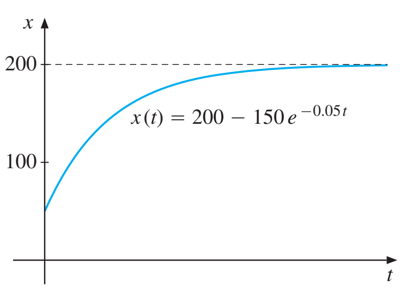


Figure 9.8.1 Growth to level 200

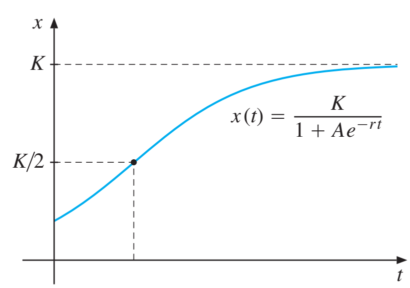


Figure 9.8.2 Logistic growth up to level K

Logistic Growth

Instead of the differential equation $(*)$, a more realistic assumption is that the relative rate of increase is approximately constant while the population is small, but that it converges to zero as the population approaches its carrying capacity K. A special form of this assumption is expressed by the equation

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) \tag{9.8.6}$$

Indeed, when the population $x(t)$ is small in proportion to K, so that $x(t)/K$ is small, then $\dot{x}(t) \approx rx(t)$, which implies that $x(t)$ increases (approximately) exponentially. As $x(t)$ becomes larger, however, the factor $1 - x(t)/K$ increases in significance. In general, we claim that if $x(t)$ satisfies (9.8.6) and is not identically equal to 0, then $x(t)$ must have the form

$$x(t) = \frac{K}{1 + Ae^{-rt}} \tag{9.8.7}$$

for some constant A. The function x given in (9.8.7) is called a *logistic function*.

In order to prove Eq. (9.8.7), we use a little trick:

Suppose that $x = x(t)$ is not 0 and introduce the new variable $u = u(t) = -1 + K/x$. Then $\dot{u} = -K\dot{x}/x^2 = -Kr/x + r = -r(-1 + K/x) = -ru$. Hence $u = u(t) = Ae^{-rt}$ for some constant A. But then $-1 + K/x(t) = Ae^{-rt}$, and solving this equation for $x(t)$ yields (9.8.7).

Suppose the population consists of x_0 individuals at time $t = 0$, and thus $x(0) = x_0$. Then Eq. (9.8.7) gives $x_0 = K/(1 + A)$, so that $A = (K - x_0)/x_0$. All in all, we have shown that the unique solution to (9.8.6) with $x(0) = x_0$ is

$$x(t) = \frac{K}{1 + Ae^{-rt}}, \quad \text{where} \quad A = \frac{K - x_0}{x_0} \tag{9.8.8}$$

If $0 < x_0 < K$, it follows from (9.8.8) that $x(t)$ is strictly increasing and that $x(t) \to K$ as $t \to \infty$, assuming $r > 0$. We say in this case that there is *logistic growth* up to the level K. The graph of the solution is shown in Fig. 9.8.2. It has an inflection point at the height $K/2$ above the t-axis. We verify this by differentiating Eq. (9.8.6) w.r.t. t. The result is $\ddot{x} = r\dot{x}(1 - x/K) + rx(-\dot{x}/K) = r\dot{x}(1 - 2x/K) = 0$. So $\ddot{x} = 0$ when $x = K/2$, and $\ddot{x}$ changes sign at this point.

Equations of type (9.8.6), and hence logistic functions of the form (9.8.7), appear in many economic models. Some of them are discussed in the problems. The simple differential equations studied here are so important that we present them and their general solutions in a form which makes it easier to see their structure. As is often done in the theory of differential equations, we suppress the symbol for time dependence.

SOLUTIONS OF SOME SIMPLE DIFFERENTIAL EQUATIONS

$$\dot{x} = ax \text{ for all } t \iff x = Ae^{at} \text{ for some constant } A \tag{9.8.9}$$

$$\dot{x} + ax = b \text{ for all } t \iff x = Ae^{-at} + \frac{b}{a} \text{ for some constant } A \tag{9.8.10}$$

$$\dot{x} + ax = bx^2 \text{ for all } t \iff x = \frac{a}{b - Ae^{at}} \text{ for some constant } A \tag{9.8.11}$$

Note that in (9.8.10) we must assume that $a \neq 0$, while in (9.8.11) the function $x(t) = 0$, for all t, is also a solution.

EXERCISES FOR SECTION 9.8

1. Which of the following functions have a constant relative rate of increase, $\dot{x}/x$?

(a) $x = 5t + 10$ (b) $x = \ln(t + 1)$ (c) $x = 5e^t$

(d) $x = -3 \cdot 2^t$ (e) $x = e^{t^2}$ (f) $x = e^t + e^{-t}$

2. Suppose that a firm's capital stock $K(t)$ satisfies the differential equation $\dot{K}(t) = I - \delta K(t)$, where investment I is constant, and $\delta K(t)$ denotes depreciation, with δ a positive constant.

(a) Find the solution of the equation if the capital stock at time $t = 0$ is K_0.

(b) Let $\delta = 0.05$ and $I = 10$. Explain what happens as $t \to \infty$ when: (i) $K_0 = 150$; (ii) $K_0 = 250$.

3. Let $N(t)$ denote the number of people in a country whose homes have broadband internet. Suppose that the rate at which new people get access is proportional to the number of people who still have no access. If the population size is P, the differential equation for $N(t)$ is, therefore, $\dot{N}(t) = k(P - N(t))$, where k is a positive constant. Find the solution of this equation if $N(0) = 0$. Then find the limit of $N(t)$ as $t \to \infty$.

4. A country's annual natural rate of population growth (births minus deaths) is 2%. In addition there is a net immigration of 40 000 persons per year. Write down a differential equation for the function $N(t)$ which denotes the number of persons in the country at time t (year). Suppose that the population at time $t = 0$ is 2 000 000. Find $N(t)$.

5. As in Examples 4.5.1 and 4.9.1, let $P(t)$ denote Europe's population in millions t years after 1960. According to UN estimates, $P(0) = 606$ and $P(10) = 657$. Suppose that $P(t)$ grows exponentially, with $P(t) = 606e^{kt}$. Compute k and then find $P(15)$, $P(40)$, and $P(55)$, which are estimates of the population in 1975, in 2000, and in 2015.

6. When a colony of bacteria is subjected to strong ultraviolet light, they die as their DNA is destroyed. In a laboratory experiment it was found that the number of living bacteria decreased approximately exponentially with the length of time they were exposed to ultraviolet light. Suppose that 70.5% of the bacteria still survive after 7 seconds of exposure. What percentage will be alive after 30 seconds? How long does it take to kill 95% of the bacteria?

7. Solve the following differential equations by using one of (9.8.9)–(9.8.11):

(a) $\dot{x} = -0.5x$ (b) $\dot{K} = 0.02K$ (c) $\dot{x} = -0.5x + 5$

(d) $\dot{K} - 0.2K = 100$ (e) $\dot{x} + 0.1x = 3x^2$ (f) $\dot{K} = K(-1 + 2K)$

8. A study of the mechanization of British agriculture from 1950 onwards estimated that y, the number of tractors in use (measured in thousands) as a function of t (measured in years, so that $t = 0$ corresponds to 1950), was approximately given by $y(t) = 250 + x(t)$, where $x = x(t)$ satisfied the logistic differential equation $\dot{x} = 0.34x(1 - x/230)$, and $x(0) = 25$.

(a) Find an expression for $y(t)$.

(b) Find the limit of $y(t)$ as $t \to \infty$, and draw the graph.

9. In a model of how influenza spreads, let $N(t)$ denote the number of persons who develop influenza t days after all members of a group of 1000 people have been in contact with a carrier of infection. Assume that $\dot{N}(t) = 0.39N(t)[1 - N(t)/1000]$, and $N(0) = 1$.

(a) Find a formula for $N(t)$. How many develop influenza after 20 days?

(b) How many days does it take until 800 people are sick?

(c) Will all 1000 people eventually get influenza?

SM 10. The logistic function (9.8.6) has been used for describing the stock of certain fish populations. Suppose such a population is harvested at a rate proportional to the stock, so that

$$\dot{x}(t) = rx(t)\left[1 - \frac{x(t)}{K}\right] - fx(t)$$

(a) Solve this equation, when the population at time $t = 0$ is x_0.

(b) Suppose $f > r$. Examine the limit of $x(t)$ as $t \to \infty$.

11. [HARDER] According to *Newton's law of cooling*, the rate at which a warm object cools is proportional to the difference between the temperature of the object and the "ambient" temperature of its surroundings. If the temperature of the object at time t and the (constant) ambient temperature is C, then $\dot{T}(t) = k(C - T(t))$. Note that this is an equation of the type given in (9.8.5). At 12 noon, the police enter a room and discover a dead body. Immediately they measure its temperature, which is 35° Celsius. At 1 pm they take the temperature again, which is now 32°. The temperature in the room is constant at 20°. When did the person die? (*Hint*: Let the temperature be $T(t)$, where t is measured in hours and 12 noon corresponds to $t = 0$.)

9.9 Separable and Linear Differential Equations

In this final section of the chapter we consider two general types of differential equation that are frequently encountered in economics. The discussion will be brief—for a more extensive treatment we refer the reader to FMEA.

Separable Equations

A differential equation of the type

$$\dot{x} = f(t)g(x) \tag{9.9.1}$$

is called *separable*. The unknown function is $x = x(t)$, and its rate of change $\dot{x}$ is given as the product of a function only of t and a function only of x. A simple case is $\dot{x} = tx$, which is obviously separable, while $\dot{x} = t + x$ is not. In fact, all the differential equations studied in the previous section were separable equations of the type $\dot{x} = g(x)$, with $f(t) \equiv 1$.

Equation (9.8.11), for instance, is separable, since $\dot{x} + ax = bx^2$ can be rewritten as $\dot{x} = g(x)$ where $g(x) = -ax + bx^2$.

The following general method for solving separable equations is justified in FMEA.

RECIPE FOR SOLVING SEPARABLE DIFFERENTIAL EQUATIONS

(i) Write Eq. (9.9.1) as

$$\frac{dx}{dt} = f(t)g(x)$$

(ii) Separate the variables:

$$\frac{1}{g(x)}\, dx = f(t)\, dt$$

(iii) Integrate each side:

$$\int \frac{1}{g(x)}\, dx = \int f(t)\, dt$$

(iv) Evaluate the two integrals, if possible, and you obtain a solution of (∗), possibly in implicit form. Solve for x, if possible.

Note that in step (ii) we divided by $g(x)$. In fact, if $g(x)$ has a zero at $x = a$, so that $g(a) = 0$, then $x(t) \equiv a$ will be a particular solution of the equation, because the right- and left-hand sides of the equation are both 0 for all t. For instance, in the logistic equation (9.8.6), both $x(t) = 0$ and $x(t) = K$ are particular solutions.

EXAMPLE 9.9.1 Solve the differential equation

$$\frac{dx}{dt} = e^t x^2$$

and find the solution curve, which is also called the *integral curve*, that passes through the point $(t, x) = (0, 1)$.

Solution: We observe first that $x(t) \equiv 0$ is one (trivial) solution. To find the other solutions we follow the last three parts of the recipe:

Separate: $(1/x^2)\, dx = e^t\, dt$;

Integrate: $\int (1/x^2)\, dx = \int e^t\, dt$;

Evaluate: $-1/x = e^t + C$.

It follows that:

$$x = \frac{-1}{e^t + C} \qquad (*)$$

To find the integral curve through $(0, 1)$, we must determine the correct value of C. Because we require $x = 1$ for $t = 0$, it follows from (∗) that $1 = -1/(1 + C)$, so $C = -2$. Thus, the integral curve passing through $(0, 1)$ is $x = 1/(2 - e^t)$.

EXAMPLE 9.9.2 **(Economic Growth[7]).** Let $X = X(t)$ denote the national product, $K = K(t)$ the capital stock, and $L = L(t)$ the number of workers in a country at time t. Suppose that, for all $t \geq 0$,

(a) $X = \sqrt{K}\sqrt{L}$ (b) $\dot{K} = 0.4X$ (c) $L = e^{0.04t}$

Derive from these equations a single differential equation for $K = K(t)$, and find the solution of that equation when $K(0) = 10\,000$.[8]

Solution: From equations (a)–(c), we derive the single differential equation

$$\dot{K} = \frac{\mathrm{d}K}{\mathrm{d}t} = 0.4\sqrt{K}\sqrt{L} = 0.4e^{0.02t}\sqrt{K}$$

This is clearly separable. Using the recipe yields the successive equations:
(ii) $\frac{1}{\sqrt{K}}\,\mathrm{d}K = 0.4e^{0.02t}\,\mathrm{d}t$; (iii) $\int \frac{1}{\sqrt{K}}\,\mathrm{d}K = \int 0.4e^{0.02t}\,\mathrm{d}t$; (iv) $2\sqrt{K} = 20e^{0.02t} + C$.
If $K = 10\,000$ for $t = 0$, then $2\sqrt{10\,000} = 20 + C$, so $C = 180$. Then $\sqrt{K} = 10e^{0.02t} + 90$, and so the required solution is

$$K(t) = (10e^{0.02t} + 90)^2 = 100(e^{0.02t} + 9)^2$$

The capital–labour ratio has a somewhat bizarre limiting value in this model: as $t \to \infty$, so

$$\frac{K(t)}{L(t)} = 100 \times \frac{(e^{0.02t} + 9)^2}{e^{0.04t}} = 100\left[\frac{e^{0.02t} + 9}{e^{0.02t}}\right]^2 = 100(1 + 9e^{-0.02t})^2 \to 100$$

EXAMPLE 9.9.3 Solve the separable differential equation $(\ln x)\dot{x} = e^{1-t}$.

Solution: Following the recipe yields

(i) $\ln x \dfrac{\mathrm{d}x}{\mathrm{d}t} = e^{1-t}$;
(ii) $\ln x\,\mathrm{d}x = e^{1-t}\,\mathrm{d}t$;
(iii) $\int \ln x\,\mathrm{d}x = \int e^{1-t}\,\mathrm{d}t$;
(iv) $x\ln x - x = -e^{1-t} + C$, using the result in Example 9.1.3.

The desired functions $x = x(t)$ are those that satisfy the last equation for all t.

We usually say that we have solved a differential equation even if the unknown function, as shown in Example 9.9.3, cannot be expressed explicitly. The important point is that we have expressed the unknown function in an equation that does not include the derivative of that function.

[7] This is a special case of the Solow–Swan growth model. See Example 5.7.3 in FMEA.
[8] In (a) we have a Cobb–Douglas production function; (b) says that aggregate investment is proportional to output; (c) implies that the labour force grows exponentially.

First-Order Linear Equations

A *first-order linear differential equation* is one that can be written in the form

$$\dot{x} + a(t)x = b(t) \tag{9.9.2}$$

where $a(t)$ and $b(t)$ denote known continuous functions of t in a certain interval, and $x = x(t)$ is the unknown function. Equation (9.9.2) is called "linear" because the left-hand side is a linear function of x and $\dot{x}$.[9]

When $a(t)$ and $b(t)$ are constants, the solution was given in (9.8.10):

$$\dot{x} + ax = b \Leftrightarrow x = Ce^{-at} + \frac{b}{a} \tag{9.9.3}$$

where C is a constant. We found the solution of this equation by introducing a new variable. In fact, the equation is separable, so the recipe for separable equations will also lead us to the solution. If we let $C = 0$ we obtain the constant solution $x(t) = b/a$. We say that $x = b/a$ is an *equilibrium state*, or a *stationary state*, for the equation. Observe how this solution can be obtained from $\dot{x} + ax = b$ by letting $\dot{x} = 0$ and then solving the resulting equation for x. If the constant a is positive, then the solution $x = Ce^{-at} + b/a$ converges to b/a as $t \to \infty$. In this case, the equation is said to be *stable*, because every solution of the equation converges to an equilibrium as t approaches infinity.[10]

EXAMPLE 9.9.4 Find the solution of $\dot{x} + 3x = -9$, and determine whether the equation is stable.

Solution: By (9.9.3), the solution is $x = Ce^{-3t} - 3$. Here the equilibrium state is $x = -3$, and the equation is stable because $a = 3 > 0$, and $x \to -3$ as $t \to \infty$.

EXAMPLE 9.9.5 **(A price adjustment mechanism).** Let $D(P) = a - bP$ and $S(P) = \alpha + \beta P$ denote, respectively, the demand and the supply of a certain commodity, when the price is P. Here a, b, α, and β are positive constants. Assume that the price $P = P(t)$ varies with time, and that $\dot{P}$ is proportional to excess demand $D(P) - S(P)$. Thus,

$$\dot{P} = \lambda[D(P) - S(P)]$$

where λ is a positive constant. Inserting the expressions for $D(P)$ and $S(P)$ into this equation gives $\dot{P} = \lambda(a - bP - \alpha - \beta P)$. Rearranging, we then obtain

$$\dot{P} + \lambda(b + \beta)P = \lambda(a - \alpha)$$

According to (9.9.3), the solution is

$$P = Ce^{-\lambda(b+\beta)t} + \frac{a - \alpha}{b + \beta}$$

Because $\lambda(b + \beta)$ is positive, as t tends to infinity, P converges to the equilibrium price $P^e = (a - \alpha)/(b + \beta)$, for which $D(P^e) = S(P^e)$. Thus, the equation is stable.

[9] It is called "first-order" because it only involves the first derivative of x, and not higher-order derivatives.

[10] Stability theory is an important issue for differential equations appearing in economics. For an extensive discussion see, for instance, FMEA.

Variable Right-Hand Side

Consider next the case where the right-hand side is not constant:

$$\dot{x} + ax = b(t) \tag{9.9.4}$$

When $b(t)$ is not constant, this equation is not separable. A clever trick helps us find the solution. We multiply each side of the equation by the positive factor e^{at}, called an *integrating factor*. This gives the equivalent equation

$$\dot{x}e^{at} + axe^{at} = b(t)e^{at} \tag{$*$}$$

This idea may not be obvious beforehand, but it works well because the left-hand side of ($*$) happens to be the exact derivative of the product xe^{at}. Thus ($*$) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}(xe^{at}) = b(t)e^{at} \tag{$**$}$$

According to the definition of the indefinite integral, Eq. ($**$) holds for all t in an interval if, and only if, $xe^{at} = \int b(t)e^{at}\,\mathrm{d}t + C$ for some constant C. Multiplying this equation by e^{-at} gives the solution for x. Briefly formulated:

$$\boxed{\dot{x} + ax = b(t) \iff x = Ce^{-at} + e^{-at}\int e^{at}b(t)\,\mathrm{d}t} \tag{9.9.5}$$

EXAMPLE 9.9.6 Find the solution of $\dot{x} + x = t$, and determine the solution curve passing through $(0, 0)$.

Solution: According to (9.9.5), with $a = 1$ and $b(t) = t$, the solution is given by

$$x = Ce^{-t} + e^{-t}\int te^t\,\mathrm{d}t = Ce^{-t} + e^{-t}(te^t - e^t) = Ce^{-t} + t - 1$$

where, following Example 9.5.1, we used integration by parts to evaluate $\int te^t\,\mathrm{d}t$. If $x = 0$ when $t = 0$, we get $0 = C - 1$, so $C = 1$ and the required solution is $x = e^{-t} + t - 1$. ∎

EXAMPLE 9.9.7 **(Economic growth).** Consider the following model of economic growth:

(a) $X(t) = 0.2K(t)$ (b) $\dot{K}(t) = 0.1X(t) + H(t)$ (c) $N(t) = 50e^{0.03t}$

This model is meant to capture the features of a developing country. Here, $X(t)$ is annual GDP, $K(t)$ is capital stock, $H(t)$ is the net inflow of foreign investment per year, and $N(t)$ is the size of the population, all measured at time t. In (i) we assume that the volume of production is simply proportional to the capital stock, with the factor of proportionality 0.2 being called the *average productivity of capital*. In (ii) we assume that the total growth of capital per year is equal to internal savings plus net foreign investment. We assume that savings are proportional to production, with the factor of proportionality 0.1 being called the *savings rate*. Finally, (iii) tells us that population increases at a constant proportional rate of growth 0.03.

Assume that $H(t) = 10e^{0.04t}$ and derive from these equations a differential equation for $K(t)$. Find its solution given that $K(0) = 200$. Find also an expression for $x(t) = X(t)/N(t)$, which is domestic product per capita.

Solution: From (a) and (b), it follows that $K(t)$ must satisfy the linear equation

$$\dot{K}(t) - 0.02K(t) = 10e^{0.04t}$$

Using (9.9.5) we obtain

$$\begin{aligned} K(t) &= Ce^{0.02t} + e^{0.02t}\int e^{-0.02t}10e^{0.04t}\,\mathrm{d}t \;=\; Ce^{0.02t} + 10e^{0.02t}\int e^{0.02t}\,\mathrm{d}t \\ &= Ce^{0.02t} + (10/0.02)e^{0.04t} \;=\; Ce^{0.02t} + 500e^{0.04t} \end{aligned}$$

For $t = 0$, $K(0) = 200 = C + 500$, so $C = -300$. Thus, the solution is

$$K(t) = 500e^{0.04t} - 300e^{0.02t} \tag{$*$}$$

Per capita production is $x(t) = X(t)/N(t) = 0.2K(t)/50e^{0.03t} = 2e^{0.01t} - 1.2e^{-0.01t}$. ▮

The solution procedure for the general linear differential equation (9.9.2) is somewhat more complicated, and again we refer the interested reader to FMEA for a detailed treatment. For the moment, notice that if x evolves according to Eq. (9.9.2), and we have a function $A(t)$ such that $A'(t) = a(t)$, we can apply the same trick as before: by multiplying both sides of (9.9.2) by $e^{A(t)}$, we get

$$\dot{x}e^{A(t)} + a(t)xe^{A(t)} = b(t)e^{A(t)}$$

Since the left-hand side of this equation is the derivative of $xe^{A(t)}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(xe^{A(t)}) = b(t)e^{A(t)}$$

Hence $xe^{A(t)} = C + \int b(t)e^{A(t)}\,\mathrm{d}t$, implying that

$$x = Ce^{-A(t)} + e^{-A(t)}\int e^{A(t)}b(t)\,\mathrm{d}t$$

which is a generalization of Eq. (9.9.5).

EXERCISES FOR SECTION 9.9

1. Solve the equation $x^4\dot{x} = 1 - t$. Find the integral curve through $(t, x) = (1, 1)$.

SM 2. Solve the following differential equations:

(a) $\dot{x} = e^{2t}/x^2$ (b) $\dot{x} = e^{-t+x}$ (c) $\dot{x} - 3x = 18$

(d) $\dot{x} = (1+t)^6/x^6$ (e) $\dot{x} - 2x = -t$ (f) $\dot{x} + 3x = te^{t^2-3t}$

3. Suppose that $y = \alpha k e^{\beta t}$ denotes production as a function of capital k, where the factor $e^{\beta t}$ is due to technical progress. Suppose that a constant fraction $s \in (0, 1)$ is saved, and that capital accumulation is equal to savings, so that we have the separable differential equation

$$\dot{k} = s\alpha k e^{\beta t}, \quad k(0) = k_0$$

The constants α, β, and k_0 are positive. Find the solution.

4. Suppose $Y = Y(t)$ is GDP, $C(t)$ is consumption, and $\bar{I}$ is investment, which is constant. Suppose $\dot{Y} = \alpha(C + \bar{I} - Y)$ and $C = aY + b$, where a, b, and α are positive constants with $a < 1$.

(a) Derive a differential equation for Y.

(b) Find its solution when $Y(0) = Y_0$ is given. What happens to $Y(t)$ as $t \to \infty$?

SM 5. In a growth model, production Q is a function of capital, K, and labour, L. Suppose that: (i) $\dot{K} = \gamma Q$; (ii) $Q = K^\alpha L$; and (iii) $\dot{L}/L = \beta$. Assuming that $L(0) = L_0$, $\beta \neq 0$ and $\alpha \in (0, 1)$, derive a differential equation for K and solve this equation when $K(0) = K_0$.

6. Find $x(t)$, when $\mathrm{El}_t\, x(t) = a$ for all t, where $\mathrm{El}_t\, x(t)$ was introduced in Section 7.7 to denote the elasticity of $x(t)$ w.r.t. t. Assume that both t and x are positive and that a is a constant.

REVIEW EXERCISES

1. Find the following integrals:

(a) $\int (-16)\, dx$ (b) $\int 5^5\, dx$ (c) $\int (3 - y)\, dy$ (d) $\int (r - 4r^{1/4})\, dr$

(e) $\int x^8\, dx$ (f) $\int x^2\sqrt{x}\, dx$ (g) $\int \frac{1}{p^5}\, dp$ (h) $\int (x^3 + x)\, dx$

2. Find the following integrals:

(a) $\int 2e^{2x}\, dx$ (b) $\int (x - 5e^{\frac{2}{5}x})\, dx$ (c) $\int (e^{-3x} + e^{3x})\, dx$ (d) $\int \frac{2}{x+5}\, dx$

3. Evaluate the following integrals:

(a) $\int_0^{12} 50\, dx$ (b) $\int_0^2 (x - \frac{1}{2}x^2)\, dx$ (c) $\int_{-3}^{3} (u+1)^2\, du$

(d) $\int_1^5 \frac{2}{z}\, dz$ (e) $\int_2^{12} \frac{3\, dt}{t+4}\, dt$ (f) $\int_0^4 v\sqrt{v^2 + 9}\, dv$

SM **4.** Find the following integrals:

(a) $\displaystyle\int_1^{\infty} \frac{5}{x^5}\,\mathrm{d}x$ (b) $\displaystyle\int_0^1 x^3(1+x^4)^4\mathrm{d}x$ (c) $\displaystyle\int_0^{\infty} \frac{-5t}{e^t}\,\mathrm{d}t$ (d) $\displaystyle\int_1^e (\ln x)^2\,\mathrm{d}x$

(e) $\displaystyle\int_0^2 x^2\sqrt{x^3+1}\,\mathrm{d}x$ (f) $\displaystyle\int_{-\infty}^0 \frac{e^{3z}}{e^{3z}+5}\,\mathrm{d}z$ (g) $\displaystyle\int_{1/2}^{e/2} x^3\ln(2x)\,\mathrm{d}x$ (h) $\displaystyle\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$

SM **5.** Find the following integrals:

(a) $\displaystyle\int_0^{25} \frac{1}{9+\sqrt{x}}\,\mathrm{d}x$ (b) $\displaystyle\int_2^7 t\sqrt{t+2}\,\mathrm{d}t$ (c) $\displaystyle\int_0^1 57x^2\sqrt[3]{19x^3+8}\,\mathrm{d}x$

6. Find $F'(x)$ if: (a) $F(x)=\displaystyle\int_4^x\left(\sqrt{u}+\frac{x}{\sqrt{u}}\right)\mathrm{d}u$; (b) $F(x)=\displaystyle\int_{\sqrt{x}}^x \ln u\,\mathrm{d}u$.

7. With $C(Y)$ as the consumption function, suppose the marginal propensity to consume is $C'(Y)=0.69$, for all Y. Find $C(Y)$, if $C(0)=1000$.

8. In manufacturing a product, the marginal cost of producing x units is $C'(x)=\alpha e^{\beta x}+\gamma$, with $\beta\neq 0$, whereas fixed costs are C_0. Find the total cost function $C(x)$.

9. Suppose f and g are continuous functions on $[-1,3]$ and that $\displaystyle\int_{-1}^3 (f(x)+g(x))\,\mathrm{d}x=6$ and $\displaystyle\int_{-1}^3 (3f(x)+4g(x))\,\mathrm{d}x=9$. Find $I=\displaystyle\int_{-1}^3 (f(x)+g(x))\,\mathrm{d}x$.

SM **10.** For the following two cases, find the equilibrium price and quantity and calculate the consumer and producer surplus when the inverse demand curve is $f(Q)$ and the inverse supply curve is $g(Q)$:

(a) $f(Q)=100-0.05Q$ and $g(Q)=10+0.1Q$.

(b) $f(Q)=\dfrac{50}{Q+5}$ and $g(Q)=4.5+0.1Q$.

SM **11.** Define f for $t>0$ by $f(t)=4(\ln t)^2/t$.

(a) Find $f'(t)$ and $f''(t)$.

(b) Find possible local extreme points, and sketch the graph of f.

(c) Calculate the area below the graph of f over the interval $[1,e^2]$.

12. Solve the following differential equations:

(a) $\dot{x}=-3x$ (b) $\dot{x}+4x=12$ (c) $\dot{x}-3x=12x^2$

(d) $5\dot{x}=-x$ (e) $3\dot{x}+6x=10$ (f) $\dot{x}-\frac{1}{2}x=x^2$

SM **13.** Solve the following differential equations:

(a) $\dot{x} = tx^2$ (b) $2\dot{x} + 3x = -15$ (c) $\dot{x} - 3x = 30$

(d) $\dot{x} + 5x = 10t$ (e) $\dot{x} + \frac{1}{2}x = e^t$ (f) $\dot{x} + 3x = t^2$

14. Let $V(x)$ denote the number of litres of fuel left in an aircraft's fuel tank if it has flown x km. Suppose that $V(x)$ satisfies the following differential equation: $V'(x) = -aV(x) - b$. Here, the fuel consumption per km is a constant $b > 0$. The term $-aV(x)$, with $a > 0$, is due to the weight of the fuel.

(a) Find the solution of the equation with $V(0) = V_0$.

(b) How many km, x^*, can the plane fly if it takes off with V_0 litres in its tank?

(c) What is the minimum number of litres, V_m, needed at the outset if the plane is to fly $\hat{x}$ km?

(d) Let $b = 8$, $a = 0.001$, $V_0 = 12\,000$, and $\hat{x} = 1200$. Find x^* and V_m in this case.

15. As discussed in Section 9.4, assume that a population of n individuals has an income density function $f(r) = (1/m)e^{-r/m}$ for r in $[0, \infty)$, where m is a positive constant.

(a) Show that m is the mean income.

(b) Suppose the demand function is $D(p, r) = ar - bp$. Compute the total demand $x(p)$ when the income distribution is as above.

10 TOPICS IN FINANCIAL MATHEMATICS

I can calculate the motions of heavenly bodies, but not the madness of people.
—Isaac Newton[1]

This chapter treats some basic topics in the mathematics of finance. The main concern is how the values of investments and loans at different times are affected by interest rates. Sections 2.2 and 4.9 have already discussed some elementary calculations involving interest rates. This chapter goes a step further and considers different interest periods. It also discusses in turn effective rates of interest, continuously compounded interest, present values of future claims, annuities, mortgages, and the internal rate of return on investment projects. The calculations involve the summation formula for geometric series, which we therefore derive.

In the last section we give a brief introduction to difference equations.

10.1 Interest Periods and Effective Rates

In advertisements that offer bank loans or savings accounts, interest is usually quoted as an *annual rate*, also called a *nominal rate*, even if the actual interest period is different. This *interest period* is the time that elapses between successive dates when interest is added to the account. For some bank accounts the interest period is one year, but it has become increasingly common for financial institutions to offer other interest schemes. For instance, many US banks used to add interest daily, some others at least monthly. If a bank offers 9% annual rate of interest with interest payments each month, then $\frac{1}{12} \times 9\% = 0.75\%$ of the capital accrues at the end of each month. The annual rate must be divided by the number of interest periods to get the *periodic rate*—that is, the interest per period.

Suppose a principal (or capital) of S_0 yields interest at the rate $p\%$ per period, for example one year. As explained in Section 2.2, after t periods it will have increased to the amount $S(t) = S_0\,(1+r)^t$, where $r = p/100$, which is $p\%$. *Each period the principal increases by the factor* $1 + r$.

[1] Attributed, circa 1720. It is claimed that he said it after losing much of his wealth in the South Sea bubble.

The formula assumes that the interest is added to the principal at the end of each period. Suppose that the annual interest rate is $p\%$, but that interest is paid semiannually—that is, twice a year—at the rate $\frac{1}{2}p/\%$. Then the principal after half a year will have increased to

$$S_0 + S_0 \frac{p/2}{100} = S_0 \left(1 + \frac{r}{2}\right)$$

Each half year the principal increases by the factor $1 + r/2$. After two periods, namely one year, it will have increased to $S_0(1 + r/2)^2$, and after t years to

$$S_0 \left(1 + \frac{r}{2}\right)^{2t}$$

Note that a semiannual interest payment at the rate $\frac{1}{2}r$ is better for a lender than an annual interest payment at the rate r. This follows from the fact that $(1 + r/2)^2 = 1 + r + r^2/4 > 1 + r$.

More generally, suppose that interest at the rate $p/n\%$ is added to the principal at n different times distributed evenly over the year. For example, $n = 4$ if interest is added quarterly, $n = 12$ if it is added monthly, etc. Then, the *principal will be multiplied by a factor* $(1 + r/n)^n$ *each year*. After t years, the principal will have increased to

$$S_0 \left(1 + \frac{r}{n}\right)^{nt} \tag{10.1.1}$$

The greater is n, the faster interest accrues to the lender, as illustrated by Exercise 10.2.6.

EXAMPLE 10.1.1 A deposit of £5 000 is put into an account earning interest at the annual rate of 9%, with interest paid quarterly. How much will there be in the account after eight years?

Solution: The periodic rate r/n is $0.09/4 = 0.0225$ and the number of periods $n \times t$ is $4 \cdot 8 = 32$. So formula (10.1.1) gives:

$$5000(1 + 0.0225)^{32} \approx 10\,190.52$$

EXAMPLE 10.1.2 How long will it take for the £5 000 in Example 10.1.1, with annual interest rate 9% and interest paid quarterly, to increase to £15 000?

Solution: After t quarterly payments the account will grow to $5000(1 + 0.0225)^t$. So

$$5000(1 + 0.0225)^t = 15\,000$$

or $1.0225^t = 3$. To find t we take the natural logarithm of each side: because $\ln a^p = p \ln a$, we get $t \ln 1.0225 = \ln 3$, so

$$t = \frac{\ln 3}{\ln 1.0225} \approx 49.37$$

Thus it takes approximately 49.37 quarterly periods, that is approximately 12 years and four months, before the account has increased to £15 000.

Effective Rate of Interest

A consumer who needs a loan may receive different offers from several competing financial institutions. It is therefore important to know how to compare various offers. The concept of *effective interest rate* is often used in making such comparisons.

Consider a loan which implies an annual interest rate of 9% with interest at the rate $9/12 = 0.75\%$ added 12 times a year. If no interest is paid in the meantime, after one year an initial principal of S_0 will have grown to a debt of $S_0(1 + 0.09/12)^{12} \approx S_0 \cdot 1.094$. In fact, as long as no interest is paid, the debt will grow at a constant proportional rate that is (approximately) 9.4% per year. For this reason, we call 9.4% the effective yearly rate. More generally:

EFFECTIVE YEARLY RATE

When interest is added n times during the year at the rate r/n per period, then the effective yearly rate, R, is defined as

$$R = \left(1 + \frac{r}{n}\right)^n - 1 \tag{10.1.2}$$

The effective yearly rate is independent of the amount S_0. For a given value of $r > 0$, it is increasing in n, as implied by the result of Exercise 10.2.6.

EXAMPLE 10.1.3 What is the effective yearly rate R corresponding to an annual interest rate of 9% with interest compounded: (i) each quarter; (ii) each month?

Solution:

(a) Applying formula (10.1.2) with $r = 0.09$ and $n = 4$, the effective rate is

$$R = \left(1 + \frac{0.09}{4}\right)^4 - 1 = (1 + 0.0225)^4 - 1 \approx 0.0931$$

or 9.31%.

(b) In this case $r = 0.09$ and $n = 12$, so the effective rate is

$$R = \left(1 + \frac{0.09}{12}\right)^{12} - 1 = (1 + 0.0075)^{12} - 1 \approx 0.0938$$

or 9.38%.

A typical case in which we can use the effective rate of interest to compare different financial offers is the following.

EXAMPLE 10.1.4 When investing in a savings account, which of the following offers is better: (i) 5.9% with interest paid quarterly; or (ii) 6% with interest paid twice a year?

Solution: According to formula (10.1.2), the effective rates for the two offers are

$$R = (1 + 0.059/4)^4 - 1 \approx 0.0603$$

or

$$R = (1 + 0.06/2)^2 - 1 = 0.0609$$

The second offer is, therefore, better for the saver.

In many countries there is an official legal definition of effective interest rate which takes into account different forms of fixed or "closing" costs incurred when initiating a loan. The *effective rate of interest* is then defined as the rate which implies that the combined present value of all the costs is equal to the size of the loan. This is the internal rate of return, as defined in Section 10.7; present values are discussed in Section 10.3.

EXERCISES FOR SECTION 10.1

1. (a) What will be the size of an account after five years, if \$8 000 is invested at an annual interest rate of 5% compounded: (i) monthly; or (ii) daily (with 365 days in a year)?

   (b) How long does it take for the investment to double with monthly compounding?

2. An investment of \$5 000 earns interest at 3% per year.

   (a) What will this amount have grown to after ten years?

   (b) How long does it take for the investment to triple?

3. What annual percentage rate of growth is needed for a country's GDP to become 100 times as large after 100 years? (Use the approximation $\sqrt[100]{100} \approx 1.047$.)

4. An amount of €2 000 is invested at 7% per year.

   (a) What is the balance in the account after (i) two years; and (ii) ten years?

   (b) How long does it take, approximately, for the balance to reach €6 000?

5. Calculate the effective yearly interest if the nominal rate is 17% and interest is added: (a) semiannually; (b) quarterly; or (c) monthly.

6. Which terms are preferable for a borrower: (i) an annual interest rate of 21.5%, with interest paid yearly; or (ii) an annual interest rate of 20%, with interest paid quarterly?

7. A sum of \$12 000 is invested at 4% annual interest.

   (a) What will this amount have grown to after 15 years?

   (b) How much should you have deposited in a bank account five years ago in order to have \$50 000 today, given that the interest rate has been 5% per year over the period?

8. A credit card is offered with interest on the outstanding balance charged at 2% per month. What is the effective annual rate of interest?

9. What is the nominal yearly interest rate if the effective yearly rate is 28% and interest is compounded quarterly?

10.2 Continuous Compounding

We saw in the previous section that if interest at the rate r/n is added to the principal S_0 at n different times during the year, the principal will be multiplied by a factor $(1+r/n)^n$ each year. After t years, the principal will have increased to $S_0(1+r/n)^{nt}$. In practice, there is a limit to how frequently interest can be added to an account. However, let us examine what happens to the expression as the annual frequency n tends to infinity. We put $r/n = 1/m$. Then $n = mr$ and so

$$S_0\left(1+\frac{r}{n}\right)^{nt} = S_0\left(1+\frac{1}{m}\right)^{mrt} = S_0\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt} \tag{10.2.1}$$

With r fixed, as $n \to \infty$, so $m = n/r \to \infty$, and according to Example 7.11.2, we have $(1+1/m)^m \to e$. Hence, the expression in (10.2.1) approaches S_0e^{rt} as n tends to infinity, implying that interest is compounded more and more frequently. In the limit, we talk about *continuous compounding* of interest:

CONTINUOUS COMPOUNDING OF INTEREST

> If the annual interest rate is r and there is continuous compounding of interest, after t years a principal of S_0 will have increased to
>
> $$S(t) = S_0e^{rt} \tag{10.2.2}$$

EXAMPLE 10.2.1 Suppose the sum of £5 000 is invested in an account earning interest at an annual rate of 9%. What is the balance after eight years, if interest is compounded continuously?

Solution: Using formula (10.2.2) with $r = 9/100 = 0.09$, we see that the balance is

$$5000e^{0.09\cdot 8} = 5000e^{0.72} \approx 10\,272.17$$

This is more than in the case of quarterly compounding studied in Example 10.1.1.

If $S(t) = S_0e^{rt}$ as in (10.2.2), then differentiating by applying formula (6.10.2) gives $S'(t) = S_0re^{rt} = rS(t)$. It follows that $S'(t)/S(t) = r$. Using the terminology introduced in Section 6.4:

With continuous compounding of interest at the rate r, the principal increases at the constant relative rate r, so that $S'(t)/S(t) = r$.

From Eq. (10.2.2), we infer that $S(1) = S_0 e^r$, so that the principal increases by the factor e^r during the first year. In general, $S(t+1) = S_0 e^{r(t+1)} = S_0 e^{rt} e^r = S(t)e^r$. Hence:

With continuous compounding of interest at the rate r, the principal increases each year by a fixed factor e^r.

Comparing Different Interest Periods

Given any fixed interest rate of $p\% = 100r$ per year, continuous compounding of interest is best for the lender—see Exercise 6. For comparatively low interest rates, however, the difference between annual and continuous compounding of interest is quite small, when the number of years of compounding is relatively small.

EXAMPLE 10.2.2 Find the amount K to which one dollar increases in the course of a year when the interest rate is 8% per year and interest is added: (a) yearly; (b) semiannually; or (c) continuously.

Solution: In this case $r = 8/100 = 0.08$, and we obtain:

(a) $K = 1.08$

(b) $K = (1 + 0.08/2)^2 = 1.0816$

(c) $K = e^{0.08} \approx 1.08329$

If we increase either the interest rate or the number of years over which interest accumulates, then the difference between yearly and continuous compounding of interest increases.

In the previous section the effective yearly interest was defined by the formula $(1 + r/n)^n - 1$, when interest is compounded n times a year with rate r/n per period. Letting n approach infinity in this formula, we see that the expression approaches

$$e^r - 1 \tag{10.2.3}$$

This is called the *effective interest* rate with continuous compounding at the annual rate r.

EXERCISES FOR SECTION 10.2

1. (a) How much does \$8 000 grow to after five years if the annual interest rate is 5%, with continuous compounding?

   (b) How long does it take before the initial amount has doubled?

2. An amount \$1 000 earns interest at 5% per year. What will this amount have grown to after: (a) ten years, and (b) 50 years, when interest is compounded: (i) monthly, or (b) continuously?

3. (a) Find the effective rate corresponding to an annual rate of 10% compounded continuously.

   (b) What is the maximum amount of compound interest that can be earned at an annual rate of 10%?

4. The value v_0 of a new car depreciates continuously at the annual rate of 10%, implying that its value after t years is $v(t) = v_0 e^{-\delta t}$ where $\delta = 0.1$. How many years does it take for the car to lose 90% of its original value?

5. The value of a machine depreciates continuously at the annual rate of 6%. How many years will it take for the value of the machine to halve?

SM 6. [HARDER] The argument we used to justify Eq. (10.2.2) shows, in particular, that $(1 + r/n)^n \to e^r$ as $n \to \infty$. For each fixed $r > 0$ we claim that $(1 + r/n)^n$ is strictly increasing in n. This implies that $(1 + r/n)^n < e^r$, for $n = 1, 2, \ldots$. In words: *This shows that continuous compounding at interest rate r is more profitable for the lender than interest payments n times a year at interest rate r/n.*

To confirm these results, given any $r > 0$, define the function $g(x) = (1 + r/x)^x$ for all $x > 0$. Use logarithmic differentiation to show that

$$g'(x) = g(x)\left[\ln(1 + r/x) - \frac{r/x}{1 + r/x}\right]$$

Next, put $h(u) = \ln(1 + u) - u/(1 + u)$. Then $h(0) = 0$. Show that $h'(u) > 0$ for $u > 0$, and hence $g'(x) > 0$ for all $x > 0$. What conclusion can you draw?

10.3 Present Value

The sum of \$1 000 in your hand today is worth more than \$1 000 to be received at some future date. One important reason is that you can invest the \$1 000 and hope to earn some interest or other positive return.[2] If the interest rate is 11% per year, then after one year the original \$1 000 will have grown to the amount $1\,000(1 + 11/100) = 1\,110$, and after six years, it will have grown to $1000(1 + 11/100)^6 = 1000 \cdot (1.11)^6 \approx 1870$. This shows that, at the interest rate 11% per year, \$1 000 now has the same value as \$1 110 next year, or \$1 870 in six years' time. Accordingly, if the amount \$1 110 is due for payment one year from now and the interest rate is 11% per year, then the *present value* of this amount is \$1 000. Because \$1 000 is less than \$1 110, we often speak of \$1 000 as the *present discounted value* (or PDV) of \$1 110 next year. The ratio $1000/1110 = 1/(1 + 11/100) \approx 0.9009$ is called the (annual) *discount factor*, whose reciprocal, 1.11, is one plus the *discount rate*, making the discount rate equal to the interest rate of 11%.

Similarly, if the interest rate is 11% per year, then the PDV of \$1 870 due six years from now is \$1 000. Again, the ratio $1000/1870 \approx 0.53$ is called the *discount factor*, this time for money due in six years' time.

Suppose that an amount K is due for payment t years after the present date. What is the *present value* when the interest rate is $p\%$ per year? Equivalently, how much must be deposited today earning $p\%$ annual interest in order to have the amount K after t years?

[2] If prices are expected to increase, another reason for preferring \$1 000 today is inflation, because \$1 000 to be paid at some future date will buy less then than \$1 000 does today.

If interest is paid annually, an amount A will have increased to $A(1+p/100)^t$ after t years, so that we need $A(1+p/100)^t = K$. Thus, $A = K(1+p/100)^{-t} = K(1+r)^{-t}$, where $r = p/100$. Here the annual discount factor is $(1+r)^{-1}$, and $(1+r)^{-t}$ is the discount factor appropriate for t years.

If interest is compounded continuously, then the amount A will have increased to Ae^{rt} after t years. Hence, $Ae^{rt} = K$, or $A = Ke^{-rt}$. Here e^{-rt} is the discount factor. To summarize:

PRESENT DISCOUNTED VALUE

If the interest or discount rate is $p\%$ per year and $r = p/100$, an amount K that is payable in t years has the *present discounted value*, or PDV:

(i) with annual interest payments,

$$K(1+r)^{-t} \tag{10.3.1}$$

(ii) with continuous compounding of interest,

$$Ke^{-rt} \tag{10.3.2}$$

EXAMPLE 10.3.1 Find the present value of \$100 000 which is due for payment after 15 years, if the interest rate is 6% per year, compounded (a) annually, or (b) continuously.

Solution:

(a) According to Eq. (10.3.1), the present value is $100\,00(1+0.06)^{-15} \approx 41\,726.51$.

(b) According to Eq. (10.3.2), the dollar PDV is $100\,000e^{-0.06\cdot 15} = 100\,000e^{-0.9} \approx 40\,656.97$.

As expected, the present value with continuous compounding is the smaller, because capital increases most rapidly with continuous compounding of interest.

EXAMPLE 10.3.2 **(When to Harvest a Tree?)** Consider a tree that is planted at time $t = 0$, and let $P(t)$ be its current market value at time t, where $P(t)$ is differentiable with $P(t) > 0$ for all $t \geq 0$. Assume that the interest rate is $100r\%$ per year, and assume continuous compounding of interest.

(a) At what time t^* should this tree be cut down in order to maximize its present value?

(b) The optimal cutting time t^* depends on the interest rate r. Find dt^*/dr.

Solution:

(a) The present value is $f(t) = P(t)e^{-rt}$, whose derivative is

$$f'(t) = P'(t)e^{-rt} + P(t)(-r)e^{-rt} = e^{-rt}\left[P'(t) - rP(t)\right] \tag{$*$}$$

A necessary condition for $t^* > 0$ to maximize $f(t)$ is that $f'(t^*) = 0$. This occurs when

$$P'(t^*) = rP(t^*) \tag{$**$}$$

The tree, therefore, should be cut down at a time t^* when the relative rate of increase in the value of the tree is equal to the interest rate. Of course, some conditions have to be placed on f in order for t^* to be a maximum point. It suffices to have $P'(t) \geq rP(t)$ for $t < t^*$ and $P'(t) \leq rP(t)$ for $t > t^*$.

(b) Differentiating $(**)$ w.r.t. r yields

$$P''(t^*)\frac{\mathrm{d}t^*}{\mathrm{d}r} = P(t^*) + rP'(t^*)\frac{\mathrm{d}t^*}{\mathrm{d}r}$$

Solving for $\mathrm{d}t^*/\mathrm{d}r$,

$$\frac{\mathrm{d}t^*}{\mathrm{d}r} = \frac{P(t^*)}{P''(t^*) - rP'(t^*)} \tag{***}$$

Differentiating $(*)$ w.r.t. t yields

$$f''(t) = P''(t)e^{-rt} - rP'(t)e^{-rt} - P'(t)re^{-rt} + r^2P(t)e^{-rt}$$

where we used $(**)$ to derive the equality. But then $(***)$ implies that the second-order condition $f''(t^*) < 0$ is satisfied if, and only if,

$$e^{-rt^*}[P''(t^*) - 2rP'(t^*) + r^2P(t^*)] = e^{-rt^*}[P''(t^*) - rP'(t^*)] < 0$$

in which case $\mathrm{d}t^*/\mathrm{d}r < 0$. Thus, the optimal growing time shortens as r increases, which makes the foresters more impatient. In particular, given any $r > 0$, the optimal t^* is less than the time that maximizes current market value $P(t)$, which is optimal only if $r = 0$.

We did not consider how the land the tree grows on may be used after harvesting—for example, by planting a new tree. This generalization is studied in Exercise 10.4.8.

EXERCISES FOR SECTION 10.3

1. Find the present value of £350 000 which is due after ten years if the interest rate is 8% per year: (a) compounded annually, or (b) compounded continuously.

2. Find the present value of €50 000 which is due after five years when the interest rate is 5.75% per year, paid: (a) annually, or (b) continuously.

3. With reference to Example 10.3.2, consider the case where $f(t) = (t + 5)^2e^{-0.05t}$ for all $t \geq 0$.

   (a) Find the value of t that maximizes $f(t)$, and study the sign variation of $f'(t)$.

   (b) Find $\lim_{t\to\infty} f(t)$ and draw the graph of f.

10.4 Geometric Series

Geometric series have many applications in economics and finance. Here we shall use them to calculate annuities and mortgage payments.

EXAMPLE 10.4.1 This year a firm has an annual revenue of \$100 million that it expects to increase by 16% per year, throughout the next decade. How large is its expected revenue in the tenth year, and what is the total revenue expected over the whole decade?

Solution: The expected revenue in the second year, in millions of dollars, amounts to $100(1 + 16/100) = 100 \cdot 1.16$, and in the third year it is $100 \cdot (1.16)^2$. In the tenth year, the expected revenue is $100 \cdot (1.16)^9$. The total revenue expected during the decade is, thus,

$$100 + 100 \cdot 1.16 + 100 \cdot (1.16)^2 + \cdots + 100 \cdot (1.16)^9$$

If we use a calculator to add the ten different numbers, we find that the sum is approximately \$2 132 million.

Finding the sum in Example 10.4.1 by adding ten different numbers on a calculator was very tedious. When there are infinitely many terms, it is obviously impossible. There is an easier method, as we now explain.

Consider the n numbers $a, ak, ak^2, \ldots, ak^{n-1}$. Each term is obtained by multiplying its predecessor by a constant k. We wish to find the sum

$$s_n = a + ak + ak^2 + \cdots + ak^{n-2} + ak^{n-1} \tag{10.4.1}$$

of these numbers. We call this sum a finite *geometric series with quotient* k. The sum in Example 10.4.1 occurs in the case when $a = 100$, $k = 1.16$, and $n = 10$.

To find the sum s_n of the series, we use a trick. First multiply both sides of Eq. (10.4.1) by k, to obtain

$$ks_n = ak + ak^2 + ak^3 + \cdots + ak^{n-1} + ak^n$$

Subtracting (10.4.1) from this equation yields

$$ks_n - s_n = ak^n - a \tag{10.4.2}$$

because all the other $n - 1$ terms cancel. This is the point of the trick: First note that if $k = 1$, then all terms in (10.4.1) equal a, so the sum must be $s_n = an$. Otherwise, for $k \neq 1$, Eq. (10.4.2) implies that

$$s_n = a\frac{k^n - 1}{k - 1}$$

In conclusion:

SUMMING A FINITE GEOMETRIC SERIES

Provided that $k \neq 1$

$$a + ak + ak^2 + \cdots + ak^{n-1} = a \cdot \frac{k^n - 1}{k - 1} \tag{10.4.3}$$

EXAMPLE 10.4.2 For the sum in Example 10.4.1, we have $a = 100$, $k = 1.16$, and $n = 10$. Hence, Eq. (10.4.3) yields

$$100 + 100 \cdot 1.16 + \cdots + 100 \cdot (1.16)^9 = 100\,\frac{(1.16)^{10} - 1}{1.16 - 1}$$

Now it takes many fewer operations on the calculator than in Example 10.4.1 to show that the sum is about 2132.

Infinite Geometric Series

Consider the infinite sequence of numbers

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$$

Each term in the sequence is formed by halving its predecessor, so that the n-th term is $1/2^{n-1}$. The sum of the n first terms is a finite geometric series with quotient $k = 1/2$ and the first term $a = 1$. Hence, (10.4.3) gives

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}} \tag{$*$}$$

We now ask what is meant by the "infinite sum"

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots \tag{$**$}$$

Because all the terms are positive, and there are infinitely many of them, you might be inclined to think that the sum must be infinitely large. However, if we look at formula $(*)$, we see that the sum of the n first terms is equal to $2 - 1/2^{n-1}$. This number is never larger than 2, irrespective of our choice of n. As n increases, the term $1/2^{n-1}$ comes closer and closer to 0, and the sum in $(*)$ tends to 2 as limit. This makes it natural to *define* the infinite sum in $(**)$ as the number 2.

In general, we ask what meaning can be given to the "infinite sum"

$$a + ak + ak^2 + \cdots + ak^{n-1} + \cdots \tag{10.4.4}$$

We use the same idea as in $(**)$, and consider the sum s_n of the first n terms in (10.4.4). According to Eq. (10.4.3), when $k \neq 1$,

$$s_n = a\,\frac{1 - k^n}{1 - k}$$

What happens to this expression as n tends to infinity? The answer evidently depends on k^n, because only this term depends on n. In fact, k^n tends to 0 if $-1 < k < 1$, whereas k^n does not tend to any limit if $k > 1$ or $k \leq -1$.[3] It follows that if $|k| < 1$, then the sum s_n of the n

[3] If you are not yet convinced by this claim, study the cases $k = -2$, $k = -1$, $k = -1/2$, $k = 1/2$, and $k = 2$.

first terms in (10.4.4) will tend to the limit $a/(1-k)$, as n tends to infinity. In this case, we let the limit of (10.4.4) *define* the infinite sum, and say that the infinite series in (10.4.4) *converges*. To summarize:

SUMMING AN INFINITE GEOMETRIC SERIES

Provided that $|k| < 1$

$$a + ak + ak^2 + \cdots + ak^{n-1} + \cdots = \frac{a}{1-k} \tag{10.4.5}$$

If we extend to infinite sums the summation notation that was introduced in Section 2.8, we can write Eq. (10.4.5) as

$$\sum_{n=1}^{\infty} ak^{n-1} = \frac{a}{1-k}$$

assuming, of course, that $|k| < 1$.

If $|k| \geq 1$, we say that the infinite series (10.4.4) *diverges*. A divergent series has no finite sum. Divergence is obvious if $|k| > 1$. When $k = 1$, then $s_n = na$, which tends to $+\infty$ if $a > 0$ or to $-\infty$ if $a < 0$. When $k = -1$, then s_n is a when n is odd, but 0 when n is even; again there is no limit as $n \to \infty$, if $a \neq 0$.

EXAMPLE 10.4.3 Find the sum of the infinite series

$$1 + 0.25 + (0.25)^2 + (0.25)^3 + (0.25)^4 + \cdots$$

Solution: According to Eq. (10.4.5) with $a = 1$ and $k = 0.25$, we have

$$1 + 0.25 + (0.25)^2 + (0.25)^3 + (0.25)^4 + \cdots = \frac{1}{1-0.25} = \frac{1}{0.75} = \frac{4}{3}$$

EXAMPLE 10.4.4 A rough estimate of the total oil and gas reserves under the Norwegian continental shelf at the beginning of 1999 was $13 \cdot 10^9 = 13$ billion tons of oil equivalent. Output that year was approximately $250 \cdot 10^6 = 250$ million tons.

(a) When will the reserves be exhausted if output is kept at the same constant level?

(b) Suppose that output is reduced each year by 2% per year beginning in 1999. How long will the reserves last in this case?

Solution:

(a) The number of years for which the reserves will last is given by

$$\frac{13 \cdot 10^9}{250 \cdot 10^6} = 52$$

That is, the reserves will be exhausted around the year 2051.

(b) In 1999, output was $a = 250 \cdot 10^6$. In 2000, it would be $a - 2a/100 = a \cdot 0.98$. In 2001, it becomes $a \cdot 0.98^2$, and so on. If this continues forever, the total amount extracted will be

$$a + a \cdot 0.98 + a \cdot (0.98)^2 + \cdots + a \cdot (0.98)^{n-1} + \cdots$$

This geometric series has quotient $k = 0.98$. According to (10.4.5), the sum is

$$s = \frac{a}{1 - 0.98} = 50a$$

Since $a = 250 \cdot 10^6$, this gives $s = 50 \cdot 250 \cdot 10^6 = 12.5 \cdot 10^9$, which is less than $13 \cdot 10^9$. The reserves will last forever, therefore, leaving $0.5 \cdot 10^9 = 500$ million tons which will never be extracted.

General Series[4]

We briefly consider general infinite series that are not necessarily geometric,

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{10.4.6}$$

What does it mean to say that this infinite series converges? By analogy with the definition for geometric series, we form the "partial" sum s_n of the n first terms:

$$s_n = a_1 + a_2 + \cdots + a_n \tag{10.4.7}$$

In particular, $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, and so on. As n increases, these partial sums include more and more terms of the series. Hence, if s_n tends toward a limit s as n tends to ∞, it is reasonable to consider s as the sum of *all* the terms in the series. Then we say that the infinite series is *convergent* with sum s. If s_n does not tend to a finite limit as n tends to infinity, we say that the series is *divergent*. The series then has no sum.[5]

For geometric series, it was easy to determine when there was convergence, because we found a simple expression for s_n. Usually, it will not be possible to find such a simple formula for the sum of the first n terms in a given series, so it can be very difficult to determine whether it converges or not. Nevertheless, there are several so-called *convergence* and *divergence criteria* that will give the answer in many cases. These criteria are seldom used directly in economics.

Let us make a general observation: If the series (10.4.6) converges, then the n-th term must tend to 0 as n tends to infinity. The argument is simple: If the series is convergent, then s_n in Eq. (10.4.7) will tend to a limit s as n tends to infinity. Now $a_n = s_n - s_{n-1}$, and by the definition of convergence, s_{n-1} will also tend to s as n tends to infinity. It follows that $a_n = s_n - s_{n-1}$ must tend to $s - s = 0$ as n tends to infinity. Expressed briefly,

$$a_1 + a_2 + \cdots + a_n + \cdots \text{ converges } \Rightarrow \lim_{n\to\infty} a_n = 0 \tag{10.4.8}$$

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[4] This section can be regarded as optional.

[5] As with limits of functions, if $s_n \to \pm\infty$ as $n \to \infty$, this is not regarded as a limit.

Convergence of a_n to 0 is necessary for convergence of the series, but not sufficient. That is, a series may satisfy the condition $\lim_{n\to\infty} a_n = 0$ and yet diverge. This is shown by the following standard example:

EXAMPLE 10.4.5 The series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \tag{10.4.9}$$

is called the *harmonic series*. The n-th term is $1/n$, which tends to 0. But the series is still divergent. To see this, we group the terms together in the following way:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \tag{$*$}$$

Between the first pair of parentheses there are two terms, one greater than $1/4$ and the other equal to $1/4$, so their sum is greater than $2/4 = 1/2$. Between the second pair of parentheses there are four terms, three greater than $1/8$ and the last equal to $1/8$, so their sum is greater than $4/8 = 1/2$. Between the third pair of parentheses there are eight terms, seven greater than $1/16$ and the last equal to $1/16$, so their sum is greater than $8/16 = 1/2$. This pattern repeats itself infinitely often: between the n-th pair of parentheses there will be 2^n terms, of which $2^n - 1$ are greater than 2^{-n-1} whereas the last is equal to 2^{-n-1}, so their sum is greater than $2^n \cdot 2^{-n-1} = 1/2$. Because its sum is larger than that of an infinite number of terms all equal to $\frac{1}{2}$, we conclude that the series in $(*)$ must diverge.[6]

In general:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent } \Leftrightarrow p > 1 \tag{10.4.10}$$

You are asked to prove this in Exercise 11.

EXERCISES FOR SECTION 10.4

1. (a) Find the sum, s_n, of the following finite geometric series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}$$

   (b) When n approaches infinity, what is the limit of s_n?

   (c) Find the sum $\sum_{n=1}^{\infty}(1/3^{n-1})$.

2. Find the sums of the following geometric series:

   (a) $\frac{1}{5} + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \left(\frac{1}{5}\right)^4 + \cdots$

   (b) $0.1 + (0.1)^2 + (0.1)^3 + (0.1)^4 + \cdots$

[6] According to H.H. Goldstine (1977): “The determination of $\sum 1/n$ occupied Leibniz all his life but the solution never came within his grasp.”

(c) $517 + 517(1.1)^{-1} + 517(1.1)^{-2} + 517(1.1)^{-3} + \cdots$

(d) $a + a(1+a)^{-1} + a(1+a)^{-2} + a(1+a)^{-3} + a(1+a)^{-4} + \cdots$, for $a > 0$

(e) $5 + \dfrac{5 \cdot 3}{7} + \dfrac{5 \cdot 3^2}{7^2} + \cdots + \dfrac{5 \cdot 3^{n-1}}{7^{n-1}} + \cdots$

3. Determine whether the following series are geometric, and find the sums of those geometric series that do converge.

(a) $8 + 1 + 1/8 + 1/64 + \cdots$ (b) $-2 + 6 - 18 + 54 - \cdots$

(c) $2^{1/3} + 1 + 2^{-1/3} + 2^{-2/3} + \cdots$ (d) $1 - 1/2 + 1/3 - 1/4 + \cdots$

4. Examine the convergence of the following geometric series, and find their sums when they exist:

(a) $\dfrac{1}{p} + \dfrac{1}{p^2} + \dfrac{1}{p^3} + \cdots$ (b) $x + \sqrt{x} + 1 + \dfrac{1}{\sqrt{x}} + \cdots$ (c) $\sum_{n=1}^{\infty} x^{2n}$

5. Find the sum $\displaystyle\sum_{k=0}^{\infty} b\left(1 + \frac{p}{100}\right)^{-k}$, for $p > 0$.

SM **6.** Total world consumption of iron was approximately $794 \cdot 10^6$ tons in 1971. If consumption had increased by 5% each year and the resources available for mining in 1971 were $249 \cdot 10^9$ tons, how much longer would the world's iron resources have lasted?

7. The world's total consumption of natural gas was 1824 million tons oil equivalent (MTOE) in 1994. The reserves at the end of that year were estimated to be 128 300 MTOE. If consumption had increased by 2% in each of the coming years, and no new sources were ever discovered, how much longer would these reserves have lasted?

SM **8.** Consider Example 10.3.2. Assume that immediately after one tree is felled, a new tree of the same type is planted. If we assume that a new tree is planted at times t, $2t$, $3t$, etc., then the present value of all the trees will be $f(t) = P(t)e^{-rt} + P(t)e^{-2rt} + \cdots$.

(a) Find the sum of this infinite geometric series.

(b) Prove that if $f(t)$ has a maximum for some $t^* > 0$, then $P'(t^*)/P(t^*) = r/(1 - e^{-rt^*})$.

(c) Examine the limit of $P'(t^*)/P(t^*)$ as $r \to 0$.

9. Show that the following series diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} (101/100)^n$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(1+1/n)^n}$

10. Examine the convergence or divergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{100}{101}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.00000001}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1+n}{4n-3}$ (e) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$ (f) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{3}\right)^{1-n}$

SM **11.** Use the results in Example 9.7.2 to prove the equivalence in (10.4.10). (*Hint*: Draw the graph of $f(x) = x^{-p}$ in $[1, \infty)$, and interpret each of the sums $\sum_{n=1}^{\infty} n^{-p}$ and $\sum_{n=2}^{\infty} n^{-p}$, geometrically, as sums of the areas of an infinite number of rectangles.)

10.5 Total Present Value

Suppose that three successive annual payments are to be made, with the amount \$1 000 falling due after one year, then \$1 500 after two years, and \$2 000 after three years. How much must be deposited in an account today in order to have enough savings to cover these three payments, given that the interest rate is 11% per year? We call this amount the *present value* of the three payments.

In order to have \$1 000 after one year, we must deposit an amount x_1 today, such that $x_1(1 + 0.11) = 1000$. That is,

$$x_1 = \frac{1000}{1 + 0.11} = \frac{1000}{1.11}$$

In order to have \$1 500 after two years, we must deposit an amount x_2 today, where $x_2(1 + 0.11)^2 = 1500$, or

$$x_2 = \frac{1500}{(1 + 0.11)^2} = \frac{1500}{(1.11)^2}$$

Finally, to have \$2 000 after three years, we must deposit an amount x_3 today, where $x_3(1 + 0.11)^3 = 2000$, so that

$$x_3 = \frac{2000}{(1 + 0.11)^3} = \frac{2000}{(1.11)^3}$$

So the total present value of the three payments, which is the total amount, A, that must be deposited today in order to cover all three payments, is given by

$$A = \frac{1000}{1.11} + \frac{1500}{(1.11)^2} + \frac{2000}{(1.11)^3}$$

The total is approximately $A \approx 900.90 + 1217.43 + 1462.38 = 3580.71$.

Suppose, in general, that n successive payments $a_1, \ldots, a_n$ are to be made, with a_1 being paid after one year, a_2 after two years, and so on. How much must be deposited into an account today in order to have enough to cover all these future payments, given that the annual interest rate is r? In other words, what is the *present value* of all these payments?

In order to have a_1 after one year, we must deposit $a_1/(1 + r)$ today; to have a_2 after two years we must deposit $a_2/(1 + r)^2$ today; and so on. The total amount P_n that must be deposited today in order to cover all n payments is, therefore,

$$P_n = \frac{a_1}{1 + r} + \frac{a_2}{(1 + r)^2} + \cdots + \frac{a_n}{(1 + r)^n} \tag{10.5.1}$$

Here, P_n is the *present value* of the n instalments.

An *annuity* is a sequence of equal payments made at fixed periods of time over some time span. If $a_1 = a_2 = \cdots = a_n = a$ in Eq. (10.5.1), the equation represents the present value of an annuity. In this case the sum in (10.5.1) is a finite geometric series with n terms. The first term is $a/(1+r)$ and the quotient is $k = (1+r)^{-1}$. According to the summation formula (10.4.3) for a geometric series, the sum is

$$P_n = \frac{a}{(1+r)} \frac{1-(1+r)^{-n}}{1-(1+r)^{-1}} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

where the second equality holds because the denominator of the middle expression reduces to r. Hence, we have the following:

PRESENT VALUE OF AN ANNUITY

The present value of an annuity of a per payment period for n periods, at the rate of interest r per period, where each payment is at the end of the period, is given by

$$P_n = \frac{a}{1+r} + \cdots + \frac{a}{(1+r)^n} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right] \tag{10.5.2}$$

where $r = p/100$.

This sum is illustrated in Fig. 10.5.1.

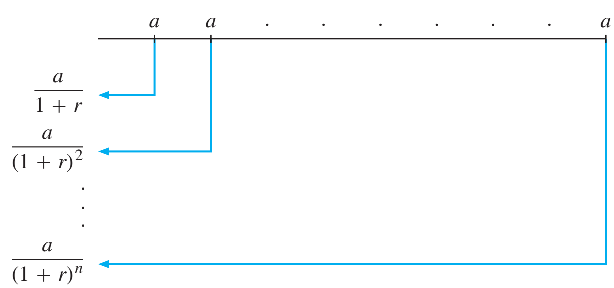

Figure 10.5.1 Present value of an annuity

Formula (10.5.2) gives the present value of n future claims, each one of a, say, dollars. If we want to find how much has accumulated in the account after n periods, immediately after the last deposit, then the *future value* F_n of the annuity is given by:

$$F_n = a + a(1+r) + a(1+r)^2 + \cdots + a(1+r)^{n-1} \tag{$*$}$$

This different sum is illustrated in Fig. 10.5.2.

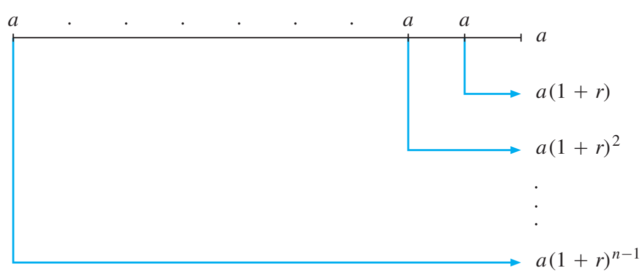

Figure 10.5.2 Future value of an annuity

The summation formula for a geometric series yields:

$$F_n = \frac{a[1-(1+r)^n]}{1-(1+r)} = \frac{a}{r}[(1+r)^n - 1]$$

We can also find the (undiscounted) future value by noticing that in the special case when $a_i = a$ for all i, the terms on the right-hand side of (∗) repeat those of the right-hand side of Eq. (10.5.1), when $a_1 = a_2 = \cdots = a_n = a$, but taken in the reverse order and multiplied by the interest factor $(1+r)^n$. Hence, $F_n = P_n(1+r)^n$, and so:

FUTURE VALUE OF AN ANNUITY

If an amount a is deposited in an account each period for n periods, earning interest at r per period, the future value of the account, immediately after the last deposit, is

$$F_n = \frac{a}{r}[(1+r)^n - 1] \tag{10.5.3}$$

EXAMPLE 10.5.1 Compute the present and the future values of a deposit of \$1 000 in each of the coming eight years if the annual interest rate is 6%.

Solution: To find the present value, we apply formula (10.5.2), with $a = 1000$, $n = 8$ and $r = 6/100 = 0.06$. This gives

$$P_8 = \frac{1000}{0.06}\left(1 - \frac{1}{(1.06)^8}\right) \approx 6209.79$$

The future value is found by applying formula (10.5.3), which gives

$$F_8 = \frac{1000}{0.06}\left[(1.06)^8 - 1\right] \approx 9897.47$$

Alternatively, $F_8 = P_8(1.06)^8 \approx 6209.79(1.06)^8 \approx 9897.47$.

If $r > 0$ and we let n approach infinity in Eq. (10.5.2), then $(1+r)^n$ approaches infinity and P_n approaches a/r. So, in the limit,

$$\frac{a}{1+r} + \frac{a}{(1+r)^2} + \cdots = \frac{a}{r} \tag{10.5.4}$$

This corresponds to the case where an investment pays a per period in perpetuity when the interest rate is $r > 0$.

EXAMPLE 10.5.2 Compute the present value of a series of deposits of \$1 000 at the end of each year in perpetuity, when the annual interest rate is 14%.

Solution: According to formula (10.5.4), we obtain

$$\frac{1000}{1+0.14} + \frac{1000}{(1+0.14)^2} + \cdots = \frac{1000}{0.14} \approx 7142.86$$

Present Value of a Continuous Income Stream

We have discussed the present value of a series of future payments made at specific discrete moments in time. It is often of interest to consider revenue as accruing continuously, like the timber yield from a large growing forest. Suppose that income is to be received continuously from time $t = 0$ to time $t = T$ at the rate of $f(t)$ dollars per year at time t. We assume that interest is compounded continuously at rate r per year.

Let $P(t)$ denote the present discounted value (PDV) of all payments made over the time interval $[0, t]$. This means that $P(T)$ represents the amount of money you would have to deposit at time $t = 0$ in order to match what results from (continuously) depositing the income stream $f(t)$ over the time interval $[0, T]$. If Δt is any number, the present value of the income received during the interval $[t, t + \Delta t]$ is $P(t + \Delta t) - P(t)$. If Δt is a small number, the income received during this interval is approximately $f(t)\,\Delta t$, and the PDV of this amount is approximately $f(t)e^{-rt}\,\Delta t$. Thus, $P(t + \Delta t) - P(t) \approx f(t)e^{-rt}\,\Delta t$ and so

$$[P(t + \Delta t) - P(t)]\,/\,\Delta t \approx f(t)e^{-rt}$$

This approximation gets better the smaller is Δt, and in the limit as $\Delta t \to 0$, we have $P'(t) = f(t)e^{-rt}$. By the definition of the definite integral, $P(T) - P(0) = \int_0^T f(t)e^{-rt}\,dt$. Because $P(0) = 0$, we have the following:

PRESENT VALUE OF A CONTINUOUS INCOME STREAM

The present discounted value, at time 0, of a continuous income stream at the rate of $f(t)$ dollars per year over the time interval $[0, T]$, with continuously compounded interest at rate r per year, is given by

$$\text{PDV} = \int_0^T f(t)e^{-rt}\,dt \tag{10.5.5}$$

Equation (10.5.5) gives the value at time 0 of an income stream $f(t)$ received during the time interval $[0, T]$. The value of this amount at time T, with continuously compounded interest at rate r, is $e^{rT}\int_0^T f(t)e^{-rt}\,dt$. Because the number e^{rT} is a constant, we can rewrite the integral as $\int_0^T f(t)e^{r(T-t)}\,dt$. This is called the future discounted value (FDV) of the income stream:

FUTURE VALUE OF A CONTINUOUS INCOME STREAM

The future discounted value, at time T, of a continuous income stream at the rate of $f(t)$ dollars per year over the time interval $[0, T]$, with continuously compounded interest at rate r per year, is given by

$$\text{FDV} = \int_0^T f(t)e^{r(T-t)}\,dt \tag{10.5.6}$$

An easy modification of Eq. 10.5.5 will give us the discounted value (DV) at any time s in $[0, T]$ of an income stream $f(t)$ received during the time interval $[s, T]$. In fact, the DV at time s of income $f(t)$ received in the small time interval $[t, t + \mathrm{d}t]$ is $f(t)e^{-r(t-s)}\mathrm{d}t$. So we have the following:

DISCOUNTED VALUE OF A CONTINUOUS INCOME STREAM

The discounted value at any time s of a continuous income stream at the rate of $f(t)$ dollars per year over the time interval $[s, T]$, with continuously compounded interest at rate r per year, is given by

$$\text{DV} = \int_s^T f(t)e^{-r(t-s)}\,\mathrm{d}t \tag{10.5.7}$$

EXAMPLE 10.5.3 Find the PDV and the FDV of a constant income stream of \$1 000 per year over the next ten years, assuming an interest rate of 8% annually, compounded continuously.

Solution:

$$\text{PDV} = \int_0^{10} 1000e^{-0.08t}\,\mathrm{d}t = \Big|_0^{10} 1000\left(-\frac{e^{-0.08t}}{0.08}\right) = \frac{1000}{0.08}(1 - e^{-0.8}) \approx 6883.39$$

while

$$\text{FDV} = e^{0.08 \cdot 10} \times \text{PDV} \approx e^{0.8} \cdot 6883.39 \approx 15\,319.27$$

EXERCISES FOR SECTION 10.5

1. What is the present value of 15 annual deposits of \$3 500 if the first deposit is after one year and the annual interest rate is 12%?

2. An account has been dormant for many years earning interest at the constant rate of 4% per year. Now the amount is \$100 000. How much was in the account ten years ago?

3. At the end of each year for four years you deposit \$10 000 into an account earning interest at a rate of 6% per year. How much is in the account at the end of the fourth year?

4. Suppose you are given the following options: (i) \$13 000 paid after ten years, or (ii) \$1 000 paid each year for ten years, first payment today. Which of these alternatives would you choose, if the annual interest rate is 6% per year for the whole period?

5. An author is to be paid royalties for publishing a book. Two alternative offers are made:

   (i) The author can be paid \$21 000 immediately.

   (ii) There can be five equal annual payments of \$4 600, the first being paid at once.

   Which of these offers will be more valuable if the interest rate is 6% per annum?

6. Compute the present value of a series of deposits of \$1 500 at the end of each year in perpetuity when the interest rate is 8% per year.

7. A trust fund is being set up with a single payment of K. This amount is to be invested at a fixed annual interest rate of r. The fund pays out a fixed annual amount. The first payment is to be made one year after the trust fund was set up. What is the largest amount that can be paid out each year if the fund is to last forever?

8. The present discounted value of a payment D growing at a constant rate g when the discount rate is r is given by

$$\frac{D}{1+r} + \frac{D(1+g)}{(1+r)^2} + \frac{D(1+g)^2}{(1+r)^3} + \cdots$$

where r and g are positive. What is the condition for convergence? Show that if the series converges with sum P_0, then $P_0 = D/(r-g)$.

9. Find the PDV and FDV of a constant income stream of \$500 per year over the next 15 years, assuming an interest rate of 6% annually, compounded continuously.

10.6 Mortgage Repayments

When a family takes out a home mortgage at a fixed interest rate, this means that, like an annuity, equal payments are due each period—say, at the end of each month. The payments continue until the loan is paid off after, say, 20 years. Each payment goes partly to pay interest on the outstanding principal, and partly to repay principal (that is, to reduce the outstanding balance). The interest part is largest in the beginning, because interest has to be paid on the whole loan for the first period, and is smallest in the last period, because by then the outstanding balance is small. For the principal repayment, which is the difference between the fixed monthly payment and the interest, it is the other way around.

EXAMPLE 10.6.1 A person borrows \$50 000 at the beginning of a year and is supposed to pay it off in five equal instalments at the end of each year, with interest at 15% compounding annually. Find the annual payment.

Solution: If the five repayments are each of amount \$$a$, their present value in dollars is

$$\frac{a}{1.15} + \frac{a}{(1.15)^2} + \frac{a}{(1.15)^3} + \frac{a}{(1.15)^4} + \frac{a}{(1.15)^5} = \frac{a}{0.15}\left[1 - \frac{1}{(1.15)^5}\right]$$

according to Eq. (10.5.2). This sum must be equal to \$50 000, so

$$\frac{a}{0.15}\left[1 - \frac{1}{(1.15)^5}\right] = 50\,000 \tag{$*$}$$

This has the solution $a \approx 14\,915.78$. Alternatively, we can calculate the sum of the future values of all repayments and equate it to the future value of the original loan. This yields the equation

$$a + a(1.15) + a(1.15)^2 + a(1.15)^3 + a(1.15)^4 = 50\,000(1.15)^5$$

which is equivalent to $(*)$.

To illustrate how the interest part and the principal repayment part of the yearly payment vary from year to year, we construct the following table:

Year	Payment	Interest	Principal repayment	Outstanding balance
1	14 915.78	7 500.00	7 415.78	42 584.22
2	14 915.78	6 387.63	8 528.15	34 056.07
3	14 915.78	5 108.41	9 807.37	24 248.70
4	14 915.78	3 637.31	11 278.47	12 970.23
5	14 915.78	1 945.55	12 970.23	0

Note that the interest payment each year is 15% of the outstanding balance from the previous year. The remainder of each annual payment of \$14 915.78 is the principal repayment that year, which is subtracted from the outstanding balance left over from the previous year. Figure 10.6.1 is a chart showing each year's interest and principal repayments.

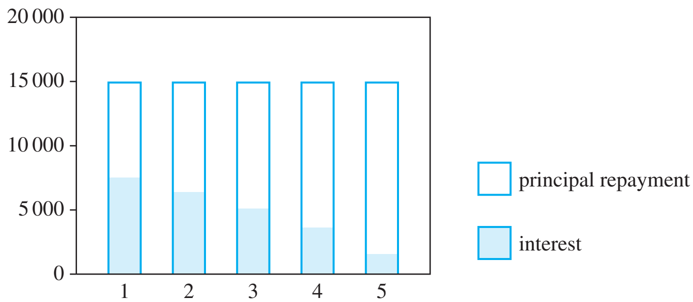

Figure 10.6.1 Interest and principal repayment in Example 10.6.1

Suppose a loan of K dollars is repaid as an annuity over n periods, at the interest rate $p\%$ per period, where the first payment a is due after one period, and the rest at equally spaced periods. According to Eq. (10.5.2), the payment a each period must satisfy

$$K = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right] = \frac{a}{r}[1-(1+r)^{-n}] \tag{10.6.1}$$

Solving for a yields

$$a = \frac{rK}{1-(1+r)^{-n}} \tag{10.6.2}$$

where $r = p/100$. A good exercise is to use this formula in Example 10.6.1.

EXAMPLE 10.6.2 Suppose that the loan in Example 10.6.1 is being repaid by equal monthly payments at the end of each month with interest at the nominal rate 15% per year, compounding monthly. Find the monthly payment.

Solution: The interest period is one month and the monthly rate is $15/12 = 1.25\%$, so that $r = 1.25/100 = 0.0125$. Also, $n = 5 \cdot 12 = 60$, so formula (10.6.2) gives:

$$a = \frac{0.0125 \cdot 5000}{1 - 1.0125^{-60}} \approx 1189.50$$

The annuities considered so far were *ordinary* annuities where each payment is made at the *end* of the payment period. If the payment each period is made at the beginning of the period, the annuity is called an *annuity due*. This kind of annuity can be handled easily by regarding it as an ordinary annuity, except that there is an immediate initial payment.

EXAMPLE 10.6.3 A person is assuming responsibility for a \$335 000 loan which should be repaid in 15 equal repayments of \$$a$, the first one immediately and the following after each of the coming 14 years. Find a if the annual interest rate is 14%.

Solution: The present value of the first payment is obviously a. The present value of the following 14 repayments is found by applying formula (10.6.1) with $r = 0.14$ and $n = 14$. The sum of the present values must be equal to \$335 000:

$$a + \frac{a}{0.14}\left[1 - \frac{1}{(1+0.14)^{14}}\right] = 335\,000$$

This reduces to $a + 6.0020715a = 335\,000$, and solving for a gives $a \approx \$47\,843$.

Some lenders prefer to specify a fixed payment each period, and let the loan run for however many periods it takes to pay off the debt. This way of paying off the loan functions essentially as an annuity. The difference is that there will be a final adjustment in the last payment in order for the present value of all the payments to be equal to the borrowed amount. In this case it is convenient to use the formula obtained by solving Eq. (10.6.1) for n. The result is

$$\frac{rK}{a} = 1 - \frac{1}{(1+r)^n} \Leftrightarrow \frac{1}{(1+r)^n} = 1 - \frac{rK}{a} = \frac{a - rK}{a} \Leftrightarrow (1+r)^n = \frac{a}{a - rK}$$

Taking the natural logarithm of each side yields $n \ln(1 + r) = \ln[a/(a - rK)]$, so:

MORTGAGE REPAYMENT PERIOD

The number of periods needed to pay off a loan of amount K at the rate a per period, when the interest rate is r per period, is given by the smallest integer n such that

$$n \geq \frac{\ln a - \ln(a - rK)}{\ln(1+r)} \tag{10.6.3}$$

If this equation holds with strict inequality, the last payment will need to be smaller than a.

EXAMPLE 10.6.4 A loan of \$50 000 is to be repaid by paying \$20 000, which covers both interest and the principal repayment, at the end of each of the coming years, until the loan is fully paid off. When is the loan paid off, and what is the final payment if the annual rate is 15%?

Solution: We begin by computing the number n of annual payments of \$20 000 which are needed to pay off \$50 000. According to Eq. (10.6.3), with $r = 0.15$, $a = 20\,000$, and $K = 50\,000$, we obtain n as the smallest integer greater than or equal to

$$\frac{\ln(20\,000) - \ln(20\,000 - 0.15 \cdot 50\,000)}{\ln(1 + 0.15)} = \frac{\ln 1.6}{\ln 1.15} \approx 3.3629$$

Thus, three payments of \$20 000 are needed, with an additional payment in the fourth year. Let us calculate the future value of the three payments of \$20 000 three years after the loan was made. This value is:

$$20\,000 \cdot (1.15)^2 + 20\,000 \cdot 1.15 + 20\,000 = \frac{20\,000}{0.15}\left[(1.15)^3 - 1\right] = 69\,450$$

The future value of the \$50 000 loan after the same three years is $\$50\,000 \cdot (1.15)^3 = \$76\,043.75$. Thus the remaining debt after the third payment is $\$76\,043.75 - \$69\,450 = \$6\,593.75$. If the remaining debt and the accumulated interest are paid one year later, the amount due is $\$6\,593.75 \cdot 1.15 = \$7\,582.81$.

Deposits within an Interest Period

Many bank accounts have an interest period of one year, or at least one month. If you deposit an amount *within* an interest period, the bank will often use simple interest, not compound interest. In this case, if you make a deposit within an interest period, then at the end of the period the amount you deposited will be multiplied by the factor $1 + rt$, where t is the remaining fraction of the interest period.

EXAMPLE 10.6.5 At the end of each quarter, beginning on March 31, 2009, a person deposits \$100 in an account on which interest is paid annually at the rate 10% per year. How much is there in the account on December 31, 2011?

Solution: The deposits during 2009 are illustrated next:

These four deposits are made within the year. In order to find the balance at the end of the year (the interest period), we use simple (i.e. not compound) interest. This gives

$$100\left(1 + 0.10 \cdot \frac{3}{4}\right) + 100\left(1 + 0.10 \cdot \frac{2}{4}\right) + 100\left(1 + 0.10 \cdot \frac{1}{4}\right) + 100 = 415$$

Doing the same for 2010 and 2011 as well, we replace the 12 original deposits by the amount \$415 at the end of each of the years 2009, 2010, and 2011.

31/12/1999	31/12/2000	31/12/2001
415	415	415

The balance is $415 \cdot (1.10)^2 + 415 \cdot 1.10 + 415 = 1373.65$. So on December 31, 2011, the person has \$1 373.65.

EXERCISES FOR SECTION 10.6

1. A person borrows \$80 000 at the beginning of one year, and is supposed to pay it off in ten equal instalments at the end of each year, with interest at 7% compounding annually.

   (a) Find the annual payment.

   (b) Suppose that the loan is to be repaid in equal instalments at the end of each month, but with interest at the nominal rate 7% compounded monthly. Find the monthly payment.

2. If you deposit \$8 000 in an account each year for six years at the rate of interest 7%, how much do you have immediately after the last deposit? How much do you have four years after the last deposit?

3. Ronald invests money in a project which triples his money in 20 years. Assuming annual compounding of interest, what is the rate of interest? What if you assume continuous compounding?

4. [HARDER] A construction firm wants to buy a building site and has the choice between three different payment schedules:

   (i) Pay \$67 000 in cash.

   (ii) Pay \$12 000 per year for eight years, where the first instalment is to be paid at once.

   (iii) Pay \$22 000 in cash and thereafter \$7 000 per year for 12 years, where the first instalment is to be paid after one year.

   Determine which schedule is least expensive if the interest rate is 11.5% and the firm has at least \$67 000 available to spend in cash. What happens if the interest rate is 12.5%?

10.7 Internal Rate of Return

Consider $n + 1$ numbers, $a_0, a_1, \dots, a_n$, which represent the returns in successive years earned by an investment project. Negative numbers represent losses, positive numbers represent profits, so each a_i is actually the *net return*. Also, we think of a_i as associated with year i, whereas a_0 is associated with the present period. In most investment projects, a_0 is a big negative number, because a large expense precedes any positive returns. If we consider an interest rate of $p\%$ per year and let $r = p/100$, then the net present value of the profits accruing from the project is given by

$$A = a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n}$$

Several different criteria are used to compare alternative investment projects. One is simply this: Choose the project whose profit stream has the largest present value. The interest rate to use could be an accepted rate for capital investments. A different criterion is based on the *internal rate of return*, defined as an interest rate that makes the present value of all payments equal to 0.

As a simple example, suppose you invest an amount a which pays back b one year later. Then the rate of return is the interest rate r that makes the present value of the investment project equal to zero. That is, r must satisfy $-a + (1+r)^{-1}b = 0$, so $r = (b/a) - 1$. For example, when $a = 1000$ and $b = 1200$, the rate of return is $r = (1200/1000) - 1 = 0.2$, or 20% per year.

For a general investment project yielding returns $a_0, a_1, \dots, a_n$, the internal rate of return is a number r such that

$$a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n} = 0 \tag{10.7.1}$$

If two investment projects both have a unique internal rate of return, then a criterion for choosing between them is to prefer the project that has the higher internal rate of return.

Note that (10.7.1) is a polynomial equation of degree n in the discount factor $(1+r)^{-1}$. In general, this equation does not have a unique positive solution r.

EXAMPLE 10.7.1 An investment project has an initial outlay of \$50 000, and at the end of the next two years has returns of \$30 000 and \$40 000, respectively. Find the associated internal rate of return.

Solution: In this case, Eq. (10.7.1) takes the form

$$-50\,000 + \frac{30\,000}{1+r} + \frac{40\,000}{(1+r)^2} = 0$$

Put $s = (1+r)^{-1}$. Then the equation becomes

$$40\,000s^2 + 30\,000s - 50\,000 = 0$$

or $4s^2 + 3s - 5 = 0$. This has only one positive solution, which is $s \approx 0.8042$. Then $r = 1/s - 1 \approx 0.243$. The internal rate of return is therefore 24.3%.

Suppose that $a_0 < 0$ and $a_1, \dots, a_n$ are all > 0. Then, Eq. (10.7.1) has a unique solution r^* satisfying $1 + r^* > 0$—that is, a unique internal rate of return $r^* > -1$. Also, the internal rate of return is positive if $\sum_{i=0}^{n} a_i > 0$. You are asked to prove these results in Exercise 3.

EXERCISES FOR SECTION 10.7

1. An investment project has an initial outlay of \$50 000 and at the end of each of the next two years has returns of \$30 000. Find the associated internal rate of return r.

2. Suppose that in Eq. (10.7.1) we have $a_0 < 0$ and $a_i = a > 0$ for $i = 1, 2, \dots$. Find an expression for the internal rate of return in the limit as $n \to \infty$.

3. Consider an investment project with an initial loss, so that $a_0 < 0$, and thereafter no losses. Suppose also that the sum of the later profits is larger than the initial loss. Prove that there is a unique internal rate of return, $r^* > -1$, and that $r^* > 0$ if $\sum_{i=0}^{n} a_i > 0$. (*Hint*: Define $f(r)$ as the expression on the left side of (10.7.1). Then study the signs of $f(r)$ and $f'(r)$ on the interval $(0, \infty)$.)

4. An investment in a certain machine is expected to earn a profit of \$400 000 each year. What is the maximum price that should be paid for the machine if it has a lifetime of seven years, the interest rate is 17.5%, and the annual profit is earned at the end of each year?

SM 5. [HARDER] An investment project has an initial outlay of \$100 000, and at the end of each of the next 20 years has a return of \$10 000. Show that there is a unique positive internal rate of return, and find its approximate value. (*Hint*: Use $s = (1 + r)^{-1}$ as a new variable. Prove that the equation you obtain for s has a unique positive solution. Verify that $s = 0.928$ is an approximate root.)

6. [HARDER] Al is obliged to pay Bob \$1 000 yearly for five years, the first payment in one year's time. Bob sells this claim to Carl for \$4 340 in cash. Find an equation that determines the rate of return p that Carl obtains from this investment. Prove that it is a little less than 5%.

10.8 A Glimpse at Difference Equations

Many of the quantities economists study, such as income, consumption, and savings, are recorded at fixed time intervals—for example, each day, week, quarter, or year. Equations that relate such quantities at different discrete moments of time are called *difference equations*. In fact difference equations can be viewed as the discrete time counterparts of the differential equations in continuous time that were studied in Sections 9.8 and 9.9.

Let $t = 0, 1, 2, \ldots$ denote different discrete time periods or moments of time. We usually call $t = 0$ the *initial period*. If $x(t)$ is a function defined for $t = 0, 1, 2, \ldots$, we often use $x_0, x_1, x_2, \ldots$ to denote $x(0), x(1), x(2), \ldots$, and in general, we write x_t for $x(t)$.

A simple example of a first-order difference equation is

$$x_{t+1} = ax_t \tag{10.8.1}$$

for $t = 0, 1, \ldots$, where a is a constant. This is a first-order equation because it relates the value of a function in period $t + 1$ to the value of the same function in the previous period t only. Suppose x_0 is given. Repeatedly applying (10.8.1) gives $x_1 = ax_0$; then, $x_2 = ax_1 = a \cdot ax_0 = a^2x_0$; then, $x_3 = ax_2 = a \cdot a^2x_0 = a^3x_0$; and so on. In general,

$$x_t = x_0a^t \tag{10.8.2}$$

The function $x_t = x_0a^t$ satisfies (10.8.1) for all t, as can be verified directly. For the given value of x_0, there is clearly no other function that satisfies the equation.

EXAMPLE 10.8.1 Consider the following difference equation, which has $x_0 = 100$: $x_{t+1} = 0.2x_t$, for $t = 0, 1, \ldots$. From Eq. (10.8.2), we have $x_t = 100(0.2)^t$.

EXAMPLE 10.8.2 Let K_t denote the balance in an account at the beginning of period t when the interest rate is r per period. Then the balance in the account at time $t + 1$ is given by the difference equation $K_{t+1} = K_t + rK_t = (1 + r)K_t$, for $t = 0, 1, \ldots$. It follows immediately from (10.8.2) that $K_t = K_0(1 + r)^t$, as is well known to us already from Section 2.2, and from earlier on in this chapter. In general, this difference equation describes growth at the constant proportional rate r each period.

EXAMPLE 10.8.3 **(A multiplier–accelerator model of economic growth)** Let Y_t denote GDP, I_t total investment, and S_t total saving, all in period t. Suppose that savings are proportional to GDP, and that investment is proportional to the change in income from period t to $t + 1$. Then, for $t = 0, 1, 2, \ldots,$

(i) $S_t = \alpha Y_t$ (ii) $I_{t+1} = \beta(Y_{t+1} - Y_t)$ (iii) $S_t = I_t$

The last equation is the equilibrium condition that saving equals investment in each period. Here α and β are positive constants, and we assume that $\beta > \alpha > 0$. Deduce a difference equation determining the path of Y_t, given Y_0, and solve it.

Solution: From (i) and (iii), $I_t = \alpha Y_t$, and so $I_{t+1} = \alpha Y_{t+1}$. Inserting this into (ii) yields $\alpha Y_{t+1} = \beta(Y_{t+1} - Y_t)$, or $(\alpha - \beta)Y_{t+1} = -\beta Y_t$. Thus,

$$Y_{t+1} = \frac{\beta}{\beta - \alpha} Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right) Y_t \tag{$*$}$$

Using (10.8.2) gives the solution

$$Y_t = \left(1 + \frac{\alpha}{\beta - \alpha}\right)^t Y_0$$

for $t = 1, 2, \ldots$.

Linear First-Order Equations with Constant Coefficients

Consider next the first-order linear difference equation

$$x_{t+1} = ax_t + b \tag{10.8.3}$$

for $t = 0, 1, 2, \ldots,$ where a and b are constants. Equation (10.8.1) is the special case where $b = 0$. Starting with a given x_0, we can calculate x_t algebraically for small t. Indeed,

$$\begin{aligned}
x_1 &= ax_0 + b \\
x_2 &= ax_1 + b = a(ax_0 + b) + b = a^2x_0 + (a + 1)b \\
x_3 &= ax_2 + b = a(a^2x_0 + (a + 1)b) + b = a^3x_0 + (a^2 + a + 1)b
\end{aligned}$$

and so on. This makes the pattern clear. In general we have

$$x_t = a^t x_0 + (a^{t-1} + a^{t-2} + \cdots + a + 1)b$$

It is straightforward to check directly that this satisfies (10.8.3). According to the summation formula for a geometric series, $1 + a + a^2 + \cdots + a^{t-1} = (1 - a^t)/(1 - a)$, for $a \neq 1$. Thus:

LINEAR FIRST-ORDER EQUATIONS WITH CONSTANT COEFFICIENTS

Given that $a \neq 1$,

$$x_{t+1} = ax_t + b \Leftrightarrow x_t = a^t\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} \quad \text{for } t = 0, 1, 2, \ldots \tag{10.8.4}$$

When $a = 1$, we have $1 + a + \cdots + a^{t-1} = t$ and $x_t = x_0 + tb$ for $t = 1, 2, \ldots$.

EXAMPLE 10.8.4 Solve the difference equation $x_{t+1} = \frac{1}{3}x_t - 8$.

Solution: Using Eq. (10.8.4), we obtain the solution $x_t = \left(\frac{1}{3}\right)^t(x_0 + 12) - 12$.

Equilibrium States and Stability

Consider the solution of $x_{t+1} = ax_t + b$ given in (10.8.4). If $x_0 = b/(1 - a)$, then $x_t = b/(1 - a)$ for all t. The constant $x^* = b/(1 - a)$ is called an *equilibrium*, or *stationary*, state for $x_{t+1} = ax_t + b$.

An alternative way of finding an equilibrium state x^* is to seek a solution of $x_{t+1} = ax_t + b$ with $x_t = x^*$ for all t. Such a solution must satisfy $x_{t+1} = x_t = x^*$ and so $x^* = ax^* + b$. Therefore, for $a \neq 1$, we get $x^* = b/(1 - a)$ as before.

Suppose the constant a in (10.8.4) is less than 1 in absolute value—that is, $-1 < a < 1$. Then $a^t \to 0$ as $t \to \infty$, so Eq. (10.8.4) implies that

$$x_t \to x^* = b/(1 - a) \quad \text{as} \quad t \to \infty$$

Hence, if $|a| < 1$, the solution converges to the equilibrium state as $t \to \infty$. The equation is then called *globally asymptotically stable*. If $|a| > 1$, then the absolute value of a^t tends to ∞ as $t \to \infty$. From (10.8.4), it follows that x_t moves farther and farther away from the equilibrium state, except when $x_0 = b/(1 - a)$. Illustrations of the different possibilities are given in Section 11.1 of FMEA.

EXAMPLE 10.8.5 The equation in Example 10.8.4 is stable because $a = 1/3$. The equilibrium state is -12. We see from the solution given in that example that $x_t \to -12$ as $t \to \infty$.

EXAMPLE 10.8.6 **(Mortgage Repayments)** A particular case of Eq. (10.8.3) occurs when a family borrows an amount K at time 0 as a home mortgage. Suppose there is a fixed interest rate r per period (usually a month rather than a year). Suppose, too, that there are equal repayments of amount a each period, until the mortgage is paid off after n periods—for example, 360 months, or 30 years. The outstanding balance or *principal* b_t on the loan in period t satisfies the difference equation $b_{t+1} = (1 + r)b_t - a$, with $b_0 = K$ and $b_n = 0$. This difference equation can be solved by using Eq. (10.8.4), which gives

$$b_t = (1 + r)^t (K - a/r) + a/r$$

But $b_t = 0$ when $t = n$, so $0 = (1 + r)^n \big(K - a/r\big) + a/r$. Solving for K yields

$$K = \frac{a}{r}\left[1 - (1 + r)^{-n}\right] = a\sum_{t=1}^{n}(1 + r)^{-t} \tag{$*$}$$

The original loan, therefore, is equal to the present discounted value of n equal repayments of amount a each period, starting in period 1. Solving for a instead yields

$$a = \frac{rK}{1 - (1 + r)^{-n}} = \frac{rK(1 + r)^n}{(1 + r)^n - 1} \tag{$**$}$$

Formulas ($*$) and ($**$) are the same as those derived by a more direct argument in Section 10.6.

EXERCISES FOR SECTION 10.8

1. Find the solutions of the following difference equations.

   (a) $x_{t+1} = -2x_t$ (b) $6x_{t+1} = 5x_t$ (c) $x_{t+1} = -0.3x_t$

2. Find the solutions of the following difference equations with the given values of x_0:

   (a) $x_{t+1} = x_t - 4, \quad x_0 = 0$ (b) $x_{t+1} = \frac{1}{2}x_t + 2, \quad x_0 = 6$

   (c) $2x_{t+1} + 6x_t + 5 = 0, \quad x_0 = 1$ (d) $x_{t+1} + x_t = 8, \quad x_0 = 2$

3. Suppose supply at price P_t is $S(P_t) = \alpha P_t - \beta$ and demand at price P_{t+1} is $D(P_{t+1}) = \gamma - \delta P_{t+1}$. Solve the difference equation $S(P_t) = D(P_{t+1})$, assuming that all constants are positive.

REVIEW EXERCISES

1. An amount \$5 000 earns interest at 3% per year.

   (a) What will this amount have grown to after ten years?

   (b) How long does it take for the \$5 000 to double?

2. An amount of €8 000 is invested at 5% per year.

   (a) What is the balance in the account after three years?

   (b) What is the balance after 13 years?

   (c) How long does it take, approximately, for the balance to reach €32 000?

3. Which is preferable for a borrower: (i) to borrow at the annual interest rate of 11% with interest paid yearly; or (ii) to borrow at annual interest rate 10% with interest paid monthly?

4. Suppose the sum of £15 000 is invested in an account earning interest at an annual rate of 7%. What is the balance after 12 years if interest is compounded continuously?

5. (a) How much has \$8 000 increased to after three years, if the annual interest rate is 6%, with continuous compounding?

   (b) How long does it take before the \$8 000 has doubled?

6. Find the sums of the following infinite series:

   (a) $44 + 44 \cdot 0.56 + 44 \cdot (0.56)^2 + \cdots$

   (b) $\sum_{n=0}^{\infty} 20 \left(\frac{1}{1.2}\right)^n$

   (c) $3 + \frac{3 \cdot 2}{5} + \frac{3 \cdot 2^2}{5^2} + \cdots + \frac{3 \cdot 2^{n-1}}{5^{n-1}} + \cdots$

   (d) $\sum_{j=-2}^{\infty} \frac{1}{20^j}$

7. A constant income stream of a dollars per year is expected over the next T years.

   (a) Find its PDV, assuming an interest rate of r annually, compounded continuously.

   (b) What is the limit of the PDV as $T \to \infty$? Compare this result with Eq. (10.5.4).

8. At the beginning of a year \$5 000 is deposited in an account earning 4% annual interest. What is the balance after four years?

SM 9. At the end of each year for four years, \$5 000 is deposited in an account earning 4% annual interest. What is the balance immediately after the fourth deposit?

SM 10. Suppose you had \$10 000 in your account on 1st January 2006. The annual interest rate is 4%. You agreed to deposit a fixed amount K each year for eight years, the first deposit on 1st January 2009. What choice of the fixed amount K will imply that you have a balance of \$70 000 immediately after the last deposit?

11. A business borrows €500 000 from a bank at the beginning of one year, and is supposed to pay it off in ten equal instalments at the end of each year, with interest at 7% compounding annually.

    (a) Find the annual payment. What is the total amount paid to the bank?

    (b) What is the total amount if the business has to pay twice a year?

12. Lucy is offered the choice between the following three options:

    (i) She gets \$3 200 each year for ten years, with the first payment due after one year.

    (ii) She gets \$7 000 today, and thereafter \$3 000 each year for five years, with the first payment after one year.

    (iii) She gets \$4 000 each year for ten years, with the first payment only due after five years.

    The annual interest rate is 8%. Calculate the present values of the three options. What would you advise Lucy to choose?

SM 13. With reference to Example 10.3.2, suppose that the market value of the tree is $P(t) = 100e^{\sqrt{t}/2}$, so that its present value is $f(t) = 100e^{\sqrt{t}/2}e^{-rt}$.

(a) Find the optimal cutting time t^*. By studying the sign variation of $f'(t)$, show that you have indeed found the maximum. What is t^* if $r = 0.05$?[7]

(b) Solve the same problem when $P(t) = 200e^{-1/t}$ and $r = 0.04$.

14. The revenue produced by a new oil well is \$1 million per year initially ($t = 0$), which is expected to rise uniformly to \$5 million per year after ten years. If we measure time in years and let $f(t)$ denote the revenue, in millions of dollars, per unit of time at time t, it follows that $f(t) = 1 + 0.4t$. If $F(t)$ denotes the total revenue that accumulates over the time interval $[0, t]$, then $F'(t) = f(t)$.

(a) Calculate the total revenue earned during the ten year period, $F(10)$.

(b) Find the present value of the revenue stream over the time interval $[0, 10]$, if we assume continuously compounded interest at the rate $r = 0.05$ per year.

15. Solve the following difference equations with the given values of x_0:

(a) $x_{t+1} = -0.1x_t$ with $x_0 = 1$

(b) $x_{t+1} = x_t - 2$ with $x_0 = 4$

(c) $2x_{t+1} - 3x_t = 2$ with $x_0 = 2$

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[7] Note that t^* decreases as r increases.

11 FUNCTIONS OF MANY VARIABLES

Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost.
—W.S. Anglin (1992)

So far, this book has been concerned almost exclusively with functions of one variable. Yet a realistic description of economic phenomena often requires considering a large number of variables. For example, one consumer's demand for a good like orange juice depends on its price, on the consumer's income, and on the prices of substitutes like other soft drinks, or complements like some kinds of food.

Previous chapters have presented important properties of functions of one variable. For functions of several variables, most of what economists need to know consists of relatively simple extensions of properties presented in the previous chapters for functions of one variable. Moreover, most of the difficulties already arise in the transition from one variable to two variables. To help readers see how to overcome these difficulties, Sections 11.1 to 11.3 deal exclusively with functions of two variables. These have graphs in three dimensions, which it is possible to represent even in two-dimensional figures—though with some difficulty. However, as the previous example of the demand for orange juice suggests, there are many interesting economic problems that can only be represented mathematically by functions of many variables. These are discussed in Sections 11.4 to 11.7. The final section, 11.8, is devoted to the economically important topic of elasticity.

11.1 Functions of Two Variables

We begin with the following definition, where D is a subset of the xy-plane.

FUNCTIONS OF TWO VARIABLES

A function f of two real variables, x and y, with domain D is a rule that assigns a specified number

$$f(x, y) \text{ to each point } (x, y) \text{ in } D \tag{11.1.1}$$

If f is a function of two variables, we often let a letter like z denote the value of f at point (x, y), so $z = f(x, y)$. Then we call x and y the *independent variables*, or the *arguments* of f, whereas z is called the *dependent variable*, because the value z, in general, depends on the values of x and y. The domain of function f is then the set of all possible pairs of the independent variables, whereas its *range* is the set of corresponding values of the dependent variable. In economics, x and y are often called the *exogenous* variables, whereas z is the *endogenous* variable.[1]

EXAMPLE 11.1.1 Consider the function f that, to every pair of numbers (x, y), assigns the number $2x + x^2y^3$. The function f is thus defined by

$$f(x, y) = 2x + x^2y^3$$

What are $f(1, 0)$, $f(0, 1)$, $f(-2, 3)$, and $f(a + 1, b)$?

Solution: First, $f(1, 0)$ is the value when $x = 1$ and $y = 0$. So $f(1, 0) = 2 \cdot 1 + 1^2 \cdot 0^3 = 2$. Similarly, we have $f(0, 1) = 2 \cdot 0 + 0^2 \cdot 1^3 = 0$, and $f(-2, 3) = 2(-2) + (-2)^2 \cdot 3^3 = -4 + 4 \cdot 27 = 104$. Finally, we find $f(a + 1, b)$ by replacing x with $a + 1$ and y with b in the formula for $f(x, y)$, giving $f(a + 1, b) = 2(a + 1) + (a + 1)^2b^3$.

EXAMPLE 11.1.2 A study of the demand for milk found the relationship

$$x = A\frac{m^{2.08}}{p^{1.5}}$$

where x is milk consumption, p is the relative price of milk, m is income per family, and A is a positive constant. This equation defines x as a function of p and m. Note that milk consumption goes up when income increases, and down when the price of milk increases—which seems reasonable.

EXAMPLE 11.1.3 A function of two variables appearing in many economic models is

$$F(x, y) = Ax^ay^b \tag{11.1.2}$$

where A, a, and b are constants. Usually, one assumes that F is defined only for $x > 0$ and $y > 0$.

[1] In economic models with several simultaneous equations, the distinction between exogenous and endogenous variables is much more nuanced.

A function F of the form (11.1.2) is generally called a *Cobb–Douglas function*.[2] It is most often used to describe certain production processes. Then x and y are called *input factors*, while $F(x, y)$ is the number of units produced, or the *output*. In this case, F is called a *production function*.

Note that the function defined in Example 11.1.2 is also a Cobb–Douglas function, because we have $x = Ap^{-1.5}m^{2.08}$.

It is important to become thoroughly familiar with standard functional notation.

EXAMPLE 11.1.4 For the function F given in Example 11.1.3, find an expression for $F(2x, 2y)$ and for $F(tx, ty)$, where t is an arbitrary positive number. Find also an expression for $F(x + h, y) - F(x, y)$. Give economic interpretations.

Solution: We find that

$$F(2x, 2y) = A(2x)^a(2y)^b = A2^a x^a 2^b y^b = 2^a 2^b A x^a y^b = 2^{a+b}F(x, y)$$

When F is a production function, this shows that if each of the input factors is doubled, then the output is 2^{a+b} times as large. For example, if $a + b = 1$, then doubling both factors of production implies doubling the output. In the general case,

$$F(tx, ty) = A(tx)^a(ty)^b = At^a x^a t^b y^b = t^a t^b A x^a y^b = t^{a+b}F(x, y) \tag{$*$}$$

(How do you formulate this result in your own words?)[3]

Finally, we see that

$$F(x + h, y) - F(x, y) = A(x + h)^a y^b - Ax^a y^b = Ay^b[(x + h)^a - x^a] \tag{$**$}$$

This shows the change in output when the first input factor is changed by h units while the other input factor is unchanged. For example, suppose $A = 100$, $a = 1/2$, and $b = 1/4$, in which case $F(x, y) = 100x^{1/2}y^{1/4}$. If we choose $x = 16$, $y = 16$, and $h = 1$, Eq. $(**)$ implies that

$$F(16 + 1, 16) - F(16, 16) = 100 \cdot 16^{1/4}[17^{1/2} - 16^{1/2}] = 100 \cdot 2[\sqrt{17} - 4] \approx 24.6$$

Hence, if we increase the input of the first factor from 16 to 17, while keeping the input of the second factor constant at 16 units, then we increase production by about 24.6 units.

Domains

For functions studied in economics, there are usually explicit or implicit restrictions on the domain where the function is defined. For instance, if $f(x, y)$ is a production function,we

[2] The function in (11.1.2) is named after American researchers C.W. Cobb and P.H. Douglas, who applied it, with $a + b = 1$, in a paper that appeared in 1927 on the estimation of production functions. The function, however, should properly be called a "Wicksell function", because Swedish economist K. Wicksell (1851–1926) introduced such production functions before 1900.

[3] Because of property $(*)$, we call function F *homogeneous of degree* $a + b$. Homogeneous functions are discussed in Sections 12.6 and 12.7.

usually assume that the input quantities are nonnegative, so $x \geq 0$ and $y \geq 0$. In economics, it is often crucially important to be clear what are the domains of the functions being used.

In the same way as for functions of one variable, we assume, unless otherwise stated, that the domain of a function defined by a formula is the largest set of points in which the formula gives a meaningful and unique value.

Sometimes it is helpful to draw a graph of the domain D in the xy-plane.

EXAMPLE 11.1.5 Determine the domains of the functions given by the following formulas, then draw the sets in the xy-plane.

(a) $f(x, y) = \sqrt{x-1} + \sqrt{y}$ (b) $g(x, y) = \dfrac{2}{(x^2 + y^2 - 4)^{1/2}} + \sqrt{9 - (x^2 + y^2)}$

Solution:

(a) We must require that $x \geq 1$ and $y \geq 0$, for only then do $\sqrt{x-1}$ and $\sqrt{y}$ have any meaning. The (unbounded) domain is indicated in Fig. 11.1.1.

(b) $(x^2 + y^2 - 4)^{1/2} = \sqrt{x^2 + y^2 - 4}$ is only defined if $x^2 + y^2 \geq 4$. Moreover, we must have $x^2 + y^2 \neq 4$; otherwise, the denominator is equal to 0. Furthermore, we must require that $9 - (x^2 + y^2) \geq 0$, or $x^2 + y^2 \leq 9$. All in all, therefore, we must have $4 < x^2 + y^2 \leq 9$. Because the graph of $x^2 + y^2 = r^2$ consists of all the points on the circle with centre at the origin and radius r, the domain is the set of points (x, y) that lie outside, but not on, the circle $x^2 + y^2 = 4$; and inside or on the circle $x^2 + y^2 = 9$. This set is shown in Fig. 11.1.2, where the solid circle is in the domain, but the dashed circle is not.

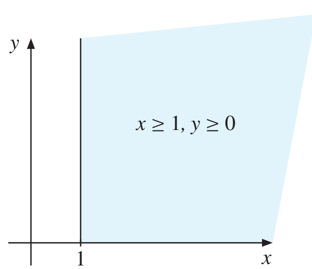

Figure 11.1.1 Domain of $\sqrt{x-1} + \sqrt{y}$

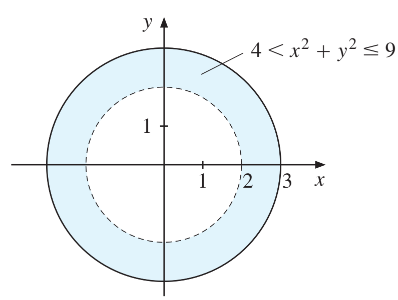

Figure 11.1.2 Domain of $2/(x^2 + y^2 - 4)^{1/2} + \sqrt{9 - (x^2 + y^2)}$

EXERCISES FOR SECTION 11.1

1. Let $f(x, y) = x + 2y$. Find $f(0, 1)$, $f(2, -1)$, $f(a, a)$, and $f(a + h, b) - f(a, b)$.

2. Let $f(x, y) = xy^2$. Find the respective values of $f(0, 1)$, $f(-1, 2)$, $f(10^4, 10^{-2})$, $f(a, a)$, $f(a + h, b)$, and $f(a, b + k) - f(a, b)$.

3. Let $f(x, y) = 3x^2 - 2xy + y^3$. Find $f(1, 1)$, $f(-2, 3)$, $f(1/x, 1/y)$, $p = [f(x + h, y) - f(x, y)]/h$, and $q = [f(x, y + k) - f(x, y)]/k$.

4. Let $f(x, y) = x^2 + 2xy + y^2$.

(a) Find $f(-1, 2)$, $f(a, a)$, and $f(a + h, b) - f(a, b)$.

(b) Prove that $f(2x, 2y) = 2^2 f(x, y)$, and that $f(tx, ty) = t^2 f(x, y)$ for all t.

5. Let $F(K, L) = 10K^{1/2}L^{1/3}$, for $K \geq 0$ and $L \geq 0$. Find $F(1, 1)$, $F(4, 27)$, $F(9, 1/27)$, $F(3, \sqrt{2})$, $F(100, 1000)$, and $F(2K, 2L)$.

6. Examine the domains of the functions given by the following formulas, and then draw them in the xy-plane:

(a) $\dfrac{x^2 + y^3}{y - x + 2}$ (b) $\sqrt{2 - (x^2 + y^2)}$ (c) $\sqrt{(4 - x^2 - y^2)(x^2 + y^2 - 1)}$

7. Find the domains of the functions defined by the following formulas:

(a) $1/(e^{x+y} - 3)$ (b) $\ln(x - a)^2 + \ln(y - b)^2$ (c) $2\ln(x - a) + 2\ln(y - b)$

11.2 Partial Derivatives with Two Variables

For a function $y = f(x)$ of one variable, the derivative $f'(x)$ is a number that measures the function's rate of change as x changes. For functions of two variables, such as $z = f(x, y)$, we also want to examine how quickly the value of the function changes w.r.t. changes in the values of the independent variables. For example, if $f(x, y)$ is a firm's profit when it uses quantities x and y of two different inputs, we want to know whether and by how much profit can increase as either x or y is varied.

Consider the function

$$z = x^3 + 2y^2 \tag{$*$}$$

Suppose first that y is held constant. Then, $2y^2$ is constant and, really, there is only one variable now. Of course, the rate of change of z w.r.t. x is given by

$$\frac{dz}{dx} = 3x^2$$

On the other hand, we can keep x fixed in $(*)$ and examine how z varies as y varies. This involves taking the derivative of z w.r.t. y while keeping x constant. The result is

$$\frac{dz}{dy} = 4y$$

Obviously, there are many other variations we could study. For example, x and y could vary simultaneously. But in this section, we restrict our attention to variations in *either* x *or* y.

When we consider functions of two variables, mathematicians (and economists) usually write $\partial z/\partial x$ instead of dz/dx for the derivative of z w.r.t. x when y is held fixed. This slight change of notation, replacing d by ∂, is intended to remind the reader that only one independent variable is changing, with the other(s) held fixed. In the same way, we write $\partial z/\partial y$ instead of $\mathrm{d}z/\mathrm{d}y$ when y varies and x is held fixed. Hence, we have

$$z = x^3 + 2y^2 \;\Rightarrow\; \frac{\partial z}{\partial x} = 3x^2 \quad \text{and} \quad \frac{\partial z}{\partial y} = 4y$$

In general, we introduce the following definitions:

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If $z = f(x, y)$, then

$$\partial z/\partial x \text{ is the derivative of } f(x, y) \text{ w.r.t. } x, \text{ when } y \text{ is held constant} \tag{11.2.1}$$

$$\partial z/\partial y \text{ is the derivative of } f(x, y) \text{ w.r.t. } y, \text{ when } x \text{ is held constant} \tag{11.2.2}$$

When $z = f(x, y)$, we also denote the derivative $\partial z/\partial x$ by $\partial f/\partial x$, and this is called the *partial derivative of z (or f) w.r.t. x*; $\partial z/\partial y = \partial f/\partial y$ is the *partial derivative of z (or f) w.r.t. y*. Note that $\partial f/\partial x$ is the rate of change of $f(x, y)$ w.r.t. x when y is constant, and correspondingly for $\partial f/\partial y$. Of course, because there are two variables, there are two partial derivatives.

It is usually easy to find the partial derivatives of a function $z = f(x, y)$. To find $\partial f/\partial x$, just think of y as a constant and differentiate $f(x, y)$ w.r.t. x as if f were a function only of x. The rules for finding derivatives of functions of one variable can all be used when we want to compute $\partial f/\partial x$. The same is true for $\partial f/\partial y$. Let us look at some further examples.

EXAMPLE 11.2.1 Find the partial derivatives of the following functions:

(a) $f(x, y) = x^3y + x^2y^2 + x + y^2$ (b) $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Solution:

(a) We find, holding y constant,

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy^2 + 1$$

while, holding x constant,

$$\frac{\partial f}{\partial y} = x^3 + 2x^2y + 2y$$

(b) For this function the quotient rule gives

$$\frac{\partial f}{\partial x} = \frac{y(x^2 + y^2) - xy \cdot 2x}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}, \qquad \frac{\partial f}{\partial y} = \frac{x^3 - y^2x}{(x^2 + y^2)^2}$$

Observe that the function is symmetric in x and y, in the sense that its value is unchanged if we interchange x and y. By interchanging x and y in the formula for $\partial f/\partial x$, therefore, we will find the correct formula for $\partial f/\partial y$.

It is a good exercise for you to find $\partial f/\partial y$ in the usual way and check that the foregoing answer is correct.

Several other forms of notation are often used to indicate the partial derivatives of $z = f(x, y)$. Some of the most common are

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z'_x = f'_x(x, y) = f'_1(x, y) = \frac{\partial f(x, y)}{\partial x} = \frac{\partial f}{\partial x}(x, y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = z'_y = f'_y(x, y) = f'_2(x, y) = \frac{\partial f(x, y)}{\partial y} = \frac{\partial f}{\partial y}(x, y)$$

Among these, we find $f'_1(x, y)$ and $f'_2(x, y)$ to be the most satisfactory. Here the numerical subscript refers to the position of the argument in the function. Thus, f'_1 indicates the partial derivative w.r.t. the first variable, and f'_2 w.r.t. the second variable. This notation also reminds us that the partial derivatives themselves are functions of x and y. Finally, $f'_1(a, b)$ and $f'_2(a, b)$ are suitable designations of the values of the partial derivatives at point (a, b) instead of at (x, y). For example, given the function $f(x, y) = x^3y + x^2y^2 + x + y^2$ in Example 11.2.1(a), one has

$$f'_1(x, y) = 3x^2y + 2xy^2 + 1, \quad f'_1(a, b) = 3a^2b + 2ab^2 + 1$$

In particular, $f'_1(0, 0) = 1$ and $f'_1(-1, 2) = 3(-1)^2 2 + 2(-1)2^2 + 1 = -1$.

We note that the alternative notation $f'_x(x, y)$ and $f'_y(x, y)$ is often used, but it is sometimes too ambiguous in connection with composite functions. For instance, what does $f'_x(x^2y, x - y)$ mean?

Remember that the numbers $f'_1(x, y)$ and $f'_2(x, y)$ measure the rate of change of f w.r.t. x and y, respectively. For example, if $f'_1(x, y) > 0$, then a small increase in x will lead to an increase in $f(x, y)$.

EXAMPLE 11.2.2 In Example 11.1.2 we studied the function $x = Ap^{-1.5}m^{2.08}$. Find the partial derivatives of x w.r.t. p and m, and discuss their signs.

Solution: We find that $\partial x/\partial p = -1.5Ap^{-2.5}m^{2.08}$ and $\partial x/\partial m = 2.08Ap^{-1.5}m^{1.08}$. Because A, p, and m are positive, $\partial x/\partial p < 0$ and $\partial x/\partial m > 0$. These signs are in accordance with the final remarks in the example.

Formal Definitions of Partial Derivatives

So far the functions have been given by explicit formulas and we have found the partial derivatives by using the ordinary rules for differentiation. If these rules cannot be used, however, we must resort to the formal definition of partial derivative. This is derived from the definition of derivative for functions of one variable in the following rather obvious way.

If $z = f(x, y)$, then with $g(x) = f(x, y)$ (y fixed), the partial derivative of $f(x, y)$ w.r.t. x is simply $g'(x)$. Now, by definition, $g'(x) = \lim_{h \to 0}[g(x + h) - g(x)]/h$. Because $f'_1(x, y) = g'(x)$, it follows that:

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Given $f(x, y)$,

$$f_1'(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{11.2.3}$$

and, similarly,

$$f_2'(x, y) = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k} \tag{11.2.4}$$

provided that the limits exist.

If the limit in (11.2.3) does not exist, we say that $f_1'(x, y)$ *does not exist*, or that z is *not differentiable* w.r.t. x at the point. Similarly, if the limit in (11.2.2) does not exist, then $f_2'(x, y)$ does not exist and z is not differentiable w.r.t. y at that point. For instance, the function $f(x, y) = |x| + |y|$ is not differentiable, w.r.t. either x or y, at the point $(x, y) = (0, 0)$.

If h is small in absolute value, then from Eq. (11.2.1) we obtain the approximation

$$f_1'(x, y) \approx \frac{f(x+h, y) - f(x, y)}{h} \tag{11.2.5}$$

Similarly, if k is small in absolute value,

$$f_2'(x, y) \approx \frac{f(x, y+k) - f(x, y)}{k} \tag{11.2.6}$$

These approximations can be interpreted as follows:

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Given $f(x, y)$:

(i) The partial derivative $f_1'(x, y)$ is approximately equal to the change in $f(x, y)$ per unit increase in x, holding y constant.

(ii) The partial derivative $f_2'(x, y)$ is approximately equal to the change in $f(x, y)$ per unit increase in y, holding x constant.

These approximations must be used with caution. Roughly speaking, they will not be too inaccurate provided that the partial derivatives do not vary too much over the actual intervals. This warning is true also for the one-variable case we first saw in Section 6.4, and then in Section 7.4. But it applies more forcefully here, as even a seemingly small variation in, say, y, can change $f_1'(x, y)$ in a significant manner. Section 12.8 and FMEA discuss approximations in detail.

EXAMPLE 11.2.3 Let $Y = F(K, L)$ be the number of units produced when K units of capital and L units of labour are used as inputs in a production process. What is the economic interpretation of $F_K'(100, 50) = 5$?

Solution: $F_K'(100, 50) = 5$ means that, starting from $K = 100$ and holding labour input fixed at 50, a small increase in K increases output by five units per unit increase in K.

Higher-Order Partial Derivatives

If $z = f(x, y)$, then $\partial f/\partial x$ and $\partial f/\partial y$ are called *first-order partial derivatives*. These partial derivatives are, in general, again functions of the two variables. From $\partial f/\partial x$, provided this derivative is itself differentiable, we can generate two new functions by taking the partial derivatives w.r.t. x and y. In the same way, we can take the partial derivatives of $\partial f/\partial y$ w.r.t. x and y. The four functions we obtain by differentiating twice in this way are called *second-order partial derivatives* of $f(x, y)$. They are expressed as

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

For brevity, we sometimes refer to the first- and second-order "partials", suppressing the word "derivatives".

EXAMPLE 11.2.4 For the function in part (a) of Example 11.2.1, we obtain

$$\frac{\partial^2 f}{\partial x^2} = 6xy + 2y^2, \quad \frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 4xy, \quad \frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 4xy, \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 2x^2 + 2$$

As with first-order partial derivatives, several other kinds of notation are also frequently used for second-order partial derivatives. For example, $\partial^2 f/\partial x^2$ is also denoted by $f''_{11}(x, y)$ or $f''_{xx}(x, y)$. In the same way, $\partial^2 f/\partial y \partial x$ can also be written as $f''_{12}(x, y)$ or $f''_{xy}(x, y)$. Note that $f''_{12}(x, y)$ means that we differentiate $f(x, y)$ first w.r.t. the first argument x and then w.r.t. the second argument y. To find $f''_{21}(x, y)$, we must differentiate in the reverse order. In Example 11.2.4, these two "cross" second-order partial derivatives (or "mixed-partials") are equal. For most functions $z = f(x, y)$, it will actually be the case that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{11.2.7}$$

Sufficient conditions for this equality are given in Theorem 11.6.1.

It is very important to note the exact meaning of the different symbols that have been introduced. If we consider Eq. (11.2.7), for example, it would be a serious mistake to believe that the two expressions are equal because $\partial x \partial y$ is the same as $\partial y \partial x$. Here the expression on the left-hand side is in fact the derivative of $\partial f/\partial y$ w.r.t. x, and the right-hand side is the derivative of $\partial f/\partial x$ w.r.t. y. It is a remarkable fact, and not a triviality, that the two are usually equal.

It is also important to observe that $\partial^2 z/\partial x^2$ is quite different from $(\partial z/\partial x)^2$. For example, if $z = x^2 + y^2$, then $\partial z/\partial x = 2x$. Therefore, $\partial^2 z/\partial x^2 = 2$, whereas $(\partial z/\partial x)^2 = 4x^2$.

Analogously, we define partial derivatives of the third, fourth, and higher orders. For example, we write $\partial^4 z/\partial x \partial y^3 = z^{(4)}_{yyyx}$ when we first differentiate z three times w.r.t. y and then differentiate the result once more w.r.t. x. Here is an additional example.

EXAMPLE 11.2.5 If $f(x, y) = x^3 e^{y^2}$, find the first- and second-order partial derivatives at the point $(x, y) = (1, 0)$.

Solution: To find $f_1'(x,y)$, we differentiate $x^3e^{y^2}$ w.r.t. x while treating y as a constant. When y is a constant, so is e^{y^2}. Hence, $f_1'(x,y) = 3x^2e^{y^2}$ and so

$$f_1'(1,0) = 3 \cdot 1^2 e^{0^2} = 3$$

To find $f_2'(x,y)$, we differentiate $f(x,y)$ w.r.t. y while treating x as a constant:

$$f_2'(x,y) = x^3 2ye^{y^2} = 2x^3ye^{y^2}$$

and so $f_2'(1,0) = 0$.

To find the second-order partial $f_{11}''(x,y)$, we must differentiate $f_1'(x,y)$ w.r.t. x once more, while treating y as a constant. Hence, $f_{11}''(x,y) = 6xe^{y^2}$ and so

$$f_{11}''(1,0) = 6 \cdot 1e^{0^2} = 6$$

To find $f_{22}''(x,y)$, we must differentiate $f_2'(x,y) = 2x^3ye^{y^2}$ w.r.t. y once more, while treating x as a constant. Because ye^{y^2} is a product of two functions, each involving y, we use the product rule to obtain

$$f_{22}''(x,y) = (2x^3)(1 \cdot e^{y^2} + y2ye^{y^2}) = 2x^3e^{y^2} + 4x^3y^2e^{y^2}$$

Evaluating this at $(1,0)$ gives $f_{22}''(1,0) = 2$. Moreover,

$$f_{12}''(x,y) = \frac{\partial}{\partial y}\left[f_1'(x,y)\right] = \frac{\partial}{\partial y}(3x^2e^{y^2}) = 3x^2 2ye^{y^2} = 6x^2ye^{y^2}$$

and

$$f_{21}''(x,y) = \frac{\partial}{\partial x}\left[f_2'(x,y)\right] = \frac{\partial}{\partial x}(2x^3ye^{y^2}) = 6x^2ye^{y^2}$$

Hence, $f_{12}''(1,0) = f_{21}''(1,0) = 0$.

EXERCISES FOR SECTION 11.2

1. Find $\partial z/\partial x$ and $\partial z/\partial y$ for the following functions:

(a) $z = 2x + 3y$ (b) $z = x^2 + y^3$ (c) $z = x^3y^4$ (d) $z = (x+y)^2$

2. Find $\partial z/\partial x$ and $\partial z/\partial y$ for the following functions:

(a) $z = x^2 + 3y^2$ (b) $z = xy$ (c) $z = 5x^4y^2 - 2xy^5$ (d) $z = e^{x+y}$

(e) $z = e^{xy}$ (f) $z = e^x/y$ (g) $z = \ln(x+y)$ (h) $z = \ln(xy)$

3. Find $f_1'(x,y)$, $f_2'(x,y)$, and $f_{12}''(x,y)$ for the following functions:

(a) $f(x,y) = x^7 - y^7$ (b) $f(x,y) = x^5 \ln y$ (c) $f(x,y) = (x^2 - 2y^2)^5$

4. Find all first- and second-order partial derivatives for the following functions:

(a) $z = 3x + 4y$ (b) $z = x^3y^2$ (c) $z = x^5 - 3x^2y + y^6$

(d) $z = x/y$ (e) $z = (x-y)/(x+y)$ (f) $z = \sqrt{x^2 + y^2}$

SM **5.** Find all the first- and second-order partial derivatives for the following functions:

(a) $z = x^2 + e^{2y}$ (b) $z = y \ln x$ (c) $z = xy^2 - e^{xy}$ (d) $z = x^y$

6. The estimated production function for a certain fishery is $F(S, E) = 2.26\, S^{0.44} E^{0.48}$, where S denotes the stock of lobsters, E the harvesting effort, and $F(S, E)$ the catch.

(a) Find $F'_S(S, E)$ and $F'_E(S, E)$.

(b) Show that $SF'_S + EF'_E = kF$ for a suitable constant k.

7. Prove that if $z = (ax + by)^2$, then $xz'_x + yz'_y = 2z$.

8. Let $z = \frac{1}{2} \ln(x^2 + y^2)$. Show that $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$.

9. Suppose that if a household consumes x units of one good and y units of a second good, its satisfaction is measured by the function $s(x, y) = 2 \ln x + 4 \ln y$. Suppose that the household presently consumes 20 units of the first good and 30 units of the second. What is the approximate increase in satisfaction from consuming one extra unit of: (a) the first good? (b) the second good?

11.3 Geometric Representation

When studying functions of one variable, we saw how useful it was to represent the function by its graph in a coordinate system in the plane. This section considers how to visualize functions of two variables as having graphs which form surfaces in a three-dimensional space. We begin by introducing a coordinate system in the space.

Recall how any point in a plane can be represented by a pair of real numbers by using two mutually orthogonal coordinate lines: a rectangular coordinate system in the plane. In a similar way, points in three-dimensional space can be represented by triples of real numbers using three mutually orthogonal coordinate lines. In Fig. 11.3.1 we have drawn such a coordinate system. The three lines that are orthogonal to each other and intersect at the point O in Fig. 11.3.1 are called *coordinate axes*. They are usually called the x-axis, y-axis, and z-axis. We choose units to measure the length along each axis, and select a positive direction on each of them as indicated by the arrows.

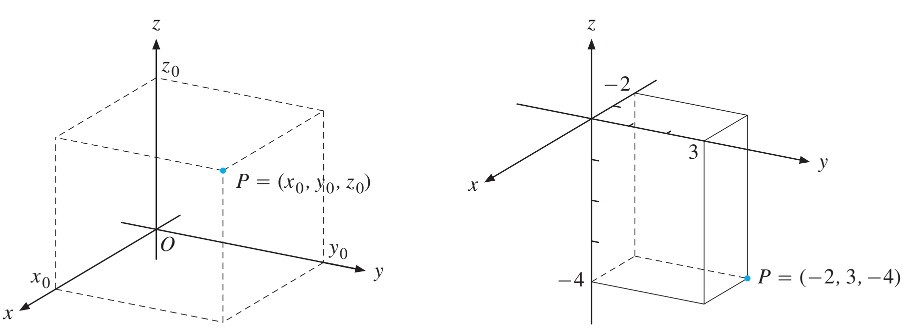

Figure 11.3.1 A coordinate system

Figure 11.3.2 $P = (-2, 3, -4)$

The equation $x = 0$ is satisfied by all points in a *coordinate plane* spanned by the y-axis and the z-axis. This is called the yz-plane. There are two other coordinate planes: the xy-plane on which $z = 0$; and the xz-plane on which $y = 0$. We often think of the xy-plane as horizontal, with the z-axis passing vertically through it.

Each coordinate plane divides the space into two *half-spaces*. For example, the xy-plane separates the space into the regions where $z > 0$, above the xy-plane, and $z < 0$, below the xy-plane. The three coordinate planes together divide up the space into eight *octants*. The octant which has $x \geq 0$, $y \geq 0$, and $z \geq 0$ is called the *nonnegative octant*.

Every point P in space now has an associated triple of numbers (x_0, y_0, z_0) that describes its location, as suggested in Fig. 11.3.1. Conversely, it is clear that every triple of numbers also represents a unique point in space in this way. Note in particular that when z_0 is negative, the point (x_0, y_0, z_0) lies below the xy-plane in which $z = 0$. In Fig. 11.3.2, we have constructed the point P with coordinates $(-2, 3, -4)$. Point P in Fig. 11.3.1 lies in the positive octant.

The Graph of a Function of Two Variables

Suppose $z = f(x, y)$ is a function of two variables defined over a domain D in the xy-plane. The *graph* of the function f is the set of all points $(x, y, f(x, y))$ in the space obtained by letting (x, y) "run through" the whole of D. If f is a sufficiently "nice" function, the graph of f will be a connected surface in the space, like the graph in Fig. 11.3.3. In particular, if (x_0, y_0) is a point in the domain D, we see how the point $P = (x_0, y_0, f(x_0, y_0))$ on the surface is obtained by letting $f(x_0, y_0)$ be the "height" of f at (x_0, y_0).

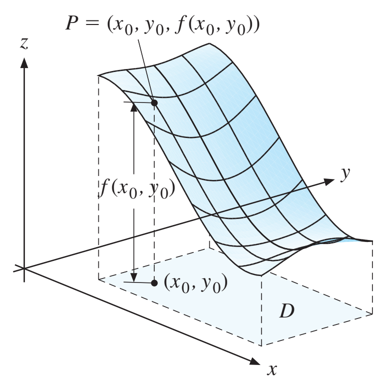

Figure 11.3.3 Graph of $y = f(x, y)$

A talented sculptor with plenty of time and resources could in principle construct this three-dimensional graph of the function $z = f(x, y)$. So could a 3D printer. Even drawing a figure like Fig. 11.3.3, which represents this graph in two dimensions, requires considerable artistic ability.[4]

We now describe a second kind of geometric representation that often does better when we are confined to two dimensions, as we are in the pages of this book.

[4] Using modern computer graphics, however, complicated functions of two variables can have their graphs drawn fairly easily, and these can be rotated or transformed to display the shape of the graph better.

Level Curves

Map makers can describe some topographical features of the earth's surface such as hills and valleys even in a plane map. The usual way of doing so is to draw a set of *level curves* or contours connecting points on the map that represent places on the earth's surface with the same elevation above sea level. For instance, there may be such contours corresponding to 100 metres above sea level, others for 200, 300, and 400 metres above sea level, and so on. Off the coast, or in places like the valley of the River Jordan, which drains into the Dead Sea, there may be contours for 100 metres below sea level, etc. Where the contours are closer together, that indicates a hill with a steeper slope. Thus, studying a contour map carefully can give a good idea how the altitude varies on the ground.

The same idea can be used to give a geometric representation of an arbitrary function $z = f(x, y)$. The graph of the function in the three-dimensional space is visualized as being cut by horizontal planes parallel to the xy-plane. The resulting intersection between each plane and the graph is then projected onto the xy-plane. If the intersecting plane is $z = c$, then the projection of the intersection onto the xy-plane is called the *level curve* at height c for f. This level curve will consist of points satisfying the equation $f(x, y) = c$. Figure 11.3.4 illustrates the construction of such a level curve.

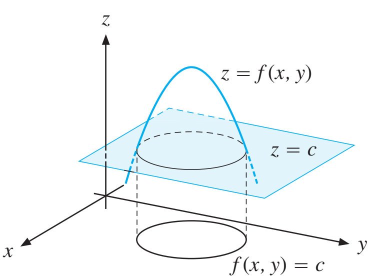

Figure 11.3.4 The graph of $z = f(x, y)$ and one of its level curves

EXAMPLE 11.3.1 Consider the function of two variables defined by the equation

$$z = x^2 + y^2 \tag{$*$}$$

What are the level curves? Draw both a set of level curves and the graph of the function.

Solution: The variable z can only assume values ≥ 0. Each level curve has the equation

$$x^2 + y^2 = c \tag{$**$}$$

for some $c \geq 0$. We see that these are circles in the xy-plane centred at the origin and with radius $\sqrt{c}$, as in Fig. 11.3.5.

As for the graph of $(*)$, all the level curves are circles. For $y = 0$, we have $z = x^2$. This shows that the graph of $(*)$ cuts the xz-plane (where $y = 0$) in a parabola. Similarly, for $x = 0$, we have $z = y^2$, which is the graph of a parabola in the yz-plane. In fact, the graph of $(*)$ is obtained by rotating the parabola $z = x^2$ around the z-axis. This surface of revolution is called a *paraboloid*, with its lowest part shown in Fig. 11.3.6. Five of the level curves in the xy-plane are also indicated.

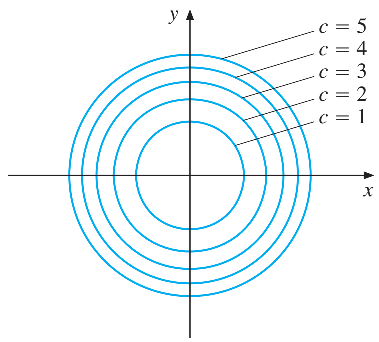

Figure 11.3.5 Solutions of $x^2 + y^2 = c$

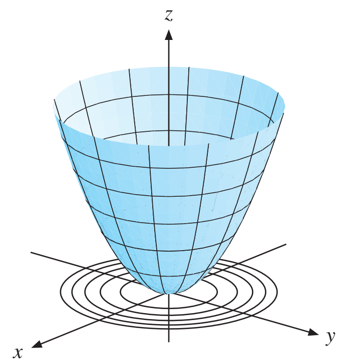

Figure 11.3.6 The graph of $z = x^2 + y^2$

EXAMPLE 11.3.2 Suppose $F(K, L)$ denotes a firm's output when its inputs of capital and labour are, respectively, K and L. A level curve for this production function is a curve in the KL-plane given by $F(K, L) = Y_0$, where Y_0 is a constant. This curve is called an *isoquant*, signifying "equal quantity". For a Cobb–Douglas function, $F(K, L) = AK^aL^b$, with $a + b < 1$ and $A > 0$, Figs 11.3.7 and 11.3.8, respectively, show a part of the graph near the origin, and three of the isoquants.

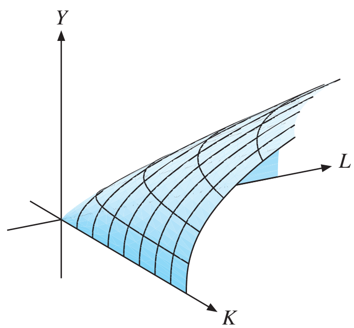

Figure 11.3.7 Graph of a Cobb–Douglas production function

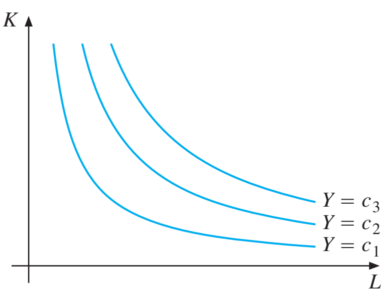

Figure 11.3.8 Isoquants of a Cobb–Douglas production function

EXAMPLE 11.3.3 Show that all points (x, y) satisfying $xy = 3$ lie on a level curve for the function

$$g(x, y) = \frac{3(xy + 1)^2}{x^4y^4 - 1}$$

Solution: By substituting $xy = 3$ in the expression for g, we find

$$g(x, y) = \frac{3(xy + 1)^2}{(xy)^4 - 1} = \frac{3(3 + 1)^2}{3^4 - 1} = \frac{48}{80} = \frac{3}{5}$$

This shows that, for all (x, y) where $xy = 3$, the value of $g(x, y)$ is a constant $3/5$. Hence, any point (x, y) satisfying $xy = 3$ is on a level curve (at height $3/5$) for g.[5]

[5] In fact, $g(x, y) = 3(c + 1)^2/(c^4 - 1)$ whenever $xy = c \neq \pm 1$, so this equation represents a level curve for g for every value of c except $c \neq \pm 1$.

Geometric Interpretations of Partial Derivatives

Partial derivatives of the first order have an interesting geometric interpretation. Let $z = f(x, y)$ be a function of two variables, with its graph as shown in Fig. 11.3.9. Let us keep the value of y fixed at y_0. The points $(x, y, f(x, y))$ on the graph of f that have $y = y_0$ are those that lie on the curve K_y indicated in the figure. The partial derivative $f'_x(x_0, y_0)$ is the derivative of $z = f(x, y_0)$ w.r.t. x at the point $x = x_0$, and is therefore the slope of the tangent line l_y to the curve K_y at $x = x_0$. In the same way, $f'_y(x_0, y_0)$ is the slope of the tangent line l_x to the curve K_x at $y = y_0$.

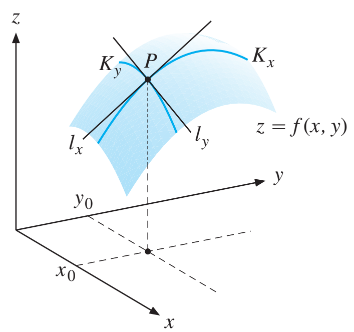

Figure 11.3.9 Partial derivatives

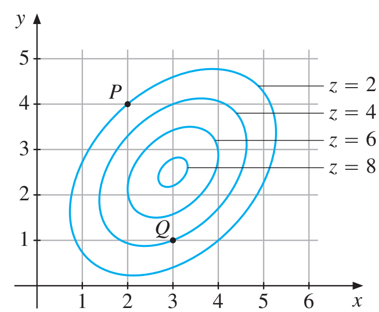

Figure 11.3.10 Level curves

This geometric interpretation of the two partial derivatives can be explained in another way. Imagine that the graph of f looks like the surface of a mountain, as in Fig. 11.3.9, and suppose that we are standing at point P with coordinates $(x_0, y_0, f(x_0, y_0))$ in three dimensions, where the height is $f(x_0, y_0)$ units above the xy-plane. The slope of the terrain at P varies as we look in different directions. In particular, suppose we look in the direction parallel to the positive x-axis. Then $f'_x(x_0, y_0)$ is a measure of the "steepness" in this direction. In the figure, $f'_x(x_0, y_0)$ is negative, because moving from P in the direction given by the positive x-axis will take us downwards. In the same way, we see that $f'_y(x_0, y_0)$ is a measure of the "steepness" in the direction parallel to the positive y-axis. We see that $f'_y(x_0, y_0)$ is positive, meaning that the slope is upward in this direction.

Let us now briefly consider the geometric interpretation of the "direct" second-order derivatives f''_{xx} and f''_{yy}. Consider the curve K_y on the graph of f in Fig. 11.3.9. It seems that along this curve, $f''_{xx}(x, y_0)$ is negative, because $f'_x(x, y_0)$ decreases as x increases. In particular, $f''_{xx}(x_0, y_0) < 0$. In the same way, we see that moving along K_x makes $f'_y(x_0, y)$ decrease as y increases, so $f''_{yy}(x_0, y) < 0$ along K_x. In particular, $f''_{yy}(x_0, y_0) < 0$. The cross-partials, f''_{xy} and f''_{yx}, do not have such easy geometric interpretations.[6]

[6] Consider again the curve K_y, and recall that its position is determined by the value of y, namely y_0, which we are keeping fixed when we compute the partial w.r.t. x. The first partial is $f'_x(x, y_0)$, which we can see as the slope of line l_y in the direction of the x-axis. Now, imagine that you increase y_0 slightly, so that the curve K_y gets pushed in the direction of the y-axis. Of course, the l_y lines gets pushed too, and its slope may change. The cross-partial f''_{xy} measures the magnitude of that change.

EXAMPLE 11.3.4 Consider Fig. 11.3.10 which shows some level curves of a function $z = f(x, y)$. On the basis of this figure, answer the following questions:

(a) What are the signs of $f'_x(x, y)$ and $f'_y(x, y)$ at the points P and Q? Estimate also the *value* of $f'_x(3, 1)$.

(b) What are the solutions of the equations: (i) $f(3, y) = 4$; and (ii) $f(x, 4) = 6$?

(c) What is the largest value that $f(x, y)$ can attain when $x = 2$, and for which y value does this maximum occur?

Solution:

(a) If you stand at P, you are on the level curve $f(x, y) = 2$. If you look in the direction of the positive x-axis, along the line $y = 4$, then you will see the terrain sloping upwards, because the nearest level curves will correspond to larger z values. Hence, $f'_x > 0$. If you stand at P and look in the direction of the positive y-axis, along $x = 2$, the terrain will slope downwards. Thus, at P, we must have $f'_y < 0$. At Q, we find similarly that $f'_x < 0$ and $f'_y > 0$. To estimate $f'_x(3, 1)$, we use $f'_x(3, 1) \approx f(4, 1) - f(3, 1) = 2 - 4 = -2$.[7]

(b) Equation (i) has the solutions $y = 1$ and $y = 4$, because the line $x = 3$ cuts the level curve $f(x, y) = 4$ at $(3, 1)$ and at $(3, 4)$. Equation (ii) has no solutions, because the line $y = 4$ does not meet the level curve $f(x, y) = 6$ at all.

(c) The highest value of c for which the level curve $f(x, y) = c$ has a point in common with the line $x = 2$ is $c = 6$. The largest value of $f(x, y)$ when $x = 2$ is therefore 6, and we see from Fig. 11.3.10 that this maximum value is attained when $y \approx 2.2$.

Gradients

We conclude this section by giving a geometric interpretation of the two partial derivatives on the xy-plane. At any point $(x, y) = (a, b)$, the two partials can be written together as the pair

$$(f'_1(a, b), f'_2(a, b)) \tag{$*$}$$

This pair, of course, can itself be represented in the plane, as in Fig. 11.3.11. In the figure, we have used $Df(a, b)$ to denote the pair $(*)$, and have indicated it by a line that connects the point with these coordinates to the origin. Suppose we add (a, b) to the pair $(*)$, giving $(a, b) + Df(a, b) = (a + f'_1(a, b), b + f'_2(a, b))$. Then we are moving the line from the origin to the point (a, b). In the figure, we have also transformed the line into an arrow, and have drawn a line that is perpendicular to it and goes through the point (a, b). We also have the level curve of function f going through that point. We will elaborate more on this in FMEA, but there are three important ideas to remember:

1. The line that is perpendicular to the arrow is also tangent to the level curve; this implies that a small change to (x, y) in the direction of that line leaves the value of the function unchanged.

[7] This approximation is actually far from exact. If we keep $y = 1$ and *decrease* x by one unit, then $f(2, 1) \approx 4$, which should give the estimate $f'_x(3, 1) \approx 4 - 4 = 0$. The "map" is not sufficiently finely graded around Q.

2. A small change to (x, y) in the direction of the arrow, on the other hand, induces the fastest possible increase in the value of the function. A step in the direction opposite to the arrow would induce the fastest possible decrease in the value of the function.
3. The length of the arrow indicates the rate of change by which the function would increase after a perturbation to (x, y) in that direction. The longer the arrow, the faster the increase.

The pair $Df(a, b) = (f_1'(a, b), f_2'(a, b))$ is a very useful object. In more advanced differential calculus, it allows our analysis to be generalized to changes in (x, y) in any direction on the plane, and not just parallel to one of the two axes. Then $Df(a, b)$ is usually thought of as an arrow, or "vector", which is called *the gradient vector of the function f at the point* (a, b).

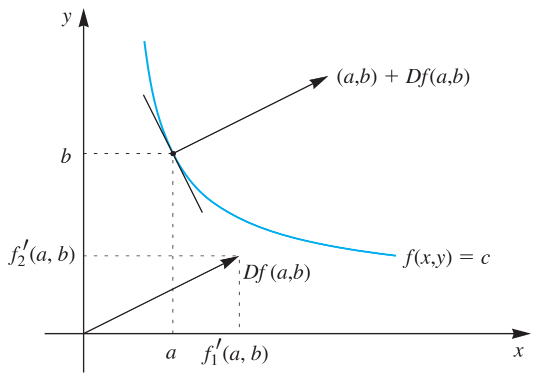

Figure 11.3.11 The gradient vector

EXERCISES FOR SECTION 11.3

1. Draw a three-dimensional coordinate system, including a box like those shown in Figs 11.3.1 and 11.3.2, and mark the points $P = (3, 0, 0)$, $Q = (0, 2, 0)$, $R = (0, 0, -1)$, and $S = (3, -2, 4)$.

2. Describe geometrically the set of points (x, y, z) in three dimensions, where: (a) $y = 2$ and $z = 3$ while x varies freely; (b) $y = x$ while z varies freely.

3. Show that $x^2 + y^2 = 6$ is a level curve of $f(x, y) = \sqrt{x^2 + y^2} - x^2 - y^2 + 2$.

4. Show that $x^2 - y^2 = c$ is a level curve of $f(x, y) = e^{x^2} e^{-y^2} + x^4 - 2x^2y^2 + y^4$ for all values of the constant c.

5. Explain why two level curves of the function $z = f(x, y)$ corresponding to different values of z cannot intersect.

6. Let $f(x)$ represent a function of one variable. If we let $g(x, y) = f(x)$, then we have defined a function of two variables, but y is not present in its formula. Explain how the graph of g is obtained from the graph of f. Illustrate with $f(x) = x$ and also with $f(x) = -x^3$.

7. Draw the graphs of the following functions in three-dimensional space, and draw a set of level curves for each of them:

(a) $z = 3 - x - y$ (b) $z = \sqrt{3 - x^2 - y^2}$

8. Figure 11.3.12 shows some level curves for the function $z = f(x, y)$.

(a) What is $f(2, 3)$? Solve the equation $f(x, 3) = 8$ for x.

(b) Find the smallest value of $z = f(x, y)$ if $x = 2$. What is the corresponding value of y?

(c) What are the signs of $f_1'(x, y)$ and $f_2'(x, y)$ at the points A, B, and C? Estimate the values of these two partial derivatives at A.

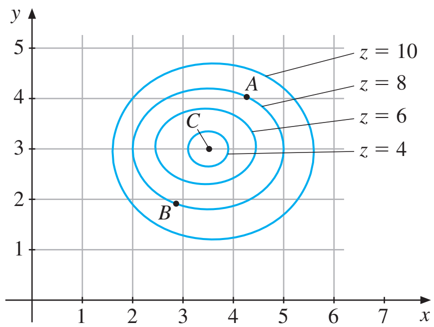

Figure 11.3.12 Exercise 8

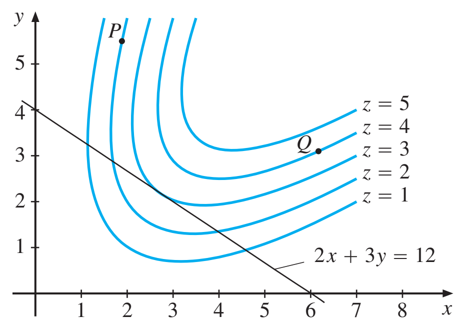

Figure 11.3.13 Exercise 9

SM 9. Figure 11.3.13 shows some level curves for $z = f(x, y)$, together with the line $2x + 3y = 12$.

(a) What are the signs of f_x' and f_y' at the points P and Q?

(b) Find possible solutions of the equations (i) $f(1, y) = 2$; and (ii) $f(x, 2) = 2$.

(c) What is the largest value of $f(x, y)$ among those (x, y) that satisfy $2x + 3y = 12$?

SM 10. [Harder] Suppose $F(x, y)$ is a function about which nothing is known except that $F(0, 0) = 0$, as well as that $F_1'(x, y) \geq 2$ for all (x, y), and $F_2'(x, y) \leq 1$ for all (x, y). What can be said about the relative sizes of $F(0, 0)$, $F(1, 0)$, $F(2, 0)$, $F(0, 1)$, and $F(1, 1)$? Write down the inequalities that have to hold between these numbers.

11.4 Surfaces and Distance

An equation in *two* variables, such as $f(x, y) = c$, can be represented by a set of points in the plane, called the graph of the equation, as in Section 5.4. In a similar way, an equation in *three* variables x, y, and z, such as $g(x, y, z) = c$, can be represented by a point set in the three-dimensional space, also called the *graph* of the equation. This graph consists of all triples (x, y, z) satisfying the equation, and will usually form what can be called a *surface* in the space.

One of the simplest types of equation in three variables is

$$ax + by + cz = d \tag{11.4.1}$$

with a, b, and c not all 0. This is the general equation for a plane in three-dimensional space. Assuming that a and b are not both 0, the graph of this equation intersects the xy-plane when $z = 0$. Then $ax + by = d$, which is a straight line in the xy-plane unless $a = b = 0$. In the same way we see that, provided at most one of a, b, and c is equal to zero, the graph intersects the two other coordinate planes in straight lines.

Let us rename the coefficients and consider the equation

$$px + qy + rz = m \tag{11.4.2}$$

where p, q, r, m are all positive. This equation can be given an economic interpretation. Suppose a household has a total budget of m to spend on three commodities, whose prices are respectively p, q, and r per unit. If the household buys x units of the first, y units of the second, and z units of the third commodity, then the total expense is $px + qy + rz$. Hence, Eq. (11.4.2) is the household's *budget equation*: only triples (x, y, z) that satisfy (11.4.2) can be bought if expenditure must equal m. The budget equation represents a *plane* in space, called the *budget plane*. Because in most cases one also has $x \geq 0$, $y \geq 0$, and $z \geq 0$, the interesting part of the plane is the triangle with vertices at $P = (m/p, 0, 0)$, $Q = (0, m/q, 0)$, and $R = (0, 0, m/r)$, as shown in Fig. 11.4.1. If we allow the household to underspend, the *budget set* is defined as

$$B = \{\, (x, y, z) : px + qy + rz \leq m,\ x \geq 0,\ y \geq 0,\ z \geq 0 \,\}$$

This represents the three-dimensional body bounded by the three coordinate planes and the budget plane. It generalizes the two-commodity budget set discussed in Example 4.4.7.

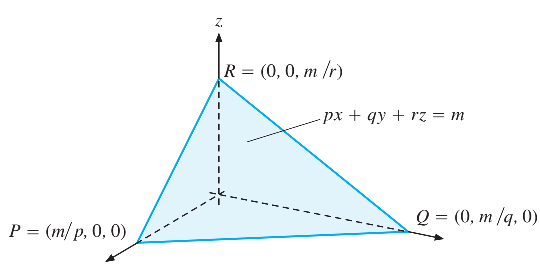

Figure 11.4.1 A budget set when there are three goods

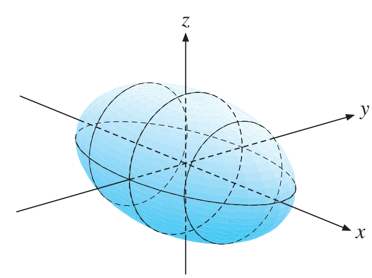

Figure 11.4.2 $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ where $a > b = c$ (a rugby ball)

Another rather interesting surface appears in Fig. 11.4.2. This surface is called an *ellipsoid*, which some readers may recognize as having the shape of a rugby ball.

The Distance Formula

In Section 5.5 we gave the formula for the distance between two points in the plane. Now we want to do the same for points in the three-dimensional space.

Consider a rectangular box with edges of length a, b, and c, as shown in Fig. 11.4.3. By Pythagoras's theorem, $(PR)^2 = a^2 + b^2$, and $(PQ)^2 = (PR)^2 + (RQ)^2 = a^2 + b^2 + c^2$, so that the box has diagonal of length $PQ = \sqrt{a^2 + b^2 + c^2}$.

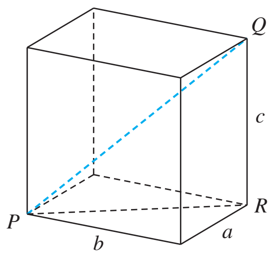

Figure 11.4.3 The distance between points P and Q, denoted PQ

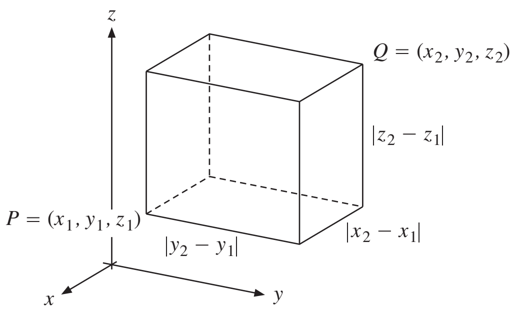

Figure 11.4.4 The distance between two typical points

Next we find the distance between two typical points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in space, as illustrated in Fig. 11.4.4. These two points lie precisely at the corners of a rectangular box with edges of lengths $a = |x_2 - x_1|$, $b = |y_2 - y_1|$, and $c = |z_2 - z_1|$. Hence

$$
\begin{aligned}
(PQ)^2 = a^2 + b^2 + c^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\
&= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2
\end{aligned}
$$

This motivates the following definition:

DISTANCE BETWEEN TWO POINTS

The *distance* between the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{11.4.3}
$$

EXAMPLE 11.4.1 Calculate the distance, d, between the points $(1, 2, -3)$ and $(-2, 4, 5)$.

Solution: According to formula (11.4.3),

$$
d = \sqrt{(-2-1)^2 + (4-2)^2 + (5-(-3))^2} = \sqrt{(-3)^2 + 2^2 + 8^2} = \sqrt{77} \approx 8.77
$$

Let (a, b, c) be a point in space. The sphere with radius r and centre at (a, b, c) is the set of all points (x, y, z) whose distance from (a, b, c) is equal to r. Using the distance formula, we obtain

$$
\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r
$$

Squaring each side yields:

EQUATION FOR A SPHERE

The equation for the *sphere* with centre at (a, b, c) and radius r is

$$
(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \tag{11.4.4}
$$

EXAMPLE 11.4.2 Find the equation for the sphere with centre at $(-2, -2, -2)$ and radius 4.

Solution: According to formula (11.4.4), the equation is

$$(x - (-2))^2 + (y - (-2))^2 + (z - (-2))^2 = 4^2$$

or

$$(x + 2)^2 + (y + 2)^2 + (z + 2)^2 = 16$$

EXAMPLE 11.4.3 How do you interpret the expression $(x + 4)^2 + (y - 3)^2 + (z + 5)^2$? As: (i) the sphere with centre at the point $(-4, 3, -5)$, (ii) the distance between the points (x, y, z) and $(-4, 3, -5)$, or (iii) the square of the distance between the points (x, y, z) and $(-4, 3, -5)$?

Solution: Only (iii) is correct.

EXERCISES FOR SECTION 11.4

1. Sketch graphs of the surfaces in space described by each of the following three equations:

   (a) $x = a$ (b) $y = b$ (c) $z = c$

2. Find the distances between the following two pairs of points:

   (a) $(-1, 2, 3)$ and $(4, -2, 0)$ (b) (a, b, c) and $(a + 1, b + 1, c + 1)$

3. Find the equation for the sphere with centre at $(2, 1, 1)$ and radius 5.

4. What is the geometric interpretation of the equation $(x + 3)^2 + (y - 3)^2 + (z - 4)^2 = 25$?

5. The graph of $z = x^2 + y^2$ is a paraboloid—see Fig. 11.3.6. If the point (x, y, z) lies on this paraboloid, interpret the expression $(x - 4)^2 + (y - 4)^2 + (z - 1/2)^2$.

11.5 Functions of More Variables

Many of the most important functions we study in economics, such as the GDP of a country, depend on a very large number of variables. Mathematicians and economists express this dependence by saying that GDP is a *function* of the different variables.

Any ordered collection of n numbers $(x_1, x_2, \ldots, x_n)$ is called an *n-vector*. To save space, n-vectors are often denoted by bold letters. For example, we write $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

FUNCTIONS OF n VARIABLES

Given a set D of n-vectors, *a function f of n variables, $x_1, \ldots, x_n$, with domain* D is a rule that assigns a specified number

$$f(\mathbf{x}) = f(x_1, \ldots, x_n) \tag{11.5.1}$$

to each n-vector $\mathbf{x} = (x_1, \ldots, x_n)$ in D.

EXAMPLE 11.5.1

(a) The demand for sugar in the USA in the period 1929–1935 was estimated to be described, approximately, by the formula

$$x = 108.83 - 6.0294p + 0.164w - 0.4217t$$

where x is the demand for sugar, p is its price, w is a production index, and t is the date (where $t = 0$ corresponds to 1929).

(b) The following formula is an estimate for the demand for beer in the UK:

$$x = 1.058x_1^{0.136}x_2^{-0.727}x_3^{0.914}x_4^{0.816}$$

Here the quantity demanded, x, is a function of four variables: x_1, the income of the individual; x_2, the price of beer; x_3, a general price index for all other commodities; and x_4, the strength of the beer.

The simpler of the functions in Example 11.5.1 is the one in part (a). The variables p, w, and t occur here only to the first power, and they are only multiplied by constants, not by each other. Such functions are called *linear*. In general,

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b \tag{11.5.2}$$

where $a_1, a_2, \ldots, a_n$, and b are constants, is a *linear function* in n variables.[8]

The function in part (b) of the example is a special case of the general Cobb–Douglas function

$$F(x_1, x_2, \ldots, x_n) = Ax_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \tag{11.5.3}$$

where $A > 0$, $a_1, \ldots, a_n$ are constants, defined for $x_1 > 0, x_2 > 0, \ldots, x_n > 0$. We use this function very often in this book.

Note that taking the natural logarithm of each side of Eq. (11.5.3) gives

$$\ln F = \ln A + a_1 \ln x_1 + a_2 \ln x_2 + \cdots + a_n \ln x_n \tag{11.5.4}$$

This shows that the Cobb–Douglas function is *log-linear* (or ln-linear), because $\ln F$ is a linear function of $\ln x_1, \ln x_2, \ldots, \ln x_n$.

EXAMPLE 11.5.2 Suppose an economist interested in the price of apples records n observations in different stores. Suppose the results are the n positive numbers $x_1, x_2, \ldots, x_n$. In statistics, several different measures for their average value are used. Some of the most common are generalized versions of the ones seen in Exercise 2.6.9:

(a) the *arithmetic* mean: $\bar{x}_A = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$

(b) the *geometric* mean: $\bar{x}_G = \sqrt[n]{x_1x_2\ldots x_n}$

---

[8] This is rather common terminology, although many mathematicians would insist that f should really be called *affine* if $b \neq 0$, and *linear* only if $b = 0$.

(c) the *harmonic* mean: $\bar{x}_H = \dfrac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$

Note that $\bar{x}_A$ is a linear function of $x_1, \ldots, x_n$, whereas $\bar{x}_G$ and $\bar{x}_H$ are nonlinear functions. ($\bar{x}_G$ is log-linear.) For example, if four observations are $x_1 = 1, x_2 = 2, x_3 = 3$, and $x_4 = 4$, then

$$\bar{x}_A = (1+2+3+4)/4 = 2.5, \quad \bar{x}_G = \sqrt[4]{1 \cdot 2 \cdot 3 \cdot 4} = \sqrt[4]{24} \approx 2.21$$
$$\text{and} \quad \bar{x}_H = [(1/1 + 1/2 + 1/3 + 1/4)/4]^{-1} = 48/25 = 1.92$$

In this case $\bar{x}_H < \bar{x}_G < \bar{x}_A$. It turns out that the corresponding weak inequalities

$$\bar{x}_H \le \bar{x}_G \le \bar{x}_A \tag{11.5.5}$$

are valid in general.[9]

EXAMPLE 11.5.3 An individual must decide what quantities of n different commodities to buy during a given time period. Consumer demand theory often assumes that the individual has a utility function $U(x_1, x_2, \ldots, x_n)$ representing preferences, and that this measures the satisfaction the individual obtains by acquiring x_1 units of good 1, x_2 units of good 2, and so on. This is an important economic example of a function of n variables, to which we return several times.

One model of consumer demand is the *linear expenditure system*, which is based on the particular utility function

$$U\left(x_1, x_2, \ldots, x_n\right) = a_1 \ln\left(x_1 - c_1\right) + a_2 \ln\left(x_2 - c_2\right) + \cdots + a_n \ln(x_n - c_n)$$

that depends on the $2n$ nonnegative parameters $a_1, a_2, \ldots, a_n$ and $c_1, c_2, \ldots, c_n$. Here, each c_i represents the quantity of the commodity numbered i that the consumer needs to survive. Some, or even all, of the constants c_i could be 0.

Because $\ln z$ is only defined when $z > 0$, we see that all n inequalities $x_1 > c_1, x_2 > c_2, \ldots, x_n > c_n$ must be satisfied if $U(x_1, x_2, \ldots, x_n)$ is to be defined. Of course, the condition $a_i > 0$ implies that the consumer prefers more of the particular good i.

Continuity

The concept of continuity for functions of one variable may be generalized to functions of several variables. Roughly speaking, a function of n variables is *continuous* if small changes in the independent variables induce small changes in the function value. Just as in the one-variable case, we have the following useful rule:

PRESERVATION OF CONTINUITY

Any function of n variables that can be constructed from continuous functions by combining the operations of addition, subtraction, multiplication, division, and functional composition is continuous wherever it is defined.

[9] Recall that Exercise 2.6.9 asked you to show Eq. (11.5.5) for the case $n = 2$.

If a function of one variable is continuous, it will also be continuous when considered as a function of several variables. For example, $f(x, y, z) = x^2$ is a continuous function of x, y, and z: small changes in x, y, and z give at most small changes in x^2.

EXAMPLE 11.5.4 Where are the functions given by the following formulas continuous?

(a) $f(x, y, z) = x^2y + 8x^2y^5z - xy + 8z$ (b) $g(x, y) = \dfrac{xy - 3}{x^2 + y^2 - 4}$

Solution:

(a) As the sum of products of positive powers, f is defined and continuous for all x, y, and z.

(b) The function g is defined and continuous for all pairs (x, y) except those that lie on the circle $x^2 + y^2 = 4$. There the denominator is zero, and so $g(x, y)$ is not defined.

Euclidean *n*-Dimensional Space

No concrete geometric interpretation is possible for functions of three or more variables. Yet we can still use *geometric language* when dealing with functions of n variables. It is usual to call the set of all possible n-vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of real numbers the *Euclidean n-dimensional space*, or *n-space*, and to denote it by $\mathbb{R}^n$. For $n = 1$, $n = 2$, and $n = 3$, we have geometric representations of $\mathbb{R}^n$ as a line, a plane, and a three-dimensional space, respectively. But for $n \geq 4$, there is no geometric representation.

If $z = f(x_1, x_2, \dots, x_n) = f(\mathbf{x})$ represents a function of n variables, we define the *graph* of f as the set of all points $(\mathbf{x}, f(\mathbf{x}))$ in $\mathbb{R}^{n+1}$ for which $\mathbf{x}$ belongs to the domain of f. We also call the graph a *surface* (or sometimes a *hypersurface*) in $\mathbb{R}^{n+1}$. For $z = z_0$ (constant), the set of points in $\mathbb{R}^n$ satisfying $f(\mathbf{x}) = z_0$ is called a *level surface* of f. When $f(x)$ is a linear function such as $a_1x_1 + a_2x_2 + \cdots + a_nx_n + b$, this surface, which would be a plane if $n = 3$, is called a *hyperplane* when $n > 3$.

In both producer and consumer theory, it is usual to give level surfaces a different name. If $x = f(\mathbf{v}) = f(v_1, v_2, \dots, v_n)$ is the amount produced when the input quantities of n different factors of production are respectively $v_1, v_2, \dots, v_n$, the level surfaces where $f(v_1, v_2, \dots, v_n) = x_0$ (constant) are called *isoquants*, as in Example 11.3.2. On the other hand, if $u = U(\mathbf{x})$ is a utility function that represents the consumer's preferences, the level surface where $U(x_1, x_2, \dots, x_n) = u_0$ is called an *indifference surface*.

EXERCISES FOR SECTION 11.5

1. Let $f(x, y, z) = xy + xz + yz$.

   (a) Find $f(-1, 2, 3)$ and $f(a + 1, b + 1, c + 1) - f(a, b, c)$.

   (b) Show that $f(tx, ty, tz) = t^2 f(x, y, z)$ for all t.

2. A study of milk production found that

$$y = 2.90\, x_1^{0.015} x_2^{0.250} x_3^{0.350} x_4^{0.408} x_5^{0.030}$$

   where y is the output of milk, and $x_1, \dots, x_5$ are the quantities of five different input factors.

(a) If all the factors of production were doubled, what would happen to y?

(b) Write the relation in log-linear form.

SM **3.** A pension fund decides to invest \$720 million in the shares of XYC Inc., a company with a volatile share price. Rather than invest all the money at once and so risk paying an unduly high price, the fund practises "dollar cost averaging" by investing \$120 million per week in six successive weeks. The prices it pays are \$50 per share in the first week, then \$60, \$45, \$40, \$75, and finally \$80 in the sixth week.

(a) How many shares in total does it buy?

(b) Which is the most accurate representation of the average price: the arithmetic mean, the geometric mean, or the harmonic mean?

4. An American bank, A, and a European bank, E, agree a currency swap. In n successive weeks $w = 1, 2, \ldots, n$, bank A will buy \$100 million worth of euros from bank E, at a price of p_w dollars per euro determined by the spot exchange rate at the end of week w. After n weeks:

(a) How many euros will bank A have bought?

(b) What is the dollar price per euro it will have paid, on average?

5. [HARDER] It is observed that three machines A, B, and C produce, respectively, 60, 80, and 40 units of a product during one workday lasting eight hours. The average output is then 60 units per day. We see that A, B, and C use, respectively, eight, six, and 12 minutes to make one unit.

(a) If all machines were equally efficient and jointly produced $60 + 80 + 40 = 180$ units during a day, then how much time would be required to produce each unit?[10]

(b) Suppose that n machines $A_1, A_2, \ldots, A_n$ produce the same product simultaneously during a time interval of length T. Given that the production times per unit are respectively $t_1, t_2, \ldots, t_n$, find the total output Q. Show that if all the machines were equally efficient and together had produced exactly the same total amount Q in the time span T, then the time needed for each machine to produce one unit would be precisely the harmonic mean $\bar{t}_H$ of $t_1, t_2, \ldots, t_n$.

11.6 Partial Derivatives with More Variables

The last section gave several economic examples of functions involving many variables. Accordingly, we need to extend the concept of partial derivative to functions of more than two variables.

PARTIAL DERIVATIVES IN n VARIABLES

If $z = f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, then $\partial f/\partial x_i$, for $i = 1, 2, \ldots, n$, means the partial derivative of $f(x_1, x_2, \ldots, x_n)$ w.r.t. x_i, when all the other variables x_j, for $j \neq i$, are held constant.

[10] Note that the answer is not $(8 + 6 + 12)/3$.

So, provided that they all exist, there are n partial derivatives of first order, one for each variable x_i, for $i = 1, \ldots, n$. Other notation used for the first-order partials of $z = f(x_1, x_2, \ldots, x_n)$ includes

$$\frac{\partial f}{\partial x_i} = \frac{\partial z}{\partial x_i} = \partial z/\partial x_i = z'_i = f'_i\left(x_1, x_2, \ldots, x_n\right)$$

EXAMPLE 11.6.1 Find the three first-order partials of $f(x_1, x_2, x_3) = 5x_1^2 + x_1x_2^3 - x_2^2x_3^2 + x_3^3$.

Solution: We find that

$$f'_1 = 10x_1 + x_2^3,\ f'_2 = 3x_1x_2^2 - 2x_2x_3^2,\ f'_3 = -2x_2^2x_3 + 3x_3^2$$

As in (11.2.5), we have the following rough approximation:

APPROXIMATE PARTIAL DERIVATIVE

The partial derivative $\partial z/\partial x_i$ is approximately equal to the per-unit change in $z = f(x_1, x_2, \ldots, x_n)$ caused by an increase in x_i, while holding constant all the other x_j for $j \neq i$.

In symbols, for small h one has

$$f'_i(x_1, \ldots, x_n) \approx \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{h} \tag{11.6.1}$$

For each of the n first-order partials of f, we have n second-order partials:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i} = z''_{ij}$$

provided that all the derivatives exist. Here both i and j may take any value 1, 2, …, n, so altogether there are n^2 second-order partials.

It is usual to display these second-order partials in an $n \times n$ square array as follows

$$f''(\mathbf{x}) = \begin{pmatrix} f''_{11}(\mathbf{x}) & f''_{12}(\mathbf{x}) & \ldots & f''_{1n}(\mathbf{x}) \\ f''_{21}(\mathbf{x}) & f''_{22}(\mathbf{x}) & \ldots & f''_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{n1}(\mathbf{x}) & f''_{n2}(\mathbf{x}) & \ldots & f''_{nn}(\mathbf{x}) \end{pmatrix} \tag{11.6.2}$$

Such rectangular arrays of numbers (or symbols) are called *matrices*, and (11.6.2) is called the *Hessian matrix* of f at $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.[11]

[11] See Chapter 15 for more discussion of matrices in general.

The n second-order partial derivatives f''_{ii} found by differentiating twice w.r.t. the same variable are called *direct second-order partials*; the others, f''_{ij} where $i \neq j$, are *mixed* or *cross* partials.

EXAMPLE 11.6.2 Find the Hessian matrix of the function f defined in Example 11.6.1.

Solution: We differentiate the first-order partials found in Example 11.6.1. The resulting Hessian is

$$\begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix} = \begin{pmatrix} 10 & 3x_2^2 & 0 \\ 3x_2^2 & 6x_1x_2 - 2x_3^2 & -4x_2x_3 \\ 0 & -4x_2x_3 & -2x_2^2 + 6x_3 \end{pmatrix}$$

Young's Theorem

If $z = f(x_1, x_2, \dots, x_n)$, then the two second-order cross-partial derivatives z''_{ij} and z''_{ji} are usually equal. That is,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

This implies that the order of differentiation does not matter. The next theorem makes precise a more general result.

THEOREM 11.6.1 (YOUNG'S THEOREM)

> Suppose that all the m-th-order partial derivatives of the function $f(x_1, x_2, \dots, x_n)$ are continuous. If any two of them involve differentiating w.r.t. each of the variables the same number of times, then they are necessarily equal.

The content of this result can be explained as follows: Let $m = m_1 + \cdots + m_n$, and suppose that $f(x_1, x_2, \dots, x_n)$ is differentiated m_1 times w.r.t. x_1, m_2 times w.r.t. x_2, …, and m_n times w.r.t. x_n.[12] Suppose that the continuity condition is satisfied for these m-th-order partial derivatives. Then we end up with the same result no matter what is the order of differentiation, because each of the final partial derivatives is equal to

$$\frac{\partial^m f}{\partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_n^{m_n}}$$

In particular, for the case when $m = 2$, for $i = 1, \dots, n$ and $j = 1, \dots, n$,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

if both these partials are continuous. A proof of Young's theorem is given in most advanced calculus books. Exercise 11 shows that the crossed partial derivatives are not always equal.

[12] Some of the integers $m_1, \dots, m_n$ can be zero, of course.

Formal Definitions of Partial Derivatives

In Section 11.2, we gave a formal definition of partial derivatives for functions of two variables. This was done by modifying the definition of the derivative for a function of one variable. The same modification works for a function of n variables.

Indeed, if $z = f(x_1, \ldots, x_n)$, then with

$$g(x_i) = f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

we have $\partial z/\partial x_i = g'(x_i)$, where we think of all the variables x_j other than x_i as constants. If we use the definition of $g'(x_i)$, as in (6.2.2), we obtain

$$\frac{\partial z}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h} \tag{11.6.3}$$

if the limit exists.

As in Section 11.2, if the limit in (11.6.3) does not exist, then we say that $\partial z/\partial x_i$ *does not exist*, or that z is not differentiable w.r.t. x_i at the point. Similarly, the approximation in (11.6.1) holds because the fraction on the right-hand side of Eq. (11.6.3) is close to the limit if $h \neq 0$ is small enough.

Virtually all the functions we consider have continuous partial derivatives everywhere in their domains. If $z = f(x_1, x_2, \ldots, x_n)$ has continuous partial derivatives of first order in a domain D, we call f *continuously differentiable* in D. In this case, f is also called a C^1 *function* on D. If all partial derivatives up to order k exist and are continuous, then f is called a C^k *function*.

EXERCISES FOR SECTION 11.6

1. Calculate $F_1'(1, 1, 1)$, $F_2'(1, 1, 1)$, and $F_3'(1, 1, 1)$ for $F(x, y, z) = x^2 e^{xz} + y^3 e^{xy}$.

SM 2. Calculate all first-order partials of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$ (b) $f(x, y, z) = 5x^2 - 3y^3 + 3z^4$ (c) $f(x, y, z) = xyz$

(d) $f(x, y, z) = x^4/yz$ (e) $f(x, y, z) = (x^2 + y^3 + z^4)^6$ (f) $f(x, y, z) = e^{xyz}$

3. Let x and y be the populations of two cities and d the distance between them. Suppose that the number of travellers T between the cities is given by $T = kxy/d^n$, where k and n are positive constants. Find $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial d$, and discuss their signs.

4. Let g be defined by

$$g(x, y, z) = 2x^2 - 4xy + 10y^2 + z^2 - 4x - 28y - z + 24$$

for all (x, y, z).

(a) Calculate $g(2, 1, 1)$, $g(3, -4, 2)$, and $g(1, 1, a + h) - g(1, 1, a)$.

(b) Find all partial derivatives of the first and second orders.

5. Let $\pi(p, r, w) = \frac{1}{4}p^2(1/r + 1/w)$. Find the partial derivatives of π w.r.t. p, r, and w.

6. Find all first- and second-order partials of $w(x, y, z) = 3xyz + x^2y - xz^3$.

7. If $f(x, y, z) = p(x) + q(y) + r(z)$, what are f_1', f_2', and f_3'?

8. Find the Hessian matrices of: (a) $f(x, y, z) = ax^2 + by^2 + cz^2$; (b) $g(x, y, z) = Ax^ay^bz^c$.

9. Prove that if $w = \left(\dfrac{x - y + z}{x + y - z}\right)^h$, then $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = 0$.

SM 10. Define the function $f(x, y, z) = x^{y^z}$ for $x > 0$, $y > 0$, and $z > 0$. Find its first-order partial derivatives by differentiating $\ln f$.

SM 11. [HARDER] Define the function $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$ when $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Find expressions for $f_1'(0, y)$ and $f_2'(x, 0)$, then show that $f_{12}''(0, 0) = -1$ and $f_{21}''(0, 0) = 1$. Check that Young's theorem is not contradicted because f_{12}'' and f_{21}'' are both discontinuous at point $(0, 0)$.

11.7 Economic Applications

This section considers several economic applications of partial derivatives.

EXAMPLE 11.7.1 Consider an agricultural production function $Y = F(K, L, T)$, where Y is the number of units produced, K is capital invested, L is labour input, and T is the area of agricultural land that is used. Then $\partial Y/\partial K = F_K'$ is called the *marginal product of capital*. It is the rate of change of output Y w.r.t. K when L and T are held fixed. Similarly, $\partial Y/\partial L = F_L'$ and $\partial Y/\partial T = F_T'$ are the *marginal products of labour and of land*, respectively. For example, if K is the value of capital equipment measured in dollars, and $\partial Y/\partial K = 5$, then increasing capital input by h units would increase output by approximately $5h$ units.

Suppose, in particular, that F is the Cobb–Douglas function $F(K, L, T) = AK^aL^bT^c$, where A, a, b, and c are positive constants. Find the marginal products, and the second-order partials. Discuss their signs.

Solution: The marginal products are

$$F_K' = AaK^{a-1}L^bT^c,\ F_L' = AbK^aL^{b-1}T^c, \text{ and } F_T' = AcK^aL^bT^{c-1}$$

Assuming K, L, and T are all positive, the marginal products are positive. Thus, an increase in capital, labour, or land will increase the number of units produced.

The cross second-order partials, also called mixed partials, are:[13]

$$F_{KL}'' = AabK^{a-1}L^{b-1}T^c,\ F_{KT}'' = AacK^{a-1}L^bT^{c-1}, \text{ and } F_{LT}'' = AbcK^aL^{b-1}T^{c-1}$$

[13] Check for yourself that F_{LK}'', F_{TK}'', and F_{TL}'' give, respectively, the same results.

Note that these partials are positive. We call each pair of factors *complementary*, because more of one increases the marginal product of the other.

The direct second-order partials are

$$F''_{KK} = Aa(a-1)K^{a-2}L^bT^c,\ F''_{LL} = Ab(b-1)K^aL^{b-2}T^c,\ F''_{TT} = Ac(c-1)K^aL^bT^{c-2}$$

For instance, F''_{KK} is the partial derivative of the marginal product of capital, F'_K, w.r.t. K. If $a < 1$, then $F''_{KK} < 0$, and there is a diminishing marginal product of capital—that is, a small increase in the capital invested will lead to a decrease in the marginal product of capital. We can interpret this to mean that, although small increases in capital cause output to rise, so that $F'_K > 0$, this rise occurs at a decreasing rate, since $F''_{KK} < 0$. Similarly for labour if $b < 1$, and for land if $c < 1$.

EXAMPLE 11.7.2 Let x be the GDP of a country, and let y be a measure of its level of pollution. If the function $u(x, y)$ purports to measure the total well-being of the society, what signs do you expect $u'_x(x, y)$ and $u'_y(x, y)$ to have? Can you guess what economists usually assume about the sign of $u''_{xy}(x, y)$?

Solution: It is reasonable to expect that well-being increases as GDP increases, but decreases as the level of pollution increases. Hence, we will usually have $u'_x(x, y) > 0$ and $u'_y(x, y) < 0$. According to (11.6.2), $u''_{xy} = (\partial/\partial y)(u'_x)$ is approximately equal to the change in u'_x per unit increase in y, the level of pollution. Moreover, u'_x is, approximately, the increase in welfare per unit increase in x.

It is often assumed that $u''_{xy} < 0$. This implies that the increase in welfare obtained by an extra unit of x will decrease when the level of pollution increases.[14] Because of Young's Theorem, 11.6.1, the inequality $u''_{xy} < 0$ also implies that $u''_{yx} < 0$. Thus the increase in welfare obtained from being exposed to one unit less pollution, which is approximately $-u'_y$, increases with consumption x. This accords with the controversial view that as people can afford to consume more, they also become less tolerant of pollution.

EXERCISES FOR SECTION 11.7

1. The demand for money, M, in the USA for the period 1929–1952 has been estimated as

$$M = 0.14Y + 76.03(r-2)^{-0.84}$$

where Y is the annual national income, and the interest rate is r% per year, with $r > 2$. Find $\partial M/\partial Y$ and $\partial M/\partial r$, then discuss their signs.

SM 2. If a and b are constants, compute the expression $KY'_K + LY'_L$ for the following:

(a) $Y = AK^a + BL^a$ (b) $Y = AK^aL^b$ (c) $Y = \dfrac{K^2L^2}{aL^3 + bK^3}$

[14] An analogy: When a confirmed nonsmoker sits in a room filled with tobacco smoke, the extra satisfaction from one more piece of cake will decrease if the concentration of smoke increases too much.

3. The demand for a product, D, depends on the price p of the product and on the price q charged by a competing producer. It is $D(p, q) = a - bpq^{-\alpha}$, where a, b, and α are positive constants with $\alpha < 1$. Find $D'_p(p, q)$ and $D'_q(p, q)$, and comment on the signs of the partial derivatives.

4. Let $F(K, L, M) = AK^a L^b M^c$. Show that $KF'_K + LF'_L + MF'_M = (a + b + c)F$.

5. Let $D(p, q)$ and $E(p, q)$ be the demands for two commodities when the prices per unit are p and q, respectively. Suppose the commodities are *substitutes* in consumption, such as butter and margarine. What are the normal signs of the partial derivatives of D and E w.r.t. p and q?

6. Find $\partial U/\partial x_i$ when $U(x_1, x_2, \ldots, x_n) = 100 - e^{-x_1} - e^{-x_2} - \cdots - e^{-x_n}$.

SM 7. [HARDER] Calculate the expression $KY'_K + LY'_L$ for the CES function $Y = Ae^{\lambda t}\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho}$.

11.8 Partial Elasticities

Section 7.7 introduced the concept of elasticity for functions of one variable. Here we study the corresponding concept for functions of several variables. This enables us to distinguish between, for instance, the price and income elasticities of demand, as well as between different price elasticities.

Two Variables

If $z = f(x, y)$, we define the partial elasticity of z w.r.t. x and y by

$$\mathrm{El}_x z = \frac{x}{z}\frac{\partial z}{\partial x}, \qquad \mathrm{El}_y z = \frac{y}{z}\frac{\partial z}{\partial y} \tag{11.8.1}$$

Often economists just refer to the elasticity rather than the partial elasticity. Thus, $\mathrm{El}_x z$ is the elasticity of z w.r.t. x when y is held constant, and $\mathrm{El}_y z$ is the elasticity of z w.r.t. y when x is held constant. The number $\mathrm{El}_x z$ is, approximately, the percentage change in z caused by a 1% increase in x when y is held constant, and $\mathrm{El}_y z$ has a corresponding interpretation.

As in Section 7.7, when all the variables are positive, elasticities can be expressed as logarithmic derivatives. Accordingly,

$$\mathrm{El}_x z = \frac{\partial \ln z}{\partial \ln x}, \quad \text{and} \quad \mathrm{El}_y z = \frac{\partial \ln z}{\partial \ln y} \tag{11.8.2}$$

EXAMPLE 11.8.1 Find the (partial) elasticities of z w.r.t. x when: (a) $z = Ax^a y^b$; (b) $z = xye^{x+y}$.

Solution:

(a) When finding the elasticity of $Ax^a y^b$ w.r.t. x, the variable y, and thus Ay^b, is held constant. From Example 7.7.1 we obtain $\mathrm{El}_x z = a$. In the same way, $\mathrm{El}_y z = b$.

(b) It is convenient here to use Eq. (11.8.2). Assuming all variables are positive, taking appropriate natural logarithms gives $\ln z = \ln x + \ln y + x + y = \ln x + \ln y + e^{\ln x} + y$. Hence $\mathrm{El}_x z = \partial \ln z / \partial \ln x = 1 + e^{\ln x} = 1 + x$.

EXAMPLE 11.8.2 The demand D_1 for potatoes in the USA, for the period 1927 to 1941, was estimated to be $D_1 = Ap^{-0.28}m^{0.34}$, where p is the price of potatoes and m is mean income. The demand for apples was estimated to be $D_2 = Bq^{-1.27}m^{1.32}$, where q is the price of apples.

Find the price elasticities of demand, $\mathrm{El}_p D_1$ and $\mathrm{El}_q D_2$, as well as the income elasticities of demand $\mathrm{El}_m D_1$ and $\mathrm{El}_m D_2$, and comment on their signs.

Solution: According to part (a) of Example 11.8.1, $\mathrm{El}_p D_1 = -0.28$. If the price of potatoes increases by 1%, demand decreases by approximately 0.28%. Furthermore, $\mathrm{El}_q D_2 = -1.27$, $\mathrm{El}_m D_1 = 0.34$, and $\mathrm{El}_m D_2 = 1.32$.

Both price elasticities $\mathrm{El}_p D_1$ and $\mathrm{El}_q D_2$ are negative, so demand decreases when the price increases in both cases, as seems reasonable. Both income elasticities $\mathrm{El}_m D_1$ and $\mathrm{El}_m D_2$ are positive, so demand increases when mean income increases—as seems reasonable. Note that the demand for apples is more sensitive to both price and income increases than is the demand for potatoes. This also seems reasonable, since at that time potatoes were a more essential commodity than apples for most consumers.

More Variables

If $z = f(x_1, x_2, \ldots, x_n) = f(\mathbf{x})$, we define the *(partial) elasticity* of z, or of f, w.r.t. x_i as the elasticity of z w.r.t. x_i when all the other variables are held constant. Thus, assuming all the variables are positive, we can write

$$\mathrm{El}_i z = \frac{x_i}{f(\mathbf{x})} \frac{\partial f(\mathbf{x})}{\partial x_i} = \frac{x_i}{z} \frac{\partial z}{\partial x_i} = \frac{\partial \ln z}{\partial \ln x_i} \tag{11.8.3}$$

The number $\mathrm{El}_i z$ is approximately equal to the percentage change in z caused by a 1% increase in x_i, the i-th variable, keeping all the other x_j constant. Among other forms of notation commonly used instead of $\mathrm{El}_i z$, we mention: $\mathrm{El}_i f(\mathbf{x})$, $\mathrm{El}_{x_i} z$, ε_i, e_i, and $\hat{z}_i$. The latter, of course, is pronounced "z hat i".

EXAMPLE 11.8.3 Suppose $D = Ax_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is defined for all $x_1 > 0$, $x_2 > 0$, …, $x_n > 0$, where $A > 0$ and $a_1, a_2, \ldots, a_n$ are constants. Find the elasticity of D w.r.t. x_i, for $i = 1, \ldots, n$.

Solution: Because all the factors except $x_i^{a_i}$ are constant, we can apply Eq. (7.7.3) to obtain the result $\mathrm{El}_i D = a_i$.

As a special case of this example, suppose that $D_i = Am^{\alpha} p_i^{-\beta} p_j^{\gamma}$, where m is income, p_i is the own price, and p_j is the price of a substitute good. Then α is the income elasticity of demand defined as in Example 11.8.2. On the other hand, $-\beta$ is the elasticity of demand w.r.t. changes in its own price p_i, so it is called the *own-price elasticity* of demand. However, because own-price elasticities of demand are usually negative, one often describes

β rather than $-\beta$ as being the own-price elasticity of demand. Finally, γ is the elasticity of demand w.r.t. the price of the specified substitute. By analogy with the cross-partial derivatives defined in Section 11.6, it is called a *cross-price elasticity* of demand.

Note that the proportion of income spent on good i is

$$\frac{p_i D_i}{m} = A m^{\alpha-1} p_i^{1-\beta} p_j^{\gamma}$$

When the income elasticity $\alpha < 1$, this proportion is a decreasing function of income. Economists describe a good with this property as a *necessity*. On the other hand, when $\alpha > 1$, the proportion of income spent on good i rises with income, in which case economists describe good i as a *luxury*. Referring back to Example 11.8.2, these definitions imply that during the period 1927–1941, which includes the years of the Great Depression, potatoes were a necessity, but apples a (relative) luxury.

Exercise 4 considers this distinction between necessities and luxuries for more general demand functions.

EXERCISES FOR SECTION 11.8

1. Find the partial elasticities of z w.r.t. x and y in the following cases:

   (a) $z = xy$ (b) $z = x^2 y^5$ (c) $z = x^n e^x y^n e^y$ (d) $z = x + y$

2. Let $z = (a x_1^d + b x_2^d + c x_3^d)^g$, where a, b, c, d, and g are constants. Find $\sum_{i=1}^{3} \mathrm{El}_i\, z$.

3. Let $z = x_1^p \cdots x_n^p \exp(a_1 x_1 + \cdots + a_n x_n)$, where $a_1, \ldots, a_n$, and p are constants. Find the partial elasticities of z w.r.t. $x_1, \ldots, x_n$.

SM 4. Let $D(p, m)$ indicate a typical consumer's demand for a particular commodity, as a function of its price p and the consumer's own income m. Show that the proportion pD/m of income spent on the commodity increases with income if $\mathrm{El}_m D > 1$ (in which case the good is a "luxury", whereas it is a "necessity" if $\mathrm{El}_m D < 1$).

REVIEW EXERCISES

1. Let $f(x, y) = 3x - 5y$. Calculate $f(0, 1)$, $f(2, -1)$, $f(a, a)$, and $f(a + h, b) - f(a, b)$.

2. Let $f(x, y) = 2x^2 - 3y^2$. Calculate $f(-1, 2)$, $f(2a, 2a)$, $f(a, b + k) - f(a, b)$, and $f(tx, ty) - t^2 f(x, y)$.

3. Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Calculate $f(3, 4, 0)$, $f(-2, 1, 3)$, and $f(tx, ty, tz)$ for $t \geq 0$.

4. Let $Y = F(K, L) = 15K^{1/5}L^{2/5}$ denote the number of units produced when K units of capital and L units of labour are used as inputs.

   (a) Compute $F(0, 0)$, $F(1, 1)$, and $F(32, 243)$.

   (b) Find an expression for $F(K + 1, L) - F(K, L)$, and give an economic interpretation.

(c) Compute $F(32+1, 243) - F(32, 243)$, and compare the result with what you get by calculating $F'_K(32, 243)$.

(d) Show that $F(tK, tL) = t^k F(K, L)$ for a constant k.

5. According to a study of industrial fishing, the annual herring catch is given by the production function $Y(K, S) = 0.06157K^{1.356}S^{0.562}$ involving the catching effort K and the herring stock S.

(a) Find $\partial Y/\partial K$ and $\partial Y/\partial S$.

(b) If K and S are both doubled, what happens to the catch?

6. For which pairs of numbers (x, y) are the functions given by the following formulas defined?

(a) $3xy^3 - 45x^4 - 3y$ (b) $\sqrt{1 - xy}$ (c) $\ln(2 - (x^2 + y^2))$

7. For which pairs of numbers (x, y) are the functions given by the following formulas defined?

(a) $1/\sqrt{x + y - 1}$ (b) $\sqrt{x^2 - y^2} + \sqrt{x^2 + y^2 - 1}$ (c) $\sqrt{y - x^2} - \sqrt{\sqrt{x} - y}$

8. Complete the following implications:

(a) $z = (x^2y^4 + 2)^5 \;\Rightarrow\; \dfrac{\partial z}{\partial x} =$

(b) $F(K, L) = (\sqrt{K} + \sqrt{L})^2 \;\Rightarrow\; \sqrt{K}\dfrac{\partial F}{\partial K} =$

(c) $F(K, L) = (K^a + L^a)^{1/a} \;\Rightarrow\; KF'_K(K, L) + LF'_L(K, L) =$

(d) $g(t, w) = \dfrac{3t}{w} + wt^2 \;\Rightarrow\; \dfrac{\partial^2 g}{\partial w \partial t} =$

(e) $g(t_1, t_2, t_3) = (t_1^2 + t_2^2 + t_3^2)^{1/2} \;\Rightarrow\; g'_3(t_1, t_2, t_3) =$

(f) $f(x, y, z) = 2x^2yz - y^3 + x^2z^2 \;\Rightarrow\; f'_1(x, y, z) =$, and $f''_{13}(x, y, z) =$

9. Let f be defined for all (x, y) by $f(x, y) = (x-2)^2(y+3)^2$.

(a) Calculate $f(0, 0)$, $f(-2, -3)$, and $f(a+2, b-3)$.

(b) Find f'_x and f'_y.

10. Verify that the two points $(-1, 5)$ and $(1, 1)$ both lie on the same level curve for the function $g(x, y) = (2x + y)^3 - 2x + 5/y$.

11. For each $c \neq 0$, verify that $x - y = c$ is a level curve for $F(x, y) = \ln(x^2 - 2xy + y^2) + e^{2x-2y}$.

SM 12. Let f be defined for all (x, y) by $f(x, y) = x^4 + 2y^2 - 4x^2y + 4y$.

(a) Find $f'_1(x, y)$ and $f'_2(x, y)$.

(b) Find all pairs (x, y) which solve both equations $f'_1(x, y) = 0$ and $f'_2(x, y) = 0$.

13. Find the partial elasticities of z w.r.t. x and y in the following cases:

(a) $z = x^3y^{-4}$ (b) $z = \ln(x^2 + y^2)$ (c) $z = e^{x+y}$ (d) $z = (x^2 + y^2)^{1/2}$

14. (a) If $F(x, y) = e^{2x}(1 - y)^2$, find $\partial F/\partial y$.

(b) If $F(K, L, M) = (\ln K)(\ln L)(\ln M)$, find F'_L and F''_{LM}.

(c) If $w = x^x y^x z^x$, with x, y, and z positive, find w'_x using logarithmic differentiation.

15. [HARDER] Compute $\partial^{p+q} z/\partial y^q \partial x^p$ at $(0, 0)$ for the following:

(a) $z = e^x \ln(1 + y)$

(b) $z = e^{x+y}(xy + y - 1)$ (*Hint:* First prove by, induction on n that $\dfrac{d^n}{du^n} e^u u = e^u(u + n)$.)

16. [HARDER] Show that, if $u = Ax^a y^b$, then $u''_{xy}/u'_x u'_y$ can be expressed as a function of u alone. Use this to prove that

$$\frac{1}{u'_x}\frac{\partial}{\partial x}\left(\frac{u''_{xy}}{u'_x u'_y}\right) = \frac{1}{u'_y}\frac{\partial}{\partial y}\left(\frac{u''_{xy}}{u'_x u'_y}\right)$$

12 TOOLS FOR COMPARATIVE STATICS

Logic merely sanctions the conquests of the intuition.
—Jacques S. Hadamard (1945)

Comparative statics is a particular technique that features very prominently in economic analysis. One question it addresses is how economic quantities, such as demand and supply, which are determined as endogenous variables that satisfy an equation system, respond to changes in exogenous parameters, like price. More generally, what happens to the solution of an optimization problem when the parameters of the problem change? Or to the solution of equations that describe an equilibrium of demand and supply? Simple examples will be studied in this chapter and the next two; more demanding problems are treated in FMEA.

Section 12.5 discusses the concept of elasticity of substitution, which is often used by economists to characterize the "curvature" of level curves.

Homogeneous and homothetic functions are important in economics. They are studied in Sections 12.6 and 12.7. The last sections of the chapter consider linear approximations, then differentials, and finally systems of equations, together with some properties that result from differentiating such systems.

12.1 A Simple Chain Rule

Many economic models involve composite functions. These are functions of one or several variables in which the variables are themselves functions of other basic variables. For example, many models of economic growth regard output as a function of capital and labour, both of which are functions of time. How does output vary with time?

More generally, what happens to the value of a composite function as its basic variables change? This is the general problem we discuss in this and the next sections.

Suppose z is a function of x and y, with $z = f(x, y)$, where x and y both are functions of a variable t, with $x = g(t)$ and $y = h(t)$. Substituting for x and y in $z = f(x, y)$ gives the composite function

$$z = F(t) = f(g(t), h(t))$$

This reduces z to a function of t alone. A change in t will in general lead to changes in both $g(t)$ and $h(t)$, and as a result, $z = F(t)$ changes. So, how does z change when t changes? For example, will a small increase in t lead to an increase or a decrease in z? Such questions would become much easier to answer if we could find an expression for $\mathrm{d}z/\mathrm{d}t$, the rate of change of z w.r.t. t. This is given by the following rule:

THE CHAIN RULE

When $z = f(x, y)$ with $x = g(t)$ and $y = h(t)$, then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f_1'(x, y)\frac{\mathrm{d}x}{\mathrm{d}t} + f_2'(x, y)\frac{\mathrm{d}y}{\mathrm{d}t} \tag{12.1.1}$$

It is important to understand the precise content of (12.1.1). It gives the derivative of $z = f(x, y)$ w.r.t. t when x and y are both differentiable functions of t. This derivative is called the *total derivative* of z w.r.t. t. According to (12.1.1), one contribution to the total derivative occurs because the first variable in $f(x, y)$, namely x, depends on t. This contribution is $f_1'(x, y)\,\mathrm{d}x/\mathrm{d}t$. A second contribution arises because the second variable in $f(x, y)$, namely y, also depends on t. This contribution is $f_2'(x, y)\,\mathrm{d}y/\mathrm{d}t$. The total derivative $\mathrm{d}z/\mathrm{d}t$ is the *sum* of the two contributions.

EXAMPLE 12.1.1 Find $\mathrm{d}z/\mathrm{d}t$ when $z = f(x, y) = x^2 + y^3$ with $x = t^2$ and $y = 2t$.

Solution: In this case $f_1'(x, y) = 2x$, $f_2'(x, y) = 3y^2$, $\mathrm{d}x/\mathrm{d}t = 2t$, and $\mathrm{d}y/\mathrm{d}t = 2$. So formula (12.1.1) gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 2x \cdot 2t + 3y^2 \cdot 2 = 4tx + 6y^2 = 4t^3 + 24t^2$$

where the last equality comes from substituting the appropriate functions of t for x and y respectively. In a simple case like this, we can verify the chain rule by substituting $x = t^2$ and $y = 2t$ in the formula for $f(x, y)$ and then differentiating w.r.t. t. The result is

$$z = x^2 + y^3 = (t^2)^2 + (2t)^3 = t^4 + 8t^3 \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t} = 4t^3 + 24t^2$$

as before.

EXAMPLE 12.1.2 Find $\mathrm{d}z/\mathrm{d}t$ when $z = f(x, y) = xe^{2y}$ with $x = \sqrt{t}$ and $y = \ln t$.

Solution: Here $f_1'(x, y) = e^{2y}$, $f_2'(x, y) = 2xe^{2y}$, $\mathrm{d}x/\mathrm{d}t = 1/2\sqrt{t}$, and $\mathrm{d}y/\mathrm{d}t = 1/t$. Now $y = \ln t$ implies that $e^{2y} = e^{2\ln t} = (e^{\ln t})^2 = t^2$, so formula (12.1.1) gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = e^{2y}\frac{1}{2\sqrt{t}} + 2xe^{2y}\frac{1}{t} = t^2\frac{1}{2\sqrt{t}} + 2\sqrt{t}t^2\frac{1}{t} = \frac{5}{2}t^{3/2}$$

As in Example 12.1.1, we can verify the chain rule directly by substituting $x = \sqrt{t}$ and $y = \ln t$ in the formula for $f(x, y)$, implying that $z = xe^{2y} = \sqrt{t} \cdot t^2 = t^{5/2}$, whose derivative is $\mathrm{d}z/\mathrm{d}t = \frac{5}{2}t^{3/2}$.

Here are some rather typical examples of ways in which economists use (12.1.1).

EXAMPLE 12.1.3 Let $D = D(p, m)$ denote the demand for a commodity as a function of price p and income m. Suppose that price p and income m vary continuously with time t, so that $p = p(t)$ and $m = m(t)$. Then demand can be determined as a function $D = D(p(t), m(t))$ of t alone. Find an expression for $\dot{D}/D$, the relative rate of growth of D.

Solution: Using (12.1.1) we obtain

$$\dot{D} = \frac{\partial D(p,m)}{\partial p}\dot{p} + \frac{\partial D(p,m)}{\partial m}\dot{m}$$

where we have denoted time derivatives by "dots". The first term on the right-hand side gives the effect on demand that arises because the price p is changing, and the second term gives the effect of the change in m. We can write the relative rate of growth of D as

$$\frac{\dot{D}}{D} = \frac{p}{D}\frac{\partial D(p,m)}{\partial p}\frac{\dot{p}}{p} + \frac{m}{D}\frac{\partial D(p,m)}{\partial m}\frac{\dot{m}}{m} = \frac{\dot{p}}{p}\,\mathrm{El}_p D + \frac{\dot{m}}{m}\,\mathrm{El}_m D$$

So the relative rate of growth is found by multiplying the relative rates of change of price and income by their respective elasticities, then adding.

EXAMPLE 12.1.4 As in Example 11.7.2, let $u(x, y)$ denote the total well-being of a society, where x denotes GDP and y denotes a measure of the level of pollution. Assume that $u'_x(x, y) > 0$ and $u'_y(x, y) < 0$. Suppose the level of pollution is some increasing function $y = h(x)$ of x, with $h'(x) > 0$. Then total well-being becomes a function

$$U(x) = u(x, h(x))$$

of x alone. Find a necessary condition for $U(x)$ to have a maximum at $x = x^* > 0$, and give this condition an economic interpretation.

Solution: By Theorem 8.1.1, a necessary condition for $U(x)$ to have a maximum at $x^* > 0$ is that $U'(x^*) = 0$. In order to find $U'(x)$, we use the chain rule (12.1.1):

$$U'(x) = u'_x(x, h(x)) \cdot 1 + u'_y(x, h(x)) \cdot h'(x)$$

So $U'(x^*) = 0$ requires that

$$u'_x(x^*, h(x^*)) = -u'_y(x^*, h(x^*))h'(x^*) \tag{$*$}$$

To illustrate this condition, consider increasing x^* by a small amount ξ, which can be positive or negative. By Eq. (11.2.5), the gain due to the increase in GDP is approximately $u'_x(x^*, h(x^*))\xi$. On the other hand, the level of pollution increases by about $h'(x^*)\xi$ units. But we lose $u'_y(x^*, h(x^*))$ in well-being per unit increase in pollution. So all in all we lose about $u'_y(x^*, h(x^*))h'(x^*)\xi$ from the extra pollution resulting from this increase in x^*. Equation $(*)$ just states that what we gain directly from increasing x^* by any small amount ξ can be neither greater nor less than what we lose indirectly through increased pollution: otherwise a small change ξ in the right direction would increase well-being slightly.

Higher-Order Derivatives

Sometimes we use the second derivative of a composite function. A general formula for d^2z/dt^2, based on formula (12.1.1), is suggested in Exercise 8. Here we derive a special case of interest in optimization theory. It concerns the function F that records what happens to f as one moves away from (x_0, y_0) in the direction (ℓ, k) or, when $t < 0$, in the reverse direction $(-\ell, -k)$. See also Fig. 13.3.3.

EXAMPLE 12.1.5 Suppose $z = f(x, y)$ where $x = x_0 + t\ell$ and $y = y_0 + tk$. Keeping (x_0, y_0) and (ℓ, k) fixed, z becomes a function only of t. So we can write $z = F(t)$. Find expressions for $F'(t)$ and $F''(t)$.

Solution: With $x = x_0 + t\ell$ and $y = y_0 + tk$, we have $F(t) = f(x, y)$. Using (12.1.1) we get

$$F'(t) = f_1'(x, y)\frac{dx}{dt} + f_2'(x, y)\frac{dy}{dt} = f_1'(x_0 + t\ell, y_0 + tk)\ell + f_2'(x_0 + t\ell, y_0 + tk)k$$

To find the second derivative $F''(t)$, we have to differentiate a second time w.r.t. t. This yields

$$F''(t) = \frac{d}{dt}f_1'(x, y)\ell + \frac{d}{dt}f_2'(x, y)k \tag{$*$}$$

To evaluate the derivatives on the right-hand side, we must use the chain rule, (12.1.1), again. This gives

$$\frac{d}{dt}f_1'(x, y) = f_{11}''(x, y)\frac{dx}{dt} + f_{12}''(x, y)\frac{dy}{dt} = f_{11}''(x, y)\ell + f_{12}''(x, y)k$$
$$\frac{d}{dt}f_2'(x, y) = f_{21}''(x, y)\frac{dx}{dt} + f_{22}''(x, y)\frac{dy}{dt} = f_{21}''(x, y)\ell + f_{22}''(x, y)k$$

Assuming that $f_{12}'' = f_{21}''$, inserting these expressions into $(*)$ gives

$$F''(t) = f_{11}''(x, y)\ell^2 + 2f_{12}''(x, y)\ell k + f_{22}''(x, y)k^2$$

where $x = x_0 + t\ell$, $y = y_0 + tk$.

A Proof of the Chain Rule

In order to show that the chain rule is valid, none of the earlier rules for derivatives can be applied. Instead, we must provide a new argument.

Suppose that $z = f(x, y)$ is continuously differentiable, while $x = g(t)$ and $y = h(t)$ are both differentiable. Fix t_0 in the domains of g and h, and denote $x_0 = g(t_0)$ and $y_0 = h(t_0)$. Using the handier notation $F(t) = f(g(t), h(t))$, we want to prove the following version of Eq. (12.1.1)

$$F'(t_0) = f_1'(x_0, y_0)g'(t_0) + f_2'(x_0, y_0)h'(t_0)$$

Define the following functions:

$$\varphi_1(x, y) = \begin{cases} \dfrac{f(x, y) - f(x_0, y)}{x - x_0} & \text{if } x \neq x_0 \\ f_1'(x_0, y) & \text{if } x = x_0 \end{cases} \quad \text{and} \quad \varphi_2(x, y) = \begin{cases} \dfrac{f(x, y) - f(x, y_0)}{y - y_0} & \text{if } y \neq y_0 \\ f_2'(x, y_0) & \text{if } y = y_0 \end{cases}$$

Define also:

$$\gamma(t) = \begin{cases} \dfrac{g(t) - x_0}{t - t_0} & \text{if } t \neq t_0 \\ g'(t_0) & \text{if } t = t_0 \end{cases} \quad \text{and} \quad \eta(t) = \begin{cases} \dfrac{h(t) - y_0}{t - t_0} & \text{if } t \neq t_0 \\ h'(t_0) & \text{if } t = t_0 \end{cases}$$

By construction, the following properties hold:

(i) for all y, $\lim_{x \to x_0} \varphi_1(x, y) = \varphi_1(x_0, y)$;
(ii) for all x, $\lim_{y \to y_0} \varphi_2(x, y) = \varphi_2(x, y_0)$;
(iii) $\lim_{t \to t_0} \gamma(t) = \gamma(t_0)$ and $\lim_{t \to t_0} \eta(t) = \eta(t_0)$;
(iv) $f(x, y) - f(x_0, y) = \varphi_1(x, y)(x - x_0)$;
(v) $f(x, y) - f(x, y_0) = \varphi_2(x, y)(y - y_0)$; and
(vi) $g(t) - x_0 = \gamma(t)(t - t_0)$ and $h(t) - y_0 = \eta(t)(t - t_0)$.

Now, for k close to 0,

$$\begin{aligned} F(t_0 + k) - F(t_0) &= f(g(t_0 + k), h(t_0 + k)) - f(g(t_0), h(t_0)) \\ &= f(g(t_0 + k), h(t_0 + k)) - f(x_0, h(t_0 + k)) + f(x_0, h(t_0 + k)) - f(x_0, y_0) \\ &= \varphi_1(g(t_0 + k), h(t_0 + k))[g(t_0 + k) - x_0] + \varphi_2(x_0, h(t_0 + k))[h(t_0 + k) - y_0] \\ &= \varphi_1(g(t_0 + k), h(t_0 + k))\gamma(t_0 + k)k + \varphi_2(x_0, h(t_0 + k))\eta(t_0 + k)k \end{aligned}$$

where the first equality is by definition, the second one just adds and subtracts the same number, the third one uses properties (iv) and (v), and the last one uses property (vi).

The Newton quotient for $F(t_0)$ is, therefore,

$$\frac{F(t_0 + k) - F(t_0)}{k} = \varphi_1(g(t_0 + k), h(t_0 + k))\gamma(t_0 + k) + \varphi_2(x_0, h(t_0 + k))\eta(t_0 + k)$$

By definition, it follows that

$$F'(t_0) = \lim_{k \to 0} [\varphi_1(g(t_0 + k), h(t_0 + k))\gamma(t_0 + k) + \varphi_2(x_0, h(t_0 + k))\eta(t_0 + k)]$$

By property (iii), $\gamma(t_0 + k) \to g'(t_0)$ and $\eta(t_0 + k) \to h'(t_0)$, as $k \to 0$. Since h is differentiable, it is also continuous and $h(t_0 + k) \to y_0$, which implies, by (ii), that $\varphi_2(x_0, h(t_0 + k)) \to f_2'(x_0, y_0)$. We only have the term $\varphi_1(g(t_0 + k), h(t_0 + k))$ left, which is slightly more complicated. Since g is continuous too, $g(t_0 + k) \to x_0$. Since f is continuously differentiable, f_1' and φ_1 are continuous functions, and, therefore,

$$\lim_{k \to 0} \varphi_1(g(t_0 + k), h(t_0 + k)) = \varphi_1(x_0, y_0) = f_1'(x_0, y_0)$$

using property (i).

EXERCISES FOR SECTION 12.1

1. In the following cases, find dz/dt by using the chain rule (12.1.1). Check the answers by first substituting the expressions for x and y and then differentiating.

(a) $F(x, y) = x + y^2, \ x = t^2, \ y = t^3$ (b) $F(x, y) = x^p y^q, \ x = at, \ y = bt$

2. Find dz/dt when: (a) $F(x, y) = x \ln y + y \ln x$, $x = t + 1$, and $y = \ln t$; (b) $F(x, y) = \ln x + \ln y$, $x = Ae^{at}$, and $y = Be^{bt}$.

3. If $z = F(t, y)$ and $y = g(t)$, find a formula for dz/dt. Consider in particular the case where $z = t^2 + ye^y$ and $y = t^2$.

4. If $Y = F(K, L)$ and $K = g(L)$, find a formula for dy/dL.

5. Let $Y = 10KL - \sqrt{K} - \sqrt{L}$. Suppose too that $K = 0.2t + 5$ and $L = 5e^{0.1t}$. Find dy/dt when $t = 0$.

SM 6. Let $x = g(t)$, $y = h(t)$, and $G(x)$ be differentiable functions. What do you get if you apply the chain rule, Eq. (12.1.1), when $f(x, y)$ is as follows? (a) $x + y$; (b) $x - y$; (c) $x \cdot y$; (d) x/y; (e) $G(x)$.

SM 7. [Harder] Consider Example 12.1.4, and let $u(x, z) = \ln(x^\alpha + z^\alpha) - \alpha \ln z$. Let $z = h(x) = \sqrt[3]{ax^4 + b}$, with the constants α, a, and b all positive. Find the optimal x^* in this case.

SM 8. [Harder] Suppose that $z = F(x, y)$, $x = g(t)$, and $y = h(t)$. Modify the solution to Example 12.1.5 in order to prove that

$$\frac{d^2z}{dt^2} = \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2$$

under appropriate assumptions on F, g, and h.

12.2 Chain Rules for Many Variables

Economists often need even more general chain rules than the simple one for two variables presented in the previous section. Exercise 11, for example, considers the example of a railway company whose fares for peak and off-peak fares are set by a regulatory authority. The costs it faces for running enough trains to carry all the passengers depend on demand for both kinds of journey. These demands are obviously affected by both peak and off-peak fares because some passengers will choose when to travel based on the fare difference. The general chain rule we are about to present allows us to work out how these costs change when either fare is increased.

Consider the general problem of this kind where $z = f(x, y)$, $x = g(t, s)$, and $y = h(t, s)$. In this case, z is a function of both t and s, with

$$z = F(t, s) = f(g(t, s), h(t, s))$$

Here it makes sense to look for both partial derivatives $\partial z/\partial t$ and $\partial z/\partial s$. If we keep s fixed, then z is a function of t alone, and we can therefore use the chain rule, (12.1.1). In the same way, by keeping t fixed, we can differentiate z w.r.t. s. The result is the following:

THE CHAIN RULE

If $z = F(x, y)$ with $x = f(t, s)$ and $y = g(t, s)$, then

$$\frac{\partial z}{\partial t} = F_1'(x, y)\frac{\partial x}{\partial t} + F_2'(x, y)\frac{\partial y}{\partial t} \tag{12.2.1}$$

and

$$\frac{\partial z}{\partial s} = F_1'(x, y)\frac{\partial x}{\partial s} + F_2'(x, y)\frac{\partial y}{\partial s} \tag{12.2.2}$$

EXAMPLE 12.2.1 Find $\partial z/\partial t$ and $\partial z/\partial s$ when $z = F(x, y) = x^2 + 2y^2$, with $x = t - s^2$ and $y = ts$.

Solution: We obtain

$$F_1'(x, y) = 2x,\ F_2'(x, y) = 4y,\ \frac{\partial x}{\partial t} = 1,\ \frac{\partial x}{\partial s} = -2s,\ \frac{\partial y}{\partial t} = s,\ \text{ and }\ \frac{\partial y}{\partial s} = t$$

Equations (12.2.1) and (12.2.2) therefore give:

$$\frac{\partial z}{\partial t} = 2x \cdot 1 + 4y \cdot s = 2(t - s^2) + 4tss = 2t - 2s^2 + 4ts^2$$
$$\frac{\partial z}{\partial s} = 2x \cdot (-2s) + 4y \cdot t = 2(t - s^2)(-2s) + 4tst = -4ts + 4s^3 + 4t^2 s$$

It is a good exercise to check these answers by first expressing z as a function of t and s, then differentiating.

EXAMPLE 12.2.2 Find $z_t'(1, 0)$ if $z = e^{x^2} + y^2 e^{xy}$, with $x = 2t + 3s$ and $y = t^2 s^3$.

Solution: We obtain

$$\frac{\partial z}{\partial x} = 2xe^{x^2} + y^3 e^{xy},\ \frac{\partial z}{\partial y} = 2ye^{xy} + xy^2 e^{xy},\ \frac{\partial x}{\partial t} = 2,\ \text{ and }\ \frac{\partial y}{\partial t} = 2ts^3$$

Using somewhat more concise notation, the chain rule gives

$$z_t'(t, s) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xe^{x^2} + y^3 e^{xy}) \cdot 2 + (2ye^{xy} + xy^2 e^{xy}) \cdot 2ts^3$$

When $t = 1$ and $s = 0$, then $x = 2$ and $y = 0$, so $z_t'(1, 0) = 4e^4 \cdot 2 = 8e^4$.

The General Case

In consumer demand theory, economists typically assume that a household's utility depends on the number of units of each good it is able to consume. The number of units consumed will depend in turn on the prices of these goods and on the household's income. Thus the household's utility is related, indirectly, to all the prices and to income. By how much, then, does utility respond to an increase in one of the prices, or to an increase in income? The following general chain rule extends to this kind of problem.

Suppose that $z = f(x_1, \dots, x_n)$, with $x_i = g_i(t_1, \dots, t_m)$, for each $i = 1, 2, \dots, n$. Substituting for all the variables x_i as functions of the variables t_j into function f expresses z as a *composite function*

$$z = F(t_1, \dots, t_m) = f(g_1(t_1, \dots, t_m), \dots, g_n(t_1, \dots, t_m))$$

In vector notation, $z = F(\mathbf{t}) = f(\mathbf{x}(\mathbf{t}))$. An obvious generalization of (12.2.1) and (12.2.2) is as follows:

THE GENERAL CHAIN RULE

If $z = f(x_1, \dots, x_n)$ is continuously differentiable, and $x_i = g_i(t_1, \dots, t_m)$, for each $i = 1, 2, \dots, n$, are all differentiable, then

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_j} \tag{12.2.3}$$

for each $j = 1, 2, \dots, m$.

This is an important formula that every economist should understand. A small change in a basic variable t_j sets off a chain reaction. First, every x_i depends on t_j in general, so it changes when t_j is changed. This affects z in turn. The contribution to the total derivative of z w.r.t. t_j that results from the change in x_i is $(\partial z/\partial x_i)(\partial x_i/\partial t_j)$. Formula (12.2.3) shows that $\partial z/\partial t_j$ is the sum of all these contributions. In alternative notation,

$$F'_j(\mathbf{t}) = f'_1(\mathbf{x}(\mathbf{t}))\frac{\partial g_1}{\partial t_j}(\mathbf{t}) + f'_2(\mathbf{x}(\mathbf{t}))\frac{\partial g_2}{\partial t_j}(\mathbf{t}) + \cdots + f'_n(\mathbf{x}(\mathbf{t}))\frac{\partial g_n}{\partial t_j}(\mathbf{t})$$

EXAMPLE 12.2.3 Example 11.7.1 considered an agricultural production function $Y = F(K, L, T)$, where Y is the size of the harvest, K is capital invested, L is labour, and T is the area of agricultural land used to grow the crop. Suppose that K, L, and T are all functions of time, which is denoted by t. Then, according to (12.2.3), one has

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\partial F}{\partial K}\frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\partial F}{\partial L}\frac{\mathrm{d}L}{\mathrm{d}t} + \frac{\partial F}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t}$$

In the special case when F is the Cobb–Douglas function $F(K, L, T) = AK^aL^bT^c$, then

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = aAK^{a-1}L^bT^c\frac{\mathrm{d}K}{\mathrm{d}t} + bAK^aL^{b-1}T^c\frac{\mathrm{d}L}{\mathrm{d}t} + cAK^aL^bT^{c-1}\frac{\mathrm{d}T}{\mathrm{d}t} \tag{$*$}$$

Denoting time derivatives by dots, and dividing each term in $(*)$ by $Y = AK^aL^bT^c$, we get

$$\frac{\dot{Y}}{Y} = a\frac{\dot{K}}{K} + b\frac{\dot{L}}{L} + c\frac{\dot{T}}{T}$$

The relative rate of change of output is, therefore, a weighted sum of the relative rates of change of capital, labour, and land. The weights are the respective powers a, b, and c.

EXERCISES FOR SECTION 12.2

1. Use (12.2.1) and (12.2.2) to find $\partial z/\partial t$ and $\partial z/\partial s$ for the following cases:

   (a) $z = F(x, y) = x + y^2$, where $x = t - s$, and $y = ts$;

   (b) $z = F(x, y) = 2x^2 + 3y^3$, where $x = t^2 - s$, and $y = t + 2s^3$.

SM 2. Using (12.2.1) and (12.2.2), find $\partial z/\partial t$ and $\partial z/\partial s$ for the following cases:

   (a) $z = xy^2$, where $x = t + s^2$, and $y = t^2 s$;

   (b) $z = \dfrac{x - y}{x + y}$, where $x = e^{t+s}$, and $y = e^{ts}$.

3. If $z = F(u, v, w)$ where $u = r^2$, $v = -2s^2$, and $w = \ln r + \ln s$, find $\partial z/\partial r$ and $\partial z/\partial s$.

4. If $z = F(x)$ and $x = f(t_1, t_2)$, find $\partial z/\partial t_1$ and $\partial z/\partial t_2$.

5. If $x = F(s, f(s), g(s, t))$, find $\partial x/\partial s$ and $\partial x/\partial t$.

6. If $z = F(u, v, w)$ where $u = f(x, y)$, $v = x^2 h(y)$ and $w = 1/y$, find $\partial z/\partial x$ and $\partial z/\partial y$.

7. Use the general chain rule, Eq. (12.2.3), to find $\partial w/\partial t$ for the following cases:

   (a) $w = xy^2 z^3$, where $x = t^2$, $y = s$, and $z = t$;

   (b) $w = x^2 + y^2 + z^2$, where $x = \sqrt{t + s}$, $y = e^{ts}$, and $z = s^3$.

8. Find expressions for $\mathrm{d}z/\mathrm{d}t$ when:

   (a) $z = F(t, t^2, t^3)$ (b) $z = F(t, f(t), g(t^2))$

9. Suppose $Z = G + Y^2 + r^2$, where Y and r are both functions of G. Find $\partial Z/\partial G$.

10. Suppose $Z = G + I(Y, r)$, where I is a differentiable function of two variables, and Y and r are both functions of G. Find $\partial Z/\partial G$.

11. Each week a suburban railway company has a long-run cost $C = aQ_1 + bQ_2 + cQ_1^2$ of providing Q_1 passenger kilometres of service during rush hours and Q_2 passenger kilometres during off-peak hours. As functions of the regulated fares p_1 and p_2 per kilometre for the rush hours and off-peak hours, respectively, the demands for the two kinds of service are $Q_1 = Ap_1^{-\alpha_1} p_2^{\beta_1}$ and $Q_2 = Bp_1^{\alpha_2} p_2^{-\beta_2}$, where the constants $A, B, \alpha_1, \alpha_2, \beta_1, \beta_2$ are all positive. Assuming that the company runs enough trains to meet the demand, find expressions for the partial derivatives of C w.r.t. p_1 and p_2.

SM 12. If $u = \ln(x^3 + y^3 + z^3 - 3xyz)$, show that

   (a) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3$ (b) $(x + y + z)\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}\right) = 3$

13. If $z = f(x^2 y)$, show that $x\dfrac{\partial z}{\partial x} = 2y\dfrac{\partial z}{\partial y}$.

14. Find a formula for $\partial u/\partial r$ when $u = f(x, y, z, w)$ and x, y, z, and w all are functions of two variables r and s.

15. Suppose $u = xyzw$, where $x = r + s$, $y = r - s$, $z = rs$, $w = r/s$. Find $\partial u/\partial r$ when $(r, s) = (2, 1)$.

12.3 Implicit Differentiation along a Level Curve

Economists often need to differentiate functions that are defined implicitly by an equation. Section 7.2 considered some simple cases; it is a good idea to review those examples now. Here we study the problem from a more general point of view.

Let F be a function of two variables, and consider the equation $F(x, y) = c$, where c is a constant. The equation represents a level curve for F, as in Section 11.3. Suppose this equation defines y implicitly as a function $y = f(x)$ of x in some interval I, as illustrated in Fig. 12.3.1. This means that

$$F(x, f(x)) = c \tag{12.3.1}$$

for all x in I. If f is differentiable, what is the derivative of $y = f(x)$? If the graph of f looks like the one given in Fig. 12.3.1, the geometric problem is to find the slope of the graph at each point like P.[1]

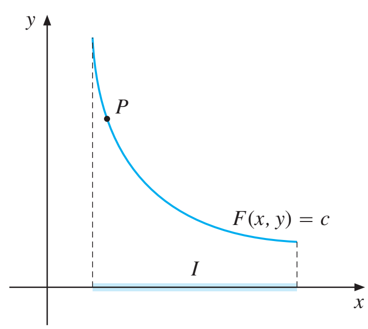

Figure 12.3.1 Differentiation along a level curve

To find an expression for the slope, introduce the auxiliary function u defined for all x in I by $u(x) = F(x, f(x))$. Then

$$u'(x) = F_1'(x, f(x)) \cdot 1 + F_2'(x, f(x)) \cdot f'(x)$$

according to the chain rule. Now, $(*)$ states that $u(x) = c$ for all x in I. The derivative of a constant is 0, so we have

$$u'(x) = F_1'(x, f(x)) + F_2'(x, f(x)) \cdot f'(x) = 0$$

[1] A particular case of this problem was Example 7.1.5.

If we replace $f(x)$ by y and solve for $f'(x) = y'$, we reach the conclusion:

SLOPE OF A LEVEL CURVE

If $F(x, y) = c$, then provided that $F_2'(x, y) \neq 0$, one has

$$y' = -\frac{F_1'(x, y)}{F_2'(x, y)} \tag{12.3.2}$$

This is an important result. Before applying this formula for y', however, recall that the pair (x, y) must satisfy the equation $F(x, y) = c$. On the other hand, note that there is no need to solve the equation $F(x, y) = c$ explicitly for y before applying (12.3.2) in order to find y'—see Example 12.3.3.

The same argument with x and y interchanged gives an analogous result to (12.3.2). Thus, if x is a continuously differentiable function of y which satisfies $F(x, y) = c$, then

$$F(x, y) = c \Rightarrow \frac{dx}{dy} = -\frac{\partial F/\partial y}{\partial F/\partial x} \tag{12.3.3}$$

provided that $\partial F/\partial x \neq 0$.

EXAMPLE 12.3.1 Use Eq. (12.3.2) to find y' when $xy = 5$.

Solution: We put $F(x, y) = xy$. Then $F_1'(x, y) = y$ and $F_2'(x, y) = x$. Hence, (12.3.2) gives

$$y' = -\frac{F_1'(x, y)}{F_2'(x, y)} = -\frac{y}{x}$$

This confirms the result in Example 7.1.1.

EXAMPLE 12.3.2 For the curve given by $x^3 + x^2y - 2y^2 - 10y = 0$, find the slope and the equation for the tangent at the point $(x, y) = (2, 1)$.

Solution: Let $F(x, y) = x^3 + x^2y - 2y^2 - 10y$. Then the given equation is equivalent to $F(x, y) = 0$, which is a level curve for F. First, we check that $F(2, 1) = 0$, so $(x, y) = (2, 1)$ is a point on the curve. Also, $F_1'(x, y) = 3x^2 + 2xy$ and $F_2'(x, y) = x^2 - 4y - 10$. So (12.3.2) implies that

$$y' = -\frac{3x^2 + 2xy}{x^2 - 4y - 10}$$

For $x = 2$ and $y = 1$ in particular, one has $y' = 8/5$. Then the point–slope formula for a line implies that the tangent at $(2, 1)$ must have the equation $y - 1 = (8/5)(x - 2)$, or $5y = 8x - 11$. See Fig. 12.3.2, in which the curve has been drawn by a computer program. Note that, for many values of x, there is more than one corresponding value of y such that (x, y) lies on the curve. For instance, $(2, 1)$ and $(2, -4)$ both lie on the curve.[2]

[2] Note that $y' = 0.4$ at $(2, -4)$. It would be good practice for you to confirm this.

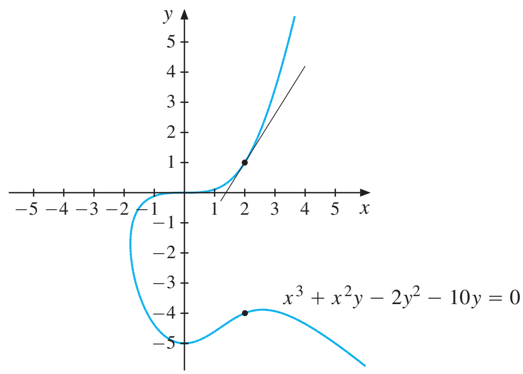

Figure 12.3.2 Example 12.3.2

EXAMPLE 12.3.3 Assume that the equation

$$e^{xy^2} - 2x - 4y = c$$

implicitly defines y as a differentiable function $y = f(x)$ of x. Find a value of the constant c such that $f(0) = 1$, and find y' at $(x, y) = (0, 1)$.

Solution: When $x = 0$ and $y = 1$, the equation becomes $1 - 4 = c$, so $c = -3$. Let $F(x, y) = e^{xy^2} - 2x - 4y$. Then $F_1'(x, y) = y^2 e^{xy^2} - 2$, and $F_2'(x, y) = 2xye^{xy^2} - 4$. Thus, from (12.3.2) we have

$$y' = -\frac{F_1'(x, y)}{F_2'(x, y)} = -\frac{y^2 e^{xy^2} - 2}{2xye^{xy^2} - 4}$$

At $(x, y) = (0, 1)$, we find $y' = -1/4$.

Note that in this example it was impossible to solve $e^{xy^2} - 2x - 4y = -3$ explicitly for y. Even so, we managed to find an explicit expression for the derivative of y w.r.t. x.

Here is an important economic example using a function defined implicitly by an equation.

EXAMPLE 12.3.4 We generalize Example 7.2.2, and assume that $D = f(t, P)$ is the demand for a commodity that depends on the price P before tax, as well as on the sales tax per unit, denoted by t. Suppose that $S = g(P)$ is the supply function. At equilibrium, when supply is equal to demand, the equilibrium price $P = P(t)$ depends on t. Indeed, $P = P(t)$ must satisfy the equation

$$f(t, P) = g(P) \tag{$*$}$$

for all t in some relevant interval. Suppose that $(*)$ defines P implicitly as a differentiable function of t. Find an expression for $\mathrm{d}P/\mathrm{d}t$, then discuss its sign.

Solution: Let $F(t,P) = f(t,P) - g(P)$. Then Eq. $(*)$ becomes $F(t,P) = 0$, so Eq. (12.3.2) yields

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{F'_t(t,P)}{F'_P(t,P)} = -\frac{f'_t(t,P)}{f'_P(t,P) - g'(P)} = \frac{f'_t(t,P)}{g'(P) - f'_P(t,P)} \tag{$**$}$$

It is reasonable to assume that $g'(P) > 0$, meaning that supply increases if price increases; and that $f'_t(t,P)$ and $f'_P(t,P)$ are both < 0, meaning that demand decreases if either the tax or the price increases. Then, $(**)$ tells us that $\mathrm{d}P/\mathrm{d}t < 0$, implying that the pre-tax price faced by suppliers decreases as the tax increases. Thus the suppliers, as well as the consumers, are adversely affected if the tax on their product rises.

Of course, we can also derive formula $(**)$ by implicitly differentiating $(*)$ w.r.t. t. This gives

$$f'_t(t,P) \cdot 1 + f'_P(t,P)\frac{\mathrm{d}P}{\mathrm{d}t} = g'(P)\frac{\mathrm{d}P}{\mathrm{d}t}$$

Solving this equation for $\mathrm{d}P/\mathrm{d}t$ yields $(**)$ again.

The Second Derivative

Suppose that Eq. (12.3.1), the equation of the level curve $F(x,y) = c$, defines the function $y = f(x)$ implicitly. Sometimes we need to know whether this function is concave or convex. One way to find out is to calculate y'', which is the derivative of $y' = -F'_1(x,y)/F'_2(x,y)$. Write $G(x) = F'_1(x,y)$ and $H(x) = F'_2(x,y)$, where y is a function of x. Our aim now is to differentiate the quotient $y' = -G(x)/H(x)$ w.r.t. x. According to the rule for differentiating quotients,

$$y'' = -\frac{G'(x)H(x) - G(x)H'(x)}{[H(x)]^2} \tag{$*$}$$

Keeping in mind that y is a function of x, both $G(x)$ and $H(x)$ are composite functions. So we differentiate them both by using the chain rule, thereby obtaining

$$G'(x) = F''_{11}(x,y) \cdot 1 + F''_{12}(x,y) \cdot y'$$
$$H'(x) = F''_{21}(x,y) \cdot 1 + F''_{22}(x,y) \cdot y'$$

Assuming that F is a C^2 function, Young's Theorem (Theorem 11.6.1) implies that $F''_{12} = F''_{21}$. Replace y' in both the preceding equations by the quotient $-F'_1/F'_2$, and then insert the results into $(*)$. After some algebraic simplification, this yields the formula

$$y'' = -\frac{1}{(F'_2)^3}\left[F''_{11}(F'_2)^2 - 2F''_{12}F'_1F'_2 + F''_{22}(F'_1)^2\right] \tag{12.3.4}$$

Occasionally, formula (12.3.4) is used in theoretical arguments, but generally it is easier to find y'' by direct differentiation, as in the examples in Section 7.1.

EXAMPLE 12.3.5 Use (12.3.4) to find y'' when $xy = 5$.

Solution: With $F(x, y) = xy$ we have $F'_1 = y$, $F'_2 = x$, $F''_{11} = 0$, $F''_{12} = 1$, and $F''_{22} = 0$. According to (12.3.4), we obtain

$$y'' = -\frac{1}{x^3}(-2 \cdot 1 \cdot y \cdot x) = \frac{2y}{x^2}$$

which is the same result we found in Example 7.1.6.

Example 16.2.1 expresses the result in (12.3.4) in a more memorable form, using the concept of determinant covered in Section 16.2.

EXERCISES FOR SECTION 12.3

1. Use formula (12.3.2) with $F(x, y) = 2x^2 + 6xy + y^2$ and $c = 18$ to find y' when y is defined implicitly by $2x^2 + 6xy + y^2 = 18$. Compare with the result in Exercise 7.1.5.

SM 2. Use Eq. (12.3.2) to find y' for the following level curves, and find y'' using (12.3.4).

   (a) $x^2y = 1$ (b) $x - y + 3xy = 2$ (c) $y^5 - x^6 = 0$

SM 3. A curve in the xy-plane is given by the equation $2x^2 + xy + y^2 - 8 = 0$.

   (a) Find y', y'', and the equation for the tangent at the point $(2, 0)$.

   (b) Which points on the curve have a horizontal tangent?

4. The equation $3x^2 - 3xy^2 + y^3 + 3y^2 = 4$ defines y implicitly as a function $h(x)$ of x in a neighbourhood of the point $(1, 1)$. Find $h'(1)$.

5. Suppose the demand $D(P, r)$ for a certain commodity (like a luxury car) depends on its price P and the interest rate r. What signs should one expect the partial derivatives of D w.r.t. P and r to have? Suppose the supply S is constant, so that in equilibrium, $D(P, r) = S$. Differentiate implicitly to find $\mathrm{d}P/\mathrm{d}r$, and comment on its sign. (Exercise 7.2.3 considers a special case.)

6. Let $D = f(R, P)$ denote the demand for a commodity when the price is P and R is advertising expenditure. What signs should one expect the partial derivatives f'_R and f'_P to have? If the supply is $S = g(P)$, equilibrium in the market requires that $f(R, P) = g(P)$. What is $\mathrm{d}P/\mathrm{d}R$? Discuss its sign.

7. Let f be a differentiable function of one variable, and let a and b be two constants. Suppose that the equation $x - az = f(y - bz)$ defines z as a differentiable function of x and y. Prove that z satisfies $az'_x + bz'_y = 1$.

12.4 More General Cases

Consider the equation $F(x, y, z) = c$, where c is a constant. In general, this equation determines a surface in three-dimensional space consisting of all the triples (x, y, z) that satisfy the equation. This we called the graph of the equation. Suppose that $z = f(x, y)$ defines implicitly a function that, for all (x, y) in some domain A, satisfies the equation $F(x, y, z) = c$. Then, for all (x, y) in such A,

$$F(x, y, f(x, y)) = c$$

Suppose F and f are differentiable. Because the function $g(x, y) = F(x, y, f(x, y))$ is equal to the constant c for all $(x, y) \in A$, the partial derivatives g'_x and g'_y must both be 0. However, $g(x, y)$ is a composite function of x and y whose partial derivatives can be found by using the general chain rule, Eq. (12.2.3). Therefore,

$$g'_x = F'_x \cdot 1 + F'_z \cdot z'_x = 0, \quad g'_y = F'_y \cdot 1 + F'_z \cdot z'_y = 0$$

Provided that $F'_z \neq 0$, this implies the following expressions for the partial derivatives of $z = f(x, y)$:

$$F(x, y, z) = c \quad \Longrightarrow \quad z'_x = -\frac{F'_x}{F'_z} \text{ and } z'_y = -\frac{F'_y}{F'_z} \tag{12.4.1}$$

Equation (12.4.1) allows z'_x and z'_y to be found even if it is impossible to solve the equation $F(x, y, z) = c$ explicitly for z as a function of x and y.

EXAMPLE 12.4.1 Equation $x - 2y - 3z + z^2 = -2$ defines z as a twice differentiable function of x and y about the point $(x, y, z) = (0, 0, 2)$. Find z'_x and z'_y, and then z''_{xx}, z''_{xy}, and z''_{yy}. Find also the values of all these partial derivatives at $(0, 0)$.

Solution: Let $F(x, y, z) = x - 2y - 3z + z^2$ and $c = -2$. Then $F'_x = 1$, $F'_y = -2$, and $F'_z = 2z - 3$. Whenever $z \neq 3/2$, we have $F'_z \neq 0$, so (12.4.1) gives

$$z'_x = -\frac{1}{2z - 3} \text{ and } z'_y = -\frac{-2}{2z - 3} = \frac{2}{2z - 3}$$

For $x = 0$, $y = 0$, and $z = 2$ in particular, we obtain $z'_x = -1$ and $z'_y = 2$.

We find z''_{xx} by differentiating the expression for z'_x partially w.r.t. x. Keeping in mind that z is a function of x and y, we get $z''_{xx} = (\partial/\partial x)(-(2z - 3)^{-1}) = (2z - 3)^{-2} 2z'_x$. Using the expression for z'_x found above, we have

$$z''_{xx} = \frac{-2}{(2z - 3)^3}$$

Correspondingly,

$$z''_{xy} = \frac{\partial}{\partial y} z'_x = \frac{\partial}{\partial y}[-(2z - 3)^{-1}] = (2z - 3)^{-2} 2z'_y = \frac{4}{(2z - 3)^3}$$

and

$$z''_{yy} = \frac{\partial}{\partial y} z'_y = \frac{\partial}{\partial y}[2(2z-3)^{-1}] = -2(2z-3)^{-2} 2z'_y = \frac{-8}{(2z-3)^3}$$

For $x = y = 0$ and $z = 2$, we get $z''_{xx} = -2$, $z''_{xy} = 4$, and $z''_{yy} = -8$.

EXAMPLE 12.4.2 A firm produces $Q = f(L)$ units of a commodity using L units of labour. We assume that $f'(L) > 0$ and $f''(L) < 0$, so f is strictly increasing and strictly concave.[3]

(a) If the firm gets P per unit produced and pays w for a unit of labour, write down the profit function, and find the first-order condition for profit maximization at $L^* > 0$.

(b) By implicit differentiation of the first-order condition, examine how changes in P and w influence the optimal choice of L^*.

Solution:

(a) The profit function is $\pi(L) = Pf(L) - wL$, so $\pi'(L) = Pf'(L) - w$. Thus, by Theorem 8.1.1, an optimal L^* must satisfy

$$Pf'(L^*) - w = 0 \tag{$*$}$$

(b) If we define $F(P, w, L^*) = Pf'(L^*) - w$, then $(*)$ is equivalent to $F(P, w, L^*) = 0$. According to (12.4.1),

$$\frac{\partial L^*}{\partial P} = -\frac{F'_P}{F'_{L^*}} = -\frac{f'(L^*)}{Pf''(L^*)}, \quad \text{while} \quad \frac{\partial L^*}{\partial w} = -\frac{F'_w}{F'_{L^*}} = -\frac{-1}{Pf''(L^*)} = \frac{1}{Pf''(L^*)}$$

The sign assumptions on f' and f'' imply that $\partial L^*/\partial P > 0$ and $\partial L^*/\partial w < 0$. Thus, the optimal labour input goes up if the price P increases, while it goes down if labour costs increase. This makes economic sense.[4]

EXAMPLE 12.4.3 **(Gains from search).** Suppose you intend to buy x^0 units of a particular commodity like flour. Right now, there is the opportunity to buy it at a price of p^0 per unit. But you expect that searching among other sellers will yield a lower price. Let $p(t)$ denote the lowest price per unit you expect to find after searching the market for t hours. It is reasonable to assume that $\dot{p}(t) < 0$. Moreover, since it is usually harder to find lower prices as the search progresses, we assume that $\ddot{p}(t) > 0$. Suppose your hourly wage is w. By searching for t hours, you save $p^0 - p(t)$ dollars for each unit you buy. Since you are buying x^0 units, total savings are $[p^0 - p(t)]x^0$. On the other hand, searching for t hours costs you wt in forgone wages. So the expected profit from searching for t hours is

$$\pi(t) = [p^0 - p(t)]x^0 - wt$$

A necessary first-order condition for $t = t^* > 0$ to maximize profit is that

$$\dot{\pi}(t^*) = -\dot{p}(t^*)x^0 - w = 0 \tag{$*$}$$

[3] See Exercise 3, where a special case is considered.

[4] Economists often prefer to use implicit differentiation rather than relying on formula (12.4.1).

This condition is also sufficient, because $\ddot{\pi}(t) = -\ddot{p}(t)x^0 < 0$ for all t.

For an economic interpretation of $(*)$, it is convenient to rewrite the condition as $-\dot{p}(t^*)x_0 = w$. Suppose now that you search for an extra small fraction, τ, of an hour. The gain expected from finding a lower price is $[p(t^*) - p(t^* + \tau)]x_0$, which is approximately $-\dot{p}(t^*)\tau x_0$. On the other hand, you lose $w\tau$ of wage income. So the first-order condition says that you should search until the marginal gain per unit of extra search time is just offset by the wage.

The optimal search time t^* depends on x^0 and w. Economists typically want to know how t^* changes as x^0 or w changes. We see that Eq. $(*)$ is similar to Eq. $(*)$ in Example 12.4.2, with $x^0 = -P$, $p = f$, and $t^* = L^*$. It follows immediately that

$$\frac{\partial t^*}{\partial x^0} = -\frac{\dot{p}(t^*)}{\ddot{p}(t^*)x^0} > 0, \quad \text{and} \quad \frac{\partial t^*}{\partial w} = -\frac{1}{\ddot{p}(t^*)x^0} < 0$$

where the signs are as indicated because $\dot{p}(t^*) < 0$, $\ddot{p}(t^*) > 0$, and $x^0 > 0$. Thus, the optimal search time rises as the quantity to be bought increases, and falls as the wage rate rises.

These qualitative results can easily be obtained by a geometric argument. Figure 12.4.1 illustrates the optimal search time t^*. It is the value of t at which the tangent to the curve $R = [p^0 - p(t)]x^0$ has slope w, and so is parallel to the line $C = wt$. If x^0 increases, the R curve is magnified vertically but not horizontally, so t^* moves to the right. On the other hand, if w increases, the straight line $C = wt$ will rotate anti-clockwise about the origin, so the optimal t^* will decrease.

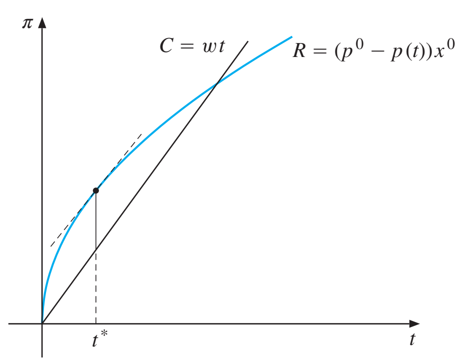

Figure 12.4.1 Optimal search

The General Case

The foregoing can be extended to any number of variables. The proof of the following result is a direct extension of the argument we gave for Eq. (12.4.1), so is left to the reader. Assuming that $\partial F/\partial z \neq 0$, we have

$$F(x_1, \dots, x_n, z) = c \;\Rightarrow\; \frac{\partial z}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z} \quad \text{for all } i = 1, 2, \dots, n \tag{12.4.2}$$

EXERCISES FOR SECTION 12.4

1. Use (12.4.1) to find $\partial z/\partial x$ for the following equations:

   (a) $3x + y - z = 0$ (b) $xyz + xz^3 - xy^2z^5 = 1$ (c) $e^{xyz} = 3xyz$

2. Find z'_x, z'_y, and z''_{xy} when $x^3 + y^3 + z^3 - 3z = 0$.

SM 3. Consider the problem of Example 12.4.2.

   (a) Suppose that $Q = f(L) = \sqrt{L}$. Write down Eq. $(*)$ in this case and find an explicit expression for L^* as a function of P and w. Find the partial derivatives of L^* w.r.t. P and w. Then verify the signs obtained in the example.

   (b) Suppose the profit function is replaced by $\pi(L) = Pf(L) - C(L, w)$, where $C(L, w)$ is the "cost function". What is the first-order condition for L^* to be optimal in this case? Find the partial derivatives of L^* w.r.t. P and w.

4. The equation $x^y + y^z + z^x = k$, where k is a positive constant, defines z as a positive-valued function of x and y, for $x > 0$ and $y > 0$. Find the partial derivatives of z w.r.t. x and y.

5. Consider the model of Exercise 12.3.6, applied to the market for an agricultural crop. Replace $S = g(P)$ by $S = g(w, P)$, where w is an index for how favourable the weather has been. Assume $g'_w(w, P) > 0$. Equilibrium now requires $f(R, P) = g(w, P)$. Assume that this equation defines P implicitly as a differentiable function of R and w. Find an expression for P'_w, and comment on its sign.

SM 6. The function F is defined for all x and y by $F(x, y) = xe^{y-3} + xy^2 - 2y$. Show that the point $(1, 3)$ lies on the level curve $F(x, y) = 4$, and find the equation for the tangent line to the curve at the point $(1, 3)$.

SM 7. The Nerlove–Ringstad production function $y = y(K, L)$ is defined implicitly by

$$y^{1+c\ln y} = AK^{\alpha}L^{\beta}$$

   where A, α, and β are positive constants. Find the marginal productivities of capital and labour, namely $\partial y/\partial K$ and $\partial y/\partial L$. (*Hint*: Take the logarithm of each side and then differentiate implicitly.)

12.5 Elasticity of Substitution

Economists are often interested in the slope of the tangent to a level curve at a particular point. Often, the level curve is downwards sloping, but economists prefer a positive answer. So, inspired by Example 7.1.5, we change the sign of the slope defined by (12.3.2), and use a special name:

MARGINAL RATE OF SUBSTITUTION

$$R_{yx} = \frac{F'_x(x,y)}{F'_y(x,y)} \tag{12.5.1}$$

is known as *the marginal rate of substitution of y for x*, abbreviated as MRS.

Note that $R_{yx} = -y' \approx -\Delta y/\Delta x$ when we move along the level curve $F(x,y) = c$. If $\Delta x = -1$ in particular, then $R_{yx} \approx \Delta y$. Thus, R_{yx} is approximately the quantity of y we must add per unit of x removed, if we are to stay on the same level curve.

EXAMPLE 12.5.1 Let $F(K,L) = 100$ be an isoquant for a production function, where K is capital input, L is labour input, and 100 is the output. Look at Fig. 12.5.1. At all the points P, Q, and R, 100 units are produced. At P a little capital input and a lot of labour input are used. The slope of the isoquant at P is approximately -4, so the MRS at P is approximately 4. This means that for each four units of labour that are taken away, adding only one unit of capital will ensure that output remains at (approximately) 100 units. Provided that units are chosen so that capital and labour have the same price, at P capital is more "valuable" than labour. At Q the MRS is approximately 1, so capital and labour are equally "valuable". Finally, at R, the MRS is approximately $1/5$, so at this point approximately five units of capital are required to compensate for the loss of one unit of labour.

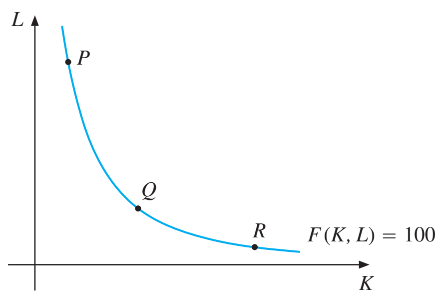

Figure 12.5.1 An isoquant

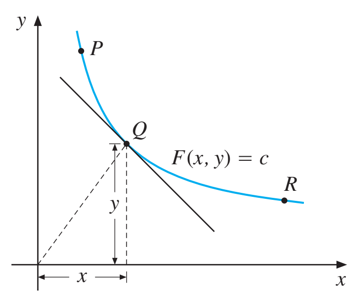

Figure 12.5.2 R_{yx} at Q

Consider a level curve $F(x,y) = c$ for a function F of two variables, as shown in Fig. 12.5.2. The MRS varies along the curve. At point P, R_{yx} is a large positive number. At Q, the number R_{yx} is about 1, and at R it is about 0.2. As we move along the level curve from left to right, R_{yx} will be strictly decreasing with values in some positive interval I. For each value of R_{yx} in I, there is a corresponding point (x,y) on the level curve $F(x,y) = c$, and thus a corresponding value of y/x. The fraction y/x is therefore a function of R_{yx}, and we define the following:

ELASTICITY OF SUBSTITUTION

When $F(x, y) = c$, the *elasticity of substitution between y and x* is

$$\sigma_{yx} = \mathrm{El}_{R_{yx}}\left(\frac{y}{x}\right) \tag{12.5.2}$$

Thus, σ_{yx} is the elasticity of the fraction y/x w.r.t. the MRS. Roughly speaking, σ_{yx} is the percentage change in the fraction y/x when we move along the level curve $F(x, y) = c$ far enough so that R_{yx} increases by 1%. Note that σ_{yx} is symmetric in x and y. In fact, $R_{xy} = 1/R_{yx}$, and so the logarithmic formula for elasticities implies that $\sigma_{xy} = \sigma_{yx}$. Also, Exercise 3 asks you to work with a (symmetric) expression for the elasticity of substitution in terms of the first- and second-order partial derivatives of F.

EXAMPLE 12.5.2 Calculate σ_{KL} for the Cobb–Douglas function $F(K, L) = AK^aL^b$.

Solution: The MRS of K for L is

$$R_{KL} = \frac{F'_L}{F'_K} = \frac{bAK^aL^{b-1}}{aAK^{a-1}L^b} = \frac{b}{a}\frac{K}{L}$$

Thus, $K/L = (a/b)R_{KL}$. The elasticity of the last expression w.r.t. R_{KL} is 1. Hence, $\sigma_{KL} = 1$ for the Cobb–Douglas function.

EXAMPLE 12.5.3 Find the elasticity of substitution for the CES function

$$F(K, L) = A(aK^{-\rho} + bL^{-\rho})^{-\mu/\rho}$$

where A, a, b, μ, and ρ are constants, $A > 0$, $a > 0$, $b > 0$, $\mu \neq 0$, $\rho > -1$, and $\rho \neq 0$.

Solution: Here

$$F'_K = A(-\mu/\rho)(aK^{-\rho} + bL^{-\rho})^{(-\mu/\rho)-1}a(-\rho)K^{-\rho-1}$$
$$F'_L = A(-\mu/\rho)(aK^{-\rho} + bL^{-\rho})^{(-\mu/\rho)-1}b(-\rho)L^{-\rho-1}$$

Hence,

$$R_{KL} = \frac{F'_L}{F'_K} = \frac{b}{a}\frac{L^{-\rho-1}}{K^{-\rho-1}} = \frac{b}{a}\left(\frac{K}{L}\right)^{\rho+1}$$

and therefore

$$\frac{K}{L} = \left(\frac{a}{b}\right)^{1/(\rho+1)}(R_{KL})^{1/(\rho+1)}$$

Recalling that the elasticity of Ax^b w.r.t. x is b, definition (12.5.2) implies that

$$\sigma_{KL} = \mathrm{El}_{R_{KL}}\left(\frac{K}{L}\right) = \frac{1}{\rho + 1}$$

We have thus shown that the function F has constant elasticity of substitution $1/(\rho + 1)$. This, of course, is the reason why F is called the "constant elasticity of substitution", CES, function.

Note that the elasticity of substitution for the CES function tends to 1 as $\rho \to 0$, which is precisely the elasticity of substitution for the Cobb–Douglas function in the previous example. This accords with the result in Example 7.12.5.

EXERCISES FOR SECTION 12.5

1. Calculate the elasticity of substitution between y and x for $F(x, y) = 10x^2 + 15y^2$.

2. Let $F(x, y) = x^a + y^a$, where a is a constant, $a \neq 0$ and $a \neq 1$.
   (a) Find the marginal rate of substitution of y for x.
   (b) Calculate the elasticity of substitution between y and x.

SM 3. The elasticity of substitution defined in (12.5.2) can be expressed in terms of the partial derivatives of the function F: if $F(x, y) = c$,

$$\sigma_{yx} = \frac{-F_1' F_2' (x F_1' + y F_2')}{xy[(F_2')^2 F_{11}'' - 2F_1' F_2' F_{12}'' + (F_1')^2 F_{22}'']}$$

Use this formula to derive the result in Example 12.5.2.

12.6 Homogeneous Functions of Two Variables

If $F(K, L)$ denotes the number of units produced when K units of capital and L units of labour are used as inputs, economists often ask: What happens to production if we double the inputs of both capital and labour? Will production rise by more or less than a factor of 2? Example 11.1.4 answered such questions for Cobb–Douglas technologies. To answer it in general, we extend the concept of *homogeneity* for functions of two variables.

HOMOGENEITY

A function f of two variables x and y defined in a domain D is said to be *homogeneous of degree k* if, for all (x, y) in D,

$$f(tx, ty) = t^k f(x, y) \tag{12.6.1}$$

for all $t > 0$. In words, this means that multiplying both variables by a positive factor t will multiply the value of the function by the factor t^k.

The degree of homogeneity of a function can be an arbitrary number—positive, zero, or negative. Earlier, we determined the degree of homogeneity for several particular functions.

For instance, we found in Example 11.1.4 that the Cobb–Douglas function F defined by $F(x, y) = Ax^a y^b$ is homogeneous of degree $a + b$. Here is an even simpler example:

EXAMPLE 12.6.1 Show that $f(x, y) = 3x^2 y - y^3$ is homogeneous of degree 3.

Solution: If we replace x by tx and y by ty in the formula for $f(x, y)$, we obtain

$$f(tx, ty) = 3(tx)^2(ty) - (ty)^3 = 3t^2x^2ty - t^3y^3 = t^3(3x^2y - y^3) = t^3 f(x, y)$$

Thus f is homogeneous of degree 3. If we let $t = 2$, then

$$f(2x, 2y) = 2^3 f(x, y) = 8f(x, y)$$

After doubling both x and y, the value of this function increases by a factor of 8.

Note that the sum of the exponents in each term of the polynomial in Example 12.6.1 is equal to 3. In general, a polynomial is homogeneous of degree k if and only if the sum of the exponents in each term is k. Other types of polynomial with different sums of exponents in different terms, such as $f(x, y) = 1 + xy$ or $g(x, y) = x^3 + xy$, are not homogeneous of any degree—see Exercise 6.

Homogeneous functions of two variables have some important properties of interest to economists. The first is:

THEOREM 12.6.1 (EULER'S THEOREM)

The function $f(x, y)$ is homogeneous of degree k if, and only if,

$$xf_1'(x, y) + yf_2'(x, y) = kf(x, y) \tag{12.6.2}$$

Here is an easy demonstration that Eq. (12.6.2) must hold when f is homogeneous of degree k:

Differentiate each side of Eq. (12.6.1) w.r.t. t, using the chain rule to differentiate the left-hand side. The result is

$$xf_1'(tx, ty) + yf_2'(tx, ty) = kt^{k-1} f(x, y)$$

Putting $t = 1$ gives $xf_1'(x, y) + yf_2'(x, y) = kf(x, y)$ immediately.

Theorem 12.7.1 in the next section proves the converse, and also considers the case of n variables.

We note three other interesting general properties of functions $f(x, y)$ that are homogeneous of degree k:

$$f_1'(x, y) \text{ and } f_2'(x, y) \text{ are both homogeneous of degree } k - 1 \tag{12.6.3}$$

$$f(x, y) = x^k f(1, y/x) = y^k f(x/y, 1) \quad \text{provided that } x > 0 \text{ and } y > 0 \tag{12.6.4}$$

and

$$x^2 f''_{11}(x, y) + 2xy f''_{12}(x, y) + y^2 f''_{22}(x, y) = k(k-1)f(x, y) \tag{12.6.5}$$

Again, these results are not difficult to demonstrate:

To prove (12.6.3), keep t and y constant and differentiate Eq. (12.6.1) partially w.r.t. x. Then $tf'_1(tx, ty) = t^k f'_1(x, y)$, so $f'_1(tx, ty) = t^{k-1} f'_1(x, y)$, thus showing that $f'_1(x, y)$ is homogeneous of degree $k - 1$. The same argument shows that $f'_2(x, y)$ is homogeneous of degree $k - 1$. One can prove the two equalities in (12.6.4) by replacing t in (12.6.1) first by $1/x$ and then by $1/y$, respectively. Finally, to show (12.6.5), assuming that $f(x, y)$ is twice continuously differentiable, we note first that because $f'_1(x, y)$ and $f'_2(x, y)$ are both homogeneous of degree $k - 1$, Euler's theorem can be applied separately to f'_1 and then to f'_2. It implies that

$$xf''_{11}(x, y) + yf''_{12}(x, y) = (k-1)f'_1(x, y) \tag{12.6.6}$$

$$xf''_{21}(x, y) + yf''_{22}(x, y) = (k-1)f'_2(x, y) \tag{12.6.7}$$

Let us now multiply the first of these equations by x, the second by y, and then add. Because f is C^2, Young's theorem (Theorem 11.6.1) implies that $f''_{12} = f''_{21}$, so the result is

$$x^2 f''_{11}(x, y) + 2xy f''_{12}(x, y) + y^2 f''_{22}(x, y) = (k-1)[xf'_1(x, y) + yf'_2(x, y)]$$

By Euler's theorem, however, $xf'_1(x, y) + yf'_2(x, y) = kf(x, y)$, so (12.6.5) is verified.

EXAMPLE 12.6.2 Check properties (12.6.2) to (12.6.5) for function $f(x, y) = 3x^2y - y^3$.

Solution: We find that $f'_1(x, y) = 6xy$ and $f'_2(x, y) = 3x^2 - 3y^2$. Hence,

$$xf'_1(x, y) + yf'_2(x, y) = 6x^2y + 3x^2y - 3y^3 = 3(3x^2y - y^3) = 3f(x, y)$$

Example 12.6.1 showed that f is homogeneous of degree 3, so this confirms (12.6.2).

Obviously, f'_1 and f'_2 are polynomials that are homogeneous of degree 2, which confirms (12.6.3). As for (12.6.4), in this case it takes the form

$$3x^2y - y^3 = x^3[3(y/x) - (y/x)^3] = y^3[3(x/y)^2 - 1]$$

Finally, to show (12.6.5), first calculate the second-order partial derivatives which are $f''_{11}(x, y) = 6y$, $f''_{12}(x, y) = 6x$, and $f''_{22}(x, y) = -6y$. Hence,

$$\begin{aligned} x^2 f''_{11}(x, y) + 2xy f''_{12}(x, y) + y^2 f''_{22}(x, y) &= 6x^2y + 12x^2y - 6y^3 \;=\; 6(3x^2y - y^3) \\ &= 3 \cdot 2f(x, y) \end{aligned}$$

which confirms (12.6.5) as well.

EXAMPLE 12.6.3 Suppose that the production function $Y = F(K, L)$ is homogeneous of degree 1. Show that one can express the output–labour ratio Y/L as a function $Y/L = f(K/L)$ of the capital–labour ratio $k = K/L$, where $f(k) = F(k, 1)$. Find the form of f when F is the Cobb–Douglas function AK^aL^b, with $a + b = 1$.

Solution: Because F is homogeneous of degree 1, as a special case of (12.6.4) one has

$$Y = F(K, L) = F(L(K/L), L \cdot 1) = LF(k, 1) = Lf(k) \text{ where } k = K/L$$

When $F(K, L) = AK^aL^{1-a}$, then $f(k) = F(k, 1) = Ak^a$.

Geometric Aspects of Homogeneous Functions

Homogeneous functions in two variables have some interesting geometric properties. Let $f(x, y)$ be homogeneous of degree k. Consider a ray in the xy-plane from the origin $(0, 0)$ through the point $(x_0, y_0) \neq (0, 0)$. An arbitrary point on this ray is of the form (tx_0, ty_0) for some positive number t. If we let $f(x_0, y_0) = c$, then $f(tx_0, ty_0) = t^k f(x_0, y_0) = t^k c$. Above any ray in the xy-plane through a point (x_0, y_0), the relevant portion of the graph of f therefore consists of the curve $z = t^k c$, where t measures the distance along the ray from the origin, and $c = f(x_0, y_0)$. A function that is homogeneous of degree k is therefore completely determined if its value is known at one point on each ray through the origin, as in Fig 12.6.1.

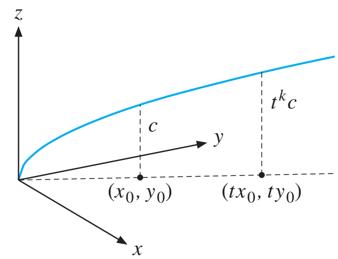

Figure 12.6.1 Function f along a ray

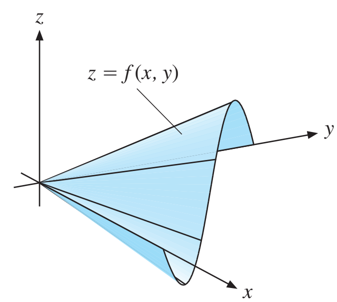

Figure 12.6.2 f is homogeneous of degree 1

In particular, let $k = 1$ so that $f(x, y)$ is homogeneous of degree 1. The curve $z = t^k c$ lying vertically above each relevant ray through the origin is then the straight line $z = tc$. Because of this, it is often said that *the graph of a homogeneous function of degree 1 is generated by straight lines through the origin*. Figure 12.6.2 illustrates this.

We have seen how, for a function $f(x, y)$ of two variables, it is often convenient to consider its level curves in the xy-plane instead of its three-dimensional graph. What can we say about the level curves of a homogeneous function? It turns out that *for a homogeneous function, even if only one of its level curves is known, then so are all its other level curves*. To see this, consider a function $f(x, y)$ that is homogeneous of degree k, and let $f(x, y) = c$ be one of its level curves, as illustrated in Fig. 12.6.3. We now explain how to construct the level curve through an arbitrary point A not lying on $f(x, y) = c$: First, draw the ray through the origin and the point A. This ray intersects the level curve $f(x, y) = c$ at a point D with coordinates (x_1, y_1). The coordinates of A will then be of the form (tx_1, ty_1) for some value of t—which in the figure is about 1.7.

In order to construct a new point on the same level curve as A, draw a new ray through the origin. Suppose this ray intersects the original level curve $f(x, y) = c$ at (x_2, y_2). Now use the value of t found earlier to determine the new point B with coordinates (tx_2, ty_2). This new point B is on the same level curve as A because $f(tx_2, ty_2) = t^k f(x_2, y_2) = t^k c = t^k f(x_1, y_1) = f(tx_1, ty_1)$. By repeating this construction for different rays through the origin that intersect the level curve $f(x, y) = c$, we can find as many points as we wish on the new level curve $f(x, y) = f(tx_1, ty_1)$.

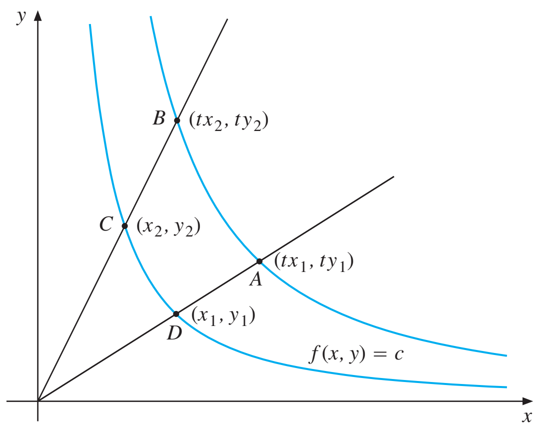

Figure 12.6.3 Level curves for a homogeneous function

The preceding argument shows that a homogeneous function $f(x, y)$ is entirely determined by any one of its level curves and by its degree of homogeneity. The shape of each level curve of a homogeneous function is often determined by specifying its elasticity of substitution, as defined in (12.5.2).

Another point worth noticing in connection with Fig. 12.6.3 is that the tangents to the level curves along each ray are parallel. We keep the assumption that f is homogeneous of degree k. If the level curve is $f(x, y) = c$, its slope at the point (x, y) is $-f_1'(x, y)/f_2'(x, y)$. At the point A in Fig. 12.6.3 the slope is

$$-\frac{f_1'(tx_1, ty_1)}{f_2'(tx_1, ty_1)} = -\frac{t^{k-1} f_1'(x_1, y_1)}{t^{k-1} f_2'(x_1, y_1)} = -\frac{f_1'(x_1, y_1)}{f_2'(x_1, y_1)} \tag{$*$}$$

where we have used Eq. (12.6.3), expressing the fact that the partial derivatives of f are homogeneous of degree $k - 1$. The equalities in $(*)$ state that the two level curves through A and D have the same slopes at those points. It follows that, at every point along a ray from the origin, the slope of the corresponding level curve will be the same. Stated differently, after removing the minus signs, $(*)$ shows that the marginal rate of substitution of y for x is a homogeneous function of degree 0.

EXERCISES FOR SECTION 12.6

1. Show that $f(x, y) = x^4 + x^2y^2$ is homogeneous of degree 4 by using definition (12.6.1).

2. Find the degree of homogeneity of $x(p, r) = Ap^{-1.5}r^{2.08}$.

SM 3. Show that $f(x, y) = xy^2 + x^3$ is homogeneous of degree 3. Verify that the four properties (12.6.2) to (12.6.5) all hold.

4. See whether the function $f(x, y) = xy/(x^2 + y^2)$ is homogeneous, and, if so, check Euler's theorem.

5. Prove that the CES function $F(K, L) = A(aK^{-\rho} + bL^{-\rho})^{-1/\rho}$ is homogeneous of degree one. Adapt the argument of Example 12.6.3 to express $F(K, L)/L$ as a function of $k = K/L$.

6. Show that $f(x, y) = x^3 + xy$ is not homogeneous of any degree. (*Hint*: Let $x = y = 1$. Apply (12.6.1) with $t = 2$ and $t = 4$ to get a contradiction.)

7. Use Eqs (12.6.6) and (12.6.7) to show that if $f(x, y)$ is homogeneous of degree 1, for $x > 0$ and $y > 0$, then $f''_{11}(x, y)f''_{22}(x, y) - [f''_{12}(x, y)]^2 = 0$.

8. Suppose that $f(x, y)$ is homogeneous of degree 2, with $f'_1(2, 3) = 4$ and $f'_2(4, 6) = 12$. Find $f(6, 9)$.

SM 9. [HARDER] Prove that if $F(x, y)$ is homogeneous of degree 1, then the elasticity of substitution can be expressed as $\sigma_{yx} = F'_1F'_2/FF''_{12}$. (*Hint:* Use Euler's theorem, together with Eqs (12.6.6) and (12.6.7), as well as the result in Exercise 12.5.3.)

12.7 Homogeneous and Homothetic Functions

Suppose that f is a function of n variables defined in a domain D. The set D is called a *cone* if, whenever $(x_1, x_2, \ldots, x_n) \in D$ and $t > 0$, the point $(tx_1, tx_2, \ldots, tx_n)$ also lies in D. When D is a cone, we say that f is *homogeneous of degree* k on D if

$$f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n) \tag{12.7.1}$$

for all $t > 0$. The constant k can be any real number—positive, zero, or negative.

EXAMPLE 12.7.1 Test the homogeneity of

$$f(x_1, x_2, x_3) = \frac{x_1 + 2x_2 + 3x_3}{x_1^2 + x_2^2 + x_3^2}$$

Solution: Here, f is defined on the set D of all points in three-dimensional space excluding the origin, which is a cone. Also,

$$f(tx_1, tx_2, tx_3) = \frac{tx_1 + 2tx_2 + 3tx_3}{(tx_1)^2 + (tx_2)^2 + (tx_3)^2} = \frac{t(x_1 + 2x_2 + 3x_3)}{t^2(x_1^2 + x_2^2 + x_3^2)} = t^{-1}f(x_1, x_2, x_3)$$

Hence, f is homogeneous of degree -1.

Euler's theorem, which we saw as Theorem 12.6.1 for the case of functions of two variables, can be generalized to functions of n variables:

THEOREM 12.7.1 (EULER'S THEOREM)

Suppose f is a differentiable function of n variables, defined in an open cone D. Then, f is homogeneous of degree k if, and only if, the following equation holds for all $\mathbf{x}$ in D:

$$\sum_{i=1}^{n} x_i f_i'(\mathbf{x}) = kf(\mathbf{x}) \tag{12.7.2}$$

The proof of this result is not difficult, and complements the argument given for Theorem 12.6.1:

Proof of Euler's Theorem: Suppose f is homogeneous of degree k, so Eq. (12.7.1) holds. Differentiating this equation w.r.t. t, with $\mathbf{x}$ fixed, yields

$$\sum_{i=1}^{n} x_i f_i'(t\mathbf{x}) = kt^{k-1} f(\mathbf{x})$$

Setting $t = 1$ gives (12.7.2) immediately.

To prove the converse, assume that Eq. (12.7.2) is valid for all $\mathbf{x}$ in cone D. Keep $\mathbf{x}$ fixed and define the function g for all $t > 0$ by $g(t) = t^{-k} f(t\mathbf{x}) - f(\mathbf{x})$. Then, differentiating gives

$$g'(t) = -kt^{-k-1} f(t\mathbf{x}) + t^{-k} \sum_{i=1}^{n} x_i f_i'(t\mathbf{x}) \tag{$*$}$$

Because $t\mathbf{x}$ lies in D, Eq. (12.7.2) must also be valid when each x_i is replaced by tx_i. It follows that $\sum_{i=1}^{n} (tx_i) f_i'(t\mathbf{x}) = kf(t\mathbf{x})$. Multiplying this equation by t^{-k-1} and using it to substitute for the last term of $(*)$ implies that, for all $t > 0$, one has

$$g'(t) = -kt^{-k-1} f(t\mathbf{x}) + t^{-k-1} kf(t\mathbf{x}) = 0$$

It follows that $g(t)$ must be a constant C. Obviously, $g(1) = 0$, so $C = 0$, implying that $g(t) = 0$ for *all* $t > 0$. According to the definition of g, this proves that $f(t\mathbf{x}) = t^k f(\mathbf{x})$, so f is indeed homogeneous of degree k.

An interesting version of the Euler equation, Eq. (12.7.2), is obtained by dividing each term of the equation by $f(\mathbf{x})$, provided this number is not 0. Recalling the definition of the partial elasticity, $\mathrm{El}_i f(\mathbf{x}) = (x_i / f(\mathbf{x})) f_i'(\mathbf{x})$, we have

$$\mathrm{El}_1 f(\mathbf{x}) + \mathrm{El}_2 f(\mathbf{x}) + \cdots + \mathrm{El}_n f(\mathbf{x}) = k \tag{12.7.3}$$

Thus, the sum of the partial elasticities of a function of n variables that is homogeneous of degree k must be equal to k.

The results in Eqs (12.6.3) to (12.6.5) can also be generalized to functions of n variables. The proofs are similar, so they can be left to the interested reader. We simply state the general versions of Eqs (12.6.3) and (12.6.5): if $f(\mathbf{x})$ is homogeneous of degree k, then: for each $i = 1, 2, \dots, n$,

$$f_i'(\mathbf{x}) \text{ is homogeneous of degree } k-1 \tag{12.7.4}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j f_{ij}''(\mathbf{x}) = k(k-1)f(\mathbf{x}) \tag{12.7.5}$$

Economic Applications

Let us consider some typical examples of homogeneous functions in economics.

EXAMPLE 12.7.2 Let $f(\mathbf{v}) = f(v_1, \dots, v_n)$ denote the output of a production process when the input quantities are $v_1, \dots, v_n$. It is often assumed that if all the input quantities are scaled by a factor t, then t times as much output as before is produced, so that for all $t > 0$,

$$f(t\mathbf{v}) = tf(\mathbf{v}) \tag{$*$}$$

This implies that f is homogeneous of degree 1. Production functions with this property are said to exhibit *constant returns to scale*.

For any fixed input vector $\mathbf{v}$, consider the function $\varphi(t) = f(t\mathbf{v})/t$. This indicates the average returns to scale—i.e. the average output per unit input when all inputs are rescaled together. For example when $t = 2$, all inputs are doubled. When $t = 3/4$, all inputs are reduced proportionally by $1/4$.

Now, when $(*)$ holds, then $\varphi(t) = f(\mathbf{v})$, independent of t. Also, a production function that is homogeneous of degree $k < 1$ has *decreasing returns to scale* because $\varphi(t) = t^{k-1}f(\mathbf{v})$ and so $\varphi'(t) < 0$. On the other hand, a production function has *increasing returns to scale* if $k > 1$ because then $\varphi'(t) > 0$.

EXAMPLE 12.7.3 The general Cobb–Douglas function, $F(v_1, \dots, v_n) = Av_1^{a_1} \cdots v_n^{a_n}$, is often used as an example of a production function. Prove that it is homogeneous, and examine when it has constant/decreasing/increasing returns to scale. Also confirm Eq. (12.7.3).

Solution: Here

$$F(t\mathbf{v}) = A(tv_1)^{a_1} \dots (tv_n)^{a_n} = At^{a_1}v_1^{a_1} \dots t^{a_n}v_n^{a_n} = t^{a_1+\dots+a_n}F(\mathbf{v})$$

So F is homogeneous of degree $a_1 + \dots + a_n$. Thus, it has constant, decreasing, or increasing returns to scale according as $a_1 + \dots + a_n$ is equal, smaller, or greater than 1. Because $\mathrm{El}_i\, F = a_i$, $i = 1, \dots, n$, we get $\sum_{i=1}^{n} \mathrm{El}_i\, F = \sum_{i=1}^{n} a_i$, which confirms (12.7.3) in this case.

EXAMPLE 12.7.4 Consider a market with three commodities with quantities denoted by x, y, and z, whose prices per unit are respectively p, q, and r. Suppose that the demand for one of the commodities by a consumer with income m is given by $D(p, q, r, m)$. Suppose that the three prices and income m are all multiplied by some $t > 0$.[5] Then the consumer's budget constraint $px + qy + rz \leq m$ becomes $tpx + tqy + trz \leq tm$, which is exactly the same constraint. The multiplicative constant t is irrelevant to the consumer. It is therefore natural to assume that the consumer's demand remains unchanged, with

$$D(tp, tq, tr, tm) = D(p, q, r, m)$$

Requiring this equation to be valid for all $t > 0$ means that the demand function D is homogeneous of degree 0. In this case, it is often said that demand is not influenced by "money illusion": a consumer with 10% more money to spend should realize that nothing has really changed if all prices have also risen by 10%.

As a specific example of a function that is common in demand analysis, consider

$$D(p, q, r, m) = \frac{mp^b}{p^{b+1} + q^{b+1} + r^{b+1}}$$

where b is a constant. Here,

$$D(tp, tq, tr, tm) = \frac{(tm)(tp)^b}{(tp)^{b+1} + (tq)^{b+1} + (tr)^{b+1}} = D(p, q, r, m)$$

since t^{b+1} cancels out.

Sometimes we encounter non-homogeneous functions of several variables that are, however, homogeneous when regarded as functions of some of the variables only, with the other variables fixed. For instance, the (minimum) cost of producing y units of a single output is often expressed as a function $C(\mathbf{w}, y)$ of y and the vector $\mathbf{w} = (w_1, \ldots, w_n)$ of prices of n different input factors. Then, if all input prices double, one expects cost to double. So economists usually assume that $C(t\mathbf{w}, y) = tC(\mathbf{w}, y)$ for all $t > 0$—i.e. that the cost function is homogeneous of degree 1 in $\mathbf{w}$, for each fixed y. See Exercise 7 for a prominent example.

Homothetic Functions

Let f be a function of n variables $\mathbf{x} = (x_1, \ldots, x_n)$ defined in a cone K. Then f is called *homothetic* if

$$\mathbf{x}, \mathbf{y} \in K,\ f(\mathbf{x}) = f(\mathbf{y}),\ t > 0 \ \Rightarrow\ f(t\mathbf{x}) = f(t\mathbf{y}) \tag{12.7.6}$$

For instance, if f is some consumer's utility function, Eq. (12.7.6) requires that whenever there is indifference between the two commodity bundles $\mathbf{x}$ and $\mathbf{y}$, then there is also indifference after they have both been magnified or shrunk by the same proportion t. For example, a consumer who is indifferent between two litres of soda and three litres of juice must also be indifferent between four litres of soda and six litres of juice. Evidently, this property may be true of some consumers, but one should not assume it of all people.

[5] Imagine, for example, that the prices of all commodities rise by 10%. Or that all prices and incomes have been converted into euros from, say, German marks.

A homogeneous function f of any degree k is homothetic. In fact, it is easy to prove a more general result:

THEOREM 12.7.2

> Suppose that function F can be written as the composition of functions H and f, so that $F(\mathbf{x}) = H(f(\mathbf{x}))$. If H is strictly increasing and f is homogeneous of any degree, then F is homothetic.

Proof: Suppose that $F(\mathbf{x}) = F(\mathbf{y})$, or equivalently, that $H(f(\mathbf{x})) = H(f(\mathbf{y}))$. Because H is strictly increasing, this implies that $f(\mathbf{x}) = f(\mathbf{y})$. Because f is homogeneous of degree k, it follows that if $t > 0$, then

$$F(t\mathbf{x}) = H(f(t\mathbf{x})) = H(t^k f(\mathbf{x})) = H(t^k f(\mathbf{y})) = H(f(t\mathbf{y})) = F(t\mathbf{y})$$

This proves that $F(\mathbf{x})$ is homothetic.

Hence, any strictly increasing function of a homogeneous function is homothetic. It is actually quite common to take this property as the definition of a homothetic function, usually with $k = 1$.[6]

The next example shows that not all homothetic functions are homogeneous.

EXAMPLE 12.7.5 Show that the function $F(x, y) = xy + 1$, which is obviously not homogeneous, is nevertheless homothetic.

Solution: Define $H(u) = u + 1$. Then H is strictly increasing. The function $f(x, y) = xy$ is homogeneous of degree 2, and $F(x, y) = xy + 1 = H(f(x, y))$. By Theorem 12.7.2, F is homothetic. Alternatively, use the definition in Eq. (12.7.6) to show directly that F is homothetic.

Suppose that $F(\mathbf{x}) = F(x_1, x_2, \dots, x_n)$ is a differentiable production function, defined for all $(x_1, \dots, x_n)$ satisfying $x_i \geq 0$, for $i = 1, \dots, n$. Recall that the marginal rate of substitution, or MRS, of factor j for factor i is defined, for $i, j = 1, 2, \dots, n$, by

$$h_{ji}(\mathbf{x}) = \frac{\partial F(\mathbf{x})}{\partial x_i} \Big/ \frac{\partial F(\mathbf{x})}{\partial x_j} \tag{12.7.7}$$

Suppose that $F(\mathbf{x}) = H(f(\mathbf{x}))$, where H is differentiable with $H'(u) > 0$ for all u in its domain. Suppose too that $f(\mathbf{x})$ is homogeneous of degree k. Then $\partial F(\mathbf{x})/\partial x_i = H'(f(\mathbf{x}))(\partial f(\mathbf{x})/\partial x_i)$, implying that

$$\frac{\partial F(\mathbf{x})}{\partial x_i} \Big/ \frac{\partial F(\mathbf{x})}{\partial x_j} = \frac{\partial f(\mathbf{x})}{\partial x_i} \Big/ \frac{\partial f(\mathbf{x})}{\partial x_j}$$

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[6] Suppose that $F(\mathbf{x})$ is any continuous homothetic function defined on the cone K of vectors $\mathbf{x}$ satisfying $x_i \geq 0, i = 1, \dots, n$. Suppose too that $F(t\mathbf{x}_0)$ is a strictly increasing function of t for each fixed $\mathbf{x}_0 \neq \mathbf{0}$ in K. Then one can prove that there exists a strictly increasing function H such that $F(\mathbf{x}) = H(f(\mathbf{x}))$, where the function $f(\mathbf{x})$ is homogeneous of degree 1. Actually, f could be made a homogeneous function of any positive degree k by a suitable modification of H, with $\tilde{H}(u) = u^k$.

wherever $f_j'(\mathbf{x}) > 0$ and so $F_j'(\mathbf{x}) > 0$. But f is homogeneous of degree k, so we can use (12.7.4) to show that, for all $t > 0$,

$$h_{ji}(t\mathbf{x}) = \frac{\partial f(t\mathbf{x})}{\partial x_i} \Bigg/ \frac{\partial f(t\mathbf{x})}{\partial x_j} = t^{k-1}\frac{\partial f(\mathbf{x})}{\partial x_i} \Bigg/ t^{k-1}\frac{\partial f(\mathbf{x})}{\partial x_j} = h_{ji}(\mathbf{x}) \tag{12.7.8}$$

Formula (12.7.8) shows that the marginal rates of substitution are homogeneous of degree 0. We have thus argued the following general result: *If F is a strictly increasing transformation of a homogeneous function, as in the premises of Theorem 12.7.2, then the function $F(\mathbf{x}) = H(f(\mathbf{x}))$ is a strictly increasing transformation of a homogeneous function, where H is differentiable with $H'(u) > 0$ for all u in its domain. Then the marginal rates of substitution of $F(\mathbf{x})$ are homogeneous of degree 0.*[7] This result generalizes the observation made for the case of two variables at the end of Section 12.6.

EXERCISES FOR SECTION 12.7

SM **1.** Find the degree of homogeneity, if there is one, for each of the following functions:

(a) $f(x, y, z) = 3x + 4y - 3z$

(b) $g(x, y, z) = 3x + 4y - 2z - 2$

(c) $h(x, y, z) = \dfrac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{x + y + z}$

(d) $G(x, y) = \sqrt{xy}\,\ln\left(\dfrac{x^2 + y^2}{xy}\right)$

(e) $H(x, y) = \ln x + \ln y$

(f) $p(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^n$

SM **2.** Find the degree of homogeneity, if there is one, for each of the following functions:

(a) $f(x_1, x_2, x_3) = \dfrac{(x_1x_2x_3)^2}{x_1^4 + x_2^4 + x_3^4}\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3}\right)$

(b) the CES function: $x(v_1, v_2, \ldots, v_n) = A\left(\delta_1 v_1^{-\rho} + \delta_2 v_2^{-\rho} + \cdots + \delta_n v_n^{-\rho}\right)^{-\mu/\rho}$

3. Examine the homogeneity of the three means, $\bar{x}_A$, $\bar{x}_G$, and $\bar{x}_H$, as defined in Example 11.5.2.

4. Consider a utility function $u(\mathbf{x}) = u(x_1, \ldots, x_n)$ whose continuous partial derivatives, for some constant a, satisfy $\displaystyle\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = a$ for all $x_1 > 0, \ldots, x_n > 0$. Show that the function $v(\mathbf{x}) = u(\mathbf{x}) - a\ln(x_1 + \cdots + x_n)$ is homogeneous of degree 0.[8] (*Hint*: Use Euler's theorem.)

SM **5.** Which of the following functions $f(x, y)$ are homothetic?

(a) $(xy)^2 + 1$

(b) $\dfrac{2(xy)^2}{(xy)^2 + 1}$

(c) $x^2 + y^3$

(d) $e^{x^2 y}$

6. [HARDER] Suppose that $f(\mathbf{x})$ and $g(\mathbf{x})$ are homogeneous of degree r and s, respectively. Examine whether the following functions h are homogeneous. Determine the degree of homogeneity in each case.

[7] Because of our previous footnote, the same must be true if F is any homothetic function with the property that $F(t\mathbf{x})$ is an increasing function of the scalar t for each fixed vector $\mathbf{x}$.

[8] This function was first studied by D.W. Katzner.

(a) $h(\mathbf{x}) = f(x_1^m, x_2^m, \dots, x_n^m)$ (b) $h(\mathbf{x}) = g(\mathbf{x})^p$ (c) $h = f + g$

(d) $h = fg$ (e) $h = f/g$

SM 7. [HARDER] The *transcendental logarithmic, or "translog", cost function* $C(\mathbf{w}, y)$ is defined implicitly by

$$\ln C(\mathbf{w}, y) = a_0 + c_1 \ln y + \sum_{i=1}^{n} a_i \ln w_i + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \ln w_i \ln w_j + \ln y \sum_{i=1}^{n} b_i \ln w_i$$

where $\mathbf{w}$ is the vector of factor prices and y is the level of output. Prove that this function is homogeneous of degree 1 in $\mathbf{w}$, for each fixed y, provided that all the following conditions are met: (i) $\sum_{i=1}^{n} a_i = 1$; (ii) $\sum_{i=1}^{n} b_i = 0$; (iii) $\sum_{j=1}^{n} a_{ij} = 0$ for all i; and (iv) $\sum_{i=1}^{n} a_{ij} = 0$ for all j.

12.8 Linear Approximations

Section 7.4 discussed the linear approximation $f(x) \approx f(a) + f'(a)(x - a)$ for a function of one variable. We will now find a similar approximation for functions f of two variables, and later for any number of variables.

For fixed values of x_0, y_0, x, and y, define the function $g(t)$ by

$$g(t) = f(x_0 + t(x - x_0), y_0 + t(y - y_0))$$

We see that $g(0) = f(x_0, y_0)$ and $g(1) = f(x, y)$. Generally, $g(t)$ is the value of f at the point $(x_0 + t(x - x_0), y_0 + t(y - y_0)) = ((1 - t)x_0 + tx, (1 - t)y_0 + ty)$, which lies on the line joining (x_0, y_0) to (x, y). According to the chain rule, the derivative $g'(t)$ equals

$$f_1'(x_0 + t(x - x_0), y_0 + t(y - y_0))(x - x_0) + f_2'(x_0 + t(x - x_0), y_0 + t(y - y_0))(y - y_0)$$

Putting $t = 0$ and using the approximation $g(1) \approx g(0) + g'(0)$, we obtain the result:

LINEAR APPROXIMATION

The linear approximation to $f(x, y)$ about (x_0, y_0) is

$$f(x, y) \approx f(x_0, y_0) + f_1'(x_0, y_0)(x - x_0) + f_2'(x_0, y_0)(y - y_0) \tag{12.8.1}$$

The usefulness of approximation (12.8.1) depends on the size of the error. Taylor's formula with remainder, presented for the case of one variable in Section 7.6, is extended to n variables in FMEA. One implication of the extended formula is that, as $x \to x_0$ and $y \to y_0$, so approximation (12.8.1) will get better, in the sense that the error will tend to 0.

EXAMPLE 12.8.1 Find the linear approximation to $f(x,y) = e^{x+y}(xy-1)$ about $(0,0)$.

Solution: Here one has $f(0,0) = -1$, as well as

$$f_1'(x,y) = e^{x+y}(xy-1) + e^{x+y}y \quad \text{and} \quad f_2'(x,y) = e^{x+y}(xy-1) + e^{x+y}x$$

So $f_1'(0,0) = -1$ and $f_2'(0,0) = -1$. Hence, Eq. (12.8.1) gives

$$e^{x+y}(xy-1) \approx -1 - x - y$$

So for x and y close to 0, the complicated function $z = e^{x+y}(xy-1)$ is approximated by the simple linear function $z = -1 - x - y$.

Formula (12.8.1) can be used to find approximate numerical values of a function near any point where the function and its derivatives are easily evaluated. Consider the following example.

EXAMPLE 12.8.2 Let $f(x,y) = xy^3 - 2x^3$. Then $f(2,3) = 38$. Using (12.8.1), find an approximate numerical value for $f(2.01, 2.98)$.

Solution: Here $f_1'(x,y) = y^3 - 6x^2$ and $f_2'(x,y) = 3xy^2$, so $f_1'(2,3) = 3$ and $f_2'(2,3) = 54$. Putting $x_0 = 2$, $y_0 = 3$, $x = 2 + 0.01$, and $y = 3 - 0.02$, we obtain

$$f(2.01, 2.98) \approx f(2,3) + f_1'(2,3) \cdot 0.01 + f_2'(2,3) \cdot (-0.02) = 38 + 3(0.01) + 54(-0.02)$$

which equals 36.95. The exact value is $f(2.01, 2.98) = 36.95061792$. The error in the approximation is, therefore, only a bit greater than -0.0006.

Approximation (12.8.1) can be generalized to functions of several variables.

LINEAR APPROXIMATION

The linear approximation to $f(\mathbf{x}) = f(x_1, \dots, x_n)$ about $\mathbf{x}^0 = (x_1^0, \dots, x_n^0)$ is given by

$$f(\mathbf{x}) \approx f(\mathbf{x}^0) + f_1'(\mathbf{x}^0)(x_1 - x_1^0) + \cdots + f_n'(\mathbf{x}^0)(x_n - x_n^0) \tag{12.8.2}$$

Exercise 8 asks you to provide a proof.

Tangent Planes

In Eq. (12.8.1), the function $z = f(x,y)$ is approximated by the *linear function*

$$z = f(x_0, y_0) + f_1'(x_0, y_0)(x - x_0) + f_2'(x_0, y_0)(y - y_0)$$

The graph of this linear function is a plane which passes through the point $P = (x_0, y_0, z_0)$, with $z_0 = f(x_0, y_0)$, on the graph of $z = f(x,y)$. This plane is called the *tangent plane* to $z = f(x,y)$ at P:

TANGENT PLANE

At the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$, the tangent plane to the graph of $z = f(x, y)$ has the equation

$$z - z_0 = f_1'(x_0, y_0)(x - x_0) + f_2'(x_0, y_0)(y - y_0) \tag{12.8.3}$$

The tangent plane is illustrated in Fig. 12.8.1. Now, does it "deserve" that name? Look back at Fig. 11.3.9, where l_x and l_y are the tangents at P to the two curves K_x and K_y that lie on the surface. Since the slope of the line l_x is $f_2'(x_0, y_0)$, the points (x, y, z) which lie on l_x are characterized by $x = x_0$ and $z - z_0 = f_2'(x_0, y_0)(y - y_0)$. But we see from (12.8.3) that these points also lie in the tangent plane. In the same way we see that the line l_y also lies in the tangent plane. Because the graph of Eq. (12.8.3) is the only plane that contains both tangent lines l_x and l_y, it makes good sense to use the term "tangent plane".

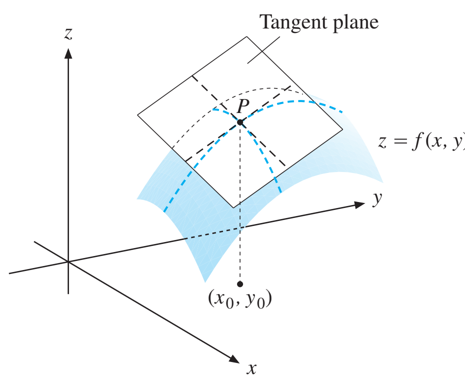

Figure 12.8.1 The graph of $z = f(x, y)$ and the tangent plane at P

EXAMPLE 12.8.3 Find the tangent plane at $P = (x_0, y_0, z_0) = (1, 1, 5)$ to the graph of

$$f(x, y) = x^2 + 2xy + 2y^2$$

Solution: Because $f(1, 1) = 5$, P lies on the graph of f. We find that $f_1'(x, y) = 2x + 2y$ and $f_2'(x, y) = 2x + 4y$. Hence, $f_1'(1, 1) = 4$ and $f_2'(1, 1) = 6$. Thus, Eq. (12.8.3) yields

$$z - 5 = 4(x - 1) + 6(y - 1)$$

or, equivalently, $z = 4x + 6y - 5$.

EXERCISES FOR SECTION 12.8

1. Find the linear approximation about (0, 0) for the following:

   (a) $f(x, y) = (x + 1)^5(y + 1)^6$ (b) $f(x, y) = \sqrt{1 + x + y}$ (c) $f(x, y) = e^x \ln(1 + y)$

2. Find the linear approximation about (x_0, y_0) for $f(x, y) = Ax^a y^b$.

3. Suppose that $g(\mu, \varepsilon) = [(1 + \mu)(1 + \varepsilon)^\alpha]^{1/(1-\beta)} - 1$, with α and β as constants. Show that if μ and ε are close to 0, then $g(\mu, \varepsilon) \approx (\mu + \alpha\varepsilon)/(1 - \beta)$.

4. Let $f(x, y) = 3x^2y + 2y^3$. Then $f(1, -1) = -5$. Use approximation (12.8.1) to estimate the value of $f(0.98, -1.01)$. How large is the error caused by this approximation?

5. Let $f(x, y) = 3x^2 + xy - y^2$.

   (a) Compute $f(1.02, 1.99)$ exactly.

   (b) Let $f(1.02, 1.99) = f(1 + 0.02, 2 - 0.01)$ and use Eq. (12.8.1) to find an approximate numerical value for $f(1.02, 1.99)$. How large is the error?

6. Suppose you have been told that a differentiable function v of two variables satisfies $v(1, 0) = -1$, $v_1'(1, 0) = -4/3$, and $v_2'(1, 0) = 1/3$. Find an approximate value for $v(1.01, 0.02)$.

SM 7. Find the tangent planes to the following surfaces at the indicated points: (a) $z = x^2 + y^2$ at (1, 2, 5); and (b) $z = (y - x^2)(y - 2x^2)$ at (1, 3, 2).

SM 8. [HARDER] Define the function

$$g(t) = f(x_1^0 + t(x_1 - x_1^0), \ldots, x_n^0 + t(x_n - x_n^0))$$

Use the approximation $g(1) \approx g(0) + g'(0)$ to derive Eq. (12.8.2).

9. [HARDER] Let $f(x, y)$ be any differentiable function. Prove that f is homogeneous of degree 1 if and only if the tangent plane at every point on its graph passes through the origin.

12.9 Differentials

Suppose that $z = f(x, y)$ is a differentiable function of two variables. Let dx and dy be any two real numbers, not necessarily small. Then, we define the *differential* of $z = f(x, y)$ at (x, y), denoted by dz or df, so that

$$z = f(x, y) \implies dz = f_1'(x, y)\,dx + f_2'(x, y)\,dy \tag{12.9.1}$$

When x is changed to $x + \mathrm{d}x$ and y is changed to $y + \mathrm{d}y$, then the actual change in the value of the function is the *increment*

$$\Delta z = f(x + \mathrm{d}x, y + \mathrm{d}y) - f(x, y)$$

If $\mathrm{d}x$ and $\mathrm{d}y$ are small in absolute value, then Δz can be approximated by $\mathrm{d}z$:

$$\Delta z \approx \mathrm{d}z = f_1'(x, y)\,\mathrm{d}x + f_2'(x, y)\,\mathrm{d}y, \text{ when } |\mathrm{d}x| \text{ and } |\mathrm{d}y| \text{ are small} \tag{12.9.2}$$

The approximation in (12.9.2) follows from (12.8.1) in the previous section. We first replace $x - x_0$ by $\mathrm{d}x$ and $y - y_0$ by $\mathrm{d}y$, and thus x by $x_0 + \mathrm{d}x$ and y by $y_0 + \mathrm{d}y$. Finally, in the formula which emerges, replace x_0 by x and y_0 by y. The approximation in (12.9.2) can be given a geometric interpretation, as illustrated in Fig. 12.9.1. The error that arises from replacing Δz by $\mathrm{d}z$ results from "following the tangent plane" from P to the point S, rather than "following the graph" to the point R.

More formally, the tangent plane at $P = (x, y, f(x, y))$ is defined as the set of points (X, Y, Z) satisfying the linear equation

$$Z - f(x, y) = f_1'(x, y)(X - x) + f_2'(x, y)(Y - y)$$

Letting $X = x + \mathrm{d}x$ and $Y = y + \mathrm{d}y$, we obtain

$$Z = f(x, y) + f_1'(x, y)\,\mathrm{d}x + f_2'(x, y)\,\mathrm{d}y = f(x, y) + \mathrm{d}z$$

The length of the line segment QS in Fig. 12.9.1 is therefore $f(x, y) + \mathrm{d}z$.

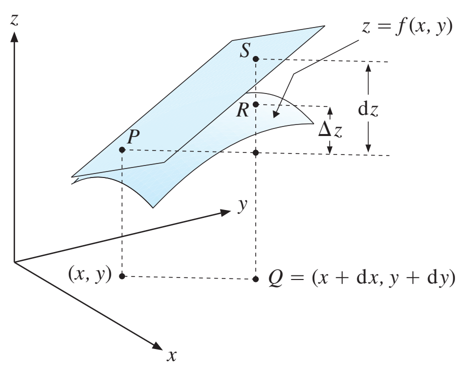

Figure 12.9.1 Δz and the differential $\mathrm{d}z$

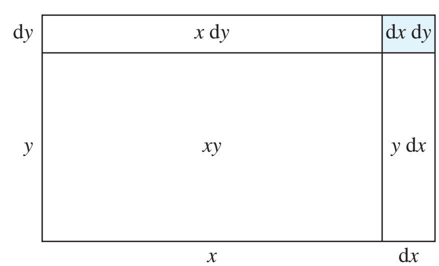

Figure 12.9.2 $\Delta z - \mathrm{d}z = \mathrm{d}x\,\mathrm{d}y$

A word of caution is worthwhile here. In the literature on mathematics for economists, a common definition of the differential $\mathrm{d}z$ in Eq. (12.9.1) requires that $\mathrm{d}x$ and $\mathrm{d}y$ be "infinitesimal", or "infinitely small". In this case, it is often claimed, Δz becomes equal to $\mathrm{d}z$. Imprecise ideas of this sort have caused confusion over the centuries since Leibniz first introduced them, and they have largely been abandoned in mathematics.[9]

[9] However, in nonstandard analysis, a respectable branch of modern mathematics, a modified version of Leibniz's ideas about infinitesimals can be made precise. There have been some interesting applications of nonstandard analysis to theoretical economics.

EXAMPLE 12.9.1 Let $z = f(x, y) = xy$. Then

$$\Delta z = f(x + \mathrm{d}x, y + \mathrm{d}y) - f(x, y) = (x + \mathrm{d}x)(y + \mathrm{d}y) - xy = y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}x\,\mathrm{d}y$$

In this case $\mathrm{d}z = f_1'(x, y)\mathrm{d}x + f_2'(x, y)\mathrm{d}y = y\,\mathrm{d}x + x\,\mathrm{d}y$, so $\Delta z - \mathrm{d}z = \mathrm{d}x\,\mathrm{d}y$. The error term is $\mathrm{d}x\,\mathrm{d}y$, and the approximation is illustrated in Fig. 12.9.2. If $\mathrm{d}x$ and $\mathrm{d}y$ are very small—for example, about 10^{-3}—then the error term $\mathrm{d}x\,\mathrm{d}y$ is "very, very small"—about 10^{-6} in this example.

EXAMPLE 12.9.2 Let $Y = F(K, L)$ be a production function with K and L as capital and labour inputs, respectively. Then F_K' and F_L' are the marginal products of capital and labour. If $\mathrm{d}K$ and $\mathrm{d}L$ are arbitrary increments in K and L, respectively, the *differential* of $Y = F(K, L)$ is $\mathrm{d}y = F_K'\,\mathrm{d}K + F_L'\,\mathrm{d}L$. The increment $\Delta Y = F(K + \mathrm{d}K, L + \mathrm{d}L) - F(K, L)$ in Y can be approximated by $\mathrm{d}y$ provided $\mathrm{d}K$ and $\mathrm{d}L$ are small in absolute value, and so

$$\Delta Y = F(K + \mathrm{d}K, L + \mathrm{d}L) - F(K, L) \approx F_K'\,\mathrm{d}K + F_L'\,\mathrm{d}L$$

Note that if $z = f(x, y)$, we can always find the differential $\mathrm{d}z = \mathrm{d}f$ by first finding the partial derivatives $f_1'(x, y)$ and $f_2'(x, y)$, and then using the definition of $\mathrm{d}z$. Conversely, once we know the differential of a function f of two variables, then we have the partial derivatives: Suppose that $\mathrm{d}z = A\,\mathrm{d}x + B\,\mathrm{d}y$ for all $\mathrm{d}x$ and $\mathrm{d}y$. By definition, $\mathrm{d}z = f_1'(x, y)\,\mathrm{d}x + f_2'(x, y)\,\mathrm{d}y$ for all $\mathrm{d}x$ and $\mathrm{d}y$. Putting $\mathrm{d}x = 1$ and $\mathrm{d}y = 0$ yields $A = f_1'(x, y)$. In the same way, putting $\mathrm{d}x = 0$ and $\mathrm{d}y = 1$ yields $B = f_2'(x, y)$. So

$$\mathrm{d}z = A\,\mathrm{d}x + B\,\mathrm{d}y \;\Rightarrow\; \frac{\partial z}{\partial x} = A \;\text{ and }\; \frac{\partial z}{\partial y} = B \tag{12.9.3}$$

Rules for Differentials

Section 7.4 developed several rules for working with differentials of functions of one variable. The same rules apply to functions of several variables. Indeed, suppose that $f(x, y)$ and $g(x, y)$ are differentiable, with differentials $\mathrm{d}f = f_1'\,\mathrm{d}x + f_2'\,\mathrm{d}y$ and $\mathrm{d}g = g_1'\,\mathrm{d}x + g_2'\,\mathrm{d}y$, respectively. If $\mathrm{d}(\,)$ denotes the differential of an expression inside the parentheses, then the following rules are exactly the same as rules (7.4.4) to (7.4.6):

RULES FOR DIFFERENTIALS

Let f and g be differentiable functions of x, and let a and b be constants. Then the following rules hold:

$$\mathrm{d}(af + bg) = a\,\mathrm{d}f + b\,\mathrm{d}g \tag{12.9.4}$$

$$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g \tag{12.9.5}$$

and, if $g \neq 0$,

$$\mathrm{d}\left(\frac{f}{g}\right) = \frac{g\,\mathrm{d}f - f\,\mathrm{d}g}{g^2} \tag{12.9.6}$$

These rules are, again, quite easy to prove. The argument for rule (12.9.5) is not very different to the one we gave for rule (7.4.5): because $(fg)(x, y) = f(x, y) \cdot g(x, y)$, we have

$$\begin{aligned} \mathrm{d}(fg) &= \frac{\partial}{\partial x}[f(x,y) \cdot g(x,y)]\,\mathrm{d}x + \frac{\partial}{\partial y}[f(x,y) \cdot g(x,y)]\,\mathrm{d}y \\ &= (f'_x \cdot g + f \cdot g'_x)\,\mathrm{d}x + (f'_y \cdot g + f \cdot g'_y)\,\mathrm{d}y \\ &= g(f'_x\,\mathrm{d}x + f'_y\,\mathrm{d}y) + f(g'_x\,\mathrm{d}x + g'_y\,\mathrm{d}y) \\ &= g\,\mathrm{d}f + f\,\mathrm{d}g \end{aligned}$$

There is also a chain rule for differentials. Suppose that $z = F(x, y) = g(f(x, y))$, where g is a differentiable function of one variable. Then,

$$\begin{aligned} \mathrm{d}z &= F'_x\,\mathrm{d}x + F'_y\,\mathrm{d}y \\ &= g'(f(x,y))f'_x\,\mathrm{d}x + g'(f(x,y))f'_y\,\mathrm{d}y \\ &= g'(f(x,y))(f'_x\,\mathrm{d}x + f'_y\,\mathrm{d}y) \\ &= g'(f(x,y))\,\mathrm{d}f \end{aligned}$$

because $F'_x = g'f'_x$, $F'_y = g'f'_y$, and $\mathrm{d}f = f'_x\,\mathrm{d}x + f'_y\,\mathrm{d}y$. Briefly formulated:

THE CHAIN RULE FOR DIFFERENTIALS

$$z = g(f(x,y)) \;\Rightarrow\; \mathrm{d}z = g'(f(x,y))\,\mathrm{d}f \tag{12.9.7}$$

EXAMPLE 12.9.3 Find an expression for $\mathrm{d}z$ in terms of $\mathrm{d}x$ and $\mathrm{d}y$ for the following functions: (a) $z = Ax^a + By^b$; (b) $z = e^{xu}$ with $u = u(x, y)$; and (c) $z = \ln(x^2 + y)$.

Solution:

(a) $\mathrm{d}z = A\,\mathrm{d}(x^a) + B\,\mathrm{d}(y^b) = Aax^{a-1}\,\mathrm{d}x + Bby^{b-1}\,\mathrm{d}y$

(b) Arguing directly, using abbreviated notation that drops (x, y) throughout, one has

$$\begin{aligned} \mathrm{d}z &= e^{xu}\,\mathrm{d}(xu) = e^{xu}(x\,\mathrm{d}u + u\,\mathrm{d}x) = e^{xu}\{x[u'_1(x,y)\,\mathrm{d}x + u'_2(x,y)\,\mathrm{d}y] + u\,\mathrm{d}x\} \\ &= e^{xu}\{[xu'_1(x,y) + u]\,\mathrm{d}x + xu'_2(x,y)\,\mathrm{d}y\} \end{aligned}$$

(c) $\mathrm{d}z = \mathrm{d}\ln(x^2 + y) = \dfrac{\mathrm{d}(x^2 + y)}{x^2 + y} = \dfrac{2x\,\mathrm{d}x + \mathrm{d}y}{x^2 + y}$

Invariance of the Differential

Suppose that $z = f(x, y)$, $x = g(t, s)$, and $y = h(t, s)$ are all differentiable functions. Thus, z is a composite function of t and s together. Suppose that t and s are changed by $\mathrm{d}t$ and $\mathrm{d}s$, respectively. The differential of z is then

$$\mathrm{d}z = z'_t \,\mathrm{d}t + z'_s \,\mathrm{d}s$$

Using the expressions for z'_t and z'_s obtained from the chain rule (12.1.1), we find that

$$\begin{aligned} \mathrm{d}z &= [f'_1(x, y)x'_t + f'_2(x, y)y'_t]\,\mathrm{d}t + [f'_1(x, y)x'_s + f'_2(x, y)y'_s]\,\mathrm{d}s \\ &= f'_1(x, y)(x'_t \,\mathrm{d}t + x'_s \,\mathrm{d}s) + f'_2(x, y)(y'_t \,\mathrm{d}t + y'_s \,\mathrm{d}s) \\ &= f'_1(x, y)\,\mathrm{d}x + f'_2(x, y)\,\mathrm{d}y \end{aligned}$$

where $\mathrm{d}x$ and $\mathrm{d}y$ denote the differentials of $x = g(t, s)$ and $y = h(t, s)$, respectively, as functions of t and s.

Note especially that the final expression for $\mathrm{d}z$ is precisely the definition of the differential of $z = f(x, y)$ when x and y are changed by $\mathrm{d}x$ and $\mathrm{d}y$, respectively. Thus, *the differential of z has the same form whether x and y are free variables, or depend on other variables t and s*. This property is referred to as the *invariance* of the differential.

The Differential of a Function of *n* Variables

The differential of a function $z = f(x_1, x_2, \ldots, x_n)$ of n variables is defined in the obvious way as

$$\mathrm{d}z = \mathrm{d}f = f'_1 \,\mathrm{d}x_1 + f'_2 \,\mathrm{d}x_2 + \cdots + f'_n \,\mathrm{d}x_n \tag{12.9.8}$$

If the absolute values of $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ are all small, then again $\Delta z \approx \mathrm{d}z$, where Δz is the actual increment of z when $(x_1, \ldots, x_n)$ is changed to $(x_1 + \mathrm{d}x_1, \ldots, x_n + \mathrm{d}x_n)$.

The rules for differentials in Eqs (12.9.4) to (12.9.6), and the chain rule (12.9.7), are valid for functions of n variables. There is also a general rule for invariance of the differential: *The differential of $z = F(x_1, \ldots, x_n)$ has the same form whether $x_1, \ldots, x_n$ are free variables, or depend on other basic variables*. The proofs of these results are easy extensions of those for two variables.

EXERCISES FOR SECTION 12.9

1. Determine the differential of $z = xy^2 + x^3$ by:

(a) computing $\partial z/\partial x$ and $\partial z/\partial y$ and then using the definition of $\mathrm{d}z$;

(b) using the rules in Eqs (12.9.4) to (12.9.6).

2. Calculate the differentials of the following functions:

(a) $z = x^3 + y^3$ (b) $z = xe^{y^2}$ (c) $z = \ln(x^2 - y^2)$

3. Find $\mathrm{d}z$ expressed in terms of $\mathrm{d}x$ and $\mathrm{d}y$ when $u = u(x, y)$ and

   (a) $z = x^2 u$ (b) $z = u^2$ (c) $z = \ln(xy + yu)$

SM 4. Find an approximate value for $T = [(2.01)^2 + (2.99)^2 + (6.02)^2]^{1/2}$ by using the approximation $\Delta T \approx \mathrm{d}t$.

5. Find $\mathrm{d}U$ expressed in terms of $\mathrm{d}x$ and $\mathrm{d}y$ when $U = U(x, y)$ satisfies the equation $Ue^U = x\sqrt{y}$.

6. Differentiate the equation $X = AN^\beta e^{\rho t}$, where A, β, and ρ are constants.

7. Differentiate the equation $X = BX^E N^{1-E}$, where B and E are constants.

8. Calculate the differentials of the following functions, where $a_1, \ldots, a_n, A, \delta_1, \ldots, \delta_n$, and ρ are positive constants:

   (a) $U = a_1 u_1^2 + \cdots + a_n u_n^2$ (b) $U = A(\delta_1 u_1^{-\rho} + \cdots + \delta_n u_n^{-\rho})^{-1/\rho}$

9. Find $\mathrm{d}z$ when $z = Ax_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$, where $x_1 > 0$, $x_2 > 0$, $\ldots$, $x_n > 0$, and A, $a_1, a_2, \ldots, a_n$ are all constants with A positive. (*Hint*: First, take the natural logarithm of each side.)

10. [HARDER] The differential $\mathrm{d}z$ defined in (12.9.1) is called the *differential of first order*. If f has continuous partial derivatives of second order, we define the *differential of second order* $\mathrm{d}^2 z$ as the differential $\mathrm{d}(\mathrm{d}z)$ of $\mathrm{d}z = f_1'(x, y)\,\mathrm{d}x + f_2'(x, y)\,\mathrm{d}y$. This implies that

$$\mathrm{d}^2 z = \mathrm{d}(\mathrm{d}z) = f_{11}''(x, y)\,(\mathrm{d}x)^2 + 2f_{12}''(x, y)\,\mathrm{d}x\,\mathrm{d}y + f_{22}''(x, y)\,(\mathrm{d}y)^2$$

   (a) Calculate $\mathrm{d}^2 z$ for $z = xy + y^2$.

   (b) Suppose that $x = t$ and $y = t^2$. Express $\mathrm{d}z$ and $\mathrm{d}^2 z$ in terms of $\mathrm{d}t$, for the function in part (a). Also find $\mathrm{d}^2 z/\mathrm{d}t^2$, then show that $\mathrm{d}^2 z \neq (\mathrm{d}^2 z/\mathrm{d}t^2)(\mathrm{d}t)^2$. This result shows that there is no invariance property for the second-order differential.

12.10 Systems of Equations

Many economic models relate a large number of variables to each other through a system of simultaneous equations. To keep track of the structure of the model, the concept of *degrees of freedom* is very useful.

Let $x_1, x_2, \ldots, x_n$ be n variables. If no restrictions are placed on them then, by definition, there are n *degrees of freedom*, because all n variables can be freely chosen. If the variables are required to satisfy *one* equation of the form $f_1(x_1, x_2, \ldots, x_n) = 0$, then the number of degrees of freedom is usually reduced by one. Whenever one further "independent" restriction is introduced, the number of degrees of freedom is again reduced by one. In general, introducing $m < n$ independent restrictions on the variables $x_1, x_2, \ldots, x_n$ means that they satisfy a system of m independent equations having the form

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ \cdots\cdots\cdots & \\ f_m(x_1, x_2, \ldots, x_n) &= 0 \end{aligned} \tag{12.10.1}$$

Then, provided that $m < n$, the remaining number of degrees of freedom is $n - m$. The rule that emerges from these considerations is rather vague, especially as it is hard to explain precisely what it means for equations to be "independent". Nevertheless, the following rule is much used in economics and statistics:

THE COUNTING RULE

To find the number of degrees of freedom for a system of equations, count the number of variables, n, and the number of "independent" equations, m. In general, if $n > m$, there are $n - m$ degrees of freedom in the system. If $n < m$, there is no solution to the system.

This rule of counting variables and equations is used to justify the following economic proposition: "The number of independent targets the government can pursue cannot possibly exceed the number of available policy instruments". For example, if a national government seeks simultaneous low inflation, low unemployment, and stability of its currency's exchange rate against, say, the US dollar, then it needs at least three independent policy instruments.

It should be noted that the counting rule is not generally valid. For example, if 100 variables $x_1, \ldots, x_{100}$ are restricted to satisfy one equation, the rule says that the number of degrees of freedom should be 99. However, if the equation happens to be

$$x_1^2 + x_2^2 + \cdots + x_{100}^2 = 0$$

then there is only one solution, $x_1 = x_2 = \cdots = x_{100} = 0$, so there are no degrees of freedom. For the equation $x_1^2 + x_2^2 + \cdots + x_{100}^2 = -1$, even this one solution is lost.

It is obvious that the word "independent" cannot be dropped from the statement of the counting rule. For instance, if we just repeat an equation that has appeared before, the number of degrees of freedom will certainly not be reduced.

The concept of degrees of freedom introduced earlier needs to be generalized.

DEGREES OF FREEDOM FOR A SYSTEM OF EQUATIONS

A system of equations in n variables is said to *have k degrees of freedom* if there is a set of k variables that can be freely chosen, while the remaining $n - k$ variables are uniquely determined once the k free variables have been assigned specific values.

In order for a system to have k degrees of freedom, it suffices that *there exist* k of the variables that can be freely chosen. We do not require that *any* set of k variables can be chosen freely. If the n variables are restricted to vary within a subset A of $\mathbb{R}^n$, we say that the system *has k degrees of freedom in A*.

The counting rule claims that if the number of equations is larger than the number of variables, then the system is, in general, *inconsistent*—that is, it has no solutions. For example, the system

$$f(x,y) = 0, \quad g(x,y) = 0, \quad h(x,y) = 0$$

with two variables and three equations, is usually inconsistent. Each of the equations represents a curve in the plane, and any pair of curves will usually have at least one point in common. But if we add a third equation, the corresponding curve will seldom pass through any points where the first two curves intersect, so the system is usually inconsistent.

So far, we have discussed the two cases $m < n$ and $m > n$. What about the case $m = n$, in which the number of equations is equal to the number of unknowns? Even in the simplest case of one equation in one variable, $f(x) = 0$, such an equation might have any number of solutions. Consider, for instance, the following three different single equations in one variable:

$$x^2 + 1 = 0, \quad x - 1 = 0, \quad (x-1)(x-2)(x-3)(x-4)(x-5) = 0$$

These have zero, one and five solutions, respectively. Those of you who know something about trigonometric functions will realize that the simple equation $\sin x = 0$ has infinitely many solutions, namely $x = n\pi$ for any integer n.

In general, a system with as many equations as unknowns is usually *consistent*—that is, has solutions—but it may have several solutions. Economists, however, ideally like their models to have a system of equations that produces a unique, economically meaningful solution, because then the model purports to predict the values of particular economic variables. Based on the earlier discussion, we can at least formulate the following rough rule: *A system of quations does not, in general, have a unique solution unless there are exactly as many equations as unknowns*.

EXAMPLE 12.10.1 Consider the macroeconomic model described by the system of equations

$$\text{(i) } Y = C + I + G \qquad \text{(ii) } C = f(Y - T) \qquad \text{(iii) } I = h(r) \qquad \text{(iv) } r = m(M)$$

where f, h, and m are given functions, Y is GDP, C is consumption, I is investment, G is public expenditure, T is tax revenue, r is the interest rate, and M is the quantity of money in circulation. How many degrees of freedom are there?

Solution: The number of variables is seven and the number of equations is four, so according to the counting rule there should be $7 - 4 = 3$ degrees of freedom. Usually macroeconomists regard M, T, and G as the exogenous (free) variables. Then the system will in general determine the endogenous variables Y, C, I, and r as functions of M, T, and G.[10] ▮

[10] For a further analysis of this model, see Example 12.11.3. For a discussion of exogenous and endogenous variables, see Section 12.11.

EXAMPLE 12.10.2 Consider the alternative macroeconomic model

(i) $Y = C + I + G$ (ii) $C = f(Y - T)$ (iii) $G = \bar{G}$

whose variables have the same interpretations as in the previous example. Here the level of public expenditure is a constant, $\bar{G}$. Determine the number of degrees of freedom in the model.

Solution: There are now three equations in the five variables Y, C, I, G, and T. Hence, there are two degrees of freedom. For suitable functions f, two of the variables can be freely chosen, while allowing the remaining variables to be determined once the values of these two are fixed. It is natural to consider I and T as the two free variables. Note that G cannot be chosen as a free variable in this case because it is fixed by equation (iii).

EXERCISES FOR SECTION 12.10

1. Use the counting rule to find the number of degrees of freedom for the following systems of equations:[11]

(a) $\begin{aligned} xu^3 + v &= y^2 \\ 3uv - x &= 4 \end{aligned}$ (b) $\begin{aligned} x_2^2 - x_3^3 + 2y_1 - y_2^3 &= 1 \\ x_1^3 - x_2 + y_1^5 - y_2 &= 0 \end{aligned}$ (c) $\begin{aligned} f(y + z + w) &= x^3 \\ x^2 + y^2 + z^2 &= w^2 \\ g(x, y) - z^3 &= w^3 \end{aligned}$

2. Use the counting rule to find the number of degrees of freedom in the following macroeconomic model, where the symbols have the same interpretation as in Example 12.10.1, and where F and f are some given functions:

(i) $Y = C + I + G$ (ii) $C = F(Y, T, r)$ (iii) $I = f(Y, r)$

3. For each of the following three systems of equations, determine the number of degrees of freedom, if any, and discuss whether the counting rule applies:

(a) $\begin{aligned} 3x - y &= 2 \\ 6x - 2y &= 4 \\ 9x - 3y &= 6 \end{aligned}$ (b) $\begin{aligned} x - 2y &= 3 \\ x - 2y &= 4 \end{aligned}$ (c) $\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}$

4. For each of the following two "systems" consisting of just one equation, determine the number of degrees of freedom, if any, and discuss whether the counting rule applies:

(a) $x_1^2 + x_2^2 + \cdots + x_{100}^2 = 1$ (b) $x_1^2 + x_2^2 + \cdots + x_{100}^2 = -1$

[11] In (c) assume that f and g are specified functions.

12.11 Differentiating Systems of Equations

This section shows how using differentials can be an efficient way to find the partial derivatives of functions defined implicitly by a system of equations. We begin with three examples.

EXAMPLE 12.11.1 Consider the following system of two linear equations in four variables:

$$5u + 5v = 2x - 3y$$
$$2u + 4v = 3x - 2y$$

It has two degrees of freedom. In fact, it defines u and v as functions of x and y. Differentiate the system and then find the differentials du and dv expressed in terms of $\mathrm{d}x$ and $\mathrm{d}y$. Derive the partial derivatives of u and v w.r.t. x and y. Check the results by solving the system explicitly for u and v.

Solution: For both equations, take the differential of each side and use the rules in Section 12.9. The result is

$$5\,\mathrm{d}u + 5\,\mathrm{d}v = 2\,\mathrm{d}x - 3\,\mathrm{d}y$$
$$2\,\mathrm{d}u + 4\,\mathrm{d}v = 3\,\mathrm{d}x - 2\,\mathrm{d}y$$

Note that in a linear system like this, without any constant terms, the differentials satisfy the same equations as the variables.

Solving simultaneously for $\mathrm{d}u$ and $\mathrm{d}v$ in terms of $\mathrm{d}x$ and $\mathrm{d}y$ yields

$$\mathrm{d}u = -\frac{7}{10}\,\mathrm{d}x - \frac{1}{5}\,\mathrm{d}y, \quad \mathrm{d}v = \frac{11}{10}\,\mathrm{d}x - \frac{2}{5}\,\mathrm{d}y$$

Now we can read off the partial derivatives: $u'_x = -\frac{7}{10}$, $u'_y = -\frac{1}{5}$, $v'_x = \frac{11}{10}$, and $v'_y = -\frac{2}{5}$.

Suppose that instead of finding the differential, we solve the given equation system directly for u and v as functions of x and y. The result is $u = -\frac{7}{10}x - \frac{1}{5}y$ and $v = \frac{11}{10}x - \frac{2}{5}y$. From these expressions we easily confirm the values found for the partial derivatives.

EXAMPLE 12.11.2 Consider the system of two nonlinear equations:

$$u^2 + v = xy; \quad uv = -x^2 + y^2$$

(a) What has the counting rule to say about this system?

(b) Find the differentials of u and v expressed in terms of $\mathrm{d}x$ and $\mathrm{d}y$. What are the partial derivatives of u and v w.r.t. x and y?

(c) The point $P = (x, y, u, v) = (1, 0, 1, -1)$ satisfies the system. If $x = 1$ is increased by 0.01 and $y = 0$ is increased by 0.02, what is the new value of u, approximately?

(d) Calculate u''_{12} at the point P.

Solution:

(a) There are four variables and two equations, so there should be two degrees of freedom. Suppose we choose fixed values for x and y. Then there are two equations for determining the two remaining variables, u and v. For example, if $x = 1$ and $y = 0$, then the system reduces to $u^2 = -v$ and $uv = -1$, from which we find that $u^3 = 1$, so $u = 1$ and $v = -1$. For other values of x and y, it is more difficult to find solutions for u and v. However, it seems reasonable to assume that the system defines $u = u(x, y)$ and $v = v(x, y)$ as differentiable functions of x and y, at least if the domain of the pair (x, y) is suitably restricted.

(b) The left- and right-hand sides of each equation in the system must be equal functions of x and y. So we can equate the differentials of each side to obtain $\mathrm{d}(u^2 + v) = \mathrm{d}(xy)$ and $\mathrm{d}(uv) = \mathrm{d}(-x^2 + y^2)$. Using the rules for differentials, we obtain

$$\begin{aligned} 2u\,\mathrm{d}u + \mathrm{d}v &= y\,\mathrm{d}x + x\,\mathrm{d}y \\ v\,\mathrm{d}u + u\,\mathrm{d}v &= -2x\,\mathrm{d}x + 2y\,\mathrm{d}y \end{aligned}$$

Note that by the invariance property of the differential stated in Section 12.9, this system is valid no matter which pair of variables are independent.

We want to solve the system for $\mathrm{d}u$ and $\mathrm{d}v$. There are two equations in the two unknowns $\mathrm{d}u$ and $\mathrm{d}v$ of the form

$$\begin{aligned} A\,\mathrm{d}u + B\,\mathrm{d}v &= C \\ D\,\mathrm{d}u + E\,\mathrm{d}v &= F \end{aligned}$$

where, for instance, $A = 2u$, $C = y\,\mathrm{d}x + x\,\mathrm{d}y$, and so on. Using Eq. (3.6.3), or standard elimination, provided that $v \neq 2u^2$, we find that

$$\mathrm{d}u = \frac{2x + yu}{2u^2 - v}\,\mathrm{d}x + \frac{xu - 2y}{2u^2 - v}\,\mathrm{d}y, \qquad \mathrm{d}v = \frac{-4xu - yv}{2u^2 - v}\,\mathrm{d}x + \frac{4uy - xv}{2u^2 - v}\,\mathrm{d}y$$

From the first of these two equations, we obtain immediately that

$$u_1' = \frac{2x + yu}{2u^2 - v} \quad \text{and} \quad u_2' = \frac{xu - 2y}{2u^2 - v}$$

Similarly, the partial derivatives of v w.r.t. x and y are the coefficients of $\mathrm{d}x$ and $\mathrm{d}y$ in the expression for $\mathrm{d}v$. So we have found all the first-order partial derivatives.

(c) We use the approximation $u(x + \mathrm{d}x, y + \mathrm{d}y) \approx u(x, y) + \mathrm{d}u$. At point P, where $(x, y, u, v) = (1, 0, 1, -1)$, and so $u_1' = \frac{2}{3}$, $u_2' = \frac{1}{3}$, we obtain

$$\begin{aligned} u(1 + 0.01, 0 + 0.02) &\approx u(1, 0) + u_1'(1, 0) \cdot 0.01 + u_2'(1, 0) \cdot 0.02 \\ &= 1 + \tfrac{2}{3} \cdot 0.01 + \tfrac{1}{3} \cdot 0.02 \\ &= 1 + \tfrac{4}{3} \cdot 0.01 \\ &\approx 1.0133 \end{aligned}$$

Note that in this case, it is not easy to find the exact value of $u(1.01, 0.02)$.

(d) We find u''_{12} by using the chain rule as follows:

$$u''_{12} = \frac{\partial}{\partial y}(u'_1) = \frac{\partial}{\partial y}\left(\frac{2x+yu}{2u^2-v}\right) = \frac{(yu'_2+u)(2u^2-v)-(2x+yu)(4uu'_2-v'_2)}{(2u^2-v)^2}$$

At the point P where $(x, y, u, v) = (1, 0, 1, -1)$, we obtain $u''_{12} = 1/9$.

EXAMPLE 12.11.3 Consider again the macroeconomic model of Example 12.10.1. If we assume that f, h, and m are differentiable functions with $0 < f' < 1$, $h' < 0$, and $m' < 0$, then these equations will determine Y, C, I, and r as differentiable functions of M, T, and G.

(a) Differentiate the system and express the differentials of Y, C, I, and r in terms of the differentials of M, T, and G. Find $\partial Y/\partial T$ and $\partial C/\partial T$, and comment on their signs.

(b) Suppose moreover that $P_0 = (M_0, T_0, G_0, Y_0, C_0, I_0, r_0)$ is an initial equilibrium point for the system. If the money supply M, tax revenue T, and public expenditure G are all slightly changed as a result of government policy or central bank intervention, find the approximate changes in national income Y and in consumption C.

Solution: Taking differentials of Eqs (i)-(iv) in Example 12.1.1 yields

$$\mathrm{d}Y = \mathrm{d}C + \mathrm{d}I + \mathrm{d}G \tag{v}$$

$$\mathrm{d}C = f'(Y-T)(\mathrm{d}Y - \mathrm{d}T) \tag{vi}$$

$$\mathrm{d}I = h'(r)\,\mathrm{d}r \tag{vii}$$

$$\mathrm{d}r = m'(M)\,\mathrm{d}M \tag{viii}$$

We wish to solve this linear system for the differential changes $\mathrm{d}y$, $\mathrm{d}C$, $\mathrm{d}I$, and $\mathrm{d}r$ in the endogenous variables Y, C, I, and r, expressing these differentials in terms of the differentials of the exogenous policy variables $\mathrm{d}M$, $\mathrm{d}T$, and $\mathrm{d}G$. From Eqs (vii) and (viii), we can find $\mathrm{d}I$ and $\mathrm{d}r$ immediately: $\mathrm{d}r = m'(M)\,\mathrm{d}M$ and $\mathrm{d}I = h'(r)m'(M)\,\mathrm{d}M$. Inserting the expression for $\mathrm{d}I$ into (v), while also rearranging (vi), we obtain the system:

$$\mathrm{d}Y - \mathrm{d}C = h'(r)m'(M)\,\mathrm{d}M + \mathrm{d}G; \quad f'(Y-T)\,\mathrm{d}Y - \mathrm{d}C = f'(Y-T)\,\mathrm{d}T$$

These are two equations to determine the two unknowns $\mathrm{d}Y$ and $\mathrm{d}C$. Solving for $\mathrm{d}Y$ and $\mathrm{d}C$, using a simplified notation, we get

$$\mathrm{d}Y = \frac{h'm'}{1-f'}\,\mathrm{d}M - \frac{f'}{1-f'}\,\mathrm{d}T + \frac{1}{1-f'}\,\mathrm{d}G \tag{ix}$$

$$\mathrm{d}C = \frac{f'h'm'}{1-f'}\,\mathrm{d}M - \frac{f'}{1-f'}\,\mathrm{d}T + \frac{f'}{1-f'}\,\mathrm{d}G \tag{x}$$

which express the differentials $\mathrm{d}Y$ and $\mathrm{d}C$ as linear functions of the differentials $\mathrm{d}M$, $\mathrm{d}T$, and $\mathrm{d}G$. Moreover, the solution is valid because $f' < 1$ by assumption.

From the four equations (vii)–(x), it is easy to find the partial derivatives of Y, C, I, and r w.r.t. M, T, and G. For example, $\partial Y/\partial T = \partial C/\partial T = -f'/(1-f')$ and $\partial r/\partial T = 0$. Note that because $0 < f' < 1$, we have $\partial Y/\partial T = \partial C/\partial T < 0$. Thus, a small increase in the tax level, keeping M and G constant, decreases GDP, unless the extra tax revenue is all spent by the government. For if $\mathrm{d}T = \mathrm{d}G = \mathrm{d}x$ (and $\mathrm{d}M = 0$), then $\mathrm{d}Y = \mathrm{d}x$ and $\mathrm{d}C = \mathrm{d}I = \mathrm{d}r = 0$.

If $\mathrm{d}M$, $\mathrm{d}T$, and $\mathrm{d}G$ are small in absolute value, then

$$\Delta Y = Y(M_0 + \mathrm{d}M, T_0 + \mathrm{d}T, G_0 + \mathrm{d}G) - Y(M_0, T_0, G_0) \approx \mathrm{d}Y$$

When computing $\mathrm{d}Y$, the partial derivatives are evaluated at the equilibrium point P_0.

Some textbooks recommend that students should express macro models like the one in the previous example as a matrix equation and then either use Cramer's rule or matrix inversion to find the solution. Elimination is vastly simpler and drastically reduces the risk of making errors.

EXAMPLE 12.11.4 Suppose that the two equations

$$(z + 2w)^5 + xy^2 = 2z - yw$$
$$(1 + z^2)^3 - z^2 w = 8x + y^5 w^2$$

define z and w as differentiable functions $z = \varphi(x, y)$ and $w = \psi(x, y)$ of x and y in a neighbourhood around $(x, y, z, w) = (1, 1, 1, 0)$.

(a) Compute $\partial z/\partial x$, $\partial z/\partial y$, $\partial w/\partial x$, and $\partial w/\partial y$ at $(1, 1, 1, 0)$.

(b) Use the results in (a) to find an approximate value of $\varphi(1 + 0.1, 1 + 0.2)$.

Solution:

(a) Equating the differentials of each side of the two equations, treated as functions of (x, y), we obtain

$$5(z + 2w)^4(\mathrm{d}z + 2\,\mathrm{d}w) + y^2\,\mathrm{d}x + 2xy\,\mathrm{d}y = 2\,\mathrm{d}z - w\,\mathrm{d}y - y\,\mathrm{d}w$$
$$3(1 + z^2)^2 2z\,\mathrm{d}z - 2zw\,\mathrm{d}z - z^2\,\mathrm{d}w = 8\,\mathrm{d}x + 5y^4w^2\,\mathrm{d}y + 2y^5w\,\mathrm{d}w$$

At the particular point $(x, y, z, w) = (1, 1, 1, 0)$ this system reduces to:

$$3\,\mathrm{d}z + 11\,\mathrm{d}w = -\mathrm{d}x - 2\,\mathrm{d}y; \quad 24\,\mathrm{d}z - \mathrm{d}w = 8\,\mathrm{d}x$$

Solving these two equations simultaneously for $\mathrm{d}z$ and $\mathrm{d}w$ in terms of $\mathrm{d}x$ and $\mathrm{d}y$ yields

$$\mathrm{d}z = \frac{29}{89}\,\mathrm{d}x - \frac{2}{267}\,\mathrm{d}y \quad \text{and} \quad \mathrm{d}w = -\frac{16}{89}\,\mathrm{d}x - \frac{16}{89}\,\mathrm{d}y$$

Hence, $\partial z/\partial x = 29/89$, $\partial z/\partial y = -2/267$, $\partial w/\partial x = -16/89$, and $\partial w/\partial y = -16/89$.

(b) If $x = 1$ is increased by $\mathrm{d}x = 0.1$ and $y = 1$ is increased by $\mathrm{d}y = 0.2$, the associated change in $z = \varphi(x, y)$ is approximately $\mathrm{d}z = (29/89) \cdot 0.1 - (2/267) \cdot 0.2 \approx 0.03$. Hence $\varphi(1 + 0.1, 1 + 0.2) \approx \varphi(1, 1) + \mathrm{d}z \approx 1 + 0.03 = 1.03$.

The General Case

When economists deal with systems of equations, notably in comparative static analysis, the variables are usually divided a priori into two types: *endogenous* variables, which the model is intended to determine; and *exogenous* variables, which are supposed to be determined by "forces" outside the economic model such as government policy, consumers' tastes, or technical progress. This classification depends on the model in question. Public expenditure, for example, is often treated as exogenous in public finance theory, which seeks to understand how tax changes affect the economy. But it is often endogenous in a "political economy" model which tries to explain how political variables like public expenditure emerge from the political system.

Economic models often give rise to a general system of *structural equations* having the form

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \\ f_2(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \\ \dots\dots\dots\dots\dots\dots \\ f_m(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \end{aligned} \tag{12.11.1}$$

Here it is assumed that $x_1, \dots, x_n$ are the exogenous variables, whereas $y_1, \dots, y_m$ are the endogenous variables. An "initial equilibrium" or "status quo" solution $(\mathbf{x}^0, \mathbf{y}^0) = (x_1^0, \dots, x_n^0, y_1^0, \dots, y_m^0)$ is frequently known, or else assumed to exist. This *equilibrium* might, for instance, represent a state in which there is equality between current supply and demand for each good.

Note that if the counting rule applies, then system (12.11.1) with m equations in $n + m$ unknowns has $n + m - m = n$ degrees of freedom. Suppose it defines all the endogenous variables $y_1, \dots, y_m$ as C^1 functions of $x_1, \dots, x_n$ in a neighbourhood of $(\mathbf{x}^0, \mathbf{y}^0)$. Then the system can be solved "in principle" for $y_1, \dots, y_m$ in terms of $x_1, \dots, x_n$ to give

$$y_1 = \varphi_1(x_1, \dots, x_n), \ \dots, \ y_m = \varphi_m(x_1, \dots, x_n) \tag{12.11.2}$$

In this case, system (12.11.2) is said to be the *reduced form* of the structural equation system (12.11.1). The endogenous variables have all been expressed as functions of the exogenous variables. The form of the functions $\varphi_1, \varphi_2, \dots, \varphi_m$ is not necessarily known.

The previous examples showed how we can often find an explicit expression for the partial derivative of any endogenous variable w.r.t. any exogenous variable. The same type of argument can be used in the general case, but a detailed discussion is left for FMEA.

EXERCISES FOR SECTION 12.1

1. Suppose that a, b, c, d, e, f, g, and h are constants satisfying $af \neq be$. Differentiate the system

$$au + bv = cx + dy$$
$$eu + fv = gx + hy$$

Then find the partial derivatives of u and v w.r.t. x and y.

2. Consider the system defined by $xu^3 + v = y^2$ and $3uv - x = 4$.

(a) Differentiate it and solve for $\mathrm{d}u$ and $\mathrm{d}v$ in terms of $\mathrm{d}x$ and $\mathrm{d}y$.

(b) Find u'_x and v'_x by using the results in part (a).

(c) The point $(x, y, u, v) = (0, 1, 4/3, 1)$ satisfies the system. Find u'_x and v'_x at this point.

SM 3. Suppose y_1 and y_2 are implicitly defined as differentiable functions of x_1 and x_2 by the following system:

$$3x_1 + x_2^2 - y_1 - 3y_2^3 = 0$$
$$x_1^3 - 2x_2 + 2y_1^3 - y_2 = 0$$

Find $\partial y_1/\partial x_1$ and $\partial y_2/\partial x_1$.

SM 4. A version of the "IS–LM" macroeconomic model leads to the system of equations $I(r) = S(Y)$ and $aY + L(r) = M$, where a is a positive parameter, while I, S, and L are given, continuously differentiable functions.[12] Suppose that the system defines Y and r implicitly as differentiable functions of a and M. Find expressions for $\partial Y/\partial M$ and $\partial r/\partial M$.

5. Find u''_{xx} when u and v are defined as functions of x and y by the equations $xy + uv = 1$ and $xu + yv = 0$.

6. Consider the macroeconomic model

(i) $Y = C + I + G$ (ii) $C = F(Y, T, r)$ (iii) $I = f(Y, r)$

where F and f are continuously differentiable functions, with $F'_Y > 0$, $F'_T < 0$, $F'_r < 0$, $f'_Y > 0$, $f'_r < 0$, and $F'_Y + f'_Y < 1$.

(a) Differentiate the system, and express dy in terms of dt, dG, and dr.

(b) What happens to Y if T increases? What if T and G undergo equal increases?

7. Consider the macroeconomic model

(i) $Y = C(Y, r) + I + \alpha$ (ii) $I = F(Y, r) + \beta$ (iii) $M = L(Y, r)$

where Y is GDP, r is the interest rate, I is total investment, α is public consumption, β is public investment, and M is the money supply, and where C, F, and L are given differentiable functions.

(a) Determine the number of degrees of freedom in the model.

(b) Differentiate the system. Put $d\beta = dM = 0$ and find dy, dr, and dI expressed in terms of $d\alpha$.

8. A standard macroeconomic model consists of the two equations:

(i) $M = \alpha Py + L(r)$ and (ii) $S(y, r, g) = I(y, r)$

where M, α, and P are positive constants, whereas L, S, and I are differentiable functions.

(a) By using the counting rule, explain why it is reasonable to assume that the system, in general, defines y and r as differentiable functions of g.

(b) Differentiate the system and find expressions for dy/dg and dr/dg.

[12] The first "IS" equation involves the investment function I and savings function S. The second "LM" equation involves the liquidity preference function L (the demand for money) and the money supply M. The variable Y denotes GDP and r denotes the interest rate. The IS–LM model was originally devised by J.R. Hicks.

9. The two equations $u^2v - u = x^3 + 2y^3$ and $e^{ux} = vy$ define u and v as differentiable functions of x and y around the point $P = (x, y, u, v) = (0, 1, 2, 1)$.

   (a) Find the differentials of u and v expressed in terms of the differentials of x and y. Find $\partial u/\partial y$ and $\partial v/\partial x$ at P.

   (b) If x increases by 0.1 and y decreases by 0.2 from their values at P, what are the approximate changes in u and v?

10. [HARDER] When there are two goods, consumer demand theory involves the equation system

    (i) $U_1'(x_1, x_2) = \lambda p_1$ (ii) $U_2'(x_1, x_2) = \lambda p_2$ (iii) $p_1x_1 + p_2x_2 = m$

    Here $U(x_1, x_2)$ is a given utility function. Suppose that the system defines x_1, x_2, and λ as differentiable functions of p_1, p_2, and m. Find an expression for $\partial x_1/\partial p_1$.

REVIEW EXERCISES

1. In the following cases, find $\mathrm{d}z/\mathrm{d}t$ by using the chain rule, then check the answers by first substituting the expressions for x and y and then differentiating:

   (a) $z = F(x, y) = 6x + y^3$, with $x = 2t^2$ and $y = 3t^3$

   (b) $z = F(x, y) = x^p + y^p$, with $x = at$ and $y = bt$

2. Let $z = G(u, v)$, $u = \varphi(t, s)$, and $v = \psi(s)$. Find expressions for $\partial z/\partial t$ and $\partial z/\partial s$.

3. Find expressions for $\partial w/\partial t$ and $\partial w/\partial s$ when $w = x^2 + y^3 + z^4$, $x = t + s$, $y = t - s$, and $z = st$.

SM 4. Suppose production X depends on the number of workers N according to the formula $X = Ng(\varphi(N)/N)$, where g and φ are given differentiable functions. Find expressions for $\mathrm{d}x/\mathrm{d}N$ and $\mathrm{d}^2X/\mathrm{d}N^2$.

5. Suppose that a household's demand for a commodity is a function $E(p, m) = Ap^{-a}m^b$ of the price p and income m, where A, a, and b are positive constants.

   (a) Suppose that p and m are both differentiable functions of time t. Then demand E is a function only of t. Find an expression for $\dot{E}/E$ in terms of $\dot{p}/p$ and $\dot{m}/m$.

   (b) Put $p = p_0(1.06)^t$ and $m = m_0(1.08)^t$, where p_0 is the price and m_0 is the income at time $t = 0$. Show that in this case $\dot{E}/E = \ln Q$, where $Q = (1.08)^b/(1.06)^a$.

6. The equation $x^3 \ln x + y^3 \ln y = 2z^3 \ln z$ defines z as a differentiable function of x and y in a neighbourhood of the point $(x, y, z) = (e, e, e)$. Calculate $z_1'(e, e)$ and $z_{11}''(e, e)$.

7. What is the elasticity of substitution between y and x when $F(x, y) = x^2 - 10y^2$?

8. Find the MRS between y and x when:

   (a) $U(x, y) = 2x^{0.4}y^{0.6}$ (b) $U(x, y) = xy + y$ (c) $U(x, y) = 10(x^{-2} + y^{-2})^{-4}$

9. Find the degree of homogeneity, if there is one, for each of the following functions:

(a) $f(x, y) = 3x^3y^{-4} + 2xy^{-2}$ (b) $Y(K, L) = (K^a + L^a)^{2c}e^{K^2/L^2}$

(c) $f(x_1, x_2) = 5x_1^4 + 6x_1x_2^3$ (d) $F(x_1, x_2, x_3) = e^{x_1+x_2+x_3}$

10. What is the elasticity of substitution between y and x when $U(x, y) = 10(x^{-2} + y^{-2})^{-4}$?

SM **11.** Find the elasticity of y w.r.t. x when $y^2e^{x+1/y} = 3$.

12. Find the degree of homogeneity, if there is one, of the following functions:

(a) $f(x, y) = xg(y/x)$, where g is an arbitrary function of one variable.

(b) $F(x, y, z) = z^kf(x/z, y/z)$, where f is an arbitrary function of two variables.

(c) $G(K, L, M, N) = K^{a-b}L^{b-c}M^{c-d}N^{d-a}$, where a, b, c, and d are constants.

13. Suppose the production function $F(K, L)$ defined for $K > 0, L > 0$ is homogeneous of degree 1. If $F''_{KK} < 0$, so that the marginal productivity of capital is a strictly decreasing function of K, prove that $F''_{KL} > 0$, so that the marginal productivity of capital is strictly increasing as labour input increases.[13] (*Hint*: Use Eqs (12.6.6) and (12.6.7).)

14. Show that no generalization of the concept of a homogeneous function emerges if one replaces t^k in definitions (12.6.1) or (12.7.1) by an arbitrary function $g(t)$. (*Hint*: Differentiate the new definition w.r.t. t, and let $t = 1$. Then use Euler's theorem.)

15. The following system of equations defines $u = u(x, y)$ and $v = v(x, y)$ as differentiable functions of x and y around the point $P = (x, y, u, v) = (1, 1, -1, 0)$:

$$u + xe^y + v = e - 1$$
$$x + e^{u+v^2} - y = e^{-1}$$

Differentiate the system and find the values of u'_x, u'_y, v'_x, and v'_y at that point.

SM **16.** An equilibrium model of labour demand and output pricing leads to the following system of equations:

$$pF'(L) - w = 0$$
$$pF(L) - wL - B = 0$$

Here, F is twice differentiable with $F'(L) > 0$ and $F''(L) < 0$. All the variables are positive. Regard w and B as exogenous, so that p and L are endogenous variables which are functions of w and B.

(a) Find expressions for $\partial p/\partial w$, $\partial p/\partial B$, $\partial L/\partial w$, and $\partial L/\partial B$ by implicit differentiation.

(b) What can be said about the signs of these partial derivatives? Show, in particular, that $\partial L/\partial w < 0$.

[13] This is called *Wicksell's law*.

17. The following system of equations defines $u = u(x, y)$ and $v = v(x, y)$ as differentiable functions of x and y around the point $P = (x, y, u, v) = (1, 1, 1, 2)$:

$$u^\alpha + v^\beta = 2^\beta x + y^3$$
$$u^\alpha v^\beta - v^\beta = x - y$$

where α and β are positive constants.

(a) Differentiate the system, and find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ at the point P.

(b) Find an approximation to $u(0.99, 1.01)$.

18. A study of the demand for some commodity involves the integral

$$S = \int_0^T e^{-rx}(e^{g(T-x)} - 1)\,dx$$

where T, r and g are positive constants.

(a) Show that

$$r(r+g)S = re^{gT} + ge^{-rT} - (r+g) \tag{$*$}$$

(b) Equation $(*)$ defines T as a differentiable function of g, r, and S. Use the equation to find an expression for $\partial T/\partial g$.

SM 19. Suppose that a vintage car has an appreciating market value given by the function $V(t)$ of time t. Suppose the maintenance cost of the car per unit of time is constant, at m per year. Allowing for continuous time discounting at a rate r per year, the present discounted value from selling the car at time t is $P(t) = V(t)e^{-rt} - \int_0^t me^{-r\tau}\,d\tau$.

(a) Show that the optimal choice t^* of t must satisfy $V'(t^*) = rV(t^*) + m$, and give this condition an economic interpretation.

(b) Show that the standard second-order condition for $P(t)$ to have a strict local maximum at $t^*(r, m)$ reduces to the condition $D = V''(t^*) - rV'(t^*) < 0$.

(c) Find the partial derivatives $\partial t^*/\partial r$ and $\partial t^*/\partial m$, and use the condition derived in the answer to (b) in order to discuss how an economist would interpret their signs.

13

MULTIVARIABLE OPTIMIZATION

At first sight it is curious that a subject as pure and passionless as mathematics can have anything useful to say about that messy, ill-structured, chancy world in which we live.

Fortunately we find that whenever we comprehend what was previously mysterious, there is at the centre of everything order, pattern and common sense.
—Patrick (B.H.P.) Rivett (1978)

Chapter 8 was concerned with optimization problems involving functions of one variable. Most interesting economic optimization problems, however, require the simultaneous choice of several variables. For example, a profit-maximizing producer of a single commodity chooses not only its output level, but also the quantities of many different inputs. A consumer chooses what quantities of the many different goods available for her to buy.

Most of the mathematical difficulties arise already in the transition from one to two variables. On the other hand, textbooks in economics often illustrate economic problems by using functions of only two variables, for which one can at least draw level curves in the plane. We therefore begin this chapter by studying the two-variable case. The first section presents the basic results, illustrated by relatively simple examples and problems. Then, we give a more systematic presentation of the theory with two variables. Subsequently we consider how the theory can be generalized to functions of several variables.

Much of economic analysis involves seeing how the solution to an optimization problem responds when the situation changes—for example, if some relevant parameters change. Thus, the theory of the firm considers how a change in the price of a good that is either an input or an output can affect the optimal quantities of all the inputs and outputs, as well as the maximum profit. Some simple results of this kind are briefly introduced at the end of the chapter.

13.1 Two Choice Variables: Necessary Conditions

Consider a differentiable function $z = f(x, y)$ defined on a set S in the xy-plane. Suppose that f attains its largest value (its maximum) at an "interior" point (x_0, y_0) of S, as indicated in Fig. 13.1.1. If we keep y fixed at y_0, then the function $g(x) = f(x, y_0)$ depends only on x

and has its maximum at $x = x_0$. Geometrically, if P is the highest point on the surface in Fig. 13.1.1, then P is certainly also the highest point on the curve through P that has $y = y_0$—i.e. on the curve which is the intersection of the surface with the plane $y = y_0$. From Theorem 8.1.1, we know that $g'(x_0) = 0$. But for all x, the derivative $g'(x)$ is exactly the same as the partial derivative $f_1'(x, y_0)$. At $x = x_0$, therefore, one has $f_1'(x_0, y_0) = 0$. In the same way, we see that (x_0, y_0) must satisfy $f_2'(x_0, y_0) = 0$, because the function $h(y) = f(x_0, y)$ has its maximum at $y = y_0$. A point (x_0, y_0) where both the partial derivatives are 0 is called a *critical* (or *stationary*) *point* of f.

If f attains its smallest value (its minimum) at an interior point (x_0, y_0) of S, a similar argument shows that the point again must be a critical point. So we have the following important result:[1]

THEOREM 13.1.1 (NECESSARY FIRST-ORDER CONDITIONS)

A differentiable function $z = f(x, y)$ can have a maximum or minimum at an interior point (x_0, y_0) of its domain only if it is a *critical point*—that is, if the point $(x, y) = (x_0, y_0)$ satisfies the two equations

$$f_1'(x, y) = 0 \quad \text{and} \quad f_2'(x, y) = 0$$

which are known as first-order conditions, or FOCs.

In Fig. 13.1.2, the three points P, Q, and R are all critical points, but only P is a maximum.[2] In the following examples and problems only the first-order conditions are considered. Section 13.2 explains how to verify that we have found the optimum.

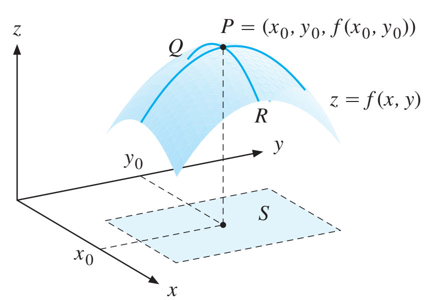

Figure 13.1.1 Maximum point, P, is critical

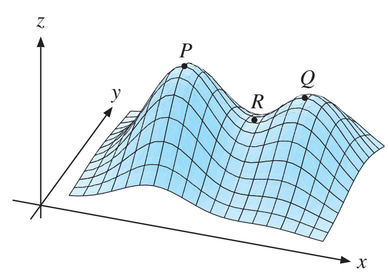

Figure 13.1.2 Only P is a maximum

EXAMPLE 13.1.1 The function f is defined for all (x, y) by

$$f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$$

Assume that f has a maximum point. Find it.

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[1] The concept of interior point is defined precisely in Section 13.5.

[2] Later, we shall call Q a *local maximum*, whereas R is a *saddle point*.

Solution: Theorem 13.1.1 applies, so a maximum point (x, y) must be a critical point, satisfying the first-order conditions:

$$f'_1(x, y) = -4x - 2y + 36 = 0 \text{ and } f'_2(x, y) = -2x - 4y + 42 = 0$$

These are two linear equations which determine x and y. We find that $(x, y) = (5, 8)$ is the only pair of numbers which satisfies both equations. Assuming there is a maximum point, these must be its coordinates. The maximum value is $f(5, 8) = 100$.[3]

EXAMPLE 13.1.2 A firm produces two different kinds, A and B, of a commodity. The daily cost of producing x units of A and y units of B is

$$C(x, y) = 0.04x^2 + 0.01xy + 0.01y^2 + 4x + 2y + 500$$

Suppose that the firm sells all its output at a price per unit of \$15 for A and \$9 for B. Find the daily production levels x and y that maximize profit per day.

Solution: Profit per day is $\pi(x, y) = 15x + 9y - C(x, y)$, so

$$\begin{aligned}\pi(x, y) &= 15x + 9y - 0.04x^2 - 0.01xy - 0.01y^2 - 4x - 2y - 500\\ &= -0.04x^2 - 0.01xy - 0.01y^2 + 11x + 7y - 500\end{aligned}$$

If $x > 0$ and $y > 0$ maximize profit, then (x, y) must satisfy

$$\frac{\partial \pi}{\partial x} = -0.08x - 0.01y + 11 = 0, \quad \frac{\partial \pi}{\partial y} = -0.01x - 0.02y + 7 = 0$$

These two linear equations in x and y have the unique solution $x = 100$, $y = 300$, with $\pi(100, 300) = 1100$. (We have not proved that this actually is a maximum. For that, see Exercise 13.2.1).

EXAMPLE 13.1.3 **(Profit maximization)** Suppose that $Q = F(K, L)$ is a production function with K as the capital input and L as the labour input. The price per unit of output is p, the cost (or rental) per unit of capital is r, and the wage rate is w. The constants p, r, and w are all positive. The profit, π, from producing and selling $F(K, L)$ units is then given by the function

$$\pi(K, L) = pF(K, L) - rK - wL$$

If F is differentiable and π has a maximum with $K > 0$, $L > 0$, then the FOCS are

$$\pi'_K(K, L) = pF'_K(K, L) - r = 0 \text{ and } \pi'_L(K, L) = pF'_L(K, L) - w = 0$$

Thus, a necessary condition for profit to be a maximum when $K = K^*$ and $L = L^*$ is that

$$pF'_K(K^*, L^*) = r \text{ and } pF'_L(K^*, L^*) = w \tag{$*$}$$

[3] In Example 13.2.2 we prove that $(5, 8)$ *really is* a maximum point.

The first equation says that r, the cost of capital, must equal the value, at price p per unit, of the marginal product of capital. The second equation has a similar interpretation.

Suppose we think of increasing capital input from the level K^* by k units. How much would be gained? Production would increase by approximately $F'_K(K^*, L^*)k$ units. Because each extra unit is priced at p, the revenue gain is approximately $pF'_K(K^*, L^*)k$. How much is lost? The answer is rk, because r is the cost of each unit of capital. These two must be equal.

The second equation in $(*)$ has a similar interpretation: Increasing labour input by ℓ units from level L^* will lead to the approximate gain $pF'_L(K^*, L^*)\ell$ in revenue, whereas the extra labour cost is $w\ell$. The profit-maximizing pair (K^*, L^*) thus has the property that the extra revenue from increasing either input is just offset by the extra cost.

Economists often divide the first-order conditions $(*)$ by the positive price p to reach the alternative form $F'_K(K, L) = r/p$ and $F'_L(K, L) = w/p$. So, to obtain maximum profit, the firm must choose K and L to equate the marginal productivity of capital to its "relative" price, r/p, and also to equate the marginal productivity of labour to its relative price, w/p.

Note that the conditions in $(*)$ are necessary, but generally not sufficient for an interior maximum.[4]

EXAMPLE 13.1.4 Find the only possible solution to the following special case of Example 13.1.3:

$$\max \pi(K, L) = 12K^{1/2}L^{1/4} - 1.2K - 0.6L$$

Solution: The first-order conditions are

$$\pi'_K(K, L) = 6K^{-1/2}L^{1/4} - 1.2 = 0 \quad \text{and} \quad \pi'_L(K, L) = 3K^{1/2}L^{-3/4} - 0.6 = 0$$

These equations imply that $K^{-1/2}L^{1/4} = K^{1/2}L^{-3/4} = 0.2 = 1/5$. Multiplying each side of the first equation here by $K^{1/2}L^{3/4}$ reduces it to $L = K$. Hence $K^{-1/4} = L^{-1/4} = 1/5$. It follows that $K = L = 5^4 = 625$ is the only possible solution.[5]

EXAMPLE 13.1.5 A firm is a monopolist in the domestic market, but takes as given the price, p_w, of its product in the world market. The quantities sold in the two markets are denoted by x_d and x_w, respectively. The price obtained in the domestic market, as a function of its sales, is given by the inverse demand function $p_d = P(x_d)$. The cost of producing x units is $C(x)$, regardless of how this output is distributed between the domestic and world markets.

(a) Find the profit function $\pi(x_d, x_w)$ and write down the first-order conditions for profit to be maximized at $x_d > 0$, $x_w > 0$. Give economic interpretations of these conditions.

(b) Suppose that in the domestic market the firm is faced with a demand curve whose price elasticity is constant, equal to -2. What is the relationship between the prices in the domestic and world markets?

[4] Sufficient conditions for an optimum are given in Example 13.3.3.

[5] See Example 13.2.3 for a proof that this is indeed a maximum point.

Solution:

(a) The revenue from selling x_d units in the domestic market at the price $p_d = P(x_d)$ is $P(x_d) \cdot x_d$. In the world market the revenue is $p_w x_w$. The profit function is $\pi = \pi(x_d, x_w) = P(x_d)x_d + p_w x_w - C(x_d + x_w)$. Thus, the first-order conditions are

$$\pi'_1 = p_d + P'(x_d) \cdot x_d - C'(x_d + x_w) = 0 \qquad (*)$$

$$\pi'_2 = p_w - C'(x_d + x_w) = 0 \qquad (**)$$

According to (∗∗), the marginal cost in the world market must equal the price, which is the marginal revenue in this case. In the domestic market the marginal cost must also equal the marginal revenue. Suppose the firm contemplates producing and selling a little extra in its domestic market. The extra revenue per unit increase in output equals p_d minus the loss that arises because of the induced price reduction for all domestic sales. The latter loss is approximately $P'(x_d) \cdot x_d$. Since the cost of an extra unit of output is approximately the marginal cost $C'(x_d + x_w)$, condition (∗) expresses the requirement that, per unit of extra output, the domestic revenue gain is just offset by the cost increase.

(b) The price elasticity of demand is −2, meaning that $\mathrm{El}_{p_d} x_d = (p_d/x_d)(\mathrm{d}x_d/\mathrm{d}p_d) = -2$. By the rule for differentiating inverse functions one has $\mathrm{d}p_d/\mathrm{d}x_d = 1/(\mathrm{d}x_d/\mathrm{d}p_d)$. It follows that

$$P'(x_d) \cdot x_d = \frac{\mathrm{d}p_d}{\mathrm{d}x_d} x_d = -\frac{1}{2} p_d$$

Then (∗) and (∗∗) imply that $\frac{1}{2} p_d = C'(x_d + x_w) = p_w$, so the domestic market price is twice the world market price. ▮

EXERCISES FOR SECTION 13.1

1. The function f defined for all (x, y) by $f(x, y) = -2x^2 - y^2 + 4x + 4y - 3$ has a maximum. Find the corresponding values of x and y.

2. Consider the function f defined for all (x, y) by $f(x, y) = x^2 + y^2 - 6x + 8y + 35$.

   (a) The function has a minimum point. Find it.

   (b) Show that $f(x, y)$ can be written in the form $f(x, y) = (x - 3)^2 + (y + 4)^2 + 10$. Explain why this shows that you have really found the minimum in part (a).

3. In the profit-maximizing problem of Example 13.1.3, let $p = 1$, $r = 0.65$, $w = 1.2$, and

$$F(K, L) = 80 - (K - 3)^2 - 2(L - 6)^2 - (K - 3)(L - 6)$$

   Find the only possible values of K and L that maximize profits.

4. Annual profits for a firm are given by

$$P(x, y) = -x^2 - y^2 + 22x + 18y - 102$$

   where x is the amount spent on research, and y is the amount spent on advertising.

   (a) Find the profits when $x = 10$, $y = 8$ and when $x = 12$, $y = 10$.

(b) Find the only possible values of x and y that can maximize profits, and the corresponding profit.

13.2 Two Choice Variables: Sufficient Conditions

Suppose f is a function of one variable defined in an interval I. Recall from Theorem 8.2.2 that, if f is twice differentiable, in this case a very simple sufficient condition for a critical point in I to be a maximum point is that $f''(x) \leq 0$ for all x in I. Shorthand for this sufficient condition is to say that the function f is concave.

For functions of two variables there is a corresponding test for concavity based on the second-order *partial* derivatives. Provided the function has an interior critical point, this test implies that its graph is a surface shaped like the one in Fig. 13.1.1.

Consider any curve parallel to the xz-plane which lies in the surface, like QPR in that figure. Any such curve is the graph of a concave function of one variable, implying that $f''_{11}(x, y) \leq 0$. A similar argument holds for any curve parallel to the yz-plane which lies in the surface, implying that $f''_{22}(x, y) \leq 0$. In general, however, having these two second-order partial derivatives be nonpositive is *not* sufficient on its own to ensure that the surface is shaped like the one in Fig. 13.1.1. This is clear from the next example.

EXAMPLE 13.2.1 The function $f(x, y) = 3xy - x^2 - y^2$ has $f''_{11}(x, y) = f''_{22}(x, y) = -2$. Each curve parallel to the xz-plane that lies in the surface defined by the graph has the equation $z = 3xy_0 - x^2 - y_0^2$ for some fixed y_0. It is therefore a concave parabola. So is each curve parallel to the yz-plane that lies in the surface. But along the line $y = x$ the function reduces to $f(x, x) = x^2$, whose graph is a convex rather than a concave parabola. It follows that f has no maximum (or minimum) at $(0, 0)$, which is its only critical point.

What Example 13.2.1 shows is that conditions ensuring that the graph of f looks like the one in Fig. 13.1.1 cannot ignore the second-order cross partial derivative $f''_{12}(x, y)$. The following result is analogous to Theorem 8.2.2. We leave a detailed discussion of it to FMEA, but present a proof of its local version in Section 13.3. To formulate the theorem, however, we need a new concept: a set S in the xy-plane is *convex* if for each pair of points P and Q in S, the whole line segment between P and Q lies in S.

THEOREM 13.2.1 (SUFFICIENT CONDITIONS FOR A MAXIMUM OR MINIMUM)

Suppose that (x_0, y_0) is an interior critical point for a C^2 function $f(x, y)$ defined in a convex set S in $\mathbb{R}^2$.

(a) If for all (x, y) in S, one has

$$f''_{11}(x, y) \leq 0, \quad f''_{22}(x, y) \leq 0, \quad \text{and}$$

$$f''_{11}(x, y) f''_{22}(x, y) - \left[f''_{12}(x, y)\right]^2 \geq 0$$

then (x_0, y_0) is a maximum point for $f(x, y)$ in S.

(b) If for all (x, y) in S, one has

$$f''_{11}(x, y) \geq 0, \;\; f''_{22}(x, y) \geq 0, \;\; \text{and}$$
$$f''_{11}(x, y)f''_{22}(x, y) - \left[f''_{12}(x, y)\right]^2 \geq 0$$

then (x_0, y_0) is a minimum point for $f(x, y)$ in S.

The conditions in part (a) of Theorem 13.2.1 are sufficient for a critical point to be a maximum point. They are far from being necessary. This is clear from the function whose graph is shown in Fig. 13.1.2, which *has* a maximum at P, but where the conditions in (a) are certainly not satisfied in the whole of its domain.

Importantly, if a twice differentiable function $z = f(x, y)$ satisfies the inequalities in (a) throughout a convex set S, it is called *concave*, whereas it is called *convex* if it satisfies the inequalities in (b) throughout S. It follows from these definitions that f is concave if and only if $-f$ is convex, just as in the one-variable case. There are more general definitions of concave and convex functions which apply to functions that are not necessarily differentiable. These are presented in FMEA.[6]

EXAMPLE 13.2.2 Show that we have found a maximum in Example 13.1.1.

Solution: We found that $f'_1(x, y) = -4x - 2y + 36$ and $f'_2(x, y) = -2x - 4y + 42$. Furthermore, $f''_{11} = -4$, $f''_{12} = -2$, and $f''_{22} = -4$. Thus, $f''_{11}(x, y) \leq 0$, $f''_{22}(x, y) \leq 0$, and

$$f''_{11}(x, y)f''_{22}(x, y) - \left[f''_{12}(x, y)\right]^2 = 16 - 4 = 12 \geq 0$$

According to part (a) in Theorem 13.2.1, these inequalities guarantee that the critical point $(5, 8)$ is a maximum point.

EXAMPLE 13.2.3 Show that we have found the maximum in Example 13.1.4.

Solution: If $K > 0$ and $L > 0$, we find that

$$\pi''_{KK} = -3K^{-3/2}L^{1/4}, \;\; \pi''_{KL} = \tfrac{3}{2}K^{-1/2}L^{-3/4}, \;\; \text{and } \pi''_{LL} = -\tfrac{9}{4}K^{1/2}L^{-7/4}$$

Clearly, $\pi''_{KK} < 0$, $\pi''_{LL} < 0$, and moreover,

$$\pi''_{KK}\pi''_{LL} - (\pi''_{KL})^2 = \tfrac{27}{4}K^{-1}L^{-3/2} - \tfrac{9}{4}K^{-1}L^{-3/2} = \tfrac{9}{2}K^{-1}L^{-3/2} > 0$$

It follows that the critical point $(K, L) = (625, 625)$ maximizes profit.

This section concludes with two examples of optimization problems where the choice of variables is subject to constraints. Nevertheless, a simple transformation can be used to convert the problem into the form we have been discussing, without any constraints.

[6] The one-variable case was briefly discussed in Section 8.7.

EXAMPLE 13.2.4 Suppose that any production by the firm in Example 13.1.2 creates pollution, so it is legally restricted to produce a total of 320 units of the two kinds of output. The firm's problem is then

$$\max -0.04x^2 - 0.01xy - 0.01y^2 + 11x + 7y - 500 \text{ subject to } x + y = 320$$

What are the optimal quantities of the two kinds of output now?

Solution: The firm still wants to maximize its profits. But because of the restriction $y = 320 - x$, the new profit function is

$$\hat{\pi}(x) = -0.04x^2 - 0.01x(320 - x) - 0.01(320 - x)^2 + 11x + 7(320 - x) - 500$$

We easily find $\hat{\pi}'(x) = -0.08x + 7.2$, so $\hat{\pi}'(x) = 0$ for $x = 7.2/0.08 = 90$. Since $\hat{\pi}''(x) = -0.08 < 0$ for all x, the point $x = 90$ does maximize $\hat{\pi}$. The corresponding value of y is $y = 320 - 90 = 230$. The maximum profit is 1040.

EXAMPLE 13.2.5 A firm has three factories producing the same item. Let x, y, and z denote the respective output quantities that the three factories produce in order to fulfil an order for 2000 units in total. Hence, $x + y + z = 2000$. The cost functions for the three factories are

$$C_1(x) = 200 + \frac{1}{100}x^2, \quad C_2(y) = 200 + y + \frac{1}{300}y^3, \quad \text{and} \quad C_3(z) = 200 + 10z$$

The total cost of fulfilling the order is, thus,

$$C(x, y, z) = C_1(x) + C_2(y) + C_3(z)$$

Find the values of x, y, and z that minimize C.

Solution: Solving the equation $x + y + z = 2000$ for z yields $z = 2000 - x - y$. Substituting this expression for z in the expression for C yields, after simplifying,

$$\hat{C}(x, y) = C(x, y, 2000 - x - y) = \frac{1}{100}x^2 - 10x + \frac{1}{300}y^3 - 9y + 20\,600$$

Any critical points of $\hat{C}$ must satisfy the two equations

$$\hat{C}_1'(x, y) = \frac{1}{50}x - 10 = 0 \quad \text{and} \quad \hat{C}_2'(x, y) = \frac{1}{100}y^2 - 9 = 0$$

The only economically sensible solution is $x = 500$ and $y = 30$, implying that $z = 1470$. The corresponding value of C is 17 920.

The second-order partials are $\hat{C}_{11}''(x, y) = \frac{1}{50}$, $\hat{C}_{12}''(x, y) = 0$, and $\hat{C}_{22}''(x, y) = \frac{1}{50}y$. It follows that for all $x \ge 0$, $y \ge 0$, one has $\hat{C}_{11}''(x, y) \ge 0$, $\hat{C}_{22}''(x, y) \ge 0$, and

$$\hat{C}_{11}''(x, y)\hat{C}_{22}''(x, y) - \hat{C}_{12}''(x, y)^2 = \frac{y}{2500} \ge 0$$

Part (b) of Theorem 13.2.1 implies that $(500, 30)$ is a minimum point of $\hat{C}$ within the convex domain of points (x, y) satisfying $x \ge 0$, $y \ge 0$, and $x + y \le 2000$. It follows that $(500, 30, 1470)$ is a minimum point of C within the domain of (x, y, z) satisfying $x \ge 0$, $y \ge 0$, $z \ge 0$, and $x + y + z = 2000$.

EXERCISES FOR SECTION 13.2

1. Prove that the true maximum has been found in: (a) Example 13.1.2; (b) Exercise 13.1.1; (c) Exercise 13.1.3.

2. A firm produces two different kinds, A and B, of a commodity. The daily cost of producing x units of A and y units of B is

$$C(x, y) = 2x^2 - 4xy + 4y^2 - 40x - 20y + 514$$

Suppose that the firm sells all its output at a price per unit of \$24 for A and \$12 for B.

(a) Find the daily production levels x and y that maximize profit.

(b) The firm is required to produce exactly 54 units per day of the two kinds combined. What is the optimal production plan now?

SM 3. Maximize the utility function $U(x, y, z) = xyz$, subject to $x + 3y + 4z = 108$ and $x, y, z > 0$, by eliminating the variable x and defining an appropriate function of only y and z.

4. The demands for a monopolist's two products are determined by the equations $p = 25 - x$ and $q = 24 - 2y$, where p and q are prices per unit of the two goods, and x and y are the corresponding quantities. The costs of producing x units of the first good and y units of the other are

$$C(x, y) = 3x^2 + 3xy + y^2$$

(a) Find the monopolist's profit $\pi(x, y)$ from producing and selling x units of the first good and y units of the other.

(b) Find the values of x and y that maximize $\pi(x, y)$. Verify that you have found the maximum profit.

5. A firm produces two goods. The cost of producing x units of good 1 and y units of good 2 is

$$C(x, y) = x^2 + xy + y^2 + x + y + 14$$

Suppose that the firm sells all its output of each good at prices per unit of p and q respectively. Find the values of x and y that maximize profits, under the assumptions that $\frac{1}{2}p + \frac{1}{2} < q < 2p - 1$ and $p > 1$.

6. The profit function of a firm is $\pi(x, y) = px + qy - \alpha x^2 - \beta y^2$, where p and q are the prices per unit and $\alpha x^2 + \beta y^2$ are the costs of producing and selling x units of the first good and y units of the other. The constants are all positive.

(a) Find the values of x and y that maximize profits. Denote them by x^* and y^*. Verify that the second-order conditions are satisfied.

(b) Define $\pi^*(p, q) = \pi(x^*, y^*)$. Verify that $\partial\pi^*(p, q)/\partial p = x^*$ and $\partial\pi^*(p, q)/\partial q = y^*$. Give these results economic interpretations.

7. Find the smallest value of $x^2 + y^2 + z^2$ when we require that $4x + 2y - z = 5$.[7]

8. Let A, a, and b be positive constants, and p, q, and r be arbitrary constants. Show that the function $f(x, y) = Ax^a y^b - px - qy - r$ is concave for $x > 0$, $y > 0$ provided that $a + b \leq 1$.

13.3 Local Extreme Points

Sometimes one needs to consider *local* extreme points of a function. The point (x_0, y_0) is said to be a *local maximum point* of f in S if $f(x, y) \leq f(x_0, y_0)$ for all pairs (x, y) in S that lie sufficiently close to (x_0, y_0). More precisely, the definition is that there exists a positive number r such that $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in S that lie inside the circle with centre (x_0, y_0) and radius r. If the inequality is strict for $(x, y) \neq (x_0, y_0)$, then (x_0, y_0) is a *strict* local maximum point.

A *(strict) local minimum* point is defined in the obvious way, and it should also be clear what we mean by *local maximum and minimum values*, *local extreme points*, and *local extreme values*. Note how these definitions imply that a global extreme point is also a local extreme point; the converse is not true, of course.

In searching for maximum and minimum points, the first-order conditions were very useful. The same result also applies to the local extreme points: *Any local extreme point in the interior of the domain of a differentiable function must be critical.* This observation follows because in the argument for Theorem 13.1.1 it was sufficient to consider the behaviour of the function in a small neighbourhood of the optimal point.

These first-order conditions are necessary for a differentiable function to have a local extreme point. However, a critical point does not have to be a local extreme point. A critical point (x_0, y_0) of f which, like point R in Fig. 13.1.2, is neither a local maximum nor a local minimum point, is called a *saddle point* of f. Hence: *A saddle point* (x_0, y_0) *is a critical point with the property that there exist points* (x, y) *arbitrarily close to* (x_0, y_0) *with* $f(x, y) < f(x_0, y_0)$*, and there also exist such points with* $f(x, y) > f(x_0, y_0)$.

EXAMPLE 13.3.1 Show that $(0, 0)$ is a saddle point of $f(x, y) = x^2 - y^2$.

Solution: It is easy to check that $(0, 0)$ is a critical point at which $f(0, 0) = 0$. Moreover, $f(x, 0) = x^2$ and $f(0, y) = -y^2$, so $f(x, y)$ takes positive and negative values arbitrarily close to the origin. Hence, $(0, 0)$ is a saddle point. See the graph in Fig. 13.3.1.

Local extreme points and saddle points can be illustrated by thinking of the mountains in the Himalayas. Every summit is a local maximum, but only the highest (Mount Everest) is the (global) maximum. The deepest points of the lakes or glaciers are local minima. In every mountain pass there will be a saddle point that is the highest point in one compass

[7] Geometrically, the problem is to find the point in the plane $4x + 2y - z = 5$ which is closest to the origin.

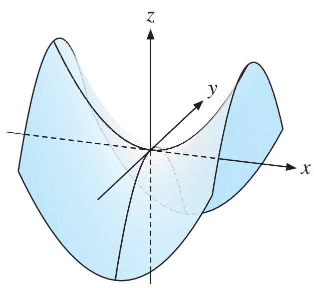

Figure 13.3.1 $z = x^2 - y^2$, with saddle point at $(0, 0)$

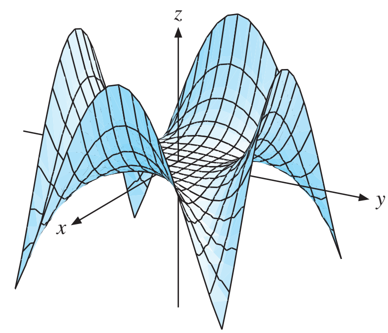

Figure 13.3.2 $z = x^4 - 3x^2y^2 + y^4$, with saddle point at $(0, 0)$

direction and the lowest in another. That said, the surface in Fig. 13.3.2 shows that not all saddle points have graphs that look as neat as the one shown in Fig. 13.3.1.

The critical points of a function thus fall into three categories: local maximum points, local minimum points, and saddle points. How do we distinguish between these three cases?

Consider first the case when $z = f(x, y)$ has a local maximum at (x_0, y_0). The functions $g(x) = f(x, y_0)$ and $h(y) = f(x_0, y)$ describe the behaviour of f along the straight lines $y = y_0$ and $x = x_0$, respectively, as in Fig. 13.1.1. These functions must achieve local maxima at x_0 and y_0, respectively, so $g''(x_0) = f''_{11}(x_0, y_0) \le 0$ and $h''(y_0) = f''_{22}(x_0, y_0) \le 0$.

On the other hand, if $g''(x_0) < 0$ and $h''(y_0) < 0$, then we know that g and h really do achieve local maxima at x_0 and y_0, respectively. Stated differently, the conditions $f''_{11}(x_0, y_0) < 0$ and $f''_{22}(x_0, y_0) < 0$ will ensure that $f(x, y)$ has a local maximum in the directions through (x_0, y_0) that are parallel to the x-axis and the y-axis. Note, however, that the signs of $f''_{11}(x_0, y_0)$ and $f''_{22}(x_0, y_0)$ on their own do not reveal much about the behaviour of the graph of $z = f(x, y)$ when we move away from (x_0, y_0) in directions other than the two mentioned. Example 13.3.1 illustrated the problem.

It turns out that in order to have a correct second-derivative test for functions f of two variables, the mixed second-order partial $f''_{12}(x_0, y_0)$ must also be considered, just as it had to be in Section 13.2. The following theorem can be used to determine the nature of the critical points in most cases. (A proof is given at the end of this section.)

THEOREM 13.3.1 (SECOND-DERIVATIVE TEST FOR LOCAL EXTREMA)

Suppose that $f(x, y)$ is a C^2 function in a domain S, and let (x_0, y_0) be an interior critical point of S. Write

$$A = f''_{11}(x_0, y_0),\ B = f''_{12}(x_0, y_0),\ \text{and}\ C = f''_{22}(x_0, y_0)$$

Now:

(a) if $A < 0$ and $AC - B^2 > 0$, then (x_0, y_0) is a strict local maximum point.

(b) if $A > 0$ and $AC - B^2 > 0$, then (x_0, y_0) is a strict local minimum point.

(c) if $AC - B^2 < 0$, then (x_0, y_0) is a saddle point.

(d) if $AC - B^2 = 0$, then (x_0, y_0) could be a local maximum, a local minimum, or a saddle point.

Note that $AC - B^2 > 0$ in (a) implies that $AC > B^2 \geq 0$, and so $AC > 0$. Thus, if $A < 0$, then also $C < 0$. The condition $C = f''_{22}(x_0, y_0) < 0$ is, thus, indirectly included in the assumptions in (a). The corresponding observation for (b) is also valid.

The conditions in (a), (b), and (c) are usually called local *second-order conditions*. Note that these are sufficient conditions for a critical point to be, respectively, a *strict local* maximum point, a *strict local* minimum point, or a saddle point. None of these conditions is necessary. The results in Exercise 5 will confirm (d), because it shows that a critical point where $AC - B^2 = 0$ can fall into any of the three categories. The second-derivative test is inconclusive in this case.

EXAMPLE 13.3.2 Find the critical points and classify them when $f(x, y) = x^3 - x^2 - y^2 + 8$.

Solution: The critical points must satisfy the two equations

$$f'_1(x, y) = 3x^2 - 2x = 0 \text{ and } f'_2(x, y) = -2y = 0$$

Because $3x^2 - 2x = x(3x - 2)$, we see that the first equation has the solutions $x = 0$ and $x = 2/3$. The second equation has the solution $y = 0$. We conclude that $(0, 0)$ and $(2/3, 0)$ are the only critical points.

Furthermore, $f''_{11}(x, y) = 6x - 2$, $f''_{12}(x, y) = 0$, and $f''_{22}(x, y) = -2$. A convenient way of classifying the critical points is to make a table like the following:

(x, y)	A	B	C	$AC - B^2$	Type of point
$(0, 0)$	-2	0	-2	4	Local maximum point
$(2/3, 0)$	2	0	-2	-4	Saddle point

with A, B, and C defined in Theorem 13.3.1.

EXAMPLE 13.3.3 Consider Example 13.1.3 and suppose that the production function F is twice differentiable. Define

$$\Delta(K, L) = F''_{KK}(K, L)F''_{LL}(K, L) - \left[F''_{KL}(K, L)\right]^2$$

and let (K^*, L^*) be an input pair satisfying the first-order conditions $(*)$ in the example.

(a) Prove that if

$$F''_{KK}(K, L) \leq 0,\ F''_{LL}(K, L) \leq 0 \text{ and } \Delta(K, L) \geq 0 \text{ for all } K \geq 0 \text{ and } L \geq 0 \qquad (*)$$

so that the product function F is concave, then (K^*, L^*) maximizes profit.

(b) Prove also that if

$$F''_{KK}(K^*, L^*) < 0 \text{ and } \Delta(K^*, L^*) > 0 \tag{13.3.1}$$

then (K^*, L^*) is a strict local maximum for the profit function.

Solution:

(a) The second-order partials of the profit function are:

$$\pi''_{KK}(K, L) = pF''_{KK}(K, L); \quad \pi''_{KL}(K, L) = pF''_{KL}(K, L); \quad \pi''_{LL}(K, L) = pF''_{LL}(K, L)$$

Since $p > 0$, the conclusion follows from part (a) in Theorem 13.2.1.

(b) In this case the conclusion follows from part (a) in Theorem 13.3.1.

Proof of the Second-Derivative Test

We now want to prove sufficiency Theorem 13.3.1, and will do this based on our understanding of the one-dimensional case studied in Theorem 8.6.2. Before doing that, it is instructive to develop some intuition by determining some *necessary conditions* for local optimization.

Let $z = f(x, y)$ be the function graphed in Fig. 13.3.3, with (x_0, y_0) as a local maximum point. For fixed values of h and k, define the function g of one variable by

$$g(t) = f(x_0 + th, y_0 + tk)$$

This function tells us what happens to f as one moves away from (x_0, y_0) in the direction (h, k) when $t > 0$, or in the reverse direction $(-h, -k)$ when $t < 0$.

If f has a local maximum at (x_0, y_0), then $g(t)$ must certainly have a local maximum at $t = 0$. From Theorem 8.1.1 and formula (8.6.3), necessary conditions for this are that $g'(0) = 0$ and $g''(0) \leq 0$. The first- and second-order derivatives of $g(t)$ can be calculated as in Example 12.1.5. At $t = 0$, the second derivative of g is

$$g''(0) = f''_{11}(x_0, y_0)h^2 + 2f''_{12}(x_0, y_0)hk + f''_{22}(x_0, y_0)k^2 \tag{13.3.2}$$

So if f has a local maximum at (x_0, y_0), the expression in (13.3.2) must be nonpositive for all choices of (h, k).

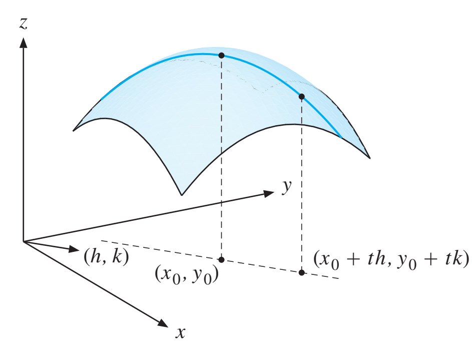

Figure 13.3.3 The second derivative test

In this way we have obtained a *necessary* condition for f to have a local maximum at (x_0, y_0). We now proceed to find *sufficient* conditions for a local maximum. For the one-variable case, we know from part (i) in Theorem 8.6.2 that the conditions $g'(0) = 0$ and $g''(0) < 0$ are sufficient for g to have a local maximum at $t = 0$. It is therefore reasonable to conjecture that we have the following result:

If $f_1'(x_0, y_0) = f_2'(x_0, y_0) = 0$ and the expression in (13.3.2) for the second derivative $g''(0)$ is negative for all directions $(h, k) \neq (0, 0)$, then (x_0, y_0) is a (strict) local maximum point for f.

This turns out to be correct, as will be proved in FMEA. Exercise 7, however, shows that the expression in (13.3.2) really must be negative for *all* directions (h, k), without exception. Relying on this result, we can prove part (a) of Theorem 13.3.1:

It suffices to verify that $A < 0$ and $AC - B^2 > 0$ imply that

$$Ah^2 + 2Bhk + Ck^2 < 0 \text{ for all } (h, k) \neq (0, 0) \tag{13.3.3}$$

To this end we complete the square:

$$Ah^2 + 2Bhk + Ck^2 = A\left[\left(h + \frac{B}{A}k\right)^2 + \frac{AC - B^2}{A^2}k^2\right] \tag{13.3.4}$$

The expression in square brackets is obviously non-negative, and equals 0 only if both $h + Bk/A = 0$ and $k = 0$, implying that $h = k = 0$. Because $A < 0$, the right-hand side of Eq. (13.3.4) is negative for all $(h, k) \neq (0, 0)$, so we have proved (13.3.3).

EXERCISES FOR SECTION 13.3

1. Consider the function f defined for all (x, y) by $f(x, y) = 5 - x^2 + 6x - 2y^2 + 8y$.

   (a) Find its partial derivatives of first and second order.

   (b) Find the only critical point and classify it by using the second-derivative test. What does Theorem 13.2.1 tell us?

2. Consider the function f defined for all (x, y) by $f(x, y) = x^2 + 2xy^2 + 2y^2$.

   (a) Find its partial derivatives of first and second order.

   (b) Show that its critical points are $(0, 0)$, $(-1, 1)$ and $(-1, -1)$, and classify them.

SM 3. Let f be a function of two variables, given by

$$f(x, y) = (x^2 - axy)e^y$$

where $a \neq 0$ is a constant.

   (a) Find the critical points of f and decide for each of them if it is a local maximum point, a local minimum point, or a saddle point.

   (b) Let (x^*, y^*) be the critical point where $x^* \neq 0$, and let $f^*(a) = f(x^*, y^*)$. Find $\mathrm{d}f^*(a)/\mathrm{d}a$. Show that if we let $\hat{f}(x, y, a) = (x^2 - axy)e^y$, then

$$\hat{f}_3'(x^*, y^*, a) = \frac{\mathrm{d}f^*(a)}{\mathrm{d}a}$$

SM **4.** Suppose in Example 10.3.2 that the market value of the tree at time t is a function $f(t,x)$ of the amount x spent on trimming the tree at time 0, as well as of t. Assuming continuous compounding at the interest rate r, the present discounted value of the profit earned on the tree is then $V(t,x) = f(t,x)e^{-rt} - x$.

(a) What are the first-order conditions for $V(t,x)$ to have a maximum at $t^* > 0$, $x^* > 0$?

(b) What are the first-order conditions if $f(t,x)$ takes the separable form $f(t,x) = g(t)h(x)$, with $g(t) > 0$ and $h(x) > 0$? (Note that in this case t^* does not depend on the function h.)

(c) In the separable case, prove that $g''(t^*) < r^2 g(t^*)$ and $h''(x^*) < 0$ are sufficient conditions for a critical point (t^*, x^*) to be a local maximum point for V.

(d) Find t^* and x^* when $g(t) = e^{\sqrt{t}}$ and $h(x) = \ln(x+1)$, and check the local second-order conditions.

5. Consider the three functions: (i) $z = -x^4 - y^4$; (ii) $z = x^4 + y^4$; (iii) $z = x^3 + y^3$.

(a) Prove that the origin is a critical point for each one of these functions, and that $AC - B^2 = 0$ at the origin in each case.

(b) By studying the functions directly, prove that the origin is respectively a maximum point for (i), a minimum point for (ii), and a saddle point for (iii).

SM **6.** [HARDER] Consider the function $f(x,y) = \ln(1 + x^2 y)$.

(a) Find its domain.

(b) Prove that the critical points are all the points on the y-axis.

(c) Show that the second-derivative test fails.

(d) Classify the critical points by looking directly at the sign of the value of $f(x,y)$.

7. [HARDER] The graph of $f(x,y) = (y - x^2)(y - 2x^2)$ intersects the xy-plane $z = 0$ in two parabolas.

(a) In the xy-plane, draw the domains where f is negative, and where f is positive. Show that $(0,0)$ is the only critical point, and that it is a saddle point.

(b) Suppose $(h,k) \neq (0,0)$ is any direction vector. Let $g(t) = f(th, tk)$ and show that g has a local minimum at $t = 0$, whatever the direction (h,k) may be.[8]

13.4 Linear Models with Quadratic Objectives

In this section we consider some other interesting economic applications of optimization theory when there are two variables. Versions of the first example have already appeared in Example 13.1.5 and Exercise 13.2.4.

EXAMPLE 13.4.1 **(Discriminating Monopolist)** Consider a firm that sells a product in two isolated geographical areas. If it wants to, it can then charge different prices in the two different

[8] Thus, although $(0,0)$ is a saddle point, the function has a local minimum at the origin in each direction.

areas because what is sold in one area cannot easily be resold in the other.[9] Suppose that such a firm also has some monopoly power to influence the different prices it faces in the two separate markets by adjusting the quantity it sells in each. Economists generally use the term "discriminating monopolist" to describe a firm having this power.

Faced with two such isolated markets, the discriminating monopolist has two independent demand curves. Suppose that, in inverse form, these are

$$P_1 = a_1 - b_1 Q_1 \text{ and } P_2 = a_2 - b_2 Q_2 \tag{$*$}$$

for market areas 1 and 2, respectively. Suppose, too, that the total cost is proportional to total production: $C(Q) = \alpha Q$, for some positive constant α.[10] As a function of Q_1 and Q_2, total profits are

$$\begin{aligned}\pi(Q_1, Q_2) &= P_1 Q_1 + P_2 Q_2 - C(Q_1 + Q_2) \\ &= (a_1 - b_1 Q_1)Q_1 + (a_2 - b_2 Q_2)Q_2 - \alpha(Q_1 + Q_2) \\ &= (a_1 - \alpha)Q_1 + (a_2 - \alpha)Q_2 - b_1 Q_1^2 - b_2 Q_2^2\end{aligned}$$

We want to find the values of $Q_1 \geq 0$ and $Q_2 \geq 0$ that maximize profits. The first-order conditions are

$$\pi'_1(Q_1, Q_2) = (a_1 - \alpha) - 2b_1 Q_1 = 0 \text{ and } \pi'_2(Q_1, Q_2) = (a_2 - \alpha) - 2b_2 Q_2 = 0$$

with the solutions $Q_1^* = (a_1 - \alpha)/2b_1$ and $Q_2^* = (a_2 - \alpha)/2b_2$. Furthermore, one has $\pi''_{11}(Q_1, Q_2) = -2b_1$, $\pi''_{12}(Q_1, Q_2) = 0$, and $\pi''_{22}(Q_1, Q_2) = -2b_2$. Hence, for all (Q_1, Q_2), it follows that

$$\pi''_{11} \leq 0,\ \pi''_{22} \leq 0, \text{ and } \pi''_{11}\pi''_{22} - (\pi''_{12})^2 = 4b_1 b_2 \geq 0$$

We conclude from Theorem 13.2.1 that if Q_1^* and Q_2^* are both positive, implying that (Q_1^*, Q_2^*) is an interior point in the domain of π, then the pair (Q_1^*, Q_2^*) really does maximize profits.

The corresponding prices can be found by inserting these values in $(*)$ to get

$$P_1^* = a_1 - b_1 Q_1^* = \tfrac{1}{2}(a_1 + \alpha) \text{ and } P_2^* = a_2 - b_2 Q_2^* = \tfrac{1}{2}(a_2 + \alpha)$$

The maximum profit is

$$\pi^* = \frac{(a_1 - \alpha)^2}{4b_1} + \frac{(a_2 - \alpha)^2}{4b_2}$$

Both sales quantities Q_1^* and Q_2^* are positive provided $a_1 > \alpha$ and $a_2 > \alpha$. In this case, P_1^* and P_2^* are both greater than α. This implies that there is no "dumping", with the price in

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[9] As an example, it seems that express mail or courier services find it possible to charge much higher prices in Europe than they can in North America. Another example is that pharmaceutical firms often charge much more for the same medication in the USA than they do in Europe or Canada.

[10] It is true that this cost function neglects transport costs. But the point to be made is that, even though supplies to the two areas are perfect substitutes in production, the monopolist will generally be able to earn higher profits by charging different prices, if this is allowed.

one market less than the cost α. Nor is there any "cross-subsidy", with the losses due to dumping in one market being subsidized out of profits in the other market. It is notable that the optimal prices are independent of b_1 and b_2. More important, note that the prices are *not* the same in the two markets, except in the special case when $a_1 = a_2$. Indeed, $P_1^* > P_2^*$ if, and only if, $a_1 > a_2$. This says that the price is higher in the market where consumers are willing to pay a higher price for each unit when the quantity is close to zero.

EXAMPLE 13.4.2 Suppose that the monopolist in Example 13.4.1 has the demand functions $P_1 = 100 - Q_1$ and $P_2 = 80 - Q_2$, and that the cost function is $C(Q) = 6Q$.

(a) How much should be sold in the two markets to maximize profits? What are the corresponding prices?

(b) How much profit is lost if it becomes illegal to discriminate?

(c) The authorities impose a tax of τ per unit sold in the first market. Discuss the consequences.

Solution:

(a) Here $a_1 = 100$, $a_2 = 80$, $b_1 = b_2 = 1$, and $\alpha = 6$. Example 13.4.1 gives the answers

$$Q_1^* = (100 - 6)/2 = 47,\ Q_2^* = 37,\ P_1^* = \tfrac{1}{2}(100 + 6) = 53, \text{ and } P_2^* = 43$$

The corresponding profit is $P_1^*Q_1^* + P_2^*Q_2^* - 6(Q_1^* + Q_2^*) = 3578$.

(b) If price discrimination is not permitted, then $P_1 = P_2 = P$, and $Q_1 = 100 - P$, $Q_2 = 80 - P$, with total demand $Q = Q_1 + Q_2 = 180 - 2P$. Then $P = 90 - \frac{1}{2}Q$, so profits are

$$\pi = \left(90 - \tfrac{1}{2}Q\right)Q - 6Q = 84Q - \tfrac{1}{2}Q^2$$

This has a maximum at $Q = 84$ when $P = 48$. The corresponding profit is now $\pi = 3528$, so the loss in profit is $3578 - 3528 = 50$.

(c) With the introduction of the tax, the new profit function is

$$\hat{\pi} = (100 - Q_1)Q_1 + (80 - Q_2)Q_2 - 6(Q_1 + Q_2) - \tau Q_1$$

We easily see that this has a maximum at $\hat{Q}_1 = 47 - \frac{1}{2}\tau$, $\hat{Q}_2 = 37$, with corresponding prices $\hat{P}_1 = 53 + \frac{1}{2}\tau$, $\hat{P}_2 = 43$. The tax therefore has no influence on the sales in market 2, while the amount sold in market 1 is lowered and the price in market 1 goes up. The optimal profit π^* is easily worked out: it equals

$$(53 + \tfrac{1}{2}\tau)(47 - \tfrac{1}{2}\tau) + 43 \cdot 37 - 6(84 - \tfrac{1}{2}\tau) - \tau(47 - \tfrac{1}{2}\tau) = 3578 - 47\tau + \tfrac{1}{4}\tau^2$$

So, compared to (a), introducing the tax makes the profit fall by $47\tau - \frac{1}{4}\tau^2$. The authorities in market 1 obtain a tax revenue which is

$$T = \tau\hat{Q}_1 = \tau(47 - \tfrac{1}{2}\tau) = 47\tau - \tfrac{1}{2}\tau^2$$

Thus we see that profits fall by $\frac{1}{4}\tau^2$ more than the tax revenue. This amount $\frac{1}{4}\tau^2$ represents the so-called deadweight loss from the tax.

A monopolistic firm faces a downward-sloping demand curve. A *discriminating monopolist* such as in Example 13.4.1 faces separate downward-sloping demand curves in two or more isolated markets. A *monopsonistic firm*, on the other hand, faces an upward-sloping supply curve for one or more of its factors of production. Then, by definition, a *discriminating monopsonist* faces two or more upward-sloping supply curves for different kinds of the same input—for example, workers of different race or gender. Of course, discrimination by race or gender is illegal in many countries. The following example, however, suggests one possible reason why firms might want to discriminate if they were allowed to.

EXAMPLE 13.4.3 **(Discriminating Monopsonist)** Consider a firm using quantities L_1 and L_2 of two kinds of labour as its only inputs in order to produce output Q according to the simple production function $Q = L_1 + L_2$. Thus, both output and labour supply are measured so that each unit of labour produces one unit of output. Note especially how the two kinds of labour are essentially indistinguishable, because each unit of each type makes an equal contribution to the firm's output. Suppose, however, that there are two segmented labour markets, with different inverse supply functions specifying the wage that must be paid to attract a given labour supply. Specifically, suppose that

$$w_1 = \alpha_1 + \beta_1 L_1; \quad w_2 = \alpha_2 + \beta_2 L_2$$

Assume moreover that the firm is competitive in its output market, taking price P as fixed. Then the firm's profits are

$$\begin{aligned}\pi(L_1, L_2) = PQ - w_1L_1 - w_2L_2 &= P(L_1 + L_2) - (\alpha_1 + \beta_1 L_1)L_1 - (\alpha_2 + \beta_2 L_2)L_2 \\ &= (P - \alpha_1)L_1 - \beta_1 L_1^2 + (P - \alpha_2)L_2 - \beta_2 L_2^2\end{aligned}$$

The firm wants to maximize profits. The first-order conditions are

$$\pi_1'(L_1, L_2) = (P - \alpha_1) - 2\beta_1 L_1 = 0 \quad \text{and} \quad \pi_2'(L_1, L_2) = (P - \alpha_2) - 2\beta_2 L_2 = 0$$

These have the solutions

$$L_1^* = \frac{P - \alpha_1}{2\beta_1} \quad \text{and} \quad L_2^* = \frac{P - \alpha_2}{2\beta_2}$$

It is easy to see that the conditions for maximum in Theorem 13.2.1 are satisfied, so that L_1^*, L_2^* really do maximize profits if $P > \alpha_1$ and $P > \alpha_2$. The maximum profit is

$$\pi^* = \frac{(P - \alpha_1)^2}{4\beta_1} + \frac{(P - \alpha_2)^2}{4\beta_2}$$

The corresponding wages are

$$w_1^* = \alpha_1 + \beta_1 L_1^* = \tfrac{1}{2}(P + \alpha_1) \quad \text{and} \quad w_2^* = \alpha_2 + \beta_2 L_2^* = \tfrac{1}{2}(P + \alpha_2)$$

Hence, $w_1^* = w_2^*$ only if $\alpha_1 = \alpha_2$. Generally, the wage is higher for the type of labour that demands a higher wage for very low levels of labour supply. Perhaps this is the type of labour with better job prospects elsewhere.

EXAMPLE 13.4.4 **(Econometrics: Linear Regression)** Empirical economics is concerned with analysing data in order to try to discern some pattern that helps in understanding the past, and possibly in predicting the future. For example, price and quantity data for a particular commodity such as natural gas may be used in order to try to estimate a demand function. This might then be used to predict how demand will respond to future price changes. The most commonly used technique for estimating such a function is *linear regression*.

Suppose it is thought that variable y depends upon variable x. Suppose that we have observations (x_t, y_t) of both variables at times $t = 1, 2, \ldots, T$. Then the technique of linear regression seeks to fit a linear function

$$y = \alpha + \beta x$$

to the data, as indicated in Fig. 13.4.1.

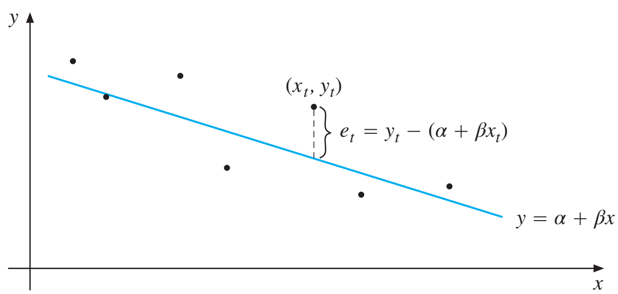

Figure 13.4.1 Linear regression

Of course, an exact fit is possible only if there exist numbers α and β for which $y_t = \alpha + \beta x_t$ for $t = 1, 2, \ldots, T$. This is rarely possible. Generally, however α and β may be chosen, one has instead

$$y_t = \alpha + \beta x_t + e_t,\ t = 1, 2, \ldots, T$$

where e_t is an *error* or *disturbance* term. Obviously, one hopes that the errors will be small, on average. So the parameters α and β are chosen to make the errors as "small as possible", somehow. One idea would be to make the sum $\sum_{t=1}^{T}(y_t - \alpha - \beta x_t)$ equal to zero. However, in this case, large positive discrepancies would cancel large negative discrepancies. Indeed, the sum of errors could be zero even though the line is very far from giving a perfect or even a good fit. We must somehow prevent large positive errors from cancelling large negative errors. Usually, this is done by minimizing the quadratic "loss" function

$$L(\alpha, \beta) = \frac{1}{T}\sum_{t=1}^{T} e_t^2 = \frac{1}{T}\sum_{t=1}^{T}\left(y_t - \alpha - \beta x_t\right)^2 \tag{$*$}$$

which equals the mean (or average) square error. Expanding the square gives

$$L(\alpha, \beta) = \frac{1}{T}\sum\nolimits_{t=1}^{T}\left(y_t^2 + \alpha^2 + \beta^2 x_t^2 - 2\alpha y_t - 2\beta x_t y_t + 2\alpha\beta x_t\right)$$

This is a quadratic function of α and β. We shall show how to derive the *ordinary least-squares* estimates of α and β. To do so it helps to introduce some standard notation. Write

$$\mu_x = \frac{x_1 + \cdots + x_T}{T} = \frac{1}{T}\sum\nolimits_{t=1}^{T} x_t, \qquad \mu_y = \frac{y_1 + \cdots + y_T}{T} = \frac{1}{T}\sum\nolimits_{t=1}^{T} y_t$$

for the *statistical means* of x_t and y_t, and

$$\sigma_{xx} = \frac{1}{T}\sum\nolimits_{t=1}^{T}(x_t - \mu_x)^2, \ \sigma_{yy} = \frac{1}{T}\sum\nolimits_{t=1}^{T}(y_t - \mu_y)^2, \ \sigma_{xy} = \frac{1}{T}\sum\nolimits_{t=1}^{T}(x_t - \mu_x)(y_t - \mu_y)$$

for their *statistical variances*, as well as their *covariance*, respectively. In what follows, we shall assume that the x_t *are not all equal*. Then, in particular, $\sigma_{xx} > 0$.

Using the result in Example 2.9.2, we have

$$\sigma_{xx} = \frac{1}{T}\sum\nolimits_{t=1}^{T} x_t^2 - \mu_x^2 \quad \text{and} \quad \sigma_{yy} = \frac{1}{T}\sum\nolimits_{t=1}^{T} y_t^2 - \mu_y^2$$

while you can verify that, similarly,

$$\sigma_{xy} = \frac{1}{T}\sum\nolimits_{t=1}^{T} x_t y_t - \mu_x \mu_y$$

Then, the expression for $L(\alpha, \beta)$ becomes

$$\begin{aligned} L(\alpha, \beta) &= \left(\sigma_{yy} + \mu_y^2\right) + \alpha^2 + \beta^2\left(\sigma_{xx} + \mu_x^2\right) - 2\alpha\mu_y - 2\beta\left(\sigma_{xy} + \mu_x\mu_y\right) + 2\alpha\beta\mu_x \\ &= \alpha^2 + \mu_y^2 + \beta^2\mu_x^2 - 2\alpha\mu_y - 2\beta\mu_x\mu_y + 2\alpha\beta\mu_x + \beta^2\sigma_{xx} - 2\beta\sigma_{xy} + \sigma_{yy} \end{aligned}$$

The first-order conditions for a minimum of $L(\alpha, \beta)$ take the form

$$\begin{aligned} L_1'(\alpha, \beta) &= 2\alpha - 2\mu_y + 2\beta\mu_x = 0 \\ L_2'(\alpha, \beta) &= 2\beta\mu_x^2 - 2\mu_x\mu_y + 2\alpha\mu_x + 2\beta\sigma_{xx} - 2\sigma_{xy} = 0 \end{aligned}$$

Note that $L_2'(\alpha, \beta) = \mu_x L_1'(\alpha, \beta) + 2\beta\sigma_{xx} - 2\sigma_{xy}$. So the unique critical point of $L(\alpha, \beta)$ is $(\hat{\alpha}, \hat{\beta})$ where

$$\hat{\beta} = \frac{\sigma_{xy}}{\sigma_{xx}} \quad \text{and} \quad \hat{\alpha} = \mu_y - \hat{\beta}\mu_x = \mu_y - \left(\frac{\sigma_{xy}}{\sigma_{xx}}\right)\mu_x \tag{$**$}$$

Furthermore, $L_{11}'' = 2$, $L_{12}'' = 2\mu_x$, $L_{22}'' = 2\mu_x^2 + 2\sigma_{xx}$. Thus $L_{11}'' \geq 0$, $L_{22}'' \geq 0$, and

$$L_{11}''L_{22}'' - (L_{12}'')^2 = 2(2\mu_x^2 + 2\sigma_{xx}) - (2\mu_x)^2 = 4\sigma_{xx} = 4T^{-1}\sum\nolimits_{t=1}^{T}(x_t - \mu_x)^2 \geq 0$$

We conclude that the conditions in part (b) of Theorem 13.2.1 are satisfied, and therefore the pair $(\hat{\alpha}, \hat{\beta})$ given by $(**)$ minimizes $L(\alpha, \beta)$. The problem is then completely solved: *The straight line that best fits the observations* (x_1, y_1), (x_2, y_2), …, (x_T, y_T), *in the sense of minimizing the mean square error in* $(*)$, *is* $y = \hat{\alpha} + \hat{\beta}x$ *where* $\hat{\alpha}$ *and* $\hat{\beta}$ *are given by* $(**)$.

Note in particular that this estimated straight line passes through the mean (μ_x, μ_y) of the observed pairs (x_t, y_t), $t = 1, \ldots, T$. Also, with a little bit of tedious algebra we obtain

$$L(\alpha, \beta) = \left(\alpha + \beta\mu_x - \mu_y\right)^2 + \sigma_{xx}\left(\beta - \sigma_{xy}/\sigma_{xx}\right)^2 + \left(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2\right)/\sigma_{xx}$$

The first two terms on the right are always nonnegative, and with $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$, they are zero, confirming that $\hat{\alpha}$ and $\hat{\beta}$ do give the minimum value of $L(\alpha, \beta)$.

EXERCISES FOR SECTION 13.4

1. Suppose that the monopolist in Example 13.4.1 faces the two inverse demand functions $P_1 = 200 - 2Q_1$ and $P_2 = 180 - 4Q_2$, and that the cost function is $C = 20(Q_1 + Q_2)$.

   (a) How much should be sold in the two markets to maximize total profit? What are the corresponding prices?

   (b) How much profit is lost if it becomes illegal to discriminate?

   (c) Discuss the consequences of imposing a tax of $\tau = 5$ per unit on the product sold in market 1.

SM 2. A firm produces and sells a product in two separate markets. When the price in market A is p per ton, and the price in market B is q per ton, the demand in tons per week in the two markets are, respectively,

$$Q_A = a - bp, \ Q_B = c - dq$$

   The cost function is $C(Q_A, Q_B) = \alpha + \beta(Q_A + Q_B)$, and all constants are positive.

   (a) Find the firm's profit π as a function of the prices p and q, and then find the pair (p^*, q^*) that maximizes profit.

   (b) Suppose it becomes unlawful to discriminate by price, so that the firm must charge the same price in the two markets. What price $\hat{p}$ will now maximize profit?

   (c) In the case $\beta = 0$, find the firm's loss of profit if it has to charge the same price in both markets. Comment.

3. In Example 13.4.1, discuss the effects of a tax imposed in market 1 of τ per unit of Q_1.

SM 4. The following table shows the Norwegian gross national product (GNP) and spending on foreign aid (FA) for the period 1970–1973, in millions of crowns:

Year	1970	1971	1972	1973
GNP	79 835	89 112	97 339	110 156
FA	274	307	436	524

   Growth of both GNP and FA was almost exponential during the period. So, approximately, one has $\text{GNP} = Ae^{a(t-t_0)}$, with $t_0 = 1970$. Define $x = t - t_0$ and $b = \ln A$. Then $\ln(\text{GNP}) = ax + b$. On the basis of the table above, one gets the following

Year	1970	1971	1972	1973
$y = \ln(\text{GNP})$	11.29	11.40	11.49	11.61

   (a) Using the method of least squares, determine the straight line $y = ax + b$ which best fits the data in the last table.

   (b) Repeat the method above to estimate c and d, where $\ln(\text{FA}) = cx + d$.

(c) The Norwegian government had a stated goal of eventually giving 1% of its GNP as foreign aid. If the time trends of the two variables had continued as they did during the years 1970–1973, when would this goal have been reached?

SM **5.** *(Duopoly)* Each of two firms A and B produces its own brand of a commodity such as mineral water in amounts denoted by x and y, which are sold at prices p and q per unit, respectively. Each firm determines its own price and produces exactly as much as is demanded. The demands for the two brands are given by

$$x = 29 - 5p + 4q \quad \text{and} \quad y = 16 + 4p - 6q$$

Firm A has total costs $5 + x$, whereas firm B has total costs $3 + 2y$. (Assume that the functions to be maximized have maxima, and at positive prices.)

(a) Initially, the two firms collude in order to maximize their combined profit, as one monopolist would. Find the prices (p, q), the production levels (x, y), and the profits of firms A and B.

(b) Then an anti-trust authority prohibits collusion, so each producer maximizes its own profit, taking the other's price as given. If q is fixed, how will A choose p as a function $p = p_A(q)$ of q? If p is fixed, how will B choose q as a function $q = q_B(p)$ of p?

(c) Under the assumptions in part (b), what constant equilibrium prices are possible? What are the production levels and profits in this case?

(d) Draw a diagram with p along the horizontal axis and q along the vertical axis, and draw the "reaction" curves $p_A(q)$ and $q_B(p)$. Show on the diagram how the two firms' prices change over time if A breaks the cooperation first by maximizing its profit, taking B's initial price as fixed, then B answers by maximizing its profit with A's price fixed, then A responds, and so on.

13.5 The Extreme Value Theorem

As with functions of one variable, it is easy to find examples of functions of several variables that do not have any maximum or minimum points. The extreme value theorem, Theorem 8.4.1, however, was very useful for providing sufficient conditions to ensure that extreme points do exist for functions of one variable. It can be directly generalized to functions of several variables. In order to formulate the theorem, however, we need a few new concepts.

For many of the results concerning functions of one variable, it was important to distinguish between different kinds of domain for the functions. For functions of several variables, the distinction between different kinds of domain is no less important. In the one-variable case, most functions were defined over intervals, and there are not many different kinds of interval. For functions of several variables, however, there are many different kinds of domain. Fortunately, the distinctions that are relevant to the extreme value theorem can be made using only the concepts of open, closed, and bounded sets.

A point (a, b) is called an *interior point* of a set S in the plane if there exists a circle centred at (a, b) such that all points strictly inside the circle lie in S. (See Fig. 13.5.1.) A set is called *open* if it consists only of interior points, as in the second set illustrated in Fig. 13.5.1, where we indicate boundary points that belong to the set by a solid curve, and those that do not by a dashed curve. The point (a, b) is called a *boundary point* of a set S if *every* circle centred at (a, b) contains points of S as well as points in its complement, as illustrated in the first figure.

A boundary point of S does not necessarily lie in S. If S contains all its boundary points, then S is called *closed*—this is the case for the third set in Fig. 13.5.1. Note that a set that contains some but not all of its boundary points, like the last of those illustrated in Fig. 13.5.1, is neither open nor closed. In fact, a set is closed if and only if its complement is open.[11]

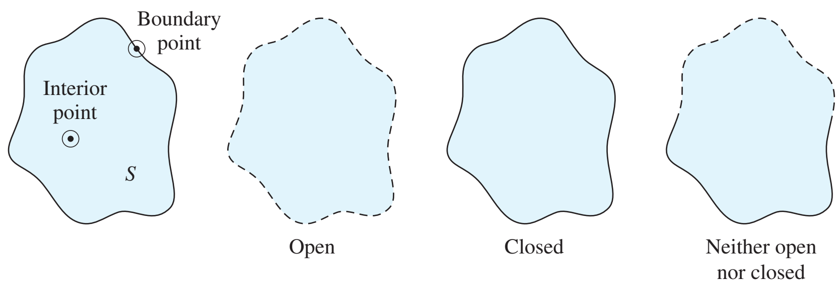

Figure 13.5.1 Open and closed sets

These illustrations give only very loose indications of what it means for a set to be either open or closed. Of course, if a set is not even precisely defined, it is impossible to decide conclusively whether it is open or closed.

In many of the optimization problems considered in economics, sets are defined by one or more inequalities, and boundary points occur where one or more of these inequalities are satisfied with equality. For instance, provided that p, q, and m are positive parameters, the "budget" set of points (x, y) that satisfy the inequalities

$$px + qy \le m, \quad x \ge 0, \quad y \ge 0 \tag{$*$}$$

is closed. This set is a triangle, as shown in Fig. 4.4.12. Its boundary consists of the three sides of the triangle. Each of the three sides corresponds to having one of the inequalities in $(*)$ be satisfied with equality. On the other hand, the set that results from replacing $\le$ by $<$ and $\ge$ by $>$ in $(*)$ is open.

In general, if $g(x, y)$ is a continuous function and c is a real number, then the sets

$$\{(x, y) : g(x, y) \ge c\}, \quad \{(x, y) : g(x, y) \le c\}, \quad \{(x, y) : g(x, y) = c\}$$

are all closed. If $\ge$ is replaced by $>$, or $\le$ is replaced by $<$, or $=$ by $\ne$, then the corresponding set becomes open.

A set in the plane is *bounded* if the whole set is contained within a sufficiently large circle. The sets in Fig. 13.5.1 and the budget triangle in Fig. 4.4.12 are all bounded. On the other hand, the set of all (x, y) satisfying $x \ge 1$ and $y \ge 0$, which appears in Fig. 11.1.1,

[11] In every day usage the words "open" and "closed" are antonyms: a shop is either open or closed. In topology, however, a set that contains some but not all its boundary points is neither open nor closed. To make matters even odder, in topology there always exist sets that are *both* open *and* closed. This is explained in FMEA.

is a closed but unbounded set. It is closed because it contains all its boundary points, but it is unbounded because no circle of finite radius can enclose it. This example shows that closed sets need not be bounded. The opposite implication does not hold true either: the set depicted in Fig. 11.1.2 is neither open nor closed, but it is bounded. Importantly, a set in the plane that is both closed and bounded is often called *compact*.

We are now ready to formulate the main result in this section.

THEOREM 13.5.1 (EXTREME VALUE THEOREM)

Suppose the function $f(x, y)$ is continuous throughout a nonempty, closed and bounded set S in the plane. Then there exist both a point (a, b) in S where f has a minimum and a point (c, d) in S where it has a maximum—that is,

$$f(a, b) \le f(x, y) \le f(c, d)$$

for all (x, y) in S

Theorem 13.5.1 is a pure existence theorem. It tells us nothing about *how to find* the extreme points. Its proof is found in most advanced calculus books and in FMEA. Also, even though the conditions of the theorem are *sufficient* to ensure the existence of extreme points, they are far from necessary, as discussed in Section 8.4.

Finding Maxima and Minima

Sections 13.1 and 13.2 presented some simple cases where we could find the maximum and minimum points of a function of two variables by finding its critical points. The procedure set out in the following frame covers many additional optimization problems.

FINDING MAXIMA AND MINIMA

In order to find the maximum and minimum values of a differentiable function $f(x, y)$ defined on a closed, bounded set S in the plane:

(i) Find all critical points of f in the interior of S.

(ii) Find the largest value and the smallest value of f on the boundary of S, along with the associated points. If it is convenient, subdivide the boundary into several pieces and find the largest and smallest value on each piece.

(iii) Compute the values of the function at all the points found in (i) and (ii). The largest function value is the maximum value of f in S; the smallest one is the minimum value of f in S.

We try out this procedure on the function whose graph is depicted in Fig. 13.5.2.[12] The function has a rectangular domain S of points (x, y) in the xy-plane. The only critical point

[12] Because the function is not specified analytically, we can only give a rough geometric argument.

of f is (x_0, y_0), which corresponds to the point P of the graph. The boundary of S consists of four straight-line segments. The point R vertically above one corner point of S represents the maximum value of f along the boundary; similarly, Q represents the minimum value of f along the boundary. The only candidates for a maximum/minimum are, therefore, the three points P, Q, and R. By comparing the values of f at these points, we see that P represents the minimum value, whereas R represents the maximum value of f in S.

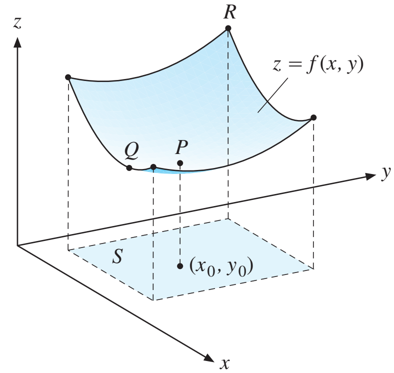

Figure 13.5.2 Finding maxima and minima

As an aspiring economist, doubtless you will be glad to hear that most optimization problems in economics, especially those appearing in textbooks, rarely create enough difficulties to call for the full recipe. Usually, there is an interior optimum that can be found by equating all the first-order partial derivatives to zero. Conditions that are sufficient for this easier approach to work were already discussed in Section 13.2. Nevertheless, we consider an example of a harder problem which illustrates how the whole recipe is sometimes needed. This recipe is also needed in several of the problems for this section. In particular, Exercise 3 gives an economic application.

EXAMPLE 13.5.1 Find the extreme values for $f(x, y)$ defined over S, when

$$f(x, y) = x^2 + y^2 + y - 1 \quad \text{and} \quad S = \{(x, y) : x^2 + y^2 \leq 1\}$$

Solution: Set S consists of all the points on or inside the circle of radius 1 centred at the origin, as shown in Fig. 13.5.3. The continuous function f will attain both a maximum and a minimum over S, by the extreme value theorem.

According to the preceding recipe, we start by finding all the critical points in the interior of S. These critical points satisfy the two equations

$$f_1'(x, y) = 2x = 0 \quad \text{and} \quad f_2'(x, y) = 2y + 1 = 0$$

It follows that $(x, y) = (0, -1/2)$ is the only critical point, and it is in the interior of S, with $f(0, -1/2) = -5/4$.

The boundary of S consists of the circle $x^2 + y^2 = 1$. Note that if (x, y) lies on this circle, then in particular both x and y lie in the interval $[-1, 1]$. Inserting $x^2 + y^2 = 1$ into the

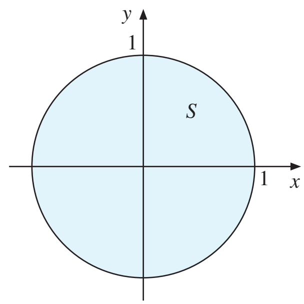

Figure 13.5.3 The domain in Example 13.5.1

expression for $f(x, y)$ shows that, *along the boundary of* S, the value of f is determined by the following function of one variable:

$$g(y) = 1 + y - 1 = y, \text{ defined for } y \in [-1, 1]$$

The maximum value of g is 1 for $y = 1$, and then $x = 0$. The minimum value is -1 when $y = -1$, and then again $x = 0$.

We have now found the only three possible candidates for extreme points, namely, $(0, -1/2)$, $(0, 1)$, and $(0, -1)$. But $f(0, -1/2) = -5/4$, $f(0, 1) = 1$, and $f(0, -1) = -1$. We conclude that the *maximum value* of f in S is 1, which is attained at $(0, 1)$, whereas the *minimum value* is $-5/4$, attained at $(0, -1/2)$.

EXERCISES FOR SECTION 13.5

1. Let $f(x, y) = 4x - 2x^2 - 2y^2$, $S = \{(x, y) : x^2 + y^2 \leq 25\}$.

   (a) Compute $f_1'(x, y)$ and $f_2'(x, y)$, then find the only critical point for f.

   (b) Find the extreme points for f over S.

SM 2. Find the maximum and minimum points for the following:

   (a) $f(x, y) = x^3 + y^3 - 9xy + 27$ subject to $0 \leq x \leq 4$ and $0 \leq y \leq 4$.

   (b) $f(x, y) = x^2 + 2y^2 - x$ subject to $x^2 + y^2 \leq 1$.

SM 3. In one study of the quantities x and y of natural gas that Western Europe should import from Norway and Siberia, respectively, it was assumed that the benefits were given by the function

$$f(x, y) = 9x + 8y - 6(x + y)^2$$

Because of capacity constraints, x and y must satisfy $0 \leq x \leq 5$ and $0 \leq y \leq 3$. Finally, for political reasons, it was felt that imports from Norway should not provide too small a fraction of total imports at the margin, so that $x \geq 2(y - 1)$, or equivalently $-x + 2y \leq 2$. In the xy-plane, draw the set S of all points satisfying the three constraints, and then find the quantities that maximize the benefits, subject to the three constraints.

4. Consider the function $f(x, y) = ax^2y + bxy + 2xy^2 + c$.

(a) Determine values of the constants a, b, and c such that f has a local minimum at the point $(2/3, 1/3)$ with local minimum value $-1/9$.

(b) With the values of a, b, and c found in part (a), find the maximum and minimum values of f over the set $S = \{(x, y) : x \geq 0, y \geq 0, 2x + y \leq 4\}$.

SM 5. Consider the function $f(x, y) = xe^{-x}(y^2 - 4y)$.

(a) Find all critical points of f and classify them by using the second-derivative test.

(b) Show that f has neither a global maximum nor a global minimum.

(c) Let $S = \{(x, y) : 0 \leq x \leq 5,\ 0 \leq y \leq 4\}$. Prove that f has global maximum and minimum points in S and find them.

(d) Find the slope of the tangent to the level curve $xe^{-x}(y^2 - 4y) = e - 4$ at the point where $x = 1$ and $y = 4 - e$.

6. Determine whether each of the following sets is open, closed, bounded, or compact:

(a) $\{(x, y) : 5x^2 + 5y^2 \leq 9\}$ (b) $\{(x, y) : x^2 + y^2 > 9\}$ (c) $\{(x, y) : x^2 + y^2 \leq 9\}$

(d) $\{(x, y) : 2x + 5y \geq 6\}$ (e) $\{(x, y) : 5x + 8y = 8\}$ (f) $\{(x, y) : 5x + 8y > 8\}$

7. [HARDER] Give an example of a discontinuous function g of one variable such that the set $\{x : g(x) \leq 1\}$ is not closed.

13.6 The General Case

So far, this chapter has considered optimization problems for functions of two variables. In order to be prepared to understand modern economic theory we need to extend the analysis to an arbitrary number of variables.

There are almost obvious extensions of the definitions of maximum and minimum points, extreme points, etc. If $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is a function defined over a set S in $\mathbb{R}^n$, then $\mathbf{c} = (c_1, \ldots, c_n)$ is a (global) *maximum point* for f in S if

$$f(\mathbf{x}) \leq f(\mathbf{c}) \quad \text{for all } \mathbf{x} \text{ in } S \tag{13.6.1}$$

If this is the case, then $-f(\mathbf{x}) \geq -f(\mathbf{c})$ for all $\mathbf{x}$ in S. Thus, $\mathbf{c}$ maximizes f over S if and only if $\mathbf{c}$ minimizes $-f$ over S. We can use this simple observation to convert maximization problems into minimization problems and vice versa.[13]

The concepts of interior and boundary points, and of open, closed, and bounded sets, are also easy to generalize. First, define the *distance* between the points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ by

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \tag{13.6.2}$$

[13] Recall Fig. 8.1.1, which illustrates this for the case of functions of one variable.

For $n = 1$, 2, and 3 this reduces to the distance concept discussed earlier. In particular, if $\mathbf{y} = \mathbf{0}$, then

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

is the distance between $\mathbf{x}$ and the origin. The number $\|\mathbf{x}\|$ is also called the *norm* or *length* of the vector $\mathbf{x}$.

The *open ball with centre at* $\mathbf{a} = (a_1, \ldots, a_n)$ *and radius* r is the set of all points $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $\|\mathbf{x} - \mathbf{a}\| < r$. The definitions in Section 13.5 of interior point, open set, boundary point, closed set, bounded set, and compact set all become valid for sets in $\mathbb{R}^n$, provided we replace the word "circle" by "ball". If A is an arbitrary set in $\mathbb{R}^n$, we define the *interior* of A as the set of interior points in A. If A is open, the interior of A is equal to the set itself.[14]

If $g(\mathbf{x}) = g(x_1, \ldots, x_n)$ is a continuous function, and c is a real number, then each of the three sets $\{\mathbf{x} : g(\mathbf{x}) \geq c\}$, $\{\mathbf{x} : g(\mathbf{x}) \leq c\}$, and $\{\mathbf{x} : g(\mathbf{x}) = c\}$ is closed. If $\geq$ is replaced by $>$, $\leq$ by $<$, or $=$ by $\neq$, the corresponding set is open.

A *critical* (or *stationary*) *point* for a function of n variables is a point where all the first-order derivatives are 0. We have the following important generalization of Theorem 13.1.1:

THEOREM 13.6.1 (NECESSARY CONDITIONS FOR INTERIOR EXTREMA)

Suppose that f is defined in a set S in $\mathbb{R}^n$ and let $\mathbf{c} = (c_1, \ldots, c_n)$ be an interior point in S where f is differentiable. A necessary condition for $\mathbf{c}$ to be a maximum or minimum point for f is that $\mathbf{c}$ is a critical point for f—that is, $\mathbf{x} = \mathbf{c}$ satisfies the n first-order conditions stating that, for each $i = 1, \ldots, n$,

$$f_i'(\mathbf{x}) = 0 \tag{13.6.3}$$

We already have everything we need to prove this theorem.

Fix $i = 1, \ldots, n$, and define the function

$$g(x) = f(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_n)$$

whose domain consists of all those x_i such that $(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_n)$ belongs to S. If $\mathbf{c} = (c_1, \ldots, c_n)$ is a maximum point for f, then the function g of one variable must attain a maximum at $x = c_i$. Because $\mathbf{c}$ is an interior point of S, it follows that c_i is also an interior point in the domain of g. Hence, according to Theorem 8.1.1, we must have $g'(c_i) = 0$. But $g'(c_i) = f_i'(c_1, \ldots, c_n)$, so the conclusion follows. The argument when $\mathbf{c}$ is a minimum is identical.

The extreme value theorem is valid also for functions of n variables:

[14] These topological definitions and results are dealt with in some detail in FMEA.

THEOREM 13.6.2 (EXTREME VALUE THEOREM)

Suppose that function f is continuous throughout a nonempty, closed and bounded set S in $\mathbb{R}^n$. Then there exist both a point $\mathbf{a}$ in S where f has a minimum and a point $\mathbf{c}$ in S where f has a maximum—that is, for all $\mathbf{x}$ in S, one has

$$f(\mathbf{a}) \leq f(\mathbf{x}) \leq f(\mathbf{c})$$

If $f(\mathbf{x})$ is defined over a set S in $\mathbb{R}^n$, then the maximum and minimum points, if there are any, must lie either in the interior of S or on the boundary of S. According to Theorem 13.6.1, if f is differentiable, then any maximum or minimum point in the interior must satisfy the first-order conditions. Consequently, the recipe in Section 13.5 is also valid for any function of n variables defined on a closed and bounded set in $\mathbb{R}^n$.

Both the local and the global second-order conditions for the two-variable case can be generalized to functions of n variables, though they become considerably more complicated. This will be discussed in FMEA.

A Useful Result

One simple result is nevertheless of considerable interest in theoretical economics. It is this: *maximizing a function is equivalent to maximizing a strictly increasing transformation of that function*. For instance, suppose we want to find all pairs (x, y) that maximize $f(x, y)$ over a set S in the xy-plane. Instead (provided the constant $a > 0$) we can find those (x, y) that maximize over S any one of the following objective functions:

(a) $af(x, y) + b$; (b) $e^{f(x,y)}$; (c) $\ln f(x, y)$ (in case $f(x, y) > 0$ throughout S).

The maximum *points* are exactly the same. But the maximum *values* are, of course, quite different. As a concrete example, because the transformation $u \mapsto \ln u$ is strictly increasing when $u > 0$, the following two problems have exactly the same solutions for x and y:

(i) $\max e^{x^2+2xy^2-y^3}$ subject to $(x, y) \in S$; (ii) $\max x^2 + 2xy^2 - y^3$ subject to $(x, y) \in S$.

In general, it is easy to prove the following result:

THEOREM 13.6.3

Suppose $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is defined over a set S in $\mathbb{R}^n$. Let F be a function of one variable defined over the range of f, and let $\mathbf{c}$ be a point in S. Define g over S by $g(\mathbf{x}) = F(f(\mathbf{x}))$.

(a) If F is increasing and $\mathbf{c}$ maximizes (minimizes) f over S, then the same point $\mathbf{c}$ also maximizes (resp. minimizes) g over S.

(b) If F is strictly increasing, then $\mathbf{c}$ maximizes (minimizes) f over S if and only if $\mathbf{c}$ maximizes (resp. minimizes) g over S.

We give a proof only for the maximization case, since the minimization case is entirely similar.

(a) Because $\mathbf{c}$ maximizes f over S, we have $f(\mathbf{x}) \le f(\mathbf{c})$ for all $\mathbf{x}$ in S. But then $g(\mathbf{x}) = F(f(\mathbf{x})) \le F(f(\mathbf{c})) = g(\mathbf{c})$ for all $\mathbf{x}$ in S, because F is increasing. It follows that $\mathbf{c}$ maximizes g over S.

(b) If F is also strictly increasing and $f(\mathbf{x}) > f(\mathbf{c})$, then it must be true that $g(\mathbf{x}) = F(f(\mathbf{x})) > F(f(\mathbf{c})) = g(\mathbf{c})$. So $g(\mathbf{x}) \le g(\mathbf{c})$ for all $\mathbf{x}$ in S implies that $f(\mathbf{x}) \le f(\mathbf{c})$ for all $\mathbf{x}$ in S.

Note how extremely simple the argument was. No continuity or differentiability assumptions were required, and, instead, the proof is based only on the concepts of maximum, and of increasing/strictly increasing functions. Some people appear to distrust such simple, direct arguments and replace them by inefficient or even insufficient arguments based on "differentiating everything in sight" in order to use first- or second-order conditions. Such distrust merely makes matters unnecessarily complicated and risks introducing errors.

EXERCISES FOR SECTION 13.6

1. Each of the following functions has a maximum point. Find it.

   (a) $f(x, y, z) = 2x - x^2 + 10y - y^2 + 3 - z^2$

   (b) $f(x, y, z) = 3 - x^2 - 2y^2 - 3z^2 - 2xy - 2xz$

2. Define $f(x) = e^{-x^2}$.

   (a) Let $F(u) = \ln u$. Verify that the two functions $f(x)$ and $F(f(x))$ have maxima at the same values of x.

   (b) Let $F(u) = 5$. Then $g(x) = F(f(x)) = 5$. Explain why this example shows that implication (a) in Theorem 13.6.3 cannot be reversed. (Recall that our definition of an increasing function is satisfied by a constant function.)

3. Suppose $g(\mathbf{x}) = F(f(\mathbf{x}))$ where $f : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are differentiable functions, with $F' \ne 0$ everywhere. Prove that $\mathbf{x}$ is a critical point for f if, and only if, it is a critical point for g.

SM 4. Find the first-order partial derivatives of the function of three variables given by

$$f(x, y, z) = -2x^3 + 15x^2 - 36x + 2y - 3z + \int_y^z e^{t^2}\, dt$$

   Then determine its eight critical points.

5. Suggest how to simplify the following problems:

   (a) $\max \frac{1}{2}[e^{x^2+y^2-2x} - e^{-(x^2+y^2-2x)}]$, subject to $(x, y) \in S$

   (b) $\max Ax_1^{a_1} \cdots x_n^{a_n}$, subject to $x_1 + x_2 + \cdots + x_n = 1$, where $A > 0$ and $x_1 > 0, \ldots, x_n > 0$

13.7 Comparative Statics and the Envelope Theorem

Optimization problems in economics usually involve maximizing or minimizing functions which depend not only on endogenous variables one can choose, but also on one or more exogenous parameters like prices, tax rates, income levels, etc. Although these parameters are held constant during the optimization, they vary according to the economic situation. For example, we may calculate a firm's profit-maximizing input and output quantities while treating the prices it faces as parameters. But then we may want to know how the optimal quantities respond to changes in those prices, or in whatever other exogenous parameters affect the problem we are considering.

Consider first the following simple problem. A function f depends on a single variable x as well as on a single parameter r. We wish to maximize $f(x, r)$ w.r.t. x while keeping r constant:[15]

$$\max_x f(x, r)$$

The value of x that maximizes f will usually depend on r, so we denote it by $x^*(r)$. Inserting $x^*(r)$ into $f(x, r)$, we obtain the *value function*:

$$f^*(r) = f(x^*(r), r)$$

What happens to the value function as r changes? Assuming that $f^*(r)$ is differentiable, the chain rule yields

$$\frac{\mathrm{d}f^*(r)}{\mathrm{d}r} = f_1'(x^*(r), r)\frac{\mathrm{d}x^*(r)}{\mathrm{d}r} + f_2'(x^*(r), r)$$

If f achieves a maximum at an interior point $x^*(r)$ in the domain of variation for x, then the FOC $f_1'(x^*(r), r) = 0$ is satisfied. It follows that

$$\frac{\mathrm{d}f^*(r)}{\mathrm{d}r} = f_2'(x^*(r), r) \tag{13.7.1}$$

Note that when r is changed, then $f^*(r)$ changes for two reasons. First, a change in r changes the value of f^* directly because r is the second variable in $f(x, r)$. Second, a change in r changes the value of the function $x^*(r)$, and hence $f(x^*(r), r)$ is changed indirectly. Equation (13.7.1) shows that the total effect is simply found by computing the partial derivative of $f(x^*(r), r)$ w.r.t. r, ignoring entirely the indirect effect of the dependence of x^* on r. At first sight, this seems very surprising. On further reflection, however, you may realize that the first-order condition for $x^*(r)$ to maximize $f(x, r)$ w.r.t. x implies that any small change in x, whether or not it is induced by a small change in r, must have a negligible effect on the value of $f(x^*, r)$.

EXAMPLE 13.7.1 Suppose that when a firm produces and sells x units of a commodity, it has revenue $R(x) = rx$, where r is a positive parameter, while the cost is $C(x) = x^2$. The profit is then

$$\pi(x, r) = R(x) - C(x) = rx - x^2$$

[15] The theory is identical for the case of minimization.

Find the optimal choice x^* of x, and verify (13.7.1) in this case.

Solution: The quadratic profit function has a maximum when $\pi'_1 = r - 2x = 0$, that is for $x^* = r/2$. So the maximum profit as a function of r is given by $\pi^*(r) = rx^* - (x^*)^2 = r(r/2) - (r/2)^2 = r^2/4$, and then $\mathrm{d}\pi^*/\mathrm{d}r = r/2$. Using formula (13.7.1) is much more direct: because $\pi'_2(x, r) = x$, it implies that $\mathrm{d}\pi^*/\mathrm{d}r = \pi'_2(x^*(r), r) = x^*(r) = \frac{1}{2}r$.

EXAMPLE 13.7.2 In Example 8.6.5 we studied a firm with the profit function $\hat{\pi}(Q, \tau) = R(Q) - C(Q) - \tau Q$, where τ denoted a tax per unit produced. Let $Q^* = Q^*(\tau)$ denote the optimal choice of Q as a function of the tax rate τ, and let $\pi^*(\tau)$ be the corresponding value function. Because $\hat{\pi}'_2 = -Q$, Eq. (13.7.1) yields

$$\frac{\mathrm{d}\pi^*(\tau)}{\mathrm{d}\tau} = \hat{\pi}'_2(Q^*(\tau), \tau) = -Q^*(\tau)$$

which is the same result found earlier.

It is easy to generalize (13.7.1) to the case with many choice variables and many parameters. We let $\mathbf{x} = (x_1, \dots, x_n)$, and $\mathbf{r} = (r_1, \dots, r_m)$. Then, assuming that the function $f(\mathbf{x}, \mathbf{r})$ is differentiable, we can formulate the following result:

THEOREM 13.7.1 (ENVELOPE THEOREM)

If $f^*(\mathbf{r}) = \max_{\mathbf{x}} f(\mathbf{x}, \mathbf{r})$ and if $\mathbf{x}^*(\mathbf{r})$ is the value of $\mathbf{x}$ that maximizes $f(\mathbf{x}, \mathbf{r})$, then

$$\frac{\partial f^*(\mathbf{r})}{\partial r_j} = \frac{\partial f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_j} \tag{13.7.2}$$

for $j = 1, \dots, m$, provided that the partial derivative exists.

Again, $f^*(\mathbf{r})$ is the *value function*. It is easy to prove Theorem 13.7.1 by using the first-order conditions to eliminate other terms, as in the argument for Eq.(13.7.1). The same equality holds if we minimize $f(\mathbf{x}, \mathbf{r})$ w.r.t. $\mathbf{x}$ instead of maximizing it, or even if $\mathbf{x}^*(\mathbf{r})$ is any critical point.

Figure 13.7.1 illustrates Eq. (13.7.2) in the case where there is only one parameter r. For each fixed value of $\mathbf{x}$ there is a curve $K_{\mathbf{x}}$ in the ry-plane, given by the equation $y = f(\mathbf{x}, r)$. Figure 13.7.1 shows some of these curves together with the graph of $f^*(r)$. For all $\tilde{\mathbf{x}}$ and all r we have

$$f(\tilde{\mathbf{x}}, r) \le \max_{\mathbf{x}} f(\mathbf{x}, r) = f^*(r)$$

It follows that none of the $K_{\mathbf{x}}$-curves can ever lie above the curve $y = f^*(r)$. On the other hand, for each value of r there is at least one value $\mathbf{x}^*(r)$ such that $f(\mathbf{x}^*(r), r) = f^*(r)$, namely a choice of $\mathbf{x}$ that solves the maximization problem for the given value of r. For instance, if we fix $r = r_0$ and let $\mathbf{x}_0$ denote $\mathbf{x}^*(r_0)$, then the curve $K_{\mathbf{x}_0}$ will touch the curve $y = f^*(r)$ at the point $(r_0, f^*(r_0))$, as in the figure. Moreover, because $K_{\mathbf{x}_0}$ can never go above this graph, it must have exactly the same tangent as the graph of f^* at the point where

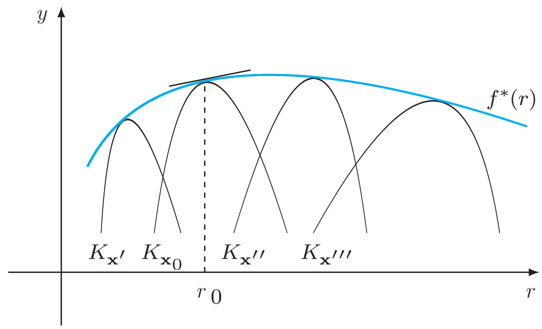

Figure 13.7.1 The curve $y = f^*(r)$ is the envelope of all the curves $y = f(\mathbf{x}, r)$

the curves touch. The slope of this common tangent, therefore, must be not only $\mathrm{d}f^*/\mathrm{d}r$, the slope of the tangent to the graph of f^* at $(r_0, f^*(r_0))$, but also $\partial f(\mathbf{x}_0, r)/\partial r$, the slope of the tangent to the curve $K_{\mathbf{x}_0}$ at the point $(r_0, f(\mathbf{x}_0, r_0))$. Equation (13.7.2) follows because $K_{\mathbf{x}_0}$ is the graph of $f(\mathbf{x}_0, r)$ when $\mathbf{x}_0$ is fixed.

As Fig. 13.7.1 suggests, the graph of $y = f^*(r)$ is the lowest curve with the property that it lies on or above all the curves $K_{\mathbf{x}}$. So its graph is like an envelope or some "cling film" that is used to enclose or wrap up all these curves. Indeed, a point is on or below the graph if and only if it lies on or below one of the curves $K_{\mathbf{x}}$. For this reason we call the graph of f^* the *envelope* of the family of $K_{\mathbf{x}}$-curves.

EXAMPLE 13.7.3 In Example 13.1.3, $Q = F(K, L)$ denoted a production function with K as capital input and L as labour input. The price per unit of the product was p, the price per unit of capital was r, and the price per unit of labour was w. The profit obtained by using K and L units of the inputs, then producing and selling $F(K, L)$ units of the product, is given by

$$\hat{\pi}(K, L, p, r, w) = pF(K, L) - rK - wL$$

Here profit has been expressed as a new function $\hat{\pi}$ of the parameters p, r, and w, as well as of the choice variables K and L. We keep p, r, and w fixed and maximize $\hat{\pi}$ w.r.t. K and L. The optimal values of K and L are functions of p, r, and w, which we denote by $K^* = K^*(p, r, w)$ and $L^* = L^*(p, r, w)$. The value function for the problem is $\hat{\pi}^*(p, r, w) = \hat{\pi}(K^*, L^*, p, r, w)$. Usually, $\hat{\pi}^*$ is called the firm's *profit function*, though it would be more accurately described as the "maximum profit function". It is found by taking prices as given and choosing the optimal quantities of all inputs and outputs.

According to Theorem 13.7.1, one has

$$\frac{\partial \hat{\pi}^*}{\partial p} = F(K^*, L^*) = Q^*, \quad \frac{\partial \hat{\pi}^*}{\partial r} = -K^*, \quad \frac{\partial \hat{\pi}^*}{\partial w} = -L^* \tag{$*$}$$

These three equalities are instances of what is known in production theory as *Hotelling's lemma*. An economic interpretation of the middle equality is this: How much profit is lost if the price of capital increases by a small amount? At the optimum the firm uses K^* units of capital, so the answer is K^* per unit increase in the price. See Exercise 4 for further interesting relationships.

EXERCISES FOR SECTION 13.7

1. A firm produces a single commodity and gets p for each unit sold. The cost of producing x units is $ax + bx^2$ and the tax per unit is t. Assume that the parameters are positive with $p > a + t$. The firm wants to maximize its profit.

   (a) Find the optimal production x^* and the optimal profit π^*.

   (b) Prove that $\partial \pi^*/\partial p = x^*$, and give an economic interpretation.

2. A firm produces $Q = \sqrt{L}$ units of a commodity when labour input is L units. The price obtained per unit of output is P, and the price per unit of labour is w, both positive.

   (a) Write down the profit function π. What choice of labour input $L = L^*$ maximizes profits?

   (b) Consider L^* as a function $L^*(P, w)$ of the two prices, and define the value function

   $$\pi^*(P, w) = \pi(L^*(P, w), P, w)$$

   Verify that $\partial \pi^*/\partial P = \pi'_P(L^*, P, w)$ and $\partial \pi^*/\partial w = \pi'_w(L^*, P, w)$, thus confirming the envelope theorem.

SM 3. A firm uses capital K, labour L, and land T to produce Q units of a commodity, where

$$Q = K^{2/3} + L^{1/2} + T^{1/3}$$

Suppose that the firm is paid a positive price p for each unit it produces, and that the positive prices it pays per unit of capital, labour, and land are r, w, and q, respectively.

   (a) Express the firm's profits as a function π of (K, L, T). Then, find the values of K, L, and T, as functions of the four prices, that maximize the firm's profits—assuming a maximum exists.

   (b) Let Q^* denote the optimal number of units produced and K^* the optimal capital stock. Show that $\partial Q^*/\partial r = -\partial K^*/\partial p$.

4. With reference to Example 13.7.3, assuming that F is a C^2 function, prove the symmetry relations:

$$\frac{\partial Q^*}{\partial r} = -\frac{\partial K^*}{\partial p}; \quad \frac{\partial Q^*}{\partial w} = -\frac{\partial L^*}{\partial p}; \quad \frac{\partial L^*}{\partial r} = \frac{\partial K^*}{\partial w}$$

(*Hint*: First establish that $\dfrac{\partial Q^*}{\partial r} = \dfrac{\partial}{\partial r}\left(\dfrac{\partial \hat{\pi}^*}{\partial p}\right) = \dfrac{\partial}{\partial p}\left(\dfrac{\partial \hat{\pi}^*}{\partial r}\right)$ by combining the first result in Example 13.7.3 with Young's theorem. Then use the other results in Example 13.7.3.)

SM 5. With reference to Example 13.1.3, we want to study the factor demand functions—in particular, how the optimal choices of capital and labour respond to price changes.

   (a) Differentiate the first-order conditions $(*)$ in Example 13.1.3 to verify that

   $$F'_K(K^*, L^*)\,\mathrm{d}p + pF''_{KK}(K^*, L^*)\,\mathrm{d}K + pF''_{KL}(K^*, L^*)\,\mathrm{d}L = \mathrm{d}r$$
   $$F'_L(K^*, L^*)\,\mathrm{d}p + pF''_{LK}(K^*, L^*)\,\mathrm{d}K + pF''_{LL}(K^*, L^*)\,\mathrm{d}L = \mathrm{d}w$$

   (b) Use this system to find the partials of K^* and L^* w.r.t. p, r, and w. (*Hint*: You might find it easier first to find $\partial K^*/\partial p$ and $\partial L^*/\partial p$ by putting $\mathrm{d}r = \mathrm{d}w = 0$, etc. in (a).)

   (c) Assume that the local second-order conditions (13.3.1) are satisfied. What can you say about the signs of the partial derivatives? In particular, show that the factor demand curves are downward sloping as functions of their own factor prices. Verify that $\partial K^*/\partial w = \partial L^*/\partial r$.

SM **6.** A profit-maximizing monopolist produces two commodities whose quantities are denoted by x_1 and x_2. Good 1 is subsidized at the rate of σ per unit and good 2 is taxed at τ per unit. The monopolist's profit function is therefore given by

$$\pi(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2) + \sigma x_1 - \tau x_2$$

where R and C are the firm's revenue and cost functions, respectively. Assume that the partial derivatives of these functions have the following signs

$$R'_1 > 0, \;\; R'_2 > 0, \;\; R''_{11} < 0, \;\; R''_{12} = R''_{21} < 0, \;\; R''_{22} < 0$$
$$C'_1 > 0, \;\; C'_2 > 0, \;\; C''_{11} > 0, \;\; C''_{12} = C''_{21} > 0, \;\; C''_{22} > 0$$

everywhere in their domains.

(a) Find the first-order conditions for maximum profits.

(b) Write down the local second-order conditions for maximum profits.

(c) Suppose that $x_1^* = x_1^*(\sigma, \tau)$, $x_2^* = x_2^*(\sigma, \tau)$ solve the problem. Find the signs of $\partial x_1^*/\partial \sigma$, $\partial x_1^*/\partial \tau$, $\partial x_2^*/\partial \sigma$, and $\partial x_2^*/\partial \tau$, assuming that the local second-order conditions are satisfied.

(d) Show that $\partial x_1^*/\partial \tau = -\partial x_2^*/\partial \sigma$.

REVIEW EXERCISES

1. The function f defined for all (x, y) by $f(x, y) = -2x^2 + 2xy - y^2 + 18x - 14y + 4$ has a maximum. Find the corresponding values of x and y. Use Theorem 13.2.1 to prove that it is a maximum point.

SM **2.** A firm produces two different kinds, A and B, of a commodity. The daily cost of producing Q_1 units of A and Q_2 units of B is $C(Q_1, Q_2) = 0.1(Q_1^2 + Q_1 Q_2 + Q_2^2)$. Suppose that the firm sells all its output at a price per unit of $P_1 = 120$ for A and $P_2 = 90$ for B.

(a) Find the daily production levels that maximize profits.

(b) If P_2 remains unchanged at 90, what new price P_1 per unit of A would imply that the optimal daily production level for A is 400 units?

3. The profit obtained by a firm from producing and selling x and y units of two brands of a commodity is given by $P(x, y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$.

(a) Find the production levels that maximize profits.

(b) A key raw material is rationed so that total production must be restricted to 200 units. Find the production levels that now maximize profits.

SM **4.** Find the critical points of the following functions of (x, y):

(a) $x^3 - x^2 y + y^2$ (b) $xye^{4x^2 - 5xy + y^2}$ (c) $4y^3 + 12x^2 y - 24x^2 - 24y^2$

5. Define $f(x, y, a) = ax^2 - 2x + y^2 - 4ay$, where a is a parameter. For each fixed $a \neq 0$, find the unique critical point $(x^*(a), y^*(a))$ of the function f w.r.t. (x, y). Find also the value function $f^*(a) = f(x^*(a), y^*(a), a)$, and verify the envelope theorem in this case.

SM **6.** Suppose the production function in Exercise 13.7.3 is replaced by $Q = K^a + L^b + T^c$, for parameters $a, b, c \in (0, 1)$.

(a) Assuming that a maximum exists, find the values of K, L, and T that maximize the firm's profits.

(b) Let π^* denote the optimal profit as a function of the four prices. Compute the partial derivative $\partial\pi^*/\partial r$.

(c) Verify the envelope theorem in this case.

7. Define $f(x, y)$ for all (x, y) by $f(x, y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$.

(a) Find the first- and second-order partial derivatives of f, then show that $f(x, y)$ is convex.

(b) Find the minimum point of $f(x, y)$.

SM **8.** Consider the function $f(x, y) = x^2 - y^2 - xy - x^3$.

(a) Find and classify its critical points.

(b) Find the domain S where f is concave, and find the largest value f in S.

SM **9.** Consider the function f defined for all (x, y) by

$$f(x, y) = \tfrac{1}{2}x^2 - x + ay(x - 1) - \tfrac{1}{3}y^3 + a^2y^2$$

where a is a constant.

(a) Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a critical point of f.

(b) Verify the envelope theorem in this case.

(c) Where in the xy-plane is f convex?

10. In this problem we will generalize several of the economic examples and problems considered so far. Consider a firm that produces two different goods, A and B. If the total cost function is $C(x, y)$, then the profit is

$$\pi(x, y) = px + qy - C(x, y) \tag{i}$$

where the prices obtained per unit of A and B are p and q respectively.

(a) Suppose first that the firm has a small share in the markets for both these goods, and so takes p and q as given. Write down and interpret the first-order conditions for $x^* > 0$ and $y^* > 0$ to maximize profits.

(b) Suppose next that the firm has a monopoly in the sale of both goods. The prices are no longer fixed, but chosen by the monopolist, bearing in mind the demand functions

$$x = f(p, q) \quad \text{and} \quad y = g(p, q) \tag{ii}$$

Suppose we solve equations (ii) for p and q to obtain the inverse demand functions

$$p = F(x, y) \quad \text{and} \quad q = G(x, y) \tag{iii}$$

Then profit as a function of x and y is

$$\pi(x, y) = xF(x, y) + yG(x, y) - C(x, y) \tag{iv}$$

Write down and interpret the first-order conditions for $x^* > 0$ and $y^* > 0$ to maximize profits.

(c) Suppose $p = a - bx - cy$ and $q = \alpha - \beta x - \gamma y$, where b and γ are positive.[16] If the cost function is $C(x, y) = Px + Qy + R$, write down the first-order conditions for maximum profit.

(d) Prove that the (global) second-order conditions are satisfied provided $4\gamma b \geq (\beta + c)^2$.

[16] An increase in the price of either good decreases the demand for that good, but may increase or decrease the demand for the other good.

14

CONSTRAINED OPTIMIZATION

Mathematics is removed from this turmoil of human life, but its methods and the relations are a mirror, an incredibly pure mirror, of the relations that link facts of our existence.
—Konrad Knopp (1928)

The previous chapter introduced unconstrained optimization problems with several variables. In economics, however, the variables to be chosen must often satisfy one or more constraints. Accordingly, this chapter considers constrained optimization problems, and studies the method of Lagrange multipliers in some detail. Sections 14.1 to 14.7 treat equality constraints, with Section 14.7 presenting some comparative static results and the envelope theorem. More general constrained optimization problems allowing inequality constraints are introduced in Sections 14.8 to 14.10. A much fuller treatment of constrained optimization can be found in FMEA.

14.1 The Lagrange Multiplier Method

A typical economic example of a constrained optimization problem concerns a consumer who chooses how much of the available income m to spend on a good x whose price is p, and how much income to leave over for expenditure on other goods,which we denote by y. Note that the consumer then faces the budget constraint $px + y = m$. Suppose that preferences are represented by the utility function $u(x, y)$. In mathematical terms the consumer's problem can be expressed as

$$\max u(x, y) \quad \text{s.t.} \quad px + y = m$$

where "s.t." stands for "such that". This is a typical *constrained maximization problem*. In this case, because $y = m - px$, the same problem can be expressed as the *unconstrained maximization* of the function $h(x) = u(x, m - px)$ w.r.t. the single variable x. Indeed, this method of converting a constrained optimization problem involving two variables to a one-variable problem was used in Section 13.2.

When the constraint involves a complicated function, or when there are several equality constraints to consider, this substitution method might be difficult or even impossible to carry out in practice. In such cases, economists make much use of the *Lagrange multiplier method*.[1]

We start with the problem of maximizing a function $f(x, y)$ of two variables, when x and y are restricted to satisfy an equality constraint $g(x, y) = c$. This can be written as

$$\max f(x, y) \quad \text{s.t.} \quad g(x, y) = c \tag{14.1.1}$$

The first step of the method is to introduce a *Lagrange multiplier*, often denoted by λ, which is "associated" with the constraint $g(x, y) = c$. We do this when we define the *Lagrangian* function, $\mathcal{L}$, by

$$\mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c] \tag{14.1.2}$$

in which the expression $g(x, y) - c$, which must be 0 when the constraint is satisfied, has been multiplied by λ. For future reference, note that $\mathcal{L}(x, y) = f(x, y)$ for all (x, y) that satisfy the constraint $g(x, y) = c$.

The Lagrange multiplier is a constant, so the partial derivatives of $\mathcal{L}(x, y)$ w.r.t. x and y are

$$\mathcal{L}'_1(x, y) = f'_1(x, y) - \lambda g'_1(x, y) \quad \text{and} \quad \mathcal{L}'_2(x, y) = f'_2(x, y) - \lambda g'_2(x, y)$$

respectively. As will be explained in Section 14.4, except in rare cases a solution of problem (14.1.1) can only be a point (x, y) where, for a suitable value of λ, the first-order partial derivatives of $\mathcal{L}$ vanish, and also the constraint $g(x, y) = c$ is satisfied. As in Chapters 8 and 13, we refer to these as "first-order" conditions.

Here is a simple economic application.

EXAMPLE 14.1.1 A consumer has the utility function $u(x, y) = xy$ and faces the budget constraint $2x + y = 100$. Find the only solution candidate to the utility maximization problem.

Solution: The problem is

$$\max\ xy \quad \text{s.t.} \quad 2x + y = 100$$

so its Lagrangian is

$$\mathcal{L}(x, y) = xy - \lambda(2x + y - 100)$$

Including the constraint, the first-order conditions for the solution of the problem are

$$\mathcal{L}'_1(x, y) = y - 2\lambda = 0, \ \mathcal{L}'_2(x, y) = x - \lambda = 0, \quad \text{and} \quad 2x + y = 100$$

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[1] Named after its discoverer, the Italian-born French mathematician J. L. Lagrange (1736–1813). The Danish economist Harald Westergaard seems to have been the first to use it in economics, in 1876. As a matter of practice, this method is often used even for problems that are quite easy to express as unconstrained problems. One reason is that Lagrange multipliers have an important economic interpretation. In addition, a similar method works for many more complicated optimization problems, such as those where the constraints are expressed in terms of inequalities, as we will see later.

The first two equations imply that $y = 2\lambda$ and $x = \lambda$. So $y = 2x$. Inserting this into the constraint yields $2x + 2x = 100$. So $x = 25$ and $y = 50$, implying that $\lambda = x = 25$.

This solution can be confirmed by the substitution method. From $2x + y = 100$ we get $y = 100 - 2x$, so the problem is reduced to maximizing the unconstrained function $h(x) = x(100 - 2x) = -2x^2 + 100x$. Since $h'(x) = -4x + 100 = 0$ gives $x = 25$, and $h''(x) = -4 < 0$ for all x, this shows that $x = 25$ *is* a maximum point.

Perhaps surprisingly, in the alternative minimization problem

$$\min f(x, y) \quad \text{s.t.} \quad g(x, y) = c \tag{14.1.3}$$

function $\mathcal{L}$ is defined identically, by Eq. (14.1.2), and the relevant first-order conditions are the same. Given this, we often write

$$\max(\min) f(x, y) \quad \text{s.t.} \quad g(x, y) = c$$

when referring to both the maximization and minimization problems.[2]

Example 14.1.1 illustrates the following general method:

THE LAGRANGE MULTIPLIER METHOD

To find the only possible solutions of problems (14.1.1) and (14.1.3), proceed as follows:

(i) Write down the Lagrangian function, as in Eq. (14.1.2), where λ is a constant.

(ii) Differentiate $\mathcal{L}$ w.r.t. x and y, and equate the partial derivatives to 0.

(iii) The two equations in (ii), together with the constraint, yield the following three *first-order conditions*:

$$\begin{aligned} \mathcal{L}_1'(x, y) &= f_1'(x, y) - \lambda g_1'(x, y) = 0 \\ \mathcal{L}_2'(x, y) &= f_2'(x, y) - \lambda g_2'(x, y) = 0 \\ g(x, y) &= c \end{aligned}$$

(iv) Solve these three equations simultaneously for the three unknowns x, y, and λ. These triples (x, y, λ) are the solution *candidates*, at least one of which solves the respective problem, if it has a solution.

Importantly, if $g_1'(x, y)$ and $g_2'(x, y)$ both vanish, the method might fail to give the right answer.

Some economists prefer to consider the Lagrangian as a function $\tilde{\mathcal{L}}(x, y, \lambda)$ of three variables. Then, the first-order condition $\tilde{\mathcal{L}}_3'(x, y, \lambda) = 0$ yields the constraint of the problem,

[2] The reader may have seen expressions like $\max\min f(x, y)$ in, for instance, game theory courses. Those expressions mean something entirely different.

$g(x, y) = c$. The advantage of this method is that, written in this way, all the three necessary conditions are obtained by equating the partial derivatives of this extended Lagrangian to 0, so that the first-order conditions can be summarized by saying that we need to find a critical point of the Lagrangian. It seems unnatural, however, to rely on differentiation in order to derive such an obvious necessary condition—namely the constraint equation. Moreover, this procedure can easily lead to trouble when treating problems with inequality constraints. For these two reasons, we prefer to avoid it.

EXAMPLE 14.1.2 A single-product firm intends to produce 30 units of output as cheaply as possible. By using K units of capital and L units of labour, it can produce $\sqrt{K} + L$ units. Suppose the prices of capital and labour are, respectively, \$1 and \$20. The firm's problem is, then:

$$\min K + 20L \ \text{ s.t. } \ \sqrt{K} + L = 30$$

(a) Find the optimal choices of K and L.

(b) What is the additional cost of producing 31 rather than 30 units?

Solution:

(a) The Lagrangian is

$$\mathcal{L} = K + 20L - \lambda(\sqrt{K} + L - 30)$$

so the first-order conditions are:

$$\mathcal{L}'_K = 1 - \lambda/2\sqrt{K} = 0, \ \mathcal{L}'_L = 20 - \lambda = 0, \ \text{ and } \ \sqrt{K} + L = 30$$

The second equation gives $\lambda = 20$, which inserted into the first equation yields $1 = 20/2\sqrt{K}$. It follows that $\sqrt{K} = 10$, and hence $K = 100$. Inserted into the constraint this gives $\sqrt{100} + L = 30$, and hence $L = 20$. The 30 units are therefore produced in the cheapest way when the firm uses 100 units of capital and 20 units of labour. The associated cost is $K + 20L = 500$.[3]

(b) Solving the problem with the constraint $\sqrt{K} + L = 31$, we see that still $\lambda = 20$ and $K = 100$, while $L = 31 - 10 = 21$. The associated minimum cost is $100 + 20 \cdot 21 = 520$, so the additional cost is $520 - 500 = 20$. This is precisely equal to the Lagrange multiplier! Thus, in this case the Lagrange multiplier tells us by how much costs increase if the production requirement is increased by one unit from 30 to 31.[4]

EXAMPLE 14.1.3 A consumer who has Cobb–Douglas utility function $u(x, y) = Ax^a y^b$ faces the budget constraint $px + qy = m$, where A, a, b, p, q, and m are all positive constants. Find the only solution candidate to the consumer demand problem

$$\max Ax^a y^b \ \text{ s.t. } \ px + qy = m \tag{$*$}$$

[3] Theorem 14.5.1 will tell us that this is the constrained minimum, because $\mathcal{L}$ is convex in (K, L).

[4] Section 14.2 will tell us why this is not entirely coincidental.

Solution: The Lagrangian is $\mathcal{L}(x, y) = Ax^a y^b - \lambda(px + qy - m)$, so the first-order conditions are

$$\mathcal{L}_1'(x, y) = aAx^{a-1}y^b - \lambda p = 0,\ \ \mathcal{L}_2'(x, y) = bAx^a y^{b-1} - \lambda q = 0, \text{ and } px + qy = m$$

Solving the first two equations for λ yields

$$\lambda = \frac{aAx^{a-1}y^b}{p} = \frac{bAx^a y^{b-1}}{q}$$

Cancelling the common factor $Ax^{a-1}y^{b-1}$ from the last equality gives $ay/p = bx/q$. Solving this equation for qy yields $qy = (b/a)px$, which inserted into the budget constraint gives $px + (b/a)px = m$. From this equation we find x and then y. The results are the following *demand functions*:

$$x = x(p, q, m) = \frac{a}{a+b}\frac{m}{p} \quad \text{and} \quad y = y(p, q, m) = \frac{b}{a+b}\frac{m}{q} \tag{$**$}$$

The solution we have found makes good sense. It follows from $(**)$ that for all $t > 0$ one has $x(tp, tq, tm) = x(p, q, m)$ and $y(tp, tq, tm) = y(p, q, m)$, so the demand functions are homogeneous of degree 0. This is as one should expect because, if (p, q, m) is changed to (tp, tq, tm), then the constraint in $(*)$ is unchanged, and so the optimal choices of x and y are unchanged—as they should be, according to Example 12.7.4.

Note that in the utility function $Ax^a y^b$, the relative sizes of the coefficients a and b indicate the relative importance of x and y in the individual's preferences. For instance, if a is larger than b, then the consumer values a 1% increase in x more than a 1% increase in y. The product px is the amount spent on the first good, and $(**)$ says that the consumer should spend the fraction $a/(a + b)$ of income on this good and the fraction $b/(a + b)$ on the second good.

Formula $(**)$ can be applied immediately to find the correct answer to thousands of exam problems in mathematical economics courses given each year all over the world! But note that the utility function has to be of the Cobb–Douglas type $Ax^a y^b$.[5]

Another warning is in order here: there is an underlying assumption in problem $(*)$ that $x \geq 0$ and $y \geq 0$. Thus, we maximize a continuous function $Ax^a y^b$ over a closed bounded set $S = \{(x, y) : px + qy = m, x \geq 0, y \geq 0\}$. According to the extreme value theorem, 13.5.1, a maximum must exist. Since utility is 0 when $x = 0$ or when $y = 0$, and positive at the point given by $(**)$, this point indeed solves the problem. Without nonnegativity conditions on x and y, however, the problem might fail to have a maximum. Indeed, consider the problem $\max\ x^2 y$ s.t. $x + y = 1$. For real t, the pair $(x, y) = (-t, 1 + t)$ satisfies the constraint, yet $x^2 y = t^2(1 + t) \to \infty$ as $t \to \infty$, so there is no maximum.

EXAMPLE 14.1.4 Examine the general utility maximizing problem with two goods:

$$\max u(x, y) \ \text{ s.t. } \ px + qy = m \tag{14.1.4}$$

[5] When $u(x, y) = x^a + y^b$, for instance, the solution is *not* given by $(**)$. To check this, assuming that $0 < a < 1$, see: Exercise 9, for the case when $b = 1$; and Exercise 14.5.4, for the case when $a = b$.

Solution: The Lagrangian is $\mathcal{L}(x, y) = u(x, y) - \lambda(px + qy - m)$, so the first-order conditions are

$$\mathcal{L}'_x(x, y) = u'_x(x, y) - \lambda p = 0 \tag{i}$$

$$\mathcal{L}'_y(x, y) = u'_y(x, y) - \lambda q = 0 \tag{ii}$$

$$px + qy = m \tag{iii}$$

From equation (i) we get $\lambda = u'_x(x, y)/p$, and from (ii), $\lambda = u'_y(x, y)/q$. Hence, $u'_x(x, y)/p = u'_y(x, y)/q$, which can be rewritten as

$$\frac{u'_x(x, y)}{u'_y(x, y)} = \frac{p}{q} \tag{14.1.5}$$

The left-hand side of the last equation is the *marginal rate of substitution*, or MRS, studied in Section 12.5. Utility maximization thus requires equating the MRS to the price ratio p/q.

A geometric interpretation of Eq. (14.1.5) is that the consumer should choose the point on the budget line at which the slope of the level curve of the utility function, $-u'_x(x, y)/u'_y(x, y)$, is equal to the slope of the budget line, $-p/q$.[6] Thus, at the optimal point the budget line is tangent to a level curve of the utility function, illustrated by point P in Fig. 14.1.1. The level curves of the utility function are the *indifference curves*, along which the utility level is constant by definition. Thus, utility is maximized at a point where the budget line is tangent to an indifference curve. The fact that $\lambda = u'_x(x, y)/p = u'_y(x, y)/q$ at point P means that the marginal utility per dollar is the same for both goods. At any other point (x, y) where, for example, $u'_x(x, y)/p > u'_y(x, y)/q$, the consumer can increase utility by shifting expenditure away from y toward x. Indeed, then the increase in utility per extra dollar spent on x would equal $u'_x(x, y)/p$; this exceeds the decrease in utility per dollar reduction in the amount spent on y, which equals $u'_y(x, y)/q$.

As in Example 14.1.3, the optimal choices of x and y can be expressed as *demand functions* of (p, q, m), which must be homogeneous of degree zero in the three variables together.

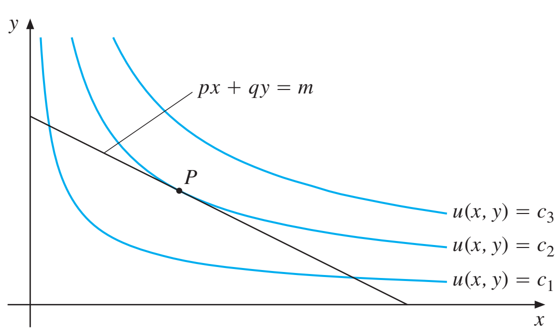

Figure 14.1.1 Assuming that $c_1 < c_2 < c_3$, the solution to problem (14.1.4) is at P

[6] See Section 12.3 to recall how to compute these slopes.

EXERCISES FOR SECTION 14.1[7]

1. Consider the problem: $\max xy$ s.t. $x + 3y = 24$.
   (a) Use Lagrange's method to find its only possible solution.
   (b) Check the solution by using the results in Example 14.1.3.

2. Use the Lagrange multiplier method to solve the problem

$$\min -40Q_1 + Q_1^2 - 2Q_1Q_2 - 20Q_2 + Q_2^2 \text{ s.t. } Q_1 + Q_2 = 15$$

3. Use the results in Example 14.1.3 to solve the following problems:
   (a) $\max 10x^{1/2}y^{1/3}$ s.t. $2x + 4y = m$
   (b) $\max x^{1/2}y^{1/2}$ s.t. $50\,000x + 0.08y = 1\,000\,000$
   (c) $\max 12x\sqrt{y}$ s.t. $3x + 4y = 12$

SM 4. Solve the following problems:
   (a) $\min f(x, y) = x^2 + y^2$ s.t. $g(x, y) = x + 2y = 4$
   (b) $\min f(x, y) = x^2 + 2y^2$ s.t. $g(x, y) = x + y = 12$
   (c) $\max f(x, y) = x^2 + 3xy + y^2$ s.t. $g(x, y) = x + y = 100$

5. A person has utility function $u(x, y) = 100xy + x + 2y$. Suppose that the price per unit of x is \$2, and that the price per unit of y is \$4. The person receives \$1 000 that all has to be spent on the two commodities x and y. Solve the utility maximization problem.

6. An individual has a Cobb–Douglas utility function $U(m, l) = Am^a l^b$, where m is income and l is leisure, and A, a, and b are positive constants, with $a + b \le 1$. A total of T_0 hours are to be allocated between work W and leisure l, so that $W + l = T_0$. If the hourly wage is w, then $m = wW$, and the individual's problem is

$$\max Am^a l^b \text{ s.t. } \frac{m}{w} + l = T_0$$

   Solve the problem by using (∗∗) in Example 14.1.3.

7. Solve part (b) of Review Exercise 13.3 by using the Lagrange method.

8. A firm produces and sells two commodities. By selling x tons of the first commodity the firm gets a price per ton given by $p = 96 - 4x$. By selling y tons of the other commodity the price per ton is given by $q = 84 - 2y$. The total cost of producing and selling x tons of the first commodity and y tons of the second is given by $C(x, y) = 2x^2 + 2xy + y^2$.
   (a) Show that the firm's profit function is $P(x, y) = -6x^2 - 3y^2 - 2xy + 96x + 84y$.
   (b) Compute the first-order partial derivatives of P, and find its only critical point.
   (c) Suppose that the firm's production activity causes so much pollution that the authorities limit its output to 11 tons in total. Solve the firm's maximization problem in this case. Verify that the production restrictions do reduce the maximum possible value of $P(x, y)$.

[7] All the following exercises have only one solution candidate, which is the optimal solution.

SM 9. Consider the utility maximization problem $\max x^a + y$ s.t. $px + y = m$, where the constants p, q, and m are positive, and the constant $a \in (0, 1)$.

(a) Find the demand functions, $x^*(p, m)$ and $y^*(p, m)$.

(b) Find the partial derivatives of the demand functions w.r.t. p and m, and check their signs.

(c) How does the optimal expenditure on the x good vary with p?[8]

(d) Put $a = 1/2$. What are the demand functions in this case? Denote the maximal utility as a function of p and m by $U^*(p, m)$, the value function, also called the indirect utility function. Verify that $\partial U^*/\partial p = -x^*(p, m)$.

SM 10. [HARDER] Consider the problem $\max U(x, y) = 100 - e^{-x} - e^{-y}$ s.t. $px + qy = m$.

(a) Write down the first-order conditions for the problem and solve them for x, y, and λ as functions of p, q, and m. What assumptions are needed for x and y to be nonnegative?

(b) Verify that x and y are homogeneous of degree 0 as functions of p, q, and m.

14.2 Interpreting the Lagrange Multiplier

Consider again the problem

$$\max(\min) f(x, y) \quad \text{s.t.} \quad g(x, y) = c$$

and suppose x^* and y^* are the values of x and y that solve this problem. In general, x^* and y^* depend on c, so we write $x^* = x^*(c)$ and $y^* = y^*(c)$. We *assume* that these solutions are differentiable functions of c. The associated value of $f(x, y)$ is then also a function of c, with

$$f^*(c) = f(x^*(c), y^*(c)) \tag{14.2.1}$$

Here $f^*(c)$ is called the (optimal) *value function* for the problem. Of course, the associated value of the Lagrange multiplier also depends on c, in general, so we write $\lambda(c)$. Provided that certain regularity conditions are satisfied, we have the remarkable result that

$$\frac{\mathrm{d}f^*(c)}{\mathrm{d}c} = \lambda(c) \tag{14.2.2}$$

Thus, *the Lagrange multiplier $\lambda = \lambda(c)$ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant c.*

In particular, if $\mathrm{d}c$ is a small change in c, then

$$f^*(c + \mathrm{d}c) - f^*(c) \approx \lambda(c)\,\mathrm{d}c \tag{14.2.3}$$

In economic applications, c often denotes the available stock of some resource, and $f(x, y)$ denotes utility or profit. Then $\lambda(c)\,\mathrm{d}c$ measures the approximate change in utility or profit that can be obtained from $\mathrm{d}c$ units more.[9] Economists call λ a *shadow price* of the resource.

[8] Check the elasticity of $px^*(p, m)$ w.r.t. p.

[9] Or $-\mathrm{d}c$ units less, when $\mathrm{d}c < 0$.

If $f^*(c)$ is the maximum profit when the resource input is c, then Eq. (14.2.3) says that λ indicates the approximate increase in profit per unit increase in the resource.

Assuming that $f^*(c)$ is differentiable, we can prove Eq. (14.2.2) as follows:

Taking the differential of the value function defined by Eq. (14.2.1) gives

$$\mathrm{d}f^*(c) = \mathrm{d}f(x^*, y^*) = f_1'(x^*, y^*)\,\mathrm{d}x^* + f_2'(x^*, y^*)\,\mathrm{d}y^* \tag{$*$}$$

But from the first-order conditions we have $f_1'(x^*, y^*) = \lambda g_1'(x^*, y^*)$ and $f_2'(x^*, y^*) = \lambda g_2'(x^*, y^*)$, so $(*)$ can be written as

$$\begin{aligned}\mathrm{d}f^*(c) &= \lambda g_1'(x^*, y^*)\,\mathrm{d}x^* + \lambda g_2'(x^*, y^*)\,\mathrm{d}y^* \\ &= \lambda[g_1'(x^*, y^*)\,\mathrm{d}x^* + g_2'(x^*, y^*)\,\mathrm{d}y^*] \qquad (**)\end{aligned}$$

Moreover, taking the differential of the identity $g(x^*(c), y^*(c)) = c$ yields

$$\mathrm{d}g(x^*, y^*) = g_1'(x^*, y^*)\,\mathrm{d}x^* + g_2'(x^*, y^*)\,\mathrm{d}y^* = \mathrm{d}c$$

Substituting the last equality in $(**)$ implies that $\mathrm{d}f^*(c) = \lambda\,\mathrm{d}c$.

EXAMPLE 14.2.1 Consider the following generalization of Example 14.1.1:

$$\max xy \quad \text{s.t.} \quad 2x + y = m$$

The first-order conditions again give $y = 2x$ with $\lambda = x$. The constraint now becomes $2x + 2x = m$, so $x = m/4$. In the notation introduced above, the solution is $x^*(m) = m/4$ and $y^*(m) = m/2$, with $\lambda(m) = m/4$. The value function is therefore $f^*(m) = (m/4)(m/2) = m^2/8$. It follows that $\mathrm{d}f^*(m)/\mathrm{d}m = m/4 = \lambda(m)$. Hence, (14.2.2) is confirmed. Suppose in particular that $m = 100$, so that $f^*(100) = 100^2/8$. If $m = 100$ increases by 1, the new value is $f^*(101) = 101^2/8$, so $f^*(101) - f^*(100) = 101^2/8 - 100^2/8 = 25.125$. Note that formula (14.2.3) with $\mathrm{d}c = 1$ gives $f^*(101) - f^*(100) \approx \lambda(100) \cdot 1 = 25 \cdot 1 = 25$, which is quite close to the exact value, 25.125.

EXAMPLE 14.2.2 Suppose $Q = F(K, L)$ denotes the output of a state-owned firm when the input of capital is K and that of labour is L. Suppose the prices of capital and labour are r and w, respectively, and that the firm is given a total budget of m to spend on the two input factors. The firm wishes to find the choice of inputs it can afford that maximizes output. So it faces the problem

$$\max F(K, L) \quad \text{s.t.} \quad rK + wL = m$$

Solving this problem by using Lagrange's method, the value of the Lagrange multiplier will tell us approximately the increase in output if m is increased by 1 dollar.

Consider, for example, the specific problem $\max 120KL$ s.t. $2K + 5L = m$. Note that this is, mathematically, a special case of the problem in Example 14.1.3.[10] From $(**)$ in Example 14.1.3, we find the solution $K^* = m/4$ and $L^* = m/10$, with $\lambda = 6m$. The optimal output is

$$Q^*(m) = 120K^*L^* = 120 \cdot \frac{1}{4}m \cdot \frac{1}{10}m = 3m^2$$

so $\mathrm{d}Q^*/\mathrm{d}m = 6m = \lambda$, and (14.2.2) is confirmed.

[10] Only the notation is different, along with the fact that the consumer has been replaced with a firm.

EXERCISES FOR SECTION 14.2

1. Verify that Eq. (14.2.2) holds for the problem $\max x^3 y$ s.t. $2x + 3y = m$.

2. With reference to Example 14.1.2:

   (a) Solve the problem $\min rK + wL$ s.t. $\sqrt{K} + L = Q$, assuming that $Q > w/2r$, where r, w, and Q are positive constants.

   (b) Verify Eq. (14.2.2).

3. Consider the problem $\min x^2 + y^2$ s.t. $x + 2y = a$, where a is a constant.

   (a) Solve the problem by transforming it into an unconstrained optimization problem with one variable.

   (b) Show that the Lagrange method leads to the same solution, and verify Eq. (14.2.2).

   (c) Explain the solution by studying the level curves of $f(x, y) = x^2 + y^2$ and the graph of the straight line $x + 2y = a$. Can you give a geometric interpretation of the problem? Does the corresponding maximization problem have a solution?

SM 4. Consider the utility maximization problem $\max U(x, y) = \sqrt{x} + y$ s.t. $x + 4y = 100$.

   (a) Using the Lagrange method, find the quantities demanded of the two goods.

   (b) Suppose income increases from 100 to 101. What is the exact increase in the optimal value of $U(x, y)$? Compare with the value found in (a) for the Lagrange multiplier.

   (c) Suppose we change the budget constraint to $px + qy = m$, but keep the same utility function. Derive the quantities demanded of the two goods if $m > q^2/4p$.

SM 5. Consider the consumer demand problem

$$\max U(x, y) = \alpha \ln(x - a) + \beta \ln(y - b) \quad \text{s.t.} \quad px + qy = m \tag{$*$}$$

where α, β, a, b, p, q, and m are positive constants, with $\alpha + \beta = 1$ and $m > ap + bq$.

   (a) Show that if x^*, y^* solve problem $(*)$, then expenditure on the two goods is given by the two linear functions

$$px^* = \alpha m + pa - \alpha(pa + qb) \quad \text{and} \quad qy^* = \beta m + qb - \beta(pa + qb) \tag{$**$}$$

   of the variables (m, p, q).[11]

   (b) Let $U^*(p, q, m) = U(x^*, y^*)$ denote the indirect utility function. Show that $\partial U^*/\partial m > 0$ and verify the so-called Roy's identities:

$$\frac{\partial U^*}{\partial p} = -\frac{\partial U^*}{\partial m} x^* \quad \text{and} \quad \frac{\partial U^*}{\partial q} = -\frac{\partial U^*}{\partial m} y^*$$

SM 6. [HARDER] An oil producer starts extracting oil from a well at time $t = 0$, and ends at a time $t = T$ that the producer chooses. Suppose that the output flow at any time t in the interval $[0, T]$ is $xt(T - t)$ barrels per unit of time, where the intensity x can also be chosen. The total amount of

[11] This is a special case of the *linear expenditure system* that the Nobel prize winning British economist Richard (J.R.N.) Stone fitted to UK data, as described in the *Economic Journal*, 1954.

oil extracted in the given time span is thus given by the function $g(x, T) = \int_0^T xt(T - t)\,\mathrm{d}t$ of x and T. Assume further that the sales price per barrel at time t is $p = 1 + t$, and that the cost per barrel extracted is equal to αT^2, where α is a positive constant. The profit per unit of time is then $(1 + t - \alpha T^2)xt(T - t)$, so that the total profit earned during the time interval $[0, T]$ is a function of x and T given by

$$f(x, T) = \int_0^T (1 + t - \alpha T^2)\, xt\,(T - t)\,\mathrm{d}t$$

If the total amount of extractable oil in the field is M barrels, the producer can choose values of x and T such that $g(x, T) = M$. The producer's problem is thus

$$\max f(x, T) \quad \text{s.t.} \quad g(x, T) = M \tag{$*$}$$

Find explicit expressions for $f(x, T)$ and $g(x, T)$ by calculating the given integrals. Then solve problem $(*)$ and verify Eq. (14.2.2).

14.3 Multiple Solution Candidates

In all our examples and problems so far, the recipe for solving constrained optimization problems has produced only one solution candidate. In this section we consider a problem where there are several of them. In such cases, we have to decide which of the candidates actually solves the problem, assuming it has any solution at all.

EXAMPLE 14.3.1 Solve the problems

$$\max(\min)\ f(x, y) = x^2 + y^2 \quad \text{s.t.} \quad g(x, y) = x^2 + xy + y^2 = 3$$

Solution: For both the maximization and the minimization problems, the Lagrangian is

$$\mathcal{L}(x, y) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$$

so the three FOCs to consider are

$$\mathcal{L}'_1(x, y) = 2x - \lambda(2x + y) = 0 \tag{i}$$

$$\mathcal{L}'_2(x, y) = 2y - \lambda(x + 2y) = 0 \tag{ii}$$

$$x^2 + xy + y^2 - 3 = 0 \tag{iii}$$

Let us eliminate λ from (i) and (ii). From (i) we get $\lambda = 2x/(2x + y)$ provided that $y \neq -2x$. Inserting this value of λ into (ii) gives

$$2y = \frac{2x}{2x + y}(x + 2y)$$

This reduces to $y^2 = x^2$, and so $y = \pm x$, which leaves us with three possibilities:

1. *Suppose, first, that* $y = x$. Then, (iii) yields $x^2 = 1$, so $x = 1$ or $x = -1$. This gives the two solution candidates $(x, y) = (1, 1)$ and $(-1, -1)$, with $\lambda = 2/3$.

2. *Alternatively, suppose* $y = -x$. Then (iii) yields $x^2 = 3$, so $x = \sqrt{3}$ or $x = -\sqrt{3}$. This gives the two solution candidates $(x, y) = (\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$, with $\lambda = 2$.
3. It only remains to consider the case $y = -2x$. Then from (i) we have $x = 0$ and so $y = 0$. But this contradicts (iii), so this case cannot occur.

We have found the only four points (x, y) that can solve the problem. Furthermore,

$$f(1, 1) = f(-1, -1) = 2 \quad \text{and} \quad f(\sqrt{3}, -\sqrt{3}) = f(-\sqrt{3}, \sqrt{3}) = 6$$

We conclude that if the problem has solutions, then $(1, 1)$ and $(-1, -1)$ solve the minimization problem, whereas $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ solve the maximization problem.

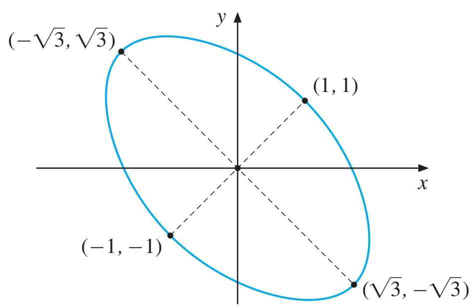

Figure 14.3.1 The constraint curve in Example 14.3.1

Geometrically, the equality constraint determines an ellipse. The problem is therefore to find what points on the ellipse are nearest to or furthest from the origin. See Fig. 14.3.1; it is "geometrically obvious" that such points exist.

EXERCISES FOR SECTION 14.3

SM 1. Solve the problems:

(a) $\max(\min)\, 3xy$ s.t. $x^2 + y^2 = 8$ (b) $\max(\min)\, x + y$ s.t. $x^2 + 3xy + 3y^2 = 3$

SM 2. Solve the problems:[12]

(a) $\max\, x^2 + y^2 - 2x + 1$ s.t. $x^2 + 4y^2 = 16$ (b) $\min\, \ln(2 + x^2) + y^2$ s.t. $x^2 + 2y = 2$

3. Consider the problem $\max(\min) f(x, y) = x + y$ s.t. $g(x, y) = x^2 + y = 1$.

(a) Find the solutions to the necessary conditions for these problems.

(b) Explain the solution geometrically by drawing appropriate level curves for $f(x, y)$ together with the graph of the parabola $x^2 + y = 1$. Does the associated minimization problem have a solution?

[12] In (b) you should take it for granted that the minimum value exists.

(c) Replace the constraint by $x^2 + y = 1.1$, and solve the problem in this case. Find the corresponding change in the optimal value of $f(x, y) = x + y$, and check to see if this change is approximately equal to $\lambda \cdot 0.1$, as suggested by Eq. (14.2.3).

SM **4.** Consider the problem $\max f(x, y) = 24x - x^2 + 16y - 2y^2$ s.t. $g(x, y) = x^2 + 2y^2 = 44$.

(a) Solve the problem.

(b) What is the approximate change in the optimal value of $f(x, y)$ if 44 is changed to 45?

14.4 Why the Lagrange Method Works

We have explained the Lagrange multiplier method for solving the problem

$$\max f(x, y) \ \text{ s.t. } \ g(x, y) = c \tag{14.4.1}$$

In this section we give a geometric as well as an analytic argument for the method.

A Geometric Argument

The maximization problem in (14.4.1) can be given a geometric interpretation, as shown in Fig. 14.4.1. Here the graph of f is like the surface of an inverted bowl, whereas the equation $g(x, y) = c$ represents a curve in the xy-plane. The curve K on the bowl is the one that lies directly above the curve $g(x, y) = c$. Maximizing $f(x, y)$ without taking the constraint into account gets us to the peak A in Fig. 14.4.1. The solution to problem (14.4.1), however, is at B, which is the highest point on the curve K. If we think of the graph of f as representing a mountain, and K as a mountain path, then we seek the highest point on the path, which is B. Analytically, the problem is to find the coordinates of B.

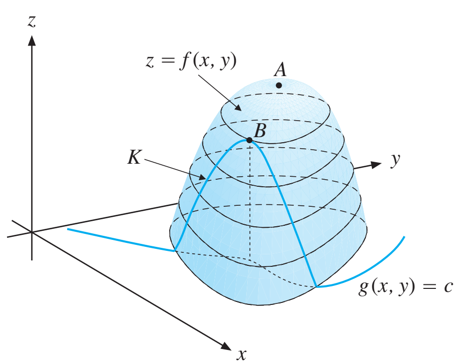

Figure 14.4.1 A constrained optimization problem

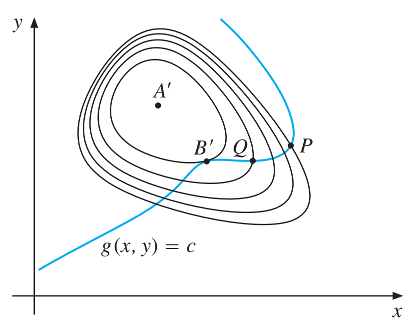

Figure 14.4.2 Geometry of the Lagrange method

Figure 14.4.2 "projects" the information of Fig. 14.4.1 into the xy-plane. The curve $g(x, y) = c$, which appeared in Fig. 14.4.1 too, is the projection of curve K. The figure also shows some of the level curves for f, and also indicates the constraint curve $g(x, y) = c$. Now A' represents the point at which $f(x, y)$ has its unconstrained maximum. The closer a level curve of f is to point A', the higher is the value of f along that level curve. We are seeking that point on the constraint curve $g(x, y) = c$ where f has its highest value. If we start at point P on the constraint curve and move along that curve toward A', we encounter level curves with higher and higher values of f.

Obviously, the point Q indicated in Fig. 14.4.2 is *not* the point on $g(x, y) = c$ at which f has its highest value, because the constraint curve passes *through* the level curve of f at that point. Therefore, we can cross a level curve to higher values of f by proceeding further along the constraint curve. However, when we reach point B', we cannot go any higher. It is intuitively clear that B' is the point where the constraint curve touches a level curve for f, *without intersecting it*. This observation implies that the slope of the tangent to the curve $g(x, y) = c$ at (x, y) is equal to the slope of the tangent to the level curve of f at that point.

Recall from Section 12.3 that the slope of the level curve $F(x, y) = c$ is given by $\mathrm{d}y/\mathrm{d}x = -F'_1(x, y)/F'_2(x, y)$. Thus, the condition that the slope of the tangent to $g(x, y) = c$ is equal to the slope of a level curve for $f(x, y)$ can be expressed analytically as:[13]

$$-\frac{g'_1(x, y)}{g'_2(x, y)} = -\frac{f'_1(x, y)}{f'_2(x, y)}$$

or, equivalently, as

$$\frac{f'_1(x, y)}{g'_1(x, y)} = \frac{f'_2(x, y)}{g'_2(x, y)} \tag{14.4.2}$$

It follows that a necessary condition for (x, y) to solve problem (14.4.1) is that the left- and right-hand sides of the last equation in (14.4.2) be equal at (x, y). Let λ denote the common value of these fractions. This is the Lagrange multiplier introduced in Section 14.1. With this definition,

$$f'_1(x, y) - \lambda g'_1(x, y) = 0 \quad \text{and} \quad f'_2(x, y) - \lambda g'_2(x, y) = 0 \tag{14.4.3}$$

Using the Lagrangian from Eq. (14.1.2), we see that (14.4.3) just tells us that the Lagrangian has a critical point. An analogous argument for the problem of minimizing $f(x, y)$ subject to $g(x, y) = c$ gives the same condition.

The geometric argument above is quite convincing. But the analytic argument that follows is easier to generalize to more than two variables.

An Analytic Argument

So far we have studied the problem of finding the largest or smallest value of $f(x, y)$ subject to the constraint $g(x, y) = c$. Sometimes we are interested in studying the corresponding local maximum (minimum), in the same sense as in Section 13.3: points (x_0, y_0) where $g(x_0, y_0) = c$ and such that $f(x, y) \le (\ge) f(x_0, y_0)$ for all pairs (x, y) that satisfy $g(x, y) = c$ and lie sufficiently close to (x_0, y_0). Graphically, possible local extrema are illustrated in Fig. 14.4.3.

[13] Disregard for the moment cases where any denominator is 0.

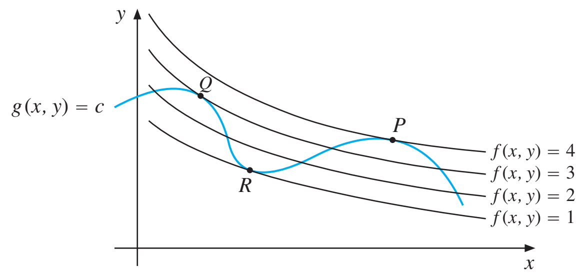

Figure 14.4.3 Q, R, and P all satisfy the first-order conditions

Point R is a local minimum point for $f(x,y)$ subject to $g(x,y)=c$, whereas Q and P are local maximum points. The global maximum of $f(x,y)$ subject to $g(x,y)=c$ is attained only at P. Each of the points P, Q, and R in Fig. 14.4.3 satisfies condition (14.4.3), so the first-order conditions are exactly as before. Let us derive them now in a way that does not rely on geometric intuition.

Except in some special cases, the equation $g(x,y)=c$ with c fixed defines y implicitly as a differentiable function of x near any local extreme point. Denote this function by $y=h(x)$. According to formula (12.3.2), provided that $g'_2(x,y)\neq 0$, one has

$$y' = h'(x) = -\frac{g'_1(x,y)}{g'_2(x,y)}$$

Now, the objective function $z=f(x,y)=f(x,h(x))$ is, in effect, a function of x alone. By calculating $\mathrm{d}z/\mathrm{d}x$ while taking into account how y depends on x, we obtain a necessary condition for local extreme points by equating $\mathrm{d}z/\mathrm{d}x$ to 0. But

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f'_1(x,y)+f'_2(x,y)y' = f'_1(x,y)+f'_2(x,y)h'(x)$$

So substituting the previous expression for $h'(x)$ gives the following necessary condition for (x,y) to solve problem (14.4.1):

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f'_1(x,y) - f'_2(x,y)\frac{g'_1(x,y)}{g'_2(x,y)} = 0 \tag{14.4.4}$$

Assuming that $g'_2(x,y)\neq 0$, and defining $\lambda = f'_2(x,y)/g'_2(x,y)$, we deduce that the two equations $f'_1(x,y)-\lambda g'_1(x,y)=0$ and $f'_2(x,y)-\lambda g'_2(x,y)=0$ must both be satisfied. Hence, the Lagrangian must have a critical point at (x,y). The same result holds, by an analogous argument, provided $g'_1(x,y)\neq 0$. To summarize, one can prove the following precise result:

THEOREM 14.4.1 (LAGRANGE'S THEOREM)

Suppose that $f(x,y)$ and $g(x,y)$ have continuous partial derivatives in a domain A of the xy-plane, and that (x_0,y_0) is both an interior point of A and a local extreme point for $f(x,y)$ subject to the constraint $g(x,y)=c$. Suppose further that $g'_1(x_0,y_0)$ and $g'_2(x_0,y_0)$ are not both 0. Then, there exists a unique number λ such that the Lagrangian has a critical point at (x_0,y_0).

Exercise 3 asks you to show how trouble can result from uncritical use of the Lagrange multiplier method, disregarding the assumptions in Theorem 14.4.1. Exercise 4 asks you to show what can go wrong if $g_1'(x_0, y_0)$ and $g_2'(x_0, y_0)$ are both 0.

In constrained optimization problems in economics, it is often implicitly assumed that the variables are nonnegative. This was certainly the case for the specific utility maximization problem in Example 14.1.3. Because the optimal solutions were positive, nothing was lost by disregarding the nonnegativity constraints. Here is an example showing that sometimes we must take greater care.

EXAMPLE 14.4.1 Consider the utility maximization problem

$$\max xy + x + 2y \text{ s.t. } 2x + y = m, \; x \geq 0 \text{ and } y \geq 0$$

where we have required that the amount of each good is nonnegative. The Lagrangian is $\mathcal{L} = xy + x + 2y - \lambda(2x + y - m)$. So the first-order conditions, disregarding the nonnegativity constraints for the moment, are

$$\mathcal{L}_1' = y + 1 - 2\lambda = 0 \text{ and } \mathcal{L}_2'(x, y) = x + 2 - \lambda = 0$$

By eliminating λ, we find that $y = 2x + 3$. Inserting this into the budget constraint gives $2x + 2x + 3 = m$, so $x = \frac{1}{4}(m - 3)$. We easily find the corresponding value of y, and the suggested solution that emerges is $x^* = \frac{1}{4}(m - 3)$, $y^* = \frac{1}{2}(m + 3)$. Note that in the case when $m < 3$, then $x^* < 0$, so that the expressions we have found for x^* and y^* do not solve the given problem. The solution in this case is, as shown below, $x^* = 0$, $y^* = m$. This implies that when income is low, this consumer would spend everything on just the second commodity.

Let us analyse the problem by converting it to one that is unconstrained. To do this, note how the constraint implies that $y = m - 2x$. In order for both x and y to be nonnegative, one must have $0 \leq x \leq m/2$ and $0 \leq y \leq m$. Substituting $y = m - 2x$ into the utility function, we obtain utility as function $U(x)$ of x alone, where

$$U(x) = x(m - 2x) + x + 2(m - 2x) = -2x^2 + (m - 3)x + 2m, \; x \in [0, m/2]$$

This is a quadratic function with $x = \frac{1}{4}(m - 3)$ as the critical point. If $m > 3$, it is an interior critical point for the concave function U, so it is a maximum point. If $m \leq 3$, then $U'(x) = -4x + (m - 3) \leq 0$ for all $x \geq 0$. Because of the constraint $x \geq 0$, it follows that $U(x)$ must have its largest value for $x = 0$.

Concerning the Lagrange multiplier method, one of the most frequently occurring errors in the economics literature—even in some leading textbooks—is the claim that it transforms a constrained optimization problem into one of finding an unconstrained optimum of the Lagrangian. Exercise 1 shows that this is wrong. What the method does instead is to transform a constrained optimization problem into one of finding the appropriate critical points of the Lagrangian. Sometimes these are maximum points, but often they are not.

To test your understanding of when the Lagrange procedure can be used, it is a good exercise to explain why it certainly works, for instance, in Exercise 14.4.1, but not in either Exercise 14.4.3 or Exercise 14.4.4.

EXERCISES FOR SECTION 14.4

1. Consider the problem $\max xy$ s.t. $x + y = 2$. Reduce it to the one-variable problem of maximizing $x(2 - x)$, and show that $(x, y) = (1, 1)$ is the only possible solution. Check that this satisfies the first-order conditions for the constrained maximization problem, with Lagrange multiplier $\lambda = 1$. Show that $(1, 1)$ does not maximize the Lagrangian $\mathcal{L}(x, y) = xy - 1 \cdot (x + y - 2)$. Does this matter?

2. The following text, which attempts to justify the Lagrange method, is taken from a book on mathematics for management. It contains *grave* errors. Sort them out.

   "Consider the general problem of finding the extreme points of $z = f(x, y)$ subject to the constraint $g(x, y) = 0$. Clearly the extreme points must satisfy the pair of equations $f'_x(x, y) = 0, f'_y(x, y) = 0$ in addition to the constraint $g(x, y) = 0$. Thus, there are three equations that must be satisfied by the pair of unknowns x, y. Because there are more equations than unknowns, the system is said to be overdetermined and, in general, is difficult to solve. In order to facilitate computation . . . "

3. [HARDER] Consider the problem $\max f(x, y) = 2x + 3y$ s.t. $g(x, y) = \sqrt{x} + \sqrt{y} = 5$.

   (a) Show that the Lagrange multiplier method suggests the solution $(x, y) = (9, 4)$. Show that this does not solve the constrained maximization problem because $f(9, 4) = 30$, and yet $f(25, 0) = 50$.

   (b) Find the true solution to the problem by studying the level curves of $f(x, y) = 2x + 3y$ along with the graph of the constraint equation. (*Hint:* See Exercise 5.4.2.)

   (c) Which assumption of Theorem 14.4.1 is violated?

SM 4. [HARDER] Solve the problem

$$\min f(x, y) = (x + 2)^2 + y^2 \text{ s.t. } g(x, y) = y^2 - x(x + 1)^2 = 0$$

Show that the Lagrange multiplier method cannot locate this minimum. (*Hint*: Draw a graph of $g(x, y) = 0$. Note in particular that $g(-1, 0) = 0$.)

14.5 Sufficient Conditions

Theorem 14.4.1 gives *necessary* conditions for the local solution of constrained optimization problems. In order to confirm that we have really found the solution, however, a more careful check is needed. The examples and problems of Section 14.3 have geometric interpretations which strongly suggest we have found the solution. Indeed, if the constraint set is closed and bounded, then Theorem 13.5.1, the extreme value theorem, guarantees that a continuous function *will* attain both maximum and minimum values over this set.[14]

[14] A case in point is Example 14.3.1. Here the constraint set, which was graphed in Fig. 14.3.1, *is* closed and bounded. The continuous function $f(x, y) = x^2 + y^2$ will therefore attain both a maximum value and a minimum value over the constraint set. Since there are four points satisfying the first-order conditions, it remains only to check which of them gives f its highest and lowest values.

Concave/Convex Lagrangian

We already know that if (x_0, y_0) solves problem

$$\max\ (\min)\ f(x, y)\ \text{ s.t. }\ g(x, y) = c \tag{14.5.1}$$

then the Lagrangian

$$\mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c] \tag{14.5.2}$$

usually has a critical point at (x_0, y_0), but $\mathcal{L}$ may not have a maximum (minimum) at (x_0, y_0). Suppose, however, that $\mathcal{L}$ happens to reach a global maximum at (x_0, y_0), in the sense that (x_0, y_0) maximizes $\mathcal{L}(x, y)$ among *all* (x, y) in the plane. Then, for all (x, y), one has

$$\mathcal{L}(x_0, y_0) = f(x_0, y_0) - \lambda[g(x_0, y_0) - c] \geq \mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c] \tag{$*$}$$

If (x_0, y_0) also satisfies the constraint $g(x_0, y_0) = c$, then $(*)$ implies that $f(x_0, y_0) \geq f(x, y)$ for all (x, y) such that $g(x, y) = c$. Hence, (x_0, y_0) really does solve the maximization problem in (14.5.1). A corresponding result is obtained for the minimization problem in (14.5.1), provided that $\mathcal{L}$ reaches a global minimum at (x_0, y_0).

Next, we recall from Theorem 13.2.1 and the definitions of concave and convex functions, that a critical point (x_0, y_0) for a concave (convex) function really does maximize (minimize) the function. Thus we have the following result:

THEOREM 14.5.1 (CONCAVE/CONVEX LAGRANGIAN)

Consider the problems in (14.5.1) and suppose (x_0, y_0) is a critical point for the Lagrangian $\mathcal{L}$, defined in (14.5.2).

(A) If the Lagrangian is concave, then (x_0, y_0) solves the maximization problem.

(B) If the Lagrangian is convex, then (x_0, y_0) solves the minimization problem.

EXAMPLE 14.5.1 Consider a firm that uses positive inputs K and L of capital and labour, respectively, to produce a single output Q according to the Cobb–Douglas production function $Q = F(K, L) = AK^aL^b$, where A, a, and b are positive parameters satisfying $a + b \leq 1$. Suppose that the unit prices per unit of capital and labour are $r > 0$ and $w > 0$, respectively. The cost-minimizing inputs of K and L must solve the problem

$$\min\ rK + wL\ \text{ s.t. }\ AK^aL^b = Q$$

Explain why the Lagrangian is convex, so that a critical point of the Lagrangian must minimize costs. (*Hint*: See Exercise 13.2.8.)

Solution: The Lagrangian is $\mathcal{L} = rK + wL - \lambda(AK^aL^b - Q)$, and the first-order conditions are $r = \lambda AaK^{a-1}L^b$ and $w = \lambda AbK^aL^{b-1}$, implying that $\lambda > 0$. From Exercise 13.2.8, we see that $-\mathcal{L}$ is concave, so $\mathcal{L}$ is convex.

Local Second-Order Conditions

Sometimes we are interested in conditions that are sufficient for (x_0, y_0) to be a local extreme point of $f(x, y)$ subject to $g(x, y) = c$. We start by looking at the expression for $\mathrm{d}z/\mathrm{d}x$ given by Eq. (14.4.4). The condition $\mathrm{d}z/\mathrm{d}x = 0$ is necessary for local optimality. If, additionally, $\mathrm{d}^2z/\mathrm{d}x^2 < 0$, then a critical point of the Lagrangian must solve the local maximization problem. The second derivative $\mathrm{d}^2z/\mathrm{d}x^2$ is just the total derivative of $\mathrm{d}z/\mathrm{d}x$ w.r.t. x. Assuming that both f and g are C^2 functions, and recalling that y is a function of x, it follows from (14.4.4) that

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = f''_{11} + f''_{12}y' - (f''_{21} + f''_{22}y')\frac{g'_1}{g'_2} - f'_2\frac{(g''_{11} + g''_{12}y')g'_2 - (g''_{21} + g''_{22}y')g'_1}{(g'_2)^2}$$

But f and g are C^2 functions, so $f''_{12} = f''_{21}$ and $g''_{12} = g''_{21}$. Moreover, $y' = -g'_1/g'_2$. Also $f'_1 = \lambda g'_1$ and $f'_2 = \lambda g'_2$, because these are the first-order conditions. Using these relationships to eliminate y' and f'_2, as well as some elementary algebra, we obtain

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = \frac{1}{(g'_2)^2}\left[(f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12})g'_1g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2\right]$$

We see that $\mathrm{d}^2z/\mathrm{d}x^2 < 0$ provided the expression in the square brackets is negative. Thus, we have the following result:

THEOREM 14.5.2 (LOCAL SECOND-ORDER CONDITIONS)

Consider the problems in Eq. (14.5.1), and suppose that (x_0, y_0) satisfies the first-order conditions $f'_1(x, y) = \lambda g'_1(x, y)$, $f'_2(x, y) = \lambda g'_2(x, y)$ and $g(x, y) = c$. Define

$$D(x, y, \lambda) = (f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12})g'_1g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2$$

Then,

(i) If $D(x_0, y_0, \lambda) < 0$, then (x_0, y_0) solves the maximization problem locally.

(ii) If $D(x_0, y_0, \lambda) > 0$, then (x_0, y_0) solves the minimization problem locally.

The conditions on the sign of $D(x_0, y_0, \lambda)$ are called the local *second-order conditions*.

EXAMPLE 14.5.2 Consider the problem

$$\max(\min)\ f(x, y) = x^2 + y^2 \quad \text{s.t.} \quad g(x, y) = x^2 + xy + y^2 = 3$$

In Example 14.3.1 we saw that the first-order conditions give the points $(1, 1)$ and $(-1, -1)$ with $\lambda = 2/3$, as well as $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ with $\lambda = 2$. Check the local second-order conditions of Theorem 14.5.2 in this case.

Solution: We find that $f''_{11} = 2, f''_{12} = 0, f''_{22} = 2, g''_{11} = 2, g''_{12} = 1$, and $g''_{22} = 2$. So

$$D(x, y, \lambda) = (2 - 2\lambda)(x + 2y)^2 + 2\lambda(2x + y)(x + 2y) + (2 - 2\lambda)(2x + y)^2$$

Hence $D(1, 1, \frac{2}{3}) = D(-1, -1, \frac{2}{3}) = 24$ and $D(\sqrt{3}, -\sqrt{3}, 2) = D(-\sqrt{3}, \sqrt{3}, 2) = -24$. From the signs of D at the four points satisfying the first-order conditions, we conclude that $(1, 1)$ and $(-1, -1)$ are local minimum points, whereas $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ are local maximum points.[15]

As with Eq. (12.3.4), using the concept of 3×3 determinants that we will study in Section 16.2, the rather lengthy expression $D(x, y, \lambda)$ can be written in a symmetric form that is easier to remember. See Example 16.2.2, in particular Eq. 16.2.4.

EXERCISES FOR SECTION 14.5

1. Use Theorem 14.5.1 to check that the optimal solution is found in part (a) of Exercise 14.1.3.

2. Consider the problem $\max \ln x + \ln y$ s.t. $px + qy = m$. Compute $D(x, y, \lambda)$, as defined in Theorem 14.5.2, and verify that the second-order conditions in that theorem are satisfied.[16]

3. Compute $D(x, y, \lambda)$ in Theorem 14.5.2 for Problem 14.2.3(a). Conclusion?

SM 4. Prove that $U(x, y) = x^a + y^a$, where $a \in (0, 1)$, is concave when $x > 0$ and $y > 0$. Then, solve the problem $\max U(x, y)$ s.t. $px + qy = m$, where p, q, and m are positive constants.

14.6 Additional Variables and Constraints

Constrained optimization problems in economics usually involve more than just two variables. The typical problem with n variables can be written in the form

$$\max(\min) f(x_1, \dots, x_n) \text{ s.t. } g(x_1, \dots, x_n) = c \tag{14.6.1}$$

The Lagrange multiplier method presented in the previous sections can be easily generalized. As before, associate a Lagrange multiplier λ with the constraint and form the Lagrangian

$$\mathcal{L}(x_1, \dots, x_n) = f(x_1, \dots, x_n) - \lambda[g(x_1, \dots, x_n) - c] \tag{14.6.2}$$

[15] In Example 14.3.1 we proved that these points were actually *global* extrema.

[16] Note that the Lagrangian is concave, as is easily checked, so the unique solution $(x, y) = (m/p, m/2q)$ to the first-order conditions is actually a global constrained maximum for this problem.

Next, find all the first-order partial derivatives of $\mathcal{L}$ and equate them to zero, so that

$$\begin{aligned} \mathcal{L}'_1 &= f'_1(x_1, \dots, x_n) - \lambda g'_1(x_1, \dots, x_n) = 0 \\ &\vdots \\ \mathcal{L}'_n &= f'_n(x_1, \dots, x_n) - \lambda g'_n(x_1, \dots, x_n) = 0 \end{aligned} \tag{14.6.3}$$

These n equations, together with the constraint, form $n + 1$ equations that should be solved simultaneously to determine the $n + 1$ unknowns: $x_1, \dots, x_n$, and λ.

This method will, in general, fail to give correct necessary conditions if all the first-order partial derivatives of $g(x_1, \dots, x_n)$ vanish at the critical point of the Lagrangian. Otherwise, the proof is an easy generalization of the analytic argument in Section 14.4 for the first-order conditions. If, say, $\partial g/\partial x_n \neq 0$, we "solve" $g(x_1, \dots, x_n) = c$ for x_n near the critical point, and thus reduce the problem to an unconstrained extremum problem in the remaining $n - 1$ variables $x_1, \dots, x_{n-1}$.

EXAMPLE 14.6.1 Solve the consumer's demand problem

$$\max U(x, y, z) = x^2y^3z \quad \text{s.t.} \quad x + y + z = 12$$

Solution: With $\mathcal{L}(x, y, z) = x^2y^3z - \lambda(x + y + z - 12)$, the first-order conditions are

$$\mathcal{L}'_1 = 2xy^3z - \lambda = 0, \ \mathcal{L}'_2 = 3x^2y^2z - \lambda = 0, \ \text{ and } \ \mathcal{L}'_3 = x^2y^3 - \lambda = 0 \tag{$*$}$$

If *any* of the variables x, y, and z is 0, then $x^2y^3z = 0$, which is *not* the maximum value. So suppose that x, y, and z are all positive. From the two first equations in $(*)$, we have $2xy^3z = 3x^2y^2z$, so $y = 3x/2$. The first and third equations in $(*)$ likewise imply that $z = x/2$. Inserting $y = 3x/2$ and $z = x/2$ into the constraint yields $x + 3x/2 + x/2 = 12$, so $x = 4$. Then $y = 6$ and $z = 2$. Thus, the only possible solution is $(x, y, z) = (4, 6, 2)$.

EXAMPLE 14.6.2 Solve the problem

$$\min f(x, y, z) = (x - 4)^2 + (y - 4)^2 + \left(z - \tfrac{1}{2}\right)^2 \quad \text{s.t.} \quad x^2 + y^2 = z$$

Can you give a geometric interpretation of the problem?

Solution: The Lagrangian is

$$\mathcal{L}(x, y, z) = (x - 4)^2 + (y - 4)^2 + \left(z - \tfrac{1}{2}\right)^2 - \lambda(x^2 + y^2 - z)$$

and the first-order conditions are:

$$\mathcal{L}'_1(x, y, z) = 2(x - 4) - 2\lambda x = 0 \tag{i}$$

$$\mathcal{L}'_2(x, y, z) = 2(y - 4) - 2\lambda y = 0 \tag{ii}$$

$$\mathcal{L}'_3(x, y, z) = 2\left(z - \tfrac{1}{2}\right) + \lambda \quad = 0 \tag{iii}$$

$$x^2 + y^2 = z \tag{iv}$$

From (i) we see that $x = 0$ is impossible. Equation (i) thus gives $\lambda = 1 - 4/x$. Inserting this into (ii) and (iii) gives $y = x$ and $z = 2/x$. Using these results, Eq. (iv) reduces to $2x^2 = 2/x$, that is, $x^3 = 1$, so $x = 1$. It follows that $(x, y, z) = (1, 1, 2)$ is the only solution candidate for the problem.

The expression $(x - 4)^2 + (y - 4)^2 + (z - 1/2)^2$ measures the square of the distance from the point $(4, 4, 1/2)$ to the point (x, y, z). The set of points (x, y, z) that satisfy $z = x^2 + y^2$ is a surface known as a paraboloid, part of which is shown in Fig. 14.6.1. The minimization problem is therefore to find that point on the paraboloid which has the smallest (square) distance from $(4, 4, 1/2)$. It is "geometrically obvious" that this problem has a solution. On the other hand, the problem of finding the largest distance from $(4, 4, 1/2)$ to a point on the paraboloid does not have a solution, because the distance can be made as large as we like.

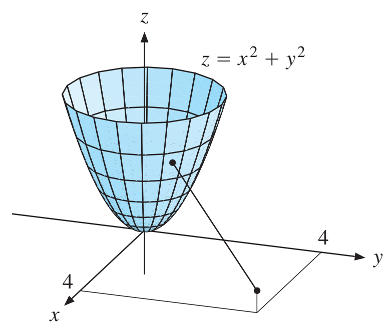

Figure 14.6.1 An illustration of Example 14.6.2

EXAMPLE 14.6.3 The general consumer optimization problem with n goods is

$$\max U(x_1, \dots, x_n) \text{ s.t. } p_1x_1 + \cdots + p_nx_n = m \tag{14.6.4}$$

where function U is defined for $x_1 \geq 0, \dots, x_n \geq 0$. Writing $\mathbf{x} = (x_1, \dots, x_n)$, the Lagrangian is

$$\mathcal{L}(\mathbf{x}) = U(\mathbf{x}) - \lambda(p_1x_1 + \cdots + p_nx_n - m)$$

The first-order conditions are

$$\mathcal{L}'_i(\mathbf{x}) = U'_i(\mathbf{x}) - \lambda p_i = 0$$

for each $i = 1, \dots, n$. By direct computation, we then have

$$\frac{U'_1(\mathbf{x})}{p_1} = \frac{U'_2(\mathbf{x})}{p_2} = \cdots = \frac{U'_n(\mathbf{x})}{p_n} = \lambda \tag{14.6.5}$$

Apart from the last equation, which serves only to determine the Lagrange multiplier λ, we have $n - 1$ equations.[17] In addition, the constraint must hold. Thus, we have n equations to

[17] For $n = 2$, there is one equation; for $n = 3$, there are two equations; and so on.

determine the values of $x_1, \dots, x_n$. From Eq. (14.6.5) it also follows that for every pair of goods j and k

$$\frac{U'_j(\mathbf{x})}{U'_k(\mathbf{x})} = \frac{p_j}{p_k} \tag{14.6.6}$$

The left-hand side is the MRS of good k for good j, whereas the right-hand side is their price ratio, or rate of exchange of good k for good j. So condition (14.6.6) equates the MRS for each pair of goods to the corresponding price ratio.

Consider the equations in (14.6.5), together with the budget constraint. Assume that this system is solved for $x_1, \dots, x_n$ and λ, as functions of $\mathbf{p} = (p_1, \dots, p_n)$ and m, giving $x_i = D_i(\mathbf{p}, m)$, for $i = 1, \dots, n$. Then $D_i(\mathbf{p}, m)$ gives the amount of the i-th commodity demanded by the individual when facing prices $\mathbf{p}$ and income m. For this reason $D_1, \dots, D_n$ are called the *individual demand functions*. By the same argument as in Examples 12.7.4 and 14.1.3, the demand functions are homogeneous of degree 0. As one check that you have correctly derived the demand functions, it is a good idea to verify that the functions you find are indeed homogeneous of degree 0, and satisfy the budget constraint.

In the case when the consumer has a Cobb–Douglas utility function, the constrained maximization problem is

$$\max A x_1^{a_1} \cdots x_n^{a_n} \quad \text{s.t.} \quad p_1 x_1 + \cdots + p_n x_n = m \tag{14.6.7}$$

where we assume that each "taste" parameter $a_i > 0$. Then as in part (a) of Exercise 8, the demand functions are

$$D_i(\mathbf{p}, m) = \frac{a_i}{a_1 + \cdots + a_n} \frac{m}{p_i} \tag{14.6.8}$$

We see how the pattern of the two-variable case in Example 14.1.3 is repeated, with a constant fraction of income m spent on each good, independent of all prices. Note also that the demand for any good i is completely unaffected by changes in the price of any other good. This is an argument against using Cobb–Douglas utility functions, because we expect realistic demand functions to depend on prices of other goods that are either complements or substitutes.

More Constraints

Occasionally economists need to consider optimization problems with more than one equality constraint, although it is much more common to have many inequality constraints. The corresponding general problem is

$$\max(\min) f(x_1, \dots, x_n) \quad \text{s.t.} \quad \begin{cases} g_1(x_1, \dots, x_n) = c_1 \\ \quad \cdots\cdots\cdots \\ g_m(x_1, \dots, x_n) = c_m \end{cases} \tag{14.6.9}$$

The Lagrange multiplier method can be extended to treat problem (14.6.9). To do so, associate a Lagrange multiplier with each constraint, then form the Lagrangian function

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j [g_j(\mathbf{x}) - c_j] \tag{14.6.10}$$

where $\mathbf{x} = (x_1, \dots, x_n)$. Except in special cases, this Lagrangian must be critical at any optimal point, both local and global. That is, its partial derivative w.r.t. each variable x_i must vanish. Hence, for each $i = 1, 2, \dots, n$,

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0 \tag{14.6.11}$$

Together with the m equality constraints, these n equations form a total of $n + m$ equations in the $n + m$ unknowns: $x_1, \dots, x_n$ and $\lambda_1, \dots, \lambda_m$.

EXAMPLE 14.6.4 Solve the problem

$$\min\ x^2 + y^2 + z^2 \ \text{ s.t. } \begin{cases} x + 2y + z = 30 \\ 2x - y - 3z = 10 \end{cases}$$

Solution: The Lagrangian is

$$\mathcal{L}(x, y, z) = x^2 + y^2 + z^2 - \lambda_1(x + 2y + z - 30) - \lambda_2(2x - y - 3z - 10)$$

The first-order conditions (14.6.11) require that

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda_1 - 2\lambda_2 = 0 \tag{i}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y - 2\lambda_1 + \lambda_2 = 0 \tag{ii}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 2z - \lambda_1 + 3\lambda_2 = 0 \tag{iii}$$

in addition to the two constraints,

$$x + 2y + z = 30 \tag{iv}$$

$$2x - y - 3z = 10 \tag{v}$$

So there are five equations, (i) to (v), to determine the five unknowns: x, y, z, λ_1, and λ_2.

Solving (i) and (ii), simultaneously, for λ_1 and λ_2 gives $\lambda_1 = \frac{2}{5}x + \frac{4}{5}y$ and $\lambda_2 = \frac{4}{5}x - \frac{2}{5}y$. Inserting these expressions for λ_1 and λ_2 into (iii) and rearranging yields

$$x - y + z = 0 \tag{vi}$$

This equation, together with (iv) and (v), constitutes a system of three linear equations in the unknowns x, y, and z. Solving this system by elimination gives $(x, y, z) = (10, 10, 0)$. The corresponding values of the multipliers are $\lambda_1 = 12$ and $\lambda_2 = 4$.

A geometric argument allows us to confirm that we have solved the minimization problem. Each of the two constraints represents a plane in $\mathbb{R}^3$, and the points satisfying both constraints consequently lie on the straight line where the two planes intersect. Now $x^2 + y^2 + z^2$ measures (the square of) the distance from the origin to a point on this straight line, which we want to make as small as possible by choosing the point on the line that is nearest to the origin. No maximum distance can possibly exist, but it is geometrically obvious that there is a minimum distance, and it must be attained at this nearest point.

An alternative method to solve this particular problem is to reduce it to a one-variable optimization problem by using the two constraints to get $y = 20 - x$ and $z = x - 10$, the equations of the straight line where the two planes intersect. Then the square of the distance from the origin is

$$x^2 + y^2 + z^2 = x^2 + (20 - x)^2 + (x - 10)^2 = 3(x - 10)^2 + 200$$

and this function is easily seen to have a minimum when $x = 10$. See also Exercise 5.

EXERCISES FOR SECTION 14.6

1. Consider the problem $\min x^2 + y^2 + z^2$ s.t. $x + y + z = 1$.

(a) Write down the Lagrangian for this problem, and find the only point (x, y, z) that satisfies the necessary conditions.

(b) Give a geometric argument for the existence of a solution. Does the corresponding maximization problem have any solution?

2. Use the result in (∗∗) in Example 14.6.3 to solve the utility maximization problem

$$\max\ 10x^{1/2}y^{1/3}z^{1/4} \quad \text{s.t.} \quad 4x + 3y + 6z = 390$$

3. A consumer's demands for three goods are chosen to maximize the utility function

$$U(x, y, z) = x + \sqrt{y} - 1/z$$

for $x \geq 0$, $y > 0$ and $z > 0$. The budget constraint is $px + qy + rz = m$, where p, q, $r > 0$ and $m \geq \sqrt{pr} + p^2/4q$.

(a) Write out the first-order conditions for a constrained maximum.

(b) Find the utility-maximizing demands for all three goods as functions of the four parameters (p, q, r, m).

(c) Show that the maximized utility is given by the indirect utility function

$$U^*(p, q, r, m) = \frac{m}{p} + \frac{p}{4q} - 2\sqrt{\frac{r}{p}}$$

(d) Find $\partial U^*/\partial m$ and comment on your answer.

4. Each week an individual consumes quantities x and y of two goods, and works for l hours. These three quantities are chosen to maximize the utility function

$$U(x, y, l) = \alpha \ln x + \beta \ln y + (1 - \alpha - \beta) \ln(L - l)$$

which is defined for $0 \leq l < L$ and for $x, y > 0$. Here α and β are positive parameters satisfying $\alpha + \beta < 1$. The individual faces the budget constraint $px + qy = wl$, where w is the wage per hour. Define $\gamma = (\alpha + \beta)/(1 - \alpha - \beta)$. Find the individual's demands x^*, y^*, and labour supply l^* as functions of p, q, and w.

5. Consider the problem in Example 14.6.4, and let $(x, y, z) = (10 + h, 10 + k, l)$. Show that if (x, y, z) satisfies both constraints, then $k = -h$ and $l = h$. Then show that $x^2 + y^2 + z^2 = 200 + 3h^2$. What do you conclude?

6. An important problem in statistics requires solving

$$\min\ a_1^2x_1^2 + a_2^2x_2^2 + \cdots + a_n^2x_n^2 \ \text{ s.t. } \ x_1 + x_2 + \cdots + x_n = 1$$

where all the constants a_i are nonzero. Solve the problem, taking it for granted that the minimum value exists. What is the solution if one of the $a_i = 0$ for some i?

SM 7. Solve the problem:

$$\max(\min)\ x + y \ \text{ s.t. } \begin{cases} x^2 + 2y^2 + z^2 = 1 \\ x + y + z = 1 \end{cases}$$

SM 8. [HARDER] Consider the consumer optimization problem in Example 14.6.3. Find the demand functions when:

(a) $U(x_1, \dots, x_n) = Ax_1^{a_1} \cdots x_n^{a_n}$, with $A > 0$, $a_1 > 0, \dots, a_n > 0$.

(b) $U(x_1, \dots, x_n) = x_1^a + \cdots + x_n^a$, with $0 < a < 1$.

14.7 Comparative Statics

Equation (14.2.2) offers an economic interpretation of the Lagrange multiplier for the case of two variables and one constraint. This can be extended to the problem with n variables and m constraints. Let us rewrite that problem in the form

$$\max(\min) f(\mathbf{x}) \ \text{ s.t. } \ g_j(\mathbf{x}) = c_j, \ \text{ for } j = 1, \dots, m \tag{14.7.1}$$

Let $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ be the values of $\mathbf{x}$ that satisfy the necessary conditions for the solution to (14.7.1). In general, $\mathbf{x}^*$ depends on the values of $\mathbf{c} = (c_1, \dots, c_m)$. We assume that each $x_i^* = x_i^*(\mathbf{c})$ is a differentiable function. The associated value function, f^*, is then a function of $\mathbf{c}$ as well:

$$f^*(\mathbf{c}) = f(\mathbf{x}^*(\mathbf{c})) \tag{14.7.2}$$

The m Lagrange multipliers associated with $\mathbf{x}^*$, namely $\lambda_1, \dots, \lambda_m$, also depend on $\mathbf{c}$. Provided that certain regularity conditions are satisfied, we have that for all $j = 1, \dots, m$,

$$\frac{\partial f^*(\mathbf{c})}{\partial c_j} = \lambda_j(\mathbf{c}) \tag{14.7.3}$$

Hence, *the Lagrange multiplier for the i-th constraint, $\lambda_i = \lambda_i(\mathbf{c})$, is the rate at which the optimal value of the objective function changes w.r.t. changes in the constant c_i*. For this reason λ_i is referred to as the imputed *shadow price* (or *marginal value*) per unit of resource i.

Suppose that we change the components of the vector $\mathbf{c} = (c_1, \dots, c_m)$ by the respective amounts $\mathrm{d}\mathbf{c} = (\mathrm{d}c_1, \dots, \mathrm{d}c_m)$. According to linear approximation (12.8.2), if $\mathrm{d}c_1, \dots, \mathrm{d}c_m$ are all small in absolute value, Eq. (14.7.3) yields

$$f^*(\mathbf{c} + \mathrm{d}\mathbf{c}) - f^*(\mathbf{c}) \approx \lambda_1(\mathbf{c})\,\mathrm{d}c_1 + \cdots + \lambda_m(\mathbf{c})\,\mathrm{d}c_m \tag{14.7.4}$$

EXAMPLE 14.7.1 Consider Example 14.6.4, and suppose we change the first constraint to $x + 2y + z = 31$ and the second constraint to $2x - y - 3z = 9$. Estimate the corresponding change in the value function by using (14.7.4). Find also the new exact value of the value function.

Solution: Using the notation introduced above and the results in Example 14.6.4, we have $c_1 = 30$, $c_2 = 10$, $\mathrm{d}c_1 = 1$, $\mathrm{d}c_2 = -1$, $\lambda_1(30, 10) = 12$, $\lambda_2(30, 10) = 4$, and $f^*(c_1, c_2) = f^*(30, 10) = 10^2 + 10^2 + 0^2 = 200$. Then, approximation (14.7.4) yields

$$\begin{aligned} f^*(30+1, 10-1) - f^*(30, 10) &\approx \lambda_1(30, 10)\,\mathrm{d}c_1 + \lambda_2(30, 10)\,\mathrm{d}c_2 \\ &= 12 \cdot 1 + 4 \cdot (-1) \;=\; 8 \end{aligned}$$

Thus, $f^*(31, 9) \approx 200 + 8 = 208$.

To find the exact value of $f^*(31, 9)$, observe that (vi) in Example 14.6.4 is still valid. Thus, we have the three equations $x + 2y + z = 31$, $2x - y - 3z = 9$, $x - y + z = 0$, whose solutions for x, y, and z are $151/15$, $31/3$, and $4/15$, respectively. We find that $f^*(31, 9) = 15\,614/75 \approx 208.19$.

The Envelope Theorem

Using vector notation with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{r} = (r_1, \ldots, r_k)$, consider the following, more general version of problem (14.7.1):

$$\max(\min)_{\mathbf{x}} f(\mathbf{x}, \mathbf{r}) \quad \text{s.t.} \quad g_j(\mathbf{x}, \mathbf{r}) = 0, \quad \text{for } j = 1, \ldots, m \tag{14.7.5}$$

Here, both the objective function f and each of the m different constraint functions g_j depend not only on the vector $\mathbf{x}$ of variables to be chosen, but also on the parameter vector $\mathbf{r}$. Suppose that $\lambda_j = \lambda_j(\mathbf{r})$, for $j = 1, \ldots, m$, are the Lagrange multipliers obtained from the first-order conditions for problem (14.7.5). Also, we use the generalized definition

$$\mathcal{L}(\mathbf{x}, \mathbf{r}) = f(\mathbf{x}, \mathbf{r}) - \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}, \mathbf{r})$$

of the corresponding Lagrangian. By analogy with Eq. (14.7.2), let $\mathbf{x}^*(\mathbf{r})$ denote the optimal choice of $\mathbf{x}$ when the parameter vector is $\mathbf{r}$, and define the value function

$$f^*(\mathbf{r}) = f(\mathbf{x}^*(\mathbf{r}), \mathbf{r}) \tag{14.7.6}$$

Then the following result holds:

THEOREM 14.7.1 (THE ENVELOPE THEOREM)

If $f^*(\mathbf{r})$ and $\mathbf{x}^*(\mathbf{r})$ are differentiable, then

$$\frac{\partial f^*(\mathbf{r})}{\partial r_h} = \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} \tag{14.7.7}$$

for each $h = 1, \ldots, k$.

This is a very useful general result that should be studied carefully. When any parameter is changed, then $f^*(\mathbf{r})$ changes for two reasons: first, a change in r_h changes the vector $\mathbf{r}$ and thus changes $f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})$ directly; and, second, a change in r_h changes, in general, all the functions $x_1^*(\mathbf{r}), \ldots, x_n^*(\mathbf{r})$, which changes $f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})$ indirectly. Theorem 14.7.1 shows that the total effect on the value function of a small change in r_h is found by computing the *partial* derivative of $\mathcal{L}(\mathbf{x}, \mathbf{r})$ w.r.t. r_h, and evaluating it at $\mathbf{x}^*(\mathbf{r})$, ignoring the indirect effect of the dependence of $\mathbf{x}^*$ on $\mathbf{r}$ altogether. The reason is that, because of FOCS (14.6.11), any small change in $\mathbf{x}$ that preserves the equality constraints of problem (14.7.5) will have a negligible effect on the value of $f(\mathbf{x}^*, \mathbf{r})$, so Eq. (14.7.7) holds.

EXAMPLE 14.7.2 In Example 14.6.3, let $U^*(\mathbf{p}, m)$ denote the *indirect utility function* whose value is the maximum utility obtainable by the consumer when prices are $\mathbf{p} = (p_1, \ldots, p_n)$ and the income is m. Let λ denote the Lagrange multiplier associated with the budget constraint. Using Eq. (14.7.3), we see that

$$\lambda = \frac{\partial U^*}{\partial m} \tag{14.7.8}$$

Thus, λ is the rate of increase in maximum utility as income increases. For this reason, λ is generally called the *marginal utility of income*.

Including the vector $(\mathbf{p}, m)$ of all parameters, the Lagrangian takes the form

$$\mathcal{L}(\mathbf{x}, \mathbf{p}, m) = U(\mathbf{x}) - \lambda(p_1 x_1 + \cdots + p_n x_n - m)$$

Obviously, $\partial\mathcal{L}/\partial m = \lambda$ and $\partial\mathcal{L}/\partial p_i = -\lambda x_i$. Hence, from (14.7.7) we get

$$\frac{\partial U^*(\mathbf{p}, m)}{\partial m} = \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{p}, m)}{\partial m} = \lambda$$

which repeats (14.7.8). Moreover,

$$\frac{\partial U^*(\mathbf{p}, m)}{\partial p_i} = \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{p}, m)}{\partial p_i} = -\lambda x_i^*$$

which is called *Roy's identity*.[18] This formula has a nice interpretation: the marginal disutility of a price increase is the marginal utility of income, λ, multiplied by the quantity demanded, x_i^*. Intuitively, this is because, for a small price change, the loss of real income is approximately equal to the change in price multiplied by the quantity demanded.

As an illustration of Roy's identity, consider the consumer optimization problem with a Cobb–Douglas utility function, as in Eq. (14.6.7). Substituting the demands given by Eq. (14.6.8) into the utility function, we obtain the indirect utility function

$$U^*(\mathbf{p}, m) = A\left(\frac{a_1 m}{a p_1}\right)^{a_1} \cdots \left(\frac{a_n m}{a p_n}\right)^{a_n} = \frac{B m^a}{P(p_1, \ldots, p_n)}$$

where we have used the notation $a = a_1 + a_2 + \cdots + a_n$, while B denotes the constant $A{a_1}^{a_1} \cdots {a_n}^{a_n}/a^a$, and $P = P(p_1, \ldots, p_n)$ denotes the function $p_1^{a_1} \cdots p_n^{a_n}$.[19]

[18] Named after the French economist René Roy (1894–1977). His name should be pronounced accordingly.

[19] Note that P is homogeneous of degree a. This *price index* is also a Cobb–Douglas function whose powers match those of the original utility function.

This formula for the indirect utility function implies that $\partial U^*/\partial m = Bam^{a-1}/P$, and also that

$$\frac{\partial U^*}{\partial p_i} = -\frac{Bm^a}{P^2}\frac{\partial P}{\partial p_i} = -\frac{Bm^a}{P^2}\frac{a_i P}{p_i} = -\frac{Bam^{a-1}}{P}\frac{a_i m}{a p_i} = -\frac{\partial U^*}{\partial m} D_i(\mathbf{p}, m)$$

This confirms Roy's identity for the case of a Cobb–Douglas utility function.

EXAMPLE 14.7.3 A firm uses K units of capital and L units of labour to produce $F(K, L)$ units of a commodity. The prices of capital and labour are r and w, respectively. Consider the cost minimization problem

$$\min C(K, L) = rK + wL \quad \text{s.t.} \quad F(K, L) = Q$$

where we want to find the values of K and L that minimize the cost of producing Q units. Let $C^* = C^*(r, w, Q)$ be the value function for the problem. Find $\partial C^*/\partial r$, $\partial C^*/\partial w$, and $\partial C^*/\partial Q$.

Solution: Including the output requirement Q and the price parameters r and w, the Lagrangian is

$$\mathcal{L}(K, L, r, w, Q) = rK + wL - \lambda(F(K, L) - Q)$$

whose partial derivatives are $\partial\mathcal{L}/\partial r = K$, $\partial\mathcal{L}/\partial w = L$, and $\partial\mathcal{L}/\partial Q = \lambda$. According to Theorem 14.7.1,

$$\frac{\partial C^*}{\partial r} = K^*, \qquad \frac{\partial C^*}{\partial w} = L^*, \qquad \text{and} \qquad \frac{\partial C^*}{\partial Q} = \lambda \tag{$*$}$$

The first two equalities are instances of *Shephard's lemma*. The last equation shows that λ must equal *marginal cost*, the rate at which minimum cost increases w.r.t. changes in output.

To conclude, we present a proof of Theorem 14.7.1.:

Using the chain rule to differentiate the right-hand side of Eq. (14.7.6) w.r.t. r_h yields

$$\frac{\partial f^*(\mathbf{r})}{\partial r_h} = \sum_{i=1}^{n} \frac{\partial f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial x_i}\frac{\partial x_i^*(\mathbf{r})}{\partial r_h} + \frac{\partial f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} \tag{i}$$

But the corresponding partial derivative of the Lagrangian, evaluated at $(\mathbf{x}^*(\mathbf{r}), \mathbf{r})$, is

$$\frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} = \frac{\partial f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} \tag{ii}$$

Subtracting each side of (ii) from the corresponding side of (i), we obtain

$$\frac{\partial f^*(\mathbf{r})}{\partial r_h} - \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} = \sum_{i=1}^{n} \frac{\partial f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial x_i}\frac{\partial x_i^*(\mathbf{r})}{\partial r_h} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} \tag{iii}$$

Differentiating each constraint $g_j(\mathbf{x}^*(\mathbf{r}), \mathbf{r}) = 0$ w.r.t. r_h, however, yields

$$\sum_{i=1}^{n} \frac{\partial g_j(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial x_i}\frac{\partial x_i^*(\mathbf{r})}{\partial r_h} + \frac{\partial g_j(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} = 0 \tag{iv}$$

Using (iv) to substitute for each term $\partial g_j(\mathbf{x}^*(\mathbf{r}), \mathbf{r})/\partial r_h$ in (iii) gives

$$\frac{\partial f^*(\mathbf{r})}{\partial r_h} - \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} = \sum_{i=1}^{n} \left\{ \left[\frac{\partial f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial x_i} \right] \frac{\partial x_i^*(\mathbf{r})}{\partial r_h} \right\} \tag{v}$$

The terms in square brackets, however, are equal to the partial derivatives $\partial \mathcal{L}/\partial x_i$ which the first-order conditions (14.6.11) require to be zero at the optimum $(\mathbf{x}^*(\mathbf{r}), \mathbf{r})$. Hence Eq. (v) reduces to

$$\frac{\partial f^*(\mathbf{r})}{\partial r_h} - \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{r}), \mathbf{r})}{\partial r_h} = 0$$

thus proving Eq. (14.7.7).

Note that this proof used only the first-order conditions (14.6.11) for the problem set out in (14.7.5). Therefore, the results in Theorem 14.7.1 are equally valid if we minimize rather than maximize $f(\mathbf{x}, \mathbf{r})$. Note also that we did *not* prove that f^* is differentiable. Sufficient conditions for this are discussed in FMEA.

EXERCISES FOR SECTION 14.7

1. Consider the utility maximization problem $\max x + a \ln y$ s.t. $px + qy = m$, where $0 \leq a < m/p$.

(a) Find the solution (x^*, y^*).

(b) Find the indirect utility function $U^*(p, q, m, a)$, and compute its partial derivatives w.r.t. p, q, m, and a.

(c) Verify the envelope theorem.

SM **2.** Consider the problem $\min x + 4y + 3z$ s.t. $x^2 + 2y^2 + \frac{1}{3}z^2 = b$, where $b > 0$. Suppose that the problem has a solution, and find it. Then verify Eq. (14.7.3).

3. A firm has L units of labour at its disposal. Its outputs are three different commodities. Producing x, y, and z units of these commodities requires αx^2, βy^2, and γz^2 units of labour, respectively.

(a) Solve the problem $\max ax + by + cz$ s.t. $\alpha x^2 + \beta y^2 + \gamma z^2 = L$, where a, b, c, α, β, and γ are positive constants.

(b) Put $a = 4$, $b = c = 1$, $\alpha = 1$, $\beta = \frac{1}{4}$, and $\gamma = \frac{1}{5}$, and show that in this case the problem in (a) has the solution $x = \frac{4}{5}\sqrt{L}$, $y = \frac{4}{5}\sqrt{L}$, and $z = \sqrt{L}$.

(c) What happens to the maximum value of $4x + y + z$ when L increases from 100 to 101? Find both the exact change and the appropriate linear approximation based on the interpretation of the Lagrange multiplier.

SM **4.** Consider the two problems[20]

$$\max(\min) f(x, y, z) = x^2 + y^2 + z \quad \text{s.t.} \quad g(x, y, z) = x^2 + 2y^2 + 4z^2 = 1$$

(a) Solve them both for the specified constraint.

(b) Suppose the constraint is changed to $x^2 + 2y^2 + 4z^2 = 1.02$. What is the approximate change in the maximum value of $f(x, y, z)$?

[20] The graph of the constraint is the surface of an ellipsoid in $\mathbb{R}^3$, a closed and bounded set.

SM **5.** With reference to Example 14.7.3, let $F(K, L) = K^{1/2}L^{1/4}$ and solve the problem, finding explicit expressions for K^*, L^*, C^*, and λ. Verify the equalities $(*)$ in the example.

6. With reference to Example 14.7.3, assuming that the cost function C^* is continuously differentiable twice, prove the symmetry relation $\partial K^*/\partial w = \partial L^*/\partial r$.

7. Consider the utility maximization problem $\max \sqrt{x} + ay$ s.t. $px + qy = m$, where $m > q^2/4a^2p$.

(a) Find the demand functions $x^*(p, q, m, a)$ and $y^*(p, q, m, a)$, as well as the indirect utility function $U^*(p, q, m, a)$.

(b) Find all four partials of $U^*(p, q, m, a) = x^* + ay^*$ and verify the envelope theorem.

14.8 Nonlinear Programming: A Simple Case

So far this chapter has considered how to maximize or minimize a function subject to equality constraints. The final two sections concern "nonlinear programming" problems which involve *inequality* constraints. Some particularly simple inequality constraints are those requiring certain variables to be nonnegative. These often have to be imposed for the solution to make economic sense. In addition, bounds on resource availability are often expressed as inequalities rather than equalities.

In this section we consider the simple *nonlinear programming problem*

$$\max f(x, y) \quad \text{s.t.} \quad g(x, y) \le c \tag{14.8.1}$$

with just one inequality constraint. Thus, we seek the largest value attained by $f(x, y)$ in the *admissible* or *feasible* set S of all pairs (x, y) satisfying $g(x, y) \le c$. Problems where one wants to minimize $f(x, y)$ subject to $(x, y) \in S$ can be handled by instead studying the problem of maximizing $-f(x, y)$ subject to $(x, y) \in S$.

Problem (14.8.1) can be solved using the methods explained in Chapter 13. This involves examining not only the critical points of f in the interior of the admissible set S, but also the behaviour of f on the boundary of S. However, since the 1950s, economists have generally tackled such problems by using an extension of the Lagrangian multiplier method due originally to H.W. Kuhn and A.W. Tucker.

To apply their method, we begin by writing down a recipe giving all the points (x, y) that can possibly solve problem (14.8.1), except in some bizarre cases. The recipe closely resembles the one we used to solve problem (14.1.3).

THE KUHN–TUCKER METHOD

To find the only possible solutions to problem (14.8.1), proceed as follows:

(i) Associate a constant Lagrange multiplier λ with the constraint $g(x, y) \le c$, and define the Lagrangian

$$\mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c]$$

(ii) Find the critical points of $\mathcal{L}(x, y)$, by equating its partial derivatives to zero:

$$\mathcal{L}'_1(x, y) = f'_1(x, y) - \lambda g'_1(x, y) = 0 \tag{14.8.2a}$$

$$\mathcal{L}'_2(x, y) = f'_2(x, y) - \lambda g'_2(x, y) = 0 \tag{14.8.2b}$$

(iii) Introduce the *complementary slackness condition*:

$$\lambda \geq 0, \quad \text{with } \lambda = 0 \text{ if } g(x, y) < c \tag{14.8.3}$$

(iv) Require (x, y) to satisfy the constraint

$$g(x, y) \leq c \tag{14.8.4}$$

(v) Find all the points (x, y) that, together with associated values of λ, satisfy all the conditions (14.8.2a) to (14.8.4). These are the solution candidates, at least one of which solves the problem, if it has a solution.

If $g = c$ and $g'_1 = g'_2 = 0$ at the maximum of the problem, this method may fail.

Note that the conditions (14.8.2a) and (14.8.2b) are exactly the same as those used in the Lagrange multiplier method of Section 14.1. Condition (14.8.4) obviously has to be satisfied, so the only new feature is condition (14.8.3), which can be rather tricky. It requires that λ be nonnegative, and moreover that $\lambda = 0$ if $g(x, y) < c$. Thus, if $\lambda > 0$, we must have $g(x, y) = c$. An alternative formulation of this condition, then, is that

$$\lambda \geq 0, \quad \text{with } \lambda \cdot [g(x, y) - c] = 0 \tag{14.8.5}$$

Later we shall see that even in nonlinear programming, the Lagrange multiplier can be interpreted as a "price" per unit associated with increasing the right-hand side c of the "resource constraint" $g(x, y) \leq c$. With this interpretation, prices are nonnegative, and if the resource constraint is not binding, because $g(x, y) < c$ at the optimum, this means that the price associated with increasing c by one unit is 0.

The two inequalities $\lambda \geq 0$ and $g(x, y) \leq c$ are *complementary* in the sense that at most one can be "slack"—that is, at most one can hold with inequality. Equivalently, at least one must be an equality. Failure to observe that it *is* possible to have *both* $\lambda = 0$ *and* $g(x, y) = c$ in the complementary slackness condition is probably the most common error when solving nonlinear programming problems.

Parts (ii) and (iii) of the method above are together called the *Kuhn–Tucker conditions*. Note that these are (essentially) *necessary* conditions for the solution of Problem (14.8.1). In general, though, they are far from sufficient: indeed, suppose that one can find a point (x_0, y_0) where f is critical and $g(x_0, y_0) < c$; then the Kuhn–Tucker conditions will automatically be satisfied by (x_0, y_0) together with the Lagrange multiplier $\lambda = 0$, yet then (x_0, y_0) could be a local or global minimum or maximum, or a saddle point.

We say that these Kuhn–Tucker conditions are only *essentially* necessary because there may not always be a Lagrange multiplier for which the Kuhn–Tucker conditions hold. The exceptions are some rather rare constrained optimization problems that fail to satisfy a special technical condition called the "constraint qualification". For details, see FMEA.

With equality constraints, setting the partial derivative $\partial\mathcal{L}/\partial\lambda$ equal to zero just recovers the constraint $g(x, y) = c$. Yet with an inequality constraint, one can have $\partial\mathcal{L}/\partial\lambda = -g(x, y) + c > 0$ if the constraint is slack or inactive at an optimum. It was for this reason that we advised against differentiating the Lagrangian w.r.t. the multiplier λ, even though many other books advocate this procedure.

In Theorem 14.5.1 we proved that if the Lagrangian is concave, then the first-order conditions in problem (14.1.1) are sufficient for optimality. The corresponding result is also valid for problem (14.8.1):

THEOREM 14.8.1 (SUFFICIENT CONDITIONS)

Consider the problem set out in (14.8.1), and suppose that (x_0, y_0) satisfies conditions (14.8.2a) to (14.8.4) for the Lagrangian

$$\mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c]$$

If the Lagrangian is concave, then (x_0, y_0) solves the problem.

The proof of this result is actually quite instructive:

Any pair (x_0, y_0) that satisfies conditions (14.8.2a) and (14.8.2b) must be a critical point of the Lagrangian. By Theorem 13.2.1, if the Lagrangian is concave, this (x_0, y_0) will give a maximum. So

$$\mathcal{L}(x_0, y_0) = f(x_0, y_0) - \lambda[g(x_0, y_0) - c] \geq \mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c]$$

Rearranging the terms, we obtain

$$f(x_0, y_0) - f(x, y) \geq \lambda[g(x_0, y_0) - g(x, y)] \tag{$*$}$$

If $g(x_0, y_0) < c$, then by (14.8.3), we have $\lambda = 0$, so $(*)$ implies that $f(x_0, y_0) \geq f(x, y)$ for all (x, y). On the other hand, if $g(x_0, y_0) = c$ and $g(x, y) \leq c$, then $\lambda[g(x_0, y_0) - g(x, y)] = \lambda[c - g(x, y)]$. Because $\lambda \geq 0$ and $c - g(x, y) \geq 0$ for all (x, y) satisfying the inequality constraint, so, again the inequality constraint $(*)$ implies that $f(x_0, y_0) \geq f(x, y)$. Hence, (x_0, y_0) solves Problem (14.8.1).

Note that, as in the argument preceding Theorem 14.5.1, this proof shows that if the Lagrangian achieves a global maximum at a point (x_0, y_0) that satisfies conditions (14.8.3) and (14.8.4) , then (x_0, y_0) solves the problem, whether or not the Lagrangian is concave. In this sense, the concavity hypothesis of Theorem 14.8.1 is a useful but unnecessarily strong sufficient condition.

EXAMPLE 14.8.1 A firm has a total of L units of labour to allocate to the production of two goods. These can be sold at fixed positive prices a and b respectively. Producing x units of the first good requires αx^2 units of labour, whereas producing y units of the second good requires

βy^2 units of labour, where α and β are positive constants. Find what output levels of the two goods maximize the revenue that the firm can earn by using this fixed amount of labour.

Solution: The firm's problem is $\max ax + by$ s.t. $\alpha x^2 + \beta y^2 \leq L$. The Lagrangian is

$$\mathcal{L}(x, y) = ax + by - \lambda(\alpha x^2 + \beta y^2 - L)$$

and the necessary conditions for (x^*, y^*) to solve the problem are

$$\mathcal{L}'_x = a - 2\lambda\alpha x^* = 0 \tag{i}$$

$$\mathcal{L}'_y = b - 2\lambda\beta y^* = 0 \tag{ii}$$

plus the complementary slackness condition

$$\lambda \geq 0, \ \text{ with } \lambda = 0 \ \text{ if } \ \alpha(x^*)^2 + \beta(y^*)^2 < L \tag{iii}$$

and the resource constraint. Because a and b are positive, we see that λ, x^*, and y^* are all positive, with

$$x^* = \frac{a}{2\alpha\lambda} \ \text{ and } \ y^* = \frac{b}{2\beta\lambda} \tag{$*$}$$

Because $\lambda > 0$, condition (iii) implies that $\alpha(x^*)^2 + \beta(y^*)^2 = L$. Inserting the expressions for x^* and y^* into the resource constraint yields $a^2/4\alpha\lambda^2 + b^2/4\beta\lambda^2 = L$. It follows that

$$\lambda = \tfrac{1}{2}L^{-1/2}\sqrt{a^2/\alpha + b^2/\beta} \tag{$**$}$$

Our recipe has produced the solution candidate with x^* and y^* given by $(*)$, and λ as in $(**)$. The Lagrangian $\mathcal{L}$ is obviously concave, so we have found the solution.

EXAMPLE 14.8.2 Solve the problem

$$\max f(x, y) = x^2 + y^2 + y - 1 \ \text{ s.t. } \ g(x, y) = x^2 + y^2 \leq 1$$

Solution: The Lagrangian is

$$\mathcal{L}(x, y) = x^2 + y^2 + y - 1 - \lambda(x^2 + y^2 - 1)$$

so the first-order conditions are:

$$\mathcal{L}'_1(x, y) = 2x - 2\lambda x = 0 \tag{i}$$

$$\mathcal{L}'_2(x, y) = 2y + 1 - 2\lambda y = 0 \tag{ii}$$

The complementary slackness condition is

$$\lambda \geq 0, \ \text{ with } \lambda = 0 \ \text{ if } \ x^2 + y^2 < 1 \tag{iii}$$

We want to find all pairs (x, y) that satisfy these conditions for some suitable value of λ.

Conditions (i) and (ii) can be written as $2x(1 - \lambda) = 0$ and $2y(1 - \lambda) = -1$, respectively. The second of these implies that $\lambda \neq 1$, so the first implies that $x = 0$.

Suppose $x^2 + y^2 = 1$ and so $y = \pm 1$ because $x = 0$. Try $y = 1$ first. Then, (ii) implies $\lambda = 3/2$ and so (iii) is satisfied. Thus, $(0, 1)$ *with* $\lambda = 3/2$ *is a first candidate for optimality*, because all the conditions (i)–(iii) are satisfied. Next, try $y = -1$. Then condition (ii) yields $\lambda = 1/2$ and (iii) is again satisfied. Thus, $(0, -1)$ *with* $\lambda = 1/2$ *is a second candidate for optimality.*

Consider, finally, the case when $x = 0$ and also $x^2 + y^2 = y^2 < 1$—that is, $-1 < y < 1$. Then (iii) implies that $\lambda = 0$, and so (ii) yields $y = -1/2$. Hence, $(0, -1/2)$ *with* $\lambda = 0$ *is a third candidate for optimality.*

We conclude that there are three candidates for optimality. Now

$$f(0,1) = 1,\ f(0,-1) = -1,\ \text{ and }\ f(0,-1/2) = -5/4$$

Because we want to maximize a continuous function over a closed, bounded set, by the extreme value theorem there is a solution to the problem. Because the only possible solutions are the three points already found, we conclude that $(x, y) = (0, 1)$ solves the maximization problem.[21]

Why Does the Kuhn–Tucker Method Work?

Suppose (x^*, y^*) solves problem (14.8.1). Then, either $g(x^*, y^*) < c$, in which case the constraint $g(x^*, y^*) \leq c$ is said to be *inactive* or *slack* at (x^*, y^*), or else $g(x^*, y^*) = c$, in which case the same inequality constraint is said to be *active* or *binding* at (x^*, y^*). The two different cases are illustrated for two different values of c in Figs 14.8.1 and 14.8.2, which both display the same four level curves of the objective function f as well. This function is assumed to increase as the level curves shrink. In Fig. 14.8.1, the solution (x^*, y^*) to problem (14.8.1) is an interior point P of the shaded admissible set. On the other hand, in Fig. 14.8.2, the solution (x^*, y^*) is at the boundary of the shaded admissible set.

In case the solution (x^*, y^*) satisfies $g(x^*, y^*) < c$, as in Fig. 14.8.1, the point (x^*, y^*) is usually an interior maximum of the function f. Then it is a critical point at which $f_1'(x^*, y^*) = f_2'(x^*, y^*) = 0$. In this case, if we set $\lambda = 0$, then conditions (14.8.2a) to (14.8.4) are all satisfied.

On the other hand, in the case when the constraint is binding at (x^*, y^*), as in Fig. 14.8.2, the point (x^*, y^*) solves the problem

$$\max f(x, y) \ \text{ s.t. } \ g(x, y) = c$$

with an equality constraint. Provided that the conditions of Theorem 14.4.1 are all satisfied, there will exist a unique Lagrange multiplier λ such that the Lagrangian satisfies the first-order conditions (14.8.2a) and (14.8.2b) at (x^*, y^*). It remains to be shown that this Lagrange multiplier λ satisfies $\lambda \geq 0$, thus ensuring that (14.8.3) is also satisfied at (x^*, y^*).

To prove that $\lambda \geq 0$, consider the following two problems

$$\max f(x, y) \ \text{ s.t. } \ g(x, y) \leq b \ \text{ and } \ \max f(x, y) \ \text{ s.t. } \ g(x, y) = b$$

[21] The point $(0, -1/2)$ solves the corresponding minimization problem. We solved both these problems in Example 13.5.1.

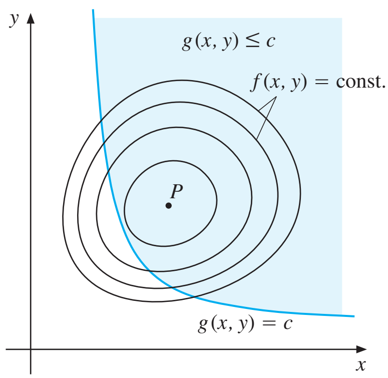

Figure 14.8.1 $P = (x^*, y^*)$ is an interior point of the admissible set

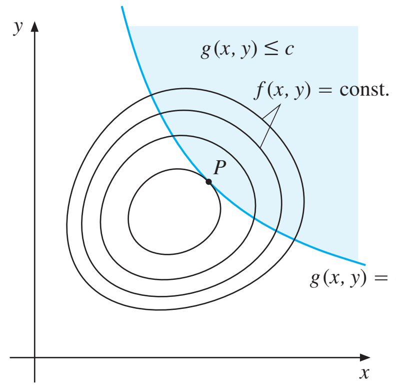

Figure 14.8.2 Constraint $g(x, y) \leq c$ is binding at $P = (x^*, y^*)$

where the constant c has been replaced by the variable parameter b. Let $v(b)$ and $f^*(b)$, respectively, be the value functions of the two problems. Recall from (14.2.2) that $\lambda = \mathrm{d}f^*(c)/\mathrm{d}c$ if f^* is differentiable at c. We shall now show that $f^*(b) \leq f^*(c)$ whenever $b \leq c$, thus implying that

$$\lambda = \lim_{b \to c} \frac{f^*(b) - f^*(c)}{b - c} = \lim_{b \to c^-} \frac{f^*(b) - f^*(c)}{b - c} \geq 0$$

at least when f^* is differentiable, because both the numerator and the denominator of the last limit are nonpositive.

Indeed, our construction implies that $f^*(b) \leq v(b)$ for all b, because the equality constraint $g(x, y) = b$ is more stringent than $g(x, y) \leq b$, and imposing a more stringent constraint never allows a higher maximum value. But also, in case $b < c$, the constraint $g(x, y) \leq b$ is more stringent than $g(x, y) \leq c$, from which it follows that $v(b) \leq v(c)$. Finally, because we are discussing the case when the constraint $g(x^*, y^*) = c$ binds at the solution to problem (14.8.1), we must have $v(c) = f^*(c)$. Thus,

$$f^*(b) \leq v(b) \leq v(c) = f^*(c)$$

is satisfied whenever $b \leq c$, as required.

EXERCISES FOR SECTION 14.8

1. Consider the problem $\max -x^2 - y^2$ s.t. $x - 3y \leq -10$.

   (a) Find the pair (x^*, y^*) that solves the problem.

   (b) The same pair (x^*, y^*) also solves the minimization problem $\min(x^2 + y^2)$ s.t. $x - 3y \leq -10$. Sketch the admissible set S and explain the solution geometrically.

2. Consider the consumer demand problem $\max \sqrt{x} + \sqrt{y}$ s.t. $px + qy \leq m$.

   (a) Find the demand functions.

   (b) Are the demand functions homogeneous of degree 0?

3. Consider the problem $\max 4 - \frac{1}{2}x^2 - 4y$ s.t. $6x - 4y \le a$.

(a) Write down the Kuhn–Tucker conditions.

(b) Solve the problem.

(c) With $V(a)$ as the value function, verify that $V'(a) = \lambda$, where λ is the Lagrange multiplier in (b).

4. Consider the problem $\max x^2 + 2y^2 - x$ s.t. $x^2 + y^2 \le 1$.

(a) Write down the Lagrangian and conditions (14.8.2a) and (14.8.2b).

(b) Find the five pairs (x, y) that satisfy all the necessary conditions.

(c) Find the solution to the problem.

SM **5.** Consider the problem $\max f(x, y) = 2 - (x - 1)^2 - e^{y^2}$ s.t. $x^2 + y^2 \le a$, where a is a positive constant.

(a) Write down the Kuhn–Tucker conditions for the solution of the problem, distinguishing between the cases $a \in (0, 1)$ and $a \ge 1$, then find the only solution candidate.

(b) Prove optimality by using Theorem 14.8.1.

(c) Let $f^*(a)$ be the value function for the problem. Verify that $\mathrm{d}f^*(a)/\mathrm{d}a = \lambda$.

6. Suppose a firm earns revenue $R(Q) = aQ - bQ^2$ and incurs cost $C(Q) = \alpha Q + \beta Q^2$ as functions of output $Q \ge 0$, where a, b, α, and β are positive parameters. The firm maximizes profit $\pi(Q) = R(Q) - C(Q)$ subject to the constraint $Q \ge 0$. Solve this one-variable problem by the Kuhn–Tucker method, and find conditions for the constraint to bind at the optimum.

14.9 Multiple Inequality Constraints

A fairly general nonlinear programming problem is the following:

$$\max f(x_1, \dots, x_n) \quad \text{s.t.} \quad \begin{cases} g_1(x_1, \dots, x_n) \le c_1 \\ \quad \dots\dots\dots \\ g_m(x_1, \dots, x_n) \le c_m \end{cases} \tag{14.9.1}$$

The set of vectors $\mathbf{x} = (x_1, \dots, x_n)$ that satisfy all the constraints is called the *admissible set* or the *feasible set*. Here is a recipe for solving this problem:

THE KUHN–TUCKER METHOD

To find the only possible solutions to problem (14.9.1), proceed as follows:

(i) Associate Lagrange multipliers $\lambda_1, \dots, \lambda_m$ with the m constraints, and then write down the Lagrangian

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j (g_j(\mathbf{x}) - c_j)$$

(ii) Find the critical points of $\mathcal{L}(\mathbf{x})$ by finding each partial derivative and then solving the equation

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0$$

for each $i = 1, \ldots, n$.

(iii) Impose the complementary slackness conditions:

$$\lambda_j \geq 0, \quad \text{with } \lambda_j = 0 \text{ if } g_j(\mathbf{x}) < c_j$$

for each $j = 1, \ldots, m$.

(iv) Require $\mathbf{x}$ to satisfy all the constraints $g_j(\mathbf{x}) \leq c_j$.

(v) Find all the vectors $\mathbf{x}$ that, together with associated values of $\lambda_1, \ldots, \lambda_m$, satisfy conditions (ii), (iii), and (v). These are the solution candidates, at least one of which solves the problem, if it has a solution.

Note that, as with Eq. (14.8.5), the constraints in steps (iii) and (iv) can be combined into the requirement that

$$\lambda_j \geq 0, \quad g_j(\mathbf{x}) \leq c_j, \quad \text{and} \quad \lambda_j[g_j(\mathbf{x}) - c_j] = 0 \tag{14.9.2}$$

This may be easier to remember and make the derivations easier to express.

If the Lagrangian $\mathcal{L}(\mathbf{x})$ is concave in $\mathbf{x}$, then conditions (iii) to (v) are sufficient for optimality.[22] If $\mathcal{L}(\mathbf{x})$ is not concave, still any vector $\mathbf{x}$ which happens to maximize the Lagrangian while also satisfying (iv) and (v) must be an optimum.[23]

Recall that minimizing $f(\mathbf{x})$ is equivalent to maximizing $-f(\mathbf{x})$. Also an inequality constraint of the form $g_j(\mathbf{x}) \geq c_j$ can be rewritten as $-g_j(\mathbf{x}) \leq -c_j$, whereas an equality constraint $g_j(\mathbf{x}) = c_j$ is equivalent to the double inequality constraint $g_j(\mathbf{x}) \leq c_j$ and $-g_j(\mathbf{x}) \leq -c_j$. In this way, most constrained optimization problems can be expressed in the form (14.9.1).

EXAMPLE 14.9.1 Consider the nonlinear programming problem

$$\max x + 3y - 4e^{-x-y} \quad \text{s.t.} \quad \begin{cases} 2 - x \geq 2y \\ x - 1 \leq -y \end{cases}$$

(a) Write down the necessary Kuhn–Tucker conditions for a point (x^*, y^*) to be a solution of the problem. Are the conditions sufficient for optimality?

(b) Solve the problem.

[22] Concavity and convexity for functions of several variables are discussed extensively in FMEA.

[23] As before, in order for the conditions to be truly necessary, a constraint qualification is needed. Once again, see FMEA for details.

Solution:

(a) The first step is to write the problem in the same form as (14.9.1):

$$\max x + 3y - 4e^{-x-y} \quad \text{s.t.} \quad \begin{cases} x + 2y \leq 2 \\ x + y \leq 1 \end{cases}$$

The Lagrangian is

$$\mathcal{L}(x, y) = x + 3y - 4e^{-x-y} - \lambda_1(x + 2y - 2) - \lambda_2(x + y - 1)$$

Hence, the Kuhn–Tucker conditions for (x^*, y^*) to solve the problem are:

$$\mathcal{L}_1' = 1 + 4e^{-x^*-y^*} - \lambda_1 - \lambda_2 = 0 \tag{i}$$

$$\mathcal{L}_2' = 3 + 4e^{-x^*-y^*} - 2\lambda_1 - \lambda_2 = 0 \tag{ii}$$

$$\lambda_1 \geq 0, \quad \text{with } \lambda_1 = 0 \text{ if } x^* + 2y^* < 2 \tag{iii}$$

$$\lambda_2 \geq 0, \quad \text{with } \lambda_2 = 0 \text{ if } x^* + y^* < 1 \tag{iv}$$

The Hessian matrix of $\mathcal{L}(x, y)$ satisfies $\mathcal{L}_{11}'' = \mathcal{L}_{22}'' = \mathcal{L}_{12}'' = -4e^{-x-y}$, implying that $\mathcal{L}_{11}'' = \mathcal{L}_{22}'' < 0$ and $\mathcal{L}_{11}''\mathcal{L}_{22}'' - (\mathcal{L}_{12}'')^2 = 0$, so the Lagrangian is concave. Hence these Kuhn–Tucker conditions are sufficient for optimality.

(b) Subtracting (ii) from (i) we get $-2 + \lambda_1 = 0$ and so $\lambda_1 = 2$. But then (iii) together with $x^* + 2y^* \leq 2$ yields $x^* + 2y^* = 2$.

Suppose $\lambda_2 = 0$. Then from (i), $4e^{-x^*-y^*} = 1$, so $-x^* - y^* = \ln(1/4)$, and then $x^* + y^* = \ln 4 > 1$, a contradiction.

Thus $\lambda_2 > 0$. Then from (iv) and $x^* + y^* \leq 1$ we deduce $x^* + y^* = 1$. Since $x^* + 2y^* = 2$, we see that $x^* = 0$ and $y^* = 1$. Inserting these values for x^* and y^* into (i) and (ii) we find that $\lambda_2 = e^{-1}(4 - e)$, which is positive. We conclude that the solution is: $x^* = 0$ and $y^* = 1$, with $\lambda_1 = 2$, $\lambda_2 = e^{-1}(4 - e)$.

EXAMPLE 14.9.2 A worker chooses both consumption c and labour supply l in order to maximize the utility function $\alpha \ln c + (1 - \alpha) \ln(1 - l)$ over consumption, c, and leisure, $1 - l$, where $0 < \alpha < 1$. The worker's budget constraint is $c \leq wl + m$, where m is unearned income. In addition, the worker must choose $l \geq 0$. Solve the worker's constrained maximization problem.

Solution: The worker's problem is

$$\max \ \alpha \ln c + (1 - \alpha) \ln(1 - l) \ \text{ s.t. } \ c \leq wl + m \ \text{ and } \ l \geq 0$$

The Lagrangian is

$$\mathcal{L}(c, l) = \alpha \ln c + (1 - \alpha) \ln(1 - l) - \lambda(c - wl - m) + \mu l$$

and the Kuhn–Tucker conditions for (c^*, l^*) to solve the problem are

$$\mathcal{L}_c' = \frac{\alpha}{c^*} - \lambda = 0 \tag{i}$$

$$\mathcal{L}_l' = \frac{-(1 - \alpha)}{1 - l^*} + \lambda w + \mu = 0 \tag{ii}$$

$$\lambda \geq 0, \quad \text{with } \lambda = 0 \text{ if } c^* < wl^* + m \tag{iii}$$

$$\mu \geq 0, \quad \text{with } \mu = 0 \text{ if } l^* > 0 \tag{iv}$$

From (i) we have $\lambda = \alpha/c^* > 0$. Then (iii) together with the first constraint yield

$$c^* = wl^* + m \tag{v}$$

Case I: $\mu = 0$. Then, from (ii) we get $l^* = 1 - (1-\alpha)/\lambda w$. Then (i) and (v) imply that $\lambda = 1/(w+m)$, so $c^* = \alpha(w+m)$ and $l^* = \alpha - (1-\alpha)m/w$. The Kuhn–Tucker conditions are all satisfied provided that $l^* \geq 0$, which holds if and only if $m \leq \alpha w/(1-\alpha)$.

Case II: $\mu > 0$. Then, $l^* = 0$, $c^* = m$, and $\lambda = \alpha/c^* = \alpha/m$. From (ii) it follows that $\mu = 1 - \alpha - \alpha w/m$, and $\mu > 0$ if, and only if, $m > \alpha w/(1-\alpha)$.

In the last two examples it was not too hard to find which constraints bind—that is, hold with equality—at the optimum. But with more complicated nonlinear programming problems, this can be harder. A general method for finding all candidates for optimality in a nonlinear programming problem with two constraints can be formulated as follows: First, examine the case where both constraints bind. Next, examine the two cases where only one constraint binds. Finally, examine the case where neither constraint binds. In each case, find all vectors **x**, with associated non-negative values of the Lagrange multipliers, that satisfy all the relevant conditions—if any do. Then calculate the value of the objective function for these values of **x**, and retain those **x** with the highest values. Except for perverse problems, this procedure will find the optimum. The next example illustrates how it works in practice.

EXAMPLE 14.9.3 Suppose your utility of consuming x_1 units of good A and x_2 units of good B is $U(x_1, x_2) = \ln x_1 + \ln x_2$, that the prices per unit of A and B are \$10 and \$5, respectively, and that you have \$350 to spend on the two goods. Suppose it takes 0.1 hours to consume one unit of A and 0.2 hours to consume one unit of B, and you have eight hours to spend on consuming the two goods. How much of each good should you buy in order to maximize your utility?

Solution: The problem is

$$\max U(x_1, x_2) = \ln x_1 + \ln x_2 \quad \text{s.t.} \quad \begin{cases} 10x_1 + 5x_2 \leq 350 \\ 0.1x_1 + 0.2x_2 \leq 8 \end{cases}$$

The Lagrangian is

$$\mathcal{L} = \ln x_1 + \ln x_2 - \lambda_1(10x_1 + 5x_2 - 350) - \lambda_2(0.1x_1 + 0.2x_2 - 8)$$

and the necessary conditions are that there exist numbers λ_1 and λ_2 such that

$$\mathcal{L}_1' = 1/x_1^* - 10\lambda_1 - 0.1\lambda_2 = 0 \tag{i}$$

$$\mathcal{L}_2' = 1/x_2^* - 5\lambda_1 - 0.2\lambda_2 = 0 \tag{ii}$$

$$\lambda_1 \geq 0, \quad \text{with} \ \lambda_1 = 0 \ \text{if} \ 10x_1^* + 5x_2^* < 350 \tag{iii}$$

$$\lambda_2 \geq 0, \quad \text{with} \ \lambda_2 = 0 \ \text{if} \ 0.1x_1^* + 0.2x_2^* < 8 \tag{iv}$$

We start the systematic procedure:

Case 1: Both constraints bind. Then

$$10x_1^* + 5x_2^* = 350 \tag{v}$$

and $0.1x_1^* + 0.2x_2^* = 8$. The solution is $(x_1^*, x_2^*) = (20, 30)$. Inserting these values into (i) and (ii) yields the system $10\lambda_1 + 0.1\lambda_2 = 1/20$ and $5\lambda_1 + 0.2\lambda_2 = 1/30$, with solution $(\lambda_1, \lambda_2) = (1/225, 1/18)$. In particular, both λ_1 and λ_2 are nonnegative. So we have found a candidate for optimality because all the Kuhn–Tucker conditions are satisfied.

Case 2: Constraint 1 binds, 2 does not. Then (v) holds and $0.1x_1^* + 0.2x_2^* < 8$. From (iv) we obtain $\lambda_2 = 0$. Now (i) and (ii) give $x_2^* = 2x_1^*$. Inserting this into (v), we get $x_1^* = 17.5$ and then $x_2^* = 2x_1^* = 35$. But then $0.1x_1^* + 0.2x_2^* = 8.75$, which violates the second constraint. So there is no candidate for optimality in this case.

Case 3: Constraint 2 binds, 1 does not. Then $10x_1^* + 5x_2^* < 350$ and $0.1x_1^* + 0.2x_2^* = 8$. From (iii), $\lambda_1 = 0$, and (i) and (ii) yield $0.1x_1^* = 0.2x_2^*$. Inserted into $0.1x_1^* + 0.2x_2^* = 8$ this yields $x_2^* = 20$ and so $x_1^* = 40$. But then $10x_1^* + 5x_2^* = 500$, violating the first constraint. So there is no candidate for optimality in this case either.

Case 4: None of the constraints bind. Then $\lambda_1 = \lambda_2 = 0$, in which case (i) and (ii) make no sense.

We conclude that there is only one candidate for optimality, which is $(x_1^*, x_2^*) = (20, 30)$. Since the Lagrangian is easily seen to be concave, we have found the solution.

Properties of the Value Function

As in previous problems, the value function of problem (14.9.1) is defined by $f^*(\mathbf{c}) = f(\mathbf{x}^*(\mathbf{c}))$, where $\mathbf{x}^*(\mathbf{c})$ is the solution to the problem and $\mathbf{c} = (c_1, \ldots, c_m)$. The following properties of f^* are very useful:

$$f^*(\mathbf{c}) \text{ is nondecreasing in each variable } c_1, \ldots, c_m \tag{14.9.3}$$

$$\text{If } \partial f^*(\mathbf{c})/\partial c_j \text{ exists, then it is equal to } \lambda_j(\mathbf{c}), \text{ for } j = 1, \ldots, m \tag{14.9.4}$$

Here, (14.9.3) follows immediately because if c_j increases, and all the other c_k are fixed, then the admissible set becomes larger; hence, $f^*(\mathbf{c})$ cannot decrease. Concerning property (14.9.4), each $\lambda_j(\mathbf{c})$ is a Lagrange multiplier coming from the Kuhn–Tucker conditions.

However, there is a catch: the value function f^* need not be differentiable. Even if f and $g_1, \ldots, g_m$ are all differentiable, the value function can have sudden changes of slope. Such cases are studied in FMEA.

EXERCISES FOR SECTION 14.9

1. Consider the problem $\max \frac{1}{2}x - y$ s.t. $x + e^{-x} \leq y$ and $x \geq 0$.

   (a) Write down the Lagrangian and the necessary Kuhn–Tucker conditions.

   (b) Find the solution to the problem.

SM 2. Solve the following consumer demand problem where, in addition to the budget constraint, there is an upper limit $\bar{x}$ which rations how much of the first good can be bought:

$$\max \alpha \ln x + (1 - \alpha) \ln y \quad \text{s.t.} \quad px + qy \leq m \;\text{ and }\; x \leq \bar{x}$$

SM 3. Consider the problem $\max x + y - e^x - e^{x+y}$ s.t. $x + y \geq 4$, $x \geq -1$ and $y \geq 1$.

   (a) Sketch the admissible set S.

   (b) Find all pairs (x, y) that satisfy all the necessary conditions.

   (c) Find the solution to the problem.

SM 4. Consider the problem $\max x + ay$ s.t. $x^2 + y^2 \leq 1$ and $x + y \geq 0$, where a is a constant.

   (a) Sketch the admissible set and write down the necessary conditions.

   (b) Find the solution for all values of the constant a.

SM 5. Solve the following problem, assuming it has a solution:

$$\max y - x^2 \quad \text{s.t.} \quad y \geq 0,\; y - x \geq -2 \text{ and } y^2 \leq x$$

SM 6. Consider the problem $\max -\left(x + \frac{1}{2}\right)^2 - \frac{1}{2}y^2$ s.t. $e^{-x} - y \leq 0$ and $y \leq \frac{2}{3}$.

   (a) Sketch the admissible set.

   (b) Write down the Kuhn–Tucker conditions, and find the solution of the problem.

7. Consider the problem $\max xz + yz$ s.t. $x^2 + y^2 + z^2 \leq 1$.

   (a) Write down the Kuhn–Tucker conditions.

   (b) Solve the problem.

14.10 Nonnegativity Constraints

Consider the general nonlinear programming problem (14.9.1) once again. Often, variables involved in economic optimization problems must be nonnegative by their very nature. It is not difficult to incorporate such constraints in the formulation of (14.9.1). If $x_1 \geq 0$, for example, this can be represented by the new constraint $h_1(x_1, \ldots, x_n) = -x_1 \leq 0$, and we introduce an additional Lagrange multiplier to go with it. But in order not to have too many Lagrange multipliers, the necessary conditions for the solution of nonlinear programming problems with nonnegativity constraints are sometimes formulated in a slightly different way.

Consider first the problem

$$\max f(x, y) \quad \text{s.t.} \quad g(x, y) \leq c, \ x \geq 0, \ \text{and } y \geq 0 \tag{14.10.1}$$

Here we introduce the functions $h_1(x, y) = -x$ and $h_2(x, y) = -y$, so that the constraints in problem (14.10.1) become $g(x, y) \leq c$, $h_1(x, y) \leq 0$, and $h_2(x, y) \leq 0$. Applying the recipe for solving (14.9.1), we introduce the Lagrangian

$$\mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c] - \mu_1(-x) - \mu_2(-y)$$

The Kuhn–Tucker conditions are

$$\mathcal{L}_1' = f_1'(x, y) - \lambda g_1'(x, y) + \mu_1 = 0 \tag{i}$$

$$\mathcal{L}_2' = f_2'(x, y) - \lambda g_2'(x, y) + \mu_2 = 0 \tag{ii}$$

$$\lambda \geq 0, \quad \text{with } \lambda = 0 \text{ if } g(x, y) < c \tag{iii}$$

$$\mu_1 \geq 0, \quad \text{with } \mu_1 = 0 \text{ if } x > 0 \tag{iv}$$

$$\mu_2 \geq 0, \quad \text{with } \mu_2 = 0 \text{ if } y > 0 \tag{v}$$

From (i), we have $f_1'(x, y) - \lambda g_1'(x, y) = -\mu_1$. From (iv), we have $-\mu_1 \leq 0$ and $-\mu_1 = 0$ if $x > 0$. Thus, (i) and (iv) are together equivalent to

$$f_1'(x, y) - \lambda g_1'(x, y) \leq 0, \ \text{with equality if } x > 0 \tag{vi}$$

In the same way, (ii) and (v) are together equivalent to

$$f_2'(x, y) - \lambda g_2'(x, y) \leq 0, \ \text{with equality if } y > 0 \tag{vii}$$

So the new Kuhn–Tucker conditions are (vi), (vii), and (iii). Note that after replacing (i) and (iv) by (vi), as well as (ii) and (v) by (vii), only the multiplier λ associated with $g(x, y) \leq c$ remains.

The same idea can obviously be extended to the n-variable problem

$$\max f(\mathbf{x}) \ \text{ s.t. } \begin{cases} g_1(\mathbf{x}) \leq c_1 \\ \ \cdots\cdots\cdots \\ g_m(\mathbf{x}) \leq c_m \end{cases} \quad x_1 \geq 0, \ \ldots, \ x_n \geq 0 \tag{14.10.2}$$

Briefly formulated, the necessary conditions for the solution of problem (14.10.2) are that, for each $i = 1, \ldots, n$,

$$\frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} \leq 0, \ \text{with equality if } x_i > 0 \tag{14.10.3}$$

and that

$$\lambda_j \geq 0, \ \text{with } \lambda_j = 0 \text{ if } g_j(\mathbf{x}) < c_j \tag{14.10.4}$$

for all $j = 1, \ldots, m$.

EXAMPLE 14.10.1 Consider the utility maximization problem

$$\max\ x + \ln(1+y) \ \text{ s.t. } \ px + y \le m,\ x \ge 0 \text{ and } y \ge 0$$

where consumption of both commodities is explicitly required to be nonnegative.

(a) Write down the necessary Kuhn–Tucker conditions for a point (x^*, y^*) to be a solution.

(b) Find the solution to the problem, for all positive values of p and m.

Solution:

(a) The Lagrangian is

$$\mathcal{L}(x,y) = x + \ln(1+y) - \lambda(px + y - m)$$

and the Kuhn–Tucker conditions for (x^*, y^*) to be a solution are that there exists a λ such that

$$\mathcal{L}_1'(x^*, y^*) = 1 - p\lambda \le 0, \ \text{with } 1 - p\lambda = 0 \text{ if } x^* > 0 \tag{i}$$

$$\mathcal{L}_2'(x^*, y^*) = \frac{1}{1+y^*} - \lambda \le 0, \ \text{with } \frac{1}{1+y^*} - \lambda = 0 \text{ if } y^* > 0 \tag{ii}$$

$$\lambda \ge 0, \ \text{with } \lambda = 0 \text{ if } px^* + y^* < m \tag{iii}$$

Also, $x^* \ge 0$, $y^* \ge 0$, and the budget constraint has to be satisfied, so $px^* + y^* \le m$.

(b) Note that the Lagrangian is concave, so a point that satisfies the Kuhn–Tucker conditions will be a maximum point. It is clear from (i) that λ cannot be 0. Therefore $\lambda > 0$, so (iii) and $px^* + y^* \le m$ imply that

$$px^* + y^* = m \tag{iv}$$

Regarding which constraints $x \ge 0$ and $y \ge 0$ bind, there are four cases to consider:

Case 1: Suppose $x^* = 0$, $y^* = 0$. Since $m > 0$, this is impossible because of (iv).

Case 2: Suppose $x^* > 0$, $y^* = 0$. From (ii) and $y^* = 0$ we get $\lambda \ge 1$. Then (i) implies that $p = 1/\lambda \le 1$. Equation (iv) gives $x^* = m/p$, so we get one candidate for a maximum point:

$$(x^*, y^*) = (m/p, 0) \text{ and } \lambda = 1/p, \ \text{if } 0 < p \le 1$$

Case 3: Suppose $x^* = 0$, $y^* > 0$. By (iv) we have $y^* = m$. Then (ii) yields $\lambda = 1/(1+y^*) = 1/(1+m)$. From (i) we get $p \ge 1/\lambda = m+1$. This gives one more candidate:

$$(x^*, y^*) = (0, m) \text{ and } \lambda = 1/(1+m), \ \text{if } p \ge m+1$$

Case 4: Suppose $x^* > 0$, $y^* > 0$. With equality in both (i) and (ii), $\lambda = 1/p = 1/(1+y^*)$. It follows that $y^* = p - 1$, and then $p > 1$ because $y^* > 0$. Equation (iv) implies that $px^* = m - y^* = m - p + 1$, so $x^* = (m+1-p)/p$. Since $x^* > 0$, we must have $p < m+1$. Thus we get one last candidate

$$(x^*, y^*) = \left(\frac{m+1-p}{p}, p-1\right) \text{ and } \lambda = 1/p, \ \text{if } 1 < p < m+1$$

Putting all this together, we see that the solution of the problem is: (a) If $0 < p \leq 1$, then $(x^*, y^*) = (m/p, 0)$, with $\lambda = 1/p$, from Case 2. (b) If $1 < p < m+1$, then $(x^*, y^*) = ((m+1-p)/p, p-1)$, with $\lambda = 1/p$, from Case 4. (c) If $p \geq m+1$, then $(x^*, y^*) = (0, m)$, with $\lambda = 1/(m+1)$, from Case 3.

Note that, except in the intermediate case (b) when $1 < p < m+1$, it is optimal to spend everything on only the cheaper of the two goods—either x in case (a), or y in case (c).

EXAMPLE 14.10.2 **(Peak load pricing)** Consider a producer who generates electricity by burning a fuel such as coal or natural gas. The demand for electricity varies between peak periods, during which all the generating capacity is used, and off-peak periods when there is spare capacity. We consider a certain time interval (say, a year) divided into n periods of equal length. Suppose the amounts of electric power sold in these n periods are $x_1, x_2, \ldots, x_n$. Assume that a regulatory authority fixes the corresponding prices at levels equal to $p_1, p_2, \ldots, p_n$. The total operating cost over all n periods is given by the function $C(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n)$, and k is the output capacity in each period. Let $D(k)$ denote the cost of maintaining output capacity at level k. The producer's total profit is then

$$\pi(\mathbf{x}, k) = \sum_{i=1}^{n} p_i x_i - C(\mathbf{x}) - D(k)$$

Because the producer cannot exceed capacity k in any period, it faces the constraints

$$x_1 \leq k, \ldots, x_n \leq k \tag{i}$$

We consider the problem of finding $x_1 \geq 0, \ldots, x_n \geq 0$ and $k \geq 0$ such that profit is maximized subject to the capacity constraints (i).

This is a nonlinear programming problem with $n+1$ variables and n constraints. The Lagrangian is

$$\mathcal{L}(\mathbf{x}, k) = \sum_{i=1}^{n} p_i x_i - C(\mathbf{x}) - D(k) - \sum_{i=1}^{n} \lambda_i (x_i - k)$$

Following (14.10.3) and (14.10.4), the choice $(\mathbf{x}^0, k^0) \geq 0$ can solve the problem only if there exist Lagrange multipliers $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - C_i'(\mathbf{x}^0) - \lambda_i \leq 0 \text{ with equality if } x_i^0 > 0, \text{ for } i = 1, \ldots, n \tag{i}$$

$$\frac{\partial \mathcal{L}}{\partial k} = -D'(k^0) + \sum_{i=1}^{n} \lambda_i \leq 0 \text{ with equality if } k^0 > 0 \tag{ii}$$

$$\lambda_i \geq 0, \text{ with } \lambda_i = 0 \text{ if } x_i^0 < k^0, \text{ for } i = 1, \ldots, n \tag{iii}$$

Suppose that i is such that $x_i^0 > 0$. Then (i) implies that

$$p_i = C_i'(\mathbf{x}^0) + \lambda_i \tag{iv}$$

If period i is an off-peak period, then $x_i^0 < k^0$ and so $\lambda_i = 0$ by (iii). From (iv) it follows that $p_i = C_i'(x_1^0, \ldots, x_n^0)$. Thus, we see that *the profit-maximizing pattern of output* $\mathbf{x}^0$ *will bring* *about equality between the regulator's price in any off-peak period and the corresponding marginal operating cost.*

On the other hand, λ_j might be positive in a peak period when $x_j^0 = k^0$. If $k^0 > 0$, it follows from (ii) that $\sum_{i=1}^n \lambda_i = D'(k^0)$. We conclude that the output pattern will be such that *in peak periods the price set by the regulator will exceed the marginal operating cost by an additional amount* λ_j, *which is really the "shadow price" of the capacity constraint* $x_j^0 \le k^0$. *The sum of these shadow prices over all peak periods is equal to the marginal capacity cost.*

EXERCISES FOR SECTION 14.10

1. Consider the utility maximization problem $\max\ x + \ln(1+y)$ s.t. $16x + y \le 495, x \ge 0, y \ge 0$.

   (a) Write down the necessary Kuhn–Tucker conditions, with nonnegativity constraints, for a point to be a solution.

   (b) Find the solution to the problem.

   (c) Estimate by how much utility will increase if income is increased from 495 to 500.

SM 2. Solve the following problem, assuming it has a solution:

$$\max\ xe^{y-x} - 2ey \quad \text{s.t.} \quad y \le 1 + x/2,\ x \ge 0,\ \text{and}\ y \ge 0$$

SM 3. Suppose that optimal capacity utilization by a firm requires that its output quantities x_1 and x_2, along with its capacity level k, should be chosen to solve the problem

$$\max\ x_1 + 3x_2 - x_1^2 - x_2^2 - k^2 \quad \text{s.t.} \quad x_1 \le k,\ x_2 \le k,\ x_1 \ge 0,\ x_2 \ge 0,\ k \ge 0$$

Show that $k = 0$ cannot be optimal, and then find the solution.

REVIEW EXERCISES

1. Consider the problem $\max f(x,y) = 3x + 4y$ s.t. $g(x,y) = x^2 + y^2 = 225$.

   (a) Solve it using the Lagrange multiplier method.

   (b) Suppose 225 is changed to 224. What is the approximate change in the optimal value of f?

2. Use result (∗∗) in Example 14.1.3 to write down the solution to the problem of maximizing $f(x,y)$ subject to $px + qy = m$, in each of the following cases, assuming $x \ge 0$ and $y \ge 0$:

   (a) $f(x,y) = 25x^2y^3$ (b) $f(x,y) = x^{1/5}y^{2/5}$ (c) $f(x,y) = 10\sqrt{x}\sqrt[3]{y}$

SM 3. By selling x tons of one commodity the firm gets a price per ton given by $p(x)$. By selling y tons of another commodity the price per ton is $q(y)$. The cost of producing and selling x tons of the first commodity and y tons of the second is given by $C(x,y)$.

(a) Write down the firm's profit function and find necessary conditions for $x^* > 0$ and $y^* > 0$ to solve the problem. Give economic interpretations of the necessary conditions.

(b) Suppose that the firm's production activity causes so much pollution that the authorities limit its output to no more than m tons of total output. Write down the necessary conditions for $\hat{x} > 0$ and $\hat{y} > 0$ to solve the problem.

4. Suppose $U(x, y)$ denotes the utility enjoyed by a person when having x hours of leisure per day (24 hours) and y units per day of other goods. The person gets an hourly wage of w and pays an average price of p per unit of the other goods, so that

$$py = w(24 - x) \tag{$*$}$$

assuming that the person spends all that is earned.

(a) Show that maximizing $U(x, y)$ subject to the constraint $(*)$ leads to the equation

$$pU_1'(x, y) = wU_2'(x, y) \tag{$**$}$$

(b) Suppose that the equations $(*)$ and $(**)$ define x and y as differentiable functions $x(p, w)$, $y(p, w)$ of p and w. Show that, with appropriate conditions on $U(x, y)$,

$$\frac{\partial x}{\partial w} = \frac{(24 - x)(wU_{22}'' - pU_{12}'') + pU_2'}{p^2 U_{11}'' - 2pwU_{12}'' + w^2 U_{22}''}$$

SM **5.** Consider the problems

$$\max(\min)\ x^2 + y^2 - 2x + 1 \ \text{ s.t. } \ \tfrac{1}{4}x^2 + y^2 = b$$

where b is a constant.[24]

(a) Solve the problem in case $b > \frac{4}{9}$.

(b) If $f^*(b)$ denotes the value function for the maximization problem, verify that $\mathrm{d}f^*(b)/\mathrm{d}b = \lambda$ when $b > \frac{4}{9}$, where λ is the corresponding Lagrange multiplier.

6. Consider the utility maximization problem (14.1.4) with a "separable" utility function $u(x, y) = v(x) + w(y)$, where $v'(x) > 0$, $w'(y) > 0$, $v''(x) \le 0$, and $w''(y) \le 0$.

(a) State the first-order conditions for utility maximization.

(b) Why are these conditions sufficient for optimality?

SM **7.** Consider the problem

$$\min\ x^2 - 2x + 1 + y^2 - 2y \ \text{ s.t. } \ (x + y)\sqrt{x + y + b} = 2\sqrt{a}$$

where a and b are positive constants and x and y are positive.

(a) Suppose that (x, y) solves the problem. Show that x and y must then satisfy the equations

$$x = y \text{ and } 2x^3 + bx^2 = a \tag{$*$}$$

(b) The equations in $(*)$ define x and y as differentiable functions of a and b. Find expressions for $\partial x/\partial a$, $\partial^2 x/\partial a^2$, and $\partial x/\partial b$.

[24] The constraint has a graph that is an ellipse in the xy-plane, so it defines a closed and bounded set.

SM **8.** For all $a > 0$, solve the problem $\max\ 10 - (x-2)^2 - (y-1)^2$ s.t. $x^2 + y^2 \leq a$.

SM **9.** Consider the nonlinear programming problem

$$\max xy \ \text{ s.t. } \begin{cases} x^2 + ry^2 \leq m \\ \qquad\quad x \geq 1 \end{cases}$$

where r and m are positive constants, with $m > 1$.

(a) Write down the necessary Kuhn–Tucker conditions for a point to be a solution of the problem.

(b) Solve the problem.

(c) Let $V(r, m)$ denote the value function. Compute $\partial V(r, m)/\partial m$, and comment on its sign.

(d) Verify that $\partial V(r, m)/\partial r = \partial \mathcal{L}/\partial r$, where $\mathcal{L}$ is the Lagrangian.

10. Suppose the firm of Example 8.5.1 earns revenue $R(Q)$ and incurs cost $C(Q)$ as functions of output $Q \geq 0$, where $R'(Q) > 0$, $C'(Q) > 0$, $R''(Q) < 0$, and $C''(Q) > 0$ for all $Q \geq 0$. The firm maximizes profit $\pi(Q) = R(Q) - C(Q)$ subject to $Q \geq 0$. Write down the first-order conditions for the solution to this problem, and find sufficient conditions for the constraint to bind at the optimum.

11. A firm uses K and L units of two inputs to produce $\sqrt{KL}$ units of a product, where $K > 0$, $L > 0$. The input factor costs are r and w per unit, respectively. The firm wants to minimize the costs of producing at least Q units.

(a) Formulate the nonlinear programming problem that emerges. Reformulate it as a maximization problem, then write down the Kuhn–Tucker conditions for the optimum. Solve these conditions to determine K^* and L^* as functions of (r, w, Q).

(b) Define the minimum cost function as $c^*(r, w, Q) = rK^* + wL^*$. Verify that $\partial c^*/\partial r = K^*$ and $\partial c^*/\partial w = L^*$, then give these results economic interpretations.

15

MATRIX AND VECTOR ALGEBRA

Indeed, models basically play the same role in economics as in fashion. They provide an articulated frame on which to show off your material to advantage... a useful role, but fraught with the dangers that the designer may get carried away by his personal inclination for the model, while the customer may forget that the model is more streamlined than reality.

—Jacques H. Drèze (1984)

Most mathematical models used by economists ultimately involve a system of several equations, which usually express how one or more endogenous variables depend on several exogenous parameters. If these equations are all linear, the study of such systems belongs to an area of mathematics called *linear algebra*. Even if the equations are nonlinear, much may be learned from linear approximations around the solution we are interested in—for example, how the solution changes in response to small shocks to the exogenous parameters.

Indeed, linear models of this kind provide the logical basis for the econometric techniques that are routinely used in most modern empirical economic analysis. Moreover, linear models become much easier to understand if we use some key mathematical concepts such as matrices, vectors, and determinants. These, as well as their application to economic models, will be introduced in this chapter and in the next.

Importantly, the usefulness of linear algebra extends far beyond its ability to solve systems of linear equations. For instance, in the theory of differential and difference equations, in linear and nonlinear optimization theory, in statistics and econometrics, the methods of linear algebra are used extensively.

15.1 Systems of Linear Equations

Section 3.6 has already introduced systems of two simultaneous linear equations in two variables. In subsequent chapters, notably Chapter 14, we have encountered up to five linear equations in five unknowns. These systems were solved in an *ad hoc* manner. It is now time to study systems of linear equations more systematically.

The first key step is to introduce suitable notation for what may be a large linear system of equations. Specifically, we consider m equations in n unknowns, where m may be greater than, equal to, or less than n. If the unknowns are denoted by $x_1, \ldots, x_n$, we usually write such a system in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{15.1.1}$$

Here $a_{11}, a_{12}, \ldots, a_{mn}$ are called the *coefficients* of the system, and $b_1, \ldots, b_m$ are called the *right-hand sides*. All are real numbers.

Note carefully the order of the subscripts. In general, a_{ij} is the coefficient in the i-th equation of the j-th variable, x_j. One or more of these coefficients may be 0—indeed, the system usually becomes easier to analyse and solve if a high proportion of the coefficients are 0.

A *solution* of system (15.1.1) is an ordered set or list of numbers $s_1, s_2, \ldots, s_n$ that satisfies all the equations simultaneously when we put $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$. Usually, a solution is written as $(s_1, s_2, \ldots, s_n)$. Note that the order in which we write the components is essential in the sense that if $(s_1, s_2, \ldots, s_n)$ satisfies (15.1.1), then $(s_n, s_{n-1}, \ldots, s_1)$, say, will usually *not* be a solution.

If system (15.1.1) has at least one solution, it is said to be *consistent*. When the system has no solution, it is said to be *inconsistent*.

EXAMPLE 15.1.1 To check your understanding of the notation used, write down the system of equations (15.1.1) when $n = m = 3$ and $a_{ij} = i + j$ for $i, j = 1, 2, 3$, while $b_i = i$ for $i = 1, 2, 3$. Verify that $(x_1, x_2, x_3) = (2, -1, 0)$ is a solution, but that $(x_1, x_2, x_3) = (2, 0, -1)$ is not.

Solution: The coefficients are $a_{11} = 1 + 1 = 2$, $a_{12} = 1 + 2 = 3$, etc. Set out in full, the system of equations is

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 1 \\ 3x_1 + 4x_2 + 5x_3 &= 2 \\ 4x_1 + 5x_2 + 6x_3 &= 3 \end{aligned}$$

Inserting $(x_1, x_2, x_3) = (2, -1, 0)$ we see that all the equations are satisfied, so this is a solution. On the other hand, if we change the order of the numbers 2, -1 and 0 to form the triple $(x_1, x_2, x_3) = (2, 0, -1)$, then $2x_1 + 3x_2 + 4x_3 = 0$, so the first equation is not satisfied, and $(2, 0, -1)$ is not a solution to the system.[1]

There are computer programs that make it easy to check whether a system like (15.1.1) is consistent, and if it is, to find possible solutions, even if there are thousands of equations and

[1] In fact, the general solution is $(x_1, x_2, x_3) = (2 + t, -1 - 2t, t)$, with t an arbitrary real number. In the terminology of Section 12.10, the system has one degree of freedom.

unknowns. Still, economists need to understand the general theory of such equation systems so that they can follow theoretical arguments and conclusions related to linear models.

EXERCISES FOR SECTION 15.1

1. Decide which of the following single equations in the variables x, y, z, and w are linear, and which are not:

   (a) $3x - y - z - w = 50$ (b) $\sqrt{3}x + 8xy - z + w = 0$

   (c) $3.33x - 4y + \dfrac{800}{3}z = 3$ (d) $3(x + y - z) = 4(x - 2y + 3z)$

   (e) $(x - y)^2 + 3z - w = -3$ (f) $2a^2x - \sqrt{|b|}\,y + (2 + \sqrt{|a|}\,)z = b^2$ where a, b are constants.

2. Let x_1, y_1, x_2, and y_2 be constants and consider the following equations in the variables a, b, c, and d.[2]

$$ax_1^2 + bx_1y_1 + cy_1^2 + d = 0$$
$$ax_2^2 + bx_2y_2 + cy_2^2 + d = 0$$

   Is this a linear system of equations in a, b, c, and d?

3. Write down the system of equations (15.1.1) in the case when $n = 4$, $m = 3$, and $a_{ij} = i + 2j + (-1)^i$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, whereas $b_i = 2^i$ for $i = 1, 2, 3$.

4. Write system (15.1.1) out in full when $n = m = 4$ and $a_{ij} = 1$ for all $i \neq j$, while $a_{ii} = 0$ for $i = 1$, 2, 3, 4. Sum the four equations to derive a simple equation for $\sum_{i=1}^{4} x_i$, then solve the whole system.

5. Consider a collection of n individuals, each of whom owns a definite quantity of m different commodities. Let a_{ij} be the number of units of commodity i owned by individual j, where $i = 1, 2, \ldots, m$, whereas $j = 1, 2, \ldots, n$.

   (a) What does the list $(a_{1j}, a_{2j}, \ldots, a_{mj})$ represent?

   (b) Explain in words what the two sums $a_{11} + a_{12} + \cdots + a_{1n}$ and $a_{i1} + a_{i2} + \cdots + a_{in}$ express.

   (c) Let p_i denote the price per unit of commodity i, for $i = 1, 2, \ldots, m$. What is the total value of the commodities owned by individual j?

SM 6. Trygve Haavelmo (1911–1999), a Norwegian Nobel prize-winning economist, devised a model of the US economy for the years 1929–1941 that is based on the following four equations:
(i) $c = 0.712y + 95.05$ (ii) $y = c + x - s$ (iii) $s = 0.158(c + x) - 34.30$ (iv) $x = 93.53$.
Here x denotes total investment, y is disposable income, s is the total saving by firms, and c is total consumption. Write the system of equations in the form (15.1.1) when the variables appear in the order x, y, s, and c. Then find the solution of the system.

[2] In *almost* all other cases in this book, a, b, c, and d denote constants!

15.2 Matrices and Matrix Operations

A *matrix* is simply a rectangular array of numbers considered as one mathematical object. When there are m *rows* and n *columns* in the array, we have an m-by-n matrix, written as $m \times n$. We usually denote a matrix with bold capital letters such as **A**, **B**, and so on. In general, an $m \times n$ matrix is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{15.2.1}$$

In this book any array like (15.2.1) will be enclosed with large parentheses surrounding the numbers. Be warned, however, that some writers replace the parentheses in (15.2.1) with large square brackets.

The matrix **A** in (15.2.1) is said to have *order* $m \times n$. The mn numbers that constitute **A** are called its *elements* or *entries*. In particular, a_{ij} denotes the element in the i-th row and the j-th column. For brevity, the $m \times n$ matrix in (15.2.1) is often expressed as $(a_{ij})_{m\times n}$, or more simply as (a_{ij}), if the order $m \times n$ is either obvious or unimportant.

A matrix with either only one row or only one column is called a *vector*. It is usual to distinguish between a *row vector*, which has only one row, and a *column vector*, which has only one column. It is usual to denote row or column vectors by small bold letters like **x** or **y** rather than capital letters.

EXAMPLE 15.2.1 The following arrays are matrices:

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 5 & 8 \end{pmatrix}, \qquad \mathbf{B} = \left(-1, 2, \sqrt{3}, 16\right), \qquad \mathbf{C} = \begin{pmatrix} -1 & 2 \\ 8 & 5 \\ 7 & 6 \\ 1 & 1 \end{pmatrix}$$

Of these, **A** is 2×2, **B** is 1×4 (and so a row vector), and **C** is 4×2. Also $a_{21} = 5$ and $c_{32} = 6$. Note that c_{23} is undefined because **C** only has two columns.

EXAMPLE 15.2.2 Construct the 4×3 matrix $\mathbf{A} = (a_{ij})_{4\times 3}$ with $a_{ij} = 2i - j$.

Solution: The matrix **A** has $4 \cdot 3 = 12$ entries. Because $a_{ij} = 2i - j$, it follows that $a_{11} = 2 \cdot 1 - 1 = 1$, $a_{12} = 2 \cdot 1 - 2 = 0$, $a_{13} = 2 \cdot 1 - 3 = -1$, and so on. The complete matrix is

$$\mathbf{A} = \begin{pmatrix} 2\cdot 1 - 1 & 2\cdot 1 - 2 & 2\cdot 1 - 3 \\ 2\cdot 2 - 1 & 2\cdot 2 - 2 & 2\cdot 2 - 3 \\ 2\cdot 3 - 1 & 2\cdot 3 - 2 & 2\cdot 3 - 3 \\ 2\cdot 4 - 1 & 2\cdot 4 - 2 & 2\cdot 4 - 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \\ 7 & 6 & 5 \end{pmatrix}$$

If $m = n$, so that the matrix has the same number of columns as rows, it is called a *square matrix* of order n. If $\mathbf{A} = (a_{ij})_{n\times n}$, then the n elements $a_{11}, a_{22}, \dots, a_{nn}$ with $i = j$ constitute the *main*, or *principal, diagonal* that runs from the top left, a_{11}, to the bottom right, a_{nn}. For instance, the matrix **A** in Example 15.2.1 is a square matrix of order 2, whose main diagonal consists of the numbers 3 and 8. Note that only a square matrix can have a main diagonal.

EXAMPLE 15.2.3 Consider the general linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{15.2.2}$$

of m equations in the n unknown variables x_j, for $j = 1, 2, \ldots, n$. It is natural to represent the coefficients of these unknowns in (15.2.2) by the $m \times n$ matrix **A** that is defined by Eq. (15.2.1). Then **A** is called the *coefficient matrix* of (15.2.2). For instance, the coefficient matrix of the equation system

$$\begin{aligned} 3x_1 - 2x_2 + 6x_3 &= 5 \\ 5x_1 + x_2 + 2x_3 &= -2 \end{aligned} \quad \text{is} \quad \begin{pmatrix} 3 & -2 & 6 \\ 5 & 1 & 2 \end{pmatrix}$$

One can also represent the numbers b_i $(i = 1, 2, \ldots, m)$ on the right-hand side of (15.2.2) by an $m \times 1$ matrix, or column vector, often denoted by **B** or by **b**.

EXAMPLE 15.2.4 Consider a chain of stores with four outlets labelled B_1, B_2, B_3, and B_4, each selling eight different commodities, $V_1, V_2, \ldots, V_8$. Let a_{ij} denote the dollar value of the sales of commodity V_i at outlet B_j during a certain month. A suitable way of recording this data is in the 8×4 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \vdots & \vdots & \vdots & \vdots \\ a_{81} & a_{82} & a_{83} & a_{84} \end{pmatrix}$$

The eight rows refer to the eight commodities, whereas the four columns refer to the four outlets. For instance, if $a_{73} = 225$, this means that the sales of commodity 7 at outlet 3 were worth \$225 for the month in question.

Matrix Operations

So far, matrices have been regarded as just rectangular arrays of numbers that can be useful for storing information. The real motivation for introducing matrices, however, is that there are useful rules for manipulating them that correspond, to some extent, with the familiar rules of ordinary algebra.

First, let us agree how to define equality between matrices of the same order. If $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ are both $m \times n$ matrices, then **A** and **B** are said to be *equal*, and we write $\mathbf{A} = \mathbf{B}$, provided that $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$. Thus, two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal if they have the same order *and* if all their corresponding entries are equal. If $\mathbf{A}$ and $\mathbf{B}$ are *not* equal, then we write $\mathbf{A} \neq \mathbf{B}$.

EXAMPLE 15.2.5 Determine conditions under which $\begin{pmatrix} 3 & t-1 \\ 2t & u \end{pmatrix} = \begin{pmatrix} t & 2v \\ u+1 & t+w \end{pmatrix}$.

Solution: Both sides of the equation are 2×2 matrices. Since both have four elements, equality requires the four equations $3 = t$, $t - 1 = 2v$, $2t = u + 1$, and $u = t + w$ to be satisfied. By solving these simultaneous equations, it follows that the two matrices are equal if and only if $t = 3, v = 1, u = 5$, and $w = 2$. Then both matrices are equal, and equal to $\begin{pmatrix} 3 & 2 \\ 6 & 5 \end{pmatrix}$.

Let us return to Example 15.2.4, where the 8×4 matrix $\mathbf{A}$ represents the dollar values of total sales of the eight commodities at the four outlets in a certain month. Suppose that the dollar values of sales for the next month are given by a corresponding 8×4 matrix $\mathbf{B} = (b_{ij})_{8\times4}$. The total sales revenues from each commodity in each of the outlets in the course of these two months combined would then be given by a new 8×4 matrix $\mathbf{C} = (c_{ij})_{8\times4}$, where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, \ldots, 8$ and for $j = 1, \ldots, 4$. Matrix $\mathbf{C}$ is called the "sum" of $\mathbf{A}$ and $\mathbf{B}$, and we write $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

MATRIX ADDITION AND MULTIPLICATION BY A SCALAR

If $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ are two matrices of the same order, we define the *sum* of $\mathbf{A}$ and $\mathbf{B}$ as the $m \times n$ matrix $(a_{ij} + b_{ij})_{m\times n}$. Thus,

$$\mathbf{A} + \mathbf{B} = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n} \tag{15.2.3}$$

If α is a real number, we define $\alpha\mathbf{A}$ by

$$\alpha\mathbf{A} = \alpha(a_{ij})_{m\times n} = (\alpha a_{ij})_{m\times n} \tag{15.2.4}$$

Thus, we add two matrices of the same order by adding their corresponding entries, whereas to multiply a matrix by a scalar, multiply *each* entry in the matrix by that scalar. Returning to the chain of stores, the matrix equation $\mathbf{B} = 2\mathbf{A}$ would mean that all the entries in $\mathbf{B}$ are twice the corresponding elements in $\mathbf{A}$—that is, the sales revenue for each commodity in each of the outlets has exactly doubled from one month to the next. Equivalently, $2\mathbf{A} = \mathbf{A} + \mathbf{A}$.

EXAMPLE 15.2.6 Compute $\mathbf{A} + \mathbf{B}$, $3\mathbf{A}$, and $\left(-\frac{1}{2}\right)\mathbf{B}$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & -3 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution: Using Eqs (15.2.3) and (15.2.4)

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}1 & 3 & 2\\ 5 & -3 & 1\end{pmatrix},\quad 3\mathbf{A}=\begin{pmatrix}3 & 6 & 0\\ 12 & -9 & -3\end{pmatrix},\quad \left(-\tfrac{1}{2}\right)\mathbf{B}=\begin{pmatrix}0 & -\frac{1}{2} & -1\\ -\frac{1}{2} & 0 & -1\end{pmatrix}$$

The matrix $(-1)\mathbf{A}$ is usually denoted by $-\mathbf{A}$, and the difference $\mathbf{A}-\mathbf{B}$ between two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimension means the same as $\mathbf{A}+(-1)\mathbf{B}$. In our chain store example in Example 15.2.4, $\mathbf{B}-\mathbf{A}$ denotes the net increase in sales revenue for each commodity from each outlet between the first month and the second. Positive components represent increases and negative components represent decreases.

With the definitions given earlier, it is easy to derive the following useful rules.

RULES FOR MATRIX ADDITION AND MULTIPLICATION BY SCALARS

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be arbitrary $m\times n$ matrices, and let α and β be real numbers. Also, let $\mathbf{0}$ denote the $m\times n$ matrix consisting only of zeros, called the *zero matrix*. Then:

(a) $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$ (b) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

(c) $\mathbf{A}+\mathbf{0}=\mathbf{A}$ (d) $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$

(e) $(\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}$ (f) $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$

Each of these rules follows directly from the definitions and the corresponding rules for ordinary numbers. Because of rule (a), there is no need to put parentheses in expressions like $\mathbf{A}+\mathbf{B}+\mathbf{C}$. Note also that definitions (15.2.3) and (15.2.4) imply that $\mathbf{A}+\mathbf{A}+\mathbf{A}$ is equal to $3\mathbf{A}$.

EXERCISES FOR SECTION 15.2

1. Construct the matrix $\mathbf{A}=(a_{ij})_{3\times 3}$ where $a_{ii}=1$ for $i=1,2,3$, and $a_{ij}=0$ for $i\neq j$.

2. Evaluate $\mathbf{A}+\mathbf{B}$ and $3\mathbf{A}$ when $\mathbf{A}=\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -1\\ 5 & 2\end{pmatrix}$.

3. Determine for what values of u and v the following equality holds:

$$\begin{pmatrix}(1-u)^2 & v^2 & 3\\ v & 2u & 5\\ 6 & u & -1\end{pmatrix}=\begin{pmatrix}4 & 4 & u\\ v & -3v & u-v\\ 6 & v+5 & -1\end{pmatrix}$$

4. Evaluate $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, and $5\mathbf{A}-3\mathbf{B}$ when $\mathbf{A}=\begin{pmatrix}0 & 1 & -1\\ 2 & 3 & 7\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -1 & 5\\ 0 & 1 & 9\end{pmatrix}$.

15.3 Matrix Multiplication

The rules we just gave for adding or subtracting matrices, and for multiplying a matrix by a scalar, should seem quite natural. The rule for matrix multiplication, however, is more subtle.[3] We justify it by considering how to manipulate an equation system.

Consider, for example, the following two linear equation systems:

$$\begin{aligned} z_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ z_2 &= a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \end{aligned} \tag{i}$$

and

$$\begin{aligned} y_1 &= b_{11}x_1 + b_{12}x_2 \\ y_2 &= b_{21}x_1 + b_{22}x_2 \\ y_3 &= b_{31}x_1 + b_{32}x_2 \end{aligned} \tag{ii}$$

The matrices of coefficients appearing on the right-hand sides of these two systems of equations are, respectively,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

System (i) expresses the z variables in terms of the y's, whereas in (ii), the y's are expressed in terms of the x's. So the z variables must be related to the x variables. Indeed, take the expressions for y_1, y_2, and y_3 in (ii) and insert them into (i). The result is

$$\begin{aligned} z_1 &= a_{11}(b_{11}x_1 + b_{12}x_2) + a_{12}(b_{21}x_1 + b_{22}x_2) + a_{13}(b_{31}x_1 + b_{32}x_2) \\ z_2 &= a_{21}(b_{11}x_1 + b_{12}x_2) + a_{22}(b_{21}x_1 + b_{22}x_2) + a_{23}(b_{31}x_1 + b_{32}x_2) \end{aligned}$$

Rearranging the terms yields

$$\begin{aligned} z_1 &= (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})x_1 + (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32})x_2 \\ z_2 &= (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31})x_1 + (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})x_2 \end{aligned}$$

The coefficient matrix of this system is, therefore,

$$\mathbf{C} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$$

The matrix $\mathbf{A}$ is 2×3 and $\mathbf{B}$ is 3×2. Thus, $\mathbf{B}$ *has as many rows as* $\mathbf{A}$ *has columns*. The matrix $\mathbf{C}$ is 2×2. Note that if we let $\mathbf{C} = (c_{ij})_{2\times 2}$, then the number

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

---

[3] It is tempting to define the product of two matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ of the same dimensions this way: The product of $\mathbf{A}$ and $\mathbf{B}$ is simply the matrix $\mathbf{C} = (c_{ij})_{m\times n}$ where each element $c_{ij} = a_{ij}b_{ij}$ is obtained by multiplying the entries of the two matrices term by term. This is a respectable matrix operation and, in fact, matrix $\mathbf{C}$ is called the *Hadamard product* of $\mathbf{A}$ and $\mathbf{B}$. However, the definition of matrix multiplication that we give is by far the one most used in linear algebra.

in the first row and first column is obtained by multiplying each of the three elements in the first row of **A** by the corresponding element in the first column of **B**, and then adding these three products. We call the resulting expression $a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$ the "inner product" of the first row in **A** and the first column in **B**. Likewise, c_{12} is the inner product of the first row in **A** and the second column in **B**, and so on. Generally, each element c_{ij} is the inner product of the i-th row in **A** and the j-th column in **B**.

The matrix **C** is called the *(matrix) product* of **A** and **B**, and we write $\mathbf{C} = \mathbf{AB}$. Here is a numerical example.

EXAMPLE 15.3.1 By definition:

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 2 + 3 \cdot 6 & 1 \cdot 3 + 0 \cdot 5 + 3 \cdot 2 \\ 2 \cdot 1 + 1 \cdot 2 + 5 \cdot 6 & 2 \cdot 3 + 1 \cdot 5 + 5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 19 & 9 \\ 34 & 21 \end{pmatrix}$$

In order to extend the argument to general matrices, assume that, as in (i), the variables $z_1, \dots, z_m$ are expressed linearly in terms of $y_1, \dots, y_n$, and that, as in (ii), the variables $y_1, \dots, y_n$ are expressed linearly in terms of $x_1, \dots, x_p$. Then $z_1, \dots, z_m$ can be expressed linearly in terms of $x_1, \dots, x_p$. Provided that the matrix **B** does indeed have as many rows as **A** has columns, the result we get leads directly to the following definition:

MATRIX MULTIPLICATION

Suppose that $\mathbf{A} = (a_{ij})_{m \times n}$ and that $\mathbf{B} = (b_{ij})_{n \times p}$. Then the product $\mathbf{C} = \mathbf{AB}$ is the $m \times p$ matrix $\mathbf{C} = (c_{ij})_{m \times p}$, whose element in the i^{th} row and the j^{th} column is the inner product

$$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} + \cdots + a_{in}b_{nj} \tag{15.3.1}$$

of the i-th row of **A** and the j-th column of **B**.

Note that to get c_{ij} we multiply each component a_{ir} in the i-th row of **A** by the corresponding component b_{rj} in the j-th column of **B**, then add all the products. One way of visualizing matrix multiplication is this:

$$\begin{pmatrix} a_{11} & \dots & a_{1k} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \boxed{a_{i1} & \dots & a_{ik} & \dots & a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mk} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \dots & \boxed{b_{1j}} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \dots & b_{kj} & \dots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & \boxed{c_{ij}} & \dots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mp} \end{pmatrix}$$

It bears repeating that the matrix product **AB** is defined only if the number of columns in **A** is equal to the number of rows in **B**. Also, if **A** and **B** are two matrices, then **AB** might be defined, even if **BA** is not. For instance, if **A** is 6×3 and **B** is 3×5, then **AB** is defined as a 6×5 matrix, whereas **BA** is not defined.

EXAMPLE 15.3.2 Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Compute the matrix product **AB**. Is the product **BA** defined?

Solution: **A** is 3×3 and **B** is 3×2, so **AB** is a 3×2 matrix:[4]

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 & 2 \\ \boxed{2 \quad 3 \quad 1} \\ 4 & -1 & 6 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ \boxed{8} & 5 \\ 5 & 14 \end{pmatrix}$$

The matrix product **BA** is not defined because the number of columns in **B** is not equal to the number of rows in **A**.

Note that in the previous example, **AB** was defined but **BA** was not. Even in cases in which **AB** and **BA** are both defined, they are usually not equal, as illustrated by Exercise 1 and Example 15.4.4.

For this reason, when we write **AB**, we say that we *premultiply* **B** by **A**, whereas in **BA** we *postmultiply* **B** by **A**.

EXAMPLE 15.3.3 Initially, three firms, A, B, and C, share the market for a certain commodity. Firm A has 20% of the market, B has 60%, and C has 20%. In the course of the next year, the following changes occur: firm A keeps 85% of its customers, while losing 5% to B and 10% to C; firm B keeps 55% of its customers, while losing 10% to A and 35% to C; and firm C keeps 85% of its customers, while losing 10% to A and 5% to B.

We can represent market shares of the three firms by means of a *market share vector*, defined as a column vector **s** whose components are all nonnegative and sum to 1. Define the matrix **T** and the initial market share vector **s** by

$$\mathbf{T} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix}$$

Notice that t_{ij} is the percentage of j's customers who become i's customers in the next period. So **T** is called the *transition matrix*.

Compute the vector **Ts**, show that it is also a market share vector, and give an interpretation. What is the interpretation of **T**(**Ts**), **T**(**T**(**Ts**)), . . . ?

[4] We have indicated how the element in the second row and first column of **AB** is found. It is the inner product of the second row in **A** and the first column in **B**; this is $2 \cdot 3 + 3 \cdot 1 + 1 \cdot (-1) = 8$.

Solution: Computing directly,

$$\mathbf{Ts} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix}$$

Because $0.25 + 0.35 + 0.40 = 1$, the product $\mathbf{Ts}$ is also a market share vector. The first entry in $\mathbf{Ts}$ is obtained from the calculation

$$0.85 \cdot 0.2 + 0.10 \cdot 0.6 + 0.10 \cdot 0.2 = 0.25$$

Here $0.85 \cdot 0.2$ is A's share of the market that it retains after one year, whereas $0.10 \cdot 0.6$ is the share A gains from B, and $0.10 \cdot 0.2$ is the share A gains from C. The sum is therefore A's total share of the market after one year. The other entries in $\mathbf{Ts}$ can be interpreted similarly, so $\mathbf{Ts}$ must be the new market share vector after one year. Then $\mathbf{T}(\mathbf{Ts})$ is the market share vector after one more year—that is, after two years, and so on.[5]

Systems of Equations in Matrix Form

The definition of matrix multiplication was introduced in order to allow systems of equations to be manipulated. Indeed, it turns out that we can write linear systems of equations very compactly by means of matrix multiplication.

For instance, consider the system

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 7x_1 - 2x_2 &= 2 \end{aligned}$$

Now define

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Then we see that

$$\mathbf{Ax} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix}$$

So the original system is equivalent to the matrix equation $\mathbf{Ax} = \mathbf{B}$.

In general, consider the linear system (15.1.1) with m equations and n unknowns. Suppose we define

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

So $\mathbf{A}$ is $m \times n$ and $\mathbf{x}$ is $n \times 1$. The matrix product $\mathbf{Ax}$ is then defined and is $m \times 1$. Moreover, you can easily check that (15.1.1) can be written as $\mathbf{Ax} = \mathbf{B}$. This very concise notation turns out to be extremely useful.

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[5] Exercise 8 asks you to find $\mathbf{T}(\mathbf{Ts})$.

EXERCISES FOR SECTION 15.3

1. Compute the products **AB** and **BA**, if possible, when **A** and **B** are, respectively:

(a) $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 4 \\ 1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 3 & -2 \\ 1 & 0 & 4 \end{pmatrix}$ and $\begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 0 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 0 & -2 & 3 \end{pmatrix}$

2. Calculate $3\mathbf{A} + 2\mathbf{B} - 2\mathbf{C} + \mathbf{D}$, $\mathbf{AB}$, and $\mathbf{C}(\mathbf{AB})$, for the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}, \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

3. Find the matrices $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}(\mathbf{BC})$, and $(\mathbf{AB})\mathbf{C}$, given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$$

4. Write out three matrix equations corresponding to the following systems:

(a) $\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 + 5x_2 &= 5 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 5 \\ 2x_1 + 3x_2 - x_3 &= 1 \end{aligned}$ (c) $\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$

5. Consider the three matrices $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$, and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Find a matrix **C** satisfying $(\mathbf{A} - 2\mathbf{I})\mathbf{C} = \mathbf{I}$.

(b) Is there a matrix **D** satisfying $(\mathbf{B} - 2\mathbf{I})\mathbf{D} = \mathbf{I}$?

SM 6. If **A** is an $m \times n$ matrix and **B** is another matrix such that both products **AB** and **BA** are defined, what must be the dimensions of **B**?

SM 7. Find all matrices **B** that "commute" with $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ in the sense that $\mathbf{AB} = \mathbf{BA}$.

8. In Example 15.3.3, compute $\mathbf{T}(\mathbf{Ts})$.

15.4 Rules for Matrix Multiplication

In Section 15.3 we saw that matrix multiplication is more complicated than the rather obvious operations of matrix addition and multiplication by a scalar that had been set out in

Section 15.2. So we need to examine carefully what rules matrix multiplication does satisfy. We have already noticed that the commutative law $\mathbf{AB} = \mathbf{BA}$ does *not* hold in general. The following important rules *are* generally valid, however.

RULES FOR MATRIX MULTIPLICATION

If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices whose dimensions are such that the specified multiplication operations are defined, and if α is an arbitrary scalar, then:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{15.4.1}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{15.4.2}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \tag{15.4.3}$$

$$(\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B}) = \alpha(\mathbf{AB}) \tag{15.4.4}$$

Rule (15.4.1) is known as the *associative law*, while rules (15.4.2) and (15.4.3) are, respectively, the *left* and *right distributive laws*. Note that both the left and right distributive laws are stated here because, unlike for numbers, matrix multiplication is not *commutative*, and so $\mathbf{A}(\mathbf{B} + \mathbf{C}) \neq (\mathbf{B} + \mathbf{C})\mathbf{A}$ in general.

EXAMPLE 15.4.1 Verify rules (15.4.1) and (15.4.2), where α is an arbitrary scalar, for the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Solution: All operations of multiplication and addition are defined, with

$$\mathbf{AB} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}, \qquad (\mathbf{AB})\mathbf{C} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 7 & 5 \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} -2 & -1 \\ 7 & 5 \end{pmatrix}, \qquad \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 7 & 5 \end{pmatrix}$$

Thus, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ in this case. Moreover,

$$\mathbf{B} + \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}, \qquad \mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 6 \\ 5 & 3 \end{pmatrix}$$

and

$$\mathbf{AC} = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}, \qquad \mathbf{AB} + \mathbf{AC} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 6 \\ 5 & 3 \end{pmatrix}$$

So $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

Rules (15.4.3) and (15.4.4) can be verified similarly.

The rules for matrix multiplication can be proved simply by carefully applying the definitions of the relevant operations. To illustrate, we now prove rule (15.4.1), the associative law:

Suppose $\mathbf{A} = (a_{ij})_{m\times n}$, $\mathbf{B} = (b_{ij})_{n\times p}$, and $\mathbf{C} = (c_{ij})_{p\times q}$. It is easy to verify that these dimensions imply that $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are both defined as $m \times q$ matrices. We have to prove that their corresponding elements are all equal.

The element in row i and column ℓ of $(\mathbf{AB})\mathbf{C}$, denoted by $[(\mathbf{AB})\mathbf{C}]_{i\ell}$, is the inner product of the i-th row in $\mathbf{AB}$ and the ℓ-th column in $\mathbf{C}$. Using the notation for sums, we see that

$$[(\mathbf{AB})\mathbf{C}]_{i\ell} = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)c_{k\ell} = \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk}c_{k\ell}\right) = [\mathbf{A}(\mathbf{BC})]_{i\ell}$$

where the two double sums are equal because they both give the sum of all the np terms $a_{ij}b_{jk}c_{k\ell}$, where j runs from 1 to n and k runs from 1 to p.

To emphasize, note that proving rule (15.4.1) involves checking in detail that each element of $(\mathbf{AB})\mathbf{C}$ equals the corresponding element of $\mathbf{A}(\mathbf{BC})$. The same sort of check is required to prove the other three rules, so we leave these proofs to the reader.

Because of (15.4.1), parentheses are not required in a matrix product such as $\mathbf{ABC}$. Of course, a corresponding result is valid for products of more factors.

A useful technique in matrix algebra is to prove new results by using rules (15.4.1) to (15.4.4), repeatedly if necessary, rather than by examining individual elements. For instance, suppose we are asked to prove that if $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are both $n \times n$ matrices, then

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{AA}+\mathbf{AB}+\mathbf{BA}+\mathbf{BB} \tag{15.4.5}$$

According to rules (15.4.2) and (15.4.3),

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = (\mathbf{A}+\mathbf{B})\mathbf{A} + (\mathbf{A}+\mathbf{B})\mathbf{B} = (\mathbf{AA}+\mathbf{BA}) + (\mathbf{AB}+\mathbf{BB})$$

Finally, applying rules (a) and (b) for matrix addition from Section 15.2 yields (15.4.5).

Powers of Matrices

If $\mathbf{A}$ is a square matrix, the associative law (15.4.1) allows us to write $\mathbf{AA}$ as $\mathbf{A}^2$, and $\mathbf{AAA}$ as $\mathbf{A}^3$, and so on. In general,

$$\mathbf{A}^n = \underbrace{\mathbf{AA}\cdots\mathbf{A}}_{n \text{ times}}$$

EXAMPLE 15.4.2 For the matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, compute $\mathbf{A}^2$, $\mathbf{A}^3$, and $\mathbf{A}^4$. Then guess the general form of $\mathbf{A}^n$. Finally, confirm your guess by using the principle of mathematical induction introduced in Section 1.4.

Solution: Routine calculation shows that

$$\mathbf{A}^2 = \mathbf{AA} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \text{ and } \quad \mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

A reasonable guess, therefore, is that for all natural numbers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \tag{$*$}$$

We confirm this by induction on n. Formula $(*)$ *is* correct for $n = 1$. As the induction hypothesis, suppose that $(*)$ holds for $n = k$—that is

$$\mathbf{A}^k = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

Then

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -k-1 \\ 0 & 1 \end{pmatrix}$$

This completes the induction step showing that, if $(*)$ holds for $n = k$, then it holds for $n = k + 1$. It follows that $(*)$ holds for all natural numbers n.

EXAMPLE 15.4.3 Suppose $\mathbf{P}$ and $\mathbf{Q}$ are two $n \times n$ matrices that satisfy $\mathbf{PQ} = \mathbf{Q}^2\mathbf{P}$. Prove then that $(\mathbf{PQ})^2 = \mathbf{Q}^6\mathbf{P}^2$.

Solution: The proof is simple if we repeatedly use rule (15.4.1) and the fact that $\mathbf{PQ} = \mathbf{Q}^2\mathbf{P}$:

$$(\mathbf{PQ})^2 = (\mathbf{PQ})(\mathbf{PQ}) = (\mathbf{Q}^2\mathbf{P})(\mathbf{Q}^2\mathbf{P}) = (\mathbf{Q}^2\mathbf{P})\mathbf{Q}(\mathbf{QP}) = \mathbf{Q}^2(\mathbf{PQ})(\mathbf{QP})$$

Substituting $\mathbf{Q}^2\mathbf{P}$ for $\mathbf{QP}$ twice more gives

$$(\mathbf{PQ})^2 = \mathbf{Q}^2(\mathbf{Q}^2\mathbf{P})(\mathbf{QP}) = \mathbf{Q}^2\mathbf{Q}^2(\mathbf{PQ})\mathbf{P} = \mathbf{Q}^2\mathbf{Q}^2(\mathbf{Q}^2\mathbf{P})\mathbf{P} = \mathbf{Q}^2\mathbf{Q}^2\mathbf{Q}^2\mathbf{P}^2 = \mathbf{Q}^6\mathbf{P}^2$$

Note that it would be virtually impossible to solve the previous example by looking at individual elements. Importantly, note also that, in general $(\mathbf{PQ})^2$ is *not* equal to $\mathbf{P}^2\mathbf{Q}^2$.

The Identity Matrix

The *identity matrix* of order n, denoted by $\mathbf{I}_n$—or often just by $\mathbf{I}$—is the $n \times n$ matrix having as entries 1 along the main diagonal and 0 elsewhere. That is

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}_{n \times n}$$

If $\mathbf{A}$ is any $m \times n$ matrix, it is easy to verify that $\mathbf{AI}_n = \mathbf{A}$. Likewise, if $\mathbf{B}$ is any $n \times m$ matrix, then $\mathbf{I}_n\mathbf{B} = \mathbf{B}$. In particular, for every $n \times n$ matrix $\mathbf{A}$

$$\mathbf{AI}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A} \tag{15.4.6}$$

Thus, $\mathbf{I}_n$ is the matrix equivalent of 1 in the real number system. In fact, it is the only matrix with this property. To prove this, suppose $\mathbf{E}$ is an arbitrary $n \times n$ matrix such that $\mathbf{AE} = \mathbf{A}$ for all $n \times n$ matrices $\mathbf{A}$. Putting $\mathbf{A} = \mathbf{I}_n$ in particular yields $\mathbf{I}_n\mathbf{E} = \mathbf{I}_n$. But $\mathbf{I}_n\mathbf{E} = \mathbf{E}$ according to Eq. (15.4.6). So $\mathbf{E} = \mathbf{I}_n$.

Errors to Avoid

The rules of matrix algebra make many arguments very easy. But it is essential to avoid inventing new rules that do not work when multiplying general matrices, even if they would work for numbers—i.e., for 1×1 matrices. For example, consider Eq. (15.4.5). It is tempting to simplify the expression $\mathbf{AA} + \mathbf{AB} + \mathbf{BA} + \mathbf{BB}$ on the right-hand side to $\mathbf{AA} + 2\mathbf{AB} + \mathbf{BB}$. This is wrong! Even when $\mathbf{AB}$ and $\mathbf{BA}$ are both defined, $\mathbf{AB}$ is not necessarily equal to $\mathbf{BA}$. As the next example shows, matrix multiplication is *not* commutative.

EXAMPLE 15.4.4 Show that if $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $\mathbf{AB} \neq \mathbf{BA}$.

Solution: Direct computation shows that $\mathbf{AB} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \neq \mathbf{BA} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$.

One more result that does not extend from scalars to matrices is the following: if a and b are real numbers, then $ab = 0$ implies that either a or b is 0; the corresponding result is not true for matrices, as $\mathbf{AB}$ can be the zero matrix even if neither $\mathbf{A}$ nor $\mathbf{B}$ is the zero matrix. The following result illustrates this.

EXAMPLE 15.4.5 Compute $\mathbf{AB}$, given that $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$.

Solution: Direct computation gives $\mathbf{AB} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

For real numbers, if $ab = ac$ and $a \neq 0$, then $b = c$, because we can cancel by multiplying each side of the equation by $1/a$. An immediate implication of the previous example is that the corresponding cancellation "rule" is not valid for matrices: there, $\mathbf{AB} = \mathbf{A0}$ and $\mathbf{A} \neq \mathbf{0}$, yet $\mathbf{B} \neq \mathbf{0}$. To summarize, in general:

(i) $\mathbf{AB} \neq \mathbf{BA}$;

(ii) $\mathbf{AB} = \mathbf{0}$ does not imply that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$;

(iii) $\mathbf{AB} = \mathbf{AC}$ and $\mathbf{A} \neq \mathbf{0}$ do not imply that $\mathbf{B} = \mathbf{C}$.

Here, (i) says that matrix multiplication is not *commutative* in general, whereas (iii) shows us that the cancellation law is generally invalid for matrix multiplication.[6]

The following two examples illustrate natural applications of matrix multiplication.

EXAMPLE 15.4.6 A firm uses m different raw materials $R_1, R_2, \ldots, R_m$ in order to produce the n different commodities $V_1, V_2, \ldots, V_n$. Suppose that for each $j = 1, 2, \ldots, n$, each unit of commodity V_j requires as inputs a_{ij} units of R_i, for all $i = 1, 2, \ldots, m$. These *input coefficients* form the matrix

$$\mathbf{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

[6] The cancellation law *is valid*, however, if $\mathbf{A}$ has a so-called inverse. See Section 16.6.

Suppose that the firm plans a monthly production of u_j units of each commodity V_j, $j = 1, 2, \dots, n$. This plan can be represented by an $n \times 1$ matrix (column vector) $\mathbf{u}$, called the firm's monthly production vector:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Since a_{i1}, in particular, is the amount of raw material R_i which is needed to produce one unit of commodity V_1, it follows that $a_{i1}u_1$ is the amount of raw material R_i which is needed to produce u_1 units of commodity V_1. Similarly $a_{ij}u_j$ is the amount needed for u_j units of V_j $(j = 2, \dots, n)$. The total monthly requirement of raw material R_i is therefore

$$a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{in}u_n = \sum_{j=1}^{n} a_{ij}u_j$$

This is the inner product of the i-th row vector in $\mathbf{A}$ and the column vector $\mathbf{u}$. The firm's monthly requirement vector $\mathbf{r}$ for all raw materials is therefore given by the matrix product $\mathbf{r} = \mathbf{Au}$. Thus $\mathbf{r}$ is an $m \times 1$ matrix, or a column vector.

Suppose that the prices of the m raw materials are respectively $p_1, p_2, \dots, p_m$ per unit. If we define the price vector $\mathbf{p} = (p_1, p_2, \dots, p_m)$, then the total monthly cost K of acquiring the required raw materials to produce the vector $\mathbf{u}$ is $\sum_{i=1}^{m} p_i r_i$. This sum can also be written as the matrix product $\mathbf{pr}$. Hence, $K = \mathbf{pr} = \mathbf{p}(\mathbf{Au}) = \mathbf{pAu}$.[7]

EXAMPLE 15.4.7 Figure 15.4.1 indicates the number of daily international flights between major airports in three different countries A, B, and C. The number attached to each connecting line shows how many flights there are between the two airports. For instance, from airport b_3 in country B there are 4 flights to airport c_3 in country C each day, but none to airport c_2 in country C.

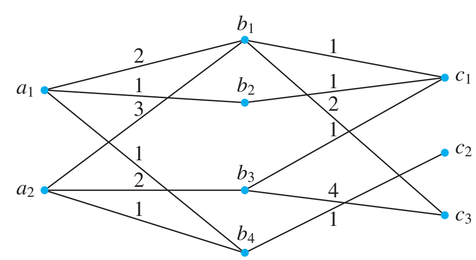

Figure 15.4.1 Flight links

[7] Recall that by rule (15.4.1) matrix multiplication is associative, so it is unnecessary to use parentheses.

The relevant data can also be represented by the two matrices

$$\mathbf{P}: \quad \begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix} \end{array} \qquad \mathbf{Q}: \quad \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Each element p_{ij} of the matrix **P** represents the number of daily flights between a_i and b_j, while each element q_{jk} of **Q** represents the number of daily flights between b_j and c_k. How many ways are there of getting from a_i to c_k using two flights, with one connection in country B? Between a_2 and c_3, for example, there are $3 \cdot 2 + 0 \cdot 0 + 2 \cdot 4 + 1 \cdot 0 = 14$ possibilities. This is the inner product of the second row vector in **P** and the third column vector in **Q**. The same reasoning applies for each a_i and c_k. So the total number of flight connections between the different airports in countries A and C is given by the matrix product

$$\mathbf{R} = \mathbf{PQ} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 \\ 5 & 1 & 14 \end{pmatrix}$$

EXERCISES FOR SECTION 15.4

1. Verify the distributive law $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 3 & -1 & 2 & 1 \end{pmatrix}, \ \text{and} \ \mathbf{C} = \begin{pmatrix} -1 & 1 & 1 & 2 \\ -2 & 2 & 0 & -1 \end{pmatrix}$$

SM 2. Compute the matrix product $\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

3. Verify, by actual multiplication, that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ if

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

4. Compute the following matrix products:

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 9 \\ 1 & 3 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. Suppose that **A** and **B** are square matrices of order n.

(a) Prove that, in general

(i) $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) \neq \mathbf{A}^2 - \mathbf{B}^2$ (ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B}) \neq \mathbf{A}^2 - 2\mathbf{AB} + \mathbf{B}^2$

(b) Find a necessary and sufficient condition for equality to hold in each case.

6. A square matrix $\mathbf{A}$ is said to be *idempotent* if $\mathbf{A}^2 = \mathbf{A}$.

(a) Show that the matrix $\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ is idempotent.

(b) Show that if $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$, then $\mathbf{A}$ and $\mathbf{B}$ are both idempotent.

(c) Show that if $\mathbf{A}$ is idempotent, then $\mathbf{A}^n = \mathbf{A}$ for all positive integers n.

7. Suppose that $\mathbf{P}$ and $\mathbf{Q}$ are $n \times n$ matrices and that $\mathbf{P}^3\mathbf{Q} = \mathbf{PQ}$. Prove that $\mathbf{P}^5\mathbf{Q} = \mathbf{PQ}$.

SM 8. [HARDER] Consider the general 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Prove that $\mathbf{A}^2 = (a + d)\mathbf{A} - (ad - bc)\mathbf{I}_2$.

(b) Use (a) to find an example of a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{0}$, but $\mathbf{A} \neq \mathbf{0}$.

(c) Use part (a) to show that if any 2×2 matrix $\mathbf{A}$ satisfies $\mathbf{A}^3 = \mathbf{0}$, then $\mathbf{A}^2 = \mathbf{0}$. (*Hint*: Multiply the equality in part (a) by $\mathbf{A}$, then use the equality $\mathbf{A}^3 = \mathbf{0}$ to derive an equation that you should then multiply by $\mathbf{A}$ once again.)

15.5 The Transpose

Consider any $m \times n$ matrix $\mathbf{A}$. The *transpose* of $\mathbf{A}$, denoted by $\mathbf{A}'$ or sometimes by $\mathbf{A}^\top$, is defined as the $n \times m$ matrix whose first column is the first row of $\mathbf{A}$, whose second column is the second row of $\mathbf{A}$, and so on. Thus,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}' = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} \tag{15.5.1}$$

So we can write $\mathbf{A}' = (a'_{ij})$, where $a'_{ij} = a_{ji}$. The subscripts i and j have to be interchanged because the j-th row of $\mathbf{A}$ becomes the j-th column of $\mathbf{A}'$, whereas the i-th column of $\mathbf{A}$ becomes the i-th row of $\mathbf{A}'$.

EXAMPLE 15.5.1 Find $\mathbf{A}'$ and $\mathbf{B}'$, for $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ 5 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \end{pmatrix}$.

Solution: Applying the definition of transpose gives

$$\mathbf{A}' = \begin{pmatrix} -1 & 2 & 5 \\ 0 & 3 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}' = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

.

The following rules apply to matrix transposition:

RULES FOR TRANSPOSITION

Given matrices $\mathbf{A}$ and $\mathbf{B}$ suitable for the following operations, and given any scalar α:

$$(\mathbf{A}')' = \mathbf{A} \tag{15.5.2}$$

$$(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}' \tag{15.5.3}$$

$$(\alpha\mathbf{A})' = \alpha\mathbf{A}' \tag{15.5.4}$$

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' \tag{15.5.5}$$

Verifying the first three rules is very easy, and you should prove them in detail, using the fact that $a'_{ij} = a_{ji}$ for each i, j. Next, we prove rule (15.5.5):

Suppose that $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$. Then $\mathbf{A}'$ is $n \times m$, $\mathbf{B}'$ is $p \times n$, $\mathbf{AB}$ is $m \times p$, so both $(\mathbf{AB})'$ and $\mathbf{B}'\mathbf{A}'$ are $p \times m$. It remains to prove that corresponding elements in the two $p \times m$ matrices are equal.

By definition of the transpose, the rs element in $(\mathbf{AB})'$ is the sr element in $\mathbf{AB}$, which is

$$a_{s1}b_{1r} + a_{s2}b_{2r} + \cdots + a_{sn}b_{nr}$$

On the other hand, the rs element in $\mathbf{B}'\mathbf{A}'$ is

$$b_{1r}a_{s1} + b_{2r}a_{s2} + \cdots + b_{nr}a_{sn}$$

Since $a_{si}b_{ir} = b_{ir}a_{si}$ for all $i = 1, 2, \ldots, n$, the two sums are clearly equal, as needed.

EXAMPLE 15.5.2 Let $\mathbf{x}$ be the column vector $(x_1, x_2, \ldots, x_n)'$. Then $\mathbf{x}'$ is a row vector of n elements. The product $\mathbf{x}'\mathbf{x}$ is $\sum_{i=1}^{n} x_i^2$, which in Eq. (13.6.2) was seen to equal $\|\mathbf{x}\|^2$, the square of the norm of $\mathbf{x}$. The reverse product $\mathbf{xx}'$, however, is an $n \times n$ matrix whose ij element is $x_i x_j$.

Symmetric Matrices

Square matrices with the property that they are symmetric about the main diagonal are called *symmetric*. For example,

$$\begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 5 \\ -1 & -3 & 2 \\ 5 & 2 & 8 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

are all symmetric. Symmetric matrices are characterized by the fact that they are equal to their own transposes:

$$\text{The matrix } \mathbf{A} \text{ is } \textit{symmetric} \iff \mathbf{A} = \mathbf{A}'$$

Hence, matrix $\mathbf{A} = (a_{ij})_{n \times n}$ is symmetric if and only if $a_{ij} = a_{ji}$ for all i, j.

EXAMPLE 15.5.3 If $\mathbf{X}$ is an arbitrary $m \times n$ matrix, show that both $\mathbf{XX}'$ and $\mathbf{X}'\mathbf{X}$ are symmetric.

Solution: First, note that $\mathbf{XX}'$ is $m \times m$, whereas $\mathbf{X}'\mathbf{X}$ is $n \times n$. Using rule (15.5.5) and then (15.5.2), we find that

$$(\mathbf{XX}')' = (\mathbf{X}')'\mathbf{X}' = \mathbf{XX}'$$

This proves that $\mathbf{XX}'$ is symmetric. There is a similar proof that $\mathbf{X}'\mathbf{X}$ is symmetric.

EXERCISES FOR SECTION 15.5

1. Find the transposes of $\mathbf{A} = \begin{pmatrix} 3 & 5 & 8 & 3 \\ -1 & 2 & 6 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 5 & 0 & -1 \end{pmatrix}$.

2. Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$, and $\alpha = -2$.

   (a) Compute $\mathbf{A}'$, $\mathbf{B}'$, $(\mathbf{A} + \mathbf{B})'$, $(\alpha\mathbf{A})'$, $\mathbf{AB}$, $(\mathbf{AB})'$, $\mathbf{B}'\mathbf{A}'$, and $\mathbf{A}'\mathbf{B}'$.

   (b) For these particular values of $\mathbf{A}$, $\mathbf{B}$, and α, verify the rules (15.5.2) to (15.5.5) for transposition.

3. Show that the matrices $\mathbf{A} = \begin{pmatrix} 3 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 4 & 8 \\ 4 & 0 & 13 \\ 8 & 13 & 0 \end{pmatrix}$ are both symmetric.

4. Determine all the values of a for which the matrix $\begin{pmatrix} a & a^2 - 1 & -3 \\ a + 1 & 2 & a^2 + 4 \\ -3 & 4a & -1 \end{pmatrix}$ is symmetric.

5. Is the product of two symmetric matrices necessarily symmetric?

SM 6. If $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are matrices for which the given products are defined, show that

$$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)' = \mathbf{A}_3'\mathbf{A}_2'\mathbf{A}_1'$$

   Generalize to products of n matrices.

7. An $n \times n$ matrix $\mathbf{P}$ is said to be *orthogonal* if $\mathbf{P}'\mathbf{P} = \mathbf{I}_n$.

   (a) For $\lambda = \pm 1/\sqrt{2}$, show that the matrix $\mathbf{P} = \begin{pmatrix} \lambda & 0 & \lambda \\ \lambda & 0 & -\lambda \\ 0 & 1 & 0 \end{pmatrix}$ is orthogonal.

   (b) Show that the 2×2 matrix $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ is orthogonal if and only if $p^2 + q^2 = 1$.

   (c) Show that the product of two orthogonal $n \times n$ matrices is orthogonal.

SM **8.** Define the two matrices $\mathbf{T} = \begin{pmatrix} p & q & 0 \\ \frac{1}{2}p & \frac{1}{2} & \frac{1}{2}q \\ 0 & p & q \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \end{pmatrix}$, and assume that $p + q = 1$.

(a) Prove that $\mathbf{T} \cdot \mathbf{S} = \mathbf{S}$, $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$, and $\mathbf{T}^3 = \frac{1}{4}\mathbf{T} + \frac{3}{4}\mathbf{S}$.

(b) Under the hypothesis that for $n = 2, 3, \ldots$ there exist constants α_n and β_n such that $\mathbf{T}^n = \alpha_n \mathbf{T} + \beta_n \mathbf{S}$, use the results of part (a) to express α_{n+1} and β_{n+1} as functions of α_n and β_n. Use these relations to conjecture formulas for the constants α_n and β_n. Then prove the formulas by induction.

15.6 Gaussian Elimination

One way of solving simultaneous equations is by eliminating unknowns, introduced as Method 2 in Example 3.6.1 for the case of two equations in two unknowns. This procedure can be extended to larger equation systems. Because it is very efficient, it is the starting point for computer programs. Consider first the following example.

EXAMPLE 15.6.1 Find all possible solutions of the system

$$\begin{aligned} 2x_2 - x_3 &= -7 \\ x_1 + x_2 + 3x_3 &= 2 \\ -3x_1 + 2x_2 + 2x_3 &= -10 \end{aligned} \tag{i}$$

Solution: The idea will be to modify the system in such a way that x_1 appears only in the first equation, then x_2 only in the first and second equations, and finally x_3 remains alone in the third equation. And we must make sure that the modified system has exactly the same solutions as the original system. We begin by interchanging the first two equations, which certainly will not alter the set of solutions. We obtain

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 2x_2 - x_3 &= -7 \\ -3x_1 + 2x_2 + 2x_3 &= -10 \end{aligned} \tag{ii}$$

This has removed x_1 from the second equation. The next step is to use the first equation in (ii) to eliminate x_1 from the third equation. This is done by adding three times the first equation to the last equation.[8] This gives

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 2x_2 - x_3 &= -7 \\ 5x_2 + 11x_3 &= -4 \end{aligned} \tag{iii}$$

[8] The same result is obtained if we solve the first equation for x_1 to obtain $x_1 = -x_2 - 3x_3 + 2$, and then substitute this into the last equation.

Having eliminated x_1, the next step in the systematic procedure is to multiply the second equation in (iii) by 1/2, so that the coefficient of x_2 becomes 1. Thus,

$$\begin{aligned} x_1 + \ x_2 + \ 3x_3 &= \ \ 2 \\ x_2 - \ \tfrac{1}{2}x_3 &= -\tfrac{7}{2} \\ 5x_2 + 11x_3 &= -4 \end{aligned} \tag{iv}$$

Next, eliminate x_2 from the last equation by multiplying the second equation by -5 and adding the result to the last equation. This gives:

$$\begin{aligned} x_1 + x_2 + 3x_2 &= \ \ 2 \\ x_2 - \tfrac{1}{2}x_3 &= -\tfrac{7}{2} \\ \tfrac{27}{2}x_3 &= \ \ \tfrac{27}{2} \end{aligned} \tag{v}$$

Finally, multiply the last equation by $2/27$ to obtain $x_3 = 1$. Now the other two unknowns can easily be found by "back substitution": Inserting $x_3 = 1$ into the second equation in (v) gives $x_2 = -3$, and the first equation in (v) subsequently yields $x_1 = 2$. Therefore the only solution of the given system is $(x_1, x_2, x_3) = (2, -3, 1)$.

Our elimination procedure led to a "staircase" in system (v), with x_1, x_2, and x_3 as *leading entries*. In matrix notation, we have

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{7}{2} \\ 1 \end{pmatrix}$$

The matrix of coefficients on the left-hand side is *upper triangular* because all entries below the main diagonal are 0. Moreover, the diagonal elements are all 1.

The solution method illustrated in this example is called *Gaussian elimination*—or sometimes the *Gauss–Jordan method*. The operations performed on the given system of equations in order to arrive at system (v) are called *elementary row operations*. These come in three different kinds:

1. Interchange any pair of rows, as in the step from (i) to (ii) in the above solution. This is indicated by a suitable two-way arrow linking the two rows.
2. Multiply any row by a scalar, as in the steps from (iii) to (iv) and from (iv) to (v) in the above solution. This is indicated by writing the scalar multiplier beside the appropriate row.
3. Add any multiple of one row to a different row, as in the steps from (ii) to (iii) and from (iv) to (v) in the above solution. This is indicated by writing the scalar multiplier beside the appropriate row, then using an arrow to link that number to the other row.

Sometimes the elementary row operations are continued until we also obtain zeros above the leading entries. In the example above, this takes three more operations of type 3. The first is as indicated in

$$\begin{array}{rcrl} x_1 + x_2 + 3x_3 &=& 2 & \\ x_2 - \tfrac{1}{2}x_3 &=& -\tfrac{7}{2} & -1 \\ x_3 &=& 1 & \end{array} \tag{15.6.1}$$

which results in

$$\begin{array}{rcrll} x_1 \qquad + \tfrac{7}{2}x_3 &=& \tfrac{11}{2} & & \\ x_2 - \tfrac{1}{2}x_3 &=& -\tfrac{7}{2} & & \\ x_3 &=& 1 & \tfrac{1}{2} & -\tfrac{7}{2} \end{array} \tag{15.6.2}$$

The above display indicates the next *two* operations, affecting rows 1 and 2 respectively. The result is the simple equation system $x_1 = 2$, $x_2 = -3$, and $x_3 = 1$.

Let us apply this method to another example.

EXAMPLE 15.6.2 Find all possible solutions of the following system of equations:

$$\begin{array}{rcl} x_1 + 3x_2 - x_3 &=& 4 \\ 2x_1 + x_2 + x_3 &=& 7 \\ 2x_1 - 4x_2 + 4x_3 &=& 6 \\ 3x_1 + 4x_2 &=& 11 \end{array}$$

Solution: We begin with three operations to remove x_1 from the second, third, and fourth equations:

$$\begin{array}{rcrlll} x_1 + 3x_2 - x_3 &=& 4 & -2 & -2 & -3 \\ 2x_1 + x_2 + x_3 &=& 7 & & & \\ 2x_1 - 4x_2 + 4x_3 &=& 6 & & & \\ 3x_1 + 4x_2 \qquad &=& 11 & & & \end{array}$$

The result is:

$$\begin{array}{rcrl} x_1 + 3x_2 - x_3 &=& 4 & \\ -5x_2 + 3x_3 &=& -1 & \times(-\tfrac{1}{5}) \\ -10x_2 + 6x_3 &=& -2 & \\ -5x_2 + 3x_3 &=& -1 & \end{array}$$

where we have also indicated the next operation of multiplying row 2 by $-\frac{1}{5}$. Further operations on the result lead to

$$\begin{array}{rcrll} x_1 + 3x_2 - x_3 &=& 4 & & \\ x_2 - \tfrac{3}{5}x_3 &=& \tfrac{1}{5} & 10 & 5 \\ -10x_2 + 6x_3 &=& -2 & & \\ -5x_2 + 3x_3 &=& -1 & & \end{array}$$

and then

$$\begin{aligned} x_1 + 3x_2 - \phantom{\tfrac{3}{5}} x_3 &= 4 \quad \xleftarrow{} \\ x_2 - \tfrac{3}{5}x_3 &= \tfrac{1}{5} \quad -3 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

We have now constructed the staircase. The last two equations are superfluous, so we drop them, while applying one more row operation to create a zero above the leading entry x_2:

$$\begin{aligned} x_1 \qquad + \tfrac{4}{5}x_3 &= \tfrac{17}{5} \\ x_2 - \tfrac{3}{5}x_3 &= \tfrac{1}{5} \end{aligned}$$

or, equivalently,

$$\begin{aligned} x_1 &= -\tfrac{4}{5}x_3 + \tfrac{17}{5} \\ x_2 &= \tfrac{3}{5}x_3 + \tfrac{1}{5} \end{aligned} \tag{$*$}$$

Clearly, x_3 can be chosen freely, after which x_1 and x_2 are uniquely determined by $(*)$. Putting $x_3 = t$, we can represent the solution set as

$$(x_1, x_2, x_3) = \left(-\tfrac{4}{5}t + \tfrac{17}{5}, \tfrac{3}{5}t + \tfrac{1}{5}, t\right) \quad \text{where } t \text{ is any real number}$$

Following the terminology of Section 12.10, we say that the solution set of the system has *one degree of freedom*, since one of the variables can be freely chosen. If this variable is given a fixed value, then the other two variables are uniquely determined.

GAUSSIAN ELIMINATION METHOD

In order to solve a system of linear equations:

(i) Make a staircase with 1 as the coefficient for each nonzero leading entry.

(ii) Produce zeros above each leading entry.

(iii) The general solution is found by expressing the unknowns that occur as leading entries in terms of those unknowns that do not. The latter unknowns, if there are any, can be chosen freely.

The number of unknowns that can be chosen freely, which may be none, is the number of *degrees of freedom*.

This description of the recipe assumes that the system has solutions. However, the Gaussian elimination method can also be used to show that a linear system of equations is inconsistent—that is, it has no solutions. Before showing you an example of this, let us introduce a device that considerably reduces the amount of notation needed. Looking back at the last two examples, we realize that we only need to know the coefficients of the system of equations and the right-hand side vector, while the variables only serve to indicate in which column the different coefficients belong. Thus, Example 15.6.2 can be represented as follows by *augmented coefficient matrices* where each has the corresponding vector of right-hand sides as an extra column.

$$\begin{pmatrix}1&3&-1&4\\2&1&1&7\\2&-4&4&6\\3&4&0&11\end{pmatrix}\begin{matrix}-2 & -2 & -3\\ & & \\ & & \\ & & \end{matrix} \sim \begin{pmatrix}1&3&-1&4\\0&-5&3&-1\\0&-10&6&-2\\0&-5&3&-1\end{pmatrix} \times \left(-\tfrac{1}{5}\right)$$

$$\sim \begin{pmatrix}1&3&-1&4\\0&1&-\frac{3}{5}&\frac{1}{5}\\0&-10&6&-2\\0&-5&3&-1\end{pmatrix}\begin{matrix} & \\ 10 & 5\\ & \\ & \end{matrix} \sim \begin{pmatrix}1&3&-1&4\\0&1&-\frac{3}{5}&\frac{1}{5}\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{matrix} \\ -3\\ \\ \\ \end{matrix}$$

$$\sim \begin{pmatrix}1&0&\frac{4}{5}&\frac{17}{5}\\0&1&-\frac{3}{5}&\frac{1}{5}\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

We have performed *elementary row operations* on the different 4×4 augmented matrices, and we have used the equivalence symbol $\sim$ between two matrices when the latter has been obtained by using elementary operations on the former. This is justified because such operations do always produce an equivalent system of equations. Note carefully how the system of equations in Example 15.6.2 is represented by the first matrix, and how the last matrix represents the system of $x_1 + \frac{4}{5}x_3 = \frac{17}{5}$ and $x_2 - \frac{3}{5}x_3 = \frac{1}{5}$.

EXAMPLE 15.6.3 For what values of the numbers a, b, and c does the following system have solutions? Find the solutions when they exist.

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= a \\ x_1 + x_2 - x_3 + x_4 &= b \\ x_1 + 7x_2 - 5x_3 - x_4 &= c \end{aligned}$$

Solution: We represent the system by its augmented matrix, then perform elementary row operations as required by the Gaussian method:

$$\begin{pmatrix}1&-2&2&2&a\\1&1&-1&1&b\\1&7&-5&-1&c\end{pmatrix}\begin{matrix}-1 & -1\\ & \\ & \end{matrix} \sim \begin{pmatrix}1&-2&1&2&a\\0&3&-2&-1&b-a\\0&9&-6&-3&c-a\end{pmatrix}\begin{matrix} \\ -3\\ \\ \end{matrix}$$

$$\sim \begin{pmatrix}1&-2&1&2&a\\0&3&-2&-1&b-a\\0&0&0&0&2a-3b+c\end{pmatrix}$$

The last row represents the equation $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 2a - 3b + c$. The system therefore has solutions only if $2a - 3b + c = 0$. In this case the last row has only zeros, and we continue using elementary operations until we end up with the following matrix:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3}(a+2b) \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3}(b-a) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and, thus

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{1}{3}(a+2b)$$
$$x_2 - \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{1}{3}(b-a)$$

Here x_3 and x_4 can be freely chosen. Once they have been chosen, however, x_1 and x_2 are uniquely determined linear functions of $s = x_3$ and $t = x_4$:

$$x_1 = \tfrac{1}{3}(a+2b) + \tfrac{1}{3}s - \tfrac{4}{3}t$$
$$x_2 = \tfrac{1}{3}(b-a) \quad + \tfrac{2}{3}s + \tfrac{1}{3}t$$

where s and t are arbitrary real numbers.

EXERCISES FOR SECTION 15.6

1. Solve the following systems by Gaussian elimination.

(a) $$\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 + 5x_2 &= 5 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 5 \\ 2x_1 + 3x_2 - x_3 &= 1 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

2. Use Gaussian elimination to discuss what are the possible solutions of the following system for different values of a and b:

$$\begin{aligned} x + y - z &= 1 \\ x - y + 2z &= 2 \\ x + 2y + az &= b \end{aligned}$$

SM 3. Find the values of c for which the following system has a solution, and find the complete solution for these values of c:

$$\begin{aligned} 2w + x + 4y + 3z &= 1 \\ w + 3x + 2y - z &= 3c \\ w + x + 2y + z &= c^2 \end{aligned}$$

SM 4. Find the values of a for which the following system has a unique solution:

$$\begin{aligned} ax + y + (a+1)z &= b_1 \\ x + 2y + z &= b_2 \\ 3x + 4y + 7z &= b_3 \end{aligned}$$

SM 5. Find all solutions to the following system:

$$\begin{aligned} \tfrac{3}{4}x + y + \tfrac{7}{4}z &= b_1 \\ x + 2y + z &= b_2 \\ 3x + 4y + 7z &= b_3 \end{aligned}$$

15.7 Vectors

Recall that a matrix with only one row is also called a *row vector*, and a matrix with only one column is called a *column vector*. We refer to both types as *vectors*. As remarked in Section 15.2, vectors are typically denoted by small bold letters. Thus, if $\mathbf{a}$ is a $1 \times n$ row vector, we write

$$\mathbf{a} = (a_1, a_2, \ldots, a_n)$$

Here, the numbers $a_1, a_2, \ldots, a_n$ are called the *components*, or *coordinates*, of the vector, and a_i is its i-th component or i-th coordinate.[9] If we want to emphasize that a vector has n components, we refer to it as an *n-vector*. Alternatively, if $\mathbf{a}$ is an n-vector, then we say that it has *dimension n*.

It is clear that the row vector $(7, 13, 4)$ and the column vector $\begin{pmatrix} 7 \\ 13 \\ 4 \end{pmatrix}$ contain exactly the same information—the numbers and their order are the same, only the arrangement of the numbers is different. In fact, following the ideas presented in Chapter 11, both the row and the column vector are represented by the same point in three-dimensional space $\mathbb{R}^3$. And any n-vector is represented by a point in n-dimensional space $\mathbb{R}^n$.

Operations on Vectors

Since a vector is just a special type of matrix, the algebraic operations introduced for matrices are equally valid for vectors. So:

(i) Two n-vectors $\mathbf{a}$ and $\mathbf{b}$ are *equal* if and only if all their corresponding components are equal; we then write $\mathbf{a} = \mathbf{b}$.

(ii) If $\mathbf{a}$ and $\mathbf{b}$ are two n-vectors, their *sum*, denoted by $\mathbf{a} + \mathbf{b}$, is the n-vector obtained by adding each component of $\mathbf{a}$ to the corresponding component of $\mathbf{b}$.[10]

(iii) If $\mathbf{a}$ is an n-vector and t is a real number, we define $t\mathbf{a}$ as the n-vector whose components are t times the corresponding components in $\mathbf{a}$.

(iv) The *difference* between two n-vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}$.

---

[9] Recall that when we consider $\mathbf{a}$ as a matrix, its components $a_1, \ldots, a_n$ are called entries or elements.

[10] If two vectors do not have the same dimension, their sum is simply not defined, nor is their difference. Nor should one add a row vector to a column vector, even if they have the same number of elements.

If $\mathbf{a}$ and $\mathbf{b}$ are two n-vectors and t and s are real numbers, the n-vector $t\mathbf{a} + s\mathbf{b}$ is said to be a *linear combination* of $\mathbf{a}$ and $\mathbf{b}$. In symbols, using column vectors,

$$t\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + s\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} ta_1 + sb_1 \\ ta_2 + sb_2 \\ \vdots \\ ta_n + sb_n \end{pmatrix}$$

Linear combinations are frequently found in economics: suppose $\mathbf{a}$ and $\mathbf{b}$ are commodity vectors, whose j-th components are quantities of commodity number j. Now, if t persons all buy the same commodity vector $\mathbf{a}$ and s persons all buy commodity vector $\mathbf{b}$, then the vector $t\mathbf{a} + s\mathbf{b}$ represents the total commodity vector bought by all $t + s$ persons combined.

Of course, the rules for matrix addition and multiplication by scalars seen in Section 15.2 apply to vectors also.

The Inner Product

Let us consider four different commodities—say, apples, bananas, cherries, and dates. Suppose you buy the commodity vector $\mathbf{x} = (5, 3, 6, 7)$. This means, of course, that you buy five units—say, kilos—of the first commodity, three kilos of the second commodity, etc. Suppose the prices per kilo of these four different commodities are given by the price vector $\mathbf{p} = (4, 5, 3, 8)$, meaning that the price per kilo of the first good is \$4, that of the second is \$5, etc. Then the total value of the commodity vector you buy is $4 \cdot 5 + 5 \cdot 3 + 3 \cdot 6 + 8 \cdot 7 = 109$. The result of this operation on the two vectors $\mathbf{p}$ and $\mathbf{x}$ is often written as $\mathbf{p} \cdot \mathbf{x}$ and is called the *inner product*, *scalar product*, or *dot product* of $\mathbf{p}$ and $\mathbf{x}$. In general, we have the following definition, formulated here for row vectors:

INNER PRODUCT

The *inner product* of the pair of n-vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i \tag{15.7.1}$$

Note that the inner product of two vectors is not a vector but a *number*. It is obtained by simply multiplying all pairs (a_j, b_j), $j = 1, 2, \ldots, n$, of the corresponding components in the two vectors $\mathbf{a}$ and $\mathbf{b}$, and then finally adding the results. Note that $\mathbf{a} \cdot \mathbf{b}$ is *defined only if* $\mathbf{a}$ and $\mathbf{b}$ are both of the same dimension.

In the case when $\mathbf{p}$ is a price vector whose components are measured in dollars per kilo, and $\mathbf{x}$ is a commodity vector whose components are measured in kilos, then each product p_jx_j is an amount of money measured in dollars, as is the inner product $\mathbf{p} \cdot \mathbf{x} = \sum_{j=1}^{n} p_jx_j$.

EXAMPLE 15.7.1 If $\mathbf{a} = (1, -2, 3)$ and $\mathbf{b} = (-3, 2, 5)$, compute $\mathbf{a} \cdot \mathbf{b}$.

Solution: We get $\mathbf{a} \cdot \mathbf{b} = 1 \cdot (-3) + (-2) \cdot 2 + 3 \cdot 5 = 8$.

Note that according to the definition of the matrix product $\mathbf{AB}$, the ij element of the product is the inner product of the i-th row vector of $\mathbf{A}$ and the j-th column vector of $\mathbf{B}$.

The inner product is defined for any two n-vectors. If

$$\mathbf{A} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

both happen to be $n \times 1$ matrices, then $\mathbf{A}'$ is a $1 \times n$ matrix, and the matrix product $\mathbf{A}'\mathbf{B}$ is defined as a 1×1 matrix. In fact,

$$\mathbf{A}'\mathbf{A} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Because 1×1 matrices behave exactly as ordinary numbers with respect to addition and multiplication, we can regard the inner product of two (column) vectors $\mathbf{a}$ and $\mathbf{b}$ as the matrix product $\mathbf{a}'\mathbf{b}$.

It is usual in economics to regard a typical vector $\mathbf{x}$ as a column vector, unless otherwise specified. This is especially true if it is a quantity or commodity vector. Another common convention is to regard a price vector as a row vector, often denoted by $\mathbf{p}$. Then $\mathbf{px}$ is the 1×1 matrix whose single element is equal to the inner product $\mathbf{p} \cdot \mathbf{x}$.

Important properties of the inner product follow:

RULES FOR THE INNER PRODUCT

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{C}$ are n-vectors and α is a scalar, then

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c) $(\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b})$

(d) $\mathbf{a} \cdot \mathbf{a} > 0$ if and only if $\mathbf{a} \neq \mathbf{0}$

Here, rules (a) and (c) are easy consequences of the definition, while rule (b) follows from the distributive law for matrix multiplication, seen in Section 15.2, when $\mathbf{a}$ is $1 \times n$ whereas $\mathbf{b}$ and $\mathbf{c}$ are $n \times 1$. To prove rule (d), it suffices to note that $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + \cdots + a_n^2$. This is always nonnegative, and is zero only if all the components a_i of $\mathbf{a}$ are 0.

EXERCISES FOR SECTION 15.7

1. Compute $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$, and $-5\mathbf{a} + 2\mathbf{b}$ when $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

2. Let $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (0, 0, -3)$, and $\mathbf{c} = (-2, 4, -3)$. Find $\mathbf{a} + \mathbf{b} + \mathbf{c}$, $\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}$, and $3\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$.

3. If $3(x, y, z) + 5(-1, 2, 3) = (4, 1, 3)$, find x, y, and z.

4. If $\mathbf{x} + \mathbf{0} = \mathbf{0}$, what do you know about the components of $\mathbf{x}$?

5. If $0\mathbf{x} = \mathbf{0}$, what do you know about the components of $\mathbf{x}$?

6. Express the vector $(4, -11)$ as a linear combination of $(2, -1)$ and $(1, 4)$.

7. Solve the vector equation $4\mathbf{x} - 7\mathbf{a} = 2\mathbf{x} + 8\mathbf{b} - \mathbf{a}$ for $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

8. Find $\mathbf{a} \cdot \mathbf{a}$, $\mathbf{a} \cdot \mathbf{b}$, and $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$, and verify that $\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$, when $\mathbf{a}$ and $\mathbf{b}$ are as in Exercise 1.

9. For what values of x is the inner product of $(x, x - 1, 3)$ and $(x, x, 3x)$ equal to 0?

10. A residential construction company plans to build several houses of three different types: five of type A, seven of type B, and 12 of type C. Suppose that each house of type A requires 20 units of timber, type B requires 18 units, and type C requires 25 units.

    (a) Write down a 3-vector $\mathbf{x}$ whose coordinates give the number of houses of each type.

    (b) Write down a vector $\mathbf{u}$ that gives the different timber quantities required for one house of each of the three different types.

    (c) Find the total timber requirement by computing the inner product $\mathbf{u} \cdot \mathbf{x}$.

11. A firm produces nonnegative output quantities $z_1, z_2, \ldots, z_n$ of n different goods, using as inputs the nonnegative quantities $x_1, x_2, \ldots, x_n$ of the same n goods. For each good i, define $y_i = z_i - x_i$ as the net output of good i, and let p_i be the price of good i. Let $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and $\mathbf{z} = (z_1, \ldots, z_n)$.

    (a) Calculate the firm's revenue and its costs.

    (b) Show that the firm's profit is given by the inner product $\mathbf{p} \cdot \mathbf{y}$. What if $\mathbf{p} \cdot \mathbf{y}$ is negative?

12. A firm produces the first of two different goods as its output, using the second good as its input. Its net output vector, as defined in Exercise 11, is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. The price vector it faces is $(1, 3)$. Find the firm's input vector, output vector, costs, revenue, value of net output, and profit or loss.

15.8 Geometric Interpretation of Vectors

Vectors, unlike general matrices, are easily interpreted geometrically. Actually, the word "vector" is originally Latin and was used to mean both "carrier" and "passenger". In particular, the word is related to the act of moving a person or object from one place to another. Following this idea, a biologist is likely to think of a "vector" as a carrier of disease, such as mosquitoes are for malaria.

In the xy-plane, any shift can be described by the distance a_1 moved in the x-direction and by the distance a_2 moved in the y-direction. A movement in the plane is therefore uniquely determined by an ordered pair or 2-vector (a_1, a_2). Geometrically, such a movement can be illustrated by an arrow from the start point P to the end point Q, as shown in Fig. 15.8.1. If we make a parallel displacement of the arrow so that it starts at P' and ends at Q', the resulting arrow will represent exactly the same shift, because the x and y components are still a_1 and a_2, respectively. The vector from P to Q is denoted by $\overrightarrow{PQ}$, and we refer to it as a *geometric vector* or *directed line segment*. Two geometric vectors that have the same direction and the same length are said to be equal (in much the same way as the two fractions $2/6$ and $1/3$ are equal because they represent the same real number).

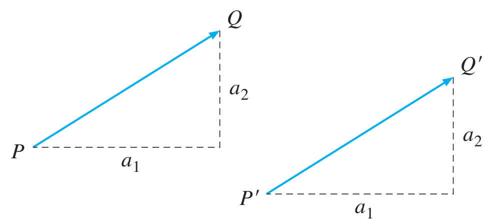

Figure 15.8.1 Vectors as movements in the plane

Suppose that the geometric vector **a** involves a movement from $P = (p_1, p_2)$ to $Q = (q_1, q_2)$. Then the pair (a_1, a_2) that describes the movement in both the x and y directions is given by $a_1 = q_1 - p_1$, $a_2 = q_2 - p_2$, or by $(a_1, a_2) = (q_1, q_2) - (p_1, p_2)$. This is illustrated in Fig. 15.8.2. On the other hand, if the pair (a_1, a_2) is given, the corresponding shift is obtained by moving a_1 units in the direction of the x-axis, as well as a_2 units in the direction of the y-axis. If we start at the point $P = (p_1, p_2)$, then we arrive at the point Q with coordinates $(q_1, q_2) = (p_1 + a_1, p_2 + a_2)$, also shown in Fig. 15.8.2.

This correspondence makes it a matter of convenience whether we think of a vector as an ordered pair of numbers (a_1, a_2), or as a directed line segment such as $\overrightarrow{PQ}$ in Fig. 15.8.2.

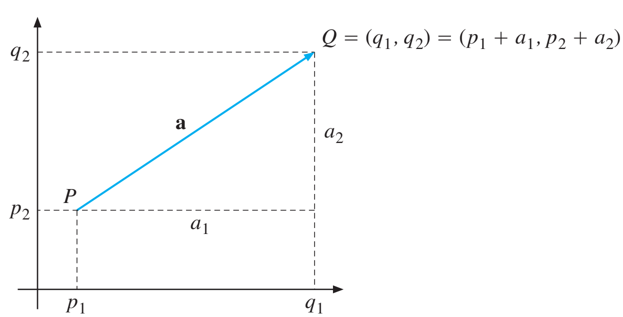

Figure 15.8.2 Vectors as ordered pairs

Vector Operations

If we represent vectors by directed line segments, the vector operations $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, and $t\mathbf{a}$ can be given interesting geometric interpretations. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ both start at the origin $(0, 0)$ of the coordinate system.

The sum $\mathbf{a}+\mathbf{b}$ shown in Fig. 15.8.3 is the diagonal in the parallelogram determined by the two sides $\mathbf{a}$ and $\mathbf{b}$. The geometric reason for this can be seen from Fig. 15.8.4, in which the two right-angled triangles OSR and PTQ are congruent. Thus, OR is parallel to PQ and has the same length, so $OPQR$ is a parallelogram.[11] The parallelogram law of addition is also illustrated in Fig. 15.8.5. One way of interpreting this figure is that if $\mathbf{a}$ takes you from O to P and $\mathbf{b}$ takes you on from P to Q, then the combined movement $\mathbf{a}+\mathbf{b}$ takes you from O to Q. Moreover, looking at Fig. 15.8.4 again, $\mathbf{b}$ takes you from O to R, whereas $\mathbf{a}$ takes you on from R to Q. So the combined movement $\mathbf{b}+\mathbf{a}$ takes you from O to Q. Of course, this verifies that $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

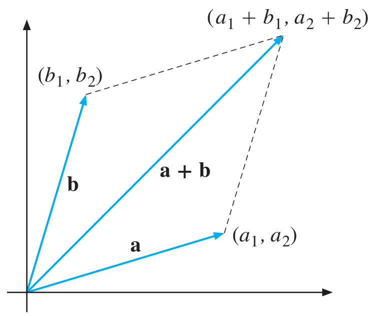

Figure 15.8.3 Vector addition

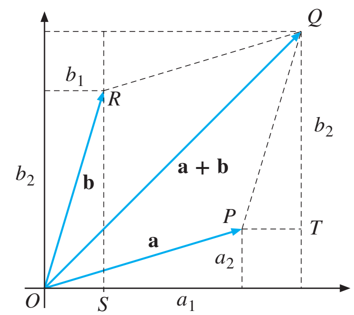

Figure 15.8.4 Geometry of vector addition

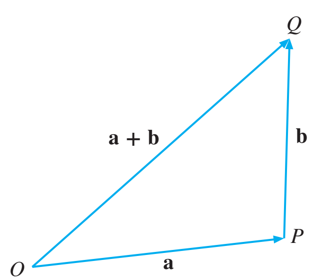

Figure 15.8.5 Vector addition

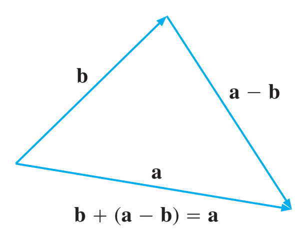

Figure 15.8.6 Vector subtraction

Figure 15.8.6 gives a geometric interpretation to the vector $\mathbf{a}-\mathbf{b}$. Note carefully the direction of the geometric vector $\mathbf{a}-\mathbf{b}$. And note that $\mathbf{b}+(\mathbf{a}-\mathbf{b})=\mathbf{a}=(\mathbf{a}-\mathbf{b})+\mathbf{b}$.

The geometric interpretation of $t\mathbf{a}$, where t is any real number, is also straightforward. If $t>0$, then $t\mathbf{a}$ is the vector with the same direction as $\mathbf{a}$ and whose length is t times the length of $\mathbf{a}$. If $t<0$, the direction is reversed and the length is multiplied by the absolute value of t. Indeed, multiplication by t is like rescaling the vector $\mathbf{a}$; that is why the number t is often called a *scalar*.

[11] This parallelogram law of adding vectors will be familiar to readers who have studied physics. For example, if $\mathbf{a}$ and $\mathbf{b}$ represent two forces acting on a particle located at the point O, then the single combined force $\mathbf{a}+\mathbf{b}$ acting on the particle will produce exactly the same physical effect.

3-Space and *n*-Space

The plane is often also called 2-space and denoted $\mathbb{R}^2$. We represent a point or a vector in a plane by a pair of real numbers using two mutually orthogonal coordinate lines. In a similar way, any point or vector in 3-space, $\mathbb{R}^3$, can be represented by a triple of real numbers using three mutually orthogonal coordinate lines, as explained in Section 11.3. Any 3-vector (a_1, a_2, a_3) can be considered in an obvious way as a geometric vector or movement in 3-space, $\mathbb{R}^3$. As with ordered pairs in the plane, there is a natural correspondence between ordered triples (a_1, a_2, a_3) and geometric vectors regarded as directed line segments. The parallelogram law of addition remains valid in $\mathbb{R}^3$, as does the geometric interpretation of the multiplication of a vector by a scalar.

The set $\mathbb{R}^n$ of all n-vectors was introduced in Section 11.5. When $n \geq 4$, it has no natural spatial interpretation. Nevertheless, geometric language is sometimes still used to discuss properties of $\mathbb{R}^n$, because many properties of $\mathbb{R}^2$ and $\mathbb{R}^3$ carry over to $\mathbb{R}^n$. In particular, the rules for addition, subtraction, and scalar multiplication of vectors remain exactly the same.

Lengths of Vectors and the Cauchy–Schwarz Inequality

If $\mathbf{a} = (a_1, a_2, \dots, a_n)$, we define the *length*, or *norm*, of the vector $\mathbf{a}$, denoted by $\|\mathbf{a}\|$, as $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$, or

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \tag{15.8.1}$$

According to (15.8.1), $\|\mathbf{a}\|$ is the distance from the origin $(0, 0, \dots, 0)$ to $(a_1, a_2, \dots, a_n)$. In Exercise 4.6.9 you were asked to prove the famous *Cauchy–Schwarz inequality*. Using the notation we have just introduced, this inequality can be expressed as $(\mathbf{a} \cdot \mathbf{b})^2 \leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2$, or equivalently, as

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\| \tag{15.8.2}$$

EXAMPLE 15.8.1 For the two vectors $\mathbf{a} = (1, -2, 3)$ and $\mathbf{b} = (-3, 2, 5)$, check the Cauchy–Schwarz inequality.

Solution: We find that

$$\|\mathbf{a}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} \quad \text{and} \quad \|\mathbf{b}\| = \sqrt{(-3)^2 + 2^2 + 5^2} = \sqrt{38}$$

In Example 15.7.1, we found the inner product of these vectors to be 8. So inequality (15.8.2) says that $8 \leq \sqrt{14}\sqrt{38}$, which is certainly true because $\sqrt{14} > 3$ and $\sqrt{38} > 6$.

Orthogonality

Consider Fig. 15.8.7, which exhibits three vectors, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - \mathbf{b}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$.

According to Pythagoras's theorem, the angle θ between the two vectors $\mathbf{a}$ and $\mathbf{b}$ is a right angle of 90° if and only if $(OA)^2 + (OB)^2 = (AB)^2$, or $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{a} - \mathbf{b}\|^2$. This implies that $\theta = 90°$ if, and only if,

$$\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \tag{$*$}$$

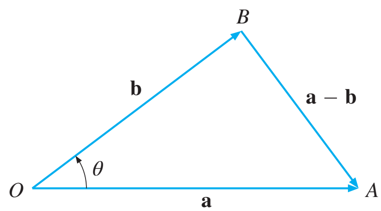

Figure 15.8.7 The angle between **a** and **b**

Because $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, equality $(*)$ requires $2\mathbf{a} \cdot \mathbf{b} = 0$, and so $\mathbf{a} \cdot \mathbf{b} = 0$. When the angle between two vectors **a** and **b** is 90°, they are said to be *orthogonal*, and we write $\mathbf{a} \perp \mathbf{b}$. Thus, we have proved that two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are orthogonal if and only if their inner product is 0. In symbols:

$$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0 \tag{15.8.3}$$

For pairs of vectors in $\mathbb{R}^n$, we *define* orthogonality between **a** and **b** by (15.8.3).

Let **a** and **b** be two nonzero vectors in $\mathbb{R}^n$. Using some elementary trigonometry, we can define the *angle* θ between them by

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|} \tag{15.8.4}$$

with $\theta \in [0, \pi]$. This definition makes sense because the Cauchy–Schwarz inequality implies that the right-hand side has an absolute value ≤ 1. Note also that according to (15.8.4), $\cos\theta = 0$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. This agrees with (15.8.3), because for $\theta \in [0, \pi]$, we have $\cos\theta = 0$ if and only if $\theta = \pi/2$.

EXAMPLE 15.8.2 Suppose we repeatedly observe a commodity's price and the quantity demanded. After n observations suppose we have the n pairs $\{(p_1, d_1), (p_2, d_2), \ldots, (p_n, d_n)\}$, where p_i represents the price and d_i is the quantity at observation i, for $i = 1, 2, \ldots, n$. Define the statistical means

$$\bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i \quad \text{and} \qquad \bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i$$

and write the deviations from these means as the vectors

$$\mathbf{a} = (p_1 - \bar{p},\ p_2 - \bar{p},\ \ldots,\ p_n - \bar{p}), \quad \mathbf{b} = (d_1 - \bar{d},\ d_2 - \bar{d},\ \ldots,\ d_n - \bar{d})$$

In statistics, the ratio $\cos\theta$ defined by Eq. (15.8.4) is called the *correlation coefficient*, often denoted by ρ. It is a measure of the degree of "correlation" between the prices and demand quantities in the data. When $\rho = 1$, there is a positive constant $\alpha > 0$ such that $d_i - \bar{d} = \alpha(p_i - \bar{p})$, implying that demand and price are *perfectly correlated*. It is more plausible, however, that $\rho = -1$ because this relationship holds for some $\alpha < 0$. Generally, if $\rho > 0$ the variables are *positively correlated*, whereas if $\rho < 0$ the variables are *negatively correlated*, and if $\rho = 0$ they are *uncorrelated*.

EXAMPLE 15.8.3 **(Orthogonality in econometrics).** In Example 13.4.4 on linear regression, the regression coefficients α and β were chosen to minimize the *mean squared error* loss function

$$L(\alpha, \beta) = \frac{1}{T}\sum_{t=1}^{T} e_t^2 = \frac{1}{T}\sum_{t=1}^{T}(y_t - \alpha - \beta x_t)^2$$

This required choosing $\hat{\alpha} = \mu_y - (\sigma_{xy}/\sigma_{xx})\mu_x$ and $\hat{\beta} = \sigma_{xy}/\sigma_{xx}$, where μ_x and μ_y denote the means of x_t and y_t respectively, whereas σ_{xx} is the variance of x_t, and σ_{xy} is the covariance of x_t with y_t. The resulting errors become $\hat{e}_t = y_t - \hat{\alpha} - \hat{\beta}x_t = y_t - \mu_y - (\sigma_{xy}/\sigma_{xx})(x_t - \mu_x)$. By definition of μ_x and μ_y, one has

$$\frac{1}{T}\sum_{t=1}^{T}\hat{e}_t = 0 \tag{$*$}$$

In addition,

$$\frac{1}{T}\sum_{t=1}^{T}\hat{e}_t x_t = \frac{1}{T}\sum_{t=1}^{T}x_t y_t - \mu_x\mu_y - \sigma_{xy}\sigma_{xx}\left(\frac{1}{T}\sum_{t=1}^{T}x_t^2 - \mu_x^2\right) = \sigma_{xy} - \frac{\sigma_{xy}}{\sigma_{xx}}\sigma_{xx} = 0 \tag{$**$}$$

Define the vectors $\mathbf{1} = (1, 1, \dots, 1)$, $\mathbf{x} = (x_1, \dots, x_T)$, and $\hat{\mathbf{e}} = (\hat{e}_1, \dots, \hat{e}_T)$. Then equation $(*)$ shows that the inner product of $\hat{\mathbf{e}}$ and $\mathbf{1}$ is 0. Moreover, equation $(**)$ shows that the inner product of $\hat{\mathbf{e}}$ and $\mathbf{x}$ is 0.

Note that $L(\alpha, \beta) = (1/T)\|\mathbf{y} - \alpha\mathbf{1} - \beta\mathbf{x}\|^2$. Geometrically, the scalars $\hat{\alpha}$ and $\hat{\beta}$ are chosen so that the vector $\hat{\mathbf{y}} = \hat{\alpha}\mathbf{1} + \hat{\beta}\mathbf{x}$ in the plane containing the vectors $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{x}$ is as close as possible to $\mathbf{y}$ in the T-dimensional space $\mathbb{R}^T$.[12] This involves having the vector $\mathbf{y} - \hat{\mathbf{y}} = \hat{\mathbf{e}}$ be orthogonal to $\mathbf{1}$ and $\mathbf{x}$, and to every other vector $\alpha\mathbf{1} + \beta\mathbf{x}$ in this plane. Accordingly, $\hat{\mathbf{y}}$ is called the *orthogonal projection* of $\mathbf{y}$ onto this plane.

EXERCISES FOR SECTION 15.8

1. Let $\mathbf{a} = (5, -1)$ and $b = (-2, 4)$. Compute $\mathbf{a} + \mathbf{b}$ and $-\frac{1}{2}\mathbf{a}$, and illustrate with geometric vectors starting at the origin.

SM 2. Let $\mathbf{a} = (3, 1)$ and $b = (-1, 2)$, and define $x = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$.

(a) Compute $\mathbf{x}$ when $\lambda = 0$, 1/4, 1/2, 3/4, and 1, and illustrate the answers.

(b) If $\lambda \in [0, 1]$, what set of points does $\mathbf{x} = \lambda\mathbf{a} + (\mathbf{1} - \lambda)\mathbf{b}$ trace out?

(c) Show that if $\lambda \in \mathbb{R}$, then $\mathbf{x}$ traces out the whole straight line through $(3, 1)$ and $(-1, 2)$.

3. Let $\mathbf{a} = (1,2,2)$, $\mathbf{b} = (0, 0, -3)$, and $\mathbf{c} = (-2, 4, -3)$. Compute $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, and $\|\mathbf{c}\|$, and verify that the Cauchy–Schwarz inequality (15.8.2) holds for $\mathbf{a}$ and $\mathbf{b}$.

4. Let $\mathbf{a} = (1, 2, 1)$ and $\mathbf{b} = (-3, 0, -2)$.

(a) Find numbers x_1 and x_2 such that $x_1\mathbf{a} + x_2\mathbf{b} = (5, 4, 4)$.

(b) Prove that there are no real numbers x_1 and x_2 satisfying $x_1\mathbf{a} + x_2\mathbf{b} = (-3, 6, 1)$.

[12] Planes are discussed in the next section.

5. Check which of the following pairs of vectors are orthogonal:

(a) $(1, 2)$ and $(-2, 1)$ (b) $(1, -1, 1)$ and $(-1, 1, -1)$ (c) $(a, -b, 1)$ and $(b, a, 0)$

6. For what values of x are $(x, -x - 8, x, x)$ and $(x, 1, -2, 1)$ orthogonal?

7. [HARDER] Show that any two different columns of an orthogonal matrix, as defined in Exercise 15.5.7, are orthogonal vectors, as are any two different rows.

8. [HARDER] If $\mathbf{a}$ and $\mathbf{b}$ are n-vectors, prove the *triangle inequality* $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$. (*Hint*: $\|\mathbf{a} + \mathbf{b}\|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$. Then use (15.8.2).)

15.9 Lines and Planes

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two distinct vectors in $\mathbb{R}^3$. We can think of them as arrows from the origin to the points with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3), respectively. The straight line L passing through these two points is shown in Fig. 15.9.1.

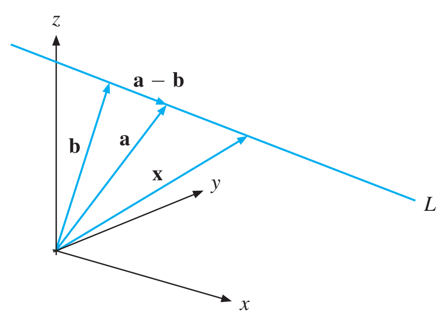

Figure 15.9.1 Line L goes through $\mathbf{a}$ and $\mathbf{b}$

Let t be a real number and put $\mathbf{x} = \mathbf{b} + t(\mathbf{a} - \mathbf{b}) = t\mathbf{a} + (1 - t)\mathbf{b}$. Then $t = 0$ gives $\mathbf{x} = \mathbf{b}$ and $t = 1$ gives $\mathbf{x} = \mathbf{a}$. As t decreases, the point $\mathbf{x}$ moves to the left in Fig. 15.9.1; as t increases, $\mathbf{x}$ moves to the right. By the natural extension from $\mathbb{R}^2$ to $\mathbb{R}^3$ of the geometric rule for adding vectors, the vector marked $\mathbf{x}$ in Fig. 15.9.1 is approximately $\mathbf{b} + 2.5(\mathbf{a} - \mathbf{b})$. As t runs through all the real numbers, so $\mathbf{x}$ describes the whole straight line L.

For $\mathbb{R}^n$, we introduce the following definition:

LINE IN n-SPACE

The line L in $\mathbb{R}^n$ through the two distinct points $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ is the set of all $\mathbf{x} = (x_1, \ldots, x_n)$ satisfying

$$\mathbf{x} = t\mathbf{a} + (1 - t)\mathbf{b} \tag{15.9.1}$$

for some real number t.

By the definitions in Section 15.7 of operations on vectors, Eq. (15.9.1) is equivalent to

$$x_1 = ta_1 + (1-t)b_1, x_2 = ta_2 + (1-t)b_2, \dots, x_n = ta_n + (1-t)b_n \tag{15.9.2}$$

EXAMPLE 15.9.1 Describe the straight line in $\mathbb{R}^3$ through the two points $(1, 2, 2)$ and $(-1, -1, 4)$. Where does it meet the x_1x_2-plane?

Solution: According to (15.9.2), the straight line is given by the equations:

$$\begin{aligned} x_1 &= t \cdot 1 + (1-t)(-1) = 2t - 1 \\ x_2 &= t \cdot 2 + (1-t)(-1) = 3t - 1 \\ x_3 &= t \cdot 2 + (1-t) \cdot 4 \quad = 4 - 2t \end{aligned}$$

This line intersects the x_1x_2-plane when $x_3 = 0$. Then $4 - 2t = 0$, so $t = 2$, implying that $x_1 = 3$ and $x_2 = 5$. It follows that the line meets the x_1x_2-plane at the point $(3, 5, 0)$, as shown in Fig. 15.9.2.

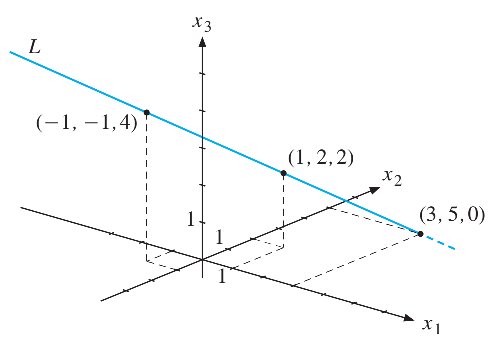

Figure 15.9.2 Line L goes through $(1, 2, 2)$ and $(-1, -1, 4)$

Suppose $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$. The straight line L passing through $\mathbf{p}$ in the same direction as the vector $\mathbf{a} = (a_1, \dots, a_n)$ is given by

$$\mathbf{x} = \mathbf{p} + t\mathbf{a} \tag{15.9.3}$$

where t is any real number.

Hyperplanes

As shown in Fig. 15.9.3, a plane $\mathcal{P}$ in $\mathbb{R}^3$ is defined by one point $\mathbf{a} = (a_1, a_2, a_3)$ in the plane, as well as one vector $\mathbf{p} = (p_1, p_2, p_3) \neq (0, 0, 0)$ which is orthogonal or perpendicular to any line in the plane. Then the vector $\mathbf{p}$ is said to be a *normal* to the plane. Thus, if $\mathbf{x} = (x_1, x_2, x_3)$ is any point in $\mathcal{P}$ other than $\mathbf{a}$, then the vector $\mathbf{x} - \mathbf{a}$ is in a direction orthogonal to $\mathbf{p}$. Therefore, the inner product of $\mathbf{p}$ and $\mathbf{x} - \mathbf{a}$ must be 0, so that

$$\mathbf{p} \cdot (\mathbf{x} - \mathbf{a}) = 0 \tag{15.9.4}$$

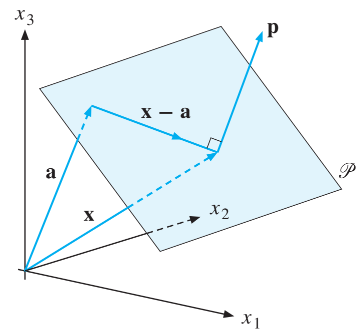

Figure 15.9.3 A hyperplane in $\mathbb{R}^3$

It follows that Eq. (15.9.4) is the general equation of a plane in $\mathbb{R}^3$ passing through the point $\mathbf{a}$ with normal $\mathbf{p} \neq \mathbf{0}$.

EXAMPLE 15.9.2 Find the equation for the plane in $\mathbb{R}^3$ through $\mathbf{a} = (2, 1, -1)$ with $\mathbf{p} = (-1, 1, 3)$ as a normal. Does the line in Example 15.9.1 intersect this plane?

Solution: Using Eq. (15.9.4), the equation is

$$-1 \cdot (x_1 - 2) + 1 \cdot (x_2 - 1) + 3(x_3 - (-1)) = 0$$

or, equivalently, $-x_1 + x_2 + 3x_3 = -4$. The line in Example 15.9.1 is given by the three equations $x_1 = 2t - 1$, $x_2 = 3t - 1$, and $x_3 = 4 - 2t$. If it meets this plane, then we must have

$$-(2t - 1) + (3t - 1) + 3(4 - 2t) = -4$$

Solving this equation for t yields $t = 16/5$, and so the point of intersection is given by $x_1 = 32/5 - 1 = 27/5$, $x_2 = 43/5$, and $x_3 = -12/5$.

Motivated by this characterization of a plane in $\mathbb{R}^3$, we introduce the following general definition in $\mathbb{R}^n$.

HYPERPLANE IN N-SPACE

The hyperplane H in $\mathbb{R}^n$ through $\mathbf{a} = (a_1, \dots, a_n)$ which is orthogonal to the nonzero vector $\mathbf{p} = (p_1, \dots, p_n)$ is the set of all points $\mathbf{x} = (x_1, \dots, x_n)$ satisfying

$$\mathbf{p} \cdot (\mathbf{x} - \mathbf{a}) = 0 \tag{15.9.5}$$

Note that if the normal vector $\mathbf{p}$ is replaced by the scalar multiple $s\mathbf{p}$, where $s \neq 0$, then precisely the same set of vectors $\mathbf{x}$ will satisfy the hyperplane equation (15.9.5).

Using the coordinate representation of the vectors, the hyperplane has the equation

$$p_1(x_1 - a_1) + p_2(x_2 - a_2) + \cdots + p_n(x_n - a_n) = 0 \tag{15.9.6}$$

or $p_1x_1 + p_2x_2 + \cdots + p_nx_n = A$, where $A = p_1a_1 + p_2a_2 + \cdots + p_na_n$.

EXAMPLE 15.9.3 A person has an amount m to spend on n different commodities, whose prices per unit are $p_1, p_2, \ldots, p_n$, respectively. She can therefore afford any commodity vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that satisfies the budget inequality

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq m \tag{15.9.7}$$

When Eq. (15.9.7) is satisfied with equality, it describes the *budget hyperplane*, whose normal is the price vector $(p_1, p_2, \ldots, p_n)$.

Usually, it is implicitly assumed that $x_1 \geq 0$, $x_2 \geq 0$, …, $x_n \geq 0$. For an example with $n = 3$ see Fig. 11.4.1, where the price vector (p, q, r) is normal to the plane.

EXERCISES FOR SECTION 15.9

1. Find the equation for the line:

   (a) that passes through the points $(3, -2, 2)$ and $(10, 2, 1)$;

   (b) that passes through the point $(1, 3, 2)$ and has the same direction as $(0, -1, 1)$.

2. The line L is given by $x_1 = -t + 2$, $x_2 = 2t - 1$, and $x_3 = t + 3$.

   (a) Verify that the point $\mathbf{a} = (2, -1, 3)$ lies on L, but that $(1, 1, 1)$ does not.

   (b) Find the equation for the plane $\mathcal{P}$ through $\mathbf{a}$ that is orthogonal to L.

   (c) Find the point P where L intersects the plane $3x_1 + 5x_2 - x_3 = 6$.

SM 3. Find the equation for the plane through the points $(1, 0, 2)$, $(5, 2, 1)$, and $(2, -1, 4)$.

4. In Example 15.9.3, suppose that the price vector is $(2, 3, 5)$ and that you can just afford the commodity vector $(10, 5, 8)$. What inequality describes your budget constraint?

5. Let $\mathbf{a} = (-2, 1, -1)$.

   (a) Show that $\mathbf{a}$ is a point in the plane $-x + 2y + 3z = 1$.

   (b) Find the equation for the normal at $\mathbf{a}$ to the plane in part (a).

REVIEW EXERCISES

1. Construct the two matrices $\mathbf{A} = (a_{ij})_{2\times 3}$, where: (a) $a_{ij} = i + j$; (b) $a_{ij} = (-1)^{i+j}$.

2. Using the matrices $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, and $\mathbf{D} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}$, calculate (where possible):

   (a) $\mathbf{A} - \mathbf{B}$ (b) $\mathbf{A} + \mathbf{B} - 2\mathbf{C}$ (c) $\mathbf{AB}$ (d) $\mathbf{C}(\mathbf{AB})$

   (e) $\mathbf{AD}$ (f) $\mathbf{DC}$ (g) $2\mathbf{A} - 3\mathbf{B}$ (h) $(\mathbf{A} - \mathbf{B})'$

   (i) $(\mathbf{C}'\mathbf{A}')\mathbf{B}'$ (j) $\mathbf{C}'(\mathbf{A}'\mathbf{B}')$ (k) $\mathbf{D}'\mathbf{D}'$ (l) $\mathbf{D}'\mathbf{D}$

3. Write the following systems of equations in matrix notation:

(a) $\begin{aligned} 2x_1 - 5x_2 &= 3 \\ 5x_1 + 8x_2 &= 5 \end{aligned}$ (b) $\begin{aligned} x + y + z + t &= a \\ x + 3y + 2z + 4t &= b \\ x + 4y + 8z \quad &= c \\ 2x \quad + z - t &= d \end{aligned}$ (c) $\begin{aligned} (a-1)x + 3y - 2z &= 5 \\ ax + 2y - z &= 2 \\ x - 2y + 3z &= 1 \end{aligned}$

4. Find the matrices $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}(\mathbf{BC})$, and $(\mathbf{AB})\mathbf{C}$, if

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 4 & 5 \\ -6 & 7 & 15 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -5 & 3 \\ 5 & 2 & -1 \\ -4 & 2 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 6 & -2 & -3 \\ 2 & 0 & 1 \\ 0 & 5 & 7 \end{pmatrix}$$

5. Find real numbers a, b, and x such that $\begin{pmatrix} a & b \\ x & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ x & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}$.

6. Let $\mathbf{A}$ denote the matrix $\begin{pmatrix} a & b & 0 \\ -b & a & b \\ 0 & -b & a \end{pmatrix}$ where a and b are arbitrary constants.

(a) Find $\mathbf{AA} = \mathbf{A}^2$.

(b) A square matrix $\mathbf{B}$ is called *skew-symmetric* if $\mathbf{B} = -\mathbf{B}'$, where $\mathbf{B}'$ is the transpose of $\mathbf{B}$. Show that if $\mathbf{C}$ is an arbitrary matrix such that $\mathbf{C}'\mathbf{BC}$ is defined, then $\mathbf{C}'\mathbf{BC}$ is skew-symmetric if $\mathbf{B}$ is. When is the matrix $\mathbf{A}$ defined in (a) skew-symmetric?

(c) If $\mathbf{A}$ is any square matrix, prove that $\mathbf{A}_1 = \frac{1}{2}(\mathbf{A}+\mathbf{A}')$ is symmetric and that $\mathbf{A}_2 = \frac{1}{2}(\mathbf{A}-\mathbf{A}')$ is skew-symmetric. Verify that $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, and explain in your own words what you have proved.

SM 7. Solve the following equation systems by Gaussian elimination.

(a) $\begin{aligned} x_1 + 4x_2 &= 1 \\ 2x_1 + 2x_2 &= 8 \end{aligned}$ (b) $\begin{aligned} 2x_1 + 2x_2 - x_3 &= 2 \\ x_1 - 3x_2 + x_3 &= 0 \\ 3x_1 + 4x_2 - x_3 &= 1 \end{aligned}$ (c) $\begin{aligned} x_1 + 3x_2 + 4x_3 &= 0 \\ 5x_1 + x_2 + x_3 &= 0 \end{aligned}$

8. Use Gaussian elimination to find for what values of a the following system has solutions. Then find all the possible solutions.

$$\begin{aligned} x + \quad ay + 2z &= 0 \\ -2x - \quad ay + z &= 4 \\ 2ax + 3a^2y + 9z &= 4 \end{aligned}$$

9. Let $\mathbf{a} = (-1, 5, 3)$, $\mathbf{b} = (1, 1, -3)$, and $\mathbf{c} = (-1, 2, 8)$. Compute $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, and $\|\mathbf{c}\|$. Then verify that the Cauchy–Schwarz inequality holds for $\mathbf{a}$ and $\mathbf{b}$.

SM 10. A firm has two plants that produce outputs of three different goods. Its total labour force is fixed. When a fraction λ of its labour force is allocated to its first plant and a fraction $1-\lambda$ to its second plant, with $0 \le \lambda \le 1$, the total output of the three different goods is given by the vector

$$\lambda(8, 4, 4) + (1-\lambda)(2, 6, 10) = (6\lambda + 2, -2\lambda + 6, -6\lambda + 10)$$

(a) Is it possible for the firm to produce either of the two output vectors $\mathbf{a} = (5, 5, 7)$ and $\mathbf{b} = (7, 5, 5)$ if output cannot be thrown away?

(b) How do your answers to part (a) change if output can be thrown away?

(c) How will the revenue-maximizing choice of the fraction λ depend upon the selling prices (p_1, p_2, p_3) of the three goods? What condition must be satisfied by these prices if both plants are to remain in use?

SM 11. If $\mathbf{P}$ and $\mathbf{Q}$ are $n \times n$ matrices with $\mathbf{PQ} - \mathbf{QP} = \mathbf{P}$, prove that $\mathbf{P}^2\mathbf{Q} - \mathbf{QP}^2 = 2\,\mathbf{P}^2$ and $\mathbf{P}^3\mathbf{Q} - \mathbf{QP}^3 = 3\,\mathbf{P}^3$. Then use induction to prove that $\mathbf{P}^k\mathbf{Q} - \mathbf{QP}^k = k\,\mathbf{P}^k$ for $k = 1, 2, \ldots$.

16

DETERMINANTS AND INVERSE MATRICES

You know we all became mathematicians for the same reason: We were lazy.
—Max Rosenlicht (1949)

This chapter continues the study of linear algebra. The first topic discussed is the *determinant* of a square matrix. It is one number that does indeed determine some key properties of the n^2 elements of an $n \times n$ matrix. Some economists regard determinants as almost obsolete because calculations that rely on them are very inefficient when the matrix is large. Nevertheless, they are important in several areas of mathematics that interest economists.

After introducing determinants, we consider the fundamentally important concept of the *inverse* of a square matrix and its main properties. Inverse matrices play a major role in the study of systems of linear equations, and in econometrics, for deriving a linear relationship that fits a data set as well as possible. Cramer's rule for the solution of a system of n linear equations and n unknowns is discussed next. Although it is not efficient for solving systems of equations with more than three unknowns, Cramer's rule is often used in theoretical studies. An important theorem on homogeneous systems of equations is also discussed. The chapter concludes with a brief introduction to the Leontief model.

16.1 Determinants of Order 2

Consider the pair of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{16.1.1}$$

with its associated coefficient matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Solving the equation system (16.1.1) in the usual way (see Section 3.6) yields

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \qquad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \tag{16.1.2}$$

The two fractions have a common denominator D, equal to $a_{11}a_{22} - a_{21}a_{12}$. The number D must be nonzero for (16.1.2) to be valid, in which case system (16.1.1) has a unique solution specified by (16.1.2). In this sense, the value of the denominator determines whether system (16.1.1) has a unique solution. In fact, $D = a_{11}a_{22} - a_{21}a_{12}$ is called the *determinant* of the matrix $\mathbf{A}$. The determinant of $\mathbf{A}$ is denoted by either $\det(\mathbf{A})$ or, more usually as in this book, by $|\mathbf{A}|$. Thus,

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{16.1.3}$$

for any 2×2 matrix $\mathbf{A}$. Such a determinant is said to have *order* 2. For the special case of order 2 determinants, the rule for calculating $|\mathbf{A}|$ is: (a) multiply the elements on the main diagonal; (b) multiply the off-diagonal elements; (c) subtract the product of the off-diagonal elements from the product of the diagonal elements.

EXAMPLE 16.1.1 By direct computation:

$$\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 4 \cdot 2 - 3 \cdot 1 = 5, \quad \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - b_2 a_{12}, \quad \text{and} \quad \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = b_2 a_{11} - b_1 a_{21}$$

Geometrically, each of the two equations in (16.1.1) represents the graph of a straight line. If $|\mathbf{A}| \neq 0$, then the two lines intersect at a unique point (x_1, x_2) given by (16.1.2). If $|\mathbf{A}| = 0$, the expressions for x_1 and x_2 become meaningless—indeed, in this case, equation system (16.1.1) either has no solution (because the two lines are parallel), or else has infinitely many solutions (because the two lines coincide).

From Example 16.1.1, we see that the *numerators* of the expressions for x_1 and x_2 in (16.1.2) can also be written as determinants. Indeed, provided that $|\mathbf{A}| \neq 0$, one has

$$x_1 = \frac{1}{|\mathbf{A}|}\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \quad \text{and} \quad x_2 = \frac{1}{|\mathbf{A}|}\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \tag{16.1.4}$$

This is a special case of a result referred to as *Cramer's rule*.[1] It is quite convenient when there are only two equations in two unknowns. But as Exercise 8 shows, it is often easier to solve macroeconomic equation systems in particular by simple substitution.

EXAMPLE 16.1.2 Use (16.1.4) to find the solutions of

$$\begin{aligned} 2x_1 + 4x_2 &= 7 \\ 2x_1 - 2x_2 &= -2 \end{aligned}$$

[1] Named after the Swiss mathematician Gabriel Cramer, 1704–1752.

Solution:

$$x_1 = \frac{\begin{vmatrix} 7 & 4 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 2 & -2 \end{vmatrix}} = \frac{-6}{-12} = \frac{1}{2} \quad \text{and} \quad x_2 = \frac{\begin{vmatrix} 2 & 7 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 2 & -2 \end{vmatrix}} = \frac{-18}{-12} = \frac{3}{2}$$

Check by substitution that $x_1 = 1/2$, $x_2 = 3/2$ really is a solution.

EXAMPLE 16.1.3 Use (16.1.4) to find Q_1^D and Q_2^D in terms of the parameters when

$$2(b+\beta_1)Q_1^D + bQ_2^D = a - \alpha_1$$
$$bQ_1^D + 2(b+\beta_2)Q_2^D = a - \alpha_2$$

Solution: The determinant of the coefficient matrix is

$$\Delta = \begin{vmatrix} 2(b+\beta_1) & b \\ b & 2(b+\beta_2) \end{vmatrix} = 4(b+\beta_1)(b+\beta_2) - b^2$$

Provided $\Delta \neq 0$, Eq. (16.1.4) tells us that the solution for Q_1^D is

$$Q_1^D = \frac{\begin{vmatrix} a-\alpha_1 & b \\ a-\alpha_2 & 2(b+\beta_2) \end{vmatrix}}{\Delta} = \frac{2(b+\beta_2)(a-\alpha_1) - b(a-\alpha_2)}{\Delta}$$

with a similar expression for Q_2^D.

In the next section, Cramer's rule is extended to three equations in three unknowns, and then in Section 16.8 to n equations in n unknowns.

A Geometric Interpretation

Determinants of order 2 have a nice geometric interpretation. If the two rows of the matrix are represented as the vectors shown in Fig. 16.1.1, then its determinant equals the shaded area of the parallelogram. If we interchange the two rows, however, the determinant becomes a negative number equal to minus this shaded area.

Figure 16.1.2 illustrates why the result claimed in Fig. 16.1.1 is true. We want to find area T. Note that the area of the whole rectangle in FIg. 16.1.2 is

$$2T_1 + 2T_2 + 2T_3 + T = (a_{11} + a_{21})(a_{12} + a_{22}) \tag{16.1.5}$$

where $T_1 = a_{12}a_{21}$, $T_2 = \frac{1}{2}a_{21}a_{22}$, and $T_3 = \frac{1}{2}a_{11}a_{12}$. Hence $T = a_{11}a_{22} - a_{21}a_{12}$, by elementary algebra.

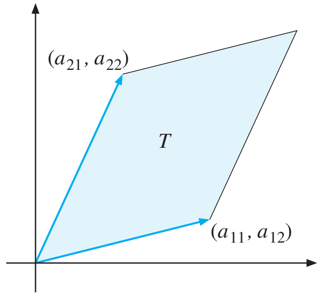

Figure 16.1.1 Area T is the absolute value of the determinant, Eq. (16.1.3)

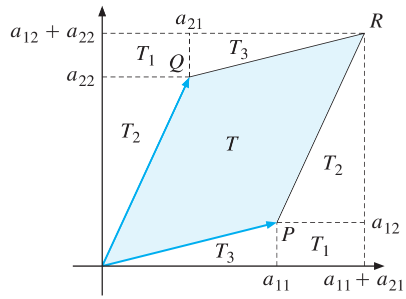

Figure 16.1.2 Illustration of Eq. (16.1.5)

EXERCISES FOR SECTION 16.1

1. Calculate the following determinants:

(a) $\begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} a & a \\ b & b \end{vmatrix}$ (c) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$ (d) $\begin{vmatrix} 3^t & 2^t \\ 3^{t-1} & 2^{t-1} \end{vmatrix}$

2. Illustrate the geometric interpretation in Fig. 16.1.1 for the determinant in Exercise 1(a).

3. Use Cramer's rule (16.1.4) to solve the following systems of equations for x and y. Test the answers by substitution.

(a) $\begin{aligned} 3x - y &= 8 \\ x - 2y &= 5 \end{aligned}$ (b) $\begin{aligned} x + 3y &= 1 \\ 3x - 2y &= 14 \end{aligned}$ (c) $\begin{aligned} ax - by &= 1 \\ bx + ay &= 2 \end{aligned}$

4. The *trace* of a square matrix $\mathbf{A}$ is the sum of its diagonal elements, denoted by $\operatorname{tr}(\mathbf{A})$. Given the matrix $\mathbf{A} = \begin{pmatrix} a & 3 \\ b & 1 \end{pmatrix}$, find numbers a and b such that $\operatorname{tr}(\mathbf{A}) = 0$ and $|\mathbf{A}| = -10$.

5. Find all the solutions to the equation $\begin{vmatrix} 2-x & 1 \\ 8 & -x \end{vmatrix} = 0.$

6. Show that $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$ for the matrices $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

7. Find two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$.

8. Let Y denote GDP and C private consumption, and suppose that investment I_0 and public expenditure G_0 are exogenous. Use Cramer's rule to solve the system of equations

$$Y = C + I_0 + G_0 \text{ and } C = a + bY$$

where a and b represent constants, with $b < 1$. Then look for an alternative simpler way of solving the equations.

SM 9. [HARDER] Consider the following macroeconomic model of two nations, $i = 1, 2$, that trade only with each other:

$$Y_1 = C_1 + A_1 + X_1 - M_1; \qquad C_1 = c_1 Y_1; \qquad M_1 = m_1 Y_1 = X_2$$
$$Y_2 = C_2 + A_2 + X_2 - M_2; \qquad C_2 = c_2 Y_2; \qquad M_2 = m_2 Y_2 = X_1$$

where Y_i is GDP, C_i is consumption, A_i is exogenous expenditure, X_i denotes exports, and M_i denotes imports of country i, for $i = 1, 2$.

(a) Interpret the two equations $M_1 = X_2$ and $M_2 = X_1$.

(b) Given the system of eight equations in eight unknowns, use substitution to reduce it to a pair of simultaneous equations in the endogenous variables Y_1 and Y_2. Then solve for the equilibrium values of Y_1, Y_2 as functions of the exogenous variables A_1, A_2.

(c) How does an increase in A_1 affect Y_2? Interpret your answer.

16.2 Determinants of Order 3

Consider the system of three linear equations in three unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \tag{16.2.1}$$

Here the coefficient matrix $\mathbf{A}$ is 3×3. If we apply the method of elimination along with some rather heavy algebraic computation, the system can be solved eventually for x_1, x_2, and x_3, except in a degenerate case. The resulting expression for x_1 is

$$x_1 = \frac{b_1a_{22}a_{33} - b_1a_{23}a_{32} - b_2a_{12}a_{33} + b_2a_{13}a_{32} + b_3a_{12}a_{23} - b_3a_{22}a_{13}}{a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}}$$

We shall not triple the demands on the reader's patience and eyesight by giving the corresponding expressions for x_2 and x_3. However, we do claim that these expressions share the same denominator as that given for x_1. This common denominator is called the *determinant* of $\mathbf{A}$, denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$, which is zero in the degenerate case. Thus, the determinant is defined as

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left\{ \begin{array}{l} a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ \qquad -a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{array} \right. \tag{16.2.2}$$

EXAMPLE 16.2.1 Suppose we want to know the second derivative of $y = f(x)$, as defined implicitly by $F(x, f(x)) = c$, as in Section 12.3. Equation (12.3.4) tells us that

$$\frac{d^2 y}{dx^2} = -\frac{1}{(F_2')^3}[F_{11}''(F_2')^2 - 2F_{12}''F_1'F_2' + F_{22}''(F_1')^2]$$

and we mentioned there that this result can be expressed in a more memorable form, if we use the concept of determinant. As promised, note that

$$\frac{d^2 y}{d x^2} = \frac{1}{(F_2')^3} \begin{vmatrix} 0 & F_1' & F_2' \\ F_1' & F_{11}'' & F_{12}'' \\ F_2' & F_{21}'' & F_{22}'' \end{vmatrix} \tag{16.2.3}$$

provided, of course, that $F_2' \neq 0$.

EXAMPLE 16.2.2 Consider the problems

$$\max(\min) f(x, y) \quad \text{s.t.} \quad g(x, y) = c$$

and let the associated Lagrangian be

$$\mathcal{L}(x, y) = f(x, y) - \lambda[g(x, y) - c]$$

as studied in Section 14.5. Theorem 14.5.2 presented the sufficient second-order conditions under which local solutions of constrained optimization problems can be found by the Lagrange method: it told us that whether the critical points of $\mathcal{L}$ are local solutions to the optimization problem can be determined by looking at the sign of

$$D(x, y, \lambda) = (f_{11}'' - \lambda g_{11}'')(g_2')^2 - 2(f_{12}'' - \lambda g_{12}'')g_1' g_2' + (f_{22}'' - \lambda g_{22}'')(g_1')^2$$

Again, we mentioned then that the rather lengthy expression $D(x, y, \lambda)$ can be written in a symmetric form that is easier to remember. That easier form is, simply, that

$$D(x, y, \lambda) = - \begin{vmatrix} 0 & g_1'(x, y) & g_2'(x, y) \\ g_1'(x, y) & \mathcal{L}_{11}''(x, y) & \mathcal{L}_{12}''(x, y) \\ g_2'(x, y) & \mathcal{L}_{21}''(x, y) & \mathcal{L}_{22}''(x, y) \end{vmatrix} \tag{16.2.4}$$

Recall from Eq. (11.6.2) that the 2×2 matrix at the bottom right of Eq. (16.2.4) is the Hessian of the Lagrangian. So the determinant $D(x, y, \lambda)$ is naturally called a *bordered Hessian*; its "borders" in the first row and first column, apart from the 0 element in the top left position, are the first-order partial derivatives of g.

Expansion by Cofactors

Consider the sum of the six terms in (16.2.2). It looks quite messy, but a method called expansion by cofactors makes it easy to write down all the terms. First, note that each of the three elements a_{11}, a_{12}, and a_{13} in the first row of $\mathbf{A}$ appears in exactly two terms of (16.2.2). In fact, $|\mathbf{A}|$ can be written as

$$|\mathbf{A}| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Applying the rule for evaluating determinants of order 2, we see that this is the same as

$$|\mathbf{A}| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{16.2.5}$$

In this way, the computation of a determinant of order 3 can be reduced to calculating three determinants of order 2. Note that a_{11} is multiplied by the second-order determinant obtained by deleting the *first* row and the *first* column of $|\mathbf{A}|$. Likewise, a_{12}, with a minus sign attached to it, is multiplied by the determinant obtained by deleting the *first* row and the *second* column of $|\mathbf{A}|$. Finally, a_{13} is multiplied by the determinant obtained by deleting the *first* row and the *third* column of $|\mathbf{A}|$.

EXAMPLE 16.2.3 Use (16.2.5) to calculate $|\mathbf{A}| = \begin{vmatrix} 3 & 0 & 2 \\ -1 & 1 & 0 \\ 5 & 2 & 3 \end{vmatrix}$.

Solution: By direct application of (16.2.5), one has

$$|\mathbf{A}| = 3 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} + 2 \cdot \begin{vmatrix} -1 & 1 \\ 5 & 2 \end{vmatrix} = 3 \cdot 3 - 0 + 2 \cdot (-2 - 5) = -5$$

EXAMPLE 16.2.4 Use (16.2.5) to prove that $|\mathbf{A}| = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$.

Solution: By direct computation,

$$|\mathbf{A}| = 1 \cdot \begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - a \cdot \begin{vmatrix} 1 & b^2 \\ 1 & c^2 \end{vmatrix} + a^2 \cdot \begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} = bc^2 - b^2c - ac^2 + ab^2 + a^2c - a^2b$$

You are not expected to "see" that these six terms can be written as $(b-a)(c-a)(c-b)$. Rather, you should expand $(b-a)[(c-a)(c-b)]$ and verify the equality that way.

A careful study of the numerator in the expression for x_1 in the beginning of this section reveals that it can also be written as a determinant. The same is true of the corresponding formulas for x_2 and x_3. In fact, if $|\mathbf{A}| \neq 0$, then one has

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|\mathbf{A}|}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|\mathbf{A}|}, \quad \text{and} \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|\mathbf{A}|} \tag{16.2.6}$$

This is Cramer's rule for the solution of (16.2.1). See Section 16.8 for a full proof of (16.2.6) for the general case of n equations in n unknowns.

Note that in the determinants appearing in the numerators of x_1, x_2, and x_3 of (16.2.6), the right-hand column in (16.2.1), which is

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

shifts from the first column when solving for x_1, to the second column when solving for x_2, and then to the third column when solving for x_3. This makes it very easy to remember Cramer's rule.

The method in (16.2.5) for calculating the value of a 3×3 determinant is called *cofactor expansion along row* 1. If we focus on the elements in row i instead of row 1, we again find that $|\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$, where for $j = 1, 2, 3$, the factor C_{ij} equals $(-1)^{i+j}$ times the determinant of the 2×2 matrix we get by deleting row i and column j from $\mathbf{A}$. Thus, we can also find the value of the determinant by cofactor expansion along row i for any $i = 1, 2, 3$. Moreover, it turns out that for $j = 1, 2$, or 3, we also have $|\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}$. In other words, we can calculate the determinant by cofactor expansion along column j. See Section 16.5 for more about cofactor expansion.

EXAMPLE 16.2.5 Use Cramer's rule to solve the following system of equations:

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= -3 \\ 4x_1 \phantom{{}+2x_2} + 2x_3 &= 8 \\ 6x_2 - 3x_3 &= -12 \end{aligned}$$

Solution: In this case, the determinant $|\mathbf{A}|$ in (16.2.6) is seen to be $|\mathbf{A}| = \begin{vmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -3 \end{vmatrix} = -24$.

As you should verify, the numerators in (16.2.6) are

$$\begin{vmatrix} -3 & 2 & -1 \\ 8 & 0 & 2 \\ -12 & 6 & -3 \end{vmatrix} = -12, \quad \begin{vmatrix} 2 & -3 & -1 \\ 4 & 8 & 2 \\ 0 & -12 & -3 \end{vmatrix} = 12 \quad \text{and} \quad \begin{vmatrix} 2 & 2 & -3 \\ 4 & 0 & 8 \\ 0 & 6 & -12 \end{vmatrix} = -72$$

Hence, (16.2.6) yields the solution $x_1 = (-12)/(-24) = 1/2$, $x_2 = 12/(-24) = -1/2$ and $x_3 = (-72)/(-24) = 3$. Inserting this into the original system of equations verifies that this is a correct answer.

A Geometric Interpretation

Like determinants of order 2, those of order 3 also have a geometric interpretation which is shown in Fig. 16.2.1. The rows of the determinant correspond to three different 3-vectors represented in the diagram. These vectors determine a "box" which is not rectangular with right-angles at each of its six corners, but a distorted "parallelepiped" which has six faces

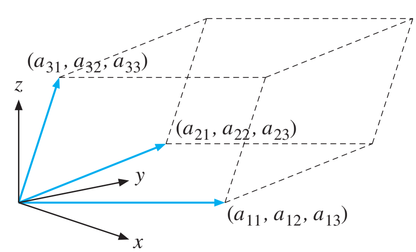

Figure 16.2.1 Parallelepiped spanned by the three row vectors in the matrix

that are all parallelograms, i.e. quadrilaterals whose opposite edges are parallel. Then the volume of this parallelepiped must equal the absolute value of the determinant $|\mathbf{A}|$, as defined by Eq. (16.2.2):

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Sarrus's Rule

Here is an alternative way of evaluating determinants of order 3 that many people find convenient. Write down the determinant twice, except that the last column in the second determinant should be omitted:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \tag{16.2.7}$$

First, multiply along the three lines falling to the right, giving all these products a plus sign:

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

Second, multiply along the three lines rising to the right, giving all these products a minus sign:

$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

The sum of all the six terms is exactly equal to formula (16.2.2) for $|\mathbf{A}|$. It is important to note that this method, known as *Sarrus's rule*, *does not generalize* to determinants of order higher than 3.

EXERCISES FOR SECTION 16.2

SM **1.** Use (16.2.5) or Sarrus's rule to calculate the following determinants:

(a) $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 0 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix}$ (d) $\begin{vmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{vmatrix}$

2. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & -1 \end{pmatrix}$.

Calculate $\mathbf{AB}$, $|\mathbf{A}|$, $|\mathbf{B}|$, $|\mathbf{A}| \cdot |\mathbf{B}|$, and $|\mathbf{AB}|$, then verify that $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$.

SM **3.** Use Cramer's rule to solve the following systems of equations. Check your answers.

(a) $\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_1 + x_2 - x_3 &= 0 \\ -x_1 - x_2 - x_3 &= -6 \end{aligned}$ (b) $\begin{aligned} x_1 - x_2 &= 0 \\ x_1 + 3x_2 + 2x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \end{aligned}$ (c) $\begin{aligned} x + 3y - 2z &= 1 \\ 3x - 2y + 5z &= 14 \\ 2x - 5y + 3z &= 1 \end{aligned}$

4. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + ab + ac + bc$.

5. Given the matrix $\mathbf{A} = \begin{pmatrix} a & 1 & 0 \\ 0 & -1 & a \\ -b & 0 & b \end{pmatrix}$, find numbers a and b such that $\operatorname{tr}(\mathbf{A}) = 0$ and $|\mathbf{A}| = 12$, where $\operatorname{tr}(\mathbf{A})$ denotes the sum of $\mathbf{A}$'s diagonal elements.

6. Solve the equation $\begin{vmatrix} 1-x & 2 & 2 \\ 2 & 1-x & 2 \\ 2 & 2 & 1-x \end{vmatrix} = 0$.

7. Define the matrix $\mathbf{A}_t = \begin{pmatrix} 1 & t & 0 \\ -2 & -2 & -1 \\ 0 & 1 & t \end{pmatrix}$.

   (a) Calculate the determinant of $\mathbf{A}_t$, and show that it is never 0.

   (b) Show that for a certain value of t, one has $\mathbf{A}_t^3 = \mathbf{I}_3$.

SM 8. Consider the simple macroeconomic model described by the three equations

$$\text{(i) } Y = C + A_0 \qquad \text{(ii) } C = a + b(Y - T) \qquad \text{(iii) } T = d + tY$$

Here Y is GDP, C is consumption, T is tax revenue, A_0 is the constant (exogenous) autonomous expenditure, and a, b, d, and t are all positive parameters. Find the equilibrium values of the endogenous variables Y, C, and T by: (a) successive elimination or substitution; (b) writing the equations in matrix form and applying Cramer's rule.

16.3 Determinants in General

This section gives a definition of $n \times n$ determinants that is particularly useful when proving general results. If you are not so interested in these proofs, you might skip this section and rely instead on expansion by cofactors (explained in Section 16.5) in all your work on determinants.

In Eq. (16.2.2), we expressed the determinant of a 3×3 matrix $\mathbf{A} = (a_{ij})_{3\times 3}$ in the form

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \tag{16.3.1}$$

Examining this expression more closely reveals a definite pattern. Each term is the product of three different elements of the matrix. Each product contains one element from each row of $\mathbf{A}$. Moreover, these elements all lie in different columns of the matrix. In fact, the three elements appearing in each of the six terms are chosen from the matrix $\mathbf{A}$ according to the pattern shown by the circles in Fig. 16.3.1—disregard the lines for a moment.

In a 3×3 matrix, there are precisely six different ways of picking three elements with one element from each row and one element from each column. All the six corresponding products appear in (16.3.1). How do we determine the sign of each term in (16.3.1)? In

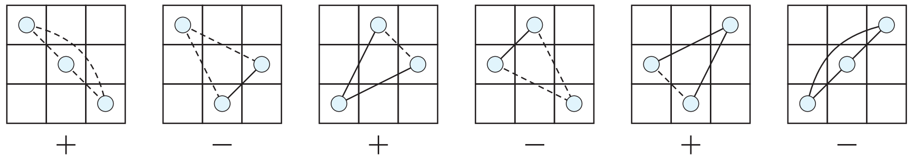

Figure 16.3.1 The terms of a 3×3 determinant

Fig. 16.3.1, we have joined each pair of circles in every box by a line, which is solid if the line rises to the right, but dashed if the line falls to the right. Using the solid lines drawn in each of the six boxes, the following rule emerges:

THE SIGN RULE

To determine the sign of any term in the sum, mark in the array all the elements appearing in that term. Join all possible pairs of these elements with lines. These lines will then either fall or rise to the right. If the number of the rising lines is even, then the corresponding term is assigned a plus sign; if it is odd, it is assigned a minus sign.

Let us apply this rule to the six boxes in Fig. 16.3.1. In the first box, for example, no lines rise to the right, so $a_{11}a_{22}a_{33}$ has a plus sign. In the fourth box, exactly one line rises to the right, so $a_{12}a_{21}a_{33}$ has a minus sign. And so on.

Suppose $\mathbf{A} = (a_{ij})_{n\times n}$ is an arbitrary $n \times n$ matrix. Suppose we pick n elements from $\mathbf{A}$, including exactly one element from each row and exactly one element from each column. Take the product of these n elements, giving an expression of the form

$$a_{1r_1} \cdot a_{2r_2} \cdot \ldots \cdot a_{nr_n}$$

where the second subscripts $r_1, r_2, \ldots, r_n$ represent a shuffling (or permutation) of the numbers $1, 2, \ldots, n$. The numbers $1, 2, \ldots, n$ can be permuted in $n! = 1 \cdot 2 \ldots (n-1)n$ different ways: for the first element, there are n choices; for each of these first choices, there are $n-1$ choices for the second element; and so on. So there are $n!$ different products of n factors to consider. Now, we define the determinant of $\mathbf{A}$, $\det(\mathbf{A})$ or $|\mathbf{A}|$, as follows:

DEFINITION OF DETERMINANT

Let $\mathbf{A}$ be an $n \times n$ matrix. Then $|\mathbf{A}|$ is a sum of $n!$ terms where:

1. Each term is the product of n elements of the matrix, with one element from each row and one element from each column. Moreover, every product of n factors, in which each row and each column is represented exactly once, must appear in this sum.
2. The sign of each term is found by applying the sign rule.

Using ($\pm$) to denote the appropriate choice of either a plus or minus sign, one can write

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum (\pm) a_{1r_1} a_{2r_2} \dots a_{nr_n} \tag{16.3.2}$$

EXAMPLE 16.3.1 Consider the determinant of an arbitrary 4×4 matrix $\mathbf{A} = \left(a_{ij}\right)_{4\times 4}$:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \boxed{a_{13}} & a_{14} \\ \boxed{a_{21}} & a_{22} & a_{23} & a_{24} \\ a_{31} & \boxed{a_{32}} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix}$$

It consists of $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ terms. One of these terms is $a_{13}a_{21}a_{32}a_{44}$, and the corresponding factors are the boxed elements in the array. What sign should this term have? According to the sign rule, the term should have the plus sign because there are two lines that rise to the right.[2] Check that the four indicated terms in the following sum have been given the correct sign:

$$|\mathbf{A}| = a_{11}a_{22}a_{33}a_{44} - a_{12}a_{21}a_{33}a_{44} + \cdots + a_{13}a_{21}a_{32}a_{44} - \cdots + a_{14}a_{23}a_{32}a_{41}$$

Note that there are 20 other terms which we have left out.

The determinant of an $n \times n$ matrix is called a *determinant of order n*. In general, it is difficult to evaluate determinants by using the definition directly, even if n is only 4 or 5. If $n > 5$, the work is usually enormous. For example, if $n = 6$, then $n! = 720$, and so there are 720 terms in the sum defining the determinant. Fortunately there are other methods based on the elementary row operations discussed in Section 15.6 that reduce the work considerably. There are several standard computer programs for evaluating determinants.

In a few special cases, it is easy to evaluate a determinant even if the order is high. For instance,

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11}a_{22} \dots a_{nn} \tag{16.3.3}$$

Here all the elements *below* the main diagonal are 0. The matrix whose determinant is given in (16.3.3) is called *upper triangular* because all the nonzero terms lie in the triangle on or above the main diagonal. Such a determinant can be evaluated by taking the product of all the elements on the main diagonal. To see why, note that in order to have a term that is not 0, we have to choose a_{11} from column 1. From column 2, we cannot choose a_{12}, because we have already picked the element a_{11} from the first row. Hence, from column 2, we have

[2] We have omitted the dashed lines, because these do not count.

to pick a_{22} in order to have a term different from 0. From the third column, we have to pick a_{33}, and so on. Thus, only the term $a_{11}a_{22}\dots a_{nn}$ can be $\neq 0$. The sign of this term is plus because no line joining any pair of elements appearing in the product rises to the right.

If a matrix is a transpose of an upper triangular matrix, so that all elements above the main diagonal are 0, then the matrix is *lower triangular*. By using essentially the same argument as for (16.3.3), we see that the determinant of a lower triangular matrix is also equal to the product of the elements on its main diagonal:

$$\begin{vmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{11}a_{22}\dots a_{nn} \tag{16.3.4}$$

EXERCISES FOR SECTION 16.3

SM 1. Use the definition of determinant to calculate the following:

(a) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & 11 \end{vmatrix}$

2. Suppose that the two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are both upper triangular. Show that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

3. The determinant of the following 5×5 matrix consists of $5! = 120$ terms. One of them is the product of the boxed elements. Write this term with its correct sign.

$$\begin{vmatrix} a_{11} & \boxed{a_{12}} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & \boxed{a_{23}} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & \boxed{a_{35}} \\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & \boxed{a_{54}} & a_{55} \end{vmatrix}$$

4. Write the term indicated by the marked boxes with its correct sign.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \boxed{a_{15}} \\ a_{21} & a_{22} & a_{23} & \boxed{a_{24}} & a_{25} \\ a_{31} & \boxed{a_{32}} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & \boxed{a_{43}} & a_{44} & a_{45} \\ \boxed{a_{51}} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}$$

5. Solve the following equation for x:

$$\begin{vmatrix} 2-x & 0 & 3 & 0 \\ 1 & 2-x & 0 & 3 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 1 & 2-x \end{vmatrix} = 0$$

16.4 Basic Rules for Determinants

The definition of the determinant of an $n \times n$ matrix $\mathbf{A}$ implies a number of important properties. All are of theoretical interest, but they also make it simpler to evaluate determinants.

THEOREM 16.4.1 (RULES FOR DETERMINANTS)

Let $\mathbf{A}$ be an $n \times n$ matrix. Then:

(i) If all the elements in a row (or column) of $\mathbf{A}$ are 0, then $|\mathbf{A}| = 0$.

(ii) $|\mathbf{A}'| = |\mathbf{A}|$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$.

(iii) If all the elements in a single row (or column) of $\mathbf{A}$ are multiplied by a number α, the determinant is multiplied by α.

(iv) If two rows (or two columns) of $\mathbf{A}$ are interchanged, the determinant changes sign, but the absolute value remains unchanged.

(v) If two of the rows (or columns) of $\mathbf{A}$ are proportional, then $|\mathbf{A}| = 0$.

(vi) The value of the determinant of $\mathbf{A}$ is unchanged if a multiple of one row (or one column) is added to a different row (or column) of $\mathbf{A}$.

(vii) The determinant of the product of two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ is the product of the determinants of each of the factors:

$$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}| \tag{16.4.1}$$

(viii) If α is a real number,

$$|\alpha\mathbf{A}| = \alpha^n |\mathbf{A}| \tag{16.4.2}$$

It should be recalled that, in general, the determinant of a sum is *not* the sum of the determinants:

$$|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}| \tag{16.4.3}$$

An example of this general inequality was asked for in Exercise 16.1.7.

Our geometric interpretations of determinants of order 2 and 3 support several of these rules. For example, rule (iii) with, say, $\alpha = 2$, reflects the fact that if one of the vectors in Figs 16.1.1 or 16.2.1 is doubled in length, then the area or volume is twice as big. A good exercise is to try to provide geometric interpretations of rules (i), (ii), (iv), (v), and (xiii).

Proofs for most of these properties are given at the end of this section. First, however, let us illustrate them for the special case of 2×2 matrices.

(i) $\begin{vmatrix} a_{11} & a_{12} \\ 0 & 0 \end{vmatrix} = a_{11} \cdot 0 - a_{12} \cdot 0 = 0.$

(ii) $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$, while $|\mathbf{A}'| = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$

(iii) $\begin{vmatrix} a_{11} & \alpha a_{12} \\ a_{21} & \alpha a_{22} \end{vmatrix} = a_{11}(\alpha a_{22}) - a_{12}(\alpha a_{21}) = \alpha(a_{11}a_{22} - a_{12}a_{21}) = \alpha \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$

(iv) $\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21}) = -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$

(v) $\begin{vmatrix} a_{11} & a_{12} \\ \beta a_{11} & \beta a_{12} \end{vmatrix} = a_{11}(\beta a_{12}) - a_{12}(\beta a_{11}) = \beta(a_{11}a_{12} - a_{11}a_{12}) = 0.$[3]

(vi) Multiply each entry in the first row of a determinant of order 2 by α and add it to the corresponding entry in the second row. We show that the determinant does not change its value.[4] Indeed,

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{array}{l} \alpha \\ \hookleftarrow \end{array} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} + \alpha a_{11} & a_{22} + \alpha a_{12} \end{vmatrix} &= a_{11}(a_{22} + \alpha a_{12}) - a_{12}(a_{21} + \alpha a_{11}) \\
&= a_{11}a_{22} + \alpha a_{11}a_{12} - a_{12}a_{21} - \alpha a_{12}a_{11} = a_{11}a_{22} - a_{12}a_{21} \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
\end{aligned}$$

(vii) Exercise 16.1.6 has already asked for a proof of this rule, for 2×2 matrices.

(viii) $\begin{vmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{vmatrix} = \alpha a_{11}\alpha a_{22} - \alpha a_{12}\alpha a_{21} = \alpha^2(a_{11}a_{22} - a_{12}a_{21}) = \alpha^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$

Theorem 16.4.1 exhibits some of the most important rules for determinants. Confidence in dealing with them comes only from doing many exercises.

Rule (vi) is particularly useful for evaluating large or complicated determinants.[5] The idea is to convert the matrix into one that is (upper or lower) triangular. This is just the same procedure as that used in the Gaussian elimination method described in Section 15.6. We give two examples involving 3×3 matrices.

EXAMPLE 16.4.1

$$\begin{aligned}
\begin{vmatrix} 1 & 5 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} \begin{array}{l} 1 \\ \hookleftarrow \\ \end{array} &= \begin{vmatrix} 1 & 5 & -1 \\ -1+1 & 1+5 & 3+(-1) \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -1 \\ 0 & 6 & 2 \\ 3 & 2 & 1 \end{vmatrix} \begin{array}{l} -3 \\ \\ \hookleftarrow \end{array} \\
&= \begin{vmatrix} 1 & 5 & -1 \\ 0 & 6 & 2 \\ 0 & -13 & 4 \end{vmatrix} \begin{array}{l} \\ \frac{13}{6} \\ \hookleftarrow \end{array} = \begin{vmatrix} 1 & 5 & -1 \\ 0 & 6 & 2 \\ 0 & 0 & \frac{25}{3} \end{vmatrix} = 1 \cdot 6 \cdot \frac{25}{3} = 50
\end{aligned}$$

Here, 1 times the first row has been added to the second row in order to obtain a zero in the first column. Then (-3) times the first row has been added to the third row, which gives a second zero in the first column. Thereafter, 13/6 times the second row has been added to the third row, which creates an extra zero in the second column. Note the way in which we have indicated these row operations. In the end, they produce an upper triangular matrix whose determinant is easy to evaluate by means of formula (16.3.3).

---

[3] This rule helps to confirm, in part, the result in Example 16.2.4. Note that the product $(b-a)(c-a)(c-b)$ is 0 if $b = a$, $c = a$, or $c = b$, and in each of these three cases, two rows of the matrix are proportional, in fact equal.

[4] Note carefully the way in which we indicate this operation—see also Section 15.6.

[5] To calculate a general 10×10 determinant using the definition directly requires no fewer than $10! - 1 = 3\,628\,799$ operations of addition or multiplication! Systematic use of rule (vi) can reduce the required number of operations to about 380.

In the next example, the first and third steps involve the simultaneous use of more than one row operation.

EXAMPLE 16.4.2

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = \begin{vmatrix} 3a+b & 3a+b & 3a+b \\ a & a+b & a \\ a & a & a+b \end{vmatrix}$$

$$= (3a+b)\begin{vmatrix} 1 & 1 & 1 \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = (3a+b)\begin{vmatrix} 1 & 1 & 1 \\ 0 & b & 0 \\ 0 & 0 & b \end{vmatrix}$$

$$= (3a+b)\cdot 1\cdot b\cdot b = b^2(3a+b)$$

EXAMPLE 16.4.3 Check that $|\mathbf{AB}| = |\mathbf{A}|\cdot|\mathbf{B}|$ when $\mathbf{A} = \begin{pmatrix} 1 & 5 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 0 \\ 5 & 2 & 3 \end{pmatrix}$.

Solution: From Example 16.4.1, $|\mathbf{A}| = 50$. You should verify that $|\mathbf{B}| = -5$. Moreover, multiplying the two matrices yields

$$\mathbf{AB} = \begin{pmatrix} -7 & 3 & -1 \\ 11 & 7 & 7 \\ 12 & 4 & 9 \end{pmatrix}$$

Using Sarrus's rule, or otherwise, we find that $|\mathbf{AB}| = -250$. Thus $|\mathbf{AB}| = |\mathbf{A}|\cdot|\mathbf{B}|$.

Finally, we discuss the argument for Theorem 16.4.1, rule by rule:

(i) Each of the $n!$ terms in the determinant must take one element from the row (or column) consisting of only zeros, so the whole determinant is 0.

(ii) Each term in $|\mathbf{A}|$ is the product of entries chosen from $\mathbf{A}$ to include exactly one element from each row and one element from each column. Exactly the same terms, therefore, must appear in $|\mathbf{A}'|$ also. One can prove that the signs are also the same, but we skip the proof.[6]

(iii) Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}$ by multiplying every element in a certain row (or column) of $\mathbf{A}$ by α. Then each term in the sum defining $|\mathbf{B}|$ is the corresponding term in the sum defining $|\mathbf{A}|$ multiplied by α. Hence, $|\mathbf{B}| = \alpha|\mathbf{A}|$.

(iv) If two rows are interchanged, or two columns, the terms involved in the definition of determinant in Section 16.3 remain the same, except that the sign of each term is reversed. Showing this, however, involves a somewhat intricate argument, so we offer only this brief explanation.

(v) By using rule (iii), the factor of proportionality can be put outside the determinant. The determinant then has two equal rows (columns). If we interchange the two rows that are equal, the determinant will be exactly the same. But according to rule (iv), the determinant has changed its sign. Hence, $|\mathbf{A}| = -|\mathbf{A}|$, which means that $2|\mathbf{A}| = 0$, and so $|\mathbf{A}| = 0$.

[6] The proof of this property and the others we leave unproved are found in most books on linear algebra.

(vi) For the case when the scalar multiple α of row i is added to row j, Eq. (16.3.2) implies that

$$\sum(\pm)a_{1r_1}\dots a_{ir_i}\dots(a_{jr_j}+\alpha a_{ir_j})\dots a_{nr_n}$$
$$= \sum(\pm)a_{1r_1}\dots a_{ir_i}\dots a_{jr_j}\dots a_{nr_n}+\alpha\sum(\pm)a_{1r_1}\dots a_{ir_i}\dots a_{ir_j}\dots a_{nr_n}$$

But the last sum is zero because it is equal to a determinant with rows i and j equal, so the right-hand side reduces to $|\mathbf{A}|+\alpha\cdot 0=|\mathbf{A}|$.

(vii) The proof of this rule for the case $n=2$ is the object of Exercise 16.1.6 . The case when $\mathbf{A}$ and $\mathbf{B}$ are both upper triangular is covered in Exercise 16.3.2. One can prove the general case by using elementary row and column operations to convert $\mathbf{A}$ and $\mathbf{B}$ as well as $\mathbf{AB}$ to upper triangular form, but we omit the proof.

(viii) The matrix $\alpha\mathbf{A}$ is obtained by multiplying *each* entry in $\mathbf{A}$ by α. By rule C, $|\alpha\mathbf{A}|$ is then equal to $\alpha^n|\mathbf{A}|$, because each of the n rows has α as a factor in each entry.

EXERCISES FOR SECTION 16.4

1. Let $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3&4\\5&6\end{pmatrix}$.

   (a) Calculate $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}'\mathbf{B}'$, and $\mathbf{B}'\mathbf{A}'$.

   (b) Show that $|\mathbf{A}|=|\mathbf{A}'|$ and $|\mathbf{AB}|=|\mathbf{A}|\cdot|\mathbf{B}|$.

   (c) Is $|\mathbf{A}'\mathbf{B}'|=|\mathbf{A}'|\cdot|\mathbf{B}'|$?

2. Given the matrix $\mathbf{A}=\begin{pmatrix}2&1&3\\1&0&1\\1&2&5\end{pmatrix}$, first determine $\mathbf{A}'$, and then show that $|\mathbf{A}|=|\mathbf{A}'|$.

3. Evaluate the following three determinants as simply as possible:

   (a) $\begin{vmatrix}3&0&1\\1&0&-1\\2&0&5\end{vmatrix}$ (b) $\begin{vmatrix}1&2&3&4\\0&-1&2&4\\0&0&3&-1\\-3&-6&-9&-12\end{vmatrix}$ (c) $\begin{vmatrix}a_1-x&a_2&a_3&a_4\\0&-x&0&0\\0&1&-x&0\\0&0&1&-x\end{vmatrix}$

4. Let $\mathbf{A}$ and $\mathbf{B}$ be 3×3 matrices with $|\mathbf{A}|=3$ and $|\mathbf{B}|=-4$.

   (a) Wherever possible, determine the unique numerical value of $|\mathbf{AB}|$, $3|\mathbf{A}|$, $|-2\mathbf{B}|$, $|4\mathbf{A}|$, $|\mathbf{A}|+|\mathbf{B}|$, and $|\mathbf{A}+\mathbf{B}|$.

   (b) Which, if any, have a numerical value that is not determined uniquely given $|\mathbf{A}|$ and $|\mathbf{B}|$?

5. If $\mathbf{A}=\begin{pmatrix}a&1&4\\2&1&a^2\\1&0&-3\end{pmatrix}$, calculate $\mathbf{A}^2$ and $|\mathbf{A}|$.

6. Prove that each of the following determinants is zero:

   (a) $\begin{vmatrix}1&2&3\\2&4&5\\3&6&8\end{vmatrix}$ (b) $\begin{vmatrix}1&a&b+c\\1&b&c+a\\1&c&a+b\end{vmatrix}$ (c) $\begin{vmatrix}x-y&x-y&x^2-y^2\\1&1&x+y\\y&1&x\end{vmatrix}$

7. Calculate $\mathbf{X}'\mathbf{X}$ and $|\mathbf{X}'\mathbf{X}|$ if $\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

8. Calculate $|\mathbf{A}_a|$ and $|\mathbf{A}_1^6|$ if $\mathbf{A}_a = \begin{pmatrix} a & 2 & 2 \\ 2 & a^2+1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

9. For an orthogonal matrix $\mathbf{P}$ defined in Exercise 15.5.7, show that its determinant must be 1 or -1.

10. A square matrix $\mathbf{A}$ of order n is called *involutive* if $\mathbf{A}^2 = \mathbf{I}_n$.

(a) Show that the determinant of an involutive matrix is 1 or -1.

(b) Show that for all a the two matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} a & 1-a^2 \\ 1 & -a \end{pmatrix}$ are both involutive.

(c) Show that $\mathbf{A}$ is involutive if and only if $(\mathbf{I}_n - \mathbf{A})(\mathbf{I}_n + \mathbf{A}) = \mathbf{0}$.

11. Determine which of the following equalities are (generally) true/false:

(a) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} a & -b \\ c & -d \end{vmatrix} = 2\begin{vmatrix} \frac{1}{2}a & \frac{1}{2}b \\ \frac{1}{2}c & \frac{1}{2}d \end{vmatrix}$
(b) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$

(c) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 & b \\ -1 & 1 & 0 \\ c & 0 & d \end{vmatrix}$
(d) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c-2a & d-2b \end{vmatrix}$

12. Let $\mathbf{B}$ be a given $n \times n$ matrix. An $n \times n$ matrix $\mathbf{P}$ is said to *commute* with $\mathbf{B}$ if $\mathbf{BP} = \mathbf{PB}$. Show that if $\mathbf{P}$ and $\mathbf{Q}$ both commute with $\mathbf{B}$, then $\mathbf{PQ}$ will also commute with $\mathbf{B}$.

13. [HARDER] Without computing the determinants, show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

SM 14. [HARDER] Prove the following useful result (which is Example 16.4.2 in the case when $n = 3$):

$$D_n = \begin{vmatrix} a+b & a & \dots & a \\ a & a+b & \dots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \dots & a+b \end{vmatrix} = b^{n-1}(na+b)$$

16.5 Expansion by Cofactors

According to the definition in Section 16.3, the determinant of an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ is a sum of $n!$ terms. Each term contains one element from each row and one element from each column. Consider in particular row i: pick out all the terms that have a_{i1} as a factor, then all the terms that have a_{i2} as a factor, and so on. Because all these terms have precisely one factor from row i, in this way we get all the terms of $|\mathbf{A}|$. So we can write

$$|\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{ij}C_{ij} + \cdots + a_{in}C_{in} \tag{16.5.1}$$

This is called the *expansion of* $|\mathbf{A}|$ *in terms of the elements of the* i-th *row*. The coefficients $C_{i1}, \ldots, C_{in}$ are the *cofactors* of the elements $a_{i1}, \ldots, a_{in}$, and Eq. (16.5.1) is called the *cofactor expansion of* $|\mathbf{A}|$ *along row* i.

Similarly, one has the *cofactor expansion of* $|\mathbf{A}|$ *down column* j, which is

$$|\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{ij}C_{ij} + \cdots + a_{nj}C_{nj} \tag{16.5.2}$$

What makes expansions (16.5.1) and (16.5.2) extremely useful is that in general each cofactor C_{ij} can be found by applying the following procedure to the determinant $|\mathbf{A}|$: First, delete row i and column j to arrive at a determinant of order $n - 1$, which is called a *minor*. Second, multiply the minor by the factor $(-1)^{i+j}$. This gives the cofactor.

In symbols, the cofactor C_{ij} is given by

$$C_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \ldots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \ldots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & \ldots & a_{i,j-1} & \boxed{a_{ij}} & a_{i,j+1} & \ldots & a_{in} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \ldots & a_{nn} \end{vmatrix} \tag{16.5.3}$$

where lines have been drawn through row i and column j, which are to be deleted from the matrix. We skip the proof, but if we look back at (16.2.5), it confirms (16.5.3) in a special case: indeed, put $|\mathbf{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$; then

$$C_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad \text{and} \quad C_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

precisely in accordance with (16.2.5).

Generally, formula (16.5.3) is rather complicated. Test your understanding of it by studying the following example.

EXAMPLE 16.5.1 Check that the cofactor of the element c in the determinant

$$|\mathbf{A}| = \begin{vmatrix} 3 & 0 & 0 & 2 \\ 6 & 1 & \boxed{c} & 2 \\ -1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 3 \end{vmatrix} \quad \text{is} \quad C_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 0 & 2 \\ -1 & 1 & 0 \\ 5 & 2 & 3 \end{vmatrix}$$

Find the value of $|\mathbf{A}|$ by using (16.5.2) and Example 16.2.3.

Solution: Because the element c is in row 2 and column 3, its cofactor has been written correctly. To find the numerical value of $|\mathbf{A}|$ we use the cofactor expansion down its *third column*, because it has many zeros. Using the answer to Example 16.2.3, this yields

$$|\mathbf{A}| = a_{23}C_{23} = c \cdot (-1)^{2+3} \begin{vmatrix} 3 & 0 & 2 \\ -1 & 1 & 0 \\ 5 & 2 & 3 \end{vmatrix} = c \cdot (-1)(-5) = 5c$$

Example 16.5.1 shows a case in which expansion by cofactors is particularly simple because there are many zeros in the third column. If the zeros are not there initially, we can often create them by appealing to rule (vi) in Theorem 16.4.1. Two examples illustrate the method.

EXAMPLE 16.5.2

$$\begin{vmatrix} 3 & -1 & 2 \\ 0 & -1 & -1 \\ 6 & 1 & 2 \end{vmatrix} \begin{array}{c} -2 \\ \big| \\ \hookleftarrow \end{array} = \begin{vmatrix} 3 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 3 & -2 \end{vmatrix} \overset{(*)}{=} 3 \begin{vmatrix} -1 & -1 \\ 3 & -2 \end{vmatrix} = 3(2+3) = 15$$

To derive the equality labelled $(*)$, expand by column 1.

EXAMPLE 16.5.3

$$\begin{vmatrix} 2 & 0 & 3 & -1 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 3 & 2 & 5 & -3 \end{vmatrix} \overset{(*)}{=} (-1)^{2+2} \cdot 4 \begin{vmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 3 & 5 & -3 \end{vmatrix} \begin{array}{c} \frac{-3}{2} \\ \big| \\ \hookleftarrow \end{array} = 4 \begin{vmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{vmatrix}$$

$$\overset{(**)}{=} 4 \cdot 2 \begin{vmatrix} -1 & 2 \\ \frac{1}{2} & -\frac{3}{2} \end{vmatrix} = 8\left(\tfrac{3}{2} - \tfrac{2}{2}\right) = 4$$

For equality $(*)$, expand by row 2. For equality $(**)$, expand by column 1.

Expansion by Alien Cofactors

According to the cofactor expansions (16.5.1) and (16.5.2), if each element a_{ij} in any row (or column) of a determinant is multiplied by the corresponding cofactor C_{ij} and then all the products are added, the result is the value of the determinant. What happens if we multiply the elements of a row by the cofactors of a different (alien) row? Or the elements of a column by the cofactors of an alien column? Consider the following example.

EXAMPLE 16.5.4 If $\mathbf{A} = (a_{ij})_{3\times 3}$, then the cofactor expansion of $|\mathbf{A}|$ along the second row is

$$|\mathbf{A}| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

Suppose we replace the elements a_{21}, a_{22}, and a_{23} of the second row by a, b, and c. Then, the corresponding cofactors C_{21}, C_{22}, and C_{23} remain unchanged, so the cofactor expansion of the new determinant along its second row is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a & b & c \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = aC_{21} + bC_{22} + cC_{23} \tag{$*$}$$

In particular, if we replace a, b, and c by a_{11}, a_{12}, and a_{13}, or by a_{31}, a_{32}, and a_{33}, then the determinant in $(*)$ is 0 because two rows are equal. Hence,

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$$

$$a_{31}C_{21} + a_{32}C_{22} + a_{33}C_{23} = 0$$

That is, the sum of the products of the elements in either row 1 or row 3 multiplied by the cofactors of the elements in row 2 is zero.

Obviously, the argument used in this example can be generalized: If we multiply the elements of any row by the cofactors of an alien row, and then add the products, the result is 0. Similarly if we multiply the elements of a column by the cofactors of an alien column, then add.

We summarize all the results in this section in the following theorem:

THEOREM 16.5.1 (COFACTOR EXPANSION OF A DETERMINANT)

Let $\mathbf{A} = (a_{ij})_{n\times n}$. Suppose that the cofactors C_{ij} are defined as in (16.5.3). Then:

(i) $a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = |\mathbf{A}|$;

(ii) if $k \neq i$, then $a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0$;

(iii) $a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = |\mathbf{A}|$; and

(iv) if $k \neq j$, then $a_{1j}C_{1k} + a_{2j}C_{2k} + \cdots + a_{nj}C_{nk} = 0$.

Theorem 16.5.1 says that an expansion of a determinant by row i in terms of the cofactors of row k vanishes when $k \neq i$, and is equal to $|\mathbf{A}|$ if $k = i$. Likewise, an expansion by column j in terms of the cofactors of column k vanishes when $k \neq j$, and is equal to $|\mathbf{A}|$ if $k = j$.

EXERCISES FOR SECTION 16.5

SM **1.** Calculate the following determinants:

(a) $\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 2 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 1 & 3 & 3 \\ 3 & 2 & 1 & 6 \\ 1 & 3 & 0 & 9 \\ 2 & 4 & 1 & 12 \end{vmatrix}$

2. Calculate the following determinants:

(a) $\begin{vmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 4 & 0 & 3 & 4 \\ 6 & 2 & 3 & 1 & 2 \end{vmatrix}$

16.6 The Inverse of a Matrix

Suppose that α is a real number different from 0. Then there is a unique number α^{-1} with the property that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$. We call α^{-1} the (multiplicative) inverse of α. We saw in Section 15.4 that the identity matrix **I**, with 1's along the main diagonal and 0's elsewhere, is the matrix equivalent of 1 in the real number system.[7] This makes the following terminology seem natural.

Given a matrix **A**, we say that **X** is an *inverse* of **A** if there exists a matrix **X** such that

$$\mathbf{AX} = \mathbf{XA} = \mathbf{I} \tag{16.6.1}$$

Then **A** is said to be *invertible*. Because $\mathbf{XA} = \mathbf{AX} = \mathbf{I}$, the matrix **A** is also an inverse of **X**—that is, **A** and **X** are inverses of each other. Note that the two matrix products **AX** and **XA** are defined and equal only if **A** and **X** are square matrices of the same order. *Thus, only square matrices can have inverses*. But not even all square matrices have inverses, as (b) in the following example shows.

EXAMPLE 16.6.1

(a) Show that the following matrices are inverses of each other:

$$\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 5 & 10 \end{pmatrix} \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{10} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

(b) Show that the following matrix has no inverse:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Solution:

(a) We simply compute, directly, that

$$\begin{pmatrix} 5 & 6 \\ 5 & 10 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{3}{10} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \frac{6}{4} & -\frac{15}{10} + \frac{6}{4} \\ \frac{5}{2} - \frac{10}{4} & -\frac{15}{10} + \frac{10}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and likewise we verify that $\mathbf{XA} = \mathbf{I}$.

(b) Observe that for all real numbers x, y, z, and w,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

Because the element in row 2 and column 2 of the last matrix is 0 and not 1, there is no way of choosing x, y, z, and w to make the product of these two matrices equal to **I**. ▮

The following questions arise: (i) *Which matrices have inverses?* (ii) *Can a given matrix have more than one inverse?* (iii) *How do we find the inverse if it exists?*

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[7] From now on, we write **I** instead of $\mathbf{I}_n$ whenever the order n of the identity matrix seems obvious.

For question (i), it is easy to find a *necessary* condition for a matrix $\mathbf{A}$ to have an inverse. In fact, from (16.6.1) and rule (vii) in Theorem 16.4.1, it follows that $|\mathbf{AX}| = |\mathbf{A}| \cdot |\mathbf{X}| = |\mathbf{I}|$. Using (16.3.3), we see that the identity matrix of any order has determinant 1. Thus, if $\mathbf{X}$ is an inverse of $\mathbf{A}$, then $|\mathbf{A}| \cdot |\mathbf{X}| = 1$. We conclude from this equation that $|\mathbf{A}| \neq 0$ is a necessary condition for $\mathbf{A}$ to have an inverse, because $|\mathbf{A}| = 0$ would lead to a contradiction.

As we shall see in the next section, the condition $|\mathbf{A}| \neq 0$ is also *sufficient* for $\mathbf{A}$ to have an inverse. Hence, for any square matrix $\mathbf{A}$,

$$\mathbf{A} \text{ has an inverse} \iff |\mathbf{A}| \neq 0 \tag{16.6.2}$$

A square matrix $\mathbf{A}$ is said to be *singular* if $|\mathbf{A}| = 0$ and *nonsingular* if $|\mathbf{A}| \neq 0$. According to (16.6.2), a matrix has an inverse if and only if it is nonsingular.

Concerning question (ii), the answer is No: a matrix cannot have more than one inverse. Indeed, suppose that $\mathbf{X}$ satisfies (16.6.1) and that $\mathbf{AY} = \mathbf{I}$ for some other square matrix $\mathbf{Y}$. Then

$$\mathbf{Y} = \mathbf{IY} = (\mathbf{XA})\mathbf{Y} = \mathbf{X}(\mathbf{AY}) = \mathbf{XI} = \mathbf{X}$$

A similar argument shows that if $\mathbf{YA} = \mathbf{I}$, then $\mathbf{Y} = \mathbf{X}$. Thus, *the inverse of* $\mathbf{A}$ *is unique, if it exists*.

If the inverse of $\mathbf{A}$ exists, it is usually written $\mathbf{A}^{-1}$. Whereas for numbers we can write $a^{-1} = 1/a$, the symbol $\mathbf{I}/\mathbf{A}$ has *no* meaning. *There are no rules for dividing matrices*. Note also that even if the product $\mathbf{A}^{-1}\mathbf{B}$ is defined, it is usually quite different from $\mathbf{BA}^{-1}$ because matrix multiplication is not commutative, in general.

The full answer to question (iii) is given in the next section. Here we only consider the case of 2×2 matrices.

EXAMPLE 16.6.2 Find the inverse of the following matrix, when it exists:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Solution: We find a 2×2 matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{I}$, after which it is easy to check that $\mathbf{XA} = \mathbf{I}$. Solving $\mathbf{AX} = \mathbf{I}$ requires finding numbers x, y, z, and w such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix multiplication implies that

$$ax + bz = 1, \quad cx + dz = 0, \quad ay + bw = 0, \quad \text{and} \quad cy + dw = 1$$

Note that we have two different systems of equations here. One is given by the two equations on the left, and the other by the two equations on the right. Both these systems have $\mathbf{A}$ as a common coefficient matrix. If $|\mathbf{A}| = ad - bc \neq 0$, solving the two pairs of simultaneous equations separately, using Cramer's rule from Section 16.1, yields

$$x = \frac{d}{ad - bc}, \quad z = \frac{-c}{ad - bc}, \quad y = \frac{-b}{ad - bc}, \quad \text{and} \quad w = \frac{a}{ad - bc}$$

Hence, we have proved the following result:

INVERSE OF A MATRIX OF ORDER 2

Provided that $|\mathbf{A}| = ad - bc \neq 0$,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{16.6.3}$$

Note that in the inverse matrix, the diagonal elements of the original 2×2 matrix are switched, whereas the off-diagonal elements just change sign.

For square matrices of order 3, one can use Cramer's rule (16.2.6) to derive a formula for the inverse. Again, the requirement for the inverse to exist is that the determinant of the coefficient matrix is not 0. Full details will be given in Section 16.7.

Some Useful Implications

If $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$, then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Actually, each of these equations implies the other, in the sense that

$$\mathbf{AX} = \mathbf{I} \Rightarrow \mathbf{X} = \mathbf{A}^{-1} \tag{16.6.4}$$

$$\mathbf{YA} = \mathbf{I} \Rightarrow \mathbf{Y} = \mathbf{A}^{-1} \tag{16.6.5}$$

To prove (16.6.4), suppose $\mathbf{AX} = \mathbf{I}$. Then $|\mathbf{A}| \cdot |\mathbf{X}| = 1$, and so $|\mathbf{A}| \neq 0$. Hence, by (16.6.2), $\mathbf{A}^{-1}$ exists. Multiplying $\mathbf{AX} = \mathbf{I}$ from the left by $\mathbf{A}^{-1}$ yields $\mathbf{X} = \mathbf{A}^{-1}$. The proof of (16.6.5) is almost the same.

Implications (16.6.4) and (16.6.5) are used repeatedly in proving properties of the inverse. Here are two examples.

EXAMPLE 16.6.3 Find the inverse of the $n \times n$ matrix $\mathbf{A}$ if $\mathbf{A} - \mathbf{A}^2 = \mathbf{I}$.

Solution: The matrix equation $\mathbf{A} - \mathbf{A}^2 = \mathbf{I}$ yields $\mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{I}$. But then it follows from (16.6.4) that $\mathbf{A}$ has the inverse $\mathbf{A}^{-1} = \mathbf{I} - \mathbf{A}$.

EXAMPLE 16.6.4 Let $\mathbf{B}$ be a $n \times n$ matrix such that $\mathbf{B}^2 = 3\mathbf{B}$. Prove that there exists a number s such that $\mathbf{I} + s\mathbf{B}$ is the inverse of $\mathbf{I} + \mathbf{B}$.

Solution: Because of (16.6.5), it suffices to find a number s such that $(\mathbf{I} + s\mathbf{B})(\mathbf{I} + \mathbf{B}) = \mathbf{I}$. Now,

$$(\mathbf{I} + s\mathbf{B})(\mathbf{I} + \mathbf{B}) = \mathbf{II} + \mathbf{IB} + s\mathbf{BI} + s\mathbf{B}^2 = \mathbf{I} + \mathbf{B} + s\mathbf{B} + 3s\mathbf{B} = \mathbf{I} + (1 + 4s)\mathbf{B}$$

which is equal to $\mathbf{I}$ provided $1 + 4s = 0$. The right choice of s is, thus, $s = -1/4$.

Properties of the Inverse

We shall now prove some useful rules for the inverse.

THEOREM 16.6.1 (PROPERTIES OF THE INVERSE)

Let $\mathbf{A}$ and $\mathbf{B}$ be invertible $n \times n$ matrices. Then:

(a) $\mathbf{A}^{-1}$ is invertible, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

(b) $\mathbf{AB}$ is invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(c) The transpose $\mathbf{A}'$ is invertible, and $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$.

(d) $(c\mathbf{A})^{-1} = c^{-1}\mathbf{A}^{-1}$ whenever c is a number $\neq 0$.

In order to prove these properties, we use (16.6.4) or (16.6.5) in each case:

(a) We have $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, so $\mathbf{A} = (\mathbf{A}^{-1})^{-1}$.

(b) To prove that $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of $\mathbf{AB}$, we just verify that $(\mathbf{AB})\mathbf{X}$ is equal to $\mathbf{I}$. In fact,

$$(\mathbf{AB})\mathbf{X} = (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

(c) Applying rule (15.5.5) for the transposition of products, with $\mathbf{B} = \mathbf{A}^{-1}$, gives $(\mathbf{A}^{-1})'\mathbf{A}' = (\mathbf{AA}^{-1})' = \mathbf{I}' = \mathbf{I}$. By (16.6.5), it follows that $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$.

(d) Here rule (15.4.4) implies that $(c\mathbf{A})(c^{-1}\mathbf{A}^{-1}) = cc^{-1}\mathbf{AA}^{-1} = 1 \cdot \mathbf{I} = \mathbf{I}$, so $c^{-1}\mathbf{A}^{-1} = (c\mathbf{A})^{-1}$.

Note that if $\mathbf{A}$ is invertible and also symmetric—that is, such that $\mathbf{A}' = \mathbf{A}$—then rule (c) implies that $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1} = \mathbf{A}^{-1}$, so $\mathbf{A}^{-1}$ is symmetric. In summary, *the inverse of a symmetric matrix is symmetric*.

Also note that rule (b) can be extended to products of several matrices. For instance, if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all invertible $n \times n$ matrices, then

$$(\mathbf{ABC})^{-1} = [(\mathbf{AB})\mathbf{C}]^{-1} = \mathbf{C}^{-1}(\mathbf{AB})^{-1} = \mathbf{C}^{-1}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

where rule (b) has been used twice. Note the assumption in (b) that $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices. In statistics and econometrics, we often consider products of the form $\mathbf{XX}'$, where $\mathbf{X}$ is $n \times m$, with $n \neq m$. Then $\mathbf{XX}'$ is $n \times n$. If the determinant $|\mathbf{XX}'|$ is not 0, then $(\mathbf{XX}')^{-1}$ exists, but (b) does not apply because $\mathbf{X}^{-1}$ and $\mathbf{X}'^{-1}$ are only defined if $n = m$.

Solving Equations by Matrix Inversion

Let $\mathbf{A}$ be any $n \times n$ matrix. If $\mathbf{B}$ is an arbitrary matrix, we consider whether there are matrices $\mathbf{X}$ and $\mathbf{Y}$ of suitable order such that $\mathbf{AX} = \mathbf{B}$ and $\mathbf{YA} = \mathbf{B}$. For the first requirement to be possible, the matrix $\mathbf{B}$ must have n rows; for the second, $\mathbf{B}$ must have n columns. Provided these conditions are satisfied, we have the following result:

THEOREM 16.6.2

Provided that $|\mathbf{A}| \neq 0$, one has:

$$\mathbf{AX} = \mathbf{B} \Rightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \tag{16.6.6}$$

$$\mathbf{YA} = \mathbf{B} \Rightarrow \mathbf{Y} = \mathbf{BA}^{-1} \tag{16.6.7}$$

The proof of this result is not difficult:

Provided that $|\mathbf{A}| \neq 0$, we can multiply each side of the equation $\mathbf{AX} = \mathbf{B}$ in (16.6.6) on the left by $\mathbf{A}^{-1}$. This yields $\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$. Because $(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$, we conclude that $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ is the only possible solution of the equation. On the other hand, by substituting $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ into $\mathbf{AX} = \mathbf{B}$, we see that it really satisfies the equation.

The proof of (16.6.7) is similar: postmultiply each side of $\mathbf{YA} = \mathbf{B}$ by $\mathbf{A}^{-1}$.

EXAMPLE 16.6.5 Solve the following system of equations by using Theorem 16.6.2:

$$\begin{aligned} 2x + y &= 3 \\ 2x + 2y &= 4 \end{aligned}$$

Solution: Suppose we define the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Then the system is equivalent to the matrix equation $\mathbf{Ax} = \mathbf{b}$. Because $|\mathbf{A}| = 2 \neq 0$, matrix $\mathbf{A}$ has an inverse, and according to Theorem 16.6.2, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where we have used Eq. (16.6.3) to find $\mathbf{A}^{-1}$. The solution is therefore $x = 1$, $y = 1$.[8]

EXERCISES FOR SECTION 16.6

1. Prove that:

(a) $\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ \frac{8}{7} & -1 & \frac{3}{7} \\ -\frac{2}{7} & 0 & \frac{1}{7} \end{pmatrix}$

[8] Check by substitution that this really is the correct solution. Clearly, it is easier to solve the system by subtracting the first equation from the second to obtain $y = 1$, and so $x = 1$.

2. Find numbers a and b that make $\mathbf{A}$ the inverse of $\mathbf{B}$ when

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -1 \\ a & 1/4 & b \\ 1/8 & 1/8 & -1/8 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{pmatrix}$$

3. Solve the following systems of equations by using Theorem 16.6.2 and then formula (16.6.3).

(a) $\begin{aligned} 2x - 3y &= 3 \\ 3x - 4y &= 5 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= 8 \\ 3x - 4y &= 11 \end{aligned}$ (c) $\begin{aligned} 2x - 3y &= 0 \\ 3x - 4y &= 0 \end{aligned}$

4. Given the matrix $\mathbf{A} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$, show that $\mathbf{A}^3 = \mathbf{I}_2$, and use this to find $\mathbf{A}^{-1}$.

5. Given the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$:

(a) Calculate $|\mathbf{A}|$, $\mathbf{A}^2$, $\mathbf{A}^3$, and $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} - \mathbf{I}_3$.

(b) Use the last calculations to show that $\mathbf{A}$ has an inverse and $\mathbf{A}^{-1} = (\mathbf{A} - \mathbf{I}_3)^2$.

(c) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^2 = \mathbf{A}$. Are there other matrices with this property?

6. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & -1 & 3 \end{pmatrix}$.

(a) Calculate $\mathbf{AA}'$, $|\mathbf{AA}'|$, and $(\mathbf{AA}')^{-1}$.

(b) The matrices $\mathbf{AA}'$ and $(\mathbf{AA}')^{-1}$ in part (a) are both symmetric. Is this a coincidence?

7. Suppose that $\mathbf{A}$, $\mathbf{P}$, and $\mathbf{D}$ are square matrices such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

(a) Show that $\mathbf{A}^2 = \mathbf{PD}^2\mathbf{P}^{-1}$.

(b) Show by induction that $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$ for any positive integer m.

SM 8. Given $\mathbf{B} = \begin{pmatrix} -1/2 & 5 \\ 1/4 & -1/2 \end{pmatrix}$, calculate $\mathbf{B}^2 + \mathbf{B}$ and $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I}$, then find $\mathbf{B}^{-1}$.

9. Suppose that $\mathbf{X}$ is an $m \times n$ matrix and that $|\mathbf{X}'\mathbf{X}| \neq 0$. Show that the matrix

$$\mathbf{A} = \mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$

is idempotent, as defined in Exercise 15.4.6—i.e., $\mathbf{A}^2 = \mathbf{A}$.

10. Find a matrix $\mathbf{X}$ that satisfies $\mathbf{AB} + \mathbf{CX} = \mathbf{D}$, where:

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} -9 & 3 \\ -8 & 17 \end{pmatrix}$$

11. Let $\mathbf{C}$ be an $n \times n$ matrix that satisfies $\mathbf{C}^2 + \mathbf{C} = \mathbf{I}$.

(a) Show that $\mathbf{C}^{-1} = \mathbf{I} + \mathbf{C}$.

(b) Show that $\mathbf{C}^3 = -\mathbf{I} + 2\mathbf{C}$ and $\mathbf{C}^4 = 2\mathbf{I} - 3\mathbf{C}$.

16.7 A General Formula for the Inverse

The previous section presents the most important facts about the inverse and its properties. As such, it contains "what every economist should know". It is perhaps less important for most economists to know much about how to calculate the inverses of large matrices, because powerful computer programs are available. Nevertheless, this section presents an explicit formula for the inverse of any nonsingular $n \times n$ matrix $\mathbf{A}$. Though this formula is extremely inefficient for computing inverses of large matrices, it does have theoretical interest. The rules for the cofactor expansion of determinants are the key to this formula.

Let $C_{11}, \dots, C_{nn}$ denote the cofactors of the elements in $\mathbf{A}$. By Theorem 16.5.1, cofactor expansion yields n^2 equations of the form

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = \begin{cases} |\mathbf{A}| & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \tag{$*$}$$

for $i, k = 1, \dots, n$. The sums on the left-hand side look very much like those appearing in matrix products. In fact, the n^2 different equations in $(*)$ reduce to the single matrix equation

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & \dots & C_{k1} & \dots & C_{n1} \\ C_{12} & \dots & C_{k2} & \dots & C_{n2} \\ \vdots & & \vdots & & \vdots \\ C_{1n} & \dots & C_{kn} & \dots & C_{nn} \end{pmatrix} = \begin{pmatrix} |\mathbf{A}| & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\mathbf{A}| \end{pmatrix}$$

Here the matrix on the right-hand equals $|\mathbf{A}| \cdot \mathbf{I}_n$. Let $\mathbf{C}^+ = (C_{ij})$ denote the matrix of cofactors. Then the second matrix in the product on the left-hand side has its row and column indices interchanged. Thus, it is the *transpose* $(\mathbf{C}^+)'$ of that matrix, which is called the *adjugate* of $\mathbf{A}$, and denoted by $\operatorname{adj}(\mathbf{A})$. Hence

$$\operatorname{adj}(\mathbf{A}) = (\mathbf{C}^+)' = \begin{pmatrix} C_{11} & \dots & C_{k1} & \dots & C_{n1} \\ C_{12} & \dots & C_{k2} & \dots & C_{n2} \\ \vdots & & \vdots & & \vdots \\ C_{1n} & \dots & C_{kn} & \dots & C_{nn} \end{pmatrix}$$

The previous equation, therefore, can be written as $\mathbf{A} \operatorname{adj}(\mathbf{A}) = |\mathbf{A}| \cdot \mathbf{I}$. In case $|\mathbf{A}| \neq 0$, this evidently implies that $\mathbf{A}^{-1} = (1/|\mathbf{A}|) \cdot \operatorname{adj}(\mathbf{A})$. We have proved the general formula for the inverse:

THEOREM 16.7.1 (GENERAL FORMULA FOR THE INVERSE)

Any square matrix $\mathbf{A} = \left(a_{ij}\right)_{n\times n}$ with determinant $|\mathbf{A}| \neq 0$ has a unique inverse $\mathbf{A}^{-1}$ satisfying $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. This is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \operatorname{adj}(\mathbf{A})$$

If $|\mathbf{A}| = 0$, then there is no matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{XA} = \mathbf{I}$.

EXAMPLE 16.7.1 Show that the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ has an inverse, and then find that inverse.

Solution: According to Theorem 16.7.1, $\mathbf{A}$ has an inverse if and only if $|\mathbf{A}| \neq 0$. Here we find that $|\mathbf{A}| = -5$, so the inverse exists. The cofactors are

$$C_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10, \quad C_{12} = -\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = -15, \quad C_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4, \quad C_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 4, \quad C_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 14, \quad C_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6$$

Hence, the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{pmatrix}$$

One can check the result by showing that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Finding Inverses by Elementary Row Operations

Theorem 16.7.1 presented a general formula for the inverse of a nonsingular matrix. Although this formula is important theoretically, it is computationally useless for matrices much larger than 2×2. An efficient way of finding the inverse of an invertible $n \times n$ matrix $\mathbf{A}$ is based on using elementary operations in a systematic way: First form the $n \times 2n$ matrix $(\mathbf{A} : \mathbf{I})$ by writing down the n columns of $\mathbf{A}$ followed by the n columns of $\mathbf{I}$. Then apply elementary row operations to this matrix in order to transform it to an $n \times 2n$ matrix $(\mathbf{I} : \mathbf{B})$ whose first n columns are all the columns of $\mathbf{I}$. It will follow that $\mathbf{B} = \mathbf{A}^{-1}$. If it is impossible to perform such row operations, then $\mathbf{A}$ has no inverse. The method is illustrated by the following example.

EXAMPLE 16.7.2 Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$.

Solution: First, write down the 3×6 matrix $(\mathbf{A} : \mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$ whose first three columns are the columns of $\mathbf{A}$ and whose next three columns are the columns of the 3×3 identity matrix.

The idea is now to use elementary operations on this matrix so that, in the end, the three first columns constitute an identity matrix. Then the last three columns will constitute the inverse of $\mathbf{A}$.

To start, we multiply the first row by -1 and add the result to the second row. This gives a zero in the second row and the first column. You should be able then to understand the other operations used and why they are chosen.

$$
\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ \hookleftarrow \\ \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ \\ \hookleftarrow \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \begin{array}{c} \hookleftarrow \\ \\ -3 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 0 & -3 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \begin{array}{c} \hookleftarrow \\ -3 \\ \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)
$$

We conclude that

$$
\mathbf{A}^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}
$$

This can be checked by using matrix multiplication to verify that $\mathbf{AA}^{-1} = \mathbf{I}$.

EXERCISES FOR SECTION 16.7

SM **1.** Use Theorem 16.7.1 to calculate the inverses of the following matrices, if they exist:

(a) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -2 & 1 \\ 4 & -16 & 8 \end{pmatrix}$

2. Find the inverse of $\mathbf{A} = \begin{pmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{pmatrix}$.

SM **3.** Find $(\mathbf{I} - \mathbf{A})^{-1}$ when $\mathbf{A} = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0 \end{pmatrix}$.

SM **4.** Repeated observations of an empirical phenomenon lead to p different systems of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_{1k} \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_{nk} \end{aligned} \tag{$*$}$$

for $k = 1, \ldots, p$, which all share the same $n \times n$ coefficient matrix $\big(a_{ij}\big)$. Explain how to find the solutions $(x_{k1}, \ldots, x_{kn})$ $(k = 1, \ldots, p)$ of all the systems simultaneously by using row operations to get

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} & b_{11} & \ldots & b_{1p} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} & b_{n1} & \ldots & b_{np} \end{pmatrix} \sim \begin{pmatrix} 1 & \ldots & 0 & b^*_{11} & \ldots & b^*_{1p} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 1 & b^*_{1n} & \ldots & b^*_{np} \end{pmatrix}$$

What then is the solution of the system of equations $(*)$ for $k = r$?

SM **5.** Use the method in Example 16.7.2 to calculate the inverses, provided that they exist, of the matrices:

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 5 & 8 \\ -9 & -6 & 3 \end{pmatrix}$

Then check each result by verifying that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

16.8 Cramer's Rule

Cramer's rule for solving n linear equations in n unknowns is a direct generalization of the same rule for systems of equations with two or three unknowns. Consider the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{16.8.1}$$

Let D_j denote the determinant obtained from $|\mathbf{A}|$ by replacing the j-th column vector with the column vector whose components are $b_1, b_2, \ldots, b_n$. Thus,

$$D_j = \begin{vmatrix} a_{11} & \ldots & a_{1,j-1} & b_1 & a_{1,j+1} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2,j-1} & b_2 & a_{2,j+1} & \ldots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{n,j-1} & b_n & a_{n,j+1} & \ldots & a_{nn} \end{vmatrix} \tag{16.8.2}$$

for $j = 1, \ldots, n$. The cofactor expansion of D_j down its j-th column gives

$$D_j = C_{1j}b_1 + C_{2j}b_2 + \cdots + C_{nj}b_n \tag{16.8.3}$$

where the cofactors C_{ij} are given by (16.5.3). Now we have the following result:

THEOREM 16.8.1 (CRAMER'S RULE)

The general linear system of equations (16.8.1) with n equations and n unknowns has a unique solution if and only if $\mathbf{A}$ is nonsingular ($|\mathbf{A}| \neq 0$). The solution is

$$x_1 = \frac{D_1}{|\mathbf{A}|}, \quad x_2 = \frac{D_2}{|\mathbf{A}|}, \ldots, \quad x_n = \frac{D_n}{|\mathbf{A}|} \tag{16.8.4}$$

where $D_1, D_2, \ldots, D_n$ are defined by (16.8.2).

An argument for the "if" part is as follows:

Suppose $|\mathbf{A}| \neq 0$. System (16.8.1) can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Using the formula for the inverse of the coefficient matrix yields

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{$*$}$$

where the cofactors C_{ij} are given by (16.5.3). From ($*$), we have

$$x_j = \frac{1}{|\mathbf{A}|}[C_{1j}b_1 + C_{2j}b_2 + \cdots + C_{nj}b_n] = \frac{D_j}{|\mathbf{A}|}$$

for each $j = 1, 2, \ldots, n$, where the last equality follows from the previous equation (16.8.3). This proves (16.8.4).

The "only if" part will be proved in detail in FMEA. Here, we merely note the following:

In case $|\mathbf{A}| = 0$, there are two possibilities. First, the equation system (16.8.1), which we write in matrix form $\mathbf{Ax} = \mathbf{b}$, may have no solutions. Second, it may have at least one particular solution $\mathbf{x}^P$. But the homogeneous system certainly has solutions of the form $\alpha\mathbf{x}^H$, where $\mathbf{x}^H$ is a nonzero vector with $\mathbf{Ax}^H = \mathbf{0}$ and α is an arbitrary real number. Then all vectors of the form $\mathbf{x}^P + \alpha\mathbf{x}$ are also solutions of the equation system. In particular, (16.8.1) has a unique solution only if $|\mathbf{A}| \neq 0$.

EXAMPLE 16.8.1 For all values of p, find the solutions of the system

$$\begin{aligned} px + \ y \quad\ &= 1 \\ x - \ y + z &= 0 \\ 2y - z &= 3 \end{aligned}$$

Solution: The coefficient matrix has determinant

$$|\mathbf{A}| = \begin{vmatrix} p & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = 1 - p$$

According to Theorem 16.8.1, the system has a unique solution if $1 - p \neq 0$—that is, if $p \neq 1$. In this case, the determinants in (16.8.2) are

$$D_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 3 & 2 & -1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} p & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & -1 \end{vmatrix}, \quad \text{and} \quad D_3 = \begin{vmatrix} p & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 3 \end{vmatrix}$$

whose numerical values are $D_1 = 2$, $D_2 = 1 - 3p$, and $D_3 = -1 - 3p$. Then, for $p \neq 1$, Eq. (16.8.4) yields

$$x = \frac{D_1}{|\mathbf{A}|} = \frac{2}{1-p}, \quad y = \frac{D_2}{|\mathbf{A}|} = \frac{1-3p}{1-p}, \quad \text{and} \quad z = \frac{D_3}{|\mathbf{A}|} = \frac{-1-3p}{1-p}$$

On the other hand, in case $p = 1$, the first equation becomes $x + y = 1$. Yet adding the last two of the original equations implies that $x + y = 3$. There is no solution to these two contradictory equations in case $p = 1$.[9]

Homogeneous Systems of Equations

Consider the special case in which the right-hand side of the system of equations (16.8.1) consists only of zeros. The system is then called *homogeneous*. A homogeneous system will always have the so-called *trivial solution* $x_1 = x_2 = \cdots = x_n = 0$. In many problems, one is interested in knowing when a homogeneous system has *nontrivial* solutions.

THEOREM 16.8.2 (NONTRIVIAL SOLUTIONS OF HOMOGENEOUS SYSTEMS)

The homogeneous linear system with n equations and n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned} \tag{16.8.5}$$

has nontrivial solutions if and only if the coefficient matrix $\mathbf{A} = (a_{ij})_{n \times n}$ is such that $|\mathbf{A}| = 0$.

As with Theorem 16.8.1, we are ready to argue one part of this result, the "only if" part in this case:

[9] It might be instructive to solve this problem by using Gaussian elimination, starting by interchanging the first two equations.

Suppose that $|\mathbf{A}| \neq 0$. Then, by Cramer's rule, $x_1, \ldots, x_n$ are given by (16.8.4). But the numerator in each of these fractions is 0, because each of the determinants $D_1, \ldots, D_n$ contains a column consisting entirely of zeros. Then the system only has the trivial solution. In other words: *System* (16.8.5) *has nontrivial solutions only if the determinant* $|\mathbf{A}|$ *vanishes.*

As for the "if" part, concepts from FMEA can be used to show that, if $|\mathbf{A}| = 0$, then the rank of $\mathbf{A}$ is less than n, so system (16.8.5) has at least one degree of freedom. That is, apart from the trivial solution, there are infinitely many nontrivial solutions which take the form $\alpha\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$, and α is an arbitrary nonzero scalar.

EXAMPLE 16.8.2 Find the values of λ for which the following system of equations has nontrivial solutions:

$$\begin{aligned} 5x + 2y + z &= \lambda x \\ 2x + \ \ y \qquad &= \lambda y \qquad (*) \\ x \qquad + z &= \lambda z \end{aligned}$$

Solution: The variables x, y, and z appear on both sides of the equations, so we start by putting the system into standard form:

$$\begin{aligned} (5-\lambda)x + \qquad 2y + \qquad\quad z &= 0 \\ 2x + (1-\lambda)y \qquad\qquad &= 0 \\ x \qquad\qquad + (1-\lambda)z &= 0 \end{aligned}$$

According to Theorem 16.8.2, this system has a nontrivial solution if and only if the coefficient matrix is singular:

$$\begin{vmatrix} 5-\lambda & 2 & 1 \\ 2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

The value of the determinant is found to be $\lambda(1-\lambda)(\lambda-6)$. Hence, system $(*)$ has nontrivial solutions if and only if $\lambda = 0$, 1, or 6.[10]

EXERCISES FOR SECTION 16.8

SM **1.** Use Cramer's rule to solve the following two systems of equations:

(a) $$\begin{aligned} x + 2y - \ z &= -5 \\ 2x - \ y + \ z &= \ \ 6 \\ x - \ y - 3z &= -3 \end{aligned}$$

(b) $$\begin{aligned} x + y \qquad\quad &= 3 \\ x \qquad + z \quad &= 2 \\ y + z + u &= 6 \\ y \qquad + u &= 1 \end{aligned}$$

[10] Using terminology explained in FMEA, this example asks us to find the eigenvalues of the coefficient matrix $\begin{pmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

2. Use Theorem 16.8.1 to prove that the following system of equations has a unique solution for all values of b_1, b_2, b_3, and find the solution.

$$\begin{aligned} 3x_1 + \ x_2 \qquad &= b_1 \\ x_1 - \ x_2 + 2x_3 &= b_2 \\ 2x_1 + 3x_2 - \ x_3 &= b_3 \end{aligned}$$

SM 3. Prove that the homogeneous system of equations

$$\begin{aligned} ax + by + cz &= 0 \\ bx + cy + az &= 0 \\ cx + ay + bz &= 0 \end{aligned}$$

has a nontrivial solution if and only if $a^3 + b^3 + c^3 - 3abc = 0$.

16.9 The Leontief Model

In order to illustrate why linear systems of equations are important in economics, we briefly discuss a simple example of the Leontief model.

EXAMPLE 16.9.1 Once upon a time, in an ancient land perhaps not too far from Norway, an economy consisted of three industries—fishing, forestry, and boat building.

(i) To produce 1 ton of fish requires the services of α fishing boats.

(ii) To produce 1 ton of timber requires β tons of fish, as extra food for the energetic foresters.

(iii) To produce 1 fishing boat requires γ tons of timber.

These are the only inputs needed for each of these three industries. Suppose there is no final (external) demand for fishing boats. Find what gross outputs each of the three industries must produce in order to meet the final demands of d_1 tons of fish to feed the general population, plus d_2 tons of timber to build houses.

Solution: Let x_1 denote the total number of tons of fish to be produced, x_2 the total number of tons of timber, and x_3 the total number of fishing boats.

Consider first the demand for fish. Because βx_2 tons of fish are needed to produce x_2 units of timber, and because the final demand for fish is d_1, we must have $x_1 = \beta x_2 + d_1$. Note that producing fishing boats does not require fish as an input, so there is no term with x_3.

In the case of timber, a similar argument shows that the equation $x_2 = \gamma x_3 + d_2$ must be satisfied. Finally, for boat building, only the fishing industry needs boats; there is no final demand in this case, and so $x_3 = \alpha x_1$. Thus, the following three equations must be satisfied:

$$\text{(i)}\ \ x_1 = \beta x_2 + d_1 \qquad \text{(ii)}\ \ x_2 = \gamma x_3 + d_2 \qquad \text{(iii)}\ \ x_3 = \alpha x_1 \tag{$*$}$$

One way to solve these equations begins by using (iii) to insert $x_3 = \alpha x_1$ into (ii). This gives $x_2 = \gamma\alpha x_1 + d_2$, which inserted into (i) yields $x_1 = \alpha\beta\gamma x_1 + \beta d_2 + d_1$. Solving this last equation for x_1 gives $x_1 = (d_1 + \beta d_2)/(1 - \alpha\beta\gamma)$. The corresponding expressions for the two other variables are easily found, and the results are:

$$x_1 = \frac{d_1 + \beta d_2}{1 - \alpha\beta\gamma}, \quad x_2 = \frac{\alpha\gamma d_1 + d_2}{1 - \alpha\beta\gamma}, \quad \text{and} \quad x_3 = \frac{\alpha d_1 + \alpha\beta d_2}{1 - \alpha\beta\gamma} \tag{$**$}$$

Clearly, this solution for (x_1, x_2, x_3) only makes sense when $\alpha\beta\gamma < 1$. In fact, if $\alpha\beta\gamma \geq 1$, it is impossible for this economy to meet any positive final demands for fish and timber—production in the economy is too inefficient.

The General Leontief Model

In Example 16.9.1 we considered a simple example of the Leontief model. More generally, the Leontief model describes an economy with n interlinked industries, each of which produces a single good using only one process of production. To produce its output good, each industry must use inputs from at least some other industries. For example, the steel industry needs goods from the iron mining and coal industries, as well as from many other industries. In addition to supplying its own good to other industries that need it, each industry also faces an external demand for its product from consumers, governments, foreigners, and so on. The amount needed to meet this external demand is called the *final demand*.

Let x_i denote the total number of units of good i that industry i is going to produce in a certain year. Furthermore, let

$$a_{ij} = \text{the number of units of good } i \text{ needed to produce one unit of good } j \tag{16.9.1}$$

We assume that input requirements are directly proportional to the amount of the output produced. Then

$$a_{ij}x_j = \text{the number of units of good } i \text{ needed to produce } x_j \text{ units of good } j \tag{16.9.2}$$

In order that x_1 units of good 1, x_2 units of good 2, ..., x_n units of good n can all be produced, industry i needs to supply a total of

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

units of good i. If we require industry i also to supply b_i units to meet final demand, then equilibrium between supply and demand requires that

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i$$

The same goes for all $i = 1, 2, \ldots, n$. So we arrive at the following system of equations:

$$\begin{aligned} x_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\ x_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n \end{aligned} \tag{16.9.3}$$

Note that in the first equation, x_1 appears on the left-hand side as well as in the first term on the right-hand side. In the second equation, x_2 appears on the left-hand side as well as in the second term on the right-hand side, and so on. Moving all terms involving $x_1, \dots, x_n$ to the left-hand side and rearranging gives the system of equations

$$\begin{array}{rcl} (1-a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n &=& b_1 \\ -a_{21}x_1 + (1-a_{22})x_2 - \cdots - a_{2n}x_n &=& b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots && \\ -a_{n1}x_1 - a_{n2}x_2 - \cdots + (1-a_{nn})x_n &=& b_n \end{array} \tag{16.9.4}$$

This system of equations is called the *Leontief system*. The numbers a_{11}, a_{12}, …, a_{nn} are called *input* (or *technical*) *coefficients*. Given any collection of final demand quantities $(b_1, b_2, \dots, b_n)$, a solution $(x_1, x_2, \dots, x_n)$ of (16.9.4) will give outputs for each industry such that the combined interindustry and final demands can just be met. Of course, only nonnegative values for x_i make sense.

It is natural to use matrix algebra to study the Leontief model. Define the following matrices:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{16.9.5}$$

The elements of the matrix **A** are the input coefficients, so it is called the *input* or *Leontief matrix*. Recall that the element a_{ij} denotes the number of units of commodity i which is needed to produce one unit of commodity j.

With these definitions, system (16.9.3) can expressed as

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{16.9.6}$$

This equation is evidently equivalent to the equation $\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{b}$. If $\mathbf{I}_n$ denotes the identity matrix of order n, then $(\mathbf{I}_n - \mathbf{A})\mathbf{x} = \mathbf{I}_n\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{x} - \mathbf{A}\mathbf{x}$, so that (16.9.3) is equivalent to

$$(\mathbf{I}_n - \mathbf{A})\mathbf{x} = \mathbf{b} \tag{16.9.7}$$

which is the matrix equivalent of system (16.9.4).[11]

Suppose now that we introduce prices into the Leontief model, with p_i denoting the price of one unit of commodity i. Because a_{ij} denotes the number of units of commodity i needed to produce one unit of commodity j, the sum $a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{nj}p_n$ is the total cost of the n commodities needed to produce one unit of commodity j. The expression

$$p_j - a_{1j}p_1 - a_{2j}p_2 - \cdots - a_{nj}p_n$$

is the difference between the price of one unit of commodity j and the cost of producing that unit. This is called *unit value added* in sector j. If we denote this unit value added by

[11] Note in particular that "$\mathbf{x} - \mathbf{A}\mathbf{x} = (1 - \mathbf{A})\mathbf{x}$" is nonsensical: $1 - \mathbf{A}$, with the number 1, is meaningless.

v_j, then for all sectors:

$$\begin{array}{c} p_1 - a_{11}p_1 - a_{21}p_2 - \cdots - a_{n1}p_n = v_1 \\ p_2 - a_{12}p_1 - a_{22}p_2 - \cdots - a_{n2}p_n = v_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_n - a_{1n}p_1 - a_{2n}p_2 - \cdots - a_{nn}p_n = v_n \end{array} \tag{16.9.8}$$

Note that the input–output coefficients a_{ij} appear in transposed order. If we define

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{16.9.9}$$

we see that (16.9.8) can be written in the matrix form $\mathbf{p} - \mathbf{A}'\mathbf{p} = \mathbf{v}$, or

$$(\mathbf{I}_n - \mathbf{A}')\mathbf{p} = \mathbf{v} \tag{16.9.10}$$

By using Eq. (15.5.4), we can express (16.9.10) in an alternative way. Transposing each side of (16.9.10) gives

$$\mathbf{p}'(\mathbf{I}_n - \mathbf{A}) = \mathbf{v}' \tag{16.9.11}$$

since $\mathbf{I}_n' = \mathbf{I}_n$ and $(\mathbf{A}')' = \mathbf{A}$. We see that the two systems (16.9.7) and (16.9.11) are closely related.

EXERCISES FOR SECTION 16.9

1. In Example 16.9.1, let $\alpha = 1/2$, $\beta = 1/4$, $\gamma = 2$, $d_1 = 100$, and $d_2 = 80$. Write down system (∗) in this case and find the solution of the system. Confirm the results by using the general formulas in (∗∗).

2. Consider an economy divided into an agricultural sector, A, and an industrial sector, I. To produce one unit in sector A requires 1/6 unit from A and 1/4 unit from I. To produce one unit in sector I requires 1/4 unit from A and 1/4 unit from I. Suppose final demands in each of the two sectors are 60 units.

   (a) Write down the Leontief system for this economy.

   (b) Find the number of units that have to be produced in each sector in order to meet the final demands.

3. Consider the Leontief model (16.9.4).

   (a) What is the interpretation of the condition that $a_{ii} = 0$ for all i?

   (b) What is the interpretation of the sum $a_{i1} + a_{i2} + \cdots + a_{in}$?

   (c) What is the interpretation of the vector of input coefficients $(a_{1j}, a_{2j}, \ldots, a_{nj})$?

   (d) Can you give any interpretation to the sum $a_{1j} + a_{2j} + \cdots + a_{nj}$?

4. Write down system (16.9.4) when $n = 2$, $a_{11} = 0.2$, $a_{12} = 0.3$, $a_{21} = 0.4$, $a_{22} = 0.1$, $b_1 = 120$, and $b_2 = 90$. What is the solution to this system?

5. Consider an input–output model with three sectors. Sector 1 is heavy industry, sector 2 is light industry, and sector 3 is agriculture. The input requirements are given by the following table:

	Heavy industry	Light industry	Agriculture
Units of heavy industry goods	$a_{11} = 0.1$	$a_{12} = 0.2$	$a_{13} = 0.1$
Units of light industry goods	$a_{21} = 0.3$	$a_{22} = 0.2$	$a_{23} = 0.2$
Units of agricultural goods	$a_{31} = 0.2$	$a_{32} = 0.2$	$a_{33} = 0.1$

Suppose the final demands for the three goods are 85, 95, and 20 units, respectively. If x_1, x_2, and x_3 denote the number of units that have to be produced in the three sectors, write down the Leontief system for the problem. Verify that $x_1 = 150$, $x_2 = 200$, and $x_3 = 100$ is a solution.

6. Write down the input matrix for the simple Leontief model of Example 16.9.1. Compare the condition for efficient production discussed in that example with the requirement that the sum of the elements of each column in the input matrix be less than 1.

7. Suppose that $\mathbf{x} = \mathbf{x}_0$ is a solution of (16.9.3) and that $\mathbf{p}' = \mathbf{p}'_0$ is a solution of (16.9.11). Prove that $\mathbf{p}'_0\mathbf{b} = \mathbf{v}'\mathbf{x}_0$.

REVIEW EXERCISES

1. Calculate the following determinants:

(a) $\begin{vmatrix} 5 & -2 \\ 3 & -2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix}$ (c) $\begin{vmatrix} (a+b)^2 & a-b \\ (a-b)^2 & a+b \end{vmatrix}$ (d) $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$

2. Calculate the following determinants, using suitable elementary row operations for (b) and (c):

(a) $\begin{vmatrix} 2 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 4 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 8 \\ 6 & 7 & 9 \end{vmatrix}$ (c) $\begin{vmatrix} 31 & 32 & 33 \\ 32 & 33 & 35 \\ 33 & 34 & 36 \end{vmatrix}$

3. Find $\mathbf{A}$ when $(\mathbf{A}^{-1} - 2\mathbf{I}_2)' = -2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

4. Let $\mathbf{A}_t = \begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(a) For what values of t does $\mathbf{A}_t$ have an inverse?

(b) Find a matrix $\mathbf{X}$ such that $\mathbf{B} + \mathbf{X}\mathbf{A}_1^{-1} = \mathbf{A}_1^{-1}$.

SM 5. Define the two 3×3 matrices $\mathbf{A} = \begin{pmatrix} q & -1 & q-2 \\ 1 & -p & 2-p \\ 2 & -1 & 0 \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Calculate $|\mathbf{A}|$ and $|\mathbf{A} + \mathbf{E}|$. For what values of p and q does $\mathbf{A} + \mathbf{E}$ have an inverse? Why does $\mathbf{BE}$ not have an inverse for any 3×3 matrix $\mathbf{B}$?

6. Use Cramer's rule to find the values of t for which the system of equations

$$\begin{aligned} -2x + 4y - tz &= t - 4 \\ -3x + y + tz &= 3 - 4t \\ (t-2)x - 7y + 4z &= 23 \end{aligned}$$

has a unique solution for the three variables x, y, and z.

7. Prove that if $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{A}^4 = \mathbf{0}$, then $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$.

SM 8. Let $\mathbf{U}$ be the $n \times n$ matrix where all the elements are 1.

(a) Show that $(\mathbf{I}_n + a\mathbf{U})(\mathbf{I}_n + b\mathbf{U}) = \mathbf{I}_n + (a + b + nab)\mathbf{U}$ for all real numbers a and b.

(b) Use the result in (a) to find the inverse of $\mathbf{A} = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}$.

9. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{X}$, and $\mathbf{Y}$ be $n \times n$ matrices, with $|\mathbf{A}| \neq 0$, which satisfy the following two equations: $\mathbf{AX} + \mathbf{Y} = \mathbf{B}$ and $\mathbf{X} + 2\mathbf{A}^{-1}\mathbf{Y} = \mathbf{C}$. Find $\mathbf{X}$ and $\mathbf{Y}$ expressed in terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

SM 10. Consider the following system of equations:

$$\begin{aligned} ax + y + 4z &= 2 \\ 2x + y + a^2 z &= 2 \\ x - 3z &= a \end{aligned}$$

(a) For what values of a does the system have one, none, or infinitely many solutions?

(b) Replace the right-hand sides of the system by b_1, b_2, and b_3, respectively. Find a necessary and sufficient condition for the new system of equations to have infinitely many solutions.

11. Let $\mathbf{A} = \begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix}$.

(a) Compute $|\mathbf{A}|$. Show that there exists a real number c such that $\mathbf{A}^2 + c\mathbf{A} = 2\mathbf{I}_2$, and then find the inverse of $\mathbf{A}$.

(b) Show that there is no 2×2 matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.

12. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are invertible $n \times n$ matrices. Show that if $\mathbf{A}'\mathbf{A} = \mathbf{I}_n$, then $(\mathbf{A}'\mathbf{B}\mathbf{A})^{-1} = \mathbf{A}'\mathbf{B}^{-1}\mathbf{A}$.

13. Examine for what values of the constants a and b the system of equations

$$\begin{aligned} ax + y &= 3 \\ x + z &= 2 \\ y + az + bu &= 6 \\ y + u &= 1 \end{aligned}$$

has a unique solution in the unknowns x, y, z, and u. When it exists, find this unique solution, expressed in terms of a and b.

14. [HARDER] The 3×3 matrix $\mathbf{B}$ satisfies the equation $\mathbf{B}^3 = -\mathbf{B}$. Show that $\mathbf{B}$ cannot have an inverse. (*Hint:* Use part (vii) of Theorem 16.4.1.)

SM 15. [HARDER] Prove that $\begin{vmatrix} a+x & b+y \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} x & y \\ c & d \end{vmatrix}$.

SM 16. [HARDER] Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are $n \times n$ matrices that differ only in the r-th row, and suppose the r-th row in $\mathbf{C}$ is obtained by adding the entries in the r-th row of $\mathbf{A}$ to the corresponding entries in the r-th row of $\mathbf{B}$. Prove that then $|\mathbf{A}| + |\mathbf{B}| = |\mathbf{C}|$. (*Hint:* Consider the cofactor expansions of the determinants along the r-th row.)

SM 17. [HARDER] Solve the equation $\begin{vmatrix} x & a & x & b \\ b & x & a & x \\ x & b & x & a \\ a & x & b & x \end{vmatrix} = 0$ for x.

17 LINEAR PROGRAMMING

If one would take statistics about which mathematical problem is using up most of the computer time in the world, then (not counting database handling problems like sorting and searching) the answer would probably be linear programming.

—László Lovász (1980)

Linear programming is the name used to describe constrained optimization problems in which the objective is to maximize or minimize a linear function subject to linear inequality constraints. Because of its extensive use in economic decision problems, all economists should know something about the basic theory of linear programming.

In principle, *any* linear programming problem, often called an LP problem, can be solved numerically, provided that a solution exists. This is because the *simplex method*, introduced by American mathematician George B. Dantzig (1914–2005) in 1947, provides a very efficient numerical algorithm that finds the solution in a finite number of steps. As the above quotation from Lovász indicates, the simplex method has made linear programming a mathematical technique of immense practical importance. It is reported that when Mobil Oil Company's multimillion-dollar computer system was installed in 1958, it paid off this huge investment in two weeks by doing linear programming.[1] That said, the simplex method will not be discussed in this book. After all, faced with a nontrivial LP problem, it is natural to use one of the great number of available LP computer programs to find the solution. In any case, it is probably more important for economists to understand the basic theory of LP than the details of the simplex method. Especially as, following important developments during the 1980s, the simplex method is no longer the state of the art for solving large linear programs numerically.[2]

Indeed, the importance of LP extends even beyond its practical applications. In particular, the duality theory of linear programming is a basis for understanding key theoretical properties of more complicated optimization problems with an even larger range of interesting economic applications.

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[1] Joel Franklin, "Mathematical methods of economics", *The American Mathematical Monthly,* 1983, Vol. 90, no. 4.

[2] For a relatively recent survey article, see Florian A. Potra and Stephen J. Wright, "Interior-point methods", *Journal of Computational and Applied Mathematics*, 2000, Vol. 124, 281–302.

17.1 A Graphical Approach

A general linear programming problem with only two decision variables involves maximizing or minimizing a linear *objective function*

$$z = c_1x_1 + c_2x_2$$

subject to m linear *inequality constraints*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\le b_1 \\ a_{21}x_1 + a_{22}x_2 &\le b_2 \\ \cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 &\le b_m \end{aligned}$$

Usually, we also impose explicit *nonnegativity constraints* on x_1 and x_2:

$$x_1 \ge 0, \; x_2 \ge 0$$

Note that having a $\le$ sign rather than $\ge$ in each inequality constraint is merely a convention because any inequality of the alternative form $ax_1 + bx_2 \ge c$ is equivalent to the inequality $-ax_1 - bx_2 \le -c$.

LP problems with only two decision variables can be solved by a simple graphical method.

EXAMPLE 17.1.1 A baker has 150 kilograms of flour, 22 kilos of sugar, and 27.5 kilos of butter with which to make two types of cake. Suppose that making one dozen A cakes requires three kilos of flour, one kilo of sugar, and one kilo of butter, whereas making one dozen B cakes requires six kilos of flour, half a kilo of sugar, and one kilo of butter. Suppose that the profit from one dozen A cakes is 20 and from one dozen B cakes is 30. How many dozen A cakes (x_1) and how many dozen B cakes (x_2) will maximize the baker's profit?

Solution: An output of x_1 dozen A cakes plus x_2 dozen B cakes needs a total of $3x_1 + 6x_2$ kilos of flour. Because there are only 150 kilos of flour, the inequality

$$3x_1 + 6x_2 \le 150 \qquad \text{(flour constraint)}$$

must hold. Similarly, for sugar

$$x_1 + 0.5x_2 \le 22 \qquad \text{(sugar constraint)}$$

and for butter,

$$x_1 + x_2 \le 27.5 \qquad \text{(butter constraint)}$$

Of course, $x_1 \geq 0$ and $x_2 \geq 0$. The profit obtained from producing x_1 dozen A cakes and x_2 dozen B cakes is $z = 20x_1 + 30x_2$. In short, the problem is

$$\max\ z = 20x_1 + 30x_2 \ \text{ subject to: } \begin{cases} 3x_1 + 6x_2 \leq 150 \\ x_1 + 0.5x_2 \leq 22 \\ x_1 + x_2 \leq 27.5 \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases} \tag{i}$$

This problem will now be solved graphically. The output pair (x_1, x_2) is called *feasible*, or *admissible*, for problem (i) if all the five inequality constraints are satisfied. Look at the flour constraint, $3x_1 + 6x_2 \leq 150$. If we use all the flour, then $3x_1 + 6x_2 = 150$, and we call the corresponding straight line the *flour border*.

We can find similar "borders" for the other two inputs. Figure 17.1.1 shows the three straight lines that represent the flour border, the sugar border, and the butter border. In order for (x_1, x_2) to be feasible, it has to be on or below (to the "south-west" of) *each* of the three borders simultaneously. Because constraints $x_1 \geq 0$ and $x_2 \geq 0$ restrict (x_1, x_2) to the nonnegative quadrant, the set of admissible pairs for problem (i) is the shaded set S shown in Fig. 17.1.2.[3]

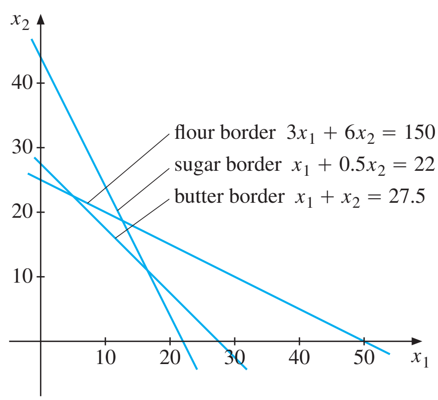


Figure 17.1.1 Borders of baker's problem

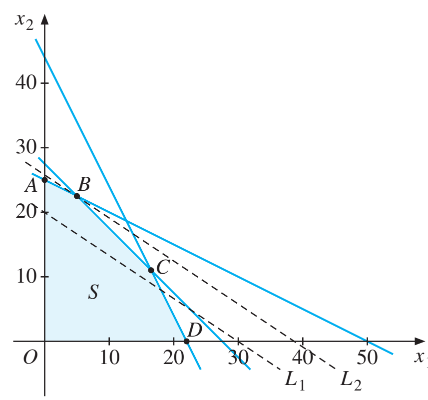


Figure 17.1.2 Feasible region for the baker

The baker might think of finding the point in the feasible region that maximizes profit by calculating $20x_1 + 30x_2$ at each point of S and picking the highest value. In practice, this is impossible because there are infinitely many feasible points.

Let us argue this way instead. Can the baker obtain a profit of 600? If so, the straight line $20x_1 + 30x_2 = 600$ must have points in common with S. This line is represented in Fig. 17.1.2 by dashed line L_1. It does have points in common with S. One of them, for instance, is $(x_1, x_2) = (0, 20)$, where no A cakes are produced, but 20 dozen B cakes are, and the profit is $20 \cdot 0 + 30 \cdot 20 = 600$.

[3] This set S is a so-called *convex polyhedron*, and the five corner points O, A, B, C, and D are called *extreme points* of the set S.

Can the baker do better? Yes. For instance, the straight line $20x_1 + 30x_2 = 601$ also has points in common with S and the profit is 601. In fact, the straight lines

$$20x_1 + 30x_2 = c$$

where c is a constant, are all parallel to $20x_1 + 30x_2 = 600$. As c increases, the line moves out farther and farther to the north-east. It is clear that the straight line that has the highest value of c and still has a point in common with S is dashed line L_2 in the figure. It touches set S at point B.

Note that B is at the intersection of the flour border and the butter border. Its coordinates, therefore, satisfy the two equations: $3x_1 + 6x_2 = 150$ and $x_1 + x_2 = 27.5$. Solving these two simultaneous equations yields $x_1 = 5$ and $x_2 = 22.5$. So the baker maximizes profit by baking 5 dozen A cakes and 22.5 dozen B cakes. This uses all the available flour and butter, but $22 - 5 - 0.5 \cdot 22.5 = 5.75$ kilos of sugar are left over. The profit earned is $20x_1 + 30x_2 = 775$.

EXAMPLE 17.1.2 A firm is producing two goods, A and B. It has two factories that jointly produce the two goods in the following quantities (per hour):

	Factory 1	Factory 2
Good A	10	20
Good B	25	25

The firm receives an order for 300 units of A and 500 units of B. The costs of operating the two factories are 10 000 and 8000 per hour. Formulate the linear programming problem of minimizing the total cost of meeting this order.

Solution: Let u_1 and u_2 be the number of hours that the two factories operate to produce the order. Then $10u_1 + 20u_2$ units of good A are produced, and $25u_1 + 25u_2$ units of good B. Because 300 units of A and 500 units of B are required, u_1 and u_2 must satisfy

$$\begin{aligned} 10u_1 + 20u_2 &\geq 300 \\ 25u_1 + 25u_2 &\geq 500 \end{aligned} \tag{i}$$

In addition, of course, $u_1 \geq 0$ and $u_2 \geq 0$. The total costs of operating the two factories for u_1 and u_2 hours, respectively, are $10\,000\,u_1 + 8000\,u_2$. The problem is, therefore,

$$\min\ 10\,000\,u_1 + 8000\,u_2 \ \text{ s.t. } \begin{cases} 10u_1 + 20u_2 \geq 300 \\ 25u_1 + 25u_2 \geq 500 \end{cases} \quad u_1 \geq 0,\ u_2 \geq 0$$

The feasible set S is shown in Fig. 17.1.3. Because the inequalities in (i) are of the $\geq$ type and all the coefficients of u_1 and u_2 are positive, the feasible set lies to the north-east. Figure 17.1.3 includes three of the level curves $10\,000u_1 + 8000u_2 = c$, marked L_1, L_2, and L_3. These three correspond to the values 100 000, 160 000, and 240 000 of the cost level c. As c increases, the level curve moves farther and farther to the north-east.

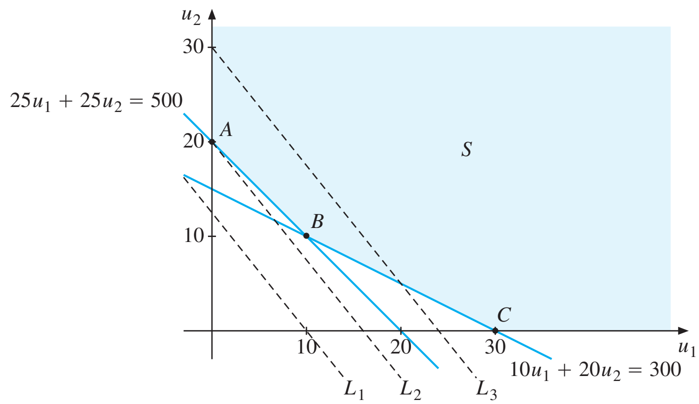


Figure 17.1.3 Feasible set, Example 17.1.2

The solution to the minimization problem is clearly the level curve that touches the feasible set S at point A with coordinates $(0, 20)$. Hence, the optimal solution is to operate factory 2 for 20 hours and not to use factory 1 at all, with minimum cost 160 000.

The graphical method of solving linear programming problems works well when there are only two decision variables. One can extend the method to the case with three decision variables. Then the feasible set is a convex polyhedron in 3-space, and the level surfaces of the objective function are planes in 3-space. However, it is not easy to visualize the solution in such cases. For more than three decision variables, no graphical method is available.[4]

Both the previous examples had optimal solutions. If the feasible region is unbounded, however, a finite optimal solution might not exist, as is the case in Exercise 4.

The General LP Problem

The general LP problem is that of maximizing or minimizing the *objective function*

$$z = c_1 x_1 + \cdots + c_n x_n \tag{17.1.1}$$

with $c_1, \ldots, c_n$ as given constants, subject to m *inequality constraints*

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &\le b_2 \\ \cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \tag{17.1.2}$$

where the elements a_{ij} and b_k are given constants. Usually, we assume explicitly that

$$x_1 \ge 0, \ \ldots, \ x_n \ge 0 \tag{17.1.3}$$

[4] By using duality theory, however, one can solve LP problems graphically when *either* the number of unknowns *or* the number of constraints is less than or equal to 3. See Section 17.5.

which are referred to as *nonnegativity constraints*. There is no essential difference between a minimization problem and a maximization problem, because the optimal solution $(x_1^*, \dots, x_n^*)$ that minimizes (17.1.1) subject to (17.1.2) and (17.1.3) also maximizes $-z$. An n-vector $(x_1, \dots, x_n)$ that satisfies (17.1.2) and (17.1.3) is called *feasible* or *admissible*.

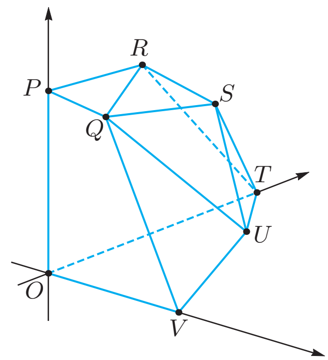


Figure 17.1.4 A convex polyhedron

The set of feasible points is a so-called *convex polyhedron* in the *nonnegative orthant* of n-space. A typical example in 3-space is shown in Fig. 17.1.4. The points O, P, Q, R, S, T, U, and V are called *extreme points*. The line segments OP, OT, OV, etc. joining two extreme points that are marked in Fig. 17.1.4 are called *edges*. These include RT, which is indicated with a dashed line because it is hidden behind the solid polyhedron. The flat portions of the boundary which are triangles or quadrilaterals lying within three or four of these edges are called *faces*. In n-space, any convex polyhedron also has extreme points, edges, and faces.

If n and m are large, the number of extreme points can be astronomical. The typical extreme point has n of the $n + m$ inequality constraints holding with equality. Thus, there can be as many as $(n + m)!/n!m!$ extreme points. For example, if $n = 50$ and $m = 60$ (which is quite small by the standards of the problems that can be solved numerically), then there can be as many as $110!/50!60!$ or more than $6 \cdot 10^{31}$ extreme points.

Nevertheless, the simplex method can solve such problems. It relies on the fact that if an LP problem has a solution, there must be a solution at an extreme point. Accordingly the method provides a procedure for moving repeatedly between adjacent extreme points of the polyhedron, along its edges, in such a way that the value of the objective function never decreases, and usually increases. The procedure terminates when it reaches an extreme point where no move to an adjacent extreme point will increase the value of the objective function. We have then reached the optimal solution.

EXERCISES FOR SECTION 17.1

1. Use the graphical method to solve the following LP problems:

(a) $\max\ 3x_1 + 4x_2$ s.t. $\begin{cases} 3x_1 + 2x_2 \le 6 \\ x_1 + 4x_2 \le 4 \end{cases}$ $\quad x_1 \ge 0,\ x_2 \ge 0$

(b) $\min\ 10u_1 + 27u_2$ s.t. $\begin{cases} u_1 + 3u_2 \geq 11 \\ 2u_1 + 5u_2 \geq 20 \end{cases}$ $\quad u_1 \geq 0,\ u_2 \geq 0$

2. Use the graphical method to solve the following LP problems:

(a) $\max\ 2x_1 + 5x_2$ s.t. $\begin{cases} -2x_1 + 3x_2 \leq 6 \\ 7x_1 - 2x_2 \leq 14 \\ x_1 + x_2 \leq 5 \end{cases}$ $\quad x_1 \geq 0,\ x_2 \geq 0$

(b) $\max\ 8x_1 + 9x_2$ s.t. $\begin{cases} x_1 + 2x_2 \leq 8 \\ 2x_1 + 3x_2 \leq 13 \\ x_1 + x_2 \leq 6 \end{cases}$ $\quad x_1 \geq 0,\ x_2 \geq 0$

(c) $\max\ -2x_1 + x_2$ s.t. $0 \leq x_1 - 3x_2 \leq 3,\ x_1 \geq 2,\ x_2 \geq 0$

SM **3.** Set A consists of all (x_1, x_2) satisfying

$$-2x_1 + x_2 \leq 2, \quad x_1 + 2x_2 \leq 8, \quad x_1 \geq 0, \quad x_2 \geq 0$$

Solve the following problems with A as the feasible set:

(a) $\max\ x_2$ (b) $\max\ x_1$ (c) $\max\ 3x_1 + 2x_2$

(d) $\min\ 2x_1 - 2x_2$ (e) $\max\ 2x_1 + 4x_2$ (f) $\min\ -3x_1 - 2x_2$

4. Consider the following problem:

$$\max\ x_1 + x_2 \ \text{ s.t. } \begin{cases} -x_1 + x_2 \leq -1 \\ -x_1 + 3x_2 \leq 3 \end{cases} \qquad x_1 \geq 0,\ x_2 \geq 0$$

(a) Is there a solution to this problem?

(b) Is there a solution if the objective function is $z = -x_1 - x_2$ instead?

5. Replace the objective function in Example 17.1.1 by $20x_1 + tx_2$. For what values of t will the maximum profit still be at $x_1 = 5$ and $x_2 = 22.5$?

6. A firm produces two types of television set, an inexpensive type (A) and an expensive type (B). The firm earns a profit of 700 from each TV of type A, and 1000 for each TV of type B. There are three stages of the production process, each requiring its own specialized kind of labour. Stage I requires three units of labour on each set of type A and five units of labour on each set of type B. The total available quantity of labour for this stage is 3900. Stage II requires one unit of labour on each set of type A and three units on each set of type B. The total labour available for this stage is 2100 units. At stage III, two units of labour are needed for each type, and 2200 units of labour are available. How many TV sets of each type should the firm produce to maximize its profit?

17.2 Introduction to Duality Theory

Confronted with an optimization problem involving scarce resources, an economist will often ask: What happens to the optimal solution if the availability of the resources changes? For linear programming problems, answers to questions like this are intimately related to the so-called duality theory of LP. As a point of departure, let us again consider the baker's problem in Example 17.1.1.

EXAMPLE 17.2.1 Suppose the baker were to stumble across an extra kilo of flour that had been hidden away in storage. How much would this extra kilo add to his maximum profit? How much would an extra kilo of sugar contribute to profit? Or an extra kilo of butter?

Solution: If the baker finds an extra kilo of flour, the flour border becomes $3x_1 + 6x_2 = 151$. It is clear from Fig. 17.1.2 that the feasible set S will expand slightly and point B will move slightly up along the butter border. The new optimal point B' will be at the intersection of the lines $3x_1 + 6x_2 = 151$ and $x_1 + x_2 = 27.5$. Solving these equations gives $x_1 = 14/3$ and $x_2 = 137/6$. The objective function attains the value $20(14/3) + 30(137/6) = 2335/3 = 775 + 10/3$. So profit rises by 10/3.

If the baker finds an extra kilo of sugar, the feasible set will expand, but the optimal point is still at B. Recall that at the optimum in the original problem, the baker had 5.75 kilos of unused sugar. There is no extra profit.

An extra kilo of butter would give a new optimal point at the intersection of the lines $3x_1 + 6x_2 = 150$ and $x_1 + x_2 = 28.5$. Solving these equations gives $x_1 = 7$ and $x_2 = 21.5$ with $20x_1 + 30x_2 = 775 + 10$. Profit rises by 10. These results can be summarized as follows: (a) an extra kilo of flour would increase the optimal z by 10/3; (b) an extra kilo of sugar would increase the optimal z by 0; and (c) an extra kilo of butter would increase the optimal z by 10.

The three numbers $u_1^* = 10/3$, $u_2^* = 0$, and $u_3^* = 10$ are related to the flour, sugar, and butter constraints, respectively. They are the *marginal* profits from an extra kilo of each ingredient. These numbers have many interesting properties that we shall now explore.

Suppose (x_1, x_2) is a feasible pair in the problem, so that the three constraints in Example 17.1.1 are satisfied. Multiply the first constraint by 10/3, the second by 0, and the third by 10. Because the multipliers are all ≥ 0, the inequalities are preserved. That is,

$$(10/3)(3x_1 + 6x_2) \leq (10/3) \cdot 150$$
$$0(x_1 + 0.5x_2) \leq 0 \cdot 22$$
$$10(x_1 + x_2) \leq 10 \cdot 27.5$$

Now add all these inequalities, using the obvious fact that if $A \leq B$, $C \leq D$, and $E \leq F$, then $A + C + E \leq B + D + F$. The result is $10x_1 + 20x_2 + 10x_1 + 10x_2 \leq \frac{10}{3} \cdot 150 + 10 \cdot 27.5$, which reduces to

$$20x_1 + 30x_2 \leq 775$$

Thus, using the "magic" numbers u_1^*, u_2^*, and u_3^* defined above, we have proved that if (x_1, x_2) is any feasible pair, then the objective function has to be less than or equal to

775. Because $x_1 = 5$ and $x_2 = 22.5$ give z the value 775, we have in this way *proved algebraically* that (5, 22.5) *is* a solution!

The Dual Problem

The pattern revealed in the last example turns up in all linear programming problems. In fact, the numbers u_1^*, u_2^*, and u_3^* are solutions to a new LP problem called the *dual*.

Recall the baker's problem, now called the *primal* and denoted by (P). It was

$$\max\ 20x_1 + 30x_2 \ \text{ s.t. } \begin{cases} 3x_1 + 6x_2 \le 150 \\ x_1 + 0.5x_2 \le 22 \\ x_1 + x_2 \le 27.5 \end{cases} \qquad x_1 \ge 0,\ x_2 \ge 0 \tag{P}$$

Suppose the baker gets tired of running the business.[5] An entrant wants to take over and buy all the ingredients. The existing baker intends to charge a price u_1 for each kilo of flour, u_2 for each kilo of sugar, and u_3 for each kilo of butter. Because one dozen A cakes requires three kilos of flour and one kilo each of sugar and butter, the baker will charge $3u_1 + u_2 + u_3$ for the ingredients needed to produce a dozen A cakes. The baker originally had a profit of 20 for each dozen A cakes, and he wants to earn at least as much from these ingredients if he quits. Hence, the baker insists that the prices (u_1, u_2, u_3) must satisfy

$$3u_1 + u_2 + u_3 \ge 20$$

Otherwise, it would be more profitable to use the ingredients himself to produce A cakes.

If the baker also wants to earn at least as much as before for the ingredients needed to produce a dozen B cakes, the requirement is

$$6u_1 + 0.5u_2 + u_3 \ge 30$$

Presumably, the entrant wants to buy the baker's resources as inexpensively as possible. The total cost of 150 kilos of flour, 22 kilos of sugar, and 27.5 kilos of butter is $150u_1 + 22u_2 + 27.5u_3$. In order to pay as little as possible while having the baker accept the offer, the entrant should suggest prices $u_1 \ge 0$, $u_2 \ge 0$, and $u_3 \ge 0$, that solve the LP problem

$$\min\ 150u_1 + 22u_2 + 27.5u_3 \ \text{ s.t. } \begin{cases} 3u_1 + u_2 + u_3 \ge 20 \\ 6u_1 + 0.5u_2 + u_3 \ge 30 \end{cases} \tag{D}$$

which is called the *dual* of the primal problem, and so labelled (D).

Suppose the baker lets the entrant take over the business and charges prices that solve (D). Will the baker earn as much as before? It turns out that the answer is yes. The solution to problem (D) is $u_1^* = 10/3$, and $u_2^* = 0$, and $u_3^* = 10$, so the amount the baker gets for selling the resources is $150u_1^* + 22u_2^* + 27.5u_3^* = 775$, which is precisely the maximum value of the objective function in problem (P). The entrant pays for each ingredient exactly the marginal profit for that ingredient which was calculated previously. In particular, the price of sugar is zero, because the baker has more than he can use optimally.

[5] After all, baking cakes this plain is hardly exciting.

The primal problem (P) and dual problem (D) turn out to be closely related. Let us explain in general how to construct the dual of an LP problem.

The General Case

Consider the general LP problem

$$\max\ c_1x_1 + \cdots + c_nx_n \quad \text{s.t.} \quad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \le b_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m \end{cases} \tag{17.2.1}$$

with nonnegativity constraints $x_1 \ge 0, \dots, x_n \ge 0$. Its *dual* is the LP problem

$$\min\ b_1u_1 + \cdots + b_mu_m \quad \text{s.t.} \quad \begin{cases} a_{11}u_1 + \cdots + a_{m1}u_m \ge c_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{1n}u_1 + \cdots + a_{mn}u_m \ge c_n \end{cases} \tag{17.2.2}$$

with nonnegativity constraints $u_1 \ge 0, \dots, u_m \ge 0$. Note that problem (17.2.2) is constructed using exactly the same coefficients $c_1, \dots, c_n$, $a_{11}, \dots, a_{mn}$, and $b_1, \dots, b_m$ as in (17.2.1).

In the *primal* problem (17.2.1), there are n variables $x_1, \dots, x_n$ and m constraints, disregarding the nonnegativity constraints. In the dual (17.2.2), there are m variables $u_1, \dots, u_m$ and n constraints. Whereas the primal is a maximization problem, the dual is a minimization problem. In both problems, all variables are nonnegative. There are m "less than or equal to" constraints in the primal problem (17.2.1), but n "greater than or equal to" constraints in the dual problem (17.2.2). The coefficients of the objective function in either problem are the right-hand side elements of the constraints in the other problem. Finally, the two matrices formed by the coefficients of the variables in the constraints in the primal and dual problems are transposes of each other, because they take the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} \tag{17.2.3}$$

Check carefully that problem (D) really is the dual of problem (P) in the sense just explained. Due to the symmetry between the two problems, we call each the dual of the other.

Matrix Formulation

Let us introduce the following column vectors (i.e. matrices with one column):

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \tag{17.2.4}$$

When $\mathbf{y}$ and $\mathbf{z}$ are vectors, the notation $\mathbf{y} \leq \mathbf{z}$ means that each component of $\mathbf{y}$ is less than or equal to the corresponding component of $\mathbf{z}$, with $\mathbf{y} \geq \mathbf{z}$ as the reverse inequality.

Then the primal can be written as follows, with $\mathbf{A}$ and $\mathbf{A}'$ given by (17.2.3):

$$\max\ \mathbf{c}'\mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \tag{17.2.5}$$

And the dual can be written as

$$\min\ \mathbf{b}'\mathbf{u} \ \text{ s.t. } \ \mathbf{A}'\mathbf{u} \geq \mathbf{c},\ \mathbf{u} \geq \mathbf{0}$$

It is more convenient, however, to write the dual in a slightly different way. Transposing $\mathbf{A}'\mathbf{u} \geq \mathbf{c}$ using the transposition rules in Eqs (15.5.2) to (15.5.5), yields one obtains $\mathbf{u}'\mathbf{A} \geq \mathbf{c}'$, and moreover $\mathbf{b}'\mathbf{u} = \mathbf{u}'\mathbf{b}$. So the dual can be written as

$$\min\ \mathbf{u}'\mathbf{b} \ \text{ s.t. } \ \mathbf{u}'\mathbf{A} \geq \mathbf{c}',\ \mathbf{u}' \geq \mathbf{0} \tag{17.2.6}$$

EXERCISES FOR SECTION 17.2

SM 1. Consider Exercise 17.1.1(a).

(a) Replace the constraint $3x_1 + 2x_2 \leq 6$ by $3x_1 + 2x_2 \leq 7$. Find the new optimal solution and compute the increase u_1^* in the objective function.

(b) Replace the constraint $x_1 + 4x_2 \leq 4$ by $x_1 + 4x_2 \leq 5$. Find the new optimal solution and compute the increase u_2^* in the objective function.

(c) By the same argument as in Example 17.2.1, prove that if (x_1, x_2) is feasible in the original problem, then the objective function can never be larger than 36/5.

2. Write down the dual to part (b) of Exercise 17.1.2.

3. Write down the duals to parts (a) and (b) of Exercise 17.1.1.

4. Consider the LP problem:

$$\max\ x_1 + x_2 \ \text{ s.t. } \begin{cases} x_1 + 2x_2 \leq 14 \\ 2x_1 + \ x_2 \leq 13 \end{cases} \qquad x_1 \geq 0,\ x_2 \geq 0$$

(a) Use the graphical method to find its solution.

(b) Write down the dual and find its solution.

17.3 The Duality Theorem

This section presents the main results relating the solution of an LP problem to that of its dual. We begin by considering the baker's problem yet again.

EXAMPLE 17.3.1 Consider problems (P) and (D) in Section 17.2. Suppose that (x_1, x_2) is an arbitrary feasible pair in (P), which means that $x_1 \geq 0$, $x_2 \geq 0$, and the three $\leq$ inequalities in (P) are all satisfied. Let (u_1, u_2, u_3) be an arbitrary feasible triple in (D). Multiply the $\leq$ inequalities in (P) by the nonnegative numbers u_1, u_2, and u_3, respectively, and then add the inequalities. The result is the new inequality

$$(3x_1 + 6x_2)u_1 + (x_1 + 0.5x_2)u_2 + (x_1 + x_2)u_3 \leq 150u_1 + 22u_2 + 27.5u_3$$

Rearranging the terms on the left-hand side yields

$$(3u_1 + u_2 + u_3)x_1 + (6u_1 + 0.5u_2 + u_3)x_2 \leq 150u_1 + 22u_2 + 27.5u_3 \tag{i}$$

Similarly, we multiply the $\geq$ inequalities in (D) by the nonnegative numbers x_1 and x_2, respectively, and add the results. This gives

$$(3u_1 + u_2 + u_3)x_1 + (6u_1 + 0.5u_2 + u_3)x_2 \geq 20x_1 + 30x_2 \tag{ii}$$

From (i) and (ii) together, it follows that

$$150u_1 + 22u_2 + 27.5u_3 \geq 20x_1 + 30x_2 \tag{iii}$$

for all feasible (x_1, x_2) in problem (P) and for all feasible (u_1, u_2, u_3) in problem (D). Thus, the objective function in the dual problem is always greater than or equal to the objective function of the primal problem, whatever feasible (x_1, x_2) and (u_1, u_2, u_3) are chosen.

The inequality (iii) is valid for the feasible pair $(x_1, x_2) = (5, 22.5)$ in particular. For each feasible triple (u_1, u_2, u_3), we therefore obtain

$$150u_1 + 22u_2 + 27.5u_3 \geq 20 \cdot 5 + 30 \cdot 22.5 = 775$$

It follows that if we can find a feasible triple (u_1^*, u_2^*, u_3^*) for problem (D) such that $150u_1^* + 22u_2^* + 27.5u_3^* = 775$, then (u_1^*, u_2^*, u_3^*) must solve problem (D), because no lower value of the objective function is obtainable. In Section 17.2, we saw that for $(u_1^*, u_2^*, u_3^*) = (10/3, 0, 10)$, the objective function in the dual did have the value 775. Hence, $(10/3, 0, 10)$ solves the dual problem.

Our analysis of this example illustrates two significant general results in LP theory. Here is the first:

THEOREM 17.3.1

If $(x_1, \ldots, x_n)$ is feasible in the primal problem (17.2.1) and $(u_1, \ldots, u_m)$ is feasible in the dual problem (17.2.2), then

$$b_1u_1 + \cdots + b_mu_m \geq c_1x_1 + \cdots + c_nx_n \tag{17.3.1}$$

So the dual objective function has a value that is always at least as large as that of the primal.

The argument for this result is not difficult:

Multiply the m inequalities in (17.2.1) by the nonnegative numbers $u_1, \ldots, u_m$, then add. Also, multiply the n inequalities in (17.2.2) by the nonnegative numbers $x_1, \ldots, x_n$, then add. These two operations yield the two inequalities

$$(a_{11}x_1 + \cdots + a_{1n}x_n)u_1 + \cdots + (a_{m1}x_1 + \cdots + a_{mn}x_n)u_m \le b_1u_1 + \cdots + b_mu_m$$
$$(a_{11}u_1 + \cdots + a_{m1}u_m)x_1 + \cdots + (a_{1n}u_1 + \cdots + a_{mn}u_m)x_n \ge c_1x_1 + \cdots + c_nx_n$$

By rearranging the terms on the left-hand side of each inequality, we see that each is equal to the double sum $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}u_ix_j$. So (17.3.1) follows immediately.

From Theorem 17.3.1 we can derive a second significant result:

THEOREM 17.3.2

Suppose that $(x_1^*, \dots, x_n^*)$ and $(u_1^*, \dots, u_m^*)$ are feasible in problems (17.2.1) and (17.2.2), respectively, and that

$$c_1x_1^* + \cdots + c_nx_n^* = b_1u_1^* + \cdots + b_mu_m^* \tag{17.3.2}$$

Then $(x_1^*, \dots, x_n^*)$ solves the primal problem (17.2.1) and $(u_1^*, \dots, u_m^*)$ solves dual problem (17.2.2).

Again, we have all the elements to prove this important result:

Let $(x_1, \dots, x_n)$ be an arbitrary feasible n-vector for problem (17.2.1). Using (17.3.1) with $u_1 = u_1^*, \dots, u_m = u_m^*$, as well as (17.3.2), yields

$$c_1x_1 + \cdots + c_nx_n \le b_1u_1^* + \cdots + b_mu_m^* = c_1x_1^* + \cdots + c_nx_n^*$$

This proves that $(x_1^*, \dots, x_n^*)$ solves (17.2.1).

Suppose that $(u_1, \dots, u_m)$ is feasible for problem (17.2.2). Then (17.3.1) and (17.3.2) together imply that

$$b_1u_1 + \cdots + b_mu_m \ge c_1x_1^* + \cdots + c_nx_n^* = b_1u_1^* + \cdots + b_mu_m^*$$

This proves that $(u_1^*, \dots, u_m^*)$ solves (17.2.2).

Theorem 17.3.2 shows that if we are able to find *feasible* solutions for problems (17.2.1) and (17.2.2) that give the same value to the relevant objective function in each of the two problems, then these feasible solutions are, in fact, *optimal* solutions.

The most important result in duality theory is the following:

THEOREM 17.3.3 (THE DUALITY THEOREM)

Suppose the primal problem (17.2.1) has a (finite) optimal solution. Then the dual problem (17.2.2) also has a (finite) optimal solution, and the corresponding values of the objective functions are equal. If the primal has no bounded optimum, then the dual has no feasible solution. Symmetrically, if the primal has no feasible solution, then the dual has no bounded optimum.

The proofs of Theorems 17.3.1 and 17.3.2 were very simple. It is much more difficult to prove the first statement in Theorem 17.3.3 concerning the existence of a solution to the dual, and we shall not attempt to do so here. The last statement in Theorem 17.3.3, however, follows readily from inequality (17.3.1). For if $(u_1, \dots, u_m)$ is any feasible solution to the dual problem, then $b_1 u_1 + \cdots + b_m u_m$ is a finite number greater than or equal to *any* number $c_1 x_1 + \cdots + c_n x_n$ when $(x_1, \dots, x_n)$ is feasible in the primal. This puts an upper bound on the possible values of $c_1 x_1 + \cdots + c_n x_n$.

Finally, an instructive exercise is to formulate and prove Theorems 17.3.1 and 17.3.2 using matrix algebra. Let us do so for Theorem 17.3.1. Suppose $\mathbf{x}$ is feasible in (17.2.5) and $\mathbf{u}$ is feasible in (17.2.6). Then $\mathbf{u}'\mathbf{b} \geq \mathbf{u}'(\mathbf{A}\mathbf{x}) = (\mathbf{u}'\mathbf{A})\mathbf{x} \geq \mathbf{c}'\mathbf{x}$. Note carefully how these inequalities correspond to those we established in the earlier proof of Theorem 17.3.1.

EXERCISES FOR SECTION 17.3

SM **1.** Consider the LP problem $\max\ 2x + 7y \text{ s.t. } \begin{cases} 4x + 5y \leq 20 \\ 3x + 7y \leq 21 \end{cases} \quad x \geq 0,\ y \geq 0.$

(a) Solve it by a graphical argument.

(b) Write down the dual and solve it by a graphical argument.

(c) Are the values of the objective functions equal?[6]

2. Write down the dual to the problem in Example 17.1.2 and solve it. Check that the optimal values of the objective functions are equal.

SM **3.** A firm produces small and medium television sets. The profit is 400 for each small and 500 for each medium television set. Each television has to be processed on three different assembly lines. Each small television requires respectively two, one, and one hour on lines 1, 2, and 3. The corresponding numbers for the medium television sets are one, four, and two. Suppose lines 1 and 2 both have a capacity of at most 16 hours per day, and line 3 has a capacity of at most 11 hours per day. Let x_1 and x_2 denote the number of small and medium television sets that are produced per day.

(a) Show that in order to maximize profits per day, one must solve the following problem:

$$\max\ 400x_1 + 500x_2 \text{ s.t. } \begin{cases} 2x_1 + x_2 \leq 16 \\ x_1 + 4x_2 \leq 16 \\ x_1 + 2x_2 \leq 11 \end{cases} \quad x_1 \geq 0,\ x_2 \geq 0$$

(b) Solve this problem graphically.

(c) Suppose the firm could increase its capacity by one hour a day on just one of its assembly lines. Which line should have its capacity increased?

[6] If not, then, according to Theorem 17.3.3, you have made a mistake in the maximized value of the objective function.

17.4 A General Economic Interpretation

This section gives an economic interpretation of the general LP problem (17.2.1) and its dual (17.2.2). Think of a firm that produces one or more different kinds of output using m different *resources* as inputs. There are n different *activities* (or processes) involved in the production process. A typical activity is characterized by the fact that running it at unit level requires a certain amount of each resource. Let a_{ij} denote the number of units of resource i that are needed to run activity j at unit level. Then the vector with components $a_{1j}, a_{2j}, \dots, a_{mj}$ expresses the m different total resource requirements for running activity j at unit level. If we run the activities at levels $x_1, \dots, x_n$, then the total resource requirement can be expressed as the column vector

$$x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

If the available resources are $b_1, \dots, b_m$, then the feasible activity levels are those that satisfy the m constraints in (17.2.1). The nonnegativity constraints reflect the fact that we cannot run the activities at negative levels.

Each activity brings a certain "reward". Let c_j denote the reward (or value) earned by running activity j at unit level. The total reward from running the n activities at levels $x_1, \dots, x_n$ is then $c_1x_1 + \cdots + c_nx_n$. So the firm faces the problem of solving the following LP problem:

Find those levels for the n activities that maximize the total reward, subject to the given resource constraints.

The baker's problem in Example 17.1.1 provides an illustration. The two activities are baking the two different types of cake, and there are three resources—flour, sugar, and butter.

Let us turn to the dual problem (17.2.2). In order to remain in business, the firm has to use some resources. Each resource, therefore, has a value or price. Let u_j be the price associated with one unit of resource j. Rather than think of u_j as a market price for resource j, we should think of it as measuring the relative contribution that one unit of resource j makes to the total economic reward. Because these are not real market prices, they are often called *shadow prices*.

Because $a_{1j}, a_{2j}, \dots, a_{mj}$ are the quantities of each of the m resources needed to run activity j at unit level, $a_{1j}u_1 + a_{2j}u_2 + \cdots + a_{mj}u_m$ is the total shadow cost of running activity j at unit level. Because c_j is the reward earned by running activity j at unit level, the difference

$$c_j - (a_{1j}u_1 + a_{2j}u_2 + \cdots + a_{mj}u_m)$$

can be regarded as the shadow *profit* from running activity j at unit level. Note that the j-th constraint in the dual problem (17.2.2) says that the shadow profit from running activity j at unit level is ≤ 0.

The objective function $Z = b_1u_1 + \cdots + b_mu_m$ in the dual LP problem measures the shadow value of the initial stock of all the resources. The dual problem is, therefore:

Among all choices of nonnegative shadow prices $u_1, \dots, u_m$ such that the profit from running each activity at unit level is non-positive, find those prices which together minimize the shadow value of the initial resources.

The Optimal Dual Variables as Shadow Prices

Consider again the primal problem (17.2.1). What happens to the optimal value of the objective function if the numbers $b_1, \dots, b_m$ change? If the changes $\Delta b_1, \dots, \Delta b_m$ are positive, then the feasible set increases and the new optimal value of the objective function cannot be smaller, and usually it increases. The following analysis also applies when some or all the changes $\Delta b_1, \dots, \Delta b_m$ are negative.

Suppose $(x_1^*, \dots, x_n^*)$ and $(x_1^* + \Delta x_1, \dots, x_n^* + \Delta x_n)$ are optimal solutions to the primal problem when the right-hand sides of the constraints are respectively $(b_1, \dots, b_m)$ and $(b_1 + \Delta b_1, \dots, b_m + \Delta b_m)$. Typically, if $\Delta b_1, \dots, \Delta b_m$ are all sufficiently small, the duals of the two problems have the same optimal solution $u_1^*, \dots, u_m^*$. Then, according to Theorem 17.3.3, one has

$$c_1 x_1^* + \cdots + c_n x_n^* = b_1 u_1^* + \cdots + b_m u_m^*$$

$$c_1(x_1^* + \Delta x_1) + \cdots + c_n(x_n^* + \Delta x_n) = (b_1 + \Delta b_1)u_1^* + \cdots + (b_m + \Delta b_m)u_m^*$$

Hence, by subtraction,

$$c_1 \Delta x_1 + \cdots + c_n \Delta x_n = u_1^* \Delta b_1 + \cdots + u_m^* \Delta b_m$$

Here the left-hand side is the change we obtain in the objective function in (17.2.1) when $b_1, \dots, b_m$ are changed by $\Delta b_1, \dots, \Delta b_m$, respectively. Denoting this change in z by Δz^*, we obtain

$$\Delta z^* = u_1^* \Delta b_1 + \cdots + u_m^* \Delta b_m \tag{17.4.1}$$

Importantly, note that an assumption underlying (17.4.1) is that the numbers b_j do not change enough to cause the optimal dual variables to change. If $\Delta b_j = 1$, while all $\Delta b_i = 0$ for $i \neq j$, then $\Delta z^* = u_j^*$. This accords with the results in Example 17.2.1.

EXERCISES FOR SECTION 17.4

1. Consider Exercise 17.3.1. We found that the optimal solution of this problem was $x^* = 0$ and $y^* = 3$, with $z^* = 2x^* + 7y^* = 21$. The optimal solution of the dual was $u_1^* = 0$ and $u_2^* = 1$. Suppose we change 20 to 20.1 and 21 to 20.8. What is the corresponding change in the maximized value of the objective function?

SM 2. A firm produces two goods A and B. The firm earns a profit of 300 from each unit of good A, and 200 from each unit of B. There are three stages of the production process. Good A requires six hours in production, then four hours in assembly, and finally five hours of packing. The corresponding numbers for B are three, six, and five, respectively. The total number of hours available for the three stages are 54, 48, and 50, respectively.

   (a) Formulate and solve the LP problem of maximizing profits subject to the given constraints.

   (b) Write down and solve the dual problem.

(c) By how much would the optimal profit increase if the firm gets two hours more preparation time and one hour more packing time?

17.5 Complementary Slackness

Consider again the baker's problem (P) in Section 17.2 and its dual (D). The solution to (P) was $x_1^* = 5$ and $x_2^* = 22.5$, with the first and the third inequalities both satisfied with equality. The solution to the dual was $u_1^* = 10/3$, $u_2^* = 0$, and $u_3^* = 10$, with both inequalities in the dual satisfied with equality. Thus, in this example

$$x_1^* > 0,\ x_2^* > 0 \ \Rightarrow \begin{cases} \text{the first and second inequalities} \\ \text{in the dual are satisfied with equality} \end{cases}$$

while

$$u_1^* > 0,\ u_3^* > 0 \ \Rightarrow \begin{cases} \text{the first and third inequalities} \\ \text{in the primal are satisfied with equality} \end{cases}$$

We interpret the second implication this way: Because the shadow prices of flour and butter are positive, the optimal solution requires all the available flour and butter to be used, but not all the available sugar, so its shadow price is zero—it is not a scarce resource.

Implications like this hold more generally. Indeed, consider the problem

$$\max\ c_1x_1 + c_2x_2 \ \text{ s.t. } \begin{cases} a_{11}x_1 + a_{12}x_2 \le b_1 \\ a_{21}x_1 + a_{22}x_2 \le b_2 \\ a_{31}x_1 + a_{32}x_2 \le b_3 \end{cases} \quad x_1 \ge 0,\ x_2 \ge 0 \tag{i}$$

and its dual

$$\min\ b_1u_1 + b_2u_2 + b_3u_3 \ \text{ s.t. } \begin{cases} a_{11}u_1 + a_{21}u_2 + a_{31}u_3 \ge c_1 \\ a_{12}u_1 + a_{22}u_2 + a_{32}u_3 \ge c_2 \end{cases} \tag{ii}$$

with u_1, u_2, and $u_3 \ge 0$. Suppose (x_1^*, x_2^*) solves (i) and (u_1^*, u_2^*, u_3^*) solves (ii). Then

$$\text{(iii)} \begin{cases} a_{11}x_1^* + a_{12}x_2^* \le b_1 \\ a_{21}x_1^* + a_{22}x_2^* \le b_2 \\ a_{31}x_1^* + a_{32}x_2^* \le b_3 \end{cases} \quad \text{and} \quad \text{(iv)} \begin{cases} a_{11}u_1^* + a_{21}u_2^* + a_{31}u_3^* \ge c_1 \\ a_{12}u_1^* + a_{22}u_2^* + a_{32}u_3^* \ge c_2 \end{cases}$$

Multiply the three inequalities in (iii) by the three nonnegative numbers u_1^*, u_2^*, and u_3^*, respectively. Then add the results. This yields the inequality

$$(a_{11}x_1^* + a_{12}x_2^*)u_1^* + (a_{21}x_1^* + a_{22}x_2^*)u_2^* + (a_{31}x_1^* + a_{32}x_2^*)u_3^* \le b_1u_1^* + b_2u_2^* + b_3u_3^* \tag{v}$$

Multiply the two inequalities in (iv) by x_1^* and x_2^*, respectively, and then add. This gives

$$(a_{11}u_1^* + a_{21}u_2^* + a_{31}u_3^*)x_1^* + (a_{12}u_1^* + a_{22}u_2^* + a_{32}u_3^*)x_2^* \ge c_1x_1^* + c_2x_2^* \tag{vi}$$

But the left-hand sides of the inequalities (v) and (vi) are rearrangements of each other. Moreover, by the duality theorem of LP, Theorem 17.3.3, their right-hand sides are the equal values of the primal and dual. Hence, both inequalities in (v) and (vi) can be replaced by *equalities*. In particular, we can rearrange the equality version of (v) to obtain

$$(a_{11}x_1^* + a_{12}x_2^* - b_1)u_1^* + (a_{21}x_1^* + a_{22}x_2^* - b_2)u_2^* + (a_{31}x_1^* + a_{32}x_2^* - b_3)u_3^* = 0$$

Because (x_1^*, x_2^*) is feasible, (iii) implies that each term in parentheses is ≤ 0. But each $u_i \ge 0$, so the left-hand side is the sum of three ≤ 0 terms. If any is negative, so is their sum. But the sum is 0, so each term is 0. Thus, for each $j = 1, 2, 3$,

$$(a_{j1}x_1^* + a_{j2}x_2^* - b_j)u_j^* = 0$$

We conclude that

$$a_{j1}x_1^* + a_{j2}x_2^* \le b_j, \text{ with } a_{j1}x_1^* + a_{j2}x_2^* = b_j \text{ if } u_j^* > 0$$

with $j = 1, 2, 3$. Using the fact that $\ge$ in (vi) can be replaced by =, and reasoning in exactly the same way as above, we also get

$$a_{1i}u_1^* + a_{2i}u_2^* + a_{3i}u_3^* \ge c_i, \text{ with } a_{1i}u_1^* + a_{2i}u_2^* + a_{3i}u_3^* = c_i \text{ if } x_i^* > 0, \quad i = 1, 2$$

These last two sets of inequalities (or equalities) are called *complementary slackness conditions*. The arguments used to show their necessity extend in a straightforward way to the general case. Furthermore, the same complementary slackness conditions are also sufficient for optimality. Here is a general statement and proof:

THEOREM 17.5.1 (COMPLEMENTARY SLACKNESS)

Suppose that the primal maximization problem (17.2.1) has an optimal solution $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$, whereas the dual minimization problem (17.2.2) has an optimal solution $\mathbf{u}^* = (u_1^*, \dots, u_m^*)$. Then for $i = 1, \dots, n$, and $j = 1, \dots, m$, one has the complementary slackness conditions

$$a_{1i}u_1^* + \cdots + a_{mi}u_m^* \ge c_i, \text{ with } a_{1i}u_1^* + \cdots + a_{mi}u_m^* = c_i \text{ if } x_i^* > 0 \tag{17.5.1}$$

$$a_{j1}x_1^* + \cdots + a_{jn}x_n^* \le b_j, \text{ with } a_{j1}x_1^* + \cdots + a_{jn}x_n^* = b_j \text{ if } u_j^* > 0 \tag{17.5.2}$$

Conversely, if $\mathbf{x}^*$ and $\mathbf{u}^*$ have all their components nonnegative and satisfy (17.5.1) and (17.5.2), then $\mathbf{x}^*$ and $\mathbf{u}^*$ solve the primal problem (17.2.1) and the dual (17.2.2), respectively.

While longer than previous arguments, the proof of this theorem is well within our reach:

Suppose $\mathbf{x}^*$ solves (17.2.1) and $\mathbf{u}^*$ solves (17.2.2). Using the matrix notation of (17.2.5) and (17.2.6), it follows that

$$\mathbf{A}\mathbf{x}^* \le \mathbf{b} \quad \text{and} \quad (\mathbf{u}^*)'\mathbf{A} \ge \mathbf{c}' \tag{i}$$

Multiplying the first inequality in (i) on the left by $(\mathbf{u}^*)' \geq \mathbf{0}$ and the second inequality on the right by $\mathbf{x}^* \geq \mathbf{0}$ yields

$$(\mathbf{u}^*)'\mathbf{A}\mathbf{x}^* \leq (\mathbf{u}^*)'\mathbf{b} \quad \text{and} \quad (\mathbf{u}^*)'\mathbf{A}\mathbf{x}^* \geq \mathbf{c}'\mathbf{x}^* \tag{ii}$$

According to Theorem 17.3.3, $(\mathbf{u}^*)'\mathbf{b} = \mathbf{c}'\mathbf{x}^*$. So both inequalities in (ii) must be equalities. They can be written as

$$(\mathbf{u}^*)'(\mathbf{A}\mathbf{x}^* - \mathbf{b}) = 0 \quad \text{and} \quad [(\mathbf{u}^*)'\mathbf{A} - \mathbf{c}']\mathbf{x}^* = 0 \tag{iii}$$

But these two equations are equivalent to the two equalities

$$\sum_{j=1}^{m} u_j^*(a_{j1}x_1^* + \cdots + a_{jn}x_n^* - b_j) = 0 \tag{iv}$$

$$\sum_{i=1}^{n} (a_{1i}u_1^* + \cdots + a_{mi}u_m^* - c_i)x_i^* = 0 \tag{v}$$

For $j = 1, \ldots, m$ one has both $u_j^* \geq 0$ and $a_{j1}x_1^* + \cdots + a_{jn}x_n^* - b_j \leq 0$. So each term in the sum (iv) is ≤ 0. If any term is negative, so is their sum; but the sum of all m terms is 0, so each term in (iv) must be 0 as well. Therefore,

$$u_j^*(a_{j1}x_1^* + \cdots + a_{jn}x_n^* - b_j) = 0, \qquad j = 1, \ldots, m \tag{vi}$$

Now (17.5.2) follows immediately. Property (17.5.1) is proved in the same way by noting how (v) implies that

$$x_i^*(a_{1i}u_1^* + \cdots + a_{mi}u_m^* - c_i) = 0, \qquad i = 1, \ldots, n \tag{vii}$$

Suppose conversely that $\mathbf{x}^*$ and $\mathbf{u}^*$ have all their components nonnegative and satisfy (17.5.1) and (17.5.2) respectively. It follows immediately that (vi) and (vii) are satisfied. So summing over j and i, respectively, we obtain (iv) and (v). These equations imply that

$$\sum\nolimits_{j=1}^{m} b_j u_j^* = \sum\nolimits_{j=1}^{m} \sum\nolimits_{i=1}^{n} a_{ji}x_i^* u_j^* \quad \text{and also} \quad \sum\nolimits_{i=1}^{n} c_i x_i^* = \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{m} a_{ji}u_j^* x_i^*$$

Because the right-hand sides of the two double sums are equal, it follows that $\sum_{j=1}^{m} b_j u_j^* = \sum_{i=1}^{n} c_i x_i^*$. So according to Theorem 17.3.2, $\mathbf{x}^*$ solves problem (17.2.1) and $\mathbf{u}^*$ solves the dual.

Using the economic interpretations we gave in Section 17.4, conditions (17.5.1) and (17.5.2) can be interpreted as follows:

(i) *If the optimal solution of the primal problem implies that activity i is in operation ($x_i^* > 0$), then the (shadow) profit from running that activity at unit level is 0.*

(ii) *If the shadow price of resource j is positive ($u_j^* > 0$), then all the available stock of resource j must be used in any optimum.*

How Complementary Slackness Can Help Solve LP Problems

If the solution to either the primal or the dual problem is known, then the complementary slackness conditions can help find the solution to the other problem by determining which constraints are slack, and so which hold with equality. Let us look at an example.

EXAMPLE 17.5.1 Write down the dual of the following LP problem and solve it by a graphical argument.

$$\max\ 3x_1 + 4x_2 + 6x_3 \ \text{ s.t. } \begin{cases} 3x_1 + \ \ x_2 + \ \ x_3 \le 2 \\ \ \ x_1 + 2x_2 + 6x_3 \le 1 \end{cases}, \quad x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \tag{i}$$

Then use complementary slackness to solve it.

Solution: The dual problem is

$$\min\ 2u_1 + u_2 \ \text{ s.t. } \begin{cases} 3u_1 + \ \ u_2 \ge 3 \\ \ \ u_1 + 2u_2 \ge 4 \\ \ \ u_1 + 6u_2 \ge 6 \end{cases}, \qquad u_1 \ge 0,\ u_2 \ge 0 \tag{ii}$$

Using the graphical solution technique shown in Example 17.1.2, we find the solution $u_1^* = 2/5$, and $u_2^* = 9/5$. Then $3u_1^* + u_2^* = 3$, and $u_1^* + 2u_2^* = 4$, and $u_1^* + 6u_2^* > 6$.

What do we know about the solution (x_1^*, x_2^*, x_3^*) to (i)? According to (17.5.2), because $u_1^* > 0$ and $u_2^* > 0$, both inequalities in (i) are satisfied with equality. So

$$3x_1^* + x_2^* + x_3^* = 2 \ \text{ and } \ x_1^* + 2x_2^* + 6x_3^* = 1 \tag{iii}$$

Next, since $u_1^* + 6u_2^* > 6$, the complementary slackness condition (17.5.1) implies that $x_3^* = 0$. Letting $x_3^* = 0$ in (iii) and solving for x_1^* and x_2^* gives

$$x_1^* = 3/5, \qquad x_2^* = 1/5, \qquad x_3^* = 0$$

This is the solution to problem (i). Note that the optimal values of the objective functions in the two problems are indeed equal: $2u_1^* + u_2^* = 13/5$ and $3x_1^* + 4x_2^* + 6x_3^* = 13/5$, just as they should be according to the duality theorem.

The Kuhn–Tucker Theorem Applied to Linear Programmes

The general linear programming problem is obviously a special case of the general nonlinear programming problem

$$\max\ f(x_1, \dots, x_n) \ \text{ s.t. } \begin{cases} g_1(x_1, \dots, x_n) \le c_1 \\ \dots\dots\dots\dots\dots\dots \\ g_m(x_1, \dots, x_n) \le c_m \end{cases}, \quad x_1 \ge 0, \dots, x_n \ge 0 \tag{17.5.3}$$

that was studied in Section 14.10.

Let us see what form the Kuhn–Tucker necessary conditions (14.10.3) and (14.10.4) take in the linear case. If we let $\lambda_j = u_j^*$ for $j = 1, \dots, m$, the conditions become

$$c_i - (a_{1i}u_1^* + \cdots + a_{mi}u_m^*) \le 0, \ \text{ with equality if } x_i^* > 0 \tag{17.5.4}$$

for each $i = 1, \dots, n$, and

$$u_j^* \ge 0 \quad (= 0 \ \text{ if } \ a_{j1}x_1^* + \cdots + a_{jn}x_n^* < b_j), \qquad j = 1, \dots, m \tag{17.5.5}$$

When combined with the requirement that $\mathbf{x}^*$ satisfy the constraints in the LP problem, these necessary conditions are precisely the complementary slackness conditions in Theorem 17.5.1.

Duality When Some Constraints are Equalities

Suppose that one of the m constraints in the primal problem is the equality

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i \tag{$*$}$$

rather than the corresponding inequality in (17.2.1). In order to put the problem into the standard form, we can replace ($*$) by the two inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n \le b_i \quad \text{and} \quad -a_{i1}x_1 - \cdots - a_{in}x_n \le -b_i \tag{$**$}$$

Constraint ($*$) thus gives rise to two dual variables u_i' and u_i''. For each $j = 1, \dots, n$ the term $a_{ij}u_i$ in the sum on the left-hand side of the constraint $\sum_{k=1}^{m} a_{kj}u_k \ge c_j$ in (17.2.2) gets replaced by $a_{ij}u_i' - a_{ij}u_i''$. Therefore, we can replace the two variables u_i' and u_i'' with the single variable $u_i = u_i' - u_i''$, but then there is no restriction on the sign of u_i. We see that *if the i-th constraint in the primal is an equality, then the i-th dual variable has an unrestricted sign*. This is consistent with the economic interpretation we have given. If we are forced to use all of resource i, then it is not surprising that the resource may have a negative shadow price—it may be something that is harmful in excess. For instance, if the baker of Example 17.1.1 was forced to include all the stock of sugar in the cakes, the best point in Fig. 17.1.2 would be C, not B. Some profit would be lost.

From the symmetry between the primal and the dual, we realize now that *if one of the variables in the primal has an unrestricted sign, then the corresponding constraint in the dual is an equality*.

EXERCISES FOR SECTION 17.5

1. Consider Exercise 17.3.1. The solution of the primal was $x^* = 0$ and $y^* = 3$, with $u_1^* = 0$ and $u_2^* = 1$ as the solution of the dual. Verify that (17.5.1) and (17.5.2) are satisfied in this case.

2. Consider the following problem:

$$\min\ y_1 + 2y_2 \quad \text{s.t.} \quad \begin{cases} y_1 + 6y_2 \ge 15 \\ y_1 + \ \ y_2 \ge 5 \\ -y_1 + \ \ y_2 \ge -5 \\ y_1 - 2y_2 \ge -20 \end{cases} \qquad y_1 \ge 0,\ y_2 \ge 0$$

   (a) Solve it graphically.

   (b) Write down the dual problem and solve it.

   (c) What happens to the optimal values of the dual variables if the constraint $y_1 + 6y_2 \ge 15$ is changed to $y_1 + 6y_2 \ge 15.1$?

SM **3.** A firm produces two commodities A and B. The firm has three factories that jointly produce both commodities in the amounts per hour given in the following table:

	Factory 1	Factory 2	Factory 3
Commodity A	10	20	20
Commodity B	20	10	20

The firm receives an order for 300 units of A and 500 units of B. The costs per hour of running factories 1, 2, and 3 are respectively 10 000, 8000, and 11 000.

(a) Let y_1, y_2, and y_3, respectively, denote the number of hours for which the three factories are used. Write down the linear programming problem of minimizing the costs of fulfilling the order.

(b) Write down the dual and solve it. Then find the solution of the problem in part (a).

(c) By how much will the minimum cost of production increase if the cost per hour in factory 1 increases by 100?

4. [HARDER] Consider the LP problem

$$\max\ 3x_1 + 2x_2 \ \text{ s.t. } \begin{cases} x_1 + \ x_2 \qquad \le 3 \\ 2x_1 + \ x_2 - \ x_3 \le 1 \\ x_1 + 2x_2 - 2x_3 \le 1 \end{cases} \qquad x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

(a) Suppose x_3 is a fixed number. Solve the problem if $x_3 = 0$ and if $x_3 = 3$.

(b) Formulate and solve the problem for any fixed value of x_3 in $[0, \infty)$. The maximal value of $3x_1 + 2x_2$ becomes a function of x_3. Find this function and maximize it.

(c) Do the results in part (b) say anything about the solution to the original problem, in which x_3 can also be chosen?

REVIEW EXERCISES

1. Consider the LP problem $\max\ x + 2y$ s.t. $\begin{cases} x + y \le 4 \\ -x + y \le 1 \\ 2x - y \le 3 \end{cases} \qquad x \ge 0,\ y \ge 0$

(a) Solve it.

(b) Formulate and solve the dual problem.

SM **2.** Consider the LP problem

$$\min\ 16y_1 + 6y_2 - 8y_3 - 15y_4 \ \text{ s.t. } \begin{cases} -y_1 + \ y_2 - 2y_3 - 4y_4 \ge -1 \\ 2y_1 - 2y_2 - \ y_3 - 5y_4 \ge \ \ 1 \end{cases}$$

where $y_i \ge 0$, for $i = 1, 2, 3, 4$.

(a) Write down the dual problem and solve it.

(b) Find the solution to the primal problem.

(c) If the first constraint in the primal is changed to $-y_1 + y_2 - 2y_3 - 4y_4 \geq k$, for what values of k will the solution of the dual occur at the same point as for $k = -1$?

3. Consider the LP problem

$$\min\ 5x + y \ \text{ s.t. } \begin{cases} 4x + \ y \geq \ 4 \\ 2x + \ y \geq \ 3 \\ 3x + 2y \geq \ 2 \\ -x + 2y \geq -2 \end{cases} \qquad x \geq 0,\ y \geq 0$$

(a) Solve it.

(b) Formulate the dual problem and solve it.

SM **4.** A firm produces x_1 cars and x_2 trucks per month. Suppose each car requires 0.04% of the capacity per month in the body division, 0.025% of the capacity per month in the motor division, and 0.05% of the capacity per month on the specialized car assembly line. The corresponding numbers for trucks are 0.03% in the body division, 0.05% in the motor division, and 0.08% on the specialized truck assembly line. The firm can therefore deliver x_1 cars and x_2 trucks per month provided the following inequalities are satisfied:

$$\begin{aligned} 0.04x_1 + 0.03x_2 &\leq 100 \\ 0.025x_1 + 0.05x_2 &\leq 100 \\ 0.05x_1 \qquad\qquad &\leq 100 \\ 0.08x_2 &\leq 100 \end{aligned} \tag{$*$}$$

with $x_1 \geq 0$, $x_2 \geq 0$. Suppose the profit per car is $500 - ax_1$, where a is a nonnegative constant, while the profit per truck is 250. The firm thus seeks to solve the problem

$$\max\ (500 - ax_1)x_1 + 250x_2$$

subject to $(*)$.

(a) Solve the problem graphically, for $a = 0$.

(b) Write down conditions (14.10.3) and (14.10.4) for the problem when $a \geq 0$.

(c) Use the conditions obtained in (b) to examine for which values of $a \geq 0$ the solution is the same as for $a = 0$.

SM **5.** The production of three goods requires using two machines. Machine 1 can be utilized for b_1 hours, while machine 2 can be utilized for b_2 hours. The time spent for the production of one unit of each good is given by the following table:

	Machine 1	Machine 2
Good 1	3	2
Good 2	1	2
Good 3	4	1

The profits per unit produced of the three goods are 6, 3, and 4, respectively.

(a) Write down the linear programming problem this leads to.

(b) Show that the dual is

$$\min\ b_1 y_1 + b_2 y_2 \ \text{ s.t. } \begin{cases} 3y_1 + 2y_2 \geq 6 \\ \ \ y_1 + 2y_2 \geq 3 \\ 4y_1 + \ \ y_2 \geq 4 \end{cases} \qquad y_1 \geq 0,\ y_2 \geq 0$$

Solve this problem graphically for $b_1 = b_2 = 100$.

(c) Solve the problem in (a) when $b_1 = b_2 = 100$.

(d) If machine 1 increases its capacity to 101, while $b_2 = 100$, what is the new maximal profit?

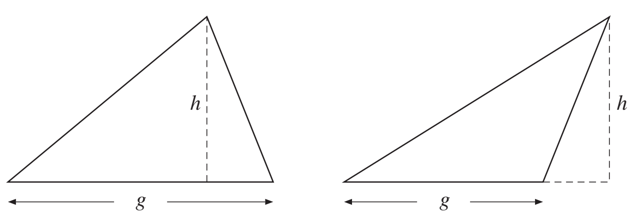
(e) The maximum value of the profit in problem (a) is a function F of b_1 and b_2. What is the degree of homogeneity of the function F?

APPENDIX

Let no one ignorant of geometry enter this door.
—Entrance to Plato's Academy

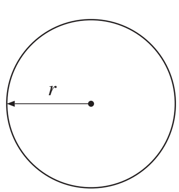
This appendix is to remind the reader about some simple formulas and results from geometry that are occasionally useful for economists, and sometimes used in this book. At the end there is also a listing of the Greek alphabet.

Triangles



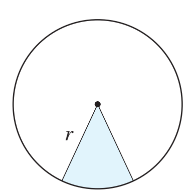
Area: $A = \frac{1}{2}gh$

Circles



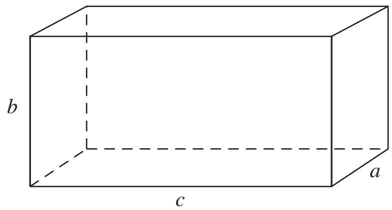
Area: $A = \pi r^2$

Circumference: $C = 2\pi r$



Area: $A = \frac{1}{2}xr$

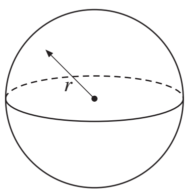
Rectangular Box



Volume: $V = abc$

Surface Area: $S = 2ab + 2ac + 2bc$

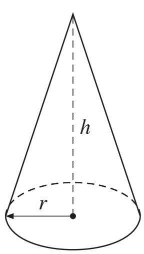
Sphere (Ball)



Volume: $V = \frac{4}{3}\pi r^3$

Surface Area: $S = 4\pi r^2$

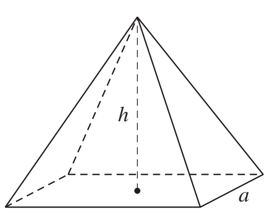
Cone



Volume: $V = \frac{1}{3}\pi r^2 h$

Surface Area: $S = \pi r^2 + \pi r\sqrt{h^2 + r^2}$

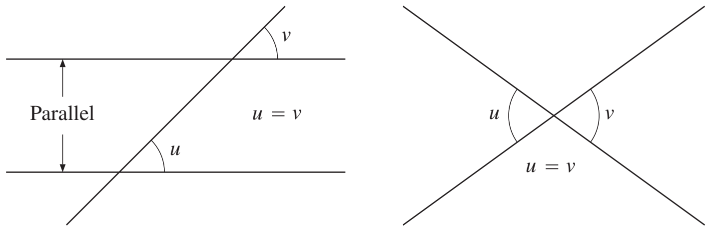
Pyramid



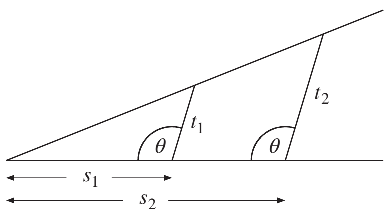
Volume: $V = \frac{1}{3}a^2 h$

Surface Area: $S = a^2 + a\sqrt{a^2 + 4h^2}$

Angles

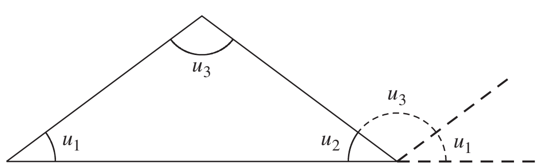


Proportions



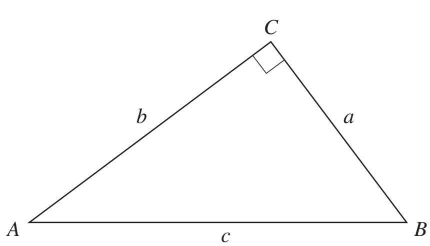
$t_1/s_1 = t_2/s_2$

Sum of Angles in a Triangle



$u_1 + u_2 + u_3 = 180°$

Pythagoras's Theorem



$\angle C = 90° \iff a^2 + b^2 = c^2$

The Greek Alphabet

A α	alpha	H η	eta	N ν	nu	T τ	tau
B β	beta	Θ θ ϑ	theta	Ξ ξ	xi	Υ υ	upsilon
Γ γ	gamma	I ι	iota	O o	omicron	Φ ϕ φ	phi
Δ δ	delta	K κ $\varkappa$	kappa	Π π	pi	X χ	chi
E ϵ ε	epsilon	Λ λ	lambda	P ρ ϱ	rho	Ψ ψ	psi
Z ζ	zeta	M μ	mu	Σ σ	sigma	Ω ω	omega

SOLUTIONS TO THE EXERCISES

Chapter 1

1.1

1. (a) $5 \in C$, $D \subseteq C$, and $B = C$ are true. The three others are false. (b) $A \cap B = \{2\}$, $A \cup B = \{2, 3, 4, 5, 6\}$, $A \setminus B = \{3, 4\}$, $B \setminus A = \{5, 6\}$, $(A \cup B) \setminus (A \cap B) = \{3, 4, 5, 6\}$, $A \cup B \cup C \cup D = \{2, 3, 4, 5, 6\}$, $A \cap B \cap C = \{2\}$, and $A \cap B \cap C \cap D = \emptyset$.

2. (a) The set $F \cap B \cap C$ consists of all female biology students in Ω who belong to the university choir; $M \cap F$ of all female mathematics students in Ω; $((M \cap B) \setminus C) \setminus T$ of all students in Ω who study both mathematics and biology but neither play tennis nor belong to the university choir.

(b) (i) $B \subseteq M$ (ii) $F \cap B \cap C \neq \emptyset$ (iii) $T \cap B = \emptyset$ (iv) $(F \setminus T) \setminus C \subseteq B$.

3. Note that $50 - 35 = 15$ liked coffee but not tea, and $40 - 35 = 5$ liked tea but not coffee.

Because 35 liked both and 10 liked neither, there were $15 + 5 + 35 + 10 = 65$ who responded.

4. The $2^3 = 8$ subsets of $\{a, b, c\}$ are the set itself, and the empty set, together with the six subsets $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. The $2^4 = 16$ subsets of $\{a, b, c, d\}$ are the eight preceding sets together with eight more sets that include d—namely $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and $\{a, b, c, d\}$. Apart from $\{a, b, c, d\}$ and the empty set, there are 14 other subsets.

5. (b) is true because $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \subseteq (A \cap B) \cup C$; the other three are generally false. Indeed:

(a) $A \setminus B \neq B \setminus A$ whenever $B \subseteq A$ with $\emptyset \neq B \neq A$;

(c) holds if and only if $A \subseteq C$;

(d) is violated when $A = \{1, 2\}$, $B = \{1\}$, and $C = \{1, 3\}$.

6. For $i = 1, 2, 3$, let S_i denote the set marked (i) in Fig. A1.1.6. Also, let S_4 denote the set of all points outside the regions marked (1), (2), or (3). Then:

(a) $(A \cup B)^c = S_4$ whereas $A^c = S_3 \cup S_4$ and $B^c = S_1 \cup S_4$, so $A^c \cap B^c = S_4$.

(b) $A \cap B = S_2$ so $(A \cap B)^c = S_1 \cup S_3 \cup S_4$, whereas $A^c \cup B^c = (S_3 \cup S_4) \cup (S_1 \cup S_4) = S_1 \cup S_3 \cup S_4$.

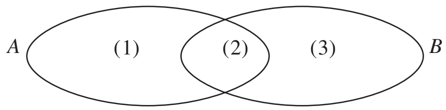


Figure A1.1.6

7. (a) Look at Fig. A1.1.6. Now, $n(A \cup B)$ is the sum of the numbers of elements in the pairwise disjoint sets labelled (1), (2), and (3) respectively—that is, $n(A \setminus B) + n(A \cap B) + n(B \setminus A)$. But $n(A) + n(B)$ is the number of elements in

(1) and (2) together, plus the number of elements in (2) and (3) together. Thus, the elements in (2) are counted twice. Hence, you must subtract $n(A \cap B)$, the number of elements in (2), to have equality. (b) Look again at Fig. A1.1.6. Now, $n(A \setminus B)$ is the number of elements in set (1), whereas $n(A) - n(A \cap B)$ is the number of elements in (1) and (2) together, minus the number of elements in (2) alone. Hence, it is the number of elements in (1).

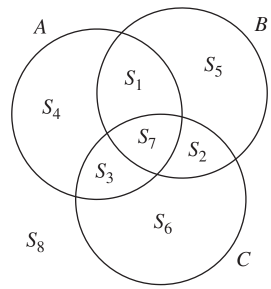


Figure A1.1.8

8. (a) Consider Fig. A1.1.8, where the circles represent the readership of the three papers. Let n_k denote the number of people in the set marked S_k, for $k = 1, 2, \ldots, 8$. Obviously $n_1 + n_2 + \cdots + n_8 = 1000$. The responses imply that: $n_1 + n_3 + n_4 + n_7 = 420$; $n_1 + n_2 + n_5 + n_7 = 316$; $n_2 + n_3 + n_6 + n_7 = 160$; $n_1 + n_7 = 116$; $n_3 + n_7 = 100$; $n_2 + n_7 = 30$; and $n_7 = 16$. From these equations we easily find $n_1 = 100$, $n_2 = 14$, $n_3 = 84$, $n_4 = 220$, $n_5 = 186$, $n_6 = 46$, $n_7 = 16$, and $n_8 = 334$. So $n_3 + n_4 = 304$ had read A but not B. (b) $n_6 = 46$. (c) $n_8 = 334$.

(d) We find $n(A \setminus B) = n_3 + n_4 = 304$, $n(C \setminus (A \cup B)) = n_6 = 46$, and $n(\Omega \setminus (A \cup B \cup C)) = n_8 = 334$. The last equality is a special case of $n(\Omega \setminus D) = n(\Omega) - n(D)$. (The number of persons who are in Ω, but not in D, is the number of persons in all of Ω minus the number of those who are in D.)

9. (Note: For the concept of set complement to make sense, it must be assumed that all the sets we consider are subsets of some "universal" set Ω. Unfortunately, the collection of everything is not a set!)
Let $\{A_i : i \in I\}$ denote the family of sets, with union $A^{\cup} = \bigcup_{i \in I} A_i$ and intersection $A^{\cap} = \bigcap_{i \in I} A_i$. Then the two statements in the problem are: (a) $(A^{\cup})^c = \bigcap_{i \in I} A_i^c$, (b) $(A^{\cap})^c = \bigcup_{i \in I} A_i^c$.

Proofs: (a) $a \in (A^{\cup})^c$ if and only if $a \notin A^{\cup}$, that is, if and only if a does not belong to any of the sets A_i, which holds if and only if $a \in A_i^c$ for all $i \in I$, and so if and only if $a \in \bigcap_{i \in I} A_i^c$.

(b) $a \in (A^{\cap})^c$ if and only if $a \notin A^{\cap}$, that is, if and only if there exists $i \in I$ such that $a \notin A_i$, which holds if and only if there exists $i \in I$ such that $a \in A_i^c$, and so if and only if $a \in \bigcup_{i \in I} A_i^c$. See SM.

1.2

1. (a) $2x - 4 = 2 \Rightarrow x = 3$ (b) $x = 3 \Rightarrow 2x - 4 = 2$ (c) $x = 1 \Rightarrow x^2 - 2x + 1 = 0$ (d) $x^2 > 4 \Leftrightarrow |x| > 2$

2. (a), (b), and (e) are all true; indeed, (e) is a common definition of what it means for two sets to be equal. For (c), suppose for example that $A = \{x\}$, $B = \{y\}$, and $C = \{z\}$, where x, y, z are all different. Then $A \cap B = A \cap C = \emptyset$, yet $B \neq C$. For (d), suppose for example that $A = \{x, y, z\}$, $B = \{y\}$, and $C = \{z\}$, where x, y, z are all different. Then $A \cup B = A \cup C = A$, yet $B \neq C$.

3. (a) $\Rightarrow$ true, $\Leftarrow$ true (b) $\Rightarrow$ true, $\Leftarrow$ false (c) $\Rightarrow$ false, $\Leftarrow$ true (d) $\Rightarrow$ true (actually both x and y are 0), $\Leftarrow$ false (e) $\Rightarrow$ true, $\Leftarrow$ true (f) $\Rightarrow$ false ($0 \cdot 5 = 0 \cdot 4$, but $5 \neq 4$), $\Leftarrow$ true

4. One has $2x + 5 \geq 13 \iff 2x \geq 8 \iff x \geq 4$, so: (a) $x \geq 0$ is necessary, but not sufficient.

(b) $x \geq 50$ is sufficient, but not necessary. (c) $x \geq 4$ is both necessary and sufficient.

5. (a) $x < 0$ or $y < 0$ (b) There exists x such that $x < a$. (c) $x < 5$ or $y < 5$, or both.

(d) There exists an $\varepsilon > 0$ such that B is not satisfied for any $\delta > 0$.

(e) There is someone who is able to resist liking cats. (f) There is someone who never loves anyone.

1.3

1. (b), (d), and (e) all express the same condition. (a) and (c) are different.

2. Logically the two statements are equivalent. The second statement may still be useful as an expressive poetic reinforcement.

3. If x and y are *not* both odd, at least one of them must be even. If, for example, $x = 2n$, where n is an integer, then $xy = 2ny$ is also even. Similarly if $y = 2m$, where m is an integer.

1.4

1. For $n = 1$, both sides are 1. Suppose $(*)$ is true for $n = k$. Then $1 + 2 + 3 + \cdots + k + (k + 1) = \frac{1}{2}k(k + 1) + (k + 1) = \frac{1}{2}(k + 1)(k + 2)$, which is $(*)$ for $n = k + 1$. Thus, by induction, $(*)$ is true for all natural numbers n.

2. For $n = 1$, both sides are $\frac{1}{2}$. Suppose $(**)$ is true for $n = k$. Then

$$\frac{1}{1 \cdot 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$

But $k(k + 2) + 1 = k^2 + 2k + 1 = (k + 1)^2$, so the last fraction simplifies to $(k + 1)/(k + 2)$. Hence $(**)$ is also true for $n = k + 1$, and it follows by induction that $(**)$ is true for all natural numbers n.

3. For $n = 1$, the sum $n^3 + (n + 1)^3 + (n + 2)^3 = 36$, which is divisible by 9. As the induction hypothesis for $n = k$, suppose that $k^3 + (k + 1)^3 + (k + 2)^3 = 9m_k$ for some natural number m_k. Then

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = -k^3 + 9m_k + (k+3)^3 = 9m_k + 9k^2 + 27k + 27 = 9(m_k + k^2 + 3k + 3)$$

Obviously, this is also divisible by 9, which confirms the induction hypothesis for $n = k + 1$.

4. The induction step breaks down for $k = 1$: Take two people A and B. Send A outside. B has the same income as himself. Bring A back, and send B outside. A has the same income as himself. But this does *not* imply that the two people have the same income! (The induction step is correct for all $k > 1$, because then the two people sent out have the same income as the others.)

Review exercises for Chapter 1

1. $A \cap B = \{1, 4\}$; $A \cup B = \{1, 3, 4, 6\}$; $A \setminus B = \{3\}$; $B \setminus A = \{6\}$; $(A \cup B) \setminus (A \cap B) = \{3, 6\}$;

$A \cup B \cup C \cup D = \{1, 2, 3, 4, 5, 6\}$; $A \cap B \cap C = \{4\}$; and $A \cap B \cap C \cap D = \emptyset$.

2. $A \cap B = \emptyset$; $A \cup B = \{1, 2, 4, 6, 11\}$; $\Omega \setminus B = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$; $A^c = \Omega \setminus A = \{2, 3, 5, 7, 8, 9, 10, 11\}$.

3. Let $n_E = 780$, $n_F = 220$, and $n_S = 52$ denote the numbers studying respectively English, French, and Spanish; then let $n_{EF} = 110$, $n_{ES} = 32$, and $n_{FS} = 15$ denote the numbers studying two of the languages, and $n_{EFS} = 10$ the number studying all three.

(a) $n_{EF} - n_{EFS} = 110 - 10 = 100$ study English and French, but not Spanish.

(b) $n_E - n_{EF} = 780 - 110 = 670$ study English but not French.

(c) The number studying at least one language can be calculated as

$$n_E + (n_F - n_{EF}) + (n_S - n_{ES} - n_{FS} + n_{EFS}) = 780 + (220 - 110) + (52 - 32 - 15 + 10) = 780 + 110 + 15 = 905$$

so there are 95 of the 1000 students who study no language.

4. (a) $\Rightarrow$ true, $\Leftarrow$ false. (b) $\Rightarrow$ false because $(-4)^2 = 16$, $\Leftarrow$ true. (c) $\Rightarrow$ true, $\Leftarrow$ false when $x = 3$.

(d) $\Rightarrow$ and $\Leftarrow$ both true.

5. (a) $(1+x)^2 = 1 + 2x + x^2 \geq 1 + 2x$ for all x since $x^2 \geq 0$.

(b) $(1+x)^3 = 1 + 3x + 3x^2 + x^3 = 1 + 3x + x^2(3+x)$. If $x \geq -3$ then $x^2(3+x) \geq 0$, implying that $(1+x)^3 \geq 1 + 3x$.

(c) We prove the result by induction on n. Evidently $(1+x)^n \geq 1 + nx$ holds with equality when $n = 1$. As the induction hypothesis, suppose that $x \geq -1$ implies that $(1+x)^k \geq 1 + kx$. Then $(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x)$ when $x \geq -1$. But $(1+kx)(1+x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x$, implying that $(1+x)^{k+1} \geq 1 + (k+1)x$. This completes the proof by induction.

Chapter 2

2.1

1. (a) True. (b) False. -5 is less than -3, so on the number line it is to the left of -3.

(c) False because all natural numbers are positive.

(d) True. Every natural number is rational. For example $5 = 5/1$. (e) False, since $3.1415 = 31\,415/10\,000$, the quotient of two integers. (Note that 3.1415 is only an approximation to the irrational number π.)

(f) False. Counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$. (g) True. (h) True.

2. In the number 1.01001000100001000001 . . . , one extra zero is added between each successive pair of ones. So there is obviously no finite sequence of digits that repeats itself indefinitely.

2.2

1. (a) $10^3 = 10 \cdot 10 \cdot 10 = 1000$ (b) $(-0.3)^2 = 0.09$ (c) $4^{-2} = 1/16$ (d) $(0.1)^{-1} = 1/0.1 = 10$

2. (a) $4 = 2^2$ (b) $1 = 2^0$ (c) $64 = 2^6$ (d) $1/16 = 2^{-4}$

3. (a) 15^3 (b) $\left(-\frac{1}{3}\right)^3$ (c) 10^{-1} (d) 10^{-7} (e) t^6 (f) $(a-b)^3$ (g) a^2b^4 (h) $(-a)^3$

4. (a) $2^5 \cdot 2^5 = 2^{5+5} = 2^{10}$ (b) $3^8 \cdot 3^{-2} \cdot 3^{-3} = 3^{8-2-3} = 3^3$ (c) $(2x)^3 = 2^3x^3 = 8x^3$

(d) $(-3xy^2)^3 = (-3)^3x^3(y^2)^3 = -27x^3y^6$ (e) $\dfrac{p^{24}p^3}{p^4p} = p^{24+3-4-1} = p^{22}$

(f) $\dfrac{a^4b^{-3}}{(a^2b^{-3})^2} = \dfrac{a^4b^{-3}}{a^4b^{-6}} = a^{4-4}b^{-3-(-6)} = b^3$ (g) $\dfrac{3^4(3^2)^6}{(-3)^{15}3^7} = \dfrac{3^4 3^{12}}{-3^{15}3^7} = -3^{-6}$ (h) $\dfrac{p^\gamma(pq)^\sigma}{p^{2\gamma+\sigma}q^{\sigma-2}} = p^{-\gamma}q^2$

5. (a) $2^6 = 64$ (b) $64/27$ (c) $8/3$ (d) x^9 (e) y^{12} (f) $8x^3y^3$ (g) $10^{-2} = 1/100$ (h) k^4 (i) $(x+1)^2$

6. (a) Because $4\pi(3r)^2 = 4\pi 3^2 r^2 = 9(4\pi r^2)$, the surface area increases by the factor 9.

(b) When r increases by 16%, it increases by a factor of 1.16, and r^2 increases by the factor $(1.16)^2 = 1.3456$, so the surface area increases by 34.56%.

7. (a) False. $a^0 = 1$. (b) True. $c^{-n} = 1/c^n$ for all $c \neq 0$. (c) True. $a^m \cdot a^m = a^{m+m} = a^{2m}$.

(d) False (unless $m = 0$ or $ab = 1$). $a^m b^m = (ab)^m$.

(e) False (unless $m = 1$ or $ab = 0$). For example, $(a+b)^2$ is equal to $a^2 + 2ab + b^2$, which is not $a^2 + b^2$ because $ab > 0$.

(f) False (unless $a^m b^n = 1$). For example, $a^2 b^3$ is not equal to $(ab)^{2+3} = (ab)^5 = a^5 b^5$.

8. (a) $x^3y^3 = (xy)^3 = 3^3 = 27$ (b) $(ab)^4 = (-2)^4 = 16$ (c) $(a^8)^0 = 1$ for all $a \neq 0$
(d) $(-1)^{2n} = [(-1)^2]^n = 1^n = 1$

9. (a) $150 \cdot 0.13 = 19.5$ (b) $2400 \cdot 0.06 = 144$ (c) $200 \cdot 0.055 = 11$

10. (a) With an interest rate of 11% per year, then in eight years, an initial investment of 50 dollars will be worth $50 \cdot (1.11)^8 \approx 115.23$ dollars. (b) Given a constant interest rate of 12% per year, then in 20 years, an initial investment of 10 000 euros will be worth $10\,000 \cdot (1.12)^{20} \approx 96\,462.93$ euros. (c) $5000 \cdot (1.07)^{-10} \approx 2541.75$ pounds is what you should have invested 10 years ago in order to have 5000 pounds today, given the constant interest rate of 7%.

11. \$1.50 cheaper, which is 15% of \$10.

12. (a) $12\,000 \cdot (1.04)^{15} \approx 21\,611.32$ (b) $50\,000 \cdot (1.06)^{-5} \approx 37\,362.91$

13. $p \approx 95.3\%$, since $(1.25)^3 \approx 1.953$.

14. (a) The profit was higher in 2010. ($(1 + 0.2)(1 - 0.17) = 1.2 \cdot 0.83 = 0.996$.)

(b) If the decrease in profits from 2011 to 2012 were $p\%$, then profits in 2010 and 2012 would be equal provided that $1.2 \cdot (1 - p/100) = 1$, or $p = 100(1 - 1/1.2) = 100/6 \approx 16.67$.

2.3

1. (a) 1 (b) 6 (c) -18 (d) -18 (e) $3x + 12$ (f) $45x - 27y$ (g) 3 (h) 0 (i) -1

2. (a) $3a^2 - 5b$ (b) $-2x^2 + 3x + 4y$ (c) t (d) $2r^3 - 6r^2s + 2s^3$

3. (a) $-3n^2 + 6n - 9$ (b) $x^5 + x^2$ (c) $4n^2 - 11n + 6$ (d) $-18a^3b^3 + 30a^3b^2$ (e) $a^3b - ab^3$

(f) $x^3 - 6x^2y + 11xy^2 - 6y^3$ (g) $acx^2 + (ad + bc)x + bd$ (h) $4 - t^4$

(i) $[(u - v)(u + v)]^2 = (u^2 - v^2)^2 = u^4 - 2u^2v^2 + v^4$

4. (a) $2t^3 - 5t^2 + 4t - 1$ (b) 4 (c) $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ (d) $4xy + 4xz$

5. (a) $x^2 + 4xy + 4y^2$ (b) $1/x^2 - 2 + x^2$ (c) $9u^2 - 30uv + 25v^2$ (d) $4z^2 - 25w^2$

6. (a) $201^2 - 199^2 = (201 + 199)(201 - 199) = 400 \cdot 2 = 800$

(b) If $u^2 - 4u + 4 = (u - 2)^2 = 1$ then $u - 2 = \pm 1$, so $u = 1$ or $u = 3$.

(c) $\dfrac{(a+1)^2 - (a-1)^2}{(b+1)^2 - (b-1)^2} = \dfrac{a^2 + 2a + 1 - (a^2 - 2a + 1)}{b^2 + 2b + 1 - (b^2 - 2b + 1)} = \dfrac{4a}{4b} = \dfrac{a}{b}$

7. $1000^2/(252^2 - 248^2) = 1000^2/(252 + 248)(252 - 248) = 1000^2/(500 \cdot 4) = 500$

8. (a) $(a + b)^3 = (a + b)^2(a + b) = (a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$

(b) $(a - b)^3 = (a - b)^2(a - b) = (a^2 - 2ab + b^2)(a - b) = a^3 - 3a^2b + 3ab^2 - b^3$

(c) and (d): Expand the right-hand sides.

9. (a) $3 \cdot 7 \cdot xxyyy$ (b) $3(x-3y+9z)$ (c) $aa(a-b)$ (d) $2 \cdot 2 \cdot 2xy(xy-2)$ (e) $2 \cdot 2 \cdot 7aabbb$ (f) $2 \cdot 2\,(x+2y-6z)$ (g) $2x(x-3y)$ (h) $2aabb(3a+2b)$ (i) $7x(x-7y)$ (j) $5xyy(1-3x)(1+3x)$ (k) $(4+b)(4-b)$ (l) $3(x+2)(x-2)$

10. (a) $(a+2b)(a+2b)$ (b) $KL(K-L)$ (c) $K^{-5}(K-L)$ (d) $(3z-4w)(3z+4w)$ (e) $-\frac{1}{5}(x-5y)(x-5y)$ (f) $(a^2-b^2)(a^2+b^2)=(a+b)(a-b)(a^2+b^2)$

11. (a) $(x-2)(x-2)$ (b) $2 \cdot 2ts(t-2s)$ (c) $2 \cdot 2(2a+b)(2a+b)$ (d) $5x(x+\sqrt{2}y)(x-\sqrt{2}y)$ (e) $(5+a)(x+y)$ (f) $u^2-v^2+3(u+v)=(u+v)(u-v)+3(u+v)=(u+v)(u-v+3)$ (g) $(P+Q)(P^2+Q^2)$ (h) $KK(K-L)$ (i) $KL(L^2+1)$ (j) $(L+K)(L-K)$ (k) $(K-L)(K-L)$ (l) $KL(K-2L)(K-2L)$

2.4

1. (a) 2/7 (b) 13/12 (c) 5/24 (d) 2/25 (e) 9/5 (Recall that the mixed numbers $3\frac{3}{5}$ and $1\frac{4}{5}$ equal $3+\frac{3}{5}$ and $1+\frac{4}{5}$, respectively.) (f) 1/2 (g) 1/2 (h) 11/27

2. (a) $3x/2$ (b) $3a/5$ (c) $1/5$ (d) $\frac{1}{12}(-5x+11)$ (e) $-1/(6b)$ (f) $1/b$

3. (a) $\dfrac{5 \cdot 5 \cdot 13}{5 \cdot 5 \cdot 5 \cdot 5}=\dfrac{13}{25}$ (b) $\dfrac{ab^2}{8c^2}$ (c) $\dfrac{2}{3}(a-b)$ (d) $\dfrac{P(P+Q)(P-Q)}{(P+Q)^2}=\dfrac{P(P-Q)}{P+Q}$

4. (a) 1/2 (b) 6 (c) 5/7 (d) 9/2

5. (a) $\dfrac{4}{x^2-4}$ (b) $\dfrac{21}{2(2x+1)}$ (c) $\dfrac{a}{a-3b}$ (d) $\dfrac{1}{4ab(a+2)}$ (e) $\dfrac{-3t^2}{t+2}$ (f) $4(1-a)$

6. (a) $\dfrac{2-3x^2}{x(x+1)}$ (b) $\dfrac{-2t}{4t^2-1}$ (c) $\dfrac{7x^2+1}{x^2-4}$ (d) $x+y$ (e) $\dfrac{y^2-x^2}{y^2+x^2}$ (f) $\dfrac{y-x}{y+x}$

7. $\dfrac{-8x}{x^2+2xy-3y^2}$

8. (a) 400 (b) $\dfrac{-n}{n-1}$ (c) 1 (d) $\dfrac{1}{(x-1)^2}$ (e) $\dfrac{-2x-h}{x^2(x+h)^2}$ (f) $\dfrac{2x}{x-1}$

2.5

1. (a) 3 (b) 40 (c) 10 (d) 5 (e) 1/6 (f) 0.7 (g) 0.1 (h) 1/5

2. (a) =. (Both expressions are equal to 20.) (b) $\neq$. In fact, $\sqrt{25+16}=\sqrt{41}\neq 9=\sqrt{25}+\sqrt{16}$. (c) $\neq$. (Put $a=b=1$.) (d) =. In fact, $(\sqrt{a+b}\,)^{-1}=[(a+b)^{1/2}]^{-1}=(a+b)^{-1/2}$.

3. (a) 81 (b) 4 (c) 623 (d) 15 (e) -1 (f) $2^x-2^{x-1}=2^{x-1}(2-1)=2^{x-1}=4$ for $x=3$.

4. (a) $\frac{6}{7}\sqrt{7}$ (b) 4 (c) $\frac{1}{8}\sqrt{6}$ (d) 1 (e) $\frac{1}{6}\sqrt{6}$ (f) $\dfrac{2\sqrt{2y}}{y}$ (g) $\dfrac{\sqrt{2x}}{2}$ (h) $x+\sqrt{x}$

5. (a) $\frac{1}{2}\big(\sqrt{7}-\sqrt{5}\big)$ (b) $4-\sqrt{15}$ (c) $-x\big(\sqrt{3}+2\big)$ (d) $\dfrac{\big(\sqrt{x}-\sqrt{y}\big)^2}{x-y}$ (e) $\sqrt{x+h}+\sqrt{x}$ (f) $\dfrac{1}{x}\big(2\sqrt{x+1}-x-2\big)$

6. (a) $\sqrt[3]{125}=5$ because $5^3=125$. (b) $(243)^{1/5}=3$ because $3^5=243$. (c) -2 (d) $\sqrt[3]{0.008}=0.2$ because $(0.2)^3=0.008$. (e) 9 (f) $(64)^{-1/3}=(4^3)^{-1/3}=4^{-1}=1/4$ (g) $(16)^{-2.25}=(2^4)^{-9/4}=2^{-9}=1/512$ (h) $\left(\frac{1}{3^{-2}}\right)^{-2}=(3^2)^{-2}=3^{-4}=1/81$

7. (a) $\sqrt[3]{55} \approx 3.80295$ (b) $(160)^{1/4} \approx 3.55656$ (c) $(2.71828)^{1/5} \approx 1.22140$ (d) $(1.0001)^{10\,000} \approx 2.718146$

8. $40(1 + p/100)^{12} = 60$ gives $(1 + p/100)^{12} = 1.5$, and therefore $1 + p/100 = (1.5)^{1/12}$.
Solving this for p yields $p = 100[(1.5)^{1/12} - 1] \approx 3.44$.

9. (a) $3x^p y^{2q} z^{4r}$ (b) $(x + 15)^{4/3-5/6} = (x + 15)^{1/2} = \sqrt{x + 15}$ (c) $\dfrac{8x^{2/3}y^{1/4}z^{-1/2}}{-2x^{1/3}y^{5/2}z^{1/2}} = -4x^{1/3}y^{-9/4}z^{-1}$

10. (a) $a^{\frac{1}{2}\frac{2}{3}\frac{3}{4}\frac{4}{5}} = a^{1/5}$ (b) $a^{\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}} = a^{163/60}$ (c) $9a^7/2$ (d) $a^{1/4}$

11. Only (b) and (c) are generally valid.

12. $x < 4$. (If $x > 0$, then $32x^{3/2} > 4x^3$ if and only if $8x^{3/2} > x^3$, which is equivalent to $8 > x^{3/2}$, and so $x < 8^{2/3} = 4$.)

2.6

1. (a), (b), (d), (f), and (h) are valid; (c), (e), and (g) are not valid.

2. (a) $x \geq -8$ (b) $x < -9$ (c) All x. (d) $x \leq 25/2$ (e) $x \leq 19/7$ (f) $x > -17/12$

3. (a) $-7 < x < -2$ (b) $n \geq 160$ or $n < 0$ (c) $0 \leq g \leq 2$ (d) $p \geq -1$ and $p \neq 2$
(e) $-4 < n < -10/3$ (f) $-1 < x < 0$ or $0 < x < 1$. (*Hint:* $x^4 - x^2 = x^2(x + 1)(x - 1)$.)

4. (a) $-2 < x < 1$ (b) $x < -4$ or $x > 3$ (c) $-5 \leq a \leq 5$ (d) $x < -4$ or $x > 1$ (e) $x > -4$ and $x \neq 1$ (f) $1 \leq x \leq 2$
(g) $x < 1$ and $x \neq 1/5$ (h) $1/5 < x < 1$ (i) $x < 0$ (j) $-3 < x < -2$ or $x > 0$ (k) $x \neq 2$ (l) $x \leq 0$

5. (a) $-41/6 < x \leq 2/3$ (b) $x < -1/5$ (c) $-1 < x < 0$

6. (a) $x(x + 3) < 0$ for x in $(-3, 0)$, so $\Rightarrow$ (b) $x^2 < 9$ for x in $(-3, 3)$, so $\Rightarrow$ (c) $\Leftarrow$ (d) $y^2 \geq 0$, so $\Rightarrow$

7. (a) Yes (b) No, put $x = \frac{1}{2}$, for example. (c) No, not for $x \leq 0$.
(d) Yes, because the inequality is equivalent to $x^2 - 2xy + y^2 \geq 0$, or $(x - y)^2 \geq 0$, which *is* satisfied for all x and y.

8. (a) We have $C = \frac{5}{9}(F - 32)$, so we must solve $4 \leq \frac{5}{9}(F - 32) \leq 6$ for F. The result is $39.2° \leq F \leq 42.8°$.
(b) Between 2.2°C and 4.4°C, approximately.

9. $\left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{ab} + b \geq 0$ yields $a + b \geq 2\sqrt{ab}$; dividing by 2 gives $m_A \geq m_G$. Because $\left(\sqrt{a} - \sqrt{b}\right)^2 = 0$ is equivalent to $a = b$, one also has $m_A > m_G$ unless $a = b$. The inequality $m_G \geq m_H$ follows easily from the hint.

2.7

1. (a) $|2 \cdot 0 - 3| = 3$, $|2 \cdot \frac{1}{2} - 3| = 2$, $|2 \cdot \frac{7}{2} - 3| = 4$ (b) $|2x - 3| = 0 \Leftrightarrow 2x - 3 = 0$, so $x = 3/2$.
(c) $|2x - 3| = 2x - 3$ for $x \geq 3/2$, and $3 - 2x$ for $x < 3/2$

2. (a) $|5 - 3(-1)| = 8$, $|5 - 3 \cdot 2| = 1$, $|5 - 3 \cdot 4| = 7$ (b) $|5 - 3x| = 5 \leftrightarrow 5 - 3x = \pm 5$, so $x = 0$ or $10/3$.
(c) $|5 - 3x| = 5 - 3x$ for $x \leq 5/3$, and $3x - 5$ for $x > 5/3$

3. (a) $x = -1$ and $x = 4$ (b) $-2 \leq x \leq 2$ (c) $1 \leq x \leq 3$ (d) $-1/4 \leq x \leq 1$ (e) $x > \sqrt{2}$ or $x < -\sqrt{2}$
(f) $-1 \leq x^2 - 2 \leq 1$, so $1 \leq x^2 \leq 3$, implying that $-\sqrt{3} \leq x \leq -1$ or $1 \leq x \leq \sqrt{3}$.

4. (a) $4.999 < x < 5.001$ (b) $|x - 5| < 0.001$

2.8

1. (a) $1+2+3+\cdots+10=55$ (b) $(5\cdot 3^0-2)+(5\cdot 3^1-3)+(5\cdot 3^2-4)+(5\cdot 3^3-5)+(5\cdot 3^4-6)=585$
(c) $1+3+5+7+9+11=36$ (d) $2^{2^0}+2^{2^1}+2^{2^2}=2^1+2^2+2^4=22$ (e) $10\cdot 2=20$
(f) $2/1+3/2+4/3+5/4=73/12$

2. (a) $2\sqrt{0}+2\sqrt{1}+2\sqrt{2}+2\sqrt{3}+2\sqrt{4}=2(3+\sqrt{2}+\sqrt{3})$
(b) $(x+0)^2+(x+2)^2+(x+4)^2+(x+6)^2=4(x^2+6x+14)$
(c) $a_{1i}b^2+a_{2i}b^3+a_{3i}b^4+\cdots+a_{ni}b^{n+1}$ (d) $f(x_0)\Delta x_0+f(x_1)\Delta x_1+f(x_2)\Delta x_2+\cdots+f(x_m)\Delta x_m$

3. (a) $\sum_{k=1}^{n}4k$ (b) $\sum_{k=1}^{n}k^3$ (c) $\sum_{k=0}^{n}(-1)^k\frac{1}{2k+1}$ (d) $\sum_{k=1}^{n}a_{ik}b_{kj}$ (e) $\sum_{n=1}^{5}3^nx^n$ (f) $\sum_{j=3}^{p}a_i^jb_{i+j}$ (g) $\sum_{k=0}^{p}a_{i+k}^{k+3}b_{i+k+3}$
(h) $\sum_{k=0}^{3}(81\,297+198k)$

4. $\dfrac{2\cdot 3+3\cdot 5+4\cdot 7}{1\cdot 3+2\cdot 5+3\cdot 7}\cdot 100=\dfrac{6+15+28}{3+10+21}\cdot 100=\dfrac{49}{34}\cdot 100\approx 144.12$

5. (a) $\sum_{k=1}^{10}(k-2)t^k=\sum_{m=-1}^{8}mt^{m+2}$ (b) $\sum_{n=0}^{N}2^{n+5}=\sum_{j=1}^{N+1}32\cdot 2^{j-1}$ (because $32=2^5$)

6. (a) The total number of people moving from nation i within the EEA.
(b) The total number of people moving to nation j within the EEA.

7. (a), (c), (d), and (e) are always true; (b) and (f) are generally not true.

2.9

1. We prove only (2.9.6); the proof of (2.9.5) is very similar, but slightly easier. Note that the last equality in (2.9.6) follows immediately from (2.9.4), so we will concentrate on proving the equality

$$1^3+2^3+3^3+\cdots+n^3=\left[\tfrac{1}{2}n(n+1)\right]^2 \tag{$*$}$$

For $n=1$ the LHS and the RHS of $(*)$ are both equal to 1. As the induction hypothesis, suppose $(*)$ is true for $n=k$. Then $\sum_{i=1}^{k+1}i^3=\sum_{i=1}^{k}i^3+(k+1)^3=[\frac{1}{2}k(k+1)]^2+(k+1)^3=(k+1)^2(\frac{1}{4}k^2+k+1)$. But this last expression is equal to $\frac{1}{4}(k+1)^2(k^2+4k+4)=[\frac{1}{2}(k+1)(k+2)]^2$, which proves that $(*)$ is true for $n=k+1$. By induction, we have proved $(*)$.

2. $\sum_{k=1}^{n}(k^2+3k+2)=\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}2=\frac{1}{6}n\,(n+1)(2n+1)+3\left[\frac{1}{2}n\,(n+1)\right]+2n=\frac{1}{3}n\,(n^2+6n+11)$.

3. $\sum_{i=0}^{n-1}(a+id)=\sum_{i=0}^{n-1}a+d\sum_{i=0}^{n-1}i=na+d\frac{1}{2}[1+(n-1)](n-1)=na+\frac{1}{2}n(n-1)d$. Using this formula, the sum that Gauss is alleged to have computed is: $100\cdot 81\,297+\frac{1}{2}100\cdot 99\cdot 198=9\,109\,800$. (One does not have to use summation signs. The sum is $a+(a+d)+(a+2d)+\cdots+(a+(n-1)d)$. There are n terms. The sum of all the a's is na. The rest is $d(1+2+\cdots+(n-1))$. Then use formula (2.9.4).)

4. (a) Writing the sum as $(a_2-a_1)+(a_3-a_2)+(a_4-a_3)+\cdots+(a_n-a_{n-1})+(a_{n+1}-a_n)$ we see that all the a_i cancel pairwise, except $-a_1$ and a_{n+1}. Actually, this is more striking if we write the sum starting with the last term and working backwards to the first: $(a_{n+1}-a_n)+(a_n-a_{n-1})+\cdots+(a_4-a_3)+(a_3-a_2)+(a_2-a_1)=a_{n+1}-a_1$.
(b) (i) $1-(1/51)=50/51$ (ii) $3^{13}-3=1\,594\,320$ (iii) $ar(r^n-1)$

2.10

1. $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. (The coefficients are those in row 6 of Pascal's triangle in the text, still calling the top row number 0.)

2. (a) $\binom{5}{3} = \dfrac{5\cdot4\cdot3}{1\cdot2\cdot3} = \dfrac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot2\cdot1} = \dfrac{5!}{3!\,2!} = \dfrac{5!}{2!\,3!}$

In general, $\binom{m}{k} = \dfrac{m(m-1)\cdots(m-k+1)}{k!} = \dfrac{m(m-1)\cdots(m-k+1)\cdot(m-k)!}{k!(m-k)!} = \dfrac{m!}{(m-k)!\,k!}$.

(b) $\binom{8}{3} = 56$. Also, $\binom{8}{8-3} = \binom{8}{5} = 56$; $\binom{8}{3} + \binom{8}{3+1} = 56 + 70 = 126$; and $\binom{8+1}{3+1} = \binom{9}{4} = 126$.

(c) $\binom{m}{k} = \dfrac{m!}{(m-k)!\,k!} = \binom{m}{m-k}$ and $\binom{m}{k} + \binom{m}{k+1} = \dfrac{m!}{(m-k)!\,k!} + \dfrac{m!}{(m-k-1)!\,(k+1)!}$

The last expression reduces to $\dfrac{m!\,(k+1+m-k)}{(m-k)!\,(k+1)!} = \dfrac{(m+1)!}{(m-k)!\,(k+1)!} = \binom{m+1}{k+1}$.

2.11

1. (a) $\sum_{i=1}^{3}\sum_{j=1}^{4} i\cdot 3^j = \sum_{i=1}^{3}(i\cdot3 + i\cdot9 + i\cdot27 + i\cdot81) = \sum_{i=1}^{3} 120i = 720$ (b) $5 + \dfrac{3113}{3600}$ (c) $\frac{1}{6}mn(2n^2 + 3n + 3m + 4)$

(d) $\frac{1}{3}m(m+1)(m+2)$

2. (a) The total number of units of good i. (b) The total number of units of all goods owned by person j.

(c) The total number of units of goods owned by the group as a whole.

3. First, $\sum_{j=1}^{i} a_{ij}$ is the sum of all the i numbers in the i-th row, so in the first double sum we sum all these m row sums. Second, $\sum_{i=j}^{m} a_{ij}$ is the sum of all the $m-j+1$ numbers in the j-th column, so in the second double sum we sum all these m column sums.

4. The mean of the n column means is $\frac{1}{n}\sum_{j=1}^{n}\bar{a}_j = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{m}\sum_{i=1}^{m}a_{ij} = \bar{a}$. Also

$$\sum_{r=1}^{m}\sum_{s=1}^{m}(a_{rj}-\bar{a})(a_{sj}-\bar{a}) = \sum_{r=1}^{m}(a_{rj}-\bar{a})\sum_{s=1}^{m}(a_{sj}-\bar{a}) = [m(\bar{a}_j-\bar{a})][m(\bar{a}_j-\bar{a})] = m^2(\bar{a}_j-\bar{a})^2$$

See SM.

Review exercises for Chapter 2

1. (a) $3(50-x)$ (b) $\dfrac{x}{y+100}$ (c) If the price before VAT is p, then the price after VAT is $p + 20p/100 = p(1+0.2) = 1.2p$. Thus $a = 1.2p$, so $p = \dfrac{a}{1.2}$. (d) $p_1x_1 + p_2x_2 + p_3x_3$ (e) $F + bx$ (f) $(F+cx)/x = F/x + c$

(g) After the $p\%$ raise, his salary is $L + pL/100 = L(1+p/100)$. A $q\%$ raise of this new salary gives the final answer: $L(1+p/100)(1+q/100)$.

2. (a) $5^3 = 5\cdot5\cdot5 = 125$ (b) $10^{-3} = 1/10^3 = 1/1000 = 0.0001$ (c) $1/3^{-3} = 3^3 = 27$ (d) -1000 (e) 3

(f) $(3^{-2})^{-3} = 3^6 = 729$ (g) -1 (h) $\left(-\frac{1}{2}\right)^{-3} = \frac{1}{(-\frac{1}{2})^3} = \frac{1}{-\frac{1}{8}} = -8$

3. (a) 1 (b) Undefined. (c) 1 (d) 1

4. (a) $2^{-6} = 1/64$ (b) $\frac{3}{2} - \frac{3}{4} = \frac{3}{4}$ (c) $-45/4$ (d) 1

5. (a) $16x^4$ (b) 4 (c) $6xyz$ (d) $a^{27}b^9$ (e) a^3 (f) x^{-15}

6. (a) $x^3y^3 = (x^{-1}y^{-1})^{-3} = 3^{-3} = 1/27$ (b) $(x^{-3})^6(x^2)^2 = x^{-18}x^4 = x^{-14} = (x^7)^{-2} = 2^{-2} = 1/4$
(c) $(z/xy)^6 = (xy/z)^{-6} = [(xy/z)^{-2}]^3 = 3^3 = 27$ (d) $(abc)^4 = (a^{-1}b^{-1}c^{-1})^{-4} = (1/4)^{-4} = 4^4 = 256$

7. (a) $0.12 \cdot 300 = 36$ (b) $0.05 \cdot 2000 = 100$ (c) $0.065 \cdot 1500 = 97.5$

8. (a) Given an interest rate of 1% per year, then in eight years, an investment of 100 million euros will grow to $100 \cdot (1.01)^8 \approx 108.3$ million euros. (b) Given an interest rate of 15% per year, then in 10 years, an initial investment of 50 000 pounds will be worth $50\,000 \cdot (1.15)^{10} \approx 202\,278$ pounds. (c) $6000 \cdot (1.03)^{-8} \approx 4736$ dollars is what you should have deposited eight years ago in order to have 6000 dollars today, given the constant interest rate of 3%.

9. (a) $100\,000(1.08)^{10} \approx 215\,892$ (b) $25\,000(1.08)^{-6} \approx 15\,754$

10. (a) $a^2 - a$ (b) $x^2 + 4x - 21$ (c) $-3 + 3\sqrt{2}$ (d) $3 - 2\sqrt{2}$ (e) $x^3 - 3x^2 + 3x - 1$ (f) $1 - b^4$ (g) $1 - x^4$
(h) $x^4 + 4x^3 + 6x^2 + 4x + 1$

11. (a) $5(5x - 1)$ (b) $xx(3 - xy)$ (c) $(\sqrt{50} - x)(\sqrt{50} + x)$ (d) $a(a - 2b)^2$

12. (a) $(5 + a)(x + 2y)$ (b) $(a + b)(c - d)$ (c) $(a + 2)(x + y)$ (d) $(2x - y)(x + 5z)$ (e) $(p - q)(p + q + 1)$
(f) $(u - v)(u - v)(u + v)$

13. (a) $16^{1/4} = \sqrt[4]{16} = 2$ (b) $243^{-1/5} = (3^5)^{-1/5} = 3^{-1} = 1/3$ (c) $5^{1/7} \cdot 5^{6/7} = 5^{1/7+6/7} = 5^1 = 5$
(d) $4^{-3/2} = 1/8$ (e) $64^{1/3} + \sqrt[3]{125} = 4 + 5 = 9$ (f) $(-8/27)^{2/3} = (\sqrt[3]{-8/27}\,)^2 = (-2/3)^2 = 4/9$
(g) $(-1/8)^{-2/3} + (1/27)^{-2/3} = (\sqrt[3]{-1/8}\,)^{-2} + (\sqrt[3]{1/27}\,)^{-2} = (-1/2)^{-2} + (1/3)^{-2} = 4 + 9 = 13$
(h) $\dfrac{1000^{-2/3}}{\sqrt[3]{5^{-3}}} = \dfrac{(\sqrt[3]{1000}\,)^{-2}}{5^{-1}} = \dfrac{10^{-2}}{5^{-1}} = \dfrac{1}{20}$

14. (a) $8 = 2^3$, so $x = 3/2$. (b) $1/81 = 3^{-4}$, so $3x + 1 = -4$ or $x = -5/3$. (c) $x^2 - 2x + 2 = 2$, so $x = 0$ or $x = 2$.

15. (a) $5 + x = 3$, so $x = -2$. (b) $3^x - 3^{x-2} = 3^{x-2}(3^2 - 1) = 3^{x-2} \cdot 8$, so $3^{x-2} = 3$, and thus $x = 3$.
(c) $3^x \cdot 3^{x-1} = 3^{2x-1} = 81 = 3^4$ provided $x = 5/2$. (d) $3^5 + 3^5 + 3^5 = 3 \cdot 3^5 = 3^6$, so $x = 6$.
(e) $4^{-6} + 4^{-6} + 4^{-6} + 4^{-6} = 4 \cdot 4^{-6} = 4^{-5}$, so $x = -5$. (f) $\dfrac{2^{26} - 2^{23}}{2^{26} + 2^{23}} = \dfrac{2^{23}(2^3 - 1)}{2^{23}(2^3 + 1)} = \dfrac{7}{9}$, so $x = 7$.

16. (a) $\dfrac{2s}{4s^2 - 1}$ (b) $\dfrac{7}{3 - x}$ (c) $\dfrac{1}{x + y}$

17. (a) $\frac{1}{5}a^2b$ (b) $x - y$ (c) $\dfrac{2a - 3b}{2a + 3b}$ (d) $\dfrac{x(x + 2)}{2 - x}$

18. (a) $x < 13/2$ (b) $y \ge -3$ (c) Valid for all x. (d) $x < 29/14$ (e) $-1 \le x \le 13/3$
(f) $-\sqrt{6} \le x \le -\sqrt{2}$ or $\sqrt{2} \le x \le \sqrt{6}$

19. (a) $30 + 0.16x$ (b) Smallest number of hours: 7.5. Largest number of hours: 10.

20. $2\pi(r + 1) - 2\pi r = 2\pi$, where r is the radius of the Earth (as an approximate sphere). So the extended rope is only about 6.28 m longer!

21. (a) Put $p/100 = r$. Then the given expression becomes $a + ar - (a + ar)r = a(1 - r^2)$, as required.
(b) $\$2000 \cdot 1.05 \cdot 0.95 = \$1995$. (c) The result is precisely the formula in (a).

(d) With the notation used in the answer to (a), we have $a - ar + (a - ar)r = a(1 - r^2)$, which is the same expression as in (a).

22. (a) No. For example, $-1 > -2$, but $(-1)^2 < (-2)^2$.

(b) Suppose $a > b$ so that $a - b > 0$. If also $a + b > 0$, then $a^2 - b^2 = (a + b)(a - b) > 0$, so $a^2 > b^2$.

23. (a) $2 > 1$ and $1/2 < 1/1$. Also, $-1 > -2$ and $1/(-1) < -1/2$. On the other hand, $2 > -1$ and $1/2 > 1/(-1)$.

(b) If $ab > 0$ and $a > b$, then $1/b - 1/a = (a - b)/ab > 0$, so $1/b > 1/a$.

(Also, if $ab < 0$ and $a > b$, then $1/b - 1/a = (a - b)/ab < 0$, so $1/b < 1/a$.)

24. (a) For any number c, one has $|c| = \sqrt{c^2}$. Then $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a| \cdot |b|$.

(b) Either $a = |a|$ or $a = -|a|$, so $-|a| \le a \le |a|$. Likewise, $-|b| \le b \le |b|$. Adding these inequalities yields $-|a| - |b| \le a + b \le |a| + |b|$, and thus $|a + b| \le |a| + |b|$.

25. Let s denote the length of each side of the equilateral triangle. Then the total area A of the triangle is the sum of the areas of three triangles with base s and heights h_1, h_2, and h_3 respectively. So $A = \frac{1}{2}sh_1 + \frac{1}{2}sh_2 + \frac{1}{2}sh_3$. It follows that $h_1 + h_2 + h_3 = 2A/s$, independent of P. See SM for a figure. (For the curious: $A = \frac{1}{4}\sqrt{3}s^2$, so $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}s$.)

26. (a) $\sum_{i=1}^{4} \dfrac{1}{i(i+2)} = \dfrac{1}{1 \cdot 3} + \dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{4 \cdot 6} = \dfrac{1}{3} + \dfrac{1}{8} + \dfrac{1}{15} + \dfrac{1}{24} = \dfrac{40 + 15 + 8 + 5}{120} = \dfrac{68}{120} = \dfrac{17}{30}$

(b) $\sum_{j=5}^{9}(2j - 8)^2 = 2^2 + 4^2 + 6^2 + 8^2 + 10^2 = 4 + 16 + 36 + 64 + 100 = 220$

(c) $\sum_{k=1}^{5} \dfrac{k-1}{k+1} = \displaystyle\sum_{k=1}^{5}\left(1 - \frac{2}{k+1}\right) = 5 - \frac{2}{2} - \frac{2}{3} - \frac{2}{4} - \frac{2}{5} - \frac{2}{6} = \frac{21}{10}$

(d) $\sum_{n=2}^{5}(n - 1)^2(n + 2) = 1^2 \cdot 4 + 2^2 \cdot 5 + 3^2 \cdot 6 + 4^2 \cdot 7 = 4 + 20 + 54 + 112 = 190$

(e) $\sum_{k=1}^{5}\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right) = \dfrac{1}{1} - \dfrac{1}{6} = \dfrac{5}{6}$

(f) $\sum_{i=-2}^{3}(i + 3)^i = 1^{-2} + 2^{-1} + 3^0 + 4^1 + 5^2 + 6^3 = 1 + \frac{1}{2} + 1 + 4 + 25 + 216 = 247\frac{1}{2}$

27. (a) $3 + 5 + 7 + \cdots + 199 + 201 = \sum_{i=1}^{100}(1 + 2i)$ (b) $\dfrac{2}{1} + \dfrac{3}{2} + \dfrac{4}{3} + \cdots + \dfrac{97}{96} = \sum_{i=1}^{96} \dfrac{1+i}{i}$

(c) $4 \cdot 6 + 5 \cdot 7 + 6 \cdot 8 + \cdots + 38 \cdot 40 = \sum_{i=4}^{38} i(i + 2)$ (d) $\dfrac{1}{x} + \dfrac{1}{x^2} + \cdots + \dfrac{1}{x^n} = \sum_{i=1}^{n} x^{-i}$

(e) $1 + \dfrac{x^2}{3} + \dfrac{x^4}{5} + \dfrac{x^6}{7} + \cdots + \dfrac{x^{32}}{33} = \sum_{i=0}^{16} \dfrac{x^{2i}}{1+2i}$ (f) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots - \dfrac{1}{80} + \dfrac{1}{81} = \sum_{i=1}^{81}(-1)^{i-1}\dfrac{1}{i}$

28. (a) and (c) are right. (b) is wrong unless the difference between the left- and right-hand sides, which is $2\sum_{i=1}^{n} a_i b_i$, happens to be zero. (d) is also wrong.

29. (a) $3 + 5 + 7 + \cdots + 197 + 199 + 201 = \sum_{i=1}^{100}(1 + 2i) = 100 + 2\sum_{i=1}^{100} i = 100 + 100 \cdot 101 = 10\,200$

(b) $1001 + 2002 + 3003 + \cdots + 8008 + 9009 + 10\,010 = 1001\sum_{i=1}^{10} i = 1001 \cdot \frac{1}{2} \cdot 10 \cdot 11 = 55\,055$

Chapter 3

3.1

1. (a) $x = 3$ (b) $x = 6$ (c) Any x is a solution. (d) $x = 1$ (e) $x = -5$. (*Hint:* $x^2 + 10x + 25 = (x + 5)^2$.)

(f) $x = -1$

2. (a) $x = -28/11$ (b) $x = 5/11$ (c) $x = 1$ (d) $x = 121$

3. (a) $x = 0$ (b) $x = -6$ (c) $x = 5$

4. (a) With x as the smallest number, one has $x + (x + 1) + (x + 2) = 10 + 2x$, so $x = 7$. The numbers are 7, 8, and 9.

(b) If x is Jane's regular hourly wage, then $38x + (48 - 38)2x = 812$. Solution: $x = 812/58 = 14$.

(c) $1500 + 12x/100 = 2100$, so $12x = 60\,000$, implying that $x = 5000$.

(d) $\frac{2}{3}x + \frac{1}{4}x + 100\,000 = x$. Solution: $x = 1\,200\,000$.

5. (a) $y = 17/23$ (b) $x = -4$ (c) $z = 4$ (d) $p = 15/16$

6. She spends $y/3$ euros on each kind of fruit. So she buys $y/9$ kilos of apples, $y/6$ kilos of bananas, and $y/18$ kilos of cherries. The total weight is $\left(\frac{1}{9} + \frac{1}{6} + \frac{1}{18}\right)y = \left(\frac{2+3+1}{18}\right)y = \frac{6}{18}y = \frac{1}{3}y$ kilos. She pays 3 euros per kilo of fruit.

3.2

1. Substituting the second equation into the first gives $Y = 750 + 0.9Y$, whose solution is $Y = 7500$. Alternatively, formula (∗∗) gives $Y = \dfrac{a}{1-b} + \dfrac{1}{1-b}\bar{I} = \dfrac{600}{1-0.9} + \dfrac{150}{1-0.9} = \dfrac{750}{1-0.9} = 7500$.

2. (a) $x = \frac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ (b) $x = \dfrac{dA - b}{a - cA}$ (c) $x = \dfrac{p^2}{4w^2}$ (d) $x = -\dfrac{1}{1+a}$ (e) $x = \pm\dfrac{b}{a}$ (f) $x = 0$

3. (a) $p = 20q/3 - 14/15$ (b) $P = (S - \alpha)/\beta$ (c) $g = 2A/h$ (d) $r = (3V/4\pi)^{1/3}$ (e) $L = (Y_0 A^{-1} K^{-\alpha})^{1/\beta}$

4. (a) $x = (a - b)/(\alpha - \beta)$ (b) $p = (3q + 5)^2/q$ (c) $Y = 100$ (d) $K = (2wQ^4/r)^{1/3}$ (e) $L = rK/2w$

(f) $K = \frac{1}{32}p^4 r^{-3} w^{-1} = p^4/(32r^3 w)$

5. (a) $s = \dfrac{tT}{T - t}$ (b) $M = \dfrac{(B + \alpha L)^2}{KL}$ (c) $z = \dfrac{4xy - x + 2y}{x + 4y}$ (d) $T = N\left(1 - \dfrac{V}{C}\right)$

3.3

1. (a) $x(15 - x) = 0$, so the solutions are $x = 0$ and $x = 15$ (b) $p = \pm 4$ (c) $q = 3$ and $q = -4$ (d) No solution.

(e) $x = 0$ and $x = 3$ (f) $x = 2$. (Note that $x^2 - 4x + 4 = (x - 2)^2$.)

2. (a) $x^2 - 5x + 6 = (x - 2)(x - 3) = 0$ for $x = 2$ and for $x = 3$. (With $x^2 - 5x = -6$, completing the square gives $x^2 - 5x + (5/2)^2 = (5/2)^2 - 6 = 25/4 - 6 = 1/4$, or $(x - 5/2)^2 = 1/4$. Hence, $x - 5/2 = \pm 1/2$.)

(b) $y^2 - y - 12 = (y - 4)(y + 3) = 0$ for $y = 4$ and for $y = -3$. (c) No solutions and no factorization.

(d) $-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2} = -\frac{1}{4}\left[x - \left(1 + \sqrt{3}\right)\right]\left[x - \left(1 - \sqrt{3}\right)\right] = 0$ for $x = 1 \pm \sqrt{3}$

(e) $m^2 - 5m - 3 = \left[m - \frac{1}{2}\left(5 + \sqrt{37}\right)\right]\left[m - \frac{1}{2}\left(5 - \sqrt{37}\right)\right] = 0$ for $m = \frac{1}{2}\left(5 \pm \sqrt{37}\right)$

(f) $0.1p^2 + p - 2.4 = 0.1(p - 2)(p + 12) = 0$ for $p = 2$ and for $p = -12$.

3. (a) $r = -13, r = 2$ (b) $p = -16, p = 1$ (c) $K = 100, K = 200$ (d) $r = -\sqrt{3}, r = \sqrt{2}$

(e) $x = -0.5, x = 0.8$ (f) $p = -1/6, p = 1/4$

4. (a) $x = 1, x = 2$ (b) $t = \frac{1}{10}\left(1 \pm \sqrt{61}\right)$ (c) $x = \frac{1}{4}\left(3 \pm \sqrt{13}\right)$ (d) $x = \frac{1}{3}\left(-7 \pm \sqrt{5}\right)$

(e) $x = -300, x = 100$ (f) $x = \frac{1}{6}\left(5 \pm \sqrt{13}\right)$

5. (a) If the sides have length x and y, then the perimeter has length $2x + 2y = 40$ and the area is $xy = 75$. Because $x + y = 20$ and $xy = 75$, it follows from the Rules for Quadratic Functions that x and y are the roots of the quadratic equation $z^2 - 20z + 75 = 0$, so the sides have length 5 and 15.

(b) If the two numbers are n and $n+1$, then $n^2 + (n+1)^2 = 13$, so $2n^2 + 2n - 12 = 0$. The roots of this equation are $n_1 = 2$, and $n_2 = -3$. But n has to be positive, so the only possibility is $n = 2$, hence the two numbers are 2 and 3. (Of course, with numbers this small it is even easier to use trial and error, starting with the smallest numbers. $1^2 + 2^2 = 5$, which is too little, but $2^2 + 3^2 = 13$, which is just right, and further along the numbers get too big, so the answer is 2 and 3.)

(c) The length x of the shortest side satisfies $x^2 + (x+14)^2 = 34^2$, or $2x^2 + 28x = 1156 - 196 = 960$, or $x^2 + 14x - 480 = 0$. The lengths are 16 cm and 30 cm.

(d) If his usual speed is s, the usual time is $80/s$ hours, or $4800/s$ minutes. Driving at a speed $s+10$, the time is $\dfrac{4800}{s+10} = \dfrac{4800}{s} - 16$. Clearing fractions gives $4800s = (4800 - 16s)(s+10)$ or $16s^2 + 160s - 48\,000 = 0$, implying that $s^2 + 10s - 3000 = 0$. The only positive root of this equation is $s = 50$. Hence the usual speed is 50 km/h.

6. (a) $x = -2,\ x = 0,\ x = 2$. $(x(x^2 - 4) = 0$ or $x(x+2)(x-2) = 0)$

(b) $x = -2,\ x = -1,\ x = 1,\ x = 2$. (Let $x^2 = u$.) (c) $z = -1/3,\ z = 1/5$. (Let $z^{-1} = u$.)

3.4

1. (a) $x = 0$ and $x = -3$ (b) $x = 0$ and $x = 1/2$ (c) $x = 1$ and $x = 3$ (d) $x = -5/2$ (e) No solutions.

(f) $x = 0$ and $x = -1$

2. (a) No solutions. (b) $x = -1$ (c) $x = -3/2$ (d) $x = 0$ and $x = 1/2$

3. (a) $z = 0$ or $z = a/(1 - a - b)$ for $a + b \neq 1$. For $a + b = 1$ the only solution is $z = 0$.

(b) $\lambda = -1$ or $\mu = 0$ or $x = y$ (c) $\lambda = 0$ and $\mu \neq \pm 1$, or $\mu = 2$ (d) $a = 2$ or $b = 0$ or $\lambda = -1$

3.5

1. $x = -1, 0$, and 1 make the equation meaningless. Multiplying each term by the common denominator $x(x-1)(x+1)$, we derive the only solution from the equivalences

$$\frac{(x+1)^2}{x(x-1)} + \frac{(x-1)^2}{x(x+1)} - 2\frac{3x+1}{x^2-1} = 0 \iff \big(x \notin \{-1,0,1\} \text{ and } 2x(x^2 - 3x + 2) = 0\big)$$
$$\iff \big(x \notin \{-1,0,1\} \text{ and } 2x(x-1)(x-2) = 0\big) \iff x = 2$$

2. (a) Squaring both sides and rearranging, $x + 2 = \sqrt{4x+13} \Rightarrow (x+2)^2 = 4x + 13 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. But $x + 2 = \sqrt{4x+13} \Rightarrow x + 2 \geq 0$. So only $x = 3$ is a solution.

(b) Squaring both sides and rearranging yields $x(x+5) = 0$. Both $x = 0$ and $x = -5$ are solutions.

(c) The equivalent equation $|x|^2 - 2|x| - 3 = 0$ gives $|x| = 3$ or $|x| = -1$. Because $|x| \geq 0$, only $x = \pm 3$ are solutions.

3. (a) No solutions. (b) $x = 20$

4. (a) $x + \sqrt{x+4} = 2 \Rightarrow \sqrt{x+4} = 2 - x \Rightarrow x + 4 = 4 - 4x + x^2 \Rightarrow x^2 - 5x = 0 \overset{\text{(i)}}{\Rightarrow} x - 5 = 0 \overset{\text{(ii)}}{\Leftarrow} x = 5$. Here implication (i) is incorrect ($x^2 - 5x = 0 \Rightarrow x - 5 = 0$ or $x = 0$). Implication (ii) is correct, but it breaks the chain of implications.

(b) $x = 0$. (After correcting implication (i), we see that the given equation implies $x = 5$ or $x = 0$. But only $x = 0$ is a solution; $x = 5$ solves $x - \sqrt{x+4} = 2$.)

3.6

1. (a) $x = 8,\ y = 3$ (b) $x = 1/2,\ y = 1/3$ (c) $x = 1.1,\ y = -0.3$

2. (a) $x = 1,\ y = -1$ (b) $x = -4,\ y = 7$ (c) $x = -7/2,\ y = 10/3$

3. (a) $p = 2,\ q = 3$ (b) $r = 2.1,\ s = 0.1$

4. (a) 39 and 13 (b) \$120 for a table and \$60 for a chair. (c) 450 of quality B and 300 of quality P.
(d) \$2 000 at 5% and \$8 000 at 7.2% interest.

Review exercises for Chapter 3

1. (a) $x = 12$ (b) $x = 3$ (c) $x = -3/2$ (d) $x = -19$ (e) $x = 11/7$ (f) $x = 39$

2. (a) $x = 0$ (b) $x = -6$ (c) $x = 5$ (d) $x = -1$

3. (a) $x = \frac{2}{3}(y-3) + y = \frac{2}{3}y - 2 + y = \frac{5}{3}y - 2$, or $\frac{5}{3}y = x + 2$, so $y = \frac{3}{5}(x+2)$.
(b) $ax - cx = b + d$, or $(a-c)x = b + d$, so $x = (b+d)/(a-c)$.
(c) $\sqrt{L} = Y_0/AK$, so squaring each side yields $L = (Y_0/AK)^2$. (d) $qy = m - px$, so $y = (m - px)/q$.
(e) Put $s = 1/(1+r)$. Then $s = (a+bc)/(1-c)$, so $r = (1/s) - 1 = [(1-a) - c(1+b)]/(a+bc)$.
(f) Multiplying by $(Px+Q)^{1/3}$ yields $Px + Px + Q = 0$, and so $x = -Q/2P$.

4. (a) $K = 225L^{2/3}$ (b) $r = 100(2^{1/t} - 1)$ (c) $x_0 = (p/ab)^{1/(b-1)}$ (d) $b = \lambda^{1/\rho}\left(c^{-\rho} - (1-\lambda)a^{-\rho}\right)^{-1/\rho}$

5. (a) $z = 0$ or $z = 8$ (b) $x = -7$ or $x = 5$ (c) $p = -7$ or $p = 2$ (d) $p = 1/4$ or $p = 1/3$ (e) $y = 4 \pm \sqrt{31}$
(f) $x = -7$ or $x = 6$

6. (a) $x = \pm 2$ or $x = 5$ (b) $x = -4$. ($x^4 + 1$ is never 0.) (c) $\lambda = 1$ or $x = y$

7. If he invested \$$x$ at 15% interest and \$$y$ at 20%, then $0.15x + 0.20y = 275$. Also, $x + y = 1500$. Solving this system yields $x = 500$, $y = 1000$.

8. (a) From the second and third equations of the model, one has $C = b(Y - tY) = b(1-t)Y$. Inserting this into the first equation and solving for Y yields $Y = \dfrac{\bar{I} + G}{1 - b(1-t)}$ and then $C = \dfrac{b(1-t)(\bar{I}+G)}{1-b(1-t)}$.
(b) Note that $0 < b(1-t) < 1$. When t increases, both Y and $1 - t$ decrease, and so therefore does $C = b(1-t)Y$.

9. $5^{3x} = 25^{y+2} = 5^{2(y+2)}$ so that $3x = 2(y+2)$. With $x - 2y = 8$ this gives $x = -2$ and $y = -5$, so $x - y = 3$.

10. (a) Let $u = 1/x$ and $v = 1/y$. Then the system reduces to $2u + 3v = 4$, $3u - 2v = 19$, with solution $u = 5$, $v = -2$, and so $x = 1/u = 1/5$, $y = 1/v = -1/2$.
(b) Let $u = \sqrt{x}$ and $v = \sqrt{y}$. Then $3u + 2v = 2$, $2u - 3v = 1/4$, with solution $u = 1/2$, $v = 1/4$, so $x = 1/4$, $y = 1/16$.
(c) With $u = x^2$ and $v = y^2$, we get $u + v = 13$, $4u - 3v = 24$, with solution $u = 9$, $v = 4$, and so $x = \pm 3$ and $y = \pm 2$.

Chapter 4

4.2

1. (a) $f(0) = 1, f(-1) = 2, f(1/2) = 5/4$, and $f(\sqrt{2}) = 3$
(b) (i) For all x. (ii) When $x = 1/2$. (iii) When $x = \pm\sqrt{1/2} = \pm\frac{1}{2}\sqrt{2}$.

2. $F(0) = F(-3) = 10$, $F(a+h) - F(a) = 10 - 10 = 0$

3. (a) $f(0) = 0, f(a) = a^2, f(-a) = a^2 - (-a-a)^2 = -3a^2$, and $f(2a) = 0$

(b) $3f(a) + f(-2a) = 3a^2 + [a^2 - (-2a-a)^2] = 3a^2 + a^2 - 9a^2 = -5a^2$

4. (a) $f(-1/10) = -10/101, f(0) = 0, f(1/\sqrt{2}) = \sqrt{2}/3, f(\sqrt{\pi}) = \sqrt{\pi}/(1+\pi), f(2) = 2/5$

(b) $f(-x) = -x/(1+(-x)^2) = -x/(1+x^2) = -f(x)$

and $f(1/x) = (1/x)/[1+(1/x)^2] = (1/x)\cdot x^2/[1+(1/x)^2]\cdot x^2 = x/(1+x^2) = f(x)$

5. $F(0) = 2$, $F(-3) = \sqrt{19}$, $F(t+1) = \sqrt{t^2+3}$

6. (a) $C(0) = 1000$, $C(100) = 41\,000$, and $C(101) - C(100) = 501$

(b) $C(x+1) - C(x) = 2x + 301$ = incremental cost of increasing production from x to $x+1$.

7. (a) $D(8) = 4$, $D(10) = 3.4$, and $D(10.22) = 3.334$ (b) $P = 10.9$

8. (a) $f(tx) = 100(tx)^2 = 100t^2x^2 = t^2 f(x)$ (b) $P(tx) = (tx)^{1/2} = t^{1/2}x^{1/2} = t^{1/2}P(x)$

9. (a) $b(0) = 0$, $b(50) = 100/11$, $b(100) = 200$ (b) $b(50+h) - b(50)$ is the additional cost of removing h% more than 50% of the impurities.

10. (a) No: $f(2+1) = f(3) = 18$, whereas $f(2) + f(1) = 8 + 2 = 10$. (b) Yes: $f(2+1) = f(2) + f(1) = -9$.

(c) No: $f(2+1) = f(3) = \sqrt{3} \approx 1.73$, whereas $f(2) + f(1) = \sqrt{2} + 1 \approx 2.41$.

11. (a) $f(a+b) = A(a+b) = Aa + Ab = f(a) + f(b)$ (b) $f(a+b) = 10^{a+b} = 10^a \cdot 10^b = f(a)\cdot f(b)$

12. See Figs A4.2.12a and A4.2.12b.

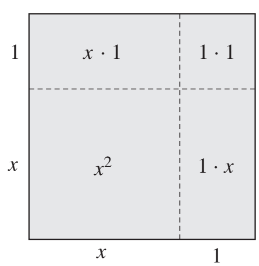


Figure A4.2.12a The area is $(x+1)^2 = x^2 + 2x + 1$

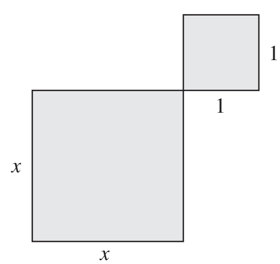


Figure A4.2.12b The area is $x^2 + 1$

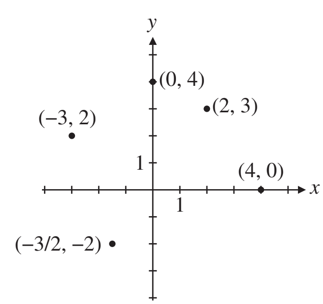


Figure A4.3.1

13. (a) $x \le 5$ (b) $x \ne 0$ and $x \ne 1$ (c) $-3 < x \le 1$ or $x > 2$

14. (a) Defined for $x \ne 2$, i.e. $D_f = (-\infty, 2) \cup (2, \infty)$. (b) $f(8) = 5$

(c) $f(x) = \dfrac{3x+6}{x-2} = 3 \iff 3x + 6 = 3(x-2) \iff 6 = -6$, which is impossible.

15. Since g obviously is defined for $x \ge -2$, $D_g = [-2, \infty)$. Note that $g(-2) = 1$, and $g(x) \le 1$ for all $x \in D_f$.

As x increases from -2 to ∞, $g(x)$ decreases from 1 to $-\infty$, so $R_g = (-\infty, 1]$.

4.3

1. See Fig. A4.3.1.

2. (a) $f(-5) = 0, f(-3) = -3, f(-2) = 0, f(0) = 2, f(3) = 4, f(4) = 0$ (b) $D_f = [-5, 4]$, $R_f = [-3, 4]$

3. (a)

x	0	1	2	3	4
$g(x) = -2x + 5$	5	3	1	-1	-3

See Fig. A4.3.3a.

(b)

x	-2	-1	0	1	2	3	4
$h(x) = x^2 - 2x - 3$	5	0	-3	-4	-3	0	5

See Fig. A4.3.3b.

(c)

x	-2	-1	0	1	2
$F(x) = 3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

See Fig. A4.3.3c.

(d)

x	-2	-1	0	1	2	3
$G(x) = 1 - 2^{-x}$	-3	-1	0	$1/2$	$3/4$	$7/8$

See Fig. A4.3.3d.

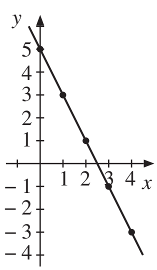


Figure A4.3.3a

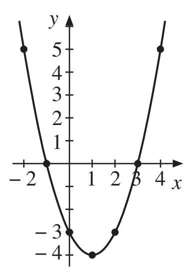


Figure A4.3.3b

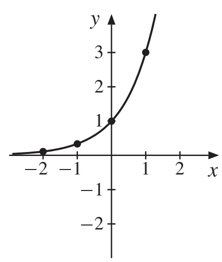


Figure A4.3.3c

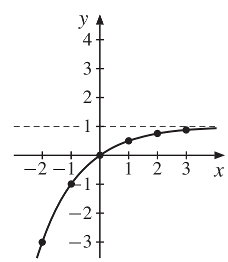


Figure A4.3.3d

4.4

1. (a) Slope = $(8 - 3)/(5 - 2) = 5/3$ (b) $-2/3$ (c) $51/5$

2. See Figs A4.4.2a, A4.4.2b, and A4.4.2c.

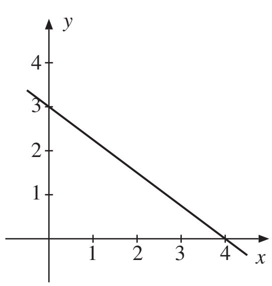


Figure A4.4.2a

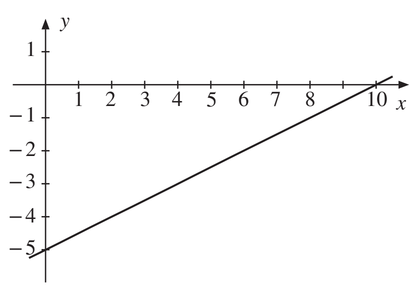


Figure A4.4.2b

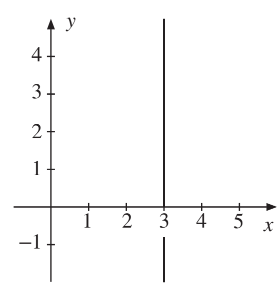


Figure A4.4.2c

3. If $D = a + bP$, then $a + 10b = 200$, and $a + 15b = 150$. Solving for a and b yields $a = 300$ and $b = -10$, so $D = 300 - 10P$.

4. (a), (b), and (d) are all linear; (c) is not, it is quadratic.

5. If P is the price of Q copies, then $P - 1400 = \dfrac{3000 - 1400}{500 - 100}(Q - 100)$ by the point–point formula, so $P = 1000 + 4Q$. The price of printing 300 copies is therefore $P = 1000 + 4 \cdot 300 = 2200$.

6. L_1: The slope is 1, and the point–slope formula with $(x_1, y_1) = (0, 2)$ and $a = 1$ gives $y = x + 2$.

L_2: Using the point–point formula with $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (5, 0)$, one obtains $y - 3 = \dfrac{0-3}{5-0}x$, or $y = -\frac{3}{5}x + 3$. L_3 is $y = 1$, with slope 0. L_4 is $y = 3x - 14$, with slope 3. L_5 is $y = \frac{1}{9}x + 2$, with slope 1/9.

7. (a) L_1: $y - 3 = 2(x - 1)$ or $y = 2x + 1$ (b) L_2: $y - 2 = \dfrac{3-2}{3-(-2)}[x - (-2)]$ or $y = x/5 + 12/5$

(c) L_3: $y = -x/2$ (d) L_4: $x/a + y/b = 1$, or $y = -bx/a + b$.

8. For (a), shown in Fig. A4.4.8a, the solution is $x = 3$, $y = -2$. For (b), shown in Fig. A4.4.8b, the solution is $x = 2$, $y = 0$. For (c), shown in Fig. A4.4.8c, there are no solutions, because the two lines are parallel.

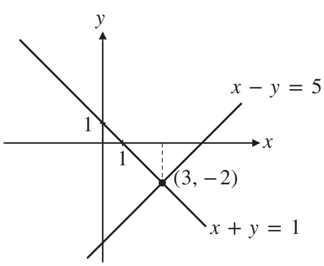


Figure A4.4.8a

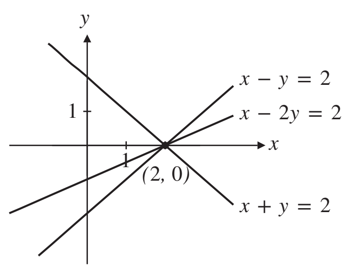


Figure A4.4.8b

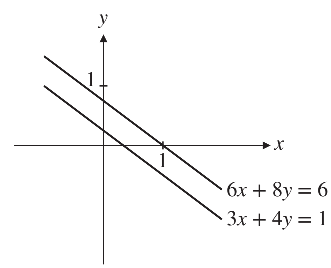


Figure A4.4.8c

9. (a) See Figs A4.4.9a, A4.4.9b, and A4.4.9c.

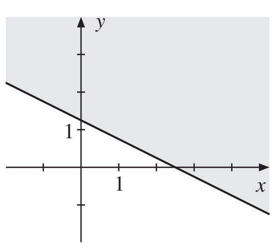


Figure A4.4.9a

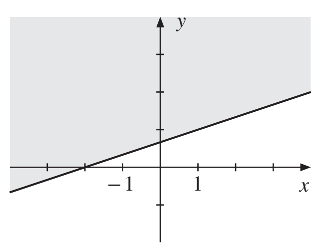


Figure A4.4.9b

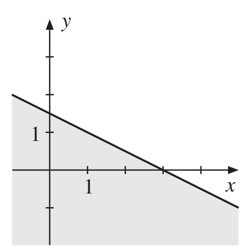


Figure A4.4.9c

10. See Fig. A4.4.10. Each small arrow points toward the side of the line where the relevant inequality is satisfied. The shaded triangle is the required set.

4.5

1. 0.78

2. (a) $75 - 3P^e = 2P^e$, and hence $P^e = 15$ (b) $P^e = 90$

3. The point–point formula gives $C - 200 = \dfrac{275 - 200}{150 - 100}(x - 100)$, or $C = \frac{3}{2}x + 50$.

4. $C = 0.8y + 100$. (With $C = ay + b$, we are told that $900 = 1000a + b$ and $a = 80/100 = 0.8$, so $b = 100$.)

5. (a) $P(t) = 20\,000 - 2000t$ (b) $W(t) = 500 - 50t$

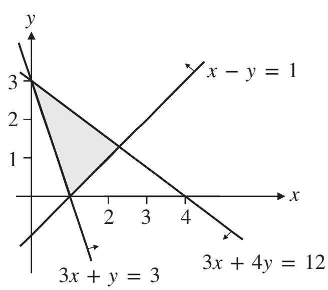


Figure A4.4.10

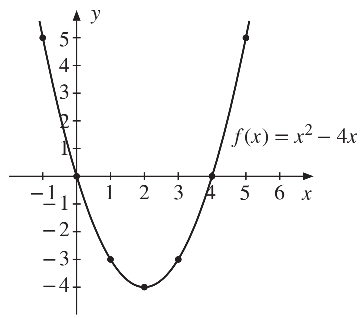


Figure A4.6.1

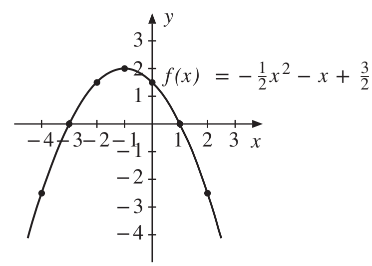


Figure A4.6.2

4.6

1. (a)

x	-1	0	1	2	3	4	5
$f(x) = x^2 - 4x$	5	0	-3	-4	-3	0	5

See Fig. A4.6.1.

(b) Minimum at $x = 2$, with $f(2) = -4$. (c) $x = 0$ and $x = 4$.

2. (a)

x	-4	-3	-2	-1	0	1	2
$f(x) = -\frac{1}{2}x^2 - x + \frac{3}{2}$	-2.5	0	1.5	2	1.5	0	-2.5

See Fig. A4.6.2.

(b) Maximum at $x = -1$ with $f(-1) = 2$. (c) $x = -3$ and $x = 1$.

(d) $f(x) > 0$ in $(-3, 1)$, $f(x) < 0$ for $x < -3$ and for $x > 1$.

3. (a) Minimum -4 for $x = -2$. (b) Minimum 9 for $x = -3$. (c) Maximum 45 for $x = 5$.
(d) Minimum -45 for $x = 1/3$. (e) Maximum 40 000 for $x = -100$. (f) Minimum $-22\,500$ for $x = -50$.

4. (a) $x(x + 4)$. Zeros 0 and -4. (b) No factoring is possible. No zeros.
(c) $-3(x - x_1)(x - x_2)$, where the zeros are $x_1 = 5 + \sqrt{15}$ and $x_2 = 5 - \sqrt{15}$.
(d) $9(x - x_1)(x - x_2)$, where the zeros are $x_1 = 1/3 + \sqrt{5}$ and $x_2 = 1/3 - \sqrt{5}$.
(e) $-(x + 300)(x - 100)$. Zeros -300 and 100. (f) $(x + 200)(x - 100)$. Zeros -200 and 100.

5. (a) $x = 2p$ and $x = p$ (b) $x = p$ and $x = q$ (c) $x = \frac{1}{2}p$ and $x = -2q$

6. Expanding gives $U(x) = -(1 + r^2)x^2 + 8(r - 1)x + 40$. By (4.6.4), $U(x)$ has a maximum for $x = 4(r - 1)/(1 + r^2)$.

7. (a) The areas when $x = 100$, 250, and 350 are $100 \cdot 400 = 40\,000$, $250 \cdot 250 = 62\,500$, and $350 \cdot 150 = 52\,500$, respectively. (b) The area is $A = (250 + x)(250 - x) = 62\,500 - x^2$, which obviously has its maximum for $x = 0$. Then the rectangle is a square.

8. (a) $\pi(Q) = (P_E - P_G - \gamma)Q = -\frac{1}{2}Q^2 + (\alpha_1 - \alpha_2 - \gamma)Q$ (b) Using (4.6.4), we see that $Q^* = \alpha_1 - \alpha_2 - \gamma$ maximizes profit if $\alpha_1 - \alpha_2 - \gamma > 0$. If $\alpha_1 - \alpha_2 - \gamma \le 0$, then $Q^* = 0$.

(c) $\pi(Q) = -\frac{1}{2}Q^2 + (\alpha_1 - \alpha_2 - \gamma - t)Q$, and $Q^* = \alpha_1 - \alpha_2 - \gamma - t$ if $\alpha_1 - \alpha_2 - \gamma - t > 0$

(d) $T = tQ^* = t(\alpha_1 - \alpha_2 - \gamma - t)$. T is a quadratic function of t; it is 0 when $t = 0$ and when $t = t_1 = \alpha_1 - \alpha_2 - \gamma$, and it is positive for t between 0 and t_1. (e) Export tax revenue is maximized when $t = \frac{1}{2}(\alpha_1 - \alpha_2 - \gamma)$.

9. (a) $361 \leq 377$ (b) If $B^2 - 4AC > 0$, then according to formula (2.3.4), the equation $f(x) = Ax^2 + Bx + C = 0$ would have two distinct solutions, contradicting $f(x) \geq 0$ for all x. Hence $B^2 - 4AC \leq 0$.

(c) (4.6.8) is equivalent to $\frac{1}{4}B^2 \leq AC$.

4.7

1. (a) $-2, -1, 1, 3$ (b) $1, -6$ (c) None. (d) $1, 2, -2$

2. (a) 1 and -2 (b) 1, 5, and -5 (c) -1

3. (a) $2x^2 + 2x + 4 + 3/(x - 1)$ (b) $x^2 + 1$ (c) $x^3 - 4x^2 + 3x + 1 - 4x/(x^2 + x + 1)$

(d) $3x^5 + 6x^3 - 3x^2 + 12x - 12 + (28x^2 - 36x + 13)/(x^3 - 2x + 1)$

4. (a) $y = \frac{1}{2}(x + 1)(x - 3)$ (b) $y = -2(x + 3)(x - 1)(x - 2)$ (c) $y = \frac{1}{2}(x + 3)(x - 2)^2$

5. (a) $x + 4$ (b) $x^2 + x + 1$ (c) $-3x^2 - 12x$

6. $c^4 + 3c^2 + 5 \geq 5 \neq 0$ for every choice of c, so the division has to leave a remainder.

7. Expand the right-hand side. (Note that $R(x) \to a/c$ as $x \to \infty$.)

8. $E = \alpha\big(x - (\beta + \gamma)\big) + \dfrac{\alpha\beta(\beta + \gamma)}{x + \beta}$

4.8

1. See Fig. A4.8.1.

2. (a) 1.6325269 (b) 36.4621596

3. (a) $2^3 = 8$, so $x = 3/2$. (b) $1/81 = 3^{-4}$, so $3x + 1 = -4$, and therefore $x = -5/3$.

(c) $x^2 - 2x + 2 = 2$, so $x^2 - 2x = 0$, implying that $x = 0$ or $x = 2$.

4. (a) C (b) D (c) E (d) B (e) A (f) F: $y = 2 - (1/2)^x$

5. (a) $3^{5t}9^t = 3^{5t}(3^2)^t = 3^{5t+2t} = 3^{7t}$ and $27 = 3^3$, so $7t = 3$, hence $t = 3/7$.

(b) $9^t = (3^2)^t = 3^{2t}$ and $(27)^{1/5}/3 = (3^3)^{1/5}/3 = 3^{3/5}/3 = 3^{-2/5}$, hence $2t = -2/5$, so $t = -1/5$.

6. $V = (4/3)\pi r^3$ implies $r^3 = 3V/4\pi$ and so $r = (3V/4\pi)^{1/3}$. Hence, $S = 4\pi r^2 = 4\pi\big(3V/4\pi\big)^{2/3} = \sqrt[3]{36\pi}\, V^{2/3}$.

4.9

1. The doubling time t^* is determined by $(1.0072)^{t^*} = 2$. Using a calculator, we find $t^* \approx 96.6$.

2. $P(t) = 1.22 \cdot 1.034^t$. The doubling time t^* is given by the equation $(1.034)^{t^*} = 2$, and we find $t^* \approx 20.7$ (years).

3. The amount of savings after t years is $100\,(1 + 12/100)^t = 100 \cdot (1.12)^t$. We have the following table:

t	1	2	5	10	20	30	50
$100 \cdot (1.12)^t$	112	125.44	176.23	310.58	964.63	2995.99	28 900.21

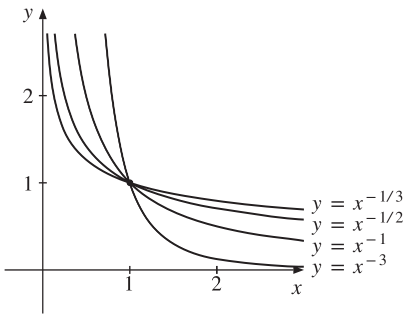


Figure A4.8.1

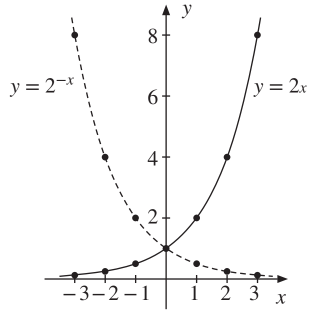


Figure A4.9.4

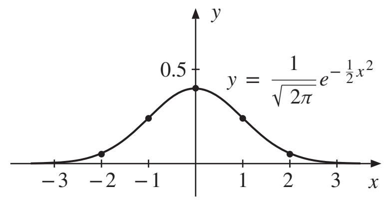


Figure A4.9.5

4. The graphs are drawn in Fig. A4.9.4. We have the following table:

x	-3	-2	-1	0	1	2	3
2^x	1/8	1/4	1/2	1	2	4	8
2^{-x}	8	4	2	1	1/2	1/4	1/8

5. The graph is drawn in Fig. A4.9.5. Here is a table:

x	-2	-1	0	1	2
$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$	0.05	0.24	0.40	0.24	0.05

6. We find $(1.035)^t = 3.91 \cdot 10^5/5.1 \approx 76\,666.67$, and using a calculator we find $t \approx 327$. So the year is $1969 + 327 = 2296$. This is when every Zimbabwean would have only 1 m^2 of land on average.

7. If the initial time is t, the doubling time t^* is given by the equation $Aa^{t+t^*} = 2Aa^t$, which implies $Aa^ta^{t^*} = 2Aa^t$, so $a^{t^*} = 2$, independent of t.

8. (b) and (d) do not define exponential functions. (In (f): $y = (1/2)^x$.)

9. (a) $16(1.19)^5 \approx 38.18$ (b) $4.40(1.19)^{10} \approx 25.06$ (c) $250\,000(1.19)^4 \approx 501\,335$

10. Suppose $y = Ab^x$, with $b > 0$. Then in (a), since the graph passes through the points $(x, y) = (0, 2)$ and $(x, y) = (2, 8)$, we get $2 = Ab^0$, or $A = 2$, and $8 = 2b^2$, so $b = 2$. Hence, $y = 2 \cdot 2^x$.

In (b), $\frac{2}{3} = Ab^{-1}$ and $6 = Ab$. It follows that $A = 2$ and $b = 3$, so $y = 2 \cdot 3^x$.

In (c), $4 = Ab^0$ and $1/4 = Ab^4$. It follows that $A = 4$ and $b^4 = 1/16$, so $b = 1/2$. Thus, $y = 4(1/2)^x$.

4.10

1. (a) $\ln 9 = \ln 3^2 = 2\ln 3$ (b) $\frac{1}{2}\ln 3$ (c) $\ln \sqrt[5]{3^2} = \ln 3^{2/5} = \frac{2}{5}\ln 3$ (d) $\ln(1/81) = \ln 3^{-4} = -4\ln 3$

2. (a) $\ln 3^x = x\ln 3 = \ln 8$, so $x = \ln 8/\ln 3$. (b) $x = e^3$ (c) $x^2 - 4x + 5 = 1$ so $(x-2)^2 = 0$. Hence, $x = 2$.
(d) $x(x-2) = 1$ or $x^2 - 2x - 1 = 0$, so $x = 1 \pm \sqrt{2}$. (e) $x = 0$ or $\ln(x+3) = 0$, so $x = 0$ or $x = -2$.
(f) $\sqrt{x} - 5 = 1$, so $x = 36$.

3. (a) $x = -\ln 2/\ln 12$ (b) $x = e^{6/7}$ (c) $x = \ln(8/3)/\ln(4/3)$ (d) $x = 4$ (e) $x = e$ (f) $x = 1/27$

4. $t = \dfrac{1}{r-s}\ln\dfrac{B}{A}$

5. The answer to Exercise 4 implies that $t \approx 22$, so the date should have been 2012.

6. (a) False. (Let $A = e$.) (b) $2 \ln \sqrt{B} = 2 \ln B^{1/2} = 2(1/2) \ln B = \ln B$
(c) $\ln A^{10} - \ln A^4 = 10 \ln A - 4 \ln A = 6 \ln A = 3 \cdot 2 \ln A = 3 \ln A^2$ (d) Wrong. (Put $A = B = C = 1$.)
(e) Correct by rule (2)(b). (f) Correct. (Use (2)(b) twice.)
(g) Wrong. (If $A = e$ and $p = 2$, then the equality becomes $0 = \ln 2$.) (h) Correct by (2)(c).
(i) Wrong. (Put $A = 2$, $B = C = 1$.)

7. (a) $\exp[\ln(x)] - \ln[\exp(x)] = e^{\ln x} - \ln e^x = x - x = 0$ (b) $\ln[x^4 \exp(-x)] = 4 \ln x - x$ (c) x^2/y^2

Review exercises for Chapter 4

1. (a) $f(0) = 3, f(-1) = 30, f(1/3) = 2, f(\sqrt[3]{2}) = 3 - 27(2^{1/3})^3 = 3 - 27 \cdot 2 = -51$
(b) $f(x) + f(-x) = 3 - 27x^3 + 3 - 27(-x)^3 = 3 - 27x^3 + 3 + 27x^3 = 6$

2. (a) $F(0) = 1$, $F(-2) = 0$, $F(2) = 2$, and $F(3) = 25/13$
(b) $F(x) = 1 + \dfrac{4}{x + 4/x}$ tends to 1 as x becomes large positive or negative. (c) See Fig. A4.R.2.

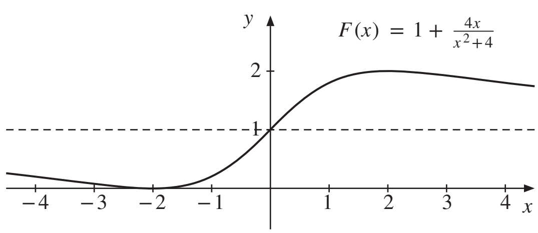


Figure A4.R.2

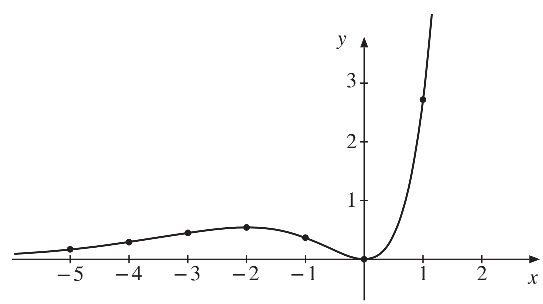


Figure A4.R.9

3. (a) $f(x) \le g(x)$ when $-2 \le x \le 3$. (b) $f(x) \le 0$ when $-1 \le x \le 3$. (c) $g(x) \ge 0$ when $x \le 3$.

4. (a) $x^2 \ge 1$, i.e. $x \ge 1$ or $x \le -1$.
(b) The square root is defined if $x \ge 4$, but $x = 4$ makes the denominator 0, so we must require $x > 4$.
(c) We must have $(x - 3)(5 - x) \ge 0$, i.e. $3 \le x \le 5$ (use a sign diagram).

5. (a) $C(0) = 100$, $C(100) = 24\,100$, and $C(101) - C(100) = 24\,542 - 24\,100 = 442$
(b) $C(x + 1) - C(x) = 4x + 42$ is the additional cost of producing one more than x units.

6. (a) Slope -4 (b) Slope $-3/4$ (c) Solving for y gives $y = b[1 - (x/a)] = b - (b/a)x$, so the slope is $-b/a$.

7. (a) The point–slope formula gives $y - 3 = -3(x + 2)$, or $y = -3x - 3$.
(b) The point–point formula gives $y - 5 = \dfrac{7 - 5}{2 - (-3)}(x - (-3))$, or $y = 2x/5 + 31/5$.
(c) The point–point formula gives $y - b = \dfrac{3b - b}{2a - a}(x - a)$, or $y = (2b/a)x - b$.

8. $f(2)=3$ and $f(-1)=-3$ give $2a+b=3$ and $-a+b=-3$, so $a=2$, $b=-1$. Hence $f(x)=2x-1$ and $f(-3)=-7$. (Or use the point–point formula.)

9.

x	-5	-4	-3	-2	-1	0	1
$y=x^2e^x$	0.17	0.29	0.45	0.54	0.37	0	2.7

The graph is drawn in Fig. A4.R.9.

10. $(1,-3)$ belongs to the graph if $a+b+c=-3$, $(0,-6)$ belongs to the graph if $c=-6$, and $(3,15)$ belongs to the graph if $9a+3b+c=15$. It follows that $a=2$, $b=1$, and $c=-6$.

11. (a) $\pi = \big(1000-\frac{1}{3}Q\big)Q-\big(800+\frac{1}{5}Q\big)Q-100Q=100Q-\frac{8}{15}Q^2$. Hence $Q=1500/16=93.75$ maximizes π.

(b) $\hat{\pi}=100Q-\frac{8}{15}Q^2-10Q=90Q-\frac{8}{15}Q^2$. So $\hat{Q}=1350/16=84.375$ maximizes $\hat{\pi}$.

12. The new profit is $\pi_t=100Q-\frac{5}{2}Q^2-tQ$, which is maximized at $Q_t=\frac{1}{5}(100-t)$.

13. (a) The profit function is $\pi(x)=100x-20x-0.25x^2=80x-0.25x^2$, which has a maximum at $x^*=160$.

(b) The profit function is $\pi_t(x)=80x-0.25x^2-10x$, which has a maximum at $x^*=140$.

(c) The profit function is $\pi_t(x)=(p-\tau-\alpha)x-\beta x^2$, which has a maximum at $x^*=(p-\alpha-\tau)/2\beta$.

14. (a) $p(x)=x(x-3)(x+4)$ (b) $q(x)=2(x-2)(x+4)(x-1/2)$

15. (a) x^3-x-1 is not 0 for $x=1$, so the division leaves a remainder.

(b) $2x^3-x-1$ is 0 for $x=1$, so the division leaves no remainder.

(c) $x^3-ax^2+bx-ab$ is 0 for $x=a$, so the division leaves no remainder.

(d) $x^{2n}-1$ is 0 for $x=-1$, so the division leaves no remainder.

16. We use (4.7.5). (a) $p(2)=8-2k=0$ for $k=4$. (b) $p(-2)=4k^2+2k-6=0$ for $k=-3/2$ and $k=1$.

(c) $p(-2)=-26+k=0$ for $k=26$. (d) $p(1)=k^2-3k-4=0$ for $k=-1$ and $k=4$.

17. $p(x)=\frac{1}{4}(x-2)(x+3)(x-5)$, so the other two roots are $x=-3$ and $x=5$.

18. $(1+p/100)^{15}=2$ gives $p=100(2^{1/15}-1)\approx 4.7$ as the percentage rate.

19. The vertical dashed line is $x=-c$, so $c<0$. Because $f(x)<0$ when x is close to $-c$ with $x<c$, one has $a>0$. Because $f(x)=0$ implies that $x>0$, one has $b<0$.

20. Because $f(x)>0$ when $|x|$ is large, one has $p>0$. Because $f(0)<0$, one has $r<0$. Because the sum of the two roots of $f(x)=0$ is evidently positive, one has $q<0$.

21. (a) Assume $F=aC+b$. Then $32=a\cdot 0+b$ and $212=a\cdot 100+b$. Therefore $a=180/100=9/5$ and $b=32$, so $F=9C/5+32$. (b) If $X=9X/5+32$, then $X=-40$.

22. (a) $\ln x=\ln e^{at+b}=at+b$, so $t=(\ln x-b)/a$. (b) $-at=\ln(1/2)=\ln 1-\ln 2=-\ln 2$, so $t=(\ln 2)/a$.

(c) $e^{-\frac{1}{2}t^2}=2^{1/2}\pi^{1/2}2^{-3}$, so $-\frac{1}{2}t^2=\frac{1}{2}\ln 2+\frac{1}{2}\ln\pi-3\ln 2=-\frac{5}{2}\ln 2+\frac{1}{2}\ln\pi$, implying that $t^2=5\ln 2-\ln\pi=\ln(32/\pi)$, and finally, $t=\pm\sqrt{\ln(32/\pi)}$.

23. (a), (b) and (c) are all obviously implied by the Rules for the Natural Logarithmic Function provided that x, y and z are all positive.

(d) When $x>0$, note that $\frac{1}{2}\ln x-\frac{3}{2}\ln(1/x)-\ln(x+1)$ reduces to

$\frac{1}{2}\ln x+\frac{3}{2}\ln x-\ln(x+1)=2\ln x-\ln(x+1)=\ln x^2-\ln(x+1)=\ln[x^2/(x+1)]$

Chapter 5

5.1

1. (a) $y = x^2 + 1$ has the graph of $y = x^2$ shifted up by 1. See Fig. A5.1.1a.

(b) $y = (x + 3)^2$ has the graph of $y = x^2$ moved three units to the left. See Fig. A5.1.1b.

(c) $y = 3 - (x + 1)^2$ has the graph of $y = x^2$ first turned upside down, then with $(0, 0)$ shifted to $(-1, 3)$. See Fig. A5.1.1c.

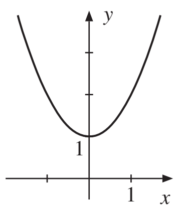


Figure A5.1.1a

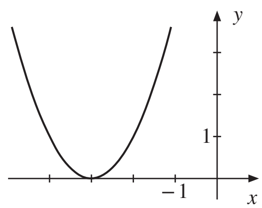


Figure A5.1.1b

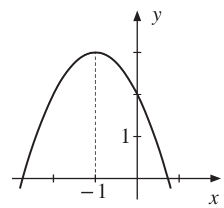


Figure A5.1.1c

2. (a) The graph of $y = f(x)$ is moved two units to the right. See Fig. A5.1.2a.

(b) The graph of $y = f(x)$ is moved downwards by two units. See Fig. A5.1.2b.

(c) The graph of $y = f(x)$ is reflected about the y-axis. See Fig. A5.1.2c.

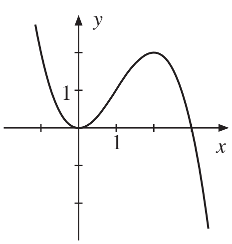


Figure A5.1.2a

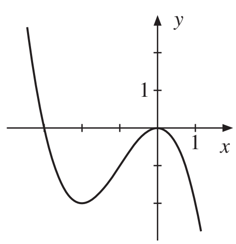


Figure A5.1.2b

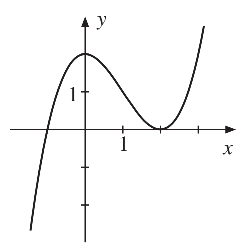


Figure A5.1.2c

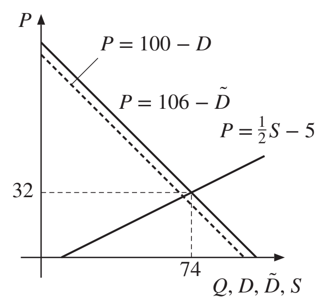


Figure A5.1.3

3. The equilibrium condition is $106 - P = 10 + 2P$, and thus $P = 32$. The corresponding quantity is $Q = 106 - 32 = 74$. See Fig. A5.1.3.

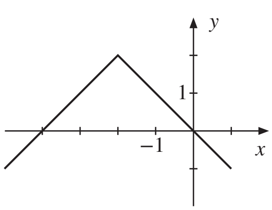


Figure A5.1.4

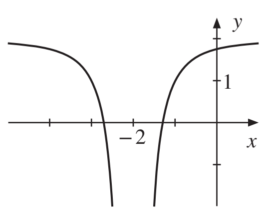


Figure A5.1.5

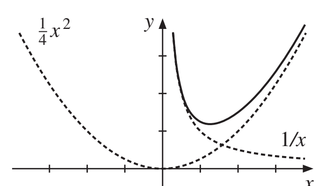


Figure A5.2.1

4. Move $y = |x|$ two units to the left. Then reflect the graph about the x-axis, and then move the graph up two units. See Fig. A5.1.4.

5. Draw the graph of $y = 1/x^2$. Move it two units to the left. Then reflect the graph about the x-axis, and then move the graph up two units to get Fig. A5.1.5.

6. $f(y^* - d) = f(y^*) - c$ gives $A(y^* - d) + B(y^* - d)^2 = Ay^* + B(y^*)^2 - c$, which expands to $Ay^* - Ad + B(y^*)^2 - 2Bdy^* + Bd^2 = Ay^* + B(y^*)^2 - c$. It follows that $y^* = [Bd^2 - Ad + c]/2Bd$.

5.2

1. See Fig. A5.2.1.

2. See Figs A5.2.2a, A5.2.2b, and A5.2.2c.

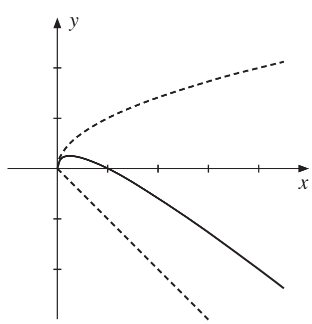


Figure A5.2.2a

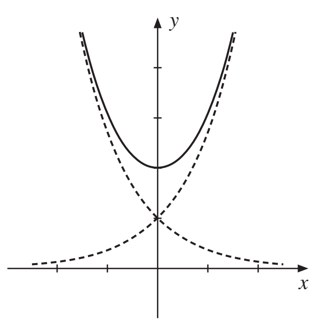


Figure A5.2.2b

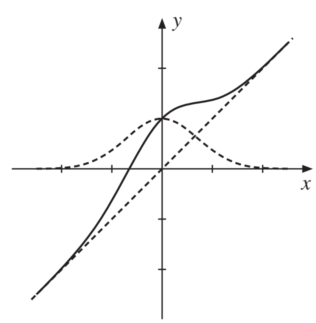


Figure A5.2.2c

3. $(f + g)(x) = 3x$, $(f - g)(x) = 3x - 2x^3$, $(fg)(x) = 3x^4 - x^6$, $(f/g)(x) = 3/x^2 - 1$, $f(g(1)) = f(1) = 2$, and $g(f(1)) = g(2) = 8$.

4. If $f(x) = 3x + 7$, then $f(f(x)) = f(3x + 7) = 3(3x + 7) + 7 = 9x + 28$. The equality $f(f(x^*)) = 100$ requires $9x^* + 28 = 100$, and so $x^* = 8$.

5. $\ln(\ln e) = \ln 1 = 0$, while $(\ln e)^2 = 1^2 = 1$.

5.3

1. $P = \frac{1}{3}(64 - 10D)$

2. $P = (157.8/D)^{10/3}$

3. (a) Domain and range: $\mathbb{R}$; inverse $x = -y/3$. (b) Domain and range: $\mathbb{R} \setminus \{0\}$; inverse $x = 1/y$.

(c) Domain and range: $\mathbb{R}$; inverse $x = y^{1/3}$. (d) Domain $[4, \infty)$, range $[0, \infty)$; inverse $x = (y^2 + 2)^2$.

4. (a) The domain of f^{-1} is $\{-4, -2, 0, 2, 4, 6, 8\}$. $f^{-1}(2) = -1$ (b) $f(x) = 2x + 4$, $f^{-1}(x) = \frac{1}{2}x - 2$

5. $f(x) = x^2$ is not one-to-one over $(-\infty, \infty)$, and therefore has no inverse. Over $[0, \infty)$, f is strictly increasing and therefore has an inverse, which is $f^{-1}(x) = \sqrt{x}$.

6. (a) $f(x) = x/2$ and $g(x) = 2x$ are inverse functions. (b) $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}(x + 2)$ are inverse functions.

(c) $C = \frac{5}{9}(F - 32)$ and $F = \frac{9}{5}C + 32$ are inverse functions.

7. $f^{-1}(C)$ determines the cost of C kilograms of carrots.

8. (a) See Fig. A5.3.8a. (b) See Fig. A5.3.8b. Triangles OBA and OBC are congruent. The point half-way between the two points A and C is $B = (\frac{1}{2}(a+b), \frac{1}{2}(a+b))$.

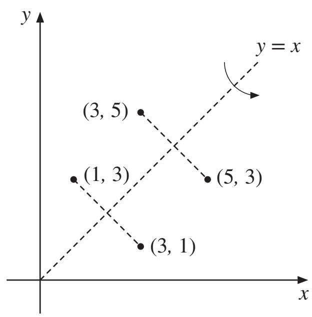


Figure A5.3.8a

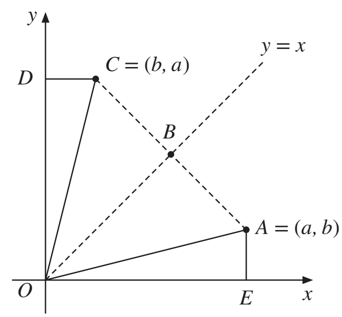


Figure A5.3.8b

9. (a) $f^{-1}(x) = (x^3+1)^{1/3}$ (b) $f^{-1}(x) = \dfrac{2x+1}{x-1}$ (c) $f^{-1}(x) = \big(1-(x-2)^5\big)^{1/3}$

10. (a) $x = \ln y - 4$, defined for $y > 0$. (b) $x = e^{y+4}$, defined for $y \in (-\infty, \infty)$.

(c) $x = 3 + \ln(e^y - 2)$, defined for $y > \ln 2$.

11. We must solve $x = \frac{1}{2}(e^y - e^{-y})$ for y. Multiply the equation by e^y to get $\frac{1}{2}e^{2y} - \frac{1}{2} = xe^y$ or $e^{2y} - 2xe^y - 1 = 0$. Letting $e^y = z$ yields $z^2 - 2xz - 1 = 0$, with solution $z = x \pm \sqrt{x^2+1}$. Choosing the minus sign would make z negative, contradicting $z = e^y$, so $z = e^y = x + \sqrt{x^2+1}$. This gives $y = \ln\big(x + \sqrt{x^2+1}\big)$ as the inverse function.

5.4

1. (a) Some solutions include $\big(0, \pm\sqrt{3}\big)$, $\big(\pm\sqrt{6}, 0\big)$, and $\big(\pm\sqrt{2}, \pm\sqrt{2}\big)$. See Fig. A5.4.1a.

(b) Some solutions include $(0, \pm 1)$, $\big(\pm 1, \pm\sqrt{2}\big)$, and $\big(\pm 3, \pm\sqrt{10}\big)$. See Fig. A5.4.1b.

2. We see that we must have $x \geq 0$ and $y \geq 0$. If (a, b) lies on the graph, so does (b, a), hence the graph is symmetric about the line $y = x$. It also includes the particular points $(25, 0)$, $(0, 25)$, and $(25/4, 25/4)$. See Fig. A5.4.2.

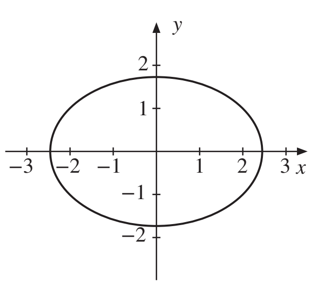


Figure A5.4.1a

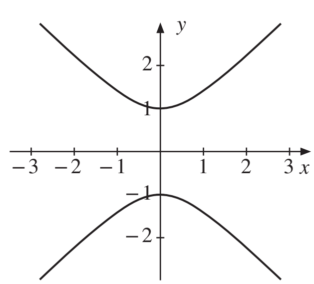


Figure A5.4.1b

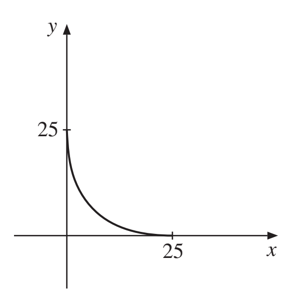


Figure A5.4.2

3. $F(100\,000) = 4070$. The graph is the thick line sketched in Fig. A5.4.3.

5.5

1. (a) $\sqrt{(2-1)^2+(4-3)^2} = \sqrt{2}$ (b) $\sqrt{5}$ (c) $\frac{1}{2}\sqrt{205}$ (d) $\sqrt{x^2+9}$ (e) $2|a|$ (f) $2\sqrt{2}$

2. $(5-2)^2 + (y-4)^2 = 13$, or $y^2 - 8y + 12 = 0$, with solutions $y = 2$ and $y = 6$. A geometric explanation is that the circle with centre at $(2, 4)$ and radius $\sqrt{13}$ intersects the line $x = 5$ at two points. See Fig. A5.5.2.

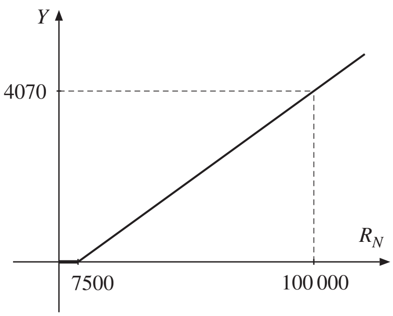


Figure A5.4.3

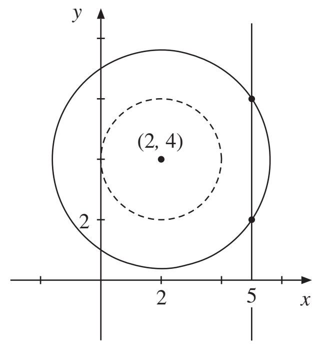


Figure A5.5.2

3. (a) 5.362 (b) $\sqrt{(2\pi)^2 + (2\pi - 1)^2} = \sqrt{8\pi^2 - 4\pi + 1} \approx 8.209$

4. (a) $(x-2)^2 + (y-3)^2 = 16$ (b) Since the circle has centre at $(2, 5)$, its equation is $(x-2)^2 + (y-5)^2 = r^2$. Since $(-1, 3)$ lies on the circle, $(-1-2)^2 + (3-5)^2 = r^2$, so $r^2 = 13$.

5. (a) Completing squares yields $(x+5)^2 + (y-3)^2 = 4$, so the circle has centre at $(-5, 3)$ and radius 2.

(b) $(x+3)^2 + (y-4)^2 = 12$, which has centre at $(-3, 4)$ and radius $\sqrt{12} = 2\sqrt{3}$.

6. The condition is that $\sqrt{(x+2)^2 + y^2} = 2\sqrt{(x-4)^2 + y^2}$, which reduces to $(x-6)^2 + y^2 = 4^2$.

7. We can write the formula as $cxy - ax + dy - b = 0$. Comparing this with (5), $A = C = 0$ and $B = c$, so $4AC < B^2$ reduces to $0 < c^2$, that is $c \neq 0$, precisely the condition assumed in Example 4.7.7.

8. If $A^2 + B^2 > 4C$, then the graph of the equation is the circle with a centre at $(-\frac{1}{2}A, -\frac{1}{4}B)$ and radius $\sqrt{C - \frac{1}{4}A^2 - \frac{1}{4}B^2}$. If $A^2 + B^2 = 4C$, then the graph is the single point set $\{(-\frac{1}{2}A, -\frac{1}{2}B)\}$. If $A^2 + B^2 < 4C$, it is the empty set.

5.6

1. Only (c) does not define a function. (Rectangles with equal areas can have different perimeters.)

2. The function in (b) is one-to-one and has an inverse: the rule mapping each youngest child alive today to his/her mother. (Though the youngest child of a mother with several children will have been different at different dates.)

The function in (d) is one-to-one and has an inverse: the rule mapping the surface area to the volume.

The function in (e) is one-to-one and has an inverse: the rule that maps (u, v) to $(u-3, v)$.

The function in (a) is many-to-one, in general, and so has no inverse.

Review exercises for Chapter 5

1. The shifts of $y = |x|$ are the same as those of $y = x^2$ in Exercise 5.1.1. See Figs A5.R.1a, A5.R.1b, and A5.R.1c.

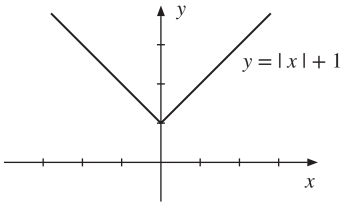


Figure A5.R.1a

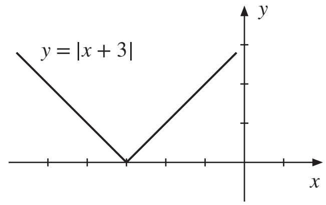


Figure A5.R.1b

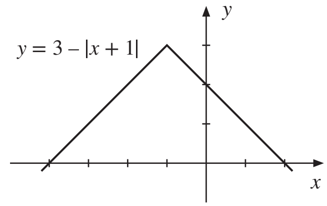


Figure A5.R.1c

2. $(f+g)(x) = x^2 - 2$, $(f-g)(x) = 2x^3 - x^2 - 2$, $(fg)(x) = x^2(1-x)(x^3-2)$, $(f/g)(x) = (x^3-2)/x^2(1-x)$, $f(g(1)) = f(0) = -2$, and $g(f(1)) = g(-1) = 2$.

3. (a) The equilibrium condition is $150 - \frac{1}{2}P^* = 20 + 2P^*$, which implies that $P^* = 52$ and $Q^* = 20 + 2P^* = 124$.

(b) $S = 20 + 2(\hat{P} - 2) = 16 + 2\hat{P}$, so $S = D$ when $5\hat{P}/2 = 134$. Hence $\hat{P} = 53.6$, $\hat{Q} = 123.2$.

(c) Before the tax, $R^* = P^*Q* = 6448$. After the tax, $\hat{R} = (\hat{P} - 2)\hat{Q} = 51.6 \cdot 123.2 = 6357.12$.

4. $P = (64 - 10D)/3$

5. $P = 24 - \frac{1}{5}D$

6. (a) $x = 50 - \frac{1}{2}y$ (b) $x = \sqrt[5]{y/2}$ (c) $x = \frac{1}{3}[2 + \ln(y/5)]$, defined for $y > 0$

7. (a) $y = \ln(2 + e^{x-3})$, defined for $x \in \mathbb{R}$ (b) $y = -\dfrac{1}{\lambda}\ln a - \dfrac{1}{\lambda}\ln\left(\dfrac{1}{x} - 1\right)$ defined for, $x \in (0, 1)$

8. (a) $\sqrt{13}$ (b) $\sqrt{17}$ (c) $\sqrt{(2-3a)^2} = |2 - 3a|$. (Note that $2 - 3a$ is the correct answer only if $2 - 3a \geq 0$, i.e. $a \leq 2/3$. Check this by putting $a = 3$.)

9. (a) $(x-2)^2 + (y+3)^2 = 25$ (b) $(x+2)^2 + (y-2)^2 = 65$

10. $(x-3)^2 + (y-2)^2 = (x-5)^2 + (y+4)^2$, which reduces to $x - 3y = 7$. See Fig. A5.R.10.

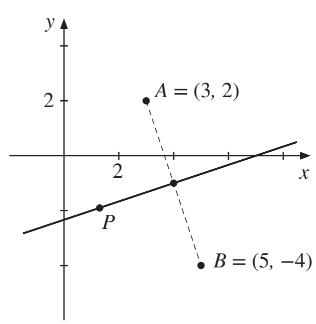


Figure A5.R.10

11. The function cannot be one-to-one, because at least two persons out of any five must have the same blood group.

Chapter 6

6.1

1. $f(3) = 2$. The tangent passes through $(0, 3)$, so has slope $-1/3$. Thus, $f'(3) = -1/3$.

2. $g(5) = 1$, $g'(5) = 1$

6.2

1. $f(5+h) - f(5) = 4(5+h)^2 - 4 \cdot 5^2 = 4(25 + 10h + h^2) - 100 = 40h + 4h^2$. So $[f(5+h) - f(5)]/h = 40 + 4h \to 40$ as $h \to 0$. Hence, $f'(5) = 40$. This accords with (6.2.6) when $a = 4$ and $b = c = 0$.

2. (a) $f'(x) = 6x + 2$ (b) $f'(0) = 2, f'(-2) = -10, f'(3) = 20$. The tangent equation is $y = 2x - 1$.

3. $\mathrm{d}D(P)/\mathrm{d}P = -b$

4. $C'(x) = 2qx$

5. $$\frac{f(x+h) - f(x)}{h} = \frac{1/(x+h) - 1/x}{h} = \frac{x - (x+h)}{hx(x+h)} = \frac{-h}{hx(x+h)} = -\frac{1}{x(x+h)} \underset{h \to 0}{\longrightarrow} -\frac{1}{x^2}$$

6. (a) $f'(0) = 3$ (b) $f'(1) = 2$ (c) $f'(3) = -1/3$ (d) $f'(0) = -2$ (e) $f'(-1) = 0$ (f) $f'(1) = 4$

7. (a) $f(x+h) - f(x) = a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c) = 2ahx + bh + ah^2$,

so $[f(x+h) - f(x)]/h = 2ax + b + ah \to 2ax + b$ as $h \to 0$. Thus $f'(x) = 2ax + b$.

(b) $f'(x) = 0$ for $x = -b/2a$. The tangent is parallel to the x-axis at the minimum/maximum point.

8. $f'(a) < 0$, $f'(b) = 0$, $f'(c) > 0$, $f'(d) < 0$

9. (a) Expand the left-hand side. (b) Rearrange the identity in (a).
(c) Letting $h \to 0$, the formula follows. (Recall that $\sqrt{x} = x^{1/2}$ and $1/\sqrt{x} = x^{-1/2}$.)

10. (a) $f'(x) = 3ax^2 + 2bx + c$. (b) Put $a = 1$ and $b = c = d = 0$ to get the result in Example 6.2.2. Then put $a = 0$ to get a quadratic expression as in part (a) of Exercise 6.2.7.

11. $$\frac{(x+h)^{1/3} - x^{1/3}}{h} = \frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}} \to \frac{1}{3x^{2/3}} \text{ as } h \to 0, \text{ and } \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$$

6.3

1. $f'(x) = 2x - 4$, so $f(x)$ is decreasing in $(-\infty, 2]$, increasing in $[2, \infty)$.

2. $f'(x) = -3x^2 + 8x - 1 = -3(x - x_0)(x - x_1)$, where $x_0 = \frac{1}{3}(4 - \sqrt{13}) \approx 0.13$ and $x_1 = \frac{1}{3}(4 + \sqrt{13}) \approx 2.54$. Then $f(x)$ is decreasing in $(-\infty, x_0]$, increasing in $[x_0, x_1]$, and decreasing in $[x_1, \infty)$.

3. The expression in square brackets is a sum of two squares, so it is never negative and it is 0 only if both $x_1 + \frac{1}{2}x_2$ and x_2 are equal to 0. This happens only when $x_1 = x_2 = 0$. Thus the bracketed expression is always positive if $x_1 \neq x_2$, and then $x_2^3 - x_1^3$ will have the same sign as $x_2 - x_1$. It follows that f is strictly increasing.

6.4

1. $C'(100) = 203$ and $C'(x) = 2x + 3$.

2. Here c is the marginal cost, and also the (constant) incremental cost of producing each additional unit, whereas $\bar{C}$ is the fixed cost.

3. (a) $S'(Y) = s$ (b) $S'(Y) = 0.1 + 0.0004Y$

4. $T'(y) = t$, so the marginal tax rate is constant.

5. The interpretation of $x'(0) = -3$ is that at time $t = 0$, the rate of extraction is three barrels per minute.

6. (a) $C'(x) = 3x^2 - 180x + 7500$ (b) By (4.6.3), the quadratic function $C'(x)$ has a minimum at $x = 180/6 = 30$.

7. (a) $\pi'(Q) = 24 - 2Q$, and $Q^* = 12$ (b) $R'(Q) = 500 - Q^2$ (c) $C'(Q) = -3Q^2 + 428.4Q - 7900$

8. (a) $C'(x) = 2a_1x + b_1$ (b) $C'(x) = 3a_1x^2$

6.5

1. (a) 3 (b) $-1/2$ (c) $13^3 = 2197$ (d) 40 (e) 1 (f) $-3/4$

2. (a) 0.6931 (b) 1.0986 (c) 0.4055. (Actually, using the result in Example 7.12.2, the precise values of these three limits are $\ln 2$, $\ln 3$, and $\ln(3/2)$, respectively.)

3. (a) We have the following table (where $*$ denotes undefined):

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$\dfrac{x^2+7x-8}{x-1}$	8.9	8.99	8.999	$*$	9.001	9.01	9.1

(b) $x^2 + 7x - 8 = (x-1)(x+8)$, so $(x^2+7x-8)/(x-1) = x + 8 \to 9$ as $x \to 1$.

4. (a) 5 (b) 1/5 (c) 1 (d) -2 (e) $3x^2$ (f) h^2

5. (a) $1/6$ (b) $-\infty$ (the limit does not exist) (c) 2 (d) $\sqrt{3}/6$ (e) $-2/3$ (f) $1/4$

6. (a) 4 (b) 5 (c) 6 (d) $2a + 2$ (e) $2a + 2$ (f) $4a + 4$

7. (a) $x^3 - 8 = (x-2)(x^2 + 2x + 4)$, so the limit is $1/6$. (b) $\lim_{h\to 0}[\sqrt[3]{27+h} - 3]/h = \lim_{u\to 3}(u-3)/(u^3 - 27)$, and $u^3 - 27 = (u-3)(u^2 + 3u + 9)$, so the limit is $\lim_{u\to 3} 1/(u^2 + 3u + 9) = 1/27$.
(c) $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$, so the limit is n.

6.6

1. (a) 0 (b) $4x^3$ (c) $90x^9$ (d) 0. (Remember that π is a constant!)

2. (a) $2g'(x)$ (b) $-\frac{1}{6}g'(x)$ (c) $\frac{1}{3}g'(x)$

3. (a) $6x^5$ (b) $33x^{10}$ (c) $50x^{49}$ (d) $28x^{-8}$ (e) x^{11} (f) $4x^{-3}$ (g) $-x^{-4/3}$ (h) $3x^{-5/2}$

4. (a) $8\pi r$ (b) $A(b+1)y^b$ (c) $(-5/2)A^{-7/2}$

5. In (6.2.1) (the definition of the derivative), choose $h = x - a$ so that $a + h$ is replaced by x, and $h \to 0$ implies $x \to a$. For $f(x) = x^2$ we get $f'(a) = 2a$.

6. (a) $F(x) = \frac{1}{3}x^3 + C$ (b) $F(x) = x^2 + 3x + C$ (c) $F(x) = x^{a+1}/(a+1) + C$. (In all cases C is an arbitrary constant.)

7. (a) With $f(x) = x^2$ and $a = 5$, one has $\displaystyle\lim_{h\to 0}\frac{(5+h)^2 - 5^2}{h} = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = f'(a) = f'(5)$.
On the other hand, $f'(x) = 2x$, so $f'(5) = 10$, so the limit is 10.
(b) Let $f(x) = x^5$. Then $f'(x) = 5x^4$, and the limit is equal to $f'(1) = 5 \cdot 1^4 = 5$.
(c) Let $f(x) = 5x^2 + 10$. Then $f'(x) = 10x$, and this is the value of the limit.

6.7

1. (a) 1 (b) $1 + 2x$ (c) $15x^4 + 8x^3$ (d) $32x^3 + x^{-1/2}$ (e) $\frac{1}{2} - 3x + 15x^2$ (f) $-21x^6$

2. (a) $\frac{6}{5}x - 14x^6 - \frac{1}{2}x^{-1/2}$ (b) $4x(3x^4 - x^2 - 1)$ (c) $10x^9 + 5x^4 + 4x^3 - x^{-2}$. (In (b) and (c), first expand and then differentiate.)

3. (a) $-6x^{-7}$ (b) $\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2}$ (c) $-\frac{3}{2}x^{-5/2}$ (d) $-2/(x-1)^2$ (e) $-4x^{-5} - 5x^{-6}$ (f) $34/(2x+8)^2$ (g) $-33x^{-12}$ (h) $(-3x^2 + 2x + 4)/(x^2 + x + 1)^2$

4. (a) $\dfrac{3}{2\sqrt{x}(\sqrt{x}+1)^2}$ (b) $\dfrac{4x}{(x^2+1)^2}$ (c) $\dfrac{-2x^2+2}{(x^2-x+1)^2}$

5. (a) $f'(L^*) < f(L^*)/L^*$. See Figure A6.7.5. The tangent at P has the slope $f'(L^*)$. We "see" that the tangent at P is less steep than the straight line from the origin to P, which has the slope $f(L^*)/L^* = g(L^*)$. (The inequality follows directly from the characterization of differentiable concave functions in FMEA, Theorem 2.4.1.)

(b) $\dfrac{d}{dL}\left(\dfrac{f(L)}{L}\right) = \dfrac{1}{L}\left[f'(L) - \dfrac{f(L)}{L}\right]$, as in Example 6.7.6.

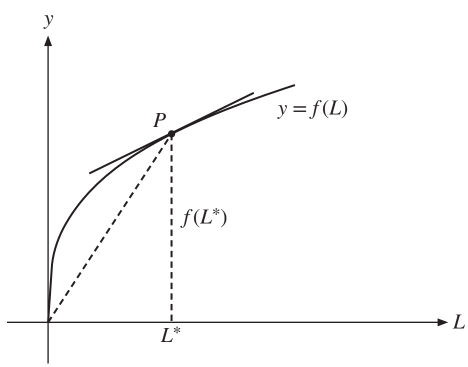


Figure A6.7.5

6. (a) $[2, \infty)$ (b) $\left[-\sqrt{3}, 0\right]$ and $\left[\sqrt{3}, \infty\right)$ (c) $\left[-\sqrt{2}, \sqrt{2}\right]$ (d) $(-\infty, \frac{1}{2}(-1-\sqrt{5})]$ and $[0, \frac{1}{2}(-1+\sqrt{5})]$

7. (a) $y = -3x + 4$ (b) $y = x - 1$ (c) $y = (17x - 19)/4$ (d) $y = -(x-3)/9$

8. $\dot{R}(t) = \dot{p}(t)x(t) + p(t)\dot{x}(t)$. Here, $R(t)$ increases for two reasons. First, $R(t)$ increases because of the price increase. This increase is proportional to the amount of extraction $x(t)$ and is equal to $\dot{p}(t)x(t)$. But $R(t)$ also rises because extraction increases. Its contribution to the rate of change of $R(t)$ must be proportional to the price, and is equal to $p(t)\dot{x}(t)$. In the end $\dot{R}(t)$, the total rate of change of $R(t)$, is the sum of these two parts.

9. (a) $(ad - bc)/(ct + d)^2$ (b) $a\left(n + \frac{1}{2}\right)t^{n-1/2} + nbt^{n-1}$ (c) $-(2at + b)/(at^2 + bt + c)^2$

10. The product rule yields $f'(x) \cdot f(x) + f(x) \cdot f'(x) = 1$, so $2f'(x) \cdot f(x) = 1$. Hence, $f'(x) = 1/2f(x) = 1/2\sqrt{x}$.

11. If $f(x) = 1/x^n$, the quotient rule yields $f'(x) = (0 \cdot x^n - 1 \cdot nx^{n-1})/(x^n)^2 = -nx^{-n-1}$, which is the power rule.

6.8

1. (a) $dy/dx = (dy/du)(du/dx) = 20u^{4-1}\, du/dx = 20(1+x^2)^3 2x = 40x(1+x^2)^3$

(b) $dy/dx = (1 - 6u^5)(du/dx) = (-1/x^2)\left(1 - 6(1+1/x)^5\right)$

2. (a) $dY/dt = (dY/dV)(dV/dt) = (-3)5(V+1)^4 t^2 = -15t^2(t^3/3 + 1)^4$

(b) $dK/dt = (dK/dL)(dL/dt) = AaL^{a-1}b = Aab(bt + c)^{a-1}$

3. (a) $y' = -5(x^2+x+1)^{-6}(2x+1)$ (b) $y' = \frac{1}{2}\big[x+(x+x^{1/2})^{1/2}\big]^{-1/2}\big(1+\frac{1}{2}(x+x^{1/2})^{-1/2}\big(1+\frac{1}{2}x^{-1/2}\big)\big)$
(c) $y' = ax^{a-1}(px+q)^b + x^a bp(px+q)^{b-1} = x^{a-1}(px+q)^{b-1}[(a+b)px+aq]$

4. $(\mathrm{d}Y/\mathrm{d}t)_{t=t_0} = (\mathrm{d}Y/\mathrm{d}K)_{t=t_0} \cdot (\mathrm{d}K/\mathrm{d}t)_{t=t_0} = Y'(K(t_0))K'(t_0)$

5. $\mathrm{d}Y/\mathrm{d}t = F'\big(h(t)\big) \cdot h'(t)$

6. $x = b - \sqrt{ap-c} = b - \sqrt{u}$, with $u = ap - c$. Then $\dfrac{\mathrm{d}x}{\mathrm{d}p} = -\dfrac{1}{2\sqrt{u}}u' = -\dfrac{a}{2\sqrt{ap-c}}$.

7. (a) $h'(x) = f'(x^2)2x$ (b) $h'(x) = f'\big(x^n g(x)\big)\big(nx^{n-1}g(x) + x^n g'(x)\big)$

8. $b(t)$ is the total fuel consumption after t hours. Then $b'(t) = B'\big(s(t)\big)s'(t)$, so the rate of fuel consumption per hour is equal to the rate per kilometre multiplied by the speed in kph.

9. $\mathrm{d}C/\mathrm{d}x = q\big(25 - \frac{1}{2}x\big)^{-1/2}$

10. (a) $y' = 5(x^4)^4 \cdot 4x^3 = 20x^{19}$ (b) $y' = 3(1-x)^2(-1) = -3 + 6x - 3x^2$

11. (a) (i) $g(5)$ is the amount accumulated if the interest rate is 5% per year, which is approximately €1 629.
(ii) $g'(5)$ is the increase in this value per unit increase in the interest rate, which is approximately €155.
(b) $g(p) = 1000(1+p/100)^{10}$, so $g(5) = 1000 \cdot 1.05^{10} = 1\,628.89$ to the nearest eurocent.
Moreover, $g'(p) = 1000 \cdot 10(1+p/100)^9 \cdot 1/100$, so $g'(5) = 100 \cdot 1.05^9 = 155.13$ to the nearest eurocent.

12. (a) $1 + f'(x)$ (b) $2f(x)f'(x) - 1$ (c) $4\big[f(x)\big]^3 f'(x)$ (d) $2xf(x) + x^2 f'(x) + 3\big[f(x)\big]^2 f'(x)$ (e) $f(x) + xf'(x)$
(f) $f'(x)/\big[2\sqrt{f(x)}\,\big]$ (g) $[2xf(x) - x^2 f'(x)]/[f(x)]^2$ (h) $[2xf(x)f'(x) - 3(f(x))^2]/x^4$

6.9

1. (a) $y'' = 20x^3 - 36x^2$ (b) $y'' = (-1/4)x^{-3/2}$
(c) $y' = 20x(1+x^2)^9$, and then $y'' = 20(1+x^2)^9 + 20x \cdot 9 \cdot 2x(1+x^2)^8 = 20(1+x^2)^8(1+19x^2)$

2. $\mathrm{d}^2y/\mathrm{d}x^2 = (1+x^2)^{-1/2} - x^2(1+x^2)^{-3/2} = (1+x^2)^{-3/2}$

3. (a) $y'' = 18x$ (b) $Y''' = 36$ (c) $\mathrm{d}^3z/\mathrm{d}t^3 = -2$ (d) $f^{(4)}(1) = 84\,000$

4. $g'(t) = \dfrac{2t(t-1)-t^2}{(t-1)^2} = \dfrac{t^2-2t}{(t-1)^2}$, and $g''(t) = \dfrac{2}{(t-1)^3}$, so $g''(2) = 2$.

5. With simplified notation: $y' = f'g + fg'$, $y'' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''$,
and $y''' = f'''g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg'''$.

6. $L = (2t-1)^{-1/2}$, so $\mathrm{d}L/\mathrm{d}t = -\frac{1}{2} \cdot 2(2t-1)^{-3/2} = -(2t-1)^{-3/2}$, and $\mathrm{d}^2L/\mathrm{d}t^2 = 3(2t-1)^{-5/2}$.

7. (a) $R = 0$ (b) $R = 1/2$ (c) $R = 3$ (d) $R = \rho$

8. Because $g(u)$ is not concave.

9. The defence secretary: $P' < 0$. Gray: $P' \geq 0$ and $P'' < 0$.

10. $\mathrm{d}^3L/\mathrm{d}t^3 > 0$

6.10

1. (a) $y' = e^x + 2x$ (b) $y' = 5e^x - 9x^2$ (c) $y' = (1 \cdot e^x - xe^x)/e^{2x} = (1-x)e^{-x}$

(d) $y' = [(1+2x)(e^x+1) - (x+x^2)e^x]/(e^x+1)^2 = [1+2x+e^x(1+x-x^2)]/(e^x+1)^2$

(e) $y' = -1 - e^x$ (f) $y' = x^2e^x(3+x)$ (g) $y' = e^x(x-2)/x^3$ (h) $y' = 2(x+e^x)(1+e^x)$

2. (a) $\mathrm{d}x/\mathrm{d}t = (b+2ct)e^t + (a+bt+ct^2)e^t = (a+b+(b+2c)t+ct^2)e^t$

(b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{3qt^2te^t - (p+qt^3)(1+t)e^t}{t^2e^{2t}} = \dfrac{-qt^4 + 2qt^3 - pt - p}{t^2e^t}$

(c) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = [2(at+bt^2)(a+2bt)e^t - (at+bt^2)^2e^t]/(e^t)^2 = [t(a+bt)(-bt^2+(4b-a)t+2a)]e^{-t}$

3. (a) $y' = -3e^{-3x}$ and $y'' = 9e^{-3x}$ (b) $y' = 6x^2e^{x^3}$ and $y'' = 6xe^{x^3}(3x^3+2)$

(c) $y' = -x^{-2}e^{1/x}$ and $y'' = x^{-4}e^{1/x}(2x+1)$ (d) $y' = 5(4x-3)e^{2x^2-3x+1}$ and $y'' = 5e^{2x^2-3x+1}(16x^2-24x+13)$

4. (a) $(-\infty,\infty)$ (b) $[0, 1/2]$ (c) $(-\infty,-1]$ and $[0,1]$

5. (a) $y' = 2xe^{-2x}(1-x)$, so y is increasing in $[0,1]$. (b) $y' = e^x(1-3e^{2x})$, so y is increasing in $(-\infty, -\frac{1}{2}\ln 3]$.

(c) $y' = (2x+3)e^{2x}/(x+2)^2$, so y is increasing in $[-3/2, \infty)$.

6. (a) $e^{e^x}e^x = e^{e^x+x}$ (b) $\frac{1}{2}(e^{t/2} - e^{-t/2})$ (c) $-\dfrac{e^t - e^{-t}}{(e^t+e^{-t})^2}$ (d) $z^2e^{z^3}(e^{z^3}-1)^{-2/3}$

7. (a) $y' = 5^x \ln 5$ (b) $y' = 2^x + x2^x \ln 2 = 2^x(1 + x\ln 2)$ (c) $y' = 2x2^{x^2}(1+x^2\ln 2)$

(d) $y' = e^x10^x + e^x10^x \ln 10 = e^x10^x(1+\ln 10)$

6.11

1. (a) $y' = 1/x + 3$ and $y'' = -1/x^2$ (b) $y' = 2x - 2/x$ and $y'' = 2 + 2/x^2$

(c) $y' = 3x^2\ln x + x^2$ and $y'' = x(6\ln x + 5)$ (d) $y' = (1-\ln x)/x^2$ and $y'' = (2\ln x - 3)/x^3$

2. (a) $x^2\ln x(3\ln x + 2)$ (b) $x(2\ln x - 1)/(\ln x)^2$ (c) $10(\ln x)^9/x$ (d) $2\ln x/x + 6\ln x + 18x + 6$

3. (a) $1/(x\ln x)$ (b) $-x/(1-x^2)$ (c) $e^x(\ln x + 1/x)$ (d) $e^{x^3}\left(3x^2\ln x^2 + 2/x\right)$ (e) $e^x/(e^x+1)$

(f) $(2x+3)/(x^2+3x-1)$ (g) $-2e^x(e^x-1)^{-2}$ (h) $(4x-1)e^{2x^2-x}$

4. (a) $x > -1$ (b) $1/3 < x < 1$ (c) $x \neq 0$

5. (a) $|x| > 1$ (b) $x > 1$ (c) $x \neq e^e$ and $x > 1$

6. (a) $(-2, 0]$. (y is defined only in $(-2,2)$, where $y' = -8x/(4-x^2)$.) (b) $[e^{-1/3}, \infty)$. ($y' = x^2(3\ln x + 1)$, $x > 0$.)

(c) $[e, e^3]$. ($y' = (1-\ln x)(\ln x - 3)/2x^2$, $x > 0$.)

7. (a) (i) $y = x - 1$ (ii) $y = 2x - 1 - \ln 2$ (iii) $y = x/e$ (b) (i) $y = x$ (ii) $y = 2ex - e$ (iii) $y = -e^{-2}x - 4e^{-2}$

8. (a) $f'(x)/f(x) = 2\ln x + 2$ (b) $f'(x)/f(x) = 1/(2x-4) + 2x/(x^2+1) + 4x^3/(x^4+6)$

(c) $f'(x)/f(x) = -2/[3(x^2-1)]$

9. (a) $(2x)^x(1 + \ln 2 + \ln x)$ (b) $x^{\sqrt{x}-\frac{1}{2}}\left(\frac{1}{2}\ln x + 1\right)$ (c) $\frac{1}{2}\left(\sqrt{x}\right)^x(\ln x + 1)$

10. $\ln y = v\ln u$, so $y'/y = v'\ln u + vu'/u$ and therefore $y' = u^v(v'\ln u + vu'/u)$.

(Alternatively, note that $y = (e^{\ln u})^v = e^{v\ln u}$, and then use the chain rule.)

11. (a) Let $f(x) = e^x - (1 + x + \frac{1}{2}x^2)$. Then $f(0) = 0$ and $f'(x) = e^x - (1 + x) > 0$ for all $x > 0$, as shown in the exercise. Hence $f(x) > 0$ for all $x > 0$, and the inequality follows.

(b) Consider the two functions $f_1(x) = \ln(1 + x) - \frac{1}{2}x$ and $f_2(x) = x - \ln(1 + x)$. For more details, see SM.

(c) Consider the function $g(x) = 2(\sqrt{x} - 1) - \ln x$. For more details, see SM.

Review exercises for Chapter 6

1. $[f(x + h) - f(x)]/h = [(x + h)^2 - (x + h) + 2 - x^2 + x - 2]/h = [2xh + h^2 - h]/h = 2x + h - 1$.

Therefore $[f(x + h) - f(x)]/h \to 2x - 1$ as $h \to 0$, so $f'(x) = 2x - 1$.

2. $[f(x + h) - f(x)]/h = -6x^2 + 2x - 6xh - 2h^2 + h \to -6x^2 + 2x$ as $h \to 0$, so $f'(x) = -6x^2 + 2x$.

3. (a) $y' = 2,\ y'' = 0$ (b) $y' = 3x^8,\ y'' = 24x^7$ (c) $y' = -x^9,\ y'' = -9x^8$ (d) $y' = 21x^6,\ y'' = 126x^5$

(e) $y' = 1/10,\ y'' = 0$ (f) $y' = 5x^4 + 5x^{-6},\ y'' = 20x^3 - 30x^{-7}$ (g) $y' = x^3 + x^2,\ y'' = 3x^2 + 2x$

(h) $y' = -x^{-2} - 3x^{-4},\ y'' = 2x^{-3} + 12x^{-5}$

4. Because $C'(1000) \approx C(1001) - C(1000)$, if $C'(1000) = 25$, the additional cost of producing slightly more than 1000 units is approximately 25 per unit. If the price per unit is fixed at 30, the extra profit from increasing output slightly above 1000 units is approximately $30 - 25 = 5$ per unit.

5. (a) $y = -3$ and $y' = -6x = -6$ at $x = 1$, so $y - (-3) = (-6)(x - 1)$, or $y = -6x + 3$.

(b) $y = -14$ and $y' = 1/2\sqrt{x} - 2x = -31/4$ at $x = 4$, so $y = -(31/4)x + 17$.

(c) $y = 0$ and $y' = (-2x^3 - 8x^2 + 6x)/(x + 3)^2 = -1/4$ at $x = 1$, so $y = (-1/4)(x - 1)$.

6. The additional cost of increasing the area by a small amount from 100 m^2 is approximately \$250 per m^2.

7. (a) $f(x) = x^3 + x$, so $f'(x) = 3x^2 + 1$. (b) $g'(w) = -5w^{-6}$ (c) $h(y) = y(y^2 - 1) = y^3 - y$, so $h'(y) = 3y^2 - 1$.

(d) $G'(t) = (-2t^2 - 2t + 6)/(t^2 + 3)^2$ (e) $\varphi'(\xi) = (4 - 2\xi^2)/(\xi^2 + 2)^2$ (f) $F'(s) = -(s^2 + 2)/(s^2 + s - 2)^2$

8. (a) $2at$ (b) $a^2 - 2t$ (c) $2x\varphi - 1/2\sqrt{\varphi}$

9. (a) $y' = 20uu' = 20(5 - x^2)(-2x) = 40x^3 - 200x$ (b) $y' = \dfrac{1}{2\sqrt{u}} \cdot u' = \dfrac{-1}{2x^2\sqrt{1/x - 1}}$

10. (a) $\mathrm{d}Z/\mathrm{d}t = (\mathrm{d}Z/\mathrm{d}u)(\mathrm{d}u/\mathrm{d}t) = 3(u^2 - 1)^2 2u3t^2 = 18t^5(t^6 - 1)^2$

(b) $\mathrm{d}K/\mathrm{d}t = (\mathrm{d}K/\mathrm{d}L)(\mathrm{d}L/\mathrm{d}t) = (1/[2\sqrt{L}\,])(-1/t^2) = -1/[2t^2\sqrt{1 + 1/t}\,]$

11. (a) $\dot{x}/x = 2\dot{a}/a + \dot{b}/b$ (b) $\dot{x}/x = \alpha\dot{a}/a + \beta\dot{b}/b$ (c) $\dot{x}/x = (\alpha + \beta)(\alpha a^{\alpha-1}\dot{a} + \beta b^{\beta-1}\dot{b})/(a^\alpha + b^\beta)$

12. $\mathrm{d}R/\mathrm{d}t = (\mathrm{d}R/\mathrm{d}S)(\mathrm{d}S/\mathrm{d}K)(\mathrm{d}K/\mathrm{d}t) = \alpha S^{\alpha-1}\beta\gamma K^{\gamma-1}Apt^{p-1} = A\alpha\beta\gamma pt^{p-1}S^{\alpha-1}K^{\gamma-1}$

13. (a) $h'(L) = apL^{a-1}(L^a + b)^{p-1}$ (b) $C'(Q) = a + 2bQ$ (c) $P'(x) = ax^{1/q-1}(ax^{1/q} + b)^{q-1}$

14. (a) $y' = -7e^x$ (b) $y' = -6xe^{-3x^2}$ (c) $y' = xe^{-x}(2 - x)$ (d) $y' = e^x[\ln(x^2 + 2) + 2x/(x^2 + 2)]$

(e) $y' = 15x^2e^{5x^3}$ (f) $y' = x^3e^{-x}(x - 4)$ (g) $y' = 10(e^x + 2x)(e^x + x^2)^9$ (h) $y' = 1/2\sqrt{x}(\sqrt{x} + 1)$

15. (a) $[1, \infty)$ (b) $[0, \infty)$ (c) $(-\infty, 1]$ and $[2, \infty)$

16. (a) $\dfrac{\mathrm{d}\pi}{\mathrm{d}Q} = P(Q) + QP'(Q) - c$ (b) $\dfrac{\mathrm{d}\pi}{\mathrm{d}L} = PF'(L) - w$

Chapter 7

7.1

1. Differentiating w.r.t. x yields $6x + 2y' = 0$, so $y' = -3x$. Solving the given equation for y yields $y = 5/2 - 3x^2/2$, verifying that $y' = -3x$.

2. Implicit differentiation yields $(*)$ $2xy + x^2(\mathrm{d}y/\mathrm{d}x) = 0$, and so $\mathrm{d}y/\mathrm{d}x = -2y/x$. Differentiating $(*)$ implicitly w.r.t. x gives $2y + 2x(\mathrm{d}y/\mathrm{d}x) + 2x(\mathrm{d}y/\mathrm{d}x) + x^2(\mathrm{d}^2y/\mathrm{d}x^2) = 0$. Inserting the result for $\mathrm{d}y/\mathrm{d}x$, then simplifying, yields $\mathrm{d}^2y/\mathrm{d}x^2 = 6y/x^2$. These results follow more easily by differentiating $y = x^{-2}$ twice.

3. (a) $y' = (1 + 3y)/(1 - 3x) = -5/(1 - 3x)^2$ and $y'' = 6y'/(1 - 3x) = -30/(1 - 3x)^3$

   (b) $y' = 6x^5/5y^4 = (6/5)x^{1/5}$ and $y'' = 6x^4y^{-4} - (144/25)x^{10}y^{-9} = (6/25)x^{-4/5}$

4. $2u + v + u(\mathrm{d}v/\mathrm{d}u) - 3v^2(\mathrm{d}v/\mathrm{d}u) = 0$, so $\mathrm{d}v/\mathrm{d}u = (2u + v)/(3v^2 - u)$. Hence $\mathrm{d}v/\mathrm{d}u = 0$ when $v = -2u$ (provided $3v^2 - u \neq 0$). Substituting for v in the original equation yields $8u^3 - u^2 = 0$. So the only point on the curve where $\mathrm{d}v/\mathrm{d}u = 0$ and $u \neq 0$ is $(u, v) = (1/8, -1/4)$.

5. Differentiating w.r.t. x yields $(*)$ $4x + 6y + 6xy' + 2yy' = 0$, so $y' = -(2x + 3y)/(3x + y) = -8/5$ at $(1, 2)$. Differentiating $(*)$ w.r.t. x yields $4 + 6y' + 6y' + 6xy'' + 2(y')^2 + 2yy'' = 0$. Substituting $x = 1$, $y = 2$, and $y' = -8/5$ yields $y'' = 126/125$.

6. (a) $2x + 2yy' = 0$, and solve for y' to get $y' = -x/y$. (b) $1/2\sqrt{x} + y'/2\sqrt{y} = 0$, and solve for y' to get $y' = -\sqrt{y/x}$.

   (c) $4x^3 - 4y^3y' = 2xy^3 + x^2 3y^2y'$, and solve for y' to get $y' = 2x(2x^2 - y^3)/y^2(3x^2 + 4y)$.

   (d) $e^{xy}(y + xy') - 2xy - x^2y' = 0$, and solve for y' to get $y' = y(2x - e^{xy})/x(e^{xy} - x)$.

7. (a) $(*)$ $2y + 2xy' - 6yy' = 0$. Inserting $x = 6$, $y = 1$ yields $2 + 12y' - 6y' = 0$, so $y' = -1/3$.
   (b) Differentiating $(*)$ w.r.t. x yields $(**)$ $2y' + 2y' + 2xy'' - 6y'y' - 6yy'' = 0$. Inserting $x = 6$, $y = 1$, and $y' = -1/3$ into $(**)$ gives $y'' = 1/3$.

8. (a) $y' = \dfrac{g'(x) - y}{x - 3y^2}$ (b) $y' = \dfrac{2x - g'(x + y)}{g'(x + y) - 2y}$ (c) $y' = \dfrac{2y[xg'(x^2y) - xy - 1]}{x[2xy + 2 - xg'(x^2y)]}$

9. Differentiating w.r.t. x yields $3x^2F(xy) + x^3F'(xy)(y + xy') + e^{xy}(y + xy') = 1$. Then put $x = 1$, $y = 0$ to obtain $y' = 1/(F'(0) + 1)$. (Note that F is a function of only one variable, with argument xy.)

10. (a) $y' = \dfrac{x[a^2 - 2(x^2 + y^2)]}{y[2(x^2 + y^2) + a^2]}$ (b) $(\pm\frac{1}{4}a\sqrt{6}, \pm\frac{1}{4}a\sqrt{2})$, with four possible sign combinations.

7.2

1. Implicit differentiation w.r.t. P, recognizing that Q is a function of P, yields $(\mathrm{d}Q/\mathrm{d}P) \cdot P^{1/2} + Q \cdot \frac{1}{2}P^{-1/2} = 0$. Thus $\mathrm{d}Q/\mathrm{d}P = -\frac{1}{2}QP^{-1} = -19/P^{3/2}$.

2. (a) $1 = C''(Q^*)(\mathrm{d}Q^*/\mathrm{d}P)$, so $\mathrm{d}Q^*/\mathrm{d}P = 1/C''(Q^*)$ (b) $\mathrm{d}Q^*/\mathrm{d}P > 0$, which is reasonable because if the price received by the producer increases, the optimal production should increase.

3. (a) Taking the natural logarithm on both sides yields $\ln A - \alpha \ln P - \beta \ln r = \ln S$. Differentiating with respect to r we have $-(\alpha/P)(\mathrm{d}P/\mathrm{d}r) - \beta/r = 0$. It follows that $\mathrm{d}P/\mathrm{d}r = -(\beta/\alpha)(P/r) < 0$.
   (b) So a rise in the interest rate depresses demand, and the equilibrium price falls to compensate.

4. (a) $Y = f(Y) + \bar{I} + \bar{X} - g(Y)$ (b) $\mathrm{d}Y/\mathrm{d}\bar{I} = 1/\big[1 - f'(Y) + g'(Y)\big] > 0$ because $f'(Y) < 1$ and $g'(Y) > 0$.

   (c) $\mathrm{d}^2Y/\mathrm{d}\bar{I}^2 = (f'' - g'')/(1 - f' + g')^3$

5. Differentiating $(*)$ w.r.t. t yields $f''(P+t)\left(\dfrac{dP}{dt}+1\right)^2 + f'(P+t)\dfrac{d^2P}{dt^2} = g''(P)\left(\dfrac{dP}{dt}\right)^2 + g'(P)\dfrac{d^2P}{dt^2}$.

With simplified notation $f''(P'+1)^2 + f'P'' = g''(P')^2 + g'P''$. Substituting $P' = f'/(g'-f')$ and solving for P'', we get $P'' = [f''(g')^2 - g''(f')^2]/(g'-f')^3$.

6. (a) Differentiating $(*)$ w.r.t. t yields $f'(P)(dP/dt) = g'((1-t)P)[-P + (1-t)(dP/dt)]$, and so

$$\frac{dP}{dt} = \frac{-Pg'((1-t)P)}{f'(P) - (1-t)g'((1-t)P)}$$

(b) Both the numerator and denominator are negative, so dP/dt is positive. Increasing the tax on producers increases the equilibrium price.

7.3

1. $f(1) = 1$ and $f'(x) = 2e^{2x-2} = 2$ for $x = 1$. So according to (7.3.2), $g'(1) = 1/f'(1) = 1/2$.

The inverse function is $g(x) = 1 + \frac{1}{2}\ln x$, so $g'(x) = 1/2x = 1/2$ for $x = 1$.

2. (a) $f'(x) = x^2\sqrt{4-x^2} + \dfrac{1}{3}x^3\dfrac{-2x}{2\sqrt{4-x^2}} = \dfrac{4x^2(3-x^2)}{3\sqrt{4-x^2}}$. So f increases in $[-\sqrt{3}, \sqrt{3}\,]$, and decreases in $[-2, -\sqrt{3}\,]$ and in $[\sqrt{3}, 2]$. See Fig. A7.3.2.

(b) f has an inverse in the interval $[0, \sqrt{3}]$ because f is strictly increasing there. $g'(\frac{1}{3}\sqrt{3}\,) = 1/f'(1) = 3\sqrt{3}/8$.

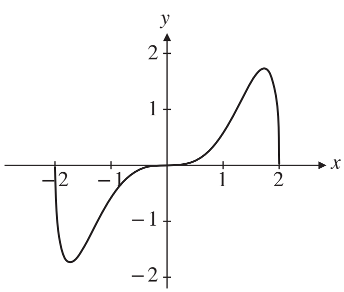


Figure A7.3.2

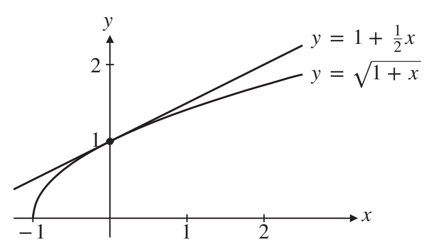


Figure A7.4.1

3. (a) $f'(x) = e^{x-3}/(e^{x-3}+2) > 0$ for all x, so f is strictly increasing.

Furthermore, $f(x) \to \ln 2$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$, so the range of f is $(\ln 2, \infty)$.

(b) $g(x) = 3 + \ln(e^x - 2)$, defined on the range of f. (c) $f'(3) = 1/g'(f(3)) = 1/3$

4. $dD/dP = -0.3 \cdot 157.8P^{-1.3} = -47.34P^{-1.3}$, so $dP/dD = 1/(dD/dP) \approx -0.021P^{1.3}$.

5. (a) $dx/dy = -e^{x+5} = -1/y$ (b) $dx/dy = -1 - 3e^x$ (c) $dx/dy = x(3y^2 - x^2)/(2 + 3x^2y - y^3)$

7.4

1. If $f(x) = \sqrt{1+x}$, then $f'(x) = 1/(2\sqrt{1+x}\,)$, so $f(0) = 1$ and $f'(0) = 1/2$. By (7.4.1), $\sqrt{1+x} \approx 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x$. See Fig. A7.4.1.

2. Here $f(0) = 1/9$ and $f'(x) = -10(5x+3)^{-3}$, so $f'(0) = -10/27$. Hence $(5x+3)^{-2} \approx 1/9 - 10x/27$.

3. (a) $(1+x)^{-1} \approx 1 - x$ (b) $(1+x)^5 \approx 1 + 5x$ (c) $(1-x)^{1/4} \approx 1 - \frac{1}{4}x$

4. $F(1) = A$ and $F'(K) = \alpha AK^{\alpha-1}$, so $F'(1) = \alpha A$. Then $F(K) \approx F(1) + F'(1)(K-1) = A + \alpha A(K-1) = A(1 + \alpha(K-1))$.

5. (a) $30x^2\,dx$ (b) $15x^2\,dx - 10x\,dx + 5\,dx$ (c) $-3x^{-4}\,dx$ (d) $(1/x)\,dx$ (e) $(px^{p-1} + qx^{q-1})\,dx$
(f) $(p+q)x^{p+q-1}\,dx$ (g) $rp(px+q)^{r-1}\,dx$ (h) $(pe^{px} + qe^{qx})dx$

6. (a) If $f(x) = (1+x)^m$, then $f(0) = 1$ and $f'(0) = m$, so $1 + mx$ is the linear approximation to $f(x)$ about $x = 0$.
(b) (i) $\sqrt[3]{1.1} = (1 + 1/10)^{1/3} \approx 1 + (1/3)(1/10) \approx 1.033$ (ii) $\sqrt[5]{33} = 2(1 + 1/32)^{1/5} \approx 2(1 + 1/160) = 2.0125$
(iii) $\sqrt[3]{9} = 2(1 + 1/8)^{1/3} \approx 2(1 + 1/24) \approx 2.083$ (iv) $(0.98)^{25} = (1 - 0.02)^{25} = (1 - 1/50)^{25} \approx 1 - 1/2 = 1/2$

7. (a) (i) $\Delta y = 0.61,\ dy = 0.6$ (ii) $\Delta y = 0.0601,\ dy = 0.06$
(b) (i) $\Delta y = 0.011494,\ dy = 0.011111$ (ii) $\Delta y = 0.001115,\ dy = 0.001111$
(c) (i) $\Delta y = 0.012461,\ dy = 0.0125$ (ii) $\Delta y = 0.002498,\ dy = 0.0025$

8. (a) $y' = -3/2$ (b) $y(x) \approx -\frac{3}{2}x + \frac{3}{2}$

9. (a) $A(r + dr) - A(r)$ is the shaded area in Fig. A7.4.9. It is approximately the circumference of the inner circle, $2\pi r$, times dr. (b) $V(r + dr) - V(r)$ is the volume of the shell between the sphere with radius $r + dr$ and the sphere with radius r. It is approximately the surface area $4\pi r^2$ of the inner sphere times the thickness dr of the shell.

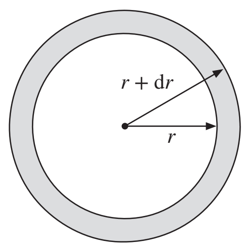


Figure A7.4.9

10. Taking logarithms, we get $\ln K_t = \ln K + t\ln(1 + p/100) \approx \ln K + tp/100$. If $K_t = 2K$, then $\ln K_t = \ln 2 + \ln K$, and with t^* as the doubling time, p must satisfy $\ln 2 \approx t^*p/100$, so $p \approx 100\ln 2/t^*$. (Using the approximation $\ln 2 \approx 0.7$, this result accords with the "Rule of 70" in Example 7.4.3.)

11. $g(0) = A - 1$ and $g'(\mu) = \big(Aa/(1+b)\big)(1+\mu)^{[a/(1+b)]-1}$, so $g'(0) = Aa/(1+b)$.
Hence, $g(\mu) \approx g(0) + g'(0)\mu = A - 1 + aA\mu/(1+b)$.

7.5

1. (a) Here $f'(x) = 5(1+x)^4$ and $f''(x) = 20(1+x)^3$. Hence $f(0) = 1, f'(0) = 5$, and $f''(0) = 20$, implying the quadratic approximation $f(x) = (1+x)^5 \approx 1 + 5x + \frac{1}{2}20x^2 = 1 + 5x + 10x^2$.
(b) $AK^\alpha \approx A + \alpha A(K-1) + \frac{1}{2}\alpha(\alpha-1)A(K-1)^2$ (c) $(1 + \frac{3}{2}\varepsilon + \frac{1}{2}\varepsilon^2)^{1/2} \approx 1 + \frac{3}{4}\varepsilon - \frac{1}{32}\varepsilon^2$
(d) Here $H'(x) = (-1)(1-x)^{-2}(-1) = (1-x)^{-2} = 1$ at $x = 0$, and $H''(x) = 2(1-x)^{-3} = 2$ at $x = 0$. It follows that $(1-x)^{-1} \approx 1 + x + x^2$.

2. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$

3. $-5 + \frac{5}{2}x - \frac{15}{8}x^2$

4. Use (7.5.1) with $f = U$, $a = y$, and $x = y + M - s$.

5. Implicit differentiation yields: (∗) $3x^2y + x^3y' + 1 = \frac{1}{2}y^{-1/2}y'$. Inserting $x = 0$ and $y = 1$ gives $1 = \left(\frac{1}{2}\right)1^{-1/2}y'$, so $y' = 2$. Differentiating (∗) once more w.r.t. x yields $6xy + 3x^2y' + 3x^2y' + x^3y'' = -\frac{1}{4}y^{-3/2}(y')^2 + \frac{1}{2}y^{-1/2}y''$. Inserting $x = 0$, $y = 1$, and $y' = 2$ gives $y'' = 2$. Hence, $y(x) \approx 1 + 2x + x^2$.

6. We find $\dot{x}(0) = 2[x(0)]^2 = 2$. Differentiating the expression for $\dot{x}(t)$ yields $\ddot{x}(t) = x(t) + t\dot{x}(t) + 4[x(t)]\dot{x}(t)$, and so $\ddot{x}(0) = x(0) + 4[x(0)]\dot{x}(0) = 1 + 4 \cdot 1 \cdot 2 = 9$. Hence, $x(t) \approx x(0) + \dot{x}(0)t + \frac{1}{2}\ddot{x}(0)t^2 = 1 + 2t + \frac{9}{2}t^2$.

7. Use (7.6.5) with $x = \sigma\sqrt{t/n}$, keeping only three terms on the right-hand side.

8. Use (7.6.2) with $f(x) = (1 + x)^n$ and $x = p/100$. Then $f'(x) = n(1 + x)^{n-1}$ and $f''(x) = n(n-1)(1 + x)^{n-2}$. The approximation follows.

9. $h'(x) = \dfrac{(px^{p-1} - qx^{q-1})(x^p + x^q) - (x^p - x^q)(px^{p-1} + qx^{q-1})}{(x^p + x^q)^2} = \dfrac{2(p-q)x^{p+q-1}}{(x^p + x^q)^2}$, so $h'(1) = \frac{1}{2}(p - q)$. Because $h(1) = 0$, this gives the approximation $h(x) \approx h(1) + h'(1)(x - 1) = \frac{1}{2}(p - q)(x - 1)$.

7.6

1. Using the answer to Exercise 7.5.2, one has $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, and $f'''(z) = 2(1 + z)^{-3}$. Then (7.6.3) gives

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x + \frac{1}{3!}f'''(z)x^3 = x - \frac{1}{2}x^2 + \frac{1}{3}(1 + z)^{-3}x^3$$

2. (a) $\sqrt[3]{25} = 3(1 - 2/27)^{1/3} \approx 3\left(1 - \dfrac{1}{3}\dfrac{2}{27} - \dfrac{1}{9}\dfrac{4}{27^2}\right) \approx 2.924$

(b) $\sqrt[5]{33} = 2(1 + 1/32)^{1/5} \approx 2\left(1 + \dfrac{1}{5 \cdot 32} - \dfrac{2}{25}\dfrac{1}{32^2}\right) \approx 2.0125$

3. $(1 + 1/8)^{1/3} = 1 + 1/24 - 1/576 + R_3(1/8)$, where $0 < R_3(1/8) < 5/(81 \cdot 8^3)$. Thus, $\sqrt[3]{9} = 2(1 + 1/8)^{1/3} \approx 2.080$, correct to three decimal places.

4. (a) $1 + \frac{1}{3}x - \frac{1}{9}x^2$ (b) $g'''(z) = \dfrac{10}{27}(1 + z)^{-8/3}$, so (7.6.2) implies that $R_3(x) = \dfrac{1}{6}\dfrac{10}{27}(1 + z)^{-8/3}x^3$ for some $z \in (0, x)$. Hence $|R_3(x)| \le \dfrac{5}{81}x^3$. For more details, see SM.

(c) First note that $\sqrt[3]{1003} = 10(1 + 3 \cdot 10^{-3})^{1/3}$. Using the approximation in part (a) gives $(1 + 3 \cdot 10^{-3})^{1/3} \approx 1.000999$, and so $\sqrt[3]{1003} \approx 10.00999$. By part (b), the error in this approximation satisfies $|R_3(x)| \le \frac{5}{3}10^{-9}$. Hence, the error in the approximation $\sqrt[3]{1003} \approx 10.00999$ is $10|R_3(x)| \le \frac{50}{3}10^{-9} < 2 \cdot 10^{-8}$, implying that the answer is correct to seven decimal places. For more details, see SM.

7.7

1. In each case, use formula (7.7.3): (a) -3 (b) 100 (c) 1/2, since $\sqrt{x} = x^{1/2}$ (d) $-3/2$, since $A/x\sqrt{x} = Ax^{-3/2}$

2. $\mathrm{El}_K T = 1.06$. A 1% increase in expenditure on road building leads to an increase in the traffic volume of approximately 1.06%.

3. (a) A 10% increase in fares leads to a decrease in passenger demand of approximately 4%.

(b) One reason could be that for long-distance travel, more people fly when rail fares go up. Another reason could be that many people may commute 60 km daily, whereas almost nobody commutes 300 km daily, and commuters' demand is likely to be less elastic.

4. (a) $\mathrm{El}_x e^{ax} = (x/e^{ax})ae^{ax} = ax$ (b) $\mathrm{El}_x \ln x = (x/\ln x)(1/x) = 1/\ln x$

(c) $\mathrm{El}_x(x^p e^{ax}) = \dfrac{x}{x^p e^{ax}}(px^{p-1}e^{ax} + x^p ae^{ax}) = p + ax$

(d) $\mathrm{El}_x(x^p \ln x) = \dfrac{x}{x^p \ln x}(px^{p-1} \ln x + x^p(1/x)) = p + 1/\ln x$

5. $\mathrm{El}_x(f(x))^p = \dfrac{x}{(f(x))^p} p(f(x))^{p-1} f'(x) = p\dfrac{x}{f(x)} f'(x) = p\ \mathrm{El}_x f(x)$

6. Using formula (7.7.3), $\mathrm{El}_r D = 1.23$. A 1% increase in income leads to an increase in demand of approximately 1.23%.

7. $\ln m = -0.02 + 0.19 \ln N$. When $N = 480\,000$, then $m \approx 11.77$.

8. (a) $\mathrm{El}_x Af(x) = \dfrac{x}{Af(x)} Af'(x) = \dfrac{x}{f(x)} f'(x) = \mathrm{El}_x f(x)$

(b) $\mathrm{El}_x \left(A + f(x)\right) = \dfrac{x}{A + f(x)} f'(x) = \dfrac{f(x)xf'(x)/f(x)}{A + f(x)} = \dfrac{f(x)\,\mathrm{El}_x f(x)}{A + f(x)}$

9. Here we prove only (d): $\mathrm{El}_x(f + g) = \dfrac{x(f' + g')}{f + g} = \dfrac{f(xf'/f) + g(xg'/g)}{f + g} = \dfrac{f\,\mathrm{El}_x f + g\,\mathrm{El}_x g}{f + g}$. For the other proofs, see SM.

10. (a) -5 (b) $\dfrac{1 + 2x}{1 + x}$ (c) $\dfrac{30x^3}{x^3 + 1}$ (d) $\mathrm{El}_x 5x^2 = 2$, so $\mathrm{El}_x(\mathrm{El}_x 5x^2) = 0$ (e) $\dfrac{2x^2}{1 + x^2}$

(f) $\mathrm{El}_x \left(\dfrac{x - 1}{x^5 + 1}\right) = \mathrm{El}_x(x - 1) - \mathrm{El}_x(x^5 + 1) = \dfrac{x\,\mathrm{El}_x x}{x - 1} - \dfrac{x^5\,\mathrm{El}_x x^5}{x^5 + 1} = \dfrac{x}{x - 1} - \dfrac{5x^5}{x^5 + 1}$

7.8

1. Only the function in (a) is not continuous.

2. f is discontinuous at $x = 0$. g is continuous at $x = 2$. The graphs of f and g are shown in Figs A7.8.2a and A7.8.2b.

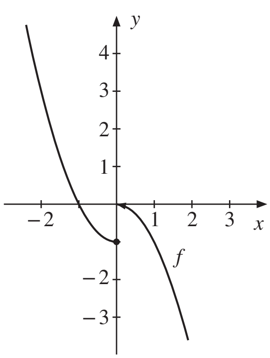


Figure A7.8.2a

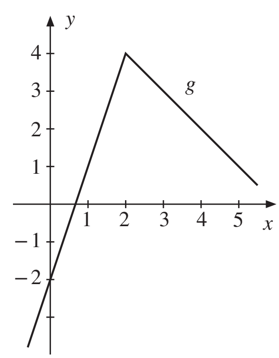


Figure A7.8.2b

3. (a) Continuous for all x. (b) Continuous for all $x \neq 1$. (c) Continuous for all $x < 2$. (d) Continuous for all x. (e) Continuous for all x where $x \neq \sqrt{3} - 1$ and $x \neq -\sqrt{3} - 1$. (f) Continuous for all $x > 0$.

4. See Fig. A7.8.4; y is discontinuous at $x = a$, where the aeroplane is vertically above the top of the overhanging cliff.

5. $a = 5$. (The line $y = ax - 1$ and parabola $y = 3x^2 + 1$ must meet when $x = 1$, which is true if and only if $a = 5$.)

6. See Fig. A7.8.6. (This example shows that the commonly seen statement: "if the inverse function exists, the original and the inverse function must both be monotonic" is wrong. This claim is correct, however, for a function which is continuous on an interval.)

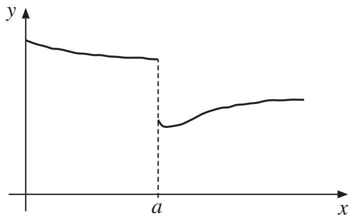


Figure A7.8.4

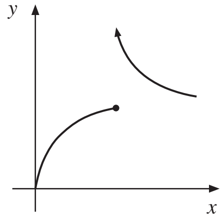


Figure A7.8.6

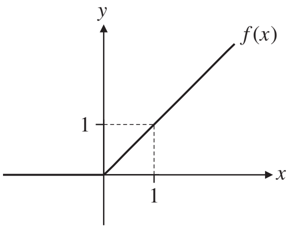


Figure A7.9.8

7.9

1. (a) A (b) A (c) B (d) 0

2. (a) -4 (b) 0 (c) 2 (d) $-\infty$ (e) ∞ (f) $-\infty$

3. (a) $\dfrac{x-3}{x^2+1} = \dfrac{1/x - 3/x^2}{1+1/x^2} \to 0$ as $x \to \infty$. (b) $\sqrt{\dfrac{2+3x}{x-1}} = \sqrt{\dfrac{3+2/x}{1-1/x}} \to \sqrt{3}$ as $x \to \infty$. (c) a^2

4. $\lim\limits_{x\to\infty} f_i(x) = \infty$ for $i = 1, 2, 3$; $\lim\limits_{x\to\infty} f_4(x) = 0$. Then: (a) ∞ (b) 0 (c) $-\infty$ (d) 1 (e) 0 (f) ∞ (g) 1 (h) ∞

5. (a) $y = x - 1$ ($x = -1$ is a vertical asymptote). (b) $y = 2x - 3$

(c) $y = 3x + 5$ ($x = 1$ is a vertical asymptote). (d) $y = 5x$ ($x = 1$ is a vertical asymptote).

6. $y = Ax + A(b - c) + d$ is an asymptote as $x \to \infty$. ($x = -c$ is not an asymptote because $x \geq 0$.)

7. (a) Neither continuous nor differentiable at $x = 1$ because $f(1) = B$. (b) Continuous but not differentiable at $x = 2$. (c) Neither continuous nor differentiable at $x = 3$. (d) Continuous but not differentiable at $x = 4$.

8. $f'(0^+) = 1$ and $f'(0^-) = 0$. See Fig. A7.9.8.

9. $f'(x) = \dfrac{3(x-1)(x+1)}{(-x^2+4x-1)^2}$. Then $f(x)$ is increasing in $(-\infty, -1]$, in $\left[1, 2+\sqrt{3}\right)$, and in $\left(2+\sqrt{3}, \infty\right)$. See SM for a sign diagram and more details.

7.10

1. (a) Let $f(x) = x^7 - 5x^5 + x^3 - 1$. Then f is continuous, $f(-1) = 2$, and $f(1) = -4$, so according to Theorem 7.10.1, the equation $f(x) = 0$ has a solution in $(-1, 1)$. Parts (b), (c), and (d) can be shown using the same method of showing that a suitable function has different signs at the two end points of the specified interval.

2. A person's height is a continuous function of time (even if growth occurs in intermittent spurts, often overnight). The intermediate value theorem (and common sense) give the conclusion.

3. Let $f(x) = x^3 - 17$. Then $f(x) = 0$ for $x = \sqrt[3]{17}$. Moreover, $f'(x) = 3x^2$. Put $x_0 = 2.5$. Then $f(x_0) = -1.375$ and $f'(x_0) = 18.75$. Formula (7.10.1) with $n = 0$ yields $x_1 = x_0 - f(x_0)/f'(x_0) = 2.5 - (-1.375)/18.75 \approx 2.573$.

4. The integer root is $x = -3$. One step of Newton's method gives -1.879, 0.347, and 1.534 for the three other roots.

5. The integer which is closest to being a solution is $x = 2$. Put $f(x) = (2x)^x - 15$. Then $f'(x) = (2x)^x[\ln(2x) + 1]$. Formula (7.10.1) with $x_0 = 2$ and $n = 0$ yields $x_1 = x_0 - f(x_0)/f'(x_0) = 2 - f(2)/f'(2) = 2 - 1/[16(\ln 4 + 1)] \approx 1.9738$.

6. If $f(x_0)$ and $f'(x_0)$ have the same sign (as in Fig. 7.10.2), then (7.10.1) implies that $x_1 < x_0$. But if they have opposite signs, then $x_1 > x_0$.

7.11

1. (a) $\alpha_n = \dfrac{(3/n)-1}{2-(1/n)} \to -\dfrac{1}{2}$ as $n \to \infty$ (b) $\beta_n = \dfrac{1+(2/n)-(1/n^2)}{3-(2/n^2)} \to \dfrac{1}{3}$ as $n \to \infty$ (c) $3(-1/2)+4(1/3) = -1/6$

(d) $(-1/2)\cdot(1/3) = -1/6$ (e) $(-1/2)\div(1/3) = -3/2$ (f) $\sqrt{(1/3)-(-1/2)} = \sqrt{5/6} = \sqrt{30}/6$

2. (a) As $n \to \infty$, so $2/n \to 0$ implying that $5 - 2/n \to 5$. (b) As $n \to \infty$, so $\dfrac{n^2-1}{n} = n - 1/n \to \infty$.

(c) As $n \to \infty$, so $\dfrac{3n}{\sqrt{2n^2-1}} = \dfrac{3n}{n\sqrt{2-1/n^2}} = \dfrac{3}{\sqrt{2-1/n^2}} \to \dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$.

3. For a fixed number x, put $x/n = 1/m$. Then $n = mx$, and as $n \to \infty$, so $m \to \infty$.

Hence $(1+x/n)^n = (1+1/m)^{mx} = [(1+1/m)^m]^x \to e^x$ as $m \to \infty$.

7.12

1. (a) $\displaystyle\lim_{x\to3}\frac{3x^2-27}{x-3} = \text{“}0/0\text{”} = \lim_{x\to3}\frac{6x}{1} = 18$ (or use $3x^2-27 = 3(x-3)(x+3)$).

(b) $\displaystyle\lim_{x\to0}\frac{e^x-1-x-\frac{1}{2}x^2}{3x^3} = \text{“}0/0\text{”} = \lim_{x\to0}\frac{e^x-1-x}{9x^2} = \text{“}0/0\text{”} = \lim_{x\to0}\frac{e^x-1}{18x} = \text{“}0/0\text{”} = \lim_{x\to0}\frac{e^x}{18} = \frac{1}{18}$

(c) $\displaystyle\lim_{x\to0}\frac{e^{-3x}-e^{-2x}+x}{x^2} = \text{“}0/0\text{”} = \lim_{x\to0}\frac{-3e^{-3x}+2e^{-2x}+1}{2x} = \text{“}0/0\text{”} = \lim_{x\to0}\frac{9e^{-3x}-4e^{-2x}}{2} = \frac{5}{2}$

2. (a) $\displaystyle\lim_{x\to a}\frac{x^2-a^2}{x-a} = \text{“}0/0\text{”} = \lim_{x\to a}\frac{2x}{1} = 2a$ (or use $x^2-a^2 = (x+a)(x-a)$).

(b) $\displaystyle\lim_{x\to0}\frac{2(1+x)^{1/2}-2-x}{2(1+x+x^2)^{1/2}-2-x} = \text{“}0/0\text{”} = \lim_{x\to0}\frac{(1+x)^{-1/2}-1}{(1+2x)(1+x+x^2)^{-1/2}-1} = \text{“}0/0\text{”}$

$\displaystyle= \lim_{x\to0}\frac{-\frac{1}{2}(1+x)^{-3/2}}{2(1+x+x^2)^{-1/2}+(1+2x)^2(-\frac{1}{2})(1+x+x^2)^{-3/2}} = \frac{-\frac{1}{2}}{2-\frac{1}{2}} = -\frac{1}{3}$

3. (a) $\frac{1}{2}$ (b) 3 (c) 2 (d) $-\frac{1}{2}$ (e) $\frac{3}{8}$ (f) -2

4. (a) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^{1/2}} = \text{“}\infty/\infty\text{”} = \lim_{x\to\infty}\frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to\infty}\frac{2}{x^{1/2}} = 0$

(b) 0. (Write $x\ln x = \dfrac{\ln x}{1/x}$, and then use l'Hôpital's rule.)

(c) $+\infty$. (Write $xe^{1/x}-x = x(e^{1/x}-1) = (e^{1/x}-1)/(1/x)$, and then use l'Hôpital's rule.)

5. The second fraction is not “0/0”. The correct limit is 5/2.

6. $\displaystyle L = \lim_{v\to0^+}\frac{1-(1+v^\beta)^{-\gamma}}{v} = \text{“}0/0\text{”} = \lim_{v\to0^+}\frac{\gamma(1+v^\beta)^{-\gamma-1}\beta v^{\beta-1}}{1}$. If $\beta = 1$, then $L = \gamma$. If $\beta > 1$, then $L = 0$, and if $\beta < 1$, then $L = \infty$.

7. Because $\dfrac{d}{d\rho}c^{1-\rho} = -c^{1-\rho}\ln c$, one has $\displaystyle\lim_{\rho\to1}\frac{c^{1-\rho}-1}{1-\rho} = \text{“}0/0\text{”} = \lim_{\rho\to1}\frac{-c^{1-\rho}\ln c}{-1} = \ln c$.

8. $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to0^+}\frac{f(1/t)}{g(1/t)} = \text{“}0/0\text{”} = \lim_{t\to0^+}\frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} = \lim_{t\to0^+}\frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$

9. Note that $L = \lim_{x\to a} \dfrac{1/g(x)}{1/f(x)} =$ "0/0" $= \lim\limits_{x\to a} \dfrac{-1/(g(x))^2}{-1/(f(x))^2} \cdot \dfrac{g'(x)}{f'(x)} = L^2 \cdot \lim\limits_{x\to a} \dfrac{g'(x)}{f'(x)}$. See SM for more details.

Review exercises for Chapter 7

1. (a) $y' = -5$, $y'' = 0$ (b) Differentiating w.r.t. x yields $y^3 + 3xy^2y' = 0$, so $y' = -y/3x$. Then differentiating $y' = -y/3x$ w.r.t. x yields $y'' = -[y'3x - 3y]/9x^2 = -[(-y/3x)3x - 3y]/9x^2 = 4y/9x^2$. Because $y = 5x^{-1/3}$, we get $y' = -(5/3)x^{-4/3}$ and $y'' = (20/9)x^{-7/3}$. The answers from differentiating $y = 5x^{-1/3}$ are the same.

(c) $2y'e^{2y} = 3x^2$, so $y' = (3x^2/2)e^{-2y}$. Then $y'' = 3xe^{-2y} + \frac{1}{2}3x^2e^{-2y}(-2y') = 3xe^{-2y} - \frac{1}{2}9x^4e^{-4y}$. From the equation $e^{2y} = x^3$ we get $2y = \ln x^3 = 3\ln x$, so $y = \frac{3}{2}\ln x$, and then $y' = \frac{3}{2}x^{-1}$, $y'' = -\frac{3}{2}x^{-2}$. By noting that $e^{-2y} = e^{-3\ln x} = (e^{\ln x})^{-3} = x^{-3}$ and $e^{-4y} = (e^{-2y})^2 = x^{-6}$, verify that the answers are the same.

2. $5y^4y' - y^2 - 2xyy' = 0$, so $y' = \dfrac{y^2}{5y^4 - 2xy} = \dfrac{y}{5y^3 - 2x}$. Because $y = 0$ makes the given equation meaningless, y' is never 0.

3. Differentiating w.r.t. x yields $3x^2 + 3y^2y' = 3y + 3xy'$. When $x = y = 3/2$, then $y' = -1$.

4. (a) Implicit differentiation yields $(*)$ $2xy + x^2y' + 9y^2y' = 0$. Inserting $x = 2$ and $y = 1$ yields $y' = -4/13$.

(b) Differentiating $(*)$ w.r.t. x yields $2y + 2xy' + 2xy' + x^2y'' + 18yy'y' + 9y^2y'' = 0$. Inserting $x = 2$, $y = 1$, and $y' = -4/13$, then solving for y'', gives the answer.

5. $\frac{1}{3}K^{-2/3}L^{1/3} + \frac{1}{3}K^{1/3}L^{-2/3}(\mathrm{d}L/\mathrm{d}K) = 0$, so $\mathrm{d}L/\mathrm{d}K = -L/K$.

6. Differentiating w.r.t. x gives $y'/y + y' = -2/x - 0.4(\ln x)/x$. Solving for y' gives $y' = \dfrac{-(2/x)(1 + \frac{1}{5}\ln x)}{1 + 1/y}$. Thus $y' = 0$ when $1 + \frac{1}{5}\ln x = 0$, implying that $\ln x = -5$ and so $x = e^{-5}$.

7. (a) Straightforward substitution.

(b) $\mathrm{d}Y/\mathrm{d}I = f'((1-\beta)Y - \alpha)(1-\beta)(\mathrm{d}Y/\mathrm{d}I) + 1$. Solving for $\mathrm{d}Y/\mathrm{d}I$ yields $\dfrac{\mathrm{d}Y}{\mathrm{d}I} = \dfrac{1}{1 - (1-\beta)f'((1-\beta)Y - \alpha)}$.

(c) Since $f' \in (0, 1)$ and $\beta \in (0, 1)$, we get $(1-\beta)f'((1-\beta)Y - \alpha) \in (0, 1)$, so $\mathrm{d}Y/\mathrm{d}I > 0$.

8. (a) Differentiating w.r.t. x yields $2x - y - xy' + 4yy' = 0$, so $y' = (y - 2x)/(4y - x)$.

(b) Horizontal tangents at $(1, 2)$ and $(-1, -2)$. ($y' = 0$ when $y = 2x$. Insert this into the given equation.) Vertical tangents at $(2\sqrt{2}, \sqrt{2}/2) \approx (2.8, 0.7)$ and at $(-2\sqrt{2}, -\sqrt{2}/2) \approx (-2.8, -0.7)$. (There is a vertical tangent when the denominator in the expression for y' is 0, i.e. when $x = 4y$.) See Fig. 7.R.2.

9. (a) $y' = \dfrac{2 - 2xy}{x^2 - 9y^2} = -\dfrac{1}{2}$ at $(-1, 1)$. (b) Vertical tangent at $(0, 0)$, $(-3, -1)$, and $(3, 1)$. (Vertical tangent requires the denominator of y' to be 0, i.e. $x = \pm 3y$. Inserting $x = 3y$ into the given equation yields $y^3 = y$, so $y = 0$, $y = 1$, or $y = -1$. The corresponding values for x are 0, 3, and -3. Inserting $y = -3x$ gives no new points.)

Horizontal tangent requires $y' = 0$, i.e. $xy = 1$. But inserting $y = 1/x$ into the given equation yields $x^4 = -3$, which has no solution. All these findings accord with Fig. 7.R.3.

10. (a) $D_f = (-1, 1)$, $R_f = (-\infty, \infty)$ (b) The inverse is $g(y) = (e^{2y} - 1)/(e^{2y} + 1)$, and then $g'(\frac{1}{2}\ln 3) = 3/4$.

11. (a) $f(e^2) = 2$ and $f(x) = \ln x(\ln x - 1)^2 = 0$ for $\ln x = 0$ and for $\ln x = 1$, so $x = 1$ or $x = e$.

(b) $f'(x) = (3/x)(\ln x - 1)(\ln x - 1/3) > 0$ for $x > e$, and so f is strictly increasing in $[e, \infty)$. It therefore has an inverse h. According to (7.3.2), because $f(e^2) = 2$, we have $h'(2) = 1/f'(e^2) = e^2/5$.

12. (a) $f(x) \approx \ln 4 + \frac{1}{2}x - \frac{1}{8}x^2$ (b) $g(x) \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2$ (c) $h(x) \approx x + 2x^2$

13. (a) $x\,\mathrm{d}x/\sqrt{1+x^2}$ (b) $8\pi r\,\mathrm{d}r$ (c) $400K^3\,\mathrm{d}K$ (d) $-3x^2\,\mathrm{d}x/(1-x^3)$

14. $\mathrm{d}f(x) = f'(x)\,\mathrm{d}x = 3x^2\,\mathrm{d}x/2\sqrt{1+x^3}$. Moreover, $\Delta f(2) \approx \mathrm{d}f(2) = 3 \cdot 2^2(0.2)/2\sqrt{1+2^3} = 0.4$.

15. Let $x = \frac{1}{2}$ and $n = 5$ and use formula (7.6.6). This gives $\sqrt{e} \approx 1.649$. A calculator shows this is correct to three decimals.

16. $y' + (1/y)y' = 1$, or $(*)$ $yy' + y' = y$. Then $y' = 1/2$ at $y = 1$. Differentiating $(*)$ w.r.t. x gives $(y')^2 + yy'' + y'' = y'$. With $y = 1$ and $y' = 1/2$, we find $y'' = 1/8$, so $y(x) \approx 1 + \frac{1}{2}x + \frac{1}{16}x^2$.

17. (a) Continuous for all $x \neq 0$. (b) Continuous for all $x > 0$. (Note that $x^2 + 2x + 2$ is never 0.)

(c) Continuous for all x in $(-2, 2)$.

18. (a) $1 = f'(y^2)2yy'$, so $y' = \dfrac{1}{2yf'(y^2)}$. (b) $y^2 + x2yy' = f'(x) - 3y^2y'$, and so $y' = \dfrac{f'(x) - y^2}{y(2x+3y)}$.

(c) $f'(2x+y)(2+y') = 1 + 2yy'$, so $y' = \dfrac{1 - 2f'(2x+y)}{f'(2x+y) - 2y}$.

19. $\mathrm{El}_r(D_{\text{marg}}) = -0.165$ and $\mathrm{El}_r(D_{\text{mah}}) = 2.39$. For each 1% increase in income, the demand for margarine decreased by approximately 0.165%, while the demand for meals away from home increased by approximately 2.39%.

20. (a) 5 (using formula (7.7.3)). (b) 1/3 (using $\sqrt[3]{x} = x^{1/3}$ and (7.7.3)).

(c) $\mathrm{El}_x(x^3 + x^5) = \dfrac{x}{x^3 + x^5}(3x^2 + 5x^4) = (5x^2 + 3)/(x^2 + 1)$, or alternatively use part (d) of Exercise 7.7.9.

(d) $2x/(x^2 - 1)$, using parts (c) and (d) of Exercise 7.7.9.

21. Put $f(x) = x^3 - x - 5$. Then $f'(x) = 3x^2 - 1$. Taking $x_0 = 2$, formula (7.10.1) with $n = 1$ gives $x_1 = 2 - f(2)/f'(2) = 2 - 1/11 \approx 1.909$.

22. f is continuous, with $f(1) = e - 3 < 0$ and $f(4) = e^2 - 3 > 0$. By Theorem 7.10.1(i), there is a zero for f in $(1, 4)$. Because $f'(x) > 0$, the solution is unique. Formula (7.10.1) yields $x_1 = 1 - f(1)/f'(1) = -1 + 6/e \approx 1.21$.

23. (a) 2 (b) Tends to $+\infty$. (c) No limit exists. (d) $-1/6$ (e) 1/5 (f) 1/16 (g) 1 (h) $-1/16$ (i) 0

24. Does not exist if $b \neq d$. If $b = d$, the limit is $(a - c)/2\sqrt{b}$.

25. $\displaystyle\lim_{x\to 0} \frac{a^x - b^x}{e^{ax} - e^{bx}} = \text{“}0/0\text{”} = \lim_{x\to 0} \frac{a^x \ln a - b^x \ln b}{ae^{ax} - be^{bx}} = \frac{\ln a - \ln b}{a - b}$

26. $x_1 = 0.9 - f(0.9)/f'(0.9) \approx 0.9247924$, $x_2 = x_1 - f(x_1)/f'(x_1) \approx 0.9279565$, $x_3 = x_2 - f(x_2)/f'(x_2) \approx 0.9280338$, and $x_4 = x_3 - f(x_3)/f'(x_3) \approx 0.9280339$. This suggests that the answer correct to three decimal places is 0.928.

Chapter 8

8.1

1. (a) $f(0) = 2$ and $f(x) \le 2$ for all x (we divide 8 by a number greater than or equal to 4), so $x = 0$ maximizes $f(x)$.

(b) $g(-2) = -3$ and $g(x) \ge -3$ for all x, so $x = -2$ minimizes $g(x)$. $g(x) \to \infty$ as $x \to \infty$, so there is no maximum.

(c) $h(x)$ has its largest value 1 when $1 + x^4$ is at its smallest, namely for $x = 0$, and $h(x)$ has its smallest value 1/2 when $1 + x^4$ is at its largest, namely for $x = \pm 1$.

(d) For all x one has $2 + x^2 \ge 2$ and so $2/(2 + x^2) \le 1$, implying that $-2/(2 + x^2) \ge -1 = F(0)$. Hence there is a minimum -1 at $x = 0$, but no maximum.

(e) Maximum 2 at $x = 1$. No minimum. (f) Minimum 99 at $x = 0$. No maximum. (When $x \to \pm\infty$, $H(x) \to 100$.)

8.2

1. $y' = 1.06 - 0.08x$. Then $y' \geq 0$ for $x \leq 13.25$ and $y' \leq 0$ for $x \geq 13.25$, so y has a maximum at $x = 13.25$.

2. $h'(x) = \dfrac{8(2-\sqrt{3}x)(2+\sqrt{3}x)}{(3x^2+4)^2}$. The function has a maximum at $x = 2\sqrt{3}/3$ and a minimum at $x = -2\sqrt{3}/3$.

3. $h'(t) = 1/2\sqrt{t} - \frac{1}{2} = (1-\sqrt{t})/2\sqrt{t}$. We see that $h'(t) \geq 0$ in $[0,1]$ and $h'(t) \leq 0$ in $[1,\infty)$. According to part (i) of Theorem 8.2.1, $t = 1$ maximizes $h(t)$.

4. $f'(x) = [4x(x^4+1) - 2x^2 4x^3]/(x^4+1)^2 = 4x(1-x^4)/(x^4+1)^2$, implying that $f(x)$ increases in $[0,1]$, but decreases in $[1,\infty)$. It follows that f has a maximum $f(1) = 1$ at $x = 1$.

5. $g'(x) = 3x^2 \ln x + x^3/x = 3x^2(\ln x + \frac{1}{3})$. So $g'(x) = 0$ when $\ln x = -\frac{1}{3}$, i.e. $x = e^{-1/3}$. We see that $g'(x) \leq 0$ in $(0, e^{-1/3}]$ and $g'(x) \geq 0$ in $[e^{-1/3}, \infty)$, so $x = e^{-1/3}$ minimizes $g(x)$. Since $g(x) \to \infty$ as $x \to \infty$, there is no maximum.

6. $f'(x) = 3e^x(e^{2x} - 2)$. Then $f'(x) = 0$ when $e^{2x} = 2$, so $x = \frac{1}{2}\ln 2$. If $x < \frac{1}{2}\ln 2$ then $f'(x) < 0$, and if $x > \frac{1}{2}\ln 2$ then $f'(x) > 0$, so $x = \frac{1}{2}\ln 2$ is a minimum point. Note that $f(x) = e^x(e^{2x} - 6)$ tends to $+\infty$ as $x \to \infty$, so f has no maximum.

7. $y' = xe^{-x}(2-x)$ is positive in $(0,2)$ and negative in $(2,4)$, so y has a maximum $4e^{-2} \approx 0.54$ at $x = 2$.

8. (a) $x = \frac{1}{3}\ln 2$ is a minimum point. (b) $x = \frac{1}{3}(a+2b)$ is a maximum point. (c) $x = \frac{1}{5}$ is a maximum point.

9. $d'(x) = 2(x-a_1) + 2(x-a_2) + \cdots + 2(x-a_n) = 2[nx - (a_1 + a_2 + \cdots + a_n)]$. So $d'(x) = 0$ for $x = \bar{x}$, where $\bar{x} = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$, the *arithmetic mean* of $a_1, a_2, \ldots, a_n$. Since $d''(x) = 2n > 0$, $\bar{x}$ minimizes $d(x)$.

10. (a) $x_0 = (1/\alpha)\ln(A\alpha/k)$ (b) Substituting for A in the expression for x_0 gives the optimal height as a function of p_0, V, δ, and k. See SM.

8.3

1. (a) $\pi(L) = 320\sqrt{L} - 40L$, so $\pi'(L) = \dfrac{160}{\sqrt{L}} - 40 = \dfrac{40(4-\sqrt{L})}{\sqrt{L}}$. We see that $\pi'(L) \geq 0$ for $0 \leq L \leq 16$, whereas $\pi'(16) = 0$, and $\pi'(L) \leq 0$ for $L \geq 16$, so $L = 16$ maximizes profits.

(b) The profit function is $\pi(L) = f(L) - wL$, so the first-order condition is $\pi'(L^*) = f'(L^*) - w = 0$.

(c) The first-order condition in (b) defines L^* as a function of w. Differentiating w.r.t. w gives $f''(L^*)(\mathrm{d}L^*/\mathrm{d}w) - 1 = 0$, or $\mathrm{d}L^*/\mathrm{d}w = 1/f''(L^*) < 0$. (If the price of labour increases, the optimal labour input decreases.)

2. (a) $Q^* = \frac{1}{2}(a-k)$, $\pi(Q^*) = \frac{1}{4}(a-k)^2$ (b) $\mathrm{d}\pi(Q^*)/\mathrm{d}k = -\frac{1}{2}(a-k) = -Q^*$ (c) $s = a - k$

3. See Figs A8.3.3a and A8.3.3b. One needs $x < 9$ to avoid cutting away everything. Differentiating V gives $V'(x) = 12(x-3)(x-9)$. So $V'(x) > 0$ if $x < 3$, but $V'(x) < 0$ if $3 < x < 9$. Theorem 8.2.1 implies that the box has maximum volume when the square cut out from each corner has sides of length 3 cm. Then the volume is $12 \times 12 \times 3 = 432\ \text{cm}^3$.

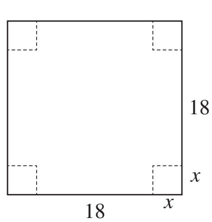


Figure A8.3.3a

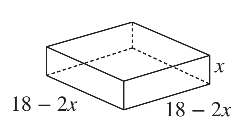


Figure A8.3.3b

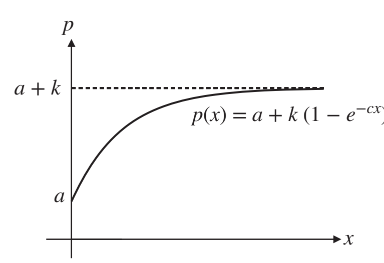


Figure A8.3.4

4. $p'(x) = kce^{-cx}$, and $p''(x) = -kc^2e^{-cx}$. No maximum exists, and $p(x) \to a + k$ as $x \to \infty$. See Fig. A8.3.4.

5. $\bar{T}'(W) = a\dfrac{pb(bW + c)^{p-1}W - (bW + c)^p}{W^2} = a(bW + c)^{p-1}\dfrac{bW(p - 1) - c}{W^2}$, which is 0 for $W^* = c/b(p - 1)$.

This must be the minimum point because $\bar{T}'(W)$ is negative for $W < W^*$ and positive for $W > W^*$.

8.4

1. $f'(x) = 8x - 40 = 0$ for $x = 5$. $f(0) = 80$, $f(5) = -20$, and $f(8) = 16$. Maximum 80 for $x = 0$. Minimum -20 for $x = 5$. See Fig. A8.4.1.

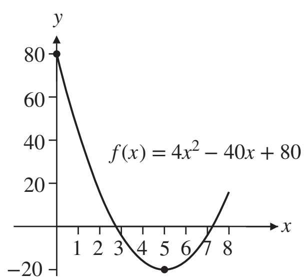


Figure A8.4.1

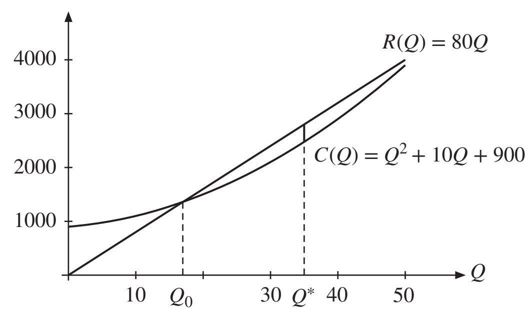


Figure A8.5.2

2. (a) Max. -1 at $x = 0$. Min. -7 at $x = 3$. (b) Max. 10 at $x = -1$ and $x = 2$. Min. 6 at $x = 1$.

(c) Max. 5/2 at $x = 1/2$ and $x = 2$. Min. 2 at $x = 1$. (d) Max. 4 at $x = -1$. Min. $-6\sqrt{3}$ at $x = \sqrt{3}$.

(e) Max. $4.5 \cdot 10^9$ at $x = 3000$. Min. 0 at $x = 0$.

3. $g'(x) = \frac{2}{5}xe^{x^2}(1 - e^{2-2x^2})$. Critical points: $x = 0$ and $x = \pm 1$. Here $x = 2$ is a maximum point, whereas $x = 1$ and $x = -1$ are minimum points. (Note that $g(2) = \frac{1}{5}(e^4 + e^{-2}) > g(0) = \frac{1}{5}(1 + e^2)$.)

4. (a) Total commission is, respectively, \$4819, \$4900, \$4800, and $C = \frac{1}{10}(60 + x)(800 - 10x) = 4800 + 20x - x^2$, for $x \in [0, 20]$. (When there are $60 + x$ passengers, the charter company earns $800 - 10x$ from each, so they earn $\$(60 + x)(800 - 10x)$. The sports club earns 1/10 of that amount.)

(b) The quadratic function C has its maximum for $x = 10$, so the maximum commission is with 70 travellers.

5. (a) $f(x) = \ln x(\ln x - 1)^2$. So $f(e^{1/3}) = 4/27$, $f(e^2) = 2$, $f(e^3) = 12$. The zeros are at $x = 1$ and $x = e$.

(b) $f'(x) = (3/x)(\ln x - 1)(\ln x - 1/3)$. Minimum 0 at $x = 1$ and at $x = e$. Maximum 12 at $x = e^3$.

(c) $f'(x) > 0$ in $[e, e^3]$, so $f(x)$ has an inverse. $g'(2) = 1/f'(e^2) = e^2/5$.

6. (a) $x^* = 3/2$ (b) $x^* = \sqrt{2}/2$ (c) $x^* = \sqrt{12}$ (d) $x^* = \sqrt{3}$

7. There is at least one point where you must be heading in the direction of the straight line joining A to B (even if that straight line hits the shore).

8. f is not continuous at $x = -1$ and $x = 1$. It has no maximum because $f(x)$ is arbitrarily close to 1 for x sufficiently close to 1. But there is no value of x for which $f(x) = 1$. Similarly, there is no minimum.

9. f has a maximum at $x = 1$ and a minimum at all $x > 1$. (Draw your own graph.) Yet the function is discontinuous at $x = 1$, and its domain of definition is neither closed nor bounded.

8.5

1. $\pi(Q) = 10Q - \dfrac{1}{1000}Q^2 - (5000 + 2Q) = 8Q - \dfrac{1}{1000}Q^2 - 5000.$

Since $\pi'(Q) = 8 - \dfrac{1}{500}Q = 0$ for $Q = 4000$, and $\pi''(Q) = -\dfrac{1}{500} < 0$, output $Q = 4000$ maximizes profits.

2. (a) See Fig. A8.5.2. (b) (i) The requirement is $\pi(Q) \geq 0$ and $Q \in [0, 50]$, that is $-Q^2 + 70Q - 900 \geq 0$ and $Q \in [0, 50]$. The firm must produce at least $Q_0 = 35 - 5\sqrt{13} \approx 17$ units. (ii) Profits are maximized at $Q^* = 35$.

3. Profits are given by $\pi(x) = -0.003x^2 + 120x - 500\,000$, which is maximized at $x = 20\,000$.

4. (a) $Q^* = 450$ (b) $Q^* = 550$ (c) $Q^* = 0$

5. (a) $\pi(Q) = QP(Q) - C(Q) = -0.01Q^2 + 14Q - 4500$, which is maximized at $Q = 700$.

(b) $\mathrm{El}_Q P(Q) = (Q/P(Q))P'(Q) = Q/(Q - 3000) = -1$ for $Q^* = 1500$.

(c) $R(Q) = QP(Q) = 18Q - 0.006Q^2$, so $R'(Q) = 18 - 0.012Q = 0$ for $Q^* = 1500$.

6. $\pi'(Q) = P - abQ^{b-1} = 0$ when $Q^{b-1} = P/ab$, or $Q = (P/ab)^{1/(b-1)}$. Moreover, $\pi''(Q) = -ab(b-1)Q^{b-2} < 0$ for all $Q > 0$, so this is a maximum point.

8.6

1. $f'(x) = 3x^2 - 12 = 0$ at $x = \pm 2$. A sign diagram shows that $x = 2$ is a local minimum point and $x = -2$ is a local maximum point. Since $f''(x) = 6x$, this is confirmed by Theorem 8.6.2.

2. (a) No local extreme points. (b) Local maximum 10 at $x = -1$. Local minimum 6 at $x = 1$.

(c) Local maximum -2 at $x = -1$. Local minimum 2 at $x = 1$.

(d) Local maximum $6\sqrt{3}$ at $x = -\sqrt{3}$. Local minimum $-6\sqrt{3}$ at $x = \sqrt{3}$.

(e) No local maximum point. Local minimum $1/2$ at $x = 3$.

(f) Local maximum 2 at $x = -2$. Local minimum -2 at $x = 0$.

3. (a) $D_f = [-6, 0) \cup (0, \infty)$, and $f(x) > 0$ in $(-6, -2) \cup (0, \infty)$.

(b) Local maximum $\frac{1}{2}\sqrt{2}$ at $x = -4$. Local minima $(8/3)\sqrt{3}$ at $x = 6$, and 0 at $x = -6$ (where $f'(x)$ is undefined).

(c) $f(x) \to -\infty$ as $x \to 0^-$, $f(x) \to \infty$ as $x \to 0^+$, $f(x) \to \infty$ as $x \to \infty$, and $f'(x) \to 0$ as $x \to \infty$. So f attains neither a maximum nor a minimum.

4. Look at the point a. Since the graph shows $f'(x)$, one has $f'(x) < 0$ to the left of a, $f'(a) = 0$, and $f'(x) > 0$ to the right of a, so a is a local minimum point. At the points b and e, $f'(x) > 0$ on both sides of each point, so they cannot be extreme points. At c, f has a local maximum, and at d it has a local minimum point.

5. (a) $f'(x) = 3x^2 + 2ax + b$, $f''(x) = 6x + 2a$. So $f'(0) = 0$ requires $b = 0$, and $f''(0) \geq 0$ requires $a \geq 0$. If $a = 0$ and $b = 0$, then $f(x) = x^3 + c$, which does not have a local minimum at $x = 0$. Hence, f has a local minimum at 0 if and only if $a > 0$ and $b = 0$.

(b) $f'(1) = 0$ and $f'(3) = 0$ require $3 + 2a + b = 0$ and $27 + 6a + b = 0$, which means that $a = -6$ and $b = 9$.

6. (a) $f'(x) = x^2e^x(3 + x)$. Use a sign diagram to show that $x = -3$ is a local (and global) minimum point. There are no local maximum points. ($x = 0$ is an inflection point (see next section)).

(b) $g'(x) = x2^x(2 + x\ln 2)$. Then $x = 0$ is a local minimum point and $x = -2/\ln 2$ is a local maximum point.

7. It is easy to see that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ if $x \to -\infty$, so by the intermediate value theorem $f(x) = 0$ for at least one x. If $a \geq 0$, then $f'(x) > 0$ for all $x \neq 0$, so f is strictly increasing in all of its domain, and the equation

$f(x) = 0$ cannot have more than one solution. If $a < 0$, then $f'(0) < 0$ and the graph of f has roughly the same shape as the graph in Fig. 4.7.4. Then f has one local maximum point and one local minimum point and it is easy to see that the graph intersects the x-axis at three different points if and only if the local maximum is greater than zero and the local minimum is less than zero. Find expressions for these two local extreme values, then find a criterion for them to have different signs. See SM.

8.7

1. (a) $f'(x) = 3x^2 + 3x - 6 = 3(x-1)(x+2)$, so $x = -2$ and $x = 1$ are critical points. A sign diagram reveals that f increases in $(-\infty, -2]$ and in $[1, \infty)$.

(b) $f''(x) = 6x + 3 = 0$ for $x = -1/2$ and $f''(x)$ changes sign around $x = -1/2$, so this is an inflection point.

2. (a) $f''(x) = 2x(x^2 - 3)/(1 + x^2)^3$. f is convex in $[-\sqrt{3}, 0]$ and in $[\sqrt{3}, \infty)$. The inflection points are $x = -\sqrt{3}, 0, \sqrt{3}$.

(b) $g''(x) = 4(1 + x)^{-3} > 0$ when $x > -1$, so g is (strictly) convex in $(-1, \infty)$. No inflection point.

(c) $h''(x) = (2 + x)e^x$, so h is convex in $[-2, \infty)$ and $x = -2$ is an inflection point.

3. (a) $x = -1$ is a local (and global) maximum point, $x = 0$ is an inflection point.

(b) $x = 1$ is a local (and global) minimum point, $x = 2$ is an inflection point.

(c) $x = 3$ is a local maximum point, $x = 0$, $3 - \sqrt{3}$, and $3 + \sqrt{3}$ are inflection points.

(d) $x = \sqrt{e}$ is a local (and global) maximum point, $x = e^{5/6}$ is an inflection point.

(e) $x = 0$ is a local (and global) minimum point, $x = -\ln 2$ is an inflection point.

(f) $x = -\sqrt{2}$ is a local minimum point, $x = \sqrt{2}$ is a local maximum point, $x = 1 - \sqrt{3}$ and $x = 1 + \sqrt{3}$ are inflection points.

4. (a) For $x > 0$ one has $R = p\sqrt{x}$, $C = wx + F$, and $\pi(x) = p\sqrt{x} - wx - F$.

(b) $\pi'(x) = 0$ when $w = p/2\sqrt{x}$. (Marginal cost = price times marginal product.) Then $x = p^2/4w^2$. Moreover, $\pi''(x) = -\frac{1}{4}px^{-3/2} < 0$ for all $x > 0$, so profit is maximized over $(0, \infty)$. When $x = p^2/4w^2$, then $\pi = p^2/2w - p^2/4w - F = p^2/4w - F$. So this is a profit maximum if $F \le p^2/4w$; otherwise, the firm does better not to start up and to choose $x = 0$.

5. $x = -2$ and $x = 4$ are minimum points, whereas $x = 2$ is a (possibly local) maximum point. Moreover, $x = 0$, $x = 1$, $x = 3$, and $x = 5$ are inflection points.

6. $a = -2/5$, $b = 3/5$. ($f(-1) = 1$ gives $-a + b = 1$. Moreover, $f'(x) = 3ax^2 + 2bx$ and $f''(x) = 6ax + 2b$, so $f''(1/2) = 0$ yields $3a + 2b = 0$.)

7. $C''(x) = 6ax + 2b$, so $C(x)$ is concave in $[0, -b/3a]$, convex in $[-b/3a, \infty)$. $x = -b/3a$ is the inflection point.

8. See Fig. A8.7.8. Use the more general definition of a concave function in Section 8.7.

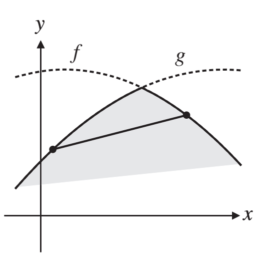


Figure A8.7.8

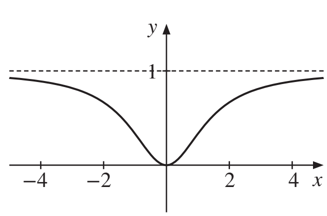


Figure A8.R.1

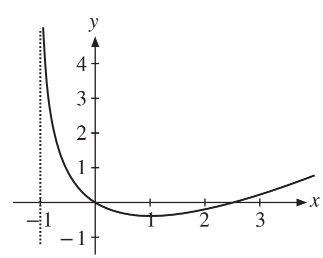


Figure A8.R.6

Review exercises for Chapter 8

1. (a) $f'(x) = \dfrac{4x}{(x^2+2)^2}$. Thus $f(x)$ decreases for $x \le 0$, and increases for $x \ge 0$.

(b) $f''(x) = 4(2-3x^2)/(x^2+2)^3$. There are inflection points at $x = \pm\frac{1}{3}\sqrt{6}$.

(c) $f(x) \to 1$ as $x \to \pm\infty$. See Fig. A8.R.1.

2. (a) $Q'(L) = 3L(8 - \frac{1}{20}L) = 0$ for $L = 160$, and $Q(L)$ is increasing in $[0, 160]$, decreasing in $[160, 200]$, so $Q^* = 160$ is the maximum value of $Q(L)$. (b) Output per worker is $Q(L)/L = 12L - \frac{1}{20}L^2$, and this quadratic function has a maximum at $L^* = 120$. $Q'(120) = Q(120)/120 = 720$. In general (see Example 6.7.6) one has $(\mathrm{d}/\mathrm{d}L)(Q(L)/L) = (1/L)(Q'(L) - Q(L)/L)$. If $L > 0$ maximizes output per worker, one must have $Q'(L) = Q(L)/L$.

3. If the side parallel to the river is y and the other side is x, then $2x + y = 1000$, so $y = 1000 - 2x$. The area of the enclosure is $xy = 1000x - 2x^2$. This quadratic function has a maximum at $x = 250$, when $y = 500$.

4. (a) $\pi = -0.0016Q^2 + 44Q - 0.0004Q^2 - 8Q - 64\,000 = -0.002Q^2 + 36Q - 64\,000$, and $Q = 9000$ maximizes π.

(b) $\mathrm{El}_Q\, C(Q) = \dfrac{Q}{C(Q)} C'(Q) = \dfrac{0.0008Q^2 + 8Q}{0.0004Q^2 + 8Q + 64\,000} \approx 0.12$ when $Q = 1000$.

This means that if Q increases from 1000 by 1%, then costs will increase by about 0.12%.

5. The profit as a function of Q is $\pi(Q) = PQ - C = (a - bQ^2)Q - \alpha + \beta Q = -bQ^3 + (a+\beta)Q - \alpha$.

Then $\pi'(Q) = -3bQ^2 + a + \beta$, which is 0 for $Q^2 = (a+\beta)/3b$, and so $Q = \sqrt{(a+\beta)/3b}$.

This value of Q maximizes the profit because $\pi''(Q) = -6bQ \le 0$ for all $Q \ge 0$.

6. (a) g is defined for $x > -1$. (b) $g'(x) = 1 - 2/(x+1) = (x-1)/(x+1)$, $g''(x) = 2/(x+1)^2$.

(c) Since $g'(x) < 0$ in $(-1, 1)$, $g'(1) = 0$ and $g'(x) > 0$ in $(1, \infty)$, $x = 1$ is a (global) minimum point.

Since $g''(x) > 0$ for all $x > -1$, the function g is convex and there are no inflection points.

When $x \to (-1)^-$, so $g(x) \to \infty$ and when $x \to \infty$, so $g(x) \to \infty$. See Fig. A8.R.6.

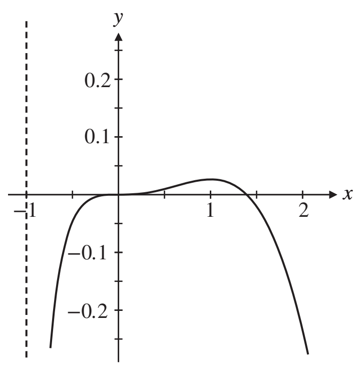


Figure A8.R.7

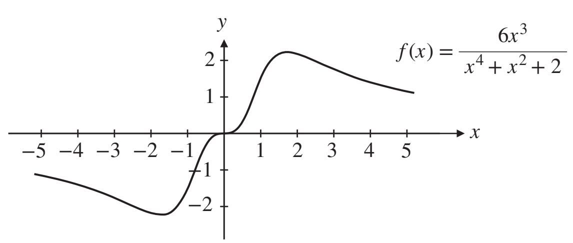


Figure A8.R.11

7. (a) $D_f = (-1, \infty)$. (b) A sign diagram shows that $f'(x) \ge 0$ in $(-1, 1]$ and $f'(x) \le 0$ in $[1, \infty)$. Hence $x = 1$ is a maximum point. f has no minimum. $f''(x) = \dfrac{-x(x^2 + x - 1)}{(x+1)^2} = 0$ for $x = 0$ and for $x = \frac{1}{2}(\sqrt{5} - 1)$.

(The point $x = \frac{1}{2}(-\sqrt{5} - 1)$ is outside the domain.) Since $f''(x)$ changes sign around these points, they are both inflection points. (c) $f(x) \to -\infty$ as $x \to (-1)^+$. See Fig. A8.R.7.

8. (a) h is increasing in $(-\infty, \frac{1}{2}\ln 2\,]$ and decreasing in $[\,\frac{1}{2}\ln 2, \infty)$, so h has a maximum at $x = \frac{1}{2}\ln 2$. It has no minimum.

(b) h is strictly increasing in $(-\infty, 0]$ (with range $(0, 1/3]$), and therefore has an inverse.

This inverse is $h^{-1}(y) = \ln(1 - \sqrt{1 - 8y^2}\,) - \ln(2y)$. See Fig. SM8.R.8 in SM.

9. (a) $f'(x) = 4e^{4x} + 8e^x - 32e^{-2x}$, $f''(x) = 16e^{4x} + 8e^x + 64e^{-2x}$ (b) $f'(x) = 4e^{-2x}(e^{3x} + 4)(e^{3x} - 2)$, so $f(x)$ is decreasing in $(-\infty, \frac{1}{3}\ln 2]$, increasing in $[\frac{1}{3}\ln 2, \infty)$. $f''(x) > 0$ for all x so f is strictly convex.

(c) $\frac{1}{3}\ln 2$ is a (global) minimum. No maximum exists because $f(x) \to \infty$ as $x \to \infty$.

10. (a) D_f is the set of all $x \neq \pm\sqrt{a}$. Also $f(x)$ is positive in $(-\sqrt{a}, 0)$ and in $(\sqrt{a}, \infty)$. The graph is symmetric about the origin because $f(-x) = -f(x)$. See the end of Section 5.2.

(b) $f(x)$ is increasing in $(-\infty, -\sqrt{3a}\,]$ and in $[\sqrt{3a}, \infty)$, but decreasing in $[-\sqrt{3a}, -\sqrt{a}\,)$, in $(-\sqrt{a}, \sqrt{a}\,)$, and in $(\sqrt{a}, \sqrt{3a}\,]$. Hence $x = -\sqrt{3a}$ is a local maximum point and $x = \sqrt{3a}$ is a local minimum point.

(c) There are inflection points at $-3\sqrt{a}$, 0, and $3\sqrt{a}$.

11. $x = \sqrt{3}$ is a maximum point, whereas $x = -\sqrt{3}$ is a minimum point, and $x = 0$ is an inflection point. See Fig. A8.R.11.

Chapter 9

9.1

1. (a) $\frac{1}{14}x^{14} + C$ (b) $\frac{2}{5}x^2\sqrt{x} + C$. ($x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$.) (c) $2\sqrt{x} + C$. ($1/\sqrt{x} = x^{-1/2}$.)

(d) $\frac{8}{15}x^{15/8} + C$. ($\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{x^{3/2}}} = \sqrt{x \cdot x^{3/4}} = \sqrt{x^{7/4}} = x^{7/8}$.) (e) $-e^{-x} + C$ (f) $4e^{\frac{1}{4}x} + C$

(g) $-\frac{3}{2}e^{-2x} + C$ (h) $(1/\ln 2)2^x + C$

2. (a) $C(x) = \frac{3}{2}x^2 + 4x + 40$. ($C(x) = \displaystyle\int (3x + 4)\,dx = \frac{3}{2}x^2 + 4x + C$. $C(0) = 40$ gives $C = 40$.)

(b) $C(x) = \frac{1}{2}ax^2 + bx + C_0$

3. (a) $\frac{1}{4}t^4 + t^2 - 3t + C$ (b) $\frac{1}{3}(x - 1)^3 + C$ (c) $\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C$ (d) $\frac{1}{4}(x + 2)^4 + C$ (e) $\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x} + e^x + C$

(f) $\frac{1}{3}x^3 - 3x + 4\ln|x| + C$

4. (a) $\frac{2}{5}y^2\sqrt{y} - \frac{8}{3}y\sqrt{y} + 8\sqrt{y} + C$ (b) $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|x + 1| + C$. (*Hint:* $x^3/(x + 1) = x^2 - x + 1 - 1/(x + 1)$.) (c) $\frac{1}{32}(1 + x^2)^{16} + C$

5. (a) and (b): Differentiate each right-hand side and check that you get the corresponding integrand. (For (a) see also Exercise 9.5.5.)

6. See Fig. A9.1.6. $f'(x) = A(x + 1)(x - 3)$ (because $f'(x)$ is 0 at $x = -1$ and at $x = 3$).

Moreover, $f'(1) = -1$. This implies that $A = 1/4$, so that $f'(x) = \frac{1}{4}(x + 1)(x - 3) = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{3}{4}$.
Integration yields $f(x) = \frac{1}{12}x^3 - \frac{1}{4}x^2 - \frac{3}{4}x + C$. Since $f(0) = 2$, one has $C = 2$.

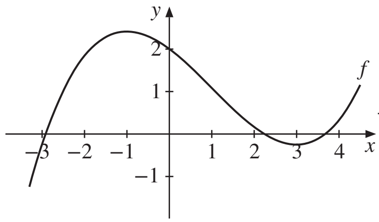


Figure A9.1.6

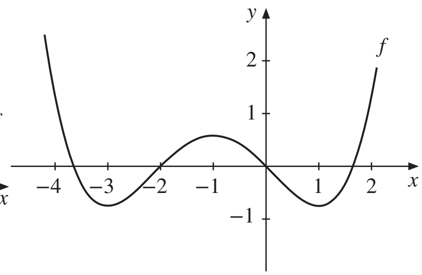


Figure A9.1.7

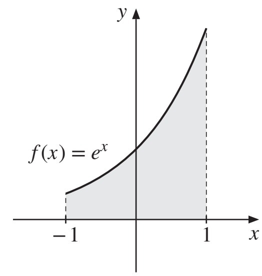


Figure A9.2.2

7. The graph of $f'(x)$ in Fig. 9.1.2 can be that of a cubic function, with roots at -3, -1, and 1, and with $f'(0) = -1$. So $f'(x) = \frac{1}{3}(x+3)(x+1)(x-1) = \frac{1}{3}x^3 + x^2 - \frac{1}{3}x - 1$. If $f(0) = 0$, integrating gives $f(x) = \frac{1}{12}x^4 + \frac{1}{3}x^3 - \frac{1}{6}x^2 - x$. Fig. A9.1.7 is the graph of this f.

8. Differentiate the right-hand side and check that you get the integrand.

9. (a) Differentiate the right-hand side. (Once we have learned integration by substitution in Section 9.6, this problem will become easy.)

   (b) (i) $\frac{1}{10}(2x+1)^5 + C$ (ii) $\frac{2}{3}(x+2)^{3/2} + C$ (iii) $-2\sqrt{4-x} + C$

   (c) (i) $F(x) = \int \left(\frac{1}{2}e^x - 2x\right)\,dx = \frac{1}{2}e^x - x^2 + C$. Then $F(0) = \frac{1}{2}$ implies $C = 0$.

   (ii) $F(x) = \int (x - x^3)\,dx = \frac{1}{2}x^2 - \frac{1}{4}x^4 + C$. Then $F(1) = \frac{5}{12}$ implies $C = \frac{1}{6}$.

10. The general form for f' is $f'(x) = \frac{1}{3}x^3 + A$, so that for f is $f(x) = \frac{1}{12}x^4 + Ax + B$. If we require that $f(0) = 1$ and $f'(0) = -1$, then $B = 1$ and $A = -1$, so $f(x) = \frac{1}{12}x^4 - x + 1$.

11. $f(x) = -\ln x + \frac{1}{20}x^5 + x^2 - x - \frac{1}{20}$

9.2

1. (a) $A = \int_0^1 x^3\,dx = \Big|_0^1 \frac{1}{4}x^4 = \frac{1}{4}1^4 - \frac{1}{4}0^4 = \frac{1}{4}$ (b) $A = \int_0^1 x^{10}\,dx = \Big|_0^1 \frac{1}{11}x^{11} = \frac{1}{11}$

2. (a) $\int_0^2 3x^2\,dx = \Big|_0^2 x^3 = 8$ (b) 1/7 (c) $e - 1/e$. (See the shaded area in Fig. A9.2.2.) (d) 9/10

3. See Fig. A9.2.3. $A = -\int_{-2}^{-1} x^{-3}\,dx = -\Big|_{-2}^{-1}\left(-\frac{1}{2}\right)x^{-2} = -\left[-\frac{1}{2} - \left(-\frac{1}{8}\right)\right] = \frac{3}{8}$

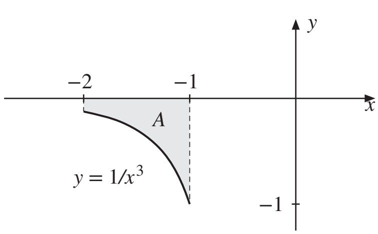


Figure A9.2.3

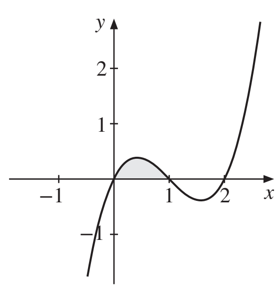


Figure A9.2.6

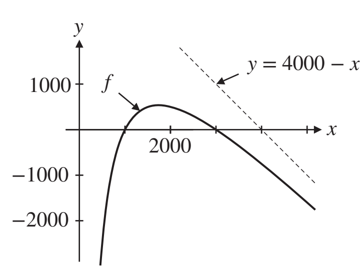


Figure A9.2.7

4. $A = \frac{1}{2}\int_{-1}^{1}(e^x + e^{-x})\,dx = \frac{1}{2}\Big|_{-1}^{1}(e^x - e^{-x}) = e - e^{-1}$

5. (a) $\int_0^1 x\,dx = \Big|_0^1 \frac{1}{2}x^2 = \frac{1}{2}$ (b) $16/3$ (c) $5/12$ (d) $-12/5$ (e) $41/2$ (f) $\ln 2 + 5/2$

6. (a) $f'(x) = 3x^2 - 6x + 2 = 0$ when $x_0 = 1 - \frac{1}{3}\sqrt{3}$ and $x_1 = 1 + \frac{1}{3}\sqrt{3}$. So $f(x)$ increases in $(-\infty, x_0)$ and in (x_1, ∞).
(b) See Fig. A9.2.6. The shaded area is $\frac{1}{4}$.

7. (a) $f'(x) = -1 + 3\,000\,000/x^2 = 0$ for $x = \sqrt{3\,000\,000} = 1000\sqrt{3}$. (Recall $x > 0$.)
Profits are maximized at $x = 1000\sqrt{3}$. See Fig. A9.2.7.
(b) $I = \frac{1}{2000}\Big|_{1000}^{3000}\big(4000x - \frac{1}{2}x^2 - 3\,000\,000\ln x\big) = 2000 - 1500\ln 3 \approx 352$

8. (a) $6/5$ (b) $26/3$ (c) $\alpha(e^\beta - 1)/\beta$ (d) $-\ln 2$

9.3

1. (a) $\Big|_0^5\left(\frac{1}{2}x^2 + \frac{1}{3}x^3\right) = 325/6$ (b) 0 (c) $\ln 9$ (d) $e - 1$ (e) -136 (f) $687/64$
(g) $\int_0^4 \frac{1}{2}x^{1/2}\,dx = \Big|_0^4 \frac{1}{2}\cdot\frac{2}{3}x^{3/2} = \frac{8}{3}$ (h) $\int_1^2 \frac{1+x^3}{x^2}\,dx = \int_1^2\left(\frac{1}{x^2} + x\right)dx = \Big|_1^2\left(-\frac{1}{x} + \frac{1}{2}x^2\right) = 2$

2. $\int_c^b f(x)\,dx = \int_a^b f(x)\,dx - \int_a^c f(x)\,dx = 8 - 4 = 4$

3. Let $A = \int_0^1 f(x)\,dx$ and $B = \int_0^1 g(x)\,dx$. Then from (i) and (ii), $A - 2B = 6$ and $2A + 2B = 9$, from which we find $A = 5$ and $B = -1/2$, and then $I = A - B = 11/2$.

4. $1/(p+q+1) + 1/(p+r+1)$

5. $f(x) = 4x^3 - 3x^2 + 5$

6. (a) $\Big|_0^3\left[\frac{1}{9}e^{3t-2} + \ln(t+2)\right] = \frac{1}{9}(e^7 - e^{-2}) + \ln(5/2)$ (b) $83/15$ (c) $2\sqrt{2} - 3/2$
(d) $A\big[b - 1 + (b-c)\ln[(b+c)/(1+c)]\big] + d\ln b$

7. From formula (9.3.6), $F'(x) = x^2 + 2$. Use formula (9.3.8) to obtain $G'(x) = [(x^2)^2 + 2]2x = 2x^5 + 4x$.

8. $H'(t) = 2tK(t^2)e^{-\rho t^2}$ (use formula (9.3.8)).

9. We use formulas (9.3.6), (9.3.7), and (9.3.8). (a) t^2 (b) $-e^{-t^2}$ (c) $2/\sqrt{t^4+1}$
(d) $(f(2) - g(2))\cdot 0 - (f(-\lambda) - g(-\lambda))\cdot(-1) = f(-\lambda) - g(-\lambda)$

10. From $y^2 = 3x$ we get $x = \frac{1}{3}y^2$. Inserting this into the other equation gives $y + 1 = (\frac{1}{3}y^2 - 1)^2$, or $y(y^3 - 6y - 9) = 0$. Here $y^3 - 6y - 9 = (y-3)(y^2 + 3y + 3)$, with $y^2 + 3y + 3$ never 0. So $(0,0)$ and $(3,3)$ are the only points of intersection. $A = \int_0^3(\sqrt{3x} - x^2 + 2x)\,dx = 6$. See Fig. A9.3.10.

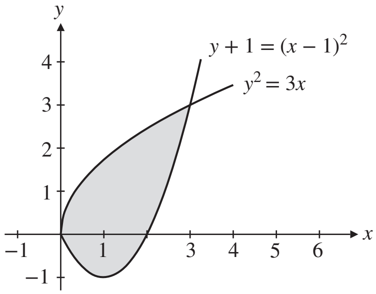


Figure A9.3.10

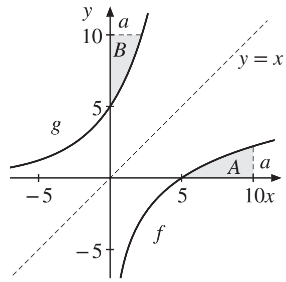


Figure A9.3.12

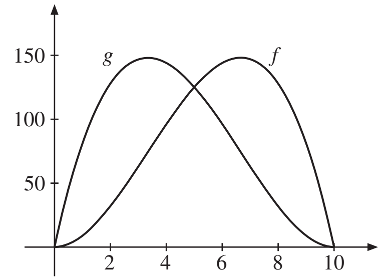


Figure A9.4.5

11. $W(T) = (K/T)\big|_0^T(-1/\rho)e^{-\rho t} = K(1 - e^{-\rho T})/\rho T$

12. (a) $g(x) = e^{x/2} + 4e^{x/4}$ defined on $(-\infty, \infty)$. (b) See Fig. A9.3.12. (c) $A = 10a + 14 - 8\sqrt{14} \approx 6.26$

9.4

1. $x(t) = K - \int_0^t \bar{u}e^{-as}\,ds = K - \bar{u}(1 - e^{-at})/a$. Note that $x(t) \to K - \bar{u}/a$ as $t \to \infty$. If $K \geq \bar{u}/a$, the well will never be exhausted.

2. (a) $m = 2b \ln 2$ (b) $x(p) = nABp^{\gamma} b^{\delta-1}(2^{\delta-1} - 1)/(\delta - 1)$

3. $T = \dfrac{1}{r}\ln(1 + rS)$. ($S = \Big|_0^T (1/r)e^{rt} = (e^{rT} - 1)/r$, so $e^{rT} - 1 = rS$, and solve for T.)

4. (a) $K(5) - K(0) = \int_0^5 (3t^2 + 2t + 5)\,dt = 175$ (b) $K(T) - K_0 = (T^3 - t_0^3) + (T^2 - t_0^2) + 5(T - t_0)$

5. (a) See Fig. A9.4.5. (b) $\int_0^t \big(g(\tau) - f(\tau)\big)\,d\tau = \int_0^t \big(2\tau^3 - 30\tau^2 + 100\tau\big)\,d\tau = \frac{1}{2}t^2(t - 10)^2 \geq 0$ for all t.

(c) $\int_0^{10} p(t)f(t)\,dt = \int_0^{10}\big(-t^3 + 9t^2 + 11t - 11 + 11/(t + 1)\big)\,dt = 940 + 11\ln 11 \approx 966.38$, whereas $\int_0^{10} p(t)g(t)\,dt = \int_0^{10}\big(t^3 - 19t^2 + 79t + 121 - 121/(t + 1)\big)\,dt = 3980/3 - 121\ln 11 \approx 1036.52$. Profile g should be chosen.

6. The equilibrium quantity is $Q^* = 600$, where $P^* = 80$. Then CS $= \int_0^{600}(120 - 0.2Q)\,dQ = 36\,000$, and PS $= \int_0^{600}(60 - 0.1Q)\,dQ = 18\,000$.

7. Equilibrium occurs when $6000/(Q^* + 50) = Q^* + 10$. The only positive solution is $Q^* = 50$, and then $P^* = 60$.

Then CS $= \displaystyle\int_0^{50}\Big[\frac{6000}{Q + 50} - 60\Big]\,dQ = \Big|_0^{50}\big[6000\ln(Q + 50) - 60Q\big] = 6000\ln 2 - 3000$,

and PS $= \displaystyle\int_0^{50}(50 - Q)\,dQ = 1250$.

9.5

1. (a) Use (9.5.1) with $f(x) = x$ and $g'(x) = e^{-x}$: $\int xe^{-x}\,dx = x(-e^{-x}) - \int 1\cdot(-e^{-x})\,dx = -xe^{-x} - e^{-x} + C$.

(b) $\frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$ (c) $-x^2e^{-x} - 2xe^{-x} - 3e^{-x} + C$ (d) $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

2. (a) $\displaystyle\int_{-1}^{1} x\ln(x+2)\,dx = \Big|_{-1}^{1} \frac{1}{2}x^2\ln(x+2) - \int_{-1}^{1} \frac{1}{2}x^2\frac{1}{x+2}\,dx = \frac{1}{2}\ln 3 - \frac{1}{2}\int_{-1}^{1}\left(x-2+\frac{4}{x+2}\right)dx = 2-\frac{3}{2}\ln 3$

(b) $8/(\ln 2) - 3/(\ln 2)^2$ (c) $e-2$ (d) $7\frac{11}{15} = \frac{116}{15}$

3. (a) $\displaystyle\int_1^4 \sqrt{t}\ln t\,dt = \int_1^4 t^{1/2}\ln t\,dt = \Big|_1^4 \frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\int_1^4 t^{3/2}(1/t)\,dt = \frac{16}{3}\ln 4 - \frac{2}{3}\Big|_1^4 \frac{2}{3}t^{3/2} = \frac{16}{3}\ln 4 - \frac{28}{9}$

(b) $\displaystyle\int_0^2 (x-2)e^{-x/2}\,dx = \Big|_0^2 (x-2)(-2)e^{-x/2} - \int_0^2 (-2)e^{-x/2}\,dt = -4-4\Big|_0^2 e^{-x/2} = -4-4(e^{-1}-1) = -4e^{-1}$

(c) $\displaystyle\int_0^3 (3-x)3^x\,dx = \Big|_0^3 (3-x)(3^x/\ln 3) - \int_0^3 (-1)(3^x/\ln 3)\,dx = 26/(\ln 3)^2 - 3/\ln 3$

4. The general formula follows from (9.5.1), and yields $\displaystyle\int \ln x\,dx = x\ln x - x + C$.

5. Use (9.5.1) with $f(x) = \ln x$ and $g'(x) = x^\rho$. (Alternatively, simply differentiate the right-hand side.)

6. (a) $br^{-2}\big[1-(1+rT)e^{-rT}\big]$ (b) $ar^{-1}(1-e^{-rT}) + br^{-2}\big[1-(1+rT)e^{-rT}\big]$
(c) $ar^{-1}\big(1-e^{-rT}\big) - br^{-2}\big[1-(1+rT)e^{-rT}\big] + cr^{-3}\big[2\big(1-e^{-rT}\big) - 2rTe^{-rT} - r^2T^2e^{-rT}\big]$

9.6

1. (a) $\frac{1}{9}(x^2+1)^9 + C$. (Substitute $u = x^2+1$, $du = 2x\,dx$.) (b) $\frac{1}{11}(x+2)^{11} + C$. (Substitute $u = x+2$.)

(c) $\ln|x^2-x+8| + C$. (Substitute $u = x^2-x+8$.)

2. (a) $\frac{1}{24}(2x^2+3)^6 + C$. (Substitute $u = 2x^2+3$, so $du = 4x\,dx$.) (b) $\frac{1}{3}e^{x^3+2} + C$. (Substitute $u = e^{x^3+2}$.)

(c) $\frac{1}{4}\big(\ln(x+2)\big)^2 + C$. (Substitute $u = \ln(x+2)$.) (d) $\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C$. (Substitute $u = \sqrt{1+x}$.)

(e) $\dfrac{-1}{2(1+x^2)} + \dfrac{1}{4(1+x^2)^2} + C$ (f) $\frac{2}{15}(4-x^3)^{5/2} - \frac{8}{9}(4-x^3)^{3/2} + C$

3. (a) With $u = \sqrt{1+x^2}$, $u^2 = 1+x^2$, so $u\,du = x\,dx$. If $x=0$, then $u=1$; if $x=1$, then $u=\sqrt{2}$. Hence, $\displaystyle\int_0^1 x\sqrt{1+x^2}\,dx = \int_1^{\sqrt{2}} u^2\,du = \Big|_1^{\sqrt{2}} \frac{1}{3}u^3 = \frac{1}{3}(2\sqrt{2}-1)$.

(b) 1/2. (Let $u = \ln y$.) (c) $\frac{1}{2}(e^2 - e^{2/3})$. (Let $u = 2/x$.)

(d) Method 1: $\displaystyle\int_5^8 \frac{x}{x-4}\,dx = \int_5^8 \frac{x-4+4}{x-4}\,dx = \int_5^8\left(1+\frac{4}{x-4}\right)dx = \Big|_5^8 [(x+4\ln(x-4)] = 3+4\ln 4$

Method 2: Performing the division $x \div (x-4)$ leads to the same result as in Method 1.

Method 3: Introduce the new variable $u = x-4$. Then $du = dx$ and $x = u+4$. When $x=5$, $u=1$, and when $x=8$, $u=4$, so $\displaystyle L = \int_1^4 \frac{u+4}{u}\,du = \int_1^4\left(1+\frac{4}{u}\right)du = \Big|_1^4 (u+4\ln u) = 3+4\ln 4$.

4. $\int_3^x \frac{2t-2}{t^2-2t}\,dt = \Big|_3^x \ln(t^2-2t) = \ln(x^2-2x) - \ln 3 = \ln \frac{1}{3}(x^2-2x)$. The equation in the exercise becomes $\ln \frac{1}{3}(x^2-2x) = \ln(\frac{2}{3}x-1) = \ln \frac{1}{3}(2x-3)$, which implies $x^2-2x = 2x-3$. Hence, $x^2-4x+3=0$, with solutions $x=1$ and $x=3$. But only $x=3$ is in the specified domain. So the solution is $x=3$.

5. Substitute $z = x(t)$. Then $dz = \dot{x}(t)\,dt$, and the result follows using (9.6.2).

6. (a) $1/70$. $((x^4-x^9)(x^5-1)^{12} = -x^4(x^5-1)^{13}.)$ (b) $2\sqrt{x}\ln x - 4\sqrt{x} + C$. (Let $u = \sqrt{x}$.) (c) $8/3$

7. (a) $2\ln(1+e^2) - 2\ln(1+e)$ (b) $\ln 2 - \ln(e^{-1/3}+1)$ (c) $7 + 2\ln 2$

8. Substitute $u = x^{1/6}$. Then $I = 6\int \frac{u^8}{1-u^2}\,du$. Here $u^8 \div (-u^2+1) = -u^6 - u^4 - u^2 - 1 + 1/(-u^2+1)$. It follows that $I = -\frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} - 2x^{1/2} - 6x^{1/6} - 3\ln|1-x^{1/6}| + 3\ln|1+x^{1/6}| + C$.

9. We find $f(x) = \frac{1}{a-b}\left[\frac{ac+d}{x-a} - \frac{bc+d}{x-b}\right]$.

(a) $$\int \frac{x\,dx}{(x+1)(x+2)} = \int \frac{-1\,dx}{x+1} + \int \frac{2\,dx}{x+2} = -\ln|x+1| + 2\ln|x+2| + C$$

(b) $$\int \frac{(1-2x)\,dx}{(x+3)(x-5)} = \int \left[-\frac{7}{8}\frac{1}{x+3} - \frac{9}{8}\frac{1}{x-5}\right] dx = -\frac{7}{8}\ln|x+3| - \frac{9}{8}\ln|x-5| + C$$

9.7

1. (a) $\int_1^b x^{-3}\,dx = \Big|_1^b \left(-\frac{1}{2}x^{-2}\right) = \frac{1}{2} - \frac{1}{2}b^{-2} \to \frac{1}{2}$ as $b \to \infty$. So $\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{2}$.

(b) $\int_1^b x^{-1/2}\,dx = \Big|_1^b 2x^{1/2} = 2b^{1/2} - 2 \to \infty$ as $b \to \infty$, so the integral diverges.

(c) 1 (d) $\int_0^a (x/\sqrt{a^2-x^2})\,dx = -\Big|_0^a \sqrt{a^2-x^2} = a$

2. (a) $$\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\Big|_a^b x = \frac{1}{b-a}(b-a) = 1$$

(b) $$\int_{-\infty}^{+\infty} xf(x)\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{2(b-a)}\Big|_a^b x^2 = \frac{1}{2(b-a)}(b^2-a^2) = \frac{1}{2}(a+b)$$

(c) $$\frac{1}{3(b-a)}\Big|_a^b x^3 = \frac{1}{3}\frac{b^3-a^3}{b-a} = \frac{1}{3}(a^2+ab+b^2)$$

3. Using a simplified notation and the result in Example 9.7.1, we have:

(a) $\int_0^\infty x\lambda e^{-\lambda x}\,dx = -\Big|_0^\infty xe^{-\lambda x} + \int_0^\infty e^{-\lambda x}\,dx = 1/\lambda$ (b) $1/\lambda^2$ (c) $2/\lambda^3$

4. The first integral diverges because $\int_0^b [x/(1+x^2)]\,dx = \Big|_0^b \frac{1}{2}\ln(1+x^2) = \frac{1}{2}\ln(1+b^2) \to \infty$ as $b \to \infty$.

On the other hand, $\int_{-b}^{b} [x/(1+x^2)]\,dx = \Big|_{-b}^{b} \dfrac{1}{2}\ln(1+x^2) = 0$ for all b, so the limit as $b \to \infty$ is 0.

5. (a) f has a maximum at $(e^{1/3}, 1/3e)$, but no minimum. (b) $\int_0^1 x^{-3}\ln x\,dx$ diverges. $\int_1^\infty x^{-3}\ln x\,dx = 1/4$.

6. $\dfrac{1}{1+x^2} \le \dfrac{1}{x^2}$ for $x \ge 1$, and $\int_1^b \dfrac{dx}{x^2} = \Big|_1^b -\dfrac{1}{x} = 1 - \dfrac{1}{b} \underset{b\to\infty}{\longrightarrow} 1$, so by Theorem 9.7.1 the integral converges.

7. Put $u = x+2$ and $v = 3-x$. Then the integral becomes

$$\int_0^5 u^{-1/2}\,du - \int_5^0 v^{-1/2}\,dv = 2\lim_{\varepsilon\to 0}\int_\varepsilon^5 u^{-1/2}\,du = 4\lim_{\varepsilon\to 0}\Big|_\varepsilon^5 u^{1/2} = 4\lim_{\varepsilon\to 0}\left(\sqrt{5}-\sqrt{\varepsilon}\right) = 4\sqrt{5}.$$

8. (a) $z = \int_0^\tau (1/\tau)e^{-rs}\,ds = (1-e^{-r\tau})/r\tau$ (b) $z = \int_0^\tau 2(\tau - s)\tau^{-2}e^{-rs}\,ds = (2/r\tau)\left[1-(1/r\tau)(1-e^{-r\tau})\right]$

9. $\int x^{-2}\,dx = -x^{-1} + C$. So evaluating $\int_{-1}^1 x^{-2}\,dx$ as $\Big|_{-1}^1 -x^{-1}$ gives the nonsensical answer -2.

The error arises because x^{-2} diverges to $+\infty$ as $x \to 0$. (In fact, $\int_{-1}^1 x^{-2}\,dx$ diverges to $+\infty$.)

10. Using the answer to Exercise 9.6.6(b), $\int_h^1 (\ln x/\sqrt{x})\,dx = \Big|_h^1 (2\sqrt{x}\ln x - 4\sqrt{x}) = -4 - (2\sqrt{h}\ln h - 4\sqrt{h}) \to -4$ as $h \to 0^+$, so the given integral converges to -4. ($\sqrt{h}\ln h = \ln h/h^{-1/2} \to 0$, by l'Hôpital's rule.)

11. $\int_1^A [k/x - k^2/(1+kx)]\,dx = k\ln[1/(1/A+k)] - k\ln[1/(1+k)] \to k\ln(1/k) - k\ln[1/(1+k)] = \ln(1+1/k)^k$ as $A \to \infty$. So $I_k = \ln(1+1/k)^k$, which tends to $\ln e = 1$ as $k \to \infty$.

12. The suggested substitution $u = (x-\mu)/\sqrt{2}\sigma$ gives $du = dx/\sigma\sqrt{2}$, and so $dx = \sigma\sqrt{2}\,du$. Hence:

(a) $\int_{-\infty}^{+\infty} f(x)\,dx = \dfrac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2}\,du = 1$, by (9.7.8).

(b) $\int_{-\infty}^{+\infty} xf(x)\,dx = \dfrac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} (\mu + \sqrt{2}\sigma u)e^{-u^2}\,du = \mu$, using part (a) and Example 9.7.3.

(c) $\int_{-\infty}^{+\infty}(x-\mu)^2 f(x)\,dx = \int_{-\infty}^{+\infty} 2\sigma^2 u^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2}\sigma\sqrt{2}\,du = \sigma^2 \frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty} u^2 e^{-u^2}\,du$. Now integration by parts yields $\int u^2 e^{-u^2}\,du = -\frac{1}{2}ue^{-u^2} + \int \frac{1}{2}e^{-u^2}\,du$, so $\int_{-\infty}^{+\infty} u^2 e^{-u^2}\,du = \frac{1}{2}\sqrt{\pi}$. Hence the integral equals σ^2.

9.8

1. The functions in (c) and (d) are the only ones that have a constant relative rate of increase. This accords with (9.8.3). (Note that $2^t = e^{(\ln 2)t}$.)

2. (a) $K(t) = (K_0 - I/\delta)e^{-\delta t} + I/\delta$ (b) (i) $K(t) = 200 - 50e^{-0.05t}$ and $K(t)$ tends to 200 from below as $t \to \infty$.
(ii) $K(t) = 200 + 50e^{-0.05t}$, and $K(t)$ tends to 200 from above as $t \to \infty$.

3. $N(t) = P(1 - e^{-kt})$. Then $N(t) \to P$ as $t \to \infty$.

4. $\dot{N}(t) = 0.02N(t) + 4 \cdot 10^4$. The solution with $N(0) = 2 \cdot 10^6$ is $N(t) = 2 \cdot 10^6(2e^{0.02t} - 1)$.

5. $P(10) = 705$ gives $641e^{10k} = 705$, or $e^{10k} = 705/641$. Taking the natural logarithm of both sides yields $10k = \ln(705/641)$, so $k = 0.1\ln(705/641) \approx 0.0095$. $P(15) \approx 739$ and $P(40) \approx 938$.

6. The percentage surviving after t seconds satisfies $p(t) = 100e^{-\delta t}$, where $p(7) = 70.5$ and so $\delta = -\ln 0.705/7 \approx 0.05$. Thus $p(30) = 100e^{-30\delta} \approx 22.3\%$ are still alive after 30 seconds. Because $100e^{-\delta t} = 5$ when $t \approx \ln 20/0.05 \approx 60$, it takes about 60 seconds to kill 95%.

7. (a) $x = Ae^{-0.5t}$ (b) $K = Ae^{0.02t}$ (c) $x = Ae^{-0.5t} + 10$ (d) $K = Ae^{0.2t} - 500$ (e) $x = 0.1/(3 - Ae^{0.1t})$ and $x \equiv 0$
(f) $K = 1/(2 - Ae^t)$ and $K \equiv 0$

8. (a) $y(t) = 250 + \dfrac{230}{1 + 8.2e^{-0.34t}}$ (b) $y(t) \to 480$ as $t \to \infty$. See Fig. A9.8.8.

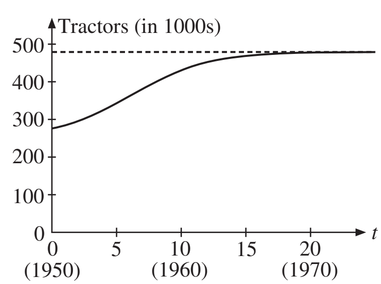


Figure A9.8.8

9. (a) Using (9.8.8) we find $N(t) = 1000/(1 + 999e^{-0.39t})$. After 20 days, $N(20) \approx 710$ have developed influenza.
(b) $800 = \dfrac{1000}{1 + 999e^{-0.39t^*}} \iff 999e^{-0.39t^*} = \dfrac{1}{4}$, so $e^{-0.39t^*} = 1/3996$, and so $0.39t^* = \ln 3996$. $t^* \approx 21$ days.
(c) After about 35 days, 999 will have or have had influenza. $N(t) \to 1000$ as $t \to \infty$.

10. (a) If $f \neq r$, the solution is $x(t) = \dfrac{(1-f/r)K}{1+\dfrac{(1-f/r)K - x_0}{x_0}e^{-(r-f)t}}$. If $f = r$, then the solution is $x = \dfrac{1}{rt/K + 1/x_0}$.

(b) If $f > r$, then $x(t) \to 0$ as $t \to \infty$. See SM for details.

11. At about 11.26. (Measuring time in hours, with $t = 0$ being 12 noon, one has $\dot{T} = k(20 - T)$ with $T(0) = 35$ and $T(1) = 32$. So the body temperature at time t is $T(t) = 20 + 15e^{-kt}$ with $k = \ln(5/4)$. Assuming that the temperature was the normal 37 degrees at the time of death t^*, then $t^* = -\ln(17/15)/\ln(5/4) \approx -0.56$ hours, or about 34 minutes before 12.00.)

9.9

1. The equation is separable: $\int x^4\,dx = \int (1-t)\,dt$, $\frac{1}{5}x^5 = t - \frac{1}{2}t^2 + C_1$, so $x^5 = 5t - \frac{5}{2}t^2 + 5C_1$, implying that $x = \sqrt[5]{5t - \frac{5}{2}t^2 + 5C_1} = \sqrt[5]{5t - \frac{5}{2}t^2 + C}$, with $C = 5C_1$. Then $x(1) = 1$ yields $C = -3/2$.

2. (a) $x = \sqrt[3]{\frac{3}{2}e^{2t} + C}$ (b) $x = -\ln(e^{-t} + C)$ (c) $x = Ce^{3t} - 6$ (d) $x = \sqrt[7]{(1+t)^7 + C}$ (e) $x = Ce^{2t} + \frac{1}{2}t + \frac{1}{4}$
(f) $x = Ce^{-3t} + \frac{1}{2}e^{t^2-3t}$

3. The equation is separable: $dk/k = s\alpha e^{\beta t}\,dt$, so $\ln k = \dfrac{s\alpha}{\beta}e^{\beta t} + C_1$, or $k = e^{(s\alpha/\beta)e^{\beta t}}e^{C_1} = Ce^{(s\alpha/\beta)e^{\beta t}}$.

With $k(0) = k_0$, we have $k_0 = Ce^{s\alpha/\beta}$, and thus $k = k_0e^{(s\alpha/\beta)(e^{\beta t}-1)}$.

4. (a) $\dot{Y} = \alpha(a-1)Y + \alpha(b + \bar{I})$ (b) $Y = \left(Y_0 - \dfrac{b+\bar{I}}{1-a}\right)e^{-\alpha(1-a)t} + \dfrac{b+\bar{I}}{1-a} \to \dfrac{b+\bar{I}}{1-a}$ as $t \to \infty$.

5. From (iii), $L = L_0e^{\beta t}$, so $\dot{K} = \gamma K^\alpha L_0 e^{\beta t}$, a separable equation with solution $K = \left[\dfrac{(1-\alpha)\gamma}{\beta}L_0(e^{\beta t} - 1) + K_0^{1-\alpha}\right]^{1/(1-\alpha)}$

6. $\dfrac{t}{x}\dfrac{dx}{dt} = a$ is separable: $\dfrac{dx}{x} = a\dfrac{dt}{t}$, so $\displaystyle\int \frac{dx}{x} = a\int \frac{dt}{t}$. Integrating yields $\ln x = a\ln t + C_1$, so $x = e^{a\ln t + C_1} = (e^{\ln t})^a e^{C_1} = Ct^a$, with $C = e^{C_1}$. This shows that the only type of function which has constant elasticity is $x = Ct^a$.

Review exercises for Chapter 9

1. (a) $-16x + C$ (b) $5^5x + C$ (c) $3y - \frac{1}{2}y^2 + C$ (d) $\frac{1}{2}r^2 - \frac{16}{5}r^{5/4} + C$ (e) $\frac{1}{9}x^9 + C$
(f) $\frac{2}{7}x^{7/2} + C$. ($x^2\sqrt{x} = x^2 \cdot x^{1/2} = x^{5/2}$.) (g) $-\frac{1}{4}p^{-4} + C$ (h) $\frac{1}{4}x^4 + \frac{1}{2}x^2 + C$

2. (a) $e^{2x} + C$ (b) $\frac{1}{2}x^2 - \frac{25}{2}e^{2x/5} + C$ (c) $-\frac{1}{3}e^{-3x} + \frac{1}{3}e^{3x} + C$ (d) $2\ln|x+5| + C$

3. (a) $\displaystyle\int_0^{12} 50\,dx = \Big|_0^{12} 50x = 600$ (b) $\displaystyle\int_0^2 (x - \tfrac{1}{2}x^2)\,dx = \Big|_0^2(\tfrac{1}{2}x^2 - \tfrac{1}{6}x^3) = \tfrac{2}{3}$

(c) $\displaystyle\int_{-3}^{3} (u+1)^2\,du = \Big|_{-3}^{3} \tfrac{1}{3}(u+1)^3 = 24$ (d) $\displaystyle\int_1^5 \frac{2}{z}\,dz = \Big|_1^5 2\ln z = 2\ln 5$ (e) $3\ln(8/3)$

(f) $I = \displaystyle\int_0^4 v\sqrt{v^2+9}\,dv = \Big|_0^4 \tfrac{1}{3}(v^2+9)^{3/2} = 98/3$. (Or introduce $z = \sqrt{v^2+9}$. Then $z^2 = v^2 + 9$ and $2z\,dz = 2v\,dv$, or $v\,dv = z\,dz$. When $v = 0$, $z = 3$, and when $v = 4$, $z = 5$, so $I = \displaystyle\int_3^5 z^2\,dz = \Big|_3^5 \frac{1}{3}z^3 = 98/3$.)

4. (a) 5/4 (b) $31/20$ (c) -5 (d) $e-2$ (e) $52/9$ (f) $\frac{1}{3}\ln(6/5)$ (g) $(1/256)(3e^4+1)$ (h) $2e^{-1}$

5. (a) $10 - 18\ln(14/9)$. (Substitute $z = 9+\sqrt{x}$.) (b) $886/15$. (Substitute $z = \sqrt{t+2}$.)
(c) 195/4. (Substitute $z = \sqrt[3]{19x^3+8}$.)

6. (a) $F'(x) = 4(\sqrt{x}-1)$. ($\int_4^x (u^{1/2} + xu^{-1/2})\,\mathrm{d}u = \big|_4^x \frac{2}{3}u^{3/2} + 2xu^{1/2} = \frac{8}{3}x^{3/2} - \frac{16}{3} - 4x$.)
(b) We use (9.3.8). $F'(x) = \ln x - (\ln\sqrt{x})(1/2\sqrt{x}) = \ln x - \ln x/4\sqrt{x}$.

7. $C(Y) = 0.69Y + 1000$

8. Integrating the marginal cost function gives $C(x) = C_0 + \int_0^x (\alpha e^{\beta u} + \gamma)\,\mathrm{d}u = C_0 + \big|_0^x \frac{\alpha}{\beta}e^{\beta u} = \frac{\alpha}{\beta}(e^{\beta x}-1) + \gamma x + C_0$.

9. Let $\displaystyle\int_{-1}^{3} f(x)\,\mathrm{d}x = A$ and $\int_{-1}^3 g(x)\,\mathrm{d}x = B$. Then $A+B=6$ and $3A+4B=9$, from which we find $A=15$ and $B=-9$. Then $I = A+B = 6$.

10. (a) $P^* = 70$, $Q^* = 600$. CS $= 9000$, PS $= 18\,000$. See Fig. A9.R.10a.
(b) $P^* = Q^* = 5$, CS $= 50\ln 2 - 25$, PS $= 1.25$. See Fig. A9.R.10b.

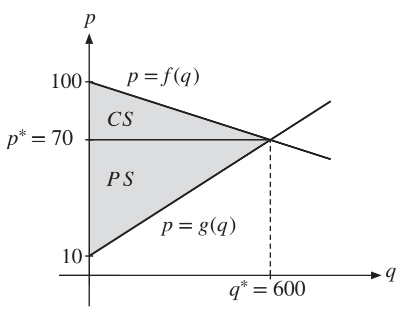


Figure A9.R.10a

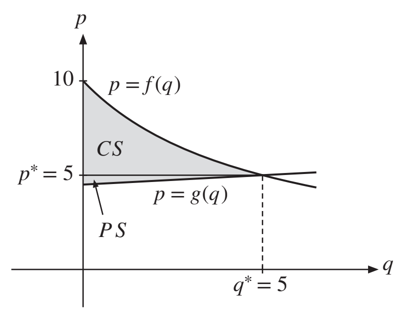


Figure A9.R.10b

11. (a) $f'(t) = 4\ln t(2-\ln t)/t^2, f''(t) = 8[(\ln t)^2 - 3\ln t + 1]/t^3$
(b) $(e^2, 16/e^2)$ is a local maximum point, $(1,0)$ is a local (and global) minimum point. See Fig. A9.R.11.
(c) Area = 32/3. (*Hint:* $\int f(t)\,\mathrm{d}t = \frac{4}{3}(\ln t)^3 + C$.)

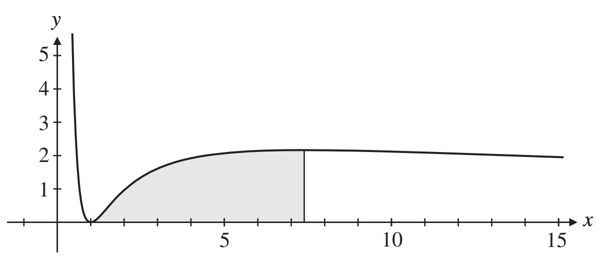


Figure A9.R.11

12. (a) $x = Ae^{-3t}$ (b) $x = Ae^{-4t} + 3$ (c) $x = 1/(Ae^{-3t} - 4)$ and $x \equiv 0$ (d) $x = Ae^{-\frac{1}{5}t}$ (e) $x = Ae^{-2t} + 5/3$ (f) $x = 1/(Ae^{-\frac{1}{2}t} - 2)$ and $x \equiv 0$

13. (a) $x = 1/(C - \frac{1}{2}t^2)$ and $x(t) \equiv 0$ (b) $x = Ce^{-3t/2} - 5$ (c) $x = Ce^{3t} - 10$ (d) $x = Ce^{-5t} + 2t - \frac{2}{5}$ (e) $x = Ce^{-t/2} + \frac{2}{3}e^t$ (f) $x = Ce^{-3t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}$

14. (a) $V(x) = (V_0 + b/a)e^{-ax} - b/a$ (b) $V(x^*) = 0$ yields $x^* = (1/a)\ln(1 + aV_0/b)$.
(c) $0 = V(\hat{x}) = (V_m + b/a)e^{-a\hat{x}} - b/a$ yields $V_m = (b/a)(e^{a\hat{x}} - 1)$.
(d) $x^* = (1/0.001)\ln(1 + 0.001 \cdot 12\,000/8) \approx 916$, and $V_m = (8/0.001)(e^{0.001\cdot 1200} - 1) = 8000(e^{1.2} - 1) \approx 18\,561$.

15. (a) $\int_0^\infty f(r)\,dr = \int_0^\infty (1/m)e^{-r/m}\,dr = 1$ (as in Example 9.7.1) and $\int_0^\infty rf(r)\,dr = \int_0^\infty r(1/m)e^{-r/m}\,dr = m$ (as in Exercise 9.7.3(a)), so the mean income is m.
(b) $x(p) = n\int_0^\infty (ar - bp)f(r)\,dr = n\left(a\int_0^\infty rf(r)\,dr - bp\int_0^\infty f(r)\,dr\right) = n(am - bp)$

Chapter 10

10.1

1. (a) (i) $8000(1 + 0.05/12)^{5\cdot 12} \approx 10\,266.87$ (ii) $8000(1 + 0.05/365)^{5\cdot 365} \approx 10\,272.03$
(b) $t = \ln 2/\ln(1 + 0.05/12) \approx 166.7$. It takes approximately $166.7/12 \approx 13.9$ years.

2. (a) $5000(1 + 0.03)^{10} \approx 6719.58$ (b) 37.17 years. ($5000(1.03)^t = 3 \cdot 5000$, so $t = \ln 3/\ln 1.03 \approx 37.17$.)

3. We solve $(1 + p/100)^{100} = 100$ for p. Raising each side to the power $1/100$, we get $1 + p/100 = \sqrt[100]{100}$, so $p = 100(\sqrt[100]{100} - 1) \approx 100(1.047 - 1) = 4.7$.

4. (a) (i) After two years: $2000(1.07)^2 = 2289.80$ (ii) After 10 years: $2000(1.07)^{10} \approx 3934.30$
(b) $2000(1.07)^t = 6000$ gives $(1.07)^t = 3$, so $t = \ln 3/\ln 1.07 \approx 16.2$ years.

5. Use formula (10.1.2). (i) $R = (1 + 0.17/2)^2 - 1 = (1 + 0.085)^2 - 1 = 0.177225$ or 17.72%
(ii) $100[(1.0425)^4 - 1] \approx 18.11\%$ (iii) $100[(1 + 0.17/12)^{12} - 1] \approx 18.39\%$

6. The effective yearly rate for alternative (ii) is $(1 + 0.2/4)^4 - 1 = 1.05^4 - 1 \approx 0.2155 > 0.215$, so (i) is (slightly) cheaper.

7. (a) $12\,000 \cdot (1.04)^{15} \approx 21\,611.32$ (b) $50\,000 \cdot (1.05)^{-5} \approx 39\,176.31$

8. $100[(1.02)^{12} - 1] \approx 26.82\%$

9. Let the nominal yearly rate be r. By (2), $0.28 = (1 + r/4)^4 - 1$, so $r = 4(\sqrt[4]{1.28} - 1) \approx 0.25$, or 25%.

10.2

1. (a) $8000e^{0.05\cdot 5} = 8000e^{0.25} \approx 10\,272.20$ (b) $8000e^{0.05t} = 16\,000$ which gives $e^{0.05t} = 2$. Hence $t = \ln 2/0.05 \approx 13.86$ years.

2. (a) (i) $1000(1+0.05/12)^{120} \approx 1647$ (ii) $1000e^{0.05\cdot 10} \approx 1649$

(b) (i) $1000(1+0.05/12)^{600} \approx 12\,119$ (ii) $1000e^{0.05\cdot 50} \approx 12\,182$

3. (a) $e^{0.1} - 1 \approx 0.105$, so the effective percentage rate is approximately 10.5. (b) Same answer.

4. If it loses 90% of its value, then $e^{-0.1t^*} = 1/10$, so $-0.1t^* = -\ln 10$, hence $t^* = (\ln 10)/0.1 \approx 23$.

5. $e^{-0.06t^*} = 1/2$, so $t^* = \ln 2/0.06 \approx 11.55$ years.

6. $g(x)$ is strictly increasing, so for all $x > 0$, one has $g(x) = (1 + r/x)^x < \lim_{x\to\infty} g(x) = e^r$. See SM.

10.3

1. (a) The present value is $350\,000 \cdot 1.08^{-10} \approx 162\,117.72$. (b) $350\,000 \cdot e^{-0.08\cdot 10} \approx 157\,265.14$

2. (a) The present value is $50\,000 \cdot 1.0575^{-5} \approx 37\,806.64$. (b) $50\,000 \cdot e^{-0.0575\cdot 5} \approx 37\,506.83$

3. (a) We find $f'(t) = 0.05(t+5)(35-t)e^{-t}$. Obviously, $f'(t) > 0$ for $t < 35$ and $f'(t) < 0$ for $t > 35$, so $t = 35$ maximizes f (with $f(35) \approx 278$). (b) $f(t) \to 0$ as $t \to \infty$. See the graph in Fig. A10.3.3.

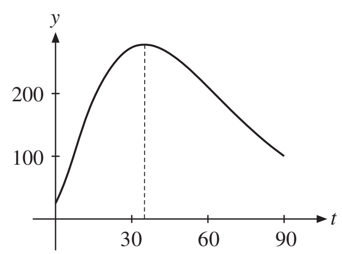


Figure A10.3.3

10.4

1. (a) $s_n = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$ (b) $s_n \to \frac{3}{2}$ as $n \to \infty$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \frac{3}{2}$.

2. We use formula (10.4.5): (a) $\dfrac{1/5}{1-1/5} = 1/4$ (b) $\dfrac{0.1}{1-0.1} = \dfrac{0.1}{0.9} = \dfrac{1}{9}$ (c) $\dfrac{517}{1-1/1.1} = 5687$

(d) $\dfrac{a}{1-1/(1+a)} = 1 + a$ (e) $\dfrac{5}{1-3/7} = \dfrac{35}{4}$

3. (a) Geometric series with quotient $1/8$. Its sum is $8/(1-\frac{1}{8}) = 64/7$.

(b) Geometric with quotient -3. It diverges. (c) Geometric, with sum $2^{1/3}/(1-2^{-1/3})$.

(d) Not geometric. (In fact, one can show that the series converges with sum $\ln 2$.)

4. (a) Quotient $k = 1/p$. Converges to $1/(p-1)$ for $|p| > 1$.

(b) Quotient $k = 1/\sqrt{x}$. Converges to $x\sqrt{x}/(\sqrt{x}-1)$ for $\sqrt{x} > 1$, that is, for $x > 1$.

(c) Quotient $k = x^2$. Converges to $x^2/(1-x^2)$ for $|x| < 1$.

5. Geometric series with quotient $(1+p/100)^{-1}$. Its sum is $b/[1-(1+p/100)^{-1}] = b(1+100/p)$.

6. The resources will be exhausted partway through the year 2028.

7. $1824 \cdot 1.02 + 1824 \cdot 1.02^2 + \cdots + 1824 \cdot 1.02^n = (1824/0.02)(1.02^{n+1} - 1.02)$ must equal 128 300. So $n \approx 43.77$. The resources will last until year 2037.

8. (a) $f(t) = \dfrac{P(t)}{e^{rt}-1}$ (b) Use $f'(t^*) = 0$. (c) $P'(t^*)/P(t^*) \to 1/t^*$ as $r \to 0$.

9. The general term does not approach 0 as $n \to \infty$ in any of these three cases, so each of the series is divergent.

10. (a) A geometric series with quotient 100/101 that converges to 100. (b) Diverges according to (10.4.10).

(c) Converges according to (10.4.10). (d) Diverges because the nth term $a_n = (1+n)/(4n-3) \to 1/4$ as $n \to \infty$.

(e) Geometric series with quotient $-1/2$ that converges to $-1/3$.

(f) Geometric series with quotient $1/\sqrt{3}$ converging to $\sqrt{3}/(\sqrt{3}-1)$.

11. See SM.

10.5

1. Use (10.5.2) with $n = 15$, $r = 0.12$, and $a = 3500$. This gives $P_{15} = \dfrac{3500}{0.12}\left(1 - \dfrac{1}{(1.12)^{15}}\right) \approx 23\,838$.

2. 10 years ago the amount was: $100\,000(1.04)^{-10} \approx 67\,556.42$.

3. $10\,000(1.06^3 + 1.06^2 + 1.06 + 1) = 10\,000(1.06^4 - 1)/(1.06 - 1)) \approx 43\,746.16$

4. The future value after 10 years of (i) is obviously \$13 000, whereas according to (10.5.3), the corresponding value of (ii) is $F_{10} = (1000/0.06)(1.06^{10} - 1) \approx 13\,180.80$. So (ii) is worth more.

5. Offer (i) is better, because the present value of (ii) is $4600\dfrac{1-(1.06)^{-5}}{1-(1.06)^{-1}} \approx 20\,539$.

6. $\dfrac{1500}{0.08} = 18\,750$ (using (10.5.4)).

7. If the largest amount is a, then according to formula (10.5.4), $a/r = K$, so that $a = rK$.

8. This is a geometric series with first term $a = D/(1+r)$ and quotient $k = (1+g)/(1+r)$. It converges if and only if $k < 1$, i.e. if and only if $g < r$. The sum is $\dfrac{a}{1-k} = \dfrac{D/(1+r)}{1-(1+g)/(1+r)} = \dfrac{D}{r-g}$.

9. $\text{PDV} = \displaystyle\int_0^{15} 500e^{-0.06t}\,dt = 500 \Big|_0^{15} (-1/0.06)e^{-0.06t} = (500/0.06)\left[1 - e^{-0.9}\right] \approx 4945.25$

$\text{FDV} = e^{0.06 \cdot 15}\text{PDV} = e^{0.9}\text{PDV} \approx 2.4596 \cdot 4945.25 \approx 12\,163.3$

10.6

1. (a) Using formula (10.6.2) we find that the annual payments are: $a = 0.07 \cdot 80\,000/(1-(1.07)^{-10}) \approx 11\,390.20$.

(b) Using (10.6.2) we get $a = (0.07/12) \cdot 80\,000/[1-(1+0.07/12)^{-120}] \approx 928.87$.

2. $(8000/0.07)[1.07^6 - 1] \approx 57\,226.33$. (Formula (10.5.3).)

Four years after the last deposit you have $57\,226.33 \cdot 1.07^4 \approx 75\,012.05$.

3. With annual compounding: $r = 3^{1/20} - 1 \approx 0.0565$, so the rate of interest is about 5.65%.

With continuous compounding: $e^{20r} = 3$, so that $r = \ln 3/20 \approx 0.0549$, so the rate of interest is about 5.49%.

4. Schedule (ii) has present value $\dfrac{12\,000 \cdot 1.115}{0.115}[1 - (1.115)^{-8}] \approx 67\,644.42$.

Schedule (iii) has present value $22\,000 + \dfrac{7000}{0.115}[1 - (1.115)^{-12}] \approx 66\,384.08$. Thus schedule (iii) is cheapest.

When the interest rate becomes 12.5%, schedules (ii) and (iii) have present values equal to 65 907.61 and 64 374.33, respectively, so (iii) is cheapest in this case too.

10.7

1. r must satisfy $-50\,000 + 30\,000/(1+r) + 30\,000/(1+r)^2 = 0$. With $s = 1/(1+r)$, this yields $s^2 + s - 5/3 = 0$, with positive solution $s = -1/2 + \sqrt{23/12} \approx 0.884$, so that $r \approx 0.13$.

2. Equation (10.7.1) is here $a/(1+r) + a/(1+r)^2 + \cdots = -a_0$, which yields $a/r = -a_0$, so $r = -a/a_0$.

3. By hypothesis, $f(0) = a_0 + a_1 + \cdots + a_n > 0$. Also, $f(r) \to a_0 < 0$ as $r \to \infty$.

Moreover, $f'(r) = -a_1(1+r)^{-2} - 2a_2(1+r)^{-3} - \cdots - na_n(1+r)^{-n-1} < 0$, so $f(r)$ is strictly decreasing.

This guarantees that there is a unique internal rate of return, with $r > 0$.

4. \$1.55 million. $(400\,000\big(1/1.175 + (1/1.175)^2 + \cdots + (1/1.175)^7\big) \approx 1\,546\,522.94.)$

5. Equation (10.7.1) reduces to $s^{21} - 11s + 10 = 0$. See SM.

6. Applying (10.5.2) with $a = 1000$ and $n = 5$ gives the equation $P_5 = (1000/r)\big[1 - 1/(1+r)^5\big] = 4340$ to be solved for r. For $r = 0.05$, the present value is \$4329.48; for $r = 0.045$, the present value is \$4389.98. Because $\mathrm{d}P_5/\mathrm{d}r < 0$, it follows that p is a little less than 5%.

10.8

1. (a) $x_t = x_0(-2)^t$ (b) $x_t = x_0(5/6)^t$ (c) $x_t = x_0(-0.3)^t$

2. (a) Equation (10.8.4) with $a = 1$ gives $x_t = -4t$. (b) $x_t = 2(1/2)^t + 4$ (c) $x_t = (13/8)(-3)^t - 5/8$ (d) $x_t = -2(-1)^t + 4$

3. Equilibrium requires $\alpha P_t - \beta = \gamma - \delta P_{t+1}$, or $P_{t+1} = -(\alpha/\delta)P_t + (\beta+\gamma)/\delta$.

Using (10.8.4) we obtain $P_t = \left(-\dfrac{\alpha}{\delta}\right)^t\left(P_0 - \dfrac{\beta+\gamma}{\alpha+\delta}\right) + \dfrac{\beta+\gamma}{\alpha+\delta}$.

Review exercises for Chapter 10

1. (a) $5000 \cdot 1.03^{10} \approx 6719.58$ (b) $5000(1.03)^{t^*} = 10\,000$, so $(1.03)^{t^*} = 2$, or $t^* = \ln 2/\ln 1.03 \approx 23.45$.

2. (a) $8000 \cdot 1.05^3 = 9261$ (b) $8000 \cdot 1.05^{13} \approx 15\,085.19$ (c) $(1.05)^{t^*} = 4$, so $t^* = \ln 4/1.05 \approx 28.5$.

3. If you borrow \$$a$ at the annual interest rate of 11% with interest paid yearly, then the debt after one year is equal to $a(1 + 11/100) = 1.11a$; if you borrow at annual interest rate 10% with interest paid monthly, your debt after one year will be $a(1 + 10/(12 \cdot 100))^{12} \approx 1.1047a$, so schedule (ii) is preferable.

4. $15\,000e^{0.07 \cdot 12} \approx 34\,745.50$

5. (a) $8000e^{0.06\cdot 3} \approx 9577.74$ (b) $t^* = \ln 2/0.06 \approx 11.6$

6. We use formula (10.4.5): (a) $\dfrac{44}{1-0.56} = 100$

(b) The first term is 20 and the quotient is $1/1.2$, so the sum is $\dfrac{20}{1-1/1.2} = 120$. (c) $\dfrac{3}{1-2/5} = 5$

(d) The first term is $(1/20)^{-2} = 400$ and the quotient is $1/20$, so the sum is $\dfrac{400}{1-1/20} = 8000/19$.

7. (a) $\int_0^T ae^{-rt}\,dt = (a/r)(1-e^{-rT})$ (b) a/r, the same as (10.5.4).

8. $5000(1.04)^4 = 5849.29$

9. 21 232.32

10. $K \approx 5990.49$

11. (a) According to formula (10.6.2), the annual payment is: $500\,000 \cdot 0.07(1.07)^{10}/(1.07^{10}-1) \approx 71\,188.80$. The total amount is $10 \cdot 71\,188.80 = 711\,888$. (b) If the person has to pay twice a year, the semiannual payment is $500\,000 \cdot 0.035(1.035)^{20}/(1.035^{20}-1) \approx 35\,180.50$. The total amount is then $20 \cdot 35\,180.50 = 703\,610.80$.

12. (i) Present value: $(3200/0.08)[1-(1.08)^{-10}] = 21\,472.26$.

(ii) Present value: $7000 + (3000/0.08)[1-(1.08)^{-5}] = 18\,978.13$.

(iii) Four years ahead the present value is $(4000/0.08)[1-(1.08)^{-10}] = 26\,840.33$. The present value when Lucy makes her choice is $26\,840.33 \cdot 1.08^{-4} = 19\,728.44$. So she should choose option (i).

13. (a) $t^* = 1/16r^2 = 25$ for $r = 0.05$. (b) $t^* = 1/\sqrt{r} = 5$ for $r = 0.04$.

14. (a) Because $F(0) = 0$, one has $F(10) = \int_0^{10} (1+0.4t)\,dt = \Big|_0^{10} (t+0.2t^2) = 30$. (b) See Example 9.5.3.

15. (a) $x_t = (-0.1)^t$ (b) $x_t = -2t+4$ (c) $x_t = 4\left(\frac{3}{2}\right)^t - 2$

Chapter 11

11.1

1. $f(0,1) = 1\cdot 0 + 2\cdot 1 = 2$, $f(2,-1) = 0$, $f(a,a) = 3a$, and $f(a+h,b) - f(a,b) = h$

2. $f(0,1) = 0$, $f(-1,2) = -4$, $f(10^4, 10^{-2}) = 1$, $f(a,a) = a^3$, $f(a+h,b) = (a+h)b^2 = ab^2 + hb^2$, and $f(a,b+k) - f(a,b) = 2abk + ak^2$.

3. $f(1,1) = 2$, $f(-2,3) = 51$, $f(1/x,1/y) = 3/x^2 - 2/xy + 1/y^3$, $p = 6x + 3h - 2y$, and $q = -2x + 3y^2 + 3yk + k^2$

4. (a) $f(-1,2) = 1$, $f(a,a) = 4a^2$, and $f(a+h,b) - f(a,b) = 2(a+b)h + h^2$

(b) $f(tx,ty) = (tx)^2 + 2(tx)(ty) + (ty)^2 = t^2(x^2+2xy+y^2) = t^2 f(x,y)$ for all t, including $t = 2$.

5. $F(1,1) = 10$, $F(4,27) = 60$, $F(9,1/27) = 10$, $F(3,\sqrt{2}) = 10\sqrt{3}\cdot\sqrt[6]{2}$, $F(100,1000) = 1000$, and $F(2K,2L) = 10\cdot 2^{5/6}K^{1/2}L^{1/3} = 2^{5/6}F(K,L)$

6. (a) The denominator must be different from 0, so the function is defined for those (x, y) where $y \neq x - 2$.

(b) Only nonnegative numbers have a square root, so we must require $2 - (x^2 + y^2) \geq 0$, i.e. $x^2 + y^2 \leq 2$.

(c) Put $a = x^2 + y^2$. We must have $(4 - a)(a - 1) \geq 0$, i.e. $1 \leq a \leq 4$. (Use a sign diagram.)

The domains in (b) and (c) are the shaded sets shown in Figs A11.1.6b and A11.1.6c respectively.

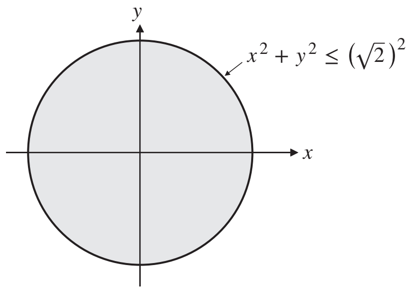


Figure A11.1.6b

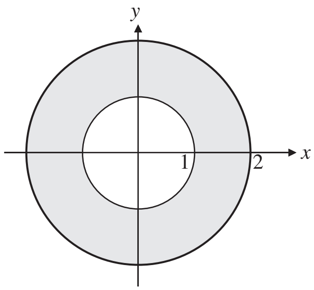


Figure A11.1.6c

7. (a) $e^{x+y} \neq 3$, that is $x + y \neq \ln 3$

(b) Since $(x - a)^2 \geq 0$ and $(y - b)^2 \geq 0$, it suffices to have $x \neq a$ and $y \neq b$, because then we take ln of positive numbers.

(c) $x > a$ and $y > b$. (Note that $\ln(x - a)^2 = 2 \ln |x - a|$, which is equal to $2 \ln(x - a)$ only if $x > a$.)

11.2

1. (a) $\partial z/\partial x = 2$, $\partial z/\partial y = 3$ (b) $\partial z/\partial x = 2x$, $\partial z/\partial y = 3y^2$ (c) $\partial z/\partial x = 3x^2y^4$, $\partial z/\partial y = 4x^3y^3$

(d) $\partial z/\partial x = \partial z/\partial y = 2(x + y)$

2. (a) $\partial z/\partial x = 2x$, $\partial z/\partial y = 6y$ (b) $\partial z/\partial x = y$, $\partial z/\partial y = x$ (c) $\partial z/\partial x = 20x^3y^2 - 2y^5$, $\partial z/\partial y = 10x^4y - 10xy^4$

(d) $\partial z/\partial x = \partial z/\partial y = e^{x+y}$ (e) $\partial z/\partial x = ye^{xy}$, $\partial z/\partial y = xe^{xy}$ (f) $\partial z/\partial x = e^x/y$, $\partial z/\partial y = -e^x/y^2$

(g) $\partial z/\partial x = \partial z/\partial y = 1/(x + y)$ (h) $\partial z/\partial x = 1/x$, $\partial z/\partial y = 1/y$

3. (a) $f_1'(x, y) = 7x^6, f_2'(x, y) = -7y^6, f_{12}'(x, y) = 0$ (b) $f_1'(x, y) = 5x^4 \ln y, f_2'(x, y) = x^5/y, f_{12}''(x, y) = 5x^4/y$

(c) $f(x, y) = (x^2 - 2y^2)^5 = u^5$, where $u = x^2 - 2y^2$. Then $f_1'(x, y) = 5u^4u_1' = 5(x^2 - 2y^2)^4 2x = 10x(x^2 - 2y^2)^4$.

In the same way, $f_2'(x, y) = 5u^4u_2' = 5(x^2 - 2y^2)^4(-4y) = -20y(x^2 - 2y^2)^4$.

Finally, $f_{12}''(x, y) = (\partial/\partial y)(10x(x^2 - 2y^2)^4) = 10x\,4(x^2 - 2y^2)^3(-4y) = -160xy(x^2 - 2y^2)^3$.

4. (a) $z_x' = 3$, $z_y' = 4$, and $z_{xx}'' = z_{xy}'' = z_{yx}'' = z_{yy}'' = 0$

(b) $z_x' = 3x^2y^2$, $z_y' = 2x^3y$, $z_{xx}'' = 6xy^2$, $z_{yy}'' = 2x^3$, and $z_{xy}'' = 6x^2y$

(c) $z_x' = 5x^4 - 6xy$, $z_y' = -3x^2 + 6y^5$, $z_{xx}'' = 20x^3 - 6y$, $z_{yy}'' = 30y^4$, and $z_{xy}'' = -6x$

(d) $z_x' = 1/y$, $z_y' = -x/y^2$, $z_{xx}'' = 0$, $z_{yy}'' = 2x/y^3$, and $z_{xy}'' = -1/y^2$

(e) $z_x' = 2y(x + y)^{-2}$, $z_y' = -2x(x + y)^{-2}$, $z_{xx}'' = -4y(x + y)^{-3}$, $z_{yy}'' = 4x(x + y)^{-3}$, and $z_{xy}'' = 2(x - y)(x + y)^{-3}$

(f) $z_x' = x(x^2 + y^2)^{-1/2}$, $z_y' = y(x^2 + y^2)^{-1/2}$, $z_{xx}'' = y^2(x^2 + y^2)^{-3/2}$, $z_{yy}'' = x^2(x^2 + y^2)^{-3/2}$, and $z_{xy}'' = -xy(x^2 + y^2)^{-3/2}$

5. (a) $z'_x = 2x$, $z'_y = 2e^{2y}$, $z''_{xx} = 2$, $z''_{yy} = 4e^{2y}$, and $z''_{xy} = 0$

(b) $z'_x = y/x$, $z'_y = \ln x$, $z''_{xx} = -y/x^2$, $z''_{yy} = 0$, and $z''_{xy} = 1/x$

(c) $z'_x = y^2 - ye^{xy}$, $z'_y = 2xy - xe^{xy}$, $z''_{xx} = -y^2e^{xy}$, $z''_{yy} = 2x - x^2e^{xy}$, and $z''_{xy} = 2y - e^{xy} - xye^{xy}$

(d) $z'_x = yx^{y-1}$, $z'_y = x^y \ln x$, $z''_{xx} = y(y-1)x^{y-2}$, $z''_{yy} = x^y(\ln x)^2$, and $z''_{xy} = x^{y-1} + yx^{y-1}\ln x$

6. (a) $F'_S = 2.26 \cdot 0.44S^{-0.56}E^{0.48} = 0.9944S^{-0.56}E^{0.48}$, $F'_E = 2.26 \cdot 0.48S^{0.44}E^{-0.52} = 1.0848S^{0.44}E^{-0.52}$

(b) $SF'_S + EF'_E = S \cdot 2.26 \cdot 0.44S^{-0.56}E^{0.48} + E \cdot 2.26 \cdot 0.48S^{0.44}E^{-0.52} = 0.44\,F + 0.48\,F = 0.92\,F$, so $k = 0.92$.

7. $xz'_x + yz'_y = x[2a(ax+by)] + y[2b(ax+by)] = (ax+by)2(ax+by) = 2(ax+by)^2 = 2z$

8. $\partial z/\partial x = x/(x^2+y^2)$, $\partial z/\partial y = y/(x^2+y^2)$, $\partial^2 z/\partial x^2 = (y^2-x^2)/(x^2+y^2)^2$, and $\partial^2 z/\partial y^2 = (x^2-y^2)/(x^2+y^2)^2$. Thus, $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$.

9. (a) $s'_x(x,y) = 2/x$, so $s'_x(20,30) = 2/20 = 1/10$. (b) $s'_y(x,y) = 4/y$, so $s'_y(20,30) = 4/30 = 2/15$.

11.3

1. See Fig. A11.3.1.

2. (a) A straight line through $(0,2,3)$ parallel to the x-axis.

(b) A plane parallel to the z-axis whose intersection with the xy-plane is the line $y = x$.

3. If $x^2 + y^2 = 6$, then $f(x,y) = \sqrt{6} - 4$, so $x^2 + y^2 = 6$ is a level curve of f at height $c = \sqrt{6} - 4$.

4. $f(x,y) = e^{x^2-y^2} + (x^2-y^2)^2 = e^c + c^2$ when $x^2 - y^2 = c$, so the last equation represents a level curve of f having height $e^c + c^2$.

5. At the point of intersection f would have two different values, which is impossible when f is a function.

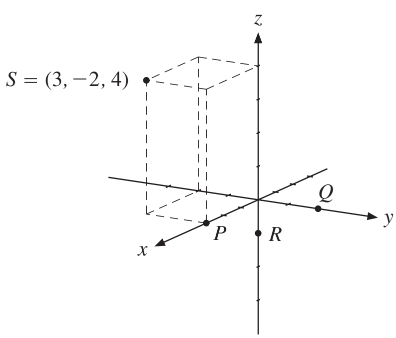


Figure A11.3.1

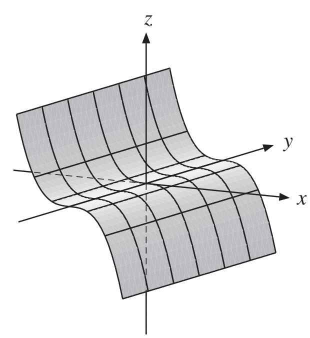


Figure A11.3.6

6. Generally, the graph of $g(x,y) = f(x)$ in 3-space consists of a surface traced out by moving the graph of $z = f(x)$ parallel to the y-axis in both directions. The graph of $g(x,y) = x$ is the plane through the y-axis at a 45° angle with the xy-plane. The graph of $g(x,y) = -x^3$ is shown in Fig. A11.3.6. (Only a portion of the unbounded graph is indicated, of course.)

7. See Figs A11.3.7a and A11.3.7b., which are both in two parts. (Note that only a portion of the graph is shown in part (a).)

8. (a) The point $(2,3)$ lies on the level curve $z = 8$, so $f(2,3) = 8$. The points $(x,3)$ are those on the line $y = 3$ parallel to the x-axis. This line intersects the level curve $z = 8$ when $x = 2$ and $x = 5$.

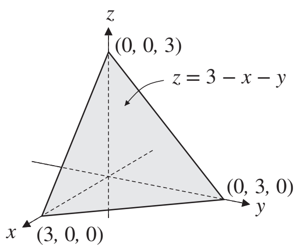


Figure A11.3.7a1

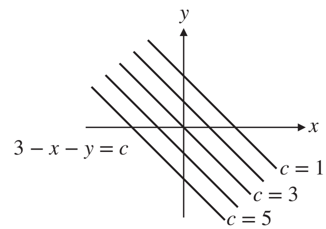


Figure A11.3.7a2

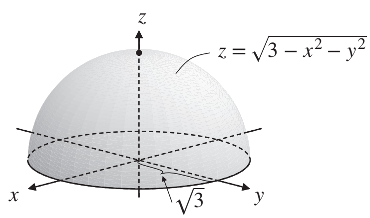


Figure A11.3.7b1

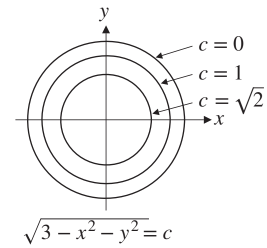


Figure A11.3.7b2

(b) As y varies with $x = 2$ fixed, the minimum of $f(2, y)$ is 8 when $y = 3$.

(c) At A, any move in the direction of increasing x with y held fixed reaches higher level curves, so $f_1'(x, y) > 0$. Similarly, any move in the direction of increasing y with x held fixed reaches higher level curves, so $f_2'(x, y) > 0$. At B: $f_1'(x, y) < 0$, $f_2'(x, y) < 0$. At C: $f_1'(x, y) = 0$, $f_2'(x, y) = 0$. Finally, to increase z by two units when moving away from A, the required increases in x and y are approximately 1 and 0.6, respectively. Hence, $f_1' \approx 2/1 = 2$ and $f_2' \approx 2/0.6 = 10/3$.

9. (a) $f_x' > 0$ and $f_y' < 0$ at P, whereas $f_x' < 0$ and $f_y' > 0$ at Q.

(b) (i) No solutions among points shown in the figure. (ii) $x \approx 2$ and $x \approx 6$

(c) The highest level curve that meets the line is $z = 3$, so 3 is the largest value.

10. See SM.

11.4

1. See Figs A11.4.1a, A11.4.1b, and A11.4.1c.

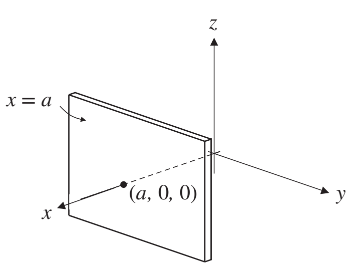


Figure A11.4.1a

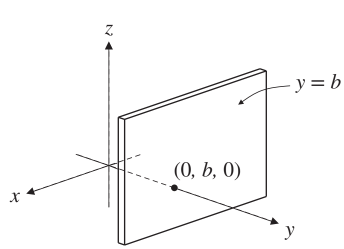


Figure A11.4.1b

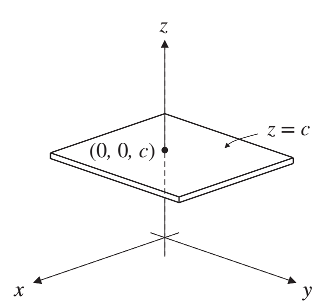


Figure A11.4.1c

2. (a) $d = \sqrt{(4-(-1))^2 + (-2-2)^2 + (0-3)^2} = \sqrt{25+16+9} = \sqrt{50} = 5\sqrt{2}$

(b) $d = \sqrt{(a+1-a)^2 + (b+1-b)^2 + (c+1-c)^2} = \sqrt{3}$

3. $(x-2)^2 + (y-1)^2 + (z-1)^2 = 25$

4. The sphere with centre at $(-3, 3, 4)$ and radius 5.

5. $(x-4)^2 + (y-4)^2 + (z-\frac{1}{2})^2$ measures the square of the distance from the point $(4, 4, \frac{1}{2})$ to the point (x, y, z) on the paraboloid.

11.5

1. (a) $f(-1, 2, 3) = 1$ and $f(a+1, b+1, c+1) - f(a, b, c) = 2a + 2b + 2c + 3$.

(b) $f(tx, ty, tz) = (tx)(ty) + (tx)(tz) + (ty)(tz) = t^2(xy + xz + yz) = t^2 f(x, y, z)$

2. (a) Because 1.053 is the sum of exponents, y would become $2^{1.053} \approx 2.07$ times as large.

(b) $\ln y = \ln 2.9 + 0.015 \ln x_1 + 0.25 \ln x_2 + 0.35 \ln x_3 + 0.408 \ln x_4 + 0.03 \ln x_5$

3. (a) In successive weeks it buys $120/50 = 2.4$, then $120/60 = 2$, $120/45 \approx 2.667$, $120/40 = 3$, $120/75 = 1.6$, and finally $120/80 = 1.5$ million shares, so about 13.167 million in total.

(b) The average price per share is about $\$720/13.167 \approx 54.68$. This is the harmonic mean price, which is almost \$4 a share lower than the arithmetic mean $\$350/6 \approx 58.67$.

4. (a) In each week w bank A will have bought $100/p_w$ million euros, for a total of $e = \sum_{w=1}^{n} 100/p_w$ million euros.

(b) Bank A will have paid $100n$ million dollars, so the price p per euro that bank A will have paid, on average, is $p = 100n/e$. It follows that $1/p = e/100n = (1/n)\sum_{w=1}^{n} 1/p_w$ dollars per euro, implying that p is the harmonic mean of $p_1, \ldots, p_n$. Since this is lower than the arithmetic mean (except in the case when p_w is the same every week), this is a supposed advantage of dollar cost averaging.

5. (a) Each machine would produce 60 units per day, so $480/60 =$ eight minutes per unit.

(b) Total output is $\sum_{i=1}^{n}(T/t_i) = T\sum_{i=1}^{n}(1/t_i)$. If all n machines were equally efficient, the time needed for each unit would be $nT/T\sum_{i=1}^{n}(1/t_i) = n/\sum_{i=1}^{n}(1/t_i)$, the harmonic mean of $t_1, \ldots, t_n$.

11.6

1. $F_1'(x, y, z) = 2xe^{xz} + x^2ze^{xz} + y^4e^{xy}$, so $F_1'(1, 1, 1) = 4e$. $F_2'(x, y, z) = 3y^2e^{xy} + xy^3e^{xy}$, so $F_2'(1, 1, 1) = 4e$. $F_3'(x, y, z) = x^3e^{xz}$, so $F_3'(1, 1, 1) = e$.

2. (a) $f_1' = 2x$, $f_2' = 3y^2$, and $f_3' = 4z^3$ (b) $f_1' = 10x$, $f_2' = -9y^2$, and $f_3' = 12z^3$

(c) $f_1' = yz$, $f_2' = xz$, and $f_3' = xy$ (d) $f_1' = 4x^3/yz$, $f_2' = -x^4/y^2z$, and $f_3' = -x^4/yz^2$

(e) $f_1' = 12x(x^2 + y^3 + z^4)^5$, $f_2' = 18y^2(x^2 + y^3 + z^4)^5$, and $f_3' = 24z^3(x^2 + y^3 + z^4)^5$

(f) $f_1' = yze^{xyz}$, $f_2' = xze^{xyz}$, and $f_3' = xye^{xyz}$

3. $\partial T/\partial x = ky/d^n$ and $\partial T/\partial y = kx/d^n$ are both positive, so that the number of travellers increases if the size of either city increases, which is reasonable. $\partial T/\partial d = -nkxy/d^{n+1}$ is negative, so that the number of travellers decreases if the distance between the cities increases, which is also reasonable.

4. (a) $g(2, 1, 1) = -2$, $g(3, -4, 2) = 352$, and $g(1, 1, a+h) - g(1, 1, a) = 2ah + h^2 - h$

(b) $g_1' = 4x - 4y - 4$, $g_2' = -4x + 20y - 28$, $g_3' = 2z - 1$. The second-order partials are: $g_{11}'' = 4$, $g_{12}'' = -4$, $g_{13}'' = 0$, $g_{21}'' = -4$, $g_{22}'' = 20$, $g_{23}'' = 0$, $g_{31}'' = 0$, $g_{32}'' = 0$, and $g_{33}'' = 2$.

5. $\partial\pi/\partial p = \frac{1}{2}p(1/r + 1/w)$, $\partial\pi/\partial r = -\frac{1}{4}p^2/r^2$, and $\partial\pi/\partial w = -\frac{1}{4}p^2/w^2$

6. First-order partials: $w_1' = 3yz + 2xy - z^3$, $w_2' = 3xz + x^2$, and $w_3' = 3xy - 3xz^2$. Second-order partials: $w_{11}'' = 2y$, $w_{12}'' = w_{21}'' = 3z + 2x$, $w_{13}'' = w_{31}'' = 3y - 3z^2$, $w_{22}'' = 0$, $w_{23}'' = w_{32}'' = 3x$, and $w_{33}'' = -6xz$.

7. $f_1' = p'(x), f_2' = q'(y), f_3' = r'(z)$

8. (a) $\begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{pmatrix}$ (b) $\begin{pmatrix} a(a-1)g/x^2 & abg/xy & acg/xz \\ abg/xy & b(b-1)g/y^2 & bcg/yz \\ acg/xz & bcg/yz & c(c-1)g/z^2 \end{pmatrix}$, in concise form.

9. Put $w = u^h$, where $u = (x - y + z)/(x + y - z)$. Then $\partial w/\partial x = hu^{h-1}\partial u/\partial x$, $\partial w/\partial y = hu^{h-1}\partial u/\partial y$, and $\partial w/\partial z = hu^{h-1}\partial u/\partial z$. With $v = x + y - z$, we get $\partial u/\partial x = (2y - 2z)/v^2$, $\partial u/\partial y = -2x/v^2$, and $\partial u/\partial z = 2x/v^2$. Hence $x\,\partial w/\partial x + y\,\partial w/\partial y + z\,\partial w/\partial z = hu^{h-1}v^{-2}[x(2y - 2z) + y(-2x) + z2x] = 0$. (In the terminology of Section 12.7, the function w is homogeneous of degree 0. Euler's theorem (Theorem 12.7.1) yields the result immediately.)

10. $f_x' = y^z x^{y^z - 1}$, $f_y' = zy^{z-1}(\ln x)x^{y^z}$, and $f_z' = y^z(\ln x)(\ln y)x^{y^z}$

11. See SM.

11.7

1. $\partial M/\partial Y = 0.14$ and $\partial M/\partial r = -0.84 \cdot 76.03(r - 2)^{-1.84} = -63.8652(r - 2)^{-1.84}$.

   So $\partial M/\partial Y$ is positive and $\partial M/\partial r$ is negative. Both signs accord with standard economic intuition.

2. (a) $KY_K' + LY_L' = aY$ (b) $KY_K' + LY_L' = (a + b)Y$ (c) $KY_K' + LY_L' = Y$

3. $D_p'(p, q) = -bq^{-\alpha}$, $D_q'(p, q) = bp\alpha q^{-\alpha-1}$. So $D_p'(p, q) < 0$, showing that demand decreases as price increases. And $D_q'(p, q) > 0$, showing that demand increases as the price of a competing product increases.

4. $F_K' = aF/K$, $F_L' = bF/L$, and $F_M' = cF/M$, so $KF_K' + LF_L' + MF_M' = (a + b + c)F$.

5. $\partial D/\partial p$ and $\partial E/\partial q$ are normally negative, because the demand for a commodity goes down when its price increases.

   If the commodities are substitutes, this means that demand increases when the price of the other good increases.

   So the usual signs are $\partial D/\partial q > 0$ and $\partial E/\partial p > 0$.

6. $\partial U/\partial x_i = e^{-x_i}$, for $i = 1, \ldots, n$

7. $KY_K' + LY_L' = \mu Y$

11.8

1. (a) $\mathrm{El}_x z = 1$ and $\mathrm{El}_y z = 1$ (b) $\mathrm{El}_x z = 2$ and $\mathrm{El}_y z = 5$ (c) $\mathrm{El}_x z = n + x$ and $\mathrm{El}_y z = n + y$

   (d) $\mathrm{El}_x z = x/(x + y)$ and $\mathrm{El}_y z = y/(x + y)$

2. Let $z = u^g$ with $u = ax_1^d + bx_2^d + cx_3^d$. Then $\mathrm{El}_1 z = \mathrm{El}_u u^g \,\mathrm{El}_1 u = g(x_1/u)adx_1^{d-1} = adgx_1^d/u$.

   Similarly, $\mathrm{El}_2 z = bdgx_2^d/u$ and $\mathrm{El}_3 z = cdgx_3^d/u$, so $\mathrm{El}_1 z + \mathrm{El}_2 z + \mathrm{El}_3 z = dg(ax_1^d + bx_2^d + cx_3^d)/u = dg$.

   (This result follows easily from the fact that z is homogeneous of degree dg and from the elasticity form (12.7.3) of the Euler equation.)

3. $\mathrm{El}_i z = p + a_i x_i$ for $i = 1, \ldots, n$

4. See SM.

Review exercises for Chapter 11

1. $f(0,1) = -5, f(2,-1) = 11, f(a,a) = -2a$, and $f(a+h,b) - f(a,b) = 3h$

2. $f(-1,2) = -10, f(2a,2a) = -4a^2, f(a,b+k) - f(a,b) = -6bk - 3k^2$, and $f(tx,ty) - t^2 f(x,y) = 0$

3. $f(3,4,0) = 5, f(-2,1,3) = \sqrt{14}$, and $f(tx,ty,tz) = \sqrt{t^2x^2 + t^2y^2 + t^2z^2} = tf(x,y,z)$

4. (a) $F(0,0) = 0$, $F(1,1) = 15$, and $F(32,243) = 15 \cdot 2 \cdot 9 = 270$

(b) $F(K+1,L) - F(K,L) = 15(K+1)^{1/5}L^{2/5} - 15K^{1/5}L^{2/5} = 15L^{2/5}[(K+1)^{1/5} - K^{1/5}]$ is the extra output from one more unit of capital, approximately equal to the marginal productivity of capital.

(c) $F(32+1,243) - F(32,243) \approx 1.667$. Moreover, $F'_K(K,L) = 3K^{-4/5}L^{2/5}$, so $F'_K(32,243) = 3 \cdot 32^{-4/5}243^{2/5} = 3 \cdot 2^{-4} \cdot 3^2 = 27/16 \approx 1.6875$. As expected, $F(32+1,243) - F(32,243)$ is close to $F'_K(32,243)$.

(d) F is homogeneous of degree $3/5$.

5. (a) $\partial Y/\partial K \approx 0.083K^{0.356}S^{0.562}$, and $\partial Y/\partial S \approx 0.035K^{1.356}S^{-0.438}$

(b) The catch becomes $2^{1.356+0.562} = 2^{1.918} \approx 3.779$ times as high.

6. (a) All (x,y) (b) For $xy \leq 1$ (c) For $x^2 + y^2 < 2$

7. (a) $x + y > 1$ (b) $x^2 \geq y^2$, and $x^2 + y^2 \geq 1$. So $x^2 + y^2 \geq 1$ and $|x| \geq |y|$.

(c) $y \geq x^2$, $x \geq 0$, and $\sqrt{x} \geq y$. So $0 \leq x \leq 1$ and $\sqrt{x} \geq y \geq x^2$.

8. (a) $\partial z/\partial x = 10xy^4(x^2y^4 + 2)^4$ (b) $\sqrt{K}(\partial F/\partial K) = 2\sqrt{K}(\sqrt{K} + \sqrt{L})(1/2\sqrt{K}) = \sqrt{K} + \sqrt{L}$

(c) $KF'_K + LF'_L = K(1/a)aK^{a-1}(K^a + L^a)^{1/a-1} + L(1/a)aL^{a-1}(K^a + L^a)^{1/a-1} = (K^a + L^a)(K^a + L^a)^{1/a-1} = F$

(d) $\partial g/\partial t = 3/w + 2wt$, so $\partial^2 g/\partial w \partial t = -3/w^2 + 2t$

(e) $g'_3 = t_3(t_1^2 + t_2^2 + t_3^2)^{-1/2}$ (f) $f'_1 = 4xyz + 2xz^2, f''_{13} = 4xy + 4xz$

9. (a) $f(0,0) = 36, f(-2,-3) = 0$, and $f(a+2,b-3) = a^2b^2$ (b) $f'_x = 2(x-2)(y+3)^2$, and $f'_y = 2(x-2)^2(y+3)$

10. Because $g(-1,5) = g(1,1) = 30$, the two points are on the same level curve.

11. If $x - y = c \neq 0$, then $F(x,y) = \ln(x-y)^2 + e^{2(x-y)} = \ln c^2 + e^{2c}$, a constant.

12. (a) $f'_1(x,y) = 4x^3 - 8xy, f'_2(x,y) = 4y - 4x^2 + 4$ (b) Critical points: $(0,-1)$, $(\sqrt{2},1)$, and $(-\sqrt{2},1)$.

13. (a) $\text{El}_x z = 3$, $\text{El}_y z = -4$ (b) $\text{El}_x z = 2x^2/(x^2+y^2)\ln(x^2+y^2)$, $\text{El}_y z = 2y^2/(x^2+y^2)\ln(x^2+y^2)$

(c) $\text{El}_x z = \text{El}_x(e^x e^y) = \text{El}_x e^x = x$, $\text{El}_y z = y$ (d) $\text{El}_x z = x^2/(x^2+y^2)$, $\text{El}_y z = y^2/(x^2+y^2)$

14. (a) $\partial F/\partial y = e^{2x}2(1-y)(-1) = -2e^{2x}(1-y)$. (b) $F'_L = (\ln K)(\ln M)/L, F'_{LK} = (\ln M)/KL$

(c) $w = x^x y^x z^x$ gives $\ln w = x\ln x + x\ln y + x\ln z$, and so by implicit differentiation, $w'_x/w = 1 \cdot \ln x + x(1/x) + \ln y + \ln z$, implying that $w'_x = w(\ln x + 1 + \ln y + \ln z) = x^x y^x z^x[\ln(xyz) + 1]$.

15. (a) Begin by differentiating w.r.t. x to obtain $\partial^p z/\partial x^p = e^x \ln(1+y)$ for any natural number p. Differentiating this repeatedly w.r.t. y yields first $\partial^{p+1}/\partial y \partial x^p = e^x(1+y)^{-1}$, then $\partial^{p+2}/\partial y^2 \partial x^p = e^x(-1)(1+y)^{-2}$, and so on. By induction on q, one has $\partial^{p+q}/\partial y^q \partial x^p = e^x(-1)^{q-1}(q-1)!(1+y)^{-q}$, which is $(-1)^{q-1}(q-1)!$ at $(x,y) = (0,0)$.

(b) Write $z = z_1 + z_2 - z_3$ where $z_1 = xe^x \cdot ye^y$, $z_2 = e^x \cdot ye^y$, and $z_3 = e^x \cdot e^y$, and note that $(\mathrm{d}/\mathrm{d}u)^n ue^u = e^u(u+n)$

for $n = 1, 2, \ldots$, as is easily proved by induction on n. Then $\partial^{p+q}z_1/\partial x^p \partial y^q = (\mathrm{d}/\mathrm{d}x)^p e^x x \cdot (\mathrm{d}/\mathrm{d}y)^q e^y y = e^x(x+p) \cdot e^y(y+q)$, whereas $\partial^{p+q}z_2/\partial x^p \partial y^q = (\mathrm{d}/\mathrm{d}x)^p e^x \cdot (\mathrm{d}/\mathrm{d}y)^q e^y y = e^x \cdot e^y(y+q)$, and $\partial^{p+q}z_3/\partial x^p \partial y^q = e^x \cdot e^y$. Gathering terms, it follows that $\partial^{p+q}z/\partial x^p \partial y^q = e^{x+y}[(x+p+1)(y+q)-1]$, which reduces to $(p+1)q-1$ at $(x,y)=(0,0)$.

16. $u'_x = au/x$ and $u'_y = bu/y$, so $u''_{xy} = au'_y/x = abu/xy$. Hence, $u''_{xy}/u'_x u'_y = 1/u$ $(u \neq 0)$. Then

$$\frac{1}{u'_x}\frac{\partial}{\partial x}\left(\frac{u''_{xy}}{u'_x u'_y}\right) = \frac{1}{u'_x} \cdot \frac{-u'_x}{u^2} = -\frac{1}{u^2} = \frac{1}{u'_y}\frac{\partial}{\partial y}\left(\frac{u''_{xy}}{u'_x u'_y}\right)$$

Chapter 12

12.1

1. (a) $\mathrm{d}z/\mathrm{d}t = F'_1(x,y)\,\mathrm{d}x/\mathrm{d}t + F'_2(x,y)\,\mathrm{d}y/\mathrm{d}t = 1 \cdot 2t + 2y \cdot 3t^2 = 2t + 6t^5$

Check: $z = t^2 + (t^3)^2 = t^2 + t^6$, so $\mathrm{d}z/\mathrm{d}t = 2t + 6t^5$.

(b) $\mathrm{d}z/\mathrm{d}t = px^{p-1}y^q a + qx^p y^{q-1} b = x^{p-1}y^{q-1}(apy + bqx) = a^p b^q (p+q) t^{p+q-1}$

Check: $z = (at)^p \cdot (bt)^q = a^p b^q t^{p+q}$, so $\mathrm{d}z/\mathrm{d}t = a^p b^q (p+q) t^{p+q-1}$.

2. (a) $\mathrm{d}z/\mathrm{d}t = (\ln y + y/x) \cdot 1 + (x/y + \ln x)(1/t) = \ln(\ln t) + \ln t/(t+1) + (t+1)/t \ln t + \ln(t+1)/t$

(b) $\mathrm{d}z/\mathrm{d}t = Aae^{at}/x + Bbe^{bt}/y = a + b$

3. $\mathrm{d}z/\mathrm{d}t = F'_1(t,y) + F'_2(t,y)g'(t)$. If $F(t,y) = t^2 + ye^y$ and $g(t) = t^2$, then $F'_1(t,y) = 2t$, $F'_2(t,y) = e^y + ye^y$, and $g'(t) = 2t$. Hence $\mathrm{d}z/\mathrm{d}t = 2t(1 + e^{t^2} + t^2 e^{t^2})$.

4. $\mathrm{d}Y/\mathrm{d}L = F'_K(K,L)g'(L) + F'_L(K,L)$

5. $\mathrm{d}Y/\mathrm{d}t = \left(10L - \frac{1}{2}K^{-1/2}\right) 0.2 + \left(10K - \frac{1}{2}L^{-1/2}\right) 0.5e^{0.1t} = 35 - 7\sqrt{5}/100$ when $t = 0$ and so $K = L = 5$.

6. The usual rules in Sections 6.7 and 6.8 for differentiating (a) a sum; (b) a difference; (c) a product; (d) a quotient; (e) a composite function of one variable.

7. $x^* = \sqrt[4]{3b/a}$

8. See SM.

12.2

1. (a) $\partial z/\partial t = F'_1(x,y)\partial x/\partial t + F'_2(x,y)\partial y/\partial t = 1 \cdot 1 + 2ys = 1 + 2ts^2$,

$\partial z/\partial s = (\partial z/\partial x)(\partial x/\partial s) + (\partial z/\partial y)(\partial y/\partial s) = 1 \cdot (-1) + 2yt = -1 + 2t^2 s$

(b) $\partial z/\partial t = 4x\,2t + 9y^2 = 8tx + 9y^2 = 8t^3 - 8ts + 9t^2 + 36ts^3 + 36s^6$,

$\partial z/\partial s = 4x(-1) + 9y^2 6s^2 = -4x + 54s^2y^2 = -4t^2 + 4s + 54t^2s^2 + 216ts^5 + 216s^8$

2. (a) $\partial z/\partial t = y^2 + 2xy2ts = 5t^4s^2 + 4t^3s^4$, $\partial z/\partial s = y^2 2s + 2xyt^2 = 2t^5 s + 4t^4 s^3$

(b) $\dfrac{\partial z}{\partial t} = \dfrac{2(1-s)e^{ts+t+s}}{(e^{t+s}+e^{ts})^2}$ and $\dfrac{\partial z}{\partial s} = \dfrac{2(1-t)e^{ts+t+s}}{(e^{t+s}+e^{ts})^2}$

3. $\partial z/\partial r = 2r\,\partial F/\partial u + (1/r)\,\partial F/\partial w$, $\partial z/\partial s = -4s\,\partial F/\partial v + (1/s)\,\partial F/\partial w$

4. $\partial z/\partial t_1 = F'(x)f_1'(t_1, t_2)$, $\partial z/\partial t_2 = F'(x)f_2'(t_1, t_2)$

5. $\partial x/\partial s = F_1' + F_2' f'(s) + F_3' g_1'(s,t)$, $\partial x/\partial t = F_3' g_2'(s,t)$

6. $\partial z/\partial x = F_1' f_1'(x,y) + F_2' 2xh(y)$, $\partial z/\partial y = F_1' f_2'(x,y) + F_2' x^2 h'(y) + F_3'(-1/y^2)$

7. (a) $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial t} = y^2z^3 \cdot 2t + 2xyz^3 \cdot 0 + 3xy^2z^2 \cdot 1 = 5s^2t^4$

(b) $\dfrac{\partial w}{\partial t} = 2x\dfrac{\partial x}{\partial t} + 2y\dfrac{\partial y}{\partial t} + 2z\dfrac{\partial z}{\partial t} = \dfrac{x}{\sqrt{t+s}} + 2sye^{ts} = 1 + 2se^{2ts}$

8. (a) We can write $z = F(u_1, u_2, u_3)$, with $u_1 = t$, $u_2 = t^2$ and $u_3 = t^3$.

Then $\dfrac{dz}{dt} = F_1'\dfrac{du_1}{dt} + F_2'\dfrac{du_2}{dt} + F_3'\dfrac{du_3}{dt} = F_1'(t,t^2,t^3) + F_2'(t,t^2,t^3)2t + F_3'(t,t^2,t^3)3t^2$.

(b) $z = F(t, f(t), g(t^2)) \implies \dfrac{dz}{dt} = F_1'(t, f(t), g(t^2)) + F_2'(t, f(t), g(t^2))f'(t) + F_3'(t, t^2, t^3)g'(t^2)2t$

9. $\partial Z/\partial G = 1 + 2Y\partial Y/\partial G + 2r\partial r/\partial G$

10. $\partial Z/\partial G = 1 + I_1'(Y,r)\partial Y/\partial G + I_2'(Y,r)\partial r/\partial G$

11. $\dfrac{\partial C}{\partial p_1} = a\dfrac{\partial Q_1}{\partial p_1} + b\dfrac{\partial Q_2}{\partial p_1} + 2cQ_1\dfrac{\partial Q_1}{\partial p_1} = -\alpha_1 A(a + 2cAp_1^{-\alpha_1}p_2^{\beta_1})p_1^{-\alpha_1-1}p_2^{\beta_1} + \alpha_2 bBp_1^{\alpha_2-1}p_2^{-\beta_2}$

$\dfrac{\partial C}{\partial p_2} = \beta_1 A(a + 2cAp_1^{-\alpha_1}p_2^{\beta_1})p_1^{-\alpha_1}p_2^{\beta_1-1} - \beta_2 bBp_1^{\alpha_2}p_2^{-\beta_2-1}$

12. See SM.

13. Follows from $\partial z/\partial x = f'(x^2y)2xy$ and $\partial z/\partial y = f'(x^2y)x^2$.

14. $\dfrac{\partial u}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial r} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial r}$

15. $\dfrac{\partial u}{\partial r} = yzw + xzw + xyws + xyz(1/s) = 28$

12.3

1. Formula (12.3.1) gives $y' = -F_1'/F_2' = -(4x + 6y)/(6x + 2y) = -(2x + 3y)/(3x + y)$.

2. (a) Put $F(x,y) = x^2y$. Then $F_1' = 2xy$, $F_2' = x^2$, $F_{11}'' = 2y$, $F_{12}'' = 2x$, $F_{22}'' = 0$, so $y' = -F_1'/F_2' = -2xy/x^2 = -2y/x$. Moreover, using Eq. (12.3.4), one has

$$y'' = -(1/(F_2')^3)\left[F_{11}''(F_2')^2 - 2F_{12}''F_1'F_2' + F_{22}''(F_1')^2\right] = -(1/x^6)[2yx^4 - 2(2x)(2xy)x^2] = 6y/x^2.$$

(See also Exercise 7.1.2.) For (b) and (c), see the answers to Exercise 7.1.3.

3. (a) $y' = -4$ and $y'' = -14$ at $(2,0)$. The tangent has the equation $y = -4x + 8$.

(b) At the two points $(a, -4a)$ and $(-a, 4a)$, where $a = 2\sqrt{7}/7$.

4. With $F(x,y) = 3x^2 - 3xy^2 + y^3 + 3y^2$, we have $F_1'(x,y) = 6x - 3y^2$ and $F_2'(x,y) = -6xy + 3y^2 + 6y$, so according to (12.3.1), $h'(x) = y' = -(6x - 3y^2)/(-6xy + 3y^2 + 6y)$. For x near 1 and so (x,y) near $(1,1)$, we have $h'(1) = -(6-3)/(-6+3+6) = -1$.

5. $D_P' < 0$ and $D_r' < 0$. Differentiating the equation w.r.t. r yields $D_P'(dP/dr) + D_r' = 0$, and so $dP/dr = -D_r'/D_P < 0$. So a rise in the interest rate depresses demand, and the price falls to compensate.

6. $\mathrm{d}P/\mathrm{d}R = f'_R(R,P)/(g'(P) - f'_P(R,P))$. It is plausible that $f'_R(R,P) > 0$ (demand increases as advertising expenditure increases), and $g'(P) > 0$, $f'_P(R,P) < 0$, so $\mathrm{d}P/\mathrm{d}R > 0$.

7. Differentiating the equation w.r.t. x gives (i) $1 - az'_x = f'(y - bz)(-bz'_x)$.

Differentiating it w.r.t. y gives (ii) $-az'_y = f'(y - bz)(1 - bz'_y)$. If $bz'_x \neq 0$, solving (i) for f' and inserting it into (ii) yields $az'_x + bz'_y = 1$. If $bz'_x = 0$, then (i) implies $az'_x = 1$. But then $z'_x \neq 0$, so $b = 0$ and then again $az'_x + bz'_y = 1$.

12.4

1. (a) With $F(x,y,z) = 3x + y - z$, the given equation is $F(x,y,z) = 0$, and then $\partial z/\partial x = -F'_1/F'_3 = -3/(-1) = 3$.

(b) $\partial z/\partial x = -(yz + z^3 - y^2z^5)/(xy + 3xz^2 - 5xy^2z^4)$

(c) With $F(x,y,z) = e^{xyz} - 3xyz$, the given equation is $F(x,y,z) = 0$. Now one has $F'_x(x,y,z) = yze^{xyz} - 3yz$ and $F'_z(x,y,z) = xye^{xyz} - 3xy$, so (12.4.1) gives

$$z'_x = -F'_x/F'_z = -(yze^{xyz} - 3yz)/(xye^{xyz} - 3xy) = -yz(e^{xyz} - 3)/xy(e^{xyz} - 3) = -z/x.$$

(Actually, the equation $e^c = 3c$ has two solutions. From $xyz = c$ (c a constant) we find z'_x much more easily.)

2. Differentiating partially w.r.t. x yields $(*)$ $3x^2 + 3z^2z'_x - 3z'_x = 0$, so $z'_x = x^2/(1 - z^2)$. By symmetry, $z'_y = y^2/(1 - z^2)$. To find z''_{xy}, differentiate $(*)$ w.r.t. y to obtain $6zz'_yz'_x + 3z^2z''_{xy} - 3z''_{xy} = 0$, so $z''_{xy} = 2zx^2y^2/(1 - z^2)^3$.

(Alternatively, differentiate $z'_x = x^2/(1 - z^2)$ w.r.t. y, treating z as a function of y and using the expression for z'_y.)

3. (a) $L^* = P^2/4w^2$, whose partial derivatives are $\partial L^*/\partial P = P/2w^2 > 0$, and $\partial L^*/\partial w = -P^2/2w^3 < 0$

(b) First-order condition: $Pf'(L^*) - C'_L(L^*, w) = 0$. Hence $\partial L^*/\partial P = -f'(L^*)/(Pf''(L^*) - C''_{LL}(L^*, w))$, $\partial L^*/\partial w = C''_{Lw}(L^*, w)/(Pf''(L^*) - C''_{LL}(L^*, w))$.

4. Using formula (12.4.1) gives $z'_x = -\dfrac{yx^{y-1} + z^x \ln z}{y^z \ln y + xz^{x-1}}$ and $z'_y = -\dfrac{x^y \ln x + zy^{z-1}}{y^z \ln y + xz^{x-1}}$

5. Implicit differentiation gives $f'_P(R,P)P'_w = g'_w(w,P) + g'_P(w,P)P'_w$. Hence $P'_w = -g'_w(w,P)/(g'_P(w,P) - f'_P(R,P)) < 0$ because $g'_w > 0$, $g'_P > 0$, and $f'_P < 0$.

6. $F(1,3) = 4$. The equation for the tangent is $y - 3 = -(F'_x(1,3)/F'_y(1,3))(x - 1)$ with $F'_x(1,3) = 10$ and $F'_y(1,3) = 5$, so $y = -2x + 5$.

7. $\partial y/\partial K = \alpha y/K(1 + 2c \ln y)$, $\partial y/\partial L = \beta y/L(1 + 2c \ln y)$

12.5

1. The marginal rate of substitution is $R_{yx} = 20x/30y$, so $y/x = (2/3)(R_{yx})^{-1}$, whose elasticity is $\sigma_{yx} = -1$.

2. (a) $R_{yx} = (x/y)^{a-1} = (y/x)^{1-a}$ (b) $\sigma_{yx} = \mathrm{El}_{R_{yx}}(y/x) = \mathrm{El}_{R_{yx}}(R_{yx})^{1/(1-a)} = 1/(1-a)$

3. See SM.

12.6

1. $f(tx,ty) = (tx)^4 + (tx)^2(ty)^2 = t^4x^4 + t^2x^2t^2y^2 = t^4(x^4 + x^2y^2) = t^4 f(x,y)$, so f is homogeneous of degree 4.

2. $x(tp,tr) = A(tp)^{-1.5}(tr)^{2.08} = At^{-1.5}p^{-1.5}\,t^{2.08}\,r^{2.08} = t^{-1.5}\,t^{2.08}Ap^{-1.5}r^{2.08} = t^{0.58}x(p,r)$, so the function is homogeneous of degree 0.58 . (Alternatively, use the result in Example 11.1.4.)

3. $f(tx,ty) = (tx)(ty)^2 + (tx)^3 = t^3(xy^2 + x^3) = t^3 f(x,y)$. So f is homogeneous of degree 3. For the rest, see SM.

4. $f(tx,ty) = (tx)(ty)/[(tx)^2 + (ty)^2] = t^2xy/t^2[x^2 + y^2] = f(x,y) = t^0 f(x,y)$, so f is homogeneous of degree 0.

Using the formulas in Example 11.2.1(b) for the partial derivatives of this function,

we get $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = \dfrac{xy^3 - x^3y + x^3y - xy^3}{(x^2+y^2)^2} = 0 = 0 \cdot f$, as claimed by Euler's theorem.

5. $F(tK,tL) = A(a(tK)^{-\rho} + b(tL)^{-\rho})^{-1/\rho} = A(t^{-\rho}aK^{-\rho} + t^{-\rho}bL^{-\rho})^{-1/\rho} = (t^{-\rho})^{-1/\rho}A(aK^{-\rho} + bL^{-\rho})^{-1/\rho}$, which reduces to $tF(K,L)$. Using Example 12.6.3, we get $F(K,L)/L = F(K/L,1) = A[a(K/L)^{-\rho} + b]^{-1/\rho}$.

6. Definition (12.6.1) requires that for some number k one has $t^3x^3 + t^2xy = t^k(x^3 + xy)$ for all $t > 0$ and all (x,y). In particular, for $x = y = 1$, we must have $t^3 + t^2 = 2t^k$. For $t = 2$, we get $12 = 2 \cdot 2^k$, or $2^k = 6$. For $t = 4$, we get $80 = 2 \cdot 4^k$, or $4^k = 40$. But $2^k = 6$ implies $4^k = 36$. So the two values of k must actually be different, implying that f is not homogeneous of any degree.

7. From Eqs (12.6.6) and (12.6.7), with $k = 1$, we get $f''_{11} = (-y/x)f''_{12}$ and $f''_{22} = (-x/y)f''_{21}$.

With $f''_{12} = f''_{21}$ we get $f''_{11}f''_{22} - (f''_{12})^2 = (-y/x)f''_{12}(-x/y)f''_{12} - (f''_{12})^2 = 0$.

8. $f'_2(4,6) = f'_2(2 \cdot 2, 2 \cdot 3) = 2f'_2(2,3)$ because $f'_2(x,y)$ is homogeneous of degree 1. (See (12.6.3).) But then $f'_2(2,3) = 12/2 = 6$. By Euler's theorem (12.6.2), $2f(2,3) = 2f'_1(2,3) + 3f'_2(2,3) = 2 \cdot 4 + 3 \cdot 6 = 26$. Hence $f(2,3) = 13$, and then $f(6,9) = f(3 \cdot 2, 3 \cdot 3) = 3^2 f(2,3) = 9 \cdot 13 = 117$, where we used definition (12.6.1).

9. See SM.

12.7

1. (a) Homogeneous of degree 1. (b) Not homogeneous. (c) Homogeneous of degree $-1/2$.

(d) Homogeneous of degree 1. (e) Not homogeneous. (f) Homogeneous of degree n.

2. (a) Homogeneous of degree 1. (b) Homogeneous of degree μ.

3. All are homogeneous of degree 1, as is easily checked by using The definition (12.7.1) directly.

4. Let s denote $x_1 + \cdots + x_n$. Then $v'_i = u'_i - a/s$, so $\sum_{i=1}^n x_i v'_i = \sum_{i=1}^n x_i u'_i - \sum_{i=1}^n ax_i/s = a - a = 0$.

By Euler's theorem, v is homogeneous of degree 0.

5. (a) Homothetic. (b) Homothetic. (c) Not homothetic. (d) Homothetic.

6. (a) $h(t\mathbf{x}) = f((tx_1)^m, \dots, (tx_n)^m) = f(t^m x_1^m, \dots, t^m x_n^m) = (t^m)^r f(x_1^m, \dots, x_n^m) = t^{mr}h(\mathbf{x})$, so h is homogeneous of degree mr. (b) Homogeneous of degree sp. (c) Homogeneous of degree r for $r = s$, not homogeneous for $r \neq s$.

(d) Homogeneous of degree $r + s$. (e) Homogeneous of degree $r - s$.

7. Routine application of the definitions. See SM.

12.8

1. We use the approximation $f(x,y) \approx f(0,0) + f'_1(0,0)x + f'_2(0,0)y$.

(a) $f'_1(x,y) = 5(x+1)^4(y+1)^6$ and $f'_2(x,y) = 6(x+1)^5(y+1)^5$, so $f'_1(0,0) = 5$ and $f'_2(0,0) = 6$.

Since $f(0,0) = 1$, $f(x,y) \approx 1 + 5x + 6y$.

(b) $f'_1(x,y) = f'_2(x,y) = \frac{1}{2}(1 + x + y)^{-1/2}$, so $f'_1(0,0) = f'_2(0,0) = 1/2$. Since $f(0,0) = 1$, one has $f(x,y) \approx 1 + \frac{1}{2}x + \frac{1}{2}$.
(c) $f'_1(x,y) = e^x \ln(1+y)$, $f'_2(x,y) = e^x/(1+y)$, so $f'_1(0,0) = 0$ and $f'_2(0,0) = 1$. Since $f(0,0) = 0$, $f(x,y) \approx y$.

2. $f(x,y) \approx Ax_0^a y_0^b + aAx_0^{a-1}y_0^b(x - x_0) + bAx_0^a y_0^{b-1}(y - y_0) = Ax_0^a y_0^b[1 + a(x - x_0)/x_0 + b(y - y_0)/y_0]$

3. Write the function in the form $g^*(\mu, \varepsilon) = (1+\mu)^a(1+\varepsilon)^{\alpha a} - 1$, where $a = 1/(1-\beta)$.

Then $\partial g^*(\mu,\varepsilon)/\partial\mu = a(1+\mu)^{a-1}(1+\varepsilon)^{\alpha a}$ and $\partial g^*(\mu,\varepsilon)/\partial\varepsilon = (1+\mu)^a \alpha a(1+\varepsilon)^{\alpha a - 1}$.

Hence, $g^*(0,0) = 0$, $\partial g^*(0,0)/\partial\mu = a$, $\partial g^*(0,0)/\partial\varepsilon = \alpha a$, and $g^*(\mu,\varepsilon) \approx a\mu + \alpha a\varepsilon = (\mu + \alpha\varepsilon)/(1-\beta)$.

4. $f(0.98, -1.01) \approx -5 - 6(-0.02) + 9(-0.01) = -4.97$. The exact value is -4.970614, so the error is 0.000614.

5. (a) $f(1.02, 1.99) = 1.1909$ (b) $f(1.02, 1.99) \approx f(1, 2) + 0.02 \cdot 8 - 0.01 \cdot (-3) = 1.19$. The error is 0.0009.

6. $v(1.01, 0.02) \approx v(1, 0) + v_1'(1, 0) \cdot 0.01 + v_2'(1, 0) \cdot 0.02 = -1 - 1/150$

7. (a) $z = 2x + 4y - 5$ (b) $z = -10x + 3y + 3$

8. Exten from two to n variables the argument used to establish the linear approximation (12.8.1). See SM.

9. The tangent plane (12.8.3) passes through $(x, y, z) = (0, 0, 0)$ if and only if $-f(x_0, y_0) = f_1'(x_0, y_0)(-x_0) + f_2'(x_0, y_0)(-y_0)$. According to Euler's theorem this equation holds for all (x_0, y_0) if and only if f is homogeneous of degree 1.

12.9

1. Both (a) and (b) give: $\mathrm{d}z = (y^2 + 3x^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y$.

2. We can either use the definition of the differential, (12.9.1), or the rules for differentials, as we do here.

(a) $\mathrm{d}z = \mathrm{d}(x^3) + \mathrm{d}(y^3) = 3x^2\,\mathrm{d}x + 3y^2\,\mathrm{d}y$

(b) $\mathrm{d}z = (\mathrm{d}x)e^{y^2} + x(\mathrm{d}e^{y^2})$. Here $\mathrm{d}(e^{y^2}) = e^{y^2}\,\mathrm{d}y^2 = e^{y^2} 2y\,\mathrm{d}y$, so $\mathrm{d}z = e^{y^2}\,\mathrm{d}x + 2xye^{y^2}\,\mathrm{d}y = e^{y^2}(\mathrm{d}x + 2xy\,\mathrm{d}y)$.

(c) $\mathrm{d}z = \mathrm{d}\ln u$, where $u = x^2 - y^2$. Then $\mathrm{d}z = \dfrac{1}{u}\,\mathrm{d}u = \dfrac{2x\,\mathrm{d}x - 2y\,\mathrm{d}y}{x^2 - y^2}$.

3. (a) $\mathrm{d}z = 2xu\,\mathrm{d}x + x^2(u_x'\,\mathrm{d}x + u_y'\,\mathrm{d}y)$ (b) $\mathrm{d}z = 2u(u_x'\,\mathrm{d}x + u_y'\,\mathrm{d}y)$

(c) $\mathrm{d}z = \dfrac{1}{xy + yu}\Big[(y + yu_x')\,\mathrm{d}x + (x + u + yu_y')\,\mathrm{d}y\Big]$

4. $T \approx 7.015714$ See SM.

5. Taking the differential of each side of the equation gives $\mathrm{d}(Ue^U) = \mathrm{d}(x\sqrt{y})$, and so $e^U\,\mathrm{d}U + Ue^U\,\mathrm{d}U = \sqrt{y}\,\mathrm{d}x + (x/2\sqrt{y})\,\mathrm{d}y$. Solving for $\mathrm{d}U$ yields $\mathrm{d}U = \sqrt{y}\,\mathrm{d}x/(e^U + Ue^U) + x\,\mathrm{d}y/2\sqrt{y}(e^U + Ue^U)$.

6. $\mathrm{d}X = A\beta N^{\beta-1}e^{\rho t}\,\mathrm{d}N + AN^{\beta}\rho e^{\rho t}\,\mathrm{d}t$

7. $\mathrm{d}X_1 = BEX^{E-1}N^{1-E}\,\mathrm{d}X + B(1-E)X^E N^{-E}\,\mathrm{d}N$

8. (a) $\mathrm{d}U = 2a_1u_1\,\mathrm{d}u_1 + \cdots + 2a_nu_n\,\mathrm{d}u_n$

(b) $\mathrm{d}U = A(\delta_1 u_1^{-\rho} + \cdots + \delta_n u_n^{-\rho})^{-1-1/\rho}(\delta_1 u_1^{-\rho-1}\mathrm{d}u_1 + \cdots + \delta_n u_n^{-\rho-1}\mathrm{d}u_n)$

9. $\mathrm{d}(\ln z) = a_1\,\mathrm{d}(\ln x_1) + \cdots + a_n\,\mathrm{d}(\ln x_n)$, so $\mathrm{d}z/z = a_1\,\mathrm{d}x_1/x_1 + a_2\,\mathrm{d}x_2/x_2 + \cdots + a_n\,\mathrm{d}x_n/x_n$.

10. (a) $\mathrm{d}^2z = 2\,\mathrm{d}x\,\mathrm{d}y + 2(\mathrm{d}y)^2$ (b) $\mathrm{d}z/\mathrm{d}t = 3t^2 + 4t^3$ and then $(\mathrm{d}^2z/\mathrm{d}t^2)(\mathrm{d}t)^2 = (6t + 12t^2)(\mathrm{d}t)^2$. On the other hand, the expression for d^2z derived from (a) is equal to $(4t + 8t^2)(\mathrm{d}t)^2$.

12.10

1. (a) $4 - 2 = 2$ (b) $5 - 2 = 3$ (c) $4 - 3 = 1$

2. There are six variables Y, C, I, G, T, and r, and three equations. So there are $6 - 3 = 3$ degrees of freedom.

3. Let m denote the number of equations and n the number of unknowns. (a) $m = 3$, $n = 2$; infinitely many solutions. (b) $m = n = 2$; no solutions. (c) $m = n = 2$; infinitely many solutions.

4. (a) $m = 1$, $n = 100$; infinitely many solutions. (b) $m = 1$, $n = 100$; no solutions. We see that the counting rule fails dramatically.

12.11

1. Differentiating yields the two equations $a\,\mathrm{d}u + b\,\mathrm{d}v = c\,\mathrm{d}x + d\,\mathrm{d}y$ and $e\,\mathrm{d}u + f\,\mathrm{d}v = g\,\mathrm{d}x + h\,\mathrm{d}y$. Solving these for $\mathrm{d}u$ and $\mathrm{d}v$ yields $\mathrm{d}u = [(cf - bg)\,\mathrm{d}x + (df - bh)\,\mathrm{d}y]/D$ and $\mathrm{d}v = [(ag - ce)\,\mathrm{d}x + (ah - de)\,\mathrm{d}y]/D$, where $D = af - be$. The required partial derivatives are then easily read off.

2. (a) Differentiating yields $u^3\,\mathrm{d}x + x3u^2\,\mathrm{d}u + \mathrm{d}v = 2y\,\mathrm{d}y$ and $3v\,\mathrm{d}u + 3u\,\mathrm{d}v - \mathrm{d}x = 0$. Solving for $\mathrm{d}u$ and $\mathrm{d}v$ with $D = 9xu^3 - 3v$ yields $\mathrm{d}u = (-3u^4 - 1)\,\mathrm{d}x/D + 6yu\,\mathrm{d}y/D$ and $\mathrm{d}v = (3xu^2 + 3u^3v)\,\mathrm{d}x/D - 6yv\,\mathrm{d}y/D$.

(b) $u'_x = (-3u^4 - 1)/D$, $v'_x = (3xu^2 + 3u^3v)/D$ (c) $u'_x = 283/81$ and $v'_x = -64/27$

3. $\partial y_1/\partial x_1 = (3 - 27x_1^2y_2^2)/J$ and $\partial y_2/\partial x_1 = (3x_1^2 + 18y_1^2)/J$ with $J = 1 + 54y_1^2y_2^2$.

4. $\partial Y/\partial M = I'(r)/[(aI'(r) + L'(r)S'(Y))]$ and $\partial r/\partial M = S'(Y)/[(aI'(r) + L'(r)S'(Y))]$.

5. Differentiating w.r.t. x yields $y + u'_x v + uv'_x = 0$ and $u + xu'_x + yv'_x = 0$. Solving for u'_x and v'_x, we get

$$u'_x = \frac{u^2 - y^2}{yv - xu} = \frac{u^2 - y^2}{2yv}, \qquad v'_x = \frac{xy - uv}{yv - xu} = \frac{2xy - 1}{2yv}$$

where we substituted $xu = -yv$ and $uv = 1 - xy$. Differentiating u'_x w.r.t. x finally yields

$$u''_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}u'_x = \frac{2uu'_x 2yv - (u^2 - y^2)2yv'_x}{4y^2v^2} = \frac{(u^2 - y^2)(4uv - 1)}{4y^2v^3}$$

(The answer to this problem can be expressed in many different ways.)

6. (a) Differentiation yields the equations: $\mathrm{d}Y = \mathrm{d}C + \mathrm{d}I + \mathrm{d}G$, $\mathrm{d}C = F'_Y\,\mathrm{d}Y + F'_T\,\mathrm{d}T + F'_r\,\mathrm{d}r$, and $\mathrm{d}I = f'_Y\,\mathrm{d}Y + f'_r\,\mathrm{d}r$. Hence, $\mathrm{d}Y = \big(F'_T\,\mathrm{d}T + \mathrm{d}G + (F'_r + f'_r)\,\mathrm{d}r\big)/(1 - F'_Y - f'_Y)$.

(b) $\partial Y/\partial T = F'_T/(1 - F'_Y - f'_Y) < 0$, so Y decreases as T increases.

But if $\mathrm{d}T = \mathrm{d}G$ with $\mathrm{d}r = 0$, then $\mathrm{d}Y = \big(1 + F'_T\big)\,\mathrm{d}T/(1 - F'_Y - f'_Y)$, which is positive provided that $F'_T > -1$.

7. (a) $6 - 3 = 3$ (b) Differentiating, then gathering all terms in $\mathrm{d}Y$, $\mathrm{d}r$, and $\mathrm{d}I$ on the left-hand side, we obtain (i) $(C'_Y - 1)\,\mathrm{d}Y + C'_r\,\mathrm{d}r + \mathrm{d}I = -\mathrm{d}\alpha$ (ii) $F'_Y\,\mathrm{d}Y + F'_r\,\mathrm{d}r - \mathrm{d}I = -\mathrm{d}\beta$ (iii) $L'_Y\,\mathrm{d}Y + L'_r\,\mathrm{d}r = \mathrm{d}M$. With $\mathrm{d}\beta = \mathrm{d}M = 0$ we get $\mathrm{d}Y = -(L'_r/D)\,\mathrm{d}\alpha$, $\mathrm{d}r = (L'_Y/D)\,\mathrm{d}\alpha$, and $\mathrm{d}I = [(F'_rL'_Y - F'_YL'_r)/D]\,\mathrm{d}\alpha$, where $D = L'_r(C'_Y + F'_Y - 1) - L'_Y(C'_r + F'_r)$.

8. (a) There are three variables and two equations, so there is (in general) one degree of freedom.

(b) Differentiation gives $0 = \alpha P\,\mathrm{d}y + L'(r)\,\mathrm{d}r$ and $S'_y\,\mathrm{d}y + S'_r\,\mathrm{d}r + S'_g\,\mathrm{d}g = I'_y\,\mathrm{d}y + I'_r\,\mathrm{d}r$.

We find $\mathrm{d}y/\mathrm{d}g = -L'(r)S'_g/D$ and $\mathrm{d}r/\mathrm{d}g = \alpha PS'_g/D$, where $D = L'(r)(S'_y - I'_y) - \alpha P(S'_r - I'_r)$.

9. (a) Differentiating yields $2uv\,\mathrm{d}u + u^2\,\mathrm{d}v - \mathrm{d}u = 3x^2\,\mathrm{d}x + 6y^2\,\mathrm{d}y$, and $e^{ux}(u\,\mathrm{d}x + x\,\mathrm{d}u) = v\,\mathrm{d}y + y\,\mathrm{d}v$.

At P these equations become $3\,\mathrm{d}u + 4\,\mathrm{d}v = 6\,\mathrm{d}y$ and $\mathrm{d}v = 2\,\mathrm{d}x - \mathrm{d}y$.

Hence $\mathrm{d}u = 2\,\mathrm{d}y - (4/3)\,\mathrm{d}v = -(8/3)\,\mathrm{d}x + (10/3)\,\mathrm{d}y$. So $\partial u/\partial y = 10/3$ and $\partial v/\partial x = 2$.
(b) $\Delta u \approx \mathrm{d}u = -(8/3)0.1 + (10/3)(-0.2) = -14/15 \approx -0.93$, $\Delta v \approx \mathrm{d}v = 2(0.1) + (-1)(-0.2) = 0.4$

10. Taking differentials and putting $\mathrm{d}p_2 = \mathrm{d}m = 0$ gives: (i) $U''_{11}\,\mathrm{d}x_1 + U''_{12}\,\mathrm{d}x_2 = p_1\,\mathrm{d}\lambda + \lambda\,\mathrm{d}p_1$;
(ii) $U''_{21}\,\mathrm{d}x_1 + U''_{22}\,\mathrm{d}x_2 = p_2\,\mathrm{d}\lambda$; (iii) $p_1\,\mathrm{d}x_1 + \mathrm{d}p_1\,x_1 + p_2\,\mathrm{d}x_2 = 0$.

Solving for $\mathrm{d}x_1$ in particular gives
$\partial x_1/\partial p_1 = [\lambda p_2^2 + x_1(p_2U''_{12} - p_1U''_{22})]/(p_1^2U''_{22} - 2p_1p_2U''_{12} + p_2^2U''_{11})$.

Review exercises for Chapter 12

1. (a) $\mathrm{d}z/\mathrm{d}t = 6\cdot 4t + 3y^2 9t^2 = 24t + 27t^2y^2 = 24t + 243t^8$ (b) $\mathrm{d}z/\mathrm{d}t = px^{p-1}a + py^{p-1}b = pt^{p-1}(a^p + b^p)$
(c) In part (a), $z = 6(2t^2) + (3t^3)^3 = 12t^2 + 27t^9$, so $\mathrm{d}z/\mathrm{d}t = 24t + 243t^8$.

In part (b), $z = (at)^p + (bt)^p = a^p t^p + b^p t^p$, so $\mathrm{d}z/\mathrm{d}t = (a^p + b^p)pt^{p-1}$.

2. $\partial z/\partial t = G_1'(u,v)\phi_1'(t,s)$, and $\partial z/\partial s = G_1'(u,v)\phi_2'(t,s) + G_2'(u,v)\psi'(s)$

3. $\partial w/\partial t = 2x\cdot 1 + 3y^2\cdot 1 + 4z^3 s = 2x + 3y^2 + 4sz^3 = 4s^4t^3 + 3s^2 + 3t^2 - 6ts + 2s + 2t$,

$\partial w/\partial s = 2x - 3y^2 + 4tz^3 = 4s^3t^4 - 3s^2 - 3t^2 + 6ts + 2s + 2t$

4. $\mathrm{d}X/\mathrm{d}N = g(u) + g'(u)(\varphi'(N) - u)$, where $u = \varphi(N)/N$; $\mathrm{d}^2X/\mathrm{d}N^2 = (1/N)g''(u)(\varphi'(N) - u)^2 + g'(u)\varphi''(N)$.

5. (a) Take the natural logarithm, $\ln E = \ln A - a\ln p + b\ln m$, and then differentiate to get $\dot{E}/E = -a(\dot{p}/p) + b(\dot{m}/m)$.

(b) $\ln p = \ln p_0 + t\ln(1.06)$, so $\dot{p}/p = \ln 1.06$. Likewise, $\dot{m}/m = \ln 1.08$.

Then $\dot{E}/E = -a\ln 1.06 + b\ln 1.08 = \ln(1.08^b/1.06^a) = \ln Q$.

6. Differentiating each side w.r.t. x while holding y constant gives $3x^2\ln x + x^2 = (6z^2\ln z + 2z^2)z_1'$. When $x = y = z = e$, this gives $z_1' = 1/2$. Differentiating a second time, $6x\ln x + 5x = (12z\ln z + 10z)(z_1')^2 + (6z^2\ln z + 2z^2)z_{11}''$. When $x = y = z = e$ and $z_1' = 1/2$, this gives $z_{11}'' = 11/16e$.

7. $R_{yx} = F_x'/F_y' = -x/10y$. Hence $y/x = -(1/10)R_{yx}^{-1}$, and so $\sigma_{yx} = \mathrm{El}_{R_{yx}}(y/x) = -1$.

8. (a) MRS = $R_{yx} = U_x'/U_y' = 2y/3x$ (b) MRS = $R_{yx} = y/(x+1)$ (c) MRS = $R_{yx} = (y/x)^3$

9. (a) -1 (b) $2ac$ (c) 4 (d) Not homogeneous. (If F were homogeneous, then by Euler's theorem, for some constant k, we would have $x_1e^{x_1+x_2+x_3} + x_2e^{x_1+x_2+x_3} + x_3e^{x_1+x_2+x_3} = ke^{x_1+x_2+x_3}$ for all positive x_1, x_2, x_3, and so $x_1 + x_2 + x_3 = k$. This is evidently impossible.)

10. Since $y/x = (R_{yx})^{1/3}$, $\sigma_{yx} = \mathrm{El}_{R_{yx}}(y/x) = 1/3$.

11. $\mathrm{El}_x\, y = xy/(1-2y)$. (*Hint:* Take the elasticity w.r.t x of $y^2e^xe^{1/y} = 3$.)

12. (a) 1 (b) k (c) 0

13. Since F is homogeneous of degree 1, according to (12.6.6), we have $KF_{KK}'' + LF_{KL}'' = 0$, so that $F_{KL}'' = -(K/L)F_{KK}'' > 0$ since $F_{KK}'' < 0$ and $K > 0$, $L > 0$.

14. Differentiate $f(tx_1,\dots,tx_n) = g(t)f(x_1,\dots,x_n)$ w.r.t. t and put $t = 1$, as in the proof of Euler's theorem (Theorem 12.7.1). This yields $\sum_{i=1}^n x_i f_i'(x_1,\dots,x_n) = g'(1)f(x_1,\dots,x_n)$. Thus, by Euler's theorem, the function f must be homogeneous of degree $g'(1)$. In fact, $g(t) = t^k$ where $k = g'(1)$.

15. $\mathrm{d}u + e^y\,\mathrm{d}x + xe^y\,\mathrm{d}y + \mathrm{d}v = 0$ and $\mathrm{d}x + e^{u+v^2}\,\mathrm{d}u + e^{u+v^2}2v\,\mathrm{d}v - \mathrm{d}y = 0$. At the given point, these equations reduce to $\mathrm{d}u + \mathrm{d}v = -e\,\mathrm{d}x - e\,\mathrm{d}y$ and $\mathrm{d}u = -e\,\mathrm{d}x + e\,\mathrm{d}y$, implying that $u_x' = -e$, $u_y' = e$, $v_x' = 0$, and $v_y' = -2e$.

16. (a) $\partial p/\partial w = L/F(L)$, $\partial p/\partial B = 1/F(L)$, $\partial L/\partial w = (F(L) - LF'(L))/pF(L)F''(L)$, and $\partial L/\partial B = -F'(L)/pF(L)F''(L)$

(b) See SM.

17. (a) $\alpha u^{\alpha-1}\,\mathrm{d}u + \beta v^{\beta-1}\,\mathrm{d}v = 2^\beta\mathrm{d}x + 3y^2\,\mathrm{d}y$, and $\alpha u^{\alpha-1}v^\beta\,\mathrm{d}u + u^\alpha\beta v^{\beta-1}\,\mathrm{d}v - \beta v^{\beta-1}\,\mathrm{d}v = \mathrm{d}x - \mathrm{d}y$.

At P we find $\partial u/\partial x = 2^{-\beta}/\alpha$, $\partial u/\partial y = -2^{-\beta}/\alpha$, $\partial v/\partial x = (2^\beta - 2^{-\beta})/\beta 2^{\beta-1}$, $\partial v/\partial y = (2^{-\beta} + 3)/\beta 2^{\beta-1}$.

(b) $u(0.99, 1.01) \approx u(1,1) + \partial u(1,1)/\partial x\cdot(-0.01) + \partial u(1,1)/\partial y\cdot 0.01 = 1 - 2^{-\beta}/100\alpha - 2^{-\beta}/100\alpha = 1 - 2^{-\beta}/50\alpha$

18. (a) $S = \int_0^T e^{-rx}(e^{gT-gx} - 1)\,dx = e^{gT}\int_0^T e^{-(r+g)x}\,dx - \int_0^T e^{-rx}\,dx = \dfrac{e^{gT} - e^{-rT}}{r+g} + \dfrac{e^{-rT} - 1}{r}$,

and therefore $r(r+g)S = re^{gT} + ge^{-rT} - (r+g)$. (b) Implicit differentiation w.r.t. g yields

$rS = re^{gT}(T + g\partial T/\partial g) + e^{-rT} + ge^{-rT}(-r\partial T/\partial g) - 1$, so $\partial T/\partial g = [rS + 1 - rTe^{gT} - e^{-rT}]/rg(e^{gT} - e^{-rT})$.

19. (a) Economic interpretation of (∗): How much do we gain by waiting one year? Approximately $V'(t^*)$. How much do we lose? Forgone interest $rV(t^*)$ plus the yearly cost m. (b) and (c) see SM.

Chapter 13

13.1

1. The first-order conditions $f_1'(x,y) = -4x + 4 = 0$ and $f_2'(x,y) = -2y + 4 = 0$ are both satisfied when $x = 1$ and $y = 2$.

2. (a) $f_1'(x,y) = 2x - 6$ and $f_2'(x,y) = 2y + 8$, which are both zero at the only critical point $(x,y) = (3,-4)$.

(b) $f(x,y) = x^2 - 6x + 3^2 + y^2 + 8y + 4^2 + 35 - 3^2 - 4^2 = (x-3)^2 + (y+4)^2 + 10 \geq 10$ for all (x,y), whereas $f(3,-4) = 10$, so $(3,-4)$ minimizes f.

3. $F_K' = -2(K-3) - (L-6)$ and $F_L' = -4(L-6) - (K-3)$,

so the first-order conditions yield $-2(K-3) - (L-6) = 0.65$, $-4(L-6) - (K-3) = 1.2$.

The only solution of these two simultaneous equations is $(K,L) = (2.8, 5.75)$.

4. (a) $P(10,8) = P(12,10) = 98$ (b) First-order conditions: $P_x' = -2x + 22 = 0$, $P_y' = -2y + 18 = 0$.

It follows that $x = 11$ and $y = 9$, where profits are $P(11,9) = 100$.

13.2

1. We check that the conditions in part (a) of Theorem 13.2.1 are satisfied in all three cases:

(a) $\partial^2\pi/\partial x^2 = -0.08 \leq 0$, $\partial^2\pi/\partial y^2 = -0.02 \leq 0$, and $(\partial^2\pi/\partial x^2)(\partial^2\pi/\partial y^2) - (\partial^2\pi/\partial x\partial y)^2 = 0.0015 \geq 0$.

(b) $f_{11}'' = -4$, $f_{12}'' = 0$, and $f_{22}'' = -2$ for all (x,y).

(c) With $\pi = F(K,L) - 0.65K - 1.2L$, one has $\pi_{KK}'' = -2$, $\pi_{KL}'' = -1$, and $\pi_{LL}'' = -4$.

2. (a) Profit: $\pi(x,y) = 24x + 12y - C(x,y) = -2x^2 - 4y^2 + 4xy + 64x + 32y - 514$. Maximum at $x = 40$, $y = 24$, with $\pi(40,24) = 1150$. Since $\pi_{11}'' = -4 \leq 0$, $\pi_{22}'' = -8 \leq 0$, and $\pi_{11}''\pi_{22}'' - (\pi_{12}'')^2 = 16 \geq 0$, this is the maximum.

(b) $x = 34$, $y = 20$. (With $y = 54 - x$, profits are $\hat{\pi} = -2x^2 - 4(54-x)^2 + 4x(54-x) + 64x + 32(54-x) - 514$, so $\hat{\pi} = -10x^2 + 680x - 10\,450$, which has a maximum at $x = 34$. Then $y = 54 - 34 = 20$. The maximum value is 1110.)

3. Maximum 3888 at $x = 36$, $y = 12$, $z = 9$.

4. (a) $\pi(x,y) = px + qy - C(x,y) = (25-x)x + (24-2y)y - (3x^2 + 3xy + y^2) = -4x^2 - 3xy - 3y^2 + 25x + 24y$.

(b) $\pi_1' = -8x - 3y + 25 = 0$ and $\pi_2' = -3x - 6y + 24 = 0$ when $(x,y) = (2,3)$. Moreover, then $\pi_{11}'' = -8 \leq 0$, $\pi_{22}'' = -6 \leq 0$, and $\pi_{11}''\pi_{22}'' - (\pi_{12}'')^2 = (-8)(-6) - (-3)^2 = 39 \geq 0$. So $(x,y) = (2,3)$ maximizes profits.

5. The profit is $\pi(x,y) = px + qy - x^2 - xy - y^2 - x - y - 14$. Profit has a critical point (x^*, y^*) where $x^* = \frac{1}{3}(2p - q - 1)$ and $y^* = \frac{1}{3}(-p + 2q - 1)$. Provided that $q < 2p - 1$ and $q > \frac{1}{2}(p+1)$, the sufficient conditions in Theorem 13.2.1 for an interior point (x^*, y^*) to maximize profits are easily seen to be satisfied.

6. (a) $x^* = p/2\alpha$, $y^* = q/2\beta$, and the second-order conditions are satisfied.

(b) $\pi^*(p,q) = px^* + qy^* - \alpha(x^*)^2 - \beta(y^*)^2 = p^2/4\alpha + q^2/2\beta$. Hence $\partial\pi^*(p,q)/\partial p = p/2\alpha = x^*$. So increasing the price p by one unit increases the optimal profit by approximately x^*, the output of the first good. The equation $\partial\pi^*(p,q)/\partial q = y^*$ has a similar interpretation.

7. The constraint implies that $z = 4x + 2y - 5$. Using this to substitute for z, we choose (x,y) to minimize $P(x,y) = x^2 + y^2 + (4x + 2y - 5)^2$ w.r.t. x and y. The first-order conditions are: $P'_1 = 34x + 16y - 40 = 0$, $P'_2 = 16x + 10y - 20 = 0$, with solution $x = 20/21$, $y = 10/21$. Since $P''_{11} = 34$, $P''_{12} = 16$, and $P''_{22} = 10$, the second-order conditions for minimum are satisfied. The minimum value is $525/441$.

8. To show that f is concave, we verify the inequalities in part (a) of Theorem 13.2.1. The second-order partial derivatives are $f''_{11} = a(a-1)Ax^{a-2}y^b$, $f''_{12} = f''_{21} = abAx^{a-1}y^{b-1}$, and $f''_{22} = b(b-1)Ax^a y^{b-2}$. Thus, $f''_{11}f''_{22} - (f''_{12})^2 = abA^2x^{2a-2}y^{2b-2}\big[1 - (a+b)\big]$. Suppose that $a + b \le 1$. Then $a \le 1$ and $b \le 1$ as well. If $x > 0$ and $y > 0$, then $f''_{11} \le 0$ and $f''_{22} \le 0$, and $f''_{11}f''_{22} - (f''_{12})^2 \ge 0$, as required.

13.3

1. (a) $f'_1 = -2x + 6$, $f'_2 = -4y + 8$, $f''_{11} = -2$, $f''_{12} = 0$, and $f''_{22} = -4$

(b) $(3,2)$ is a local maximum point, because $A = -2 < 0$ and $AC - B^2 = 8 > 0$.

Theorem 13.2.1 implies that $(3,2)$ is a (global) maximum point.

2. (a) $f'_1 = 2x + 2y^2$, $f'_2 = 4xy + 4y$, $f''_{11} = 2$, $f''_{12} = 4y$, and $f''_{22} = 4x + 4$

(b) $f'_2 = 0 \iff 4y(x+1) = 0 \iff x = -1$ or $y = 0$. If $x = -1$, then $f'_1 = 0$ for $y = \pm 1$.

If $y = 0$, then $f'_1 = 0$ for $x = 0$. Thus we get the three critical points classified in the table:

(x,y)	A	B	C	$AC - B^2$	Type of critical point:
$(0,0)$	2	0	4	8	Local minimum point
$(-1,1)$	2	4	0	-16	Saddle point
$(-1,-1)$	2	-4	0	-16	Saddle point

3. (a) $(0,0)$ is a saddle point and $(-a,-2)$ is a local minimum point. (b) $df^*(a)/da = -2ae^{-2}$

4. (a) $f'_t(t^*,x^*) = rf(t^*,x^*)$ and $f'_x(t^*,x^*) = e^{rt^*}$ (b) $g'(t^*) = rg(t^*)$ and $h'(x^*) = e^{rt^*}/g(t^*)$

(c) Verify the conditions in part (a) of Theorem 13.3.1. See SM.

(d) $t^* = 1/4r^2$, $x^* = e^{1/4r} - 1$

5. In all three cases, $(0,0)$ is a critical point where $z = 0$ and $A = B = C = 0$, so $AC - B^2 = 0$.

In case (a), $z \le 0$ for all (x,y), so the origin is a maximum point.

In case (b), $z \ge 0$ for all (x,y), so the origin is a minimum point.

In case (c), z takes both positive and negative values at points arbitrarily close to $(0,0)$, so it is a saddle point.

6. (a) f is defined for all (x,y) satisfying $1 + x^2y > 0$, or equivalently $x^2y > -1$. So it is defined for all (x,y) satisfying either (i) $x = 0$, or (ii) $x \ne 0$ and $y > -1/x^2$.

(b) $f'_1(x,y) = 2xy/(1 + x^2y)$ and $f'_2(x,y) = x^2/(1 + x^2y)$. Here $f'_1 = f'_2 = 0$ at $(0,b)$ for all $b \in \mathbb{R}$.

(c) Because $AC - B^2 = 0$ when $(x,y) = (0,b)$, the second-derivative test fails.

(d) Note that $f(0,b) = 0$ at any critical point $(0,b)$. By considering the sign of $f(x,y) = \ln(1 + x^2y)$ in the neighbourhood of any critical point, one sees that f has: a local maximum point if $b < 0$; a saddle point if $b = 0$; and a local minimum point if $b > 0$. See Fig. A13.3.6.

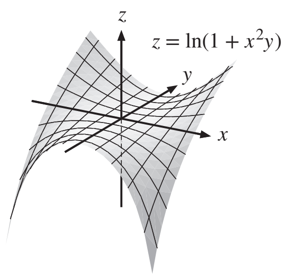


Figure A13.3.6

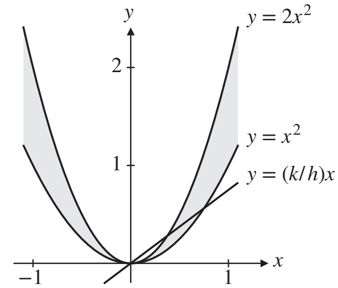


Figure A13.3.7

7. (a) See Fig. A13.3.7. The domain where $f(x, y)$ is negative is shaded. The origin is easily seen to be the only critical point, and $f(0, 0) = 0$. As the figure shows, $f(x, y)$ takes positive and negative values for points arbitrarily close to $(0, 0)$, so the origin is a saddle point.

(b) $g(t) = f(th, tk) = (tk - t^2h^2)(tk - 2t^2h^2) = 2h^4t^4 - 3h^2kt^3 + k^2t^2$, implying that $g'(t) = 8h^4t^3 - 9h^2kt^2 + 2k^2t$ and $g''(t) = 24h^4t^2 - 18h^2kt + 2k^2$. So $g'(0) = 0$ and $g''(0) = 2k^2$. It follows that $t = 0$ is a minimum point for $k \neq 0$. For $k = 0$, one has $g(t) = 2t^4h^4$, which has a minimum at $t = 0$.

13.4

1. (a) $\pi = P_1Q_1 + P_2Q_2 - C(Q_1, Q_2) = -2Q_1^2 - 4Q_2^2 + 180Q_1 + 160Q_2$, which has a maximum at $Q_1^* = 45$, $Q_2^* = 20$, with $P_1^* = 110$, $P_2^* = 100$, and $\pi^* = 5650$.

(b) Let $P = P_1 = P_2$. Then $Q_1 = 100 - \frac{1}{2}P$, $Q_2 = 45 - \frac{1}{4}P$, so profit as a function of P is $\widehat{\pi} = (P - 20)(Q_1 + Q_2) = (P - 20)(145 - \frac{3}{4}P) = -\frac{3}{4}P^2 + 160P - 2900$, which is maximized when $P = 320/3$. The corresponding profit is $16\,900/3$. The loss of profit is $5650 - 16\,900/3 = 50/3$.

(c) The new profit is $\widetilde{\pi} = -2Q_1^2 - 4Q_2^2 + 175Q_1 + 160Q_2$, with a maximum at $Q_1 = 43.75$, $Q_2 = 20$, with prices $P_1 = 112.50$ and $P_2 = 100$. The maximized profit is 5428.125. The number of units sold in market 1 goes down, the price goes up, and profits are lower. In market 2 the number of units sold and the price are unchanged.

2. (a) $\pi = -bp^2 - dq^2 + (a + \beta b)p + (c + \beta d)q - \alpha - \beta(a + c)$, $\quad p^* = (a + \beta b)/2b$, $q^* = (c + \beta d)/2d$.

The second-order conditions are obviously satisfied because $\pi''_{11} = -2b$, $\pi''_{12} = 0$, and $\pi''_{22} = -2d$.

(b) $\hat{p} = (a + c + \beta(b + d))/2(b + d)$. (c) See SM.

3. Imposing a tax of t per unit sold in market area 1 means that the new profit function is $\widehat{\pi}(Q_1, Q_2) = \pi(Q_1, Q_2) - tQ_1$. The optimal choice of production in market area 1 is then $\widehat{Q}_1 = (a_1 - \alpha - t)/2b_1$ (see the text), and the tax revenue is $T(t) = t(a_1 - \alpha - t)/2b_1 = [t(a_1 - \alpha) - t^2]/2b_1$. This quadratic function has a maximum when $T'(t) = 0$, so $t = \frac{1}{2}(a_1 - \alpha)$.

4. (a) $\hat{a} = 0.105$, and $\hat{b} = 11.29$ (b) $\hat{c} = 0.23$, and $\hat{d} = 5.575$ (c) The goal would have been reached in 1979.

5. (a) $p = 9$, $q = 8$, $x = 16$, $y = 4$. Firm A's profit is 123, whereas B's is 21.

(b) Firm A's profit is maximized at $p = p_A(q) = \frac{1}{5}(2q + 17)$. Firm B's profit is maximized at $q = q_B(p) = \frac{1}{3}(p + 7)$.

(c) Equilibrium occurs where $p = 5$, $q = 4$, $x = 20$, $y = 12$. Firm A gets 75, B gets 21. (d) See SM.

13.5

1. (a) $f_1'(x, y) = 4 - 4x$, and $f_2'(x, y) = -4y$. The only critical point is $(1, 0)$, with $f(1, 0) = 2$.

(b) $f(x, y)$ has a maximum value of 2 at $(1, 0)$ and a minimum value of -70 at $(-5, 0)$. (A maximum and a minimum exist, by the extreme value theorem. Along the boundary, the function value is $4x - 50$, with $x \in [-5, 5]$. So its maximum along the boundary is -30 at $x = 5$ and its minimum is -70 at $x = -5$.)

2. (a) Maximum 91 at $(0, 4)$ and at $(4, 0)$. Minimum 0 at $(3, 3)$.

(b) Maximum 9/4 at $(-1/2, \sqrt{3}/2)$ and at $(-1/2, -\sqrt{3}/2)$. Minimum $-1/4$ at $(1/2, 0)$.

3. See Fig. A13.5.3. No critical points in the interior. The maximum value of f is 27/8 at $(3/4, 0)$.

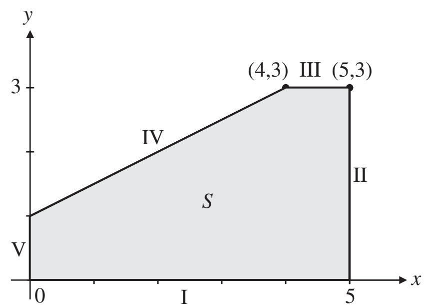


Figure A13.5.3

4. (a) The first-order conditions $2axy + by + 2y^2 = 0$ and $ax^2 + bx + 4xy = 0$ must have $(x, y) = (2/3, 1/3)$ as a solution. So $a = 1$ and $b = -2$. Also $c = 1/27$, so that $f(2/3, 1/3) = -1/9$. Then, because $A = f''_{11}(2/3, 1/3) = 2/3$, $B = f''_{12}(2/3, 1/3) = 2/3$, and $C = f''_{22}(2/3, 1/3) = 8/3$, Theorem 13.3.1 shows that this is a local minimum.

(b) Maximum 193/27 at $(2/3, 8/3)$. Minimum $-1/9$ at $(2/3, 1/3)$.

5. (a) $(1, 2)$ is a local minimum; $(0, 0)$ and $(0, 4)$ are saddle points.

(b) Note that $f(x, 1) = -3xe^{-x} \to \infty$ as $x \to -\infty$, and $f(-1, y) = -e(y^2 - 4y) \to -\infty$ as $y \to \infty$.

(c) f has a minimum value of $-4/e$ at $(1, 2)$, and a maximum value of 0 at all $(x, 0)$ and $(x, 4)$ satisfying $x \in [0, 5]$, as well as at all $(0, y)$ satisfying $y \in [0, 4]$. (d) $y' = 0$ when $x = 1$ and $y = 4 - e$.

6. (a) Closed and bounded, so compact. (b) Open and unbounded. (c) Closed and bounded, so compact.

(d) Closed and unbounded. (e) Closed and unbounded. (f) Open and unbounded.

7. Let $g(x) = 1$ in $[0, 1)$, $g(x) = 2$ in $[1, 2]$. Then g is discontinuous at $x = 1$, and the set $\{x : g(x) \leq 1\} = [0, 1)$ is not closed. (Draw your own graph of g.)

13.6

1. (a) The first-order conditions $f'_x(x, y, z) = 2 - 2x = 0$, $f'_y(x, y, z) = 10 - 2y = 0$, and $f'_z(x, y, z) = -2z = 0$ have a unique solution $(x, y, z) = (1, 5, 0)$, which must then be the maximum point.

(b) The first-order conditions are $f'_x(x, y, z) = -2x - 2y - 2z = 0$, $f'_y(x, y, z) = -4y - 2x = 0$, $f'_z(x, y, z) = -6z - 2x = 0$. From the last two equations we get $y = -\frac{1}{2}x$ and $z = -\frac{1}{3}x$. Inserting this into the first equation we get $-2x + x + \frac{2}{3}x = 0$, and thus $x = 0$, implying that $y = z = 0$. So $(x, y, z) = (0, 0, 0)$ is the maximum point.

2. (a) $f(x) = e^{-x^2}$ and $g(x) = F(f(x)) = \ln(e^{-x^2}) = -x^2$ both have a unique maximum at $x = 0$.

(b) Only $x = 0$ maximizes $f(x)$. But $g(x) = 5$ is maximized at every point x because it is a constant.

3. By the chain rule, $g'_i(\mathbf{x}) = F'(f(\mathbf{x}))f'_i(\mathbf{x})$ for $i = 1, 2, \ldots, n$. Because $F' \neq 0$ everywhere, the assertion follows.

4. $f'_x = -6x^2 + 30x - 36$, $f'_y = 2 - e^{y^2}$, and $f'_z = -3 + e^{z^2}$. There are eight critical points given by $(x, y, z) = (3, \pm\sqrt{\ln 2}, \pm\sqrt{\ln 3})$, and $(x, y, z) = (2, \pm\sqrt{\ln 2}, \pm\sqrt{\ln 3})$, where all possible sign combinations are allowed.

5. (a) Because $F(u) = \frac{1}{2}(e^u - e^{-u})$ is strictly increasing, the problem is equivalent to: $\max x^2 + y^2 - 2x$ subject to $(x, y) \in S$.

(b) The problem is equivalent to: $\max \ln A + a_1 \ln x_1 + \cdots + a_n \ln x_n$ subject to $x_1 + \cdots + x_n = 1$.

13.7

1. (a) The profit is $\pi = px - ax - bx^2 - tx$, which has a maximum at $x^* = (p - a - t)/2b$, with $\pi^* = (p - a - t)^2/4b$.

(b) $\partial\pi^*/\partial p = 2(p - a - t)/4b = x^*$. If we increase p by 1 dollar, then the optimal profit increases by x^* dollars.

(For each of the x^* units sold the revenue increases by 1 dollar.)

2. (a) The profit function is $\pi = \pi(L, P, w) = P\sqrt{L} - wL$. The value of L that maximizes profit must satisfy $\pi'_L(L, P, w) = P/2\sqrt{L} - w = 0$, which yields $L = (P/2w)^2$. Now $\pi''_{LL} = -P/4L^{3/2} < 0$ for all L. Hence profit is maximized at $L = L^*(P, w) = (P/2w)^2$.

(b) The value function is $\pi^*(P, w) = \pi(L^*, P, w) = P\sqrt{L^*} - wL^* = P(P/2w) - w(P/2w)^2 = P^2/4w$. It follows that $\partial\pi^*/\partial P = P/2w = \sqrt{L^*} = \pi'_P(L^*, P, w, a)$, and also that $\partial\pi^*/\partial w = -P^2/4w^2 = -L^* = \pi'_w(L^*, P, w)$. Thus, the envelope theorem is confirmed for this example.

3. (a) $\pi = p(K^{2/3} + L^{1/2} + T^{1/3}) - rK - wL - q$, and $K^* = \frac{8}{27}p^3r^{-3}$, $L^* = \frac{1}{4}p^2w^{-2}$, $T^* = \frac{1}{3\sqrt{3}}p^{3/2}q^{-3/2}$

(b) $Q^* = \frac{4}{9}p^2r^{-2} + \frac{1}{2}pw^{-1} + \frac{1}{\sqrt{3}}p^{1/2}q^{-1/2}$, so $\partial Q^*/\partial r = -\frac{8}{9}p^2r^{-3} = -\partial K^*/\partial p$

4. $\partial Q^*/\partial r = (\partial/\partial r)\,(\partial\hat{\pi}^*/\partial p) = (\partial/\partial p)\,(\partial\hat{\pi}^*/\partial r) = (\partial/\partial p)(-K^*) = -\partial K^*/\partial p$.

The other equalities are proved in a similar way.

5. (a) Routine application of differential formula (12.9.5).

(b) Suppressing notation indicating that the partials are all evaluated at (K^*, L^*), we get

$$\frac{\partial K^*}{\partial p} = \frac{-F'_K F''_{LL} + F'_L F''_{KL}}{p(F''_{KK}F''_{LL} - (F''_{KL})^2)}, \quad \frac{\partial L^*}{\partial p} = \frac{-F'_L F''_{KK} + F'_K F''_{LK}}{p(F''_{KK}F''_{LL} - (F''_{KL})^2)}, \quad \frac{\partial K^*}{\partial r} = \frac{F''_{LL}}{p(F''_{KK}F''_{LL} - (F''_{KL})^2)},$$

$$\frac{\partial L^*}{\partial r} = \frac{-F''_{LK}}{p(F''_{KK}F''_{LL} - (F''_{KL})^2)}, \quad \frac{\partial K^*}{\partial w} = \frac{-F''_{KL}}{p(F''_{KK}F''_{LL} - (F''_{KL})^2)}, \quad \frac{\partial L^*}{\partial w} = \frac{F''_{KK}}{p(F''_{KK}F''_{LL} - (F''_{KL})^2)}.$$

(c) We see that $\partial K^*/\partial r$ and $\partial L^*/\partial w$ are both negative. Since we have no information about the sign of F''_{KL}, the signs of the other partials are not determined by the sufficient conditions for profit maximization. We observe that $\partial K^*/\partial w = \partial L^*/\partial r$, since $F''_{KL} = F''_{LK}$.

6. (a) First-order conditions are: (i) $R'_1 - C'_1 + s = 0$, (ii) $R'_2 - C'_2 - t = 0$.

(b) $\pi''_{11} = R''_{11} - C''_{11} < 0$, and $D = \pi''_{11}\pi''_{22} - (\pi''_{12})^2 = (R''_{11} - C''_{11})(R''_{22} - C''_{22}) - (R''_{12} - C''_{12})^2 > 0$.

For (c) and (d) see SM.

Review exercises for Chapter 13

1. The first-order conditions $f'_1(x, y) = -4x + 2y + 18 = 0$ and $f'_2(x, y) = 2x - 2y - 14 = 0$ are both satisfied at $(x, y) = (2, -5)$. Moreover, $f''_{11} = -4$, $f''_{12} = 2$, and $f''_{22} = -2$, so $f''_{11}f''_{22} - (f''_{12})^2 = 4$. The conditions in (a) of Theorem 13.2.1 are satisfied.

2. (a) $(Q_1, Q_2) = (500, 200)$ (b) $P_1 = 105$

3. (a) Critical points are where $P'_1(x, y) = -0.2x - 0.2y + 47 = 0$ and $P'_2(x, y) = -0.2x - 0.4y + 48 = 0$. It follows that $x = 230$ and $y = 5$. Moreover, $P''_{11} = -0.2 \leq 0$, $P''_{12} = -0.2$, and $P''_{22} = -0.4 \leq 0$. Since also $P''_{11}P''_{22} - (P''_{12})^2 = 0.04 \geq 0$, the pair $(230, 5)$ maximizes profit.

(b) With total production $x + y = 200$, and so $y = 200 - x$, the new profit function is $\hat{\pi}(x) = f(x, 200 - x) = -0.1x^2 + 39x + 1000$. This function is easily seen to have maximum at $x = 195$. Then $y = 200 - 195 = 5$.

4. (a) The critical points are at $(0, 0)$ and $(3, 9/2)$. (b) $(0, 0)$, $(\frac{1}{2}\sqrt{2}, \sqrt{2})$, $(-\frac{1}{2}\sqrt{2}, -\sqrt{2})$

(c) $(0, 0)$, $(0, 4)$, $(2, 2)$, and $(-2, 2)$

5. Critical points are where $f'_x(x, y, a) = 2ax - 2 = 0$ and $f'_y(x, y, a) = 2y - 4a = 0$, or $x = x^*(a) = 1/a$ and $y = y^*(a) = 2a$. The value function is $f^*(a) = a(1/a)^2 - 2(1/a) + (2a)^2 - 4a(2a) = -(1/a) - 4a^2$. Thus $(d/da)f^*(a) = (1/a^2) - 8a$. On the other hand $(\partial/\partial a)f(x, y, a) = x^2 - 4y = (1/a^2) - 8a$ at $(x^*(a), y^*(a))$. This verifies the envelope theorem.

6. (a) $K^* = (ap/r)^{1/(1-a)}$, $L^* = (bp/w)^{1/(1-b)}$, and $T^* = (cp/q)^{1/(1-c)}$. For (b) and (c) see SM.

7. (a) $f'_1 = e^{x+y} + e^{x-y} - \frac{3}{2}$, $f'_2 = e^{x+y} - e^{x-y} - \frac{1}{2}$, $f''_{11} = e^{x+y} + e^{x-y}$, $f''_{12} = e^{x+y} - e^{x-y}$, and $f''_{22} = e^{x+y} + e^{x-y}$.

It follows that $f''_{11} \geq 0$, $f''_{22} \geq 0$, and $f''_{11}f''_{22} - (f''_{12})^2 = (e^{x+y} + e^{x-y})^2 - (e^{x+y} - e^{x-y})^2 = 4e^{x+y}e^{x-y} = 4e^{2x} \geq 0$, so f is convex.

(b) At a critical point, $e^{x+y} = 1$ and $e^{x-y} = \frac{1}{2}$, so $x + y = 0$ and $x - y = -\ln 2$. The only critical point is therefore $(x, y) = (-\frac{1}{2}\ln 2, \frac{1}{2}\ln 2)$, where $f(x, y) = \frac{1}{2}(3 + \ln 2)$. Because f is convex, this is the minimum.

8. (a) $(0, 0)$ is a saddle point, whereas $(5/6, -5/12)$ is a local maximum point.

(b) $f''_{11} = 2 - 6x \leq 0 \iff x \geq 1/3$, while $f''_{22} = -2 \leq 0$, and $f''_{11}f''_{22} - (f''_{12})^2 = 12x - 5 \geq 0 \iff x \geq 5/12$. We conclude that f is concave if and only if $x \geq 5/12$. The largest value of f in S is $125/432$, which is reached at $(5/6, -5/12)$.

9. (a) $f'_1(x, y) = x - 1 + ay$ and $f'_2(x, y) = a(x - 1) - y^2 + 2a^2y$, which are both 0 at $(x, y) = (1 - a^3, a^2)$.

For (b) and (c), see SM.

10. (a) $p = C'_x(x^*, y^*)$ and $q = C'_y(x^*, y^*)$ are the familiar conditions that at the optimum the price of each good should equal marginal cost.

(b) With simplified notation, at the optimum one has $\hat{\pi}'_x = F + xF'_x + yG'_x - C'_x = 0$ and $\hat{\pi}'_y = xF'_y + G + yG'_y - C'_y = 0$. The interpretation is that marginal revenue = marginal cost, as usual, with the twist that a change in output of either good affects marginal revenue in the other market as well.

(c) The profit function is $\pi = x(a - bx - cy) + y(\alpha - \beta x - \gamma y) - Px - Qy - R$, so the first-order conditions are $\partial\pi/\partial x = a - 2bx - cy - \beta y - P = 0$, and $\partial\pi/\partial y = -cx + \alpha - \beta x - 2\gamma y - Q = 0$.

(d) $\partial^2\pi/\partial x^2 = -2b$, $\partial^2\pi/\partial y^2 = -2\gamma$, $\partial^2\pi/\partial x\partial y = -(\beta + c)$. The direct partials of order 2 are negative and the cross partials satisfy $\Delta = (\partial^2\pi/\partial x^2)(\partial^2\pi/\partial y^2) - (\partial^2\pi/\partial x\partial y)^2 = 4\gamma b - (\beta + c)^2$, so the conclusion follows.

Chapter 14

14.1

1. (a) $\mathcal{L}(x, y) = xy - \lambda(x + 3y - 24)$. The first-order conditions $\mathcal{L}'_1 = y - \lambda = 0$, $\mathcal{L}'_2 = x - 3\lambda = 0$ imply that $x = 3y$. Inserted into the constraint, this yields $3y + 3y = 24$, so $y = 4$, and then $x = 12$.

(b) Using (∗∗) in Example 14.1.3 with $a = b = p = 1$, $q = 3$, and $m = 24$, we have $x = \frac{1}{2}(24/1) = 12$ and $y = \frac{1}{2}(24/3) = 4$.

2. With $\mathcal{L} = -40Q_1 + Q_1^2 - 2Q_1Q_2 - 20Q_2 + Q_2^2 - \lambda(Q_1 + Q_2 - 15)$,

the first-order conditions are $\mathcal{L}'_1 = -40 + 2Q_1 - 2Q_2 - \lambda = 0$ and $\mathcal{L}'_2 = -2Q_1 - 20 + 2Q_2 - \lambda = 0$.

It follows that $-40 + 2Q_1 - 2Q_2 = -2Q_1 - 20 + 2Q_2$, and so $Q_1 - Q_2 = 5$.

This equation and the constraint together give the solution $Q_1 = 10$, $Q_2 = 5$, with $\lambda = -30$.

3. (a) According to (∗∗) in Example 14.1.3, $x = \dfrac{3}{10}m$ and $y = \dfrac{1}{10}m$. (b) $x = 10$ and $y = 6\,250\,000$

(c) $x = 8/3$, $y = 1$

4. (a) $(x, y) = (4/5, 8/5)$ with $\lambda = 8/5$. (b) $(x, y) = (8, 4)$ with $\lambda = 16$. (c) $(x, y) = (50, 50)$ with $\lambda = 250$.

5. The budget constraint is $2x + 4y = 1000$, so with $\mathcal{L}(x, y) = 100xy + x + 2y - \lambda(2x + 4y - 1000)$, the first-order conditions are $\mathcal{L}'_1 = 100y + 1 - 2\lambda = 0$ and $\mathcal{L}'_2 = 100x + 2 - 4\lambda = 0$. Eliminating λ from these equations gives $x = 2y$. Inserting this into the constraint gives $2x + 2x = 1000$. So the solution is $x = 250$ and $y = 125$.

6. $m = awT_0/(a + b)$, $l = bT_0/(a + b)$

7. The problem is: max $-0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$ subject to $x + y = 200$.

With $\mathcal{L}(x, y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600 - \lambda(x + y - 200)$,

the first-order conditions are $\mathcal{L}'_1 = -0.2x - 0.2y + 47 - \lambda = 0$ and $\mathcal{L}'_2 = -0.2x - 0.4y + 48 - \lambda = 0$.

Eliminating x and λ yields $y = 5$, and then the budget constraint gives $x = 195$, with $\lambda = 7$.

8. (a) $P(x, y) = (96 - 4x)x + (84 - 2y)y - 2x^2 - 2xy - y^2 = -6x^2 - 3y^2 - 2xy + 96x + 84y$

(b) $P'_x(x, y) = -12x - 2y + 96$, $P'_y(x, y) = -6y - 2x + 84$. The only critical point is $(x, y) = (6, 12)$.

(c) With $\mathcal{L}(x, y) = -6x^2 - 3y^2 - 2xy + 96x + 84y - \lambda(x + y - 11)$, one has $\mathcal{L}'_1 = -12x - 2y + 96 - \lambda = 0$ and $\mathcal{L}'_2 = -6y - 2x + 84 - \lambda = 0$. Eliminating λ yields $10x - 4y = 12$. The constraint is $x + y = 11$. Solving these two equations simultaneously gives $x = 4$, $y = 7$. Since $P(4, 7) = 673 < P(6, 12) = 792$, the production restriction reduces profit by 119.

9. (a) $x^*(p, m) = a^\gamma p^{-\gamma}$ where $\gamma = 1/(1 - a)$, and $y^*(p, m) = m - a^\gamma p^{1-\gamma}$. (b)–(d) see SM.

10. (a) $x(p, q, m) = [m + q\ln(q/p)]/(p + q)$, $y(p, q, m) = [m + p\ln(p/q)]/(p + q)$ (b) Direct verification.

14.2

1. According to (∗∗) in Example 14.1.3, the solution is $x^* = 3m/8$, $y^* = m/12$, with $\lambda = 9m^3/512$.

The value function is $f^*(m) = (x^*)^3y^* = 9m^4/2048$, so we see that $\mathrm{d}f^*(m)/\mathrm{d}m = 9m^3/512 = \lambda$.

2. (a) With $\mathcal{L} = rK + wL - \lambda(\sqrt{K} + L - Q)$, the first-order conditions are $\mathcal{L}'_K = r - \lambda/2\sqrt{K^*} = 0$ and $\mathcal{L}'_L = w - \lambda = 0$. Inserting λ from the last equation into the first yields $\sqrt{K^*} = w/2r$. Then $K^* = w^2/4r^2$ and from the constraint $L^* = Q - w/2r$. (b) The value function is $C^*(Q) = rK^* + wL^* = wQ - w^2/4r$, and so $\mathrm{d}C^*(Q)/\mathrm{d}Q = w = \lambda$.

3. (a) $x + 2y = a$ yields $y = \frac{1}{2}a - \frac{1}{2}x$, and then $x^2 + y^2 = x^2 + (\frac{1}{2}a - \frac{1}{2}x)^2 = \frac{5}{4}x^2 - \frac{1}{2}ax + \frac{1}{4}a^2$. This quadratic function has a minimum at $x = a/5$, and then $y = 2a/5$.

(b) $\mathcal{L}(x, y) = x^2 + y^2 - \lambda(x + 2y - a)$. The necessary conditions are $\mathcal{L}_1' = 2x - \lambda = 0$, $\mathcal{L}_2' = 2y - 2\lambda = 0$, implying that $2x = y$. From the constraint one has $x = a/5$, and then $y = 2a/5$, with $\lambda = 2a/5$.

(c) See Fig. A14.2.3. Find the point on the straight line $x + 2y = a$ that is nearest to the origin.

The corresponding maximization problem has no solution.

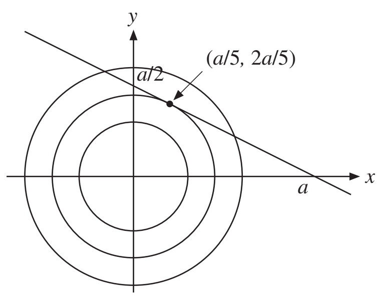


Figure A14.2.3

4. (a) $x^* = 4$, $y^* = 24$, $\lambda = 1/4$. (b) $\hat{y} = 97/4$, $\hat{x} = 4$. $\Delta U = 105/4 - 104/4 = 1/4$, the value of the Lagrange multiplier from (a). (There is exact equality here because U is linear in one of the variables.)

(c) $x^* = q^2/4p^2$, $y^* = m/q - q/4p$. (Note that $y^* > 0$ if and only if $m > q^2/4p$.)

5. (a) First-order conditions: (i) $\alpha/(x^* - a) = \lambda p$; (ii) $\beta/(y^* - b) = \lambda q$. Hence $px^* = pa + \alpha/\lambda$ and $qy^* = qb + \beta/\lambda$. Use the budget constraint to eliminate λ. The expressions for px^* and qy^* follow.

(b) $U^* = \alpha[\ln \alpha + \ln(m - (ap + bq)) - \ln p] + \beta[\ln \beta + \ln(m - (ap + bq)) - \ln q]$. The results follow.

6. $f(x, T) = -\frac{1}{6}\alpha x T^5 + \frac{1}{12} x T^4 + \frac{1}{6} x T^3$, $g(x, T) = \frac{1}{6} x T^3$. The solution of $(*)$ is $x = 384\alpha^3 M$, $T = 1/4\alpha$, and then $f^*(M) = M + M/16\alpha$, with $\lambda = 1 + 1/16\alpha$. Clearly, $\partial f^*(M)/\partial M = \lambda$, which confirms (14.2.2).

14.3

1. (a) $(2, 2)$ and $(-2, -2)$ are the only possible solutions of the maximization problem, and $(-2, 2)$ and $(2, -2)$ are the only possible solutions of the minimization problem.

(b) $(3, -1)$ solves the maximization problem and $(-3, 1)$ solves the minimization problem.

2. (a) Maximum at $(x, y, \lambda) = (-4, 0, 5/4)$, minimum at $(x, y, \lambda) = (4/3, \pm 4\sqrt{2}/3, 1/4)$.

(b) Minimum points: $(\sqrt[4]{2}, 1 - \frac{1}{2}\sqrt{2})$ and $(-\sqrt[4]{2}, 1 - \frac{1}{2}\sqrt{2})$

3. (a) $\mathcal{L} = x + y - \lambda(x^2 + y - 1)$. The equations $\mathcal{L}_1' = 1 - 2\lambda x = 0$, $\mathcal{L}_2' = 1 - \lambda = 0$, and $x^2 + y = 1$ have the solution $x = 1/2$, $y = 3/4$, and $\lambda = 1$.

(b) See Fig. A14.3.3. The minimization problem has no solution because $f(x, 1 - x^2) = x + 1 - x^2 \to -\infty$ as $x \to \infty$.

(c) New solution: $x = 0.5$ and $y = 0.85$. The change in the value function is $f^*(1.1) - f^*(1) = (0.5 + 0.85) - (0.5 + 0.75) = 0.1$. Because $\lambda = 1$, one has $\lambda \cdot dc = 1 \cdot 0.1 = 0.1$. So, in this case, (14.2.3) is satisfied with equality. (This is because of the special form of the functions f and g.)

4. (a) $x = 6$, $y = 2$ (b) The approximate change is 1.

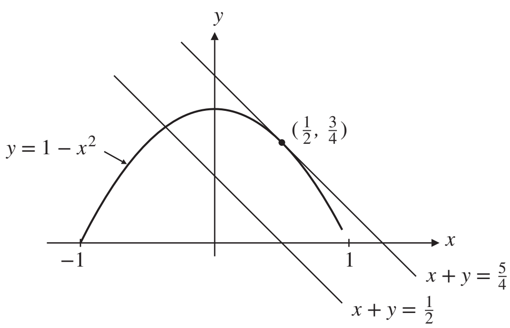


Figure A14.3.3

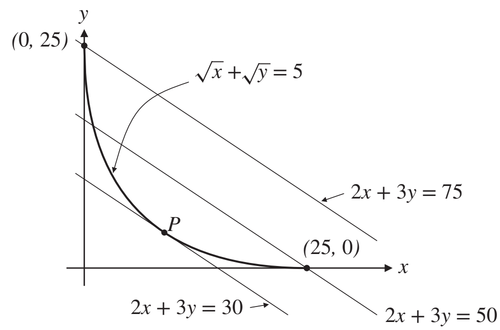


Figure A14.4.3

14.4

1. Setting $y = 2 - x$ reduces the problem to that of maximizing $x(2-x) = 2x - x^2$, which has the solution $x = 1$, and so the optimal value of y becomes $2 - x = 1$. Using the Lagrange multiplier method, with the Lagrangian $\mathcal{L}(x, y) = xy - \lambda(x + y - 2)$, the first-order conditions for the constrained problem are $y - \lambda = 0$, $x - \lambda = 0$, with the unique solution $x = y = \lambda = 1$ satisfying the constraint $x + y = 2$. Then, when $\lambda = 1$, one has $\mathcal{L}(2, 2) = 2 > \mathcal{L}(1, 1) = 1$, so $(1, 1)$ is not a maximum point for $\mathcal{L}$. (In fact, $\mathcal{L}(x, y)$ has a saddle point at $(1, 1)$.)

2. The problem with systems of three equations and two unknowns is not that they are merely difficult to solve but that they are usually inconsistent — i.e., it is *impossible* to solve them. The equations $f'_x(x, y) = f'_y(x, y) = 0$ are NOT valid at the optimal point.

3. (a) With $\mathcal{L} = 2x + 3y - \lambda(\sqrt{x} + \sqrt{y} - 5)$, the first-order conditions are $\mathcal{L}'_1(x, y) = 2 - \lambda/2\sqrt{x} = 0$ and $\mathcal{L}'_2(x, y) = 3 - \lambda/2\sqrt{y} = 0$. Thus $y = 4x/9$, so $x = 9$ and $y = 4$.

(b) See Fig. A14.4.3. Move the line $2x + 3y = c$ as far north-east as possible. So the solution is at $(x, y) = (0, 25)$.

(c) $g(x, y)$ is continuously differentiable only on the set A of (x, y) such that $x > 0$ and $y > 0$, so the theorem does not apply at the point $(x, y) = (0, 25)$.

4. The minimum is 1 at $(x, y) = (-1, 0)$.

14.5

1. The Lagrangian $\mathcal{L} = 10x^{1/2}y^{1/3} - \lambda(2x + 4y - m)$ is concave in (x, y) (Exercise 13.2.8), so Theorem 14.5.1 applies.

2. With $\mathcal{L} = \ln x + \ln y - \lambda(px + qy - m)$, $\mathcal{L}'_x = 1/x - p\lambda$, $\mathcal{L}'_y = 1/y - q\lambda$, $\mathcal{L}''_{xx} = -1/x^2$, $\mathcal{L}''_{xy} = 0$, and $\mathcal{L}''_{yy} = -1/y^2$. Moreover, $g'_x = p$ and $g'_y = q$. Hence $D(x, y, \lambda) = -q^2/x^2 - p^2/y^2 < 0$. Condition (i) in Theorem 14.5.2 is satisfied.

3. $D(x, y, \lambda) = 10$, so Theorem 14.5.2 tells us that $(a/5, 2a/5)$ is a local minimum.

4. $U''_{11}(x, y) = a(a-1)x^{a-2} \leq 0$, $U''_{22}(x, y) = a(a-1)y^{a-2} \leq 0$, and $U''_{12}(x, y) = 0$, so U is concave. The solution is $x = mp^{1/(a-1)}/R$, and $y = mq^{1/(a-1)}/R$, where $R = p^{a/(a-1)} + q^{a/(a-1)}$.

14.6

1. (a) $\mathcal{L}(x, y, z) = x^2 + y^2 + z^2 - \lambda(x + y + z - 1)$, so $\mathcal{L}'_x = 2x - \lambda = 0$, $\mathcal{L}'_y = 2y - \lambda = 0$, $\mathcal{L}'_z = 2z - \lambda = 0$. It follows that $x = y = z$. The only solution of the necessary conditions is $(1/3, 1/3, 1/3)$ with $\lambda = 2/3$.

(b) The problem is to find the shortest distance from the origin to a point in the plane $x + y + z = 1$. The corresponding maximization problem has no solution.

2. $x = \dfrac{1/2}{1/2+1/3+1/4}\dfrac{390}{4} = 45$, $y = \dfrac{1/3}{1/2+1/3+1/4}\dfrac{390}{3} = 40$, $z = \dfrac{1/4}{1/2+1/3+1/4}\dfrac{390}{6} = 15$

3. (a) With the Lagrangian $\mathcal{L} = x + \sqrt{y} - 1/z - \lambda(px + qy + rz - m)$, the first-order conditions are:

(i) $\partial\mathcal{L}/\partial x = 1 - \lambda p = 0$; (ii) $\partial\mathcal{L}/\partial y = \frac{1}{2}y^{-1/2} - \lambda q = 0$; (iii) $\partial\mathcal{L}/\partial z = z^{-2} - \lambda r = 0$.

(b) From the equations in (a) we get $\lambda = 1/p$, then $\frac{1}{2}y^{-1/2} = q/p$, so $y = p^2/4q^2$, and finally $z = \sqrt{p/r}$.

Inserting these values of y and z into the budget constraint and solving for x gives $x = m/p - p/4q - \sqrt{r/p}$.

(c) Straightforward substitution. (d) $\partial U^*/\partial m = 1/p = \lambda$, as expected from Section 14.2.

4. The Lagrangian is $\mathcal{L} = \alpha \ln x + \beta \ln y + (1 - \alpha - \beta) \ln(L - l) - \lambda(px + qy - wl)$. It has a critical point at the solution (x^*, y^*, z^*) to the equations

(i) $\mathcal{L}'_x = \alpha/x^* - \lambda p = 0$; (ii) $\mathcal{L}'_y = \beta/y^* - \lambda q = 0$; (iii) $\mathcal{L}'_l = -(1-\alpha-\beta)/(L - l^*) + \lambda w = 0$.

From (i) and (ii), $qy^* = (\beta/\alpha)px^*$, while (i) and (iii) yield $l^* = L - [(1-\alpha-\beta)/w\alpha]px^*$.

Using the budget constraint, then solving for x^*, yields $x^* = \alpha wL/p$, $y^* = \beta wL/q$, and $l^* = (\alpha + \beta)L$.

5. The constraints reduce to $h + 2k + l = 0$ and $2h - k - 3l = 0$, so $k = -h$ and $l = h$.

But then $x^2 + y^2 + z^2 = 200 + 3h^2 \geq 200$ for all h, so f is maximized for $h = 0$.

Then $k = l = 0$ also, and we conclude that $(x, y, z) = (10, 10, 0)$ solves the minimization problem.

6. Here $\mathcal{L} = a_1^2x_1^2 + \cdots + a_n^2x_n^2 - \lambda(x_1 + \cdots + x_n - 1)$. Necessary conditions are that $\mathcal{L}'_j = 2a_j^2x_j - \lambda = 0, j = 1, \ldots, n$, and so $x_j = \lambda/2a_j^2$. Inserted into the constraint, this implies that $1 = \frac{1}{2}\lambda(1/a_1^2 + \cdots + 1/a_n^2)$. Thus, for $j = 1, \ldots, n$, we have $x_j = (1/a_j^2)\Big/(1/a_1^2 + \cdots + 1/a_n^2) = (1/a_j^2)\Big/\sum_{i=1}^n(1/a_i^2)$. If at least one a_i is 0, the minimum value is 0, which is attained by letting a corresponding x_i be 1, with the other x_j all equal to 0.

7. The point $(x, y, z) = (0, 0, 1)$ with $\lambda = -1/2$ and $\mu = 1$ yields the minimum, whereas $(x, y, z) = (4/5, 2/5, -1/5)$ with $\lambda = 1/2$ and $\mu = 1/5$ yields the maximum.

8. (a) $x_j = a_jm/p_j(a_1 + \cdots + a_n)$ for $k = 1, \ldots, n$. (b) $x_j = mp_j^{-1/(1-a)}\Big/\sum_{i=1}^n p_i^{-a/(1-a)}$ for $j = 1, 2, \ldots, n$.

14.7

1. (a) With $\mathcal{L} = x + a\ln y - \lambda(px + qy - m)$, one has $\mathcal{L}'_1 = 1 - \lambda p = 0$, $\mathcal{L}'_2 = a/y^* - \lambda q = 0$. Thus $\lambda = 1/p$.

Inserting this into the second equality yields $y^* = ap/q$. From the budget constraint we get $x^* = m/p - a$.

The Lagrangian is concave, so this is the solution.

(b) $U^* = x^* + a\ln y^* = m/p - a + a\ln a + a\ln p - a\ln q$.

Then $\partial U^*/\partial p = -m/p^2 + a/p$, $\partial U^*/\partial q = -a/q$, $\partial U^*/\partial m = 1/p$, and $\partial U^*/\partial a = \ln a + \ln p - \ln q$.

(c) $\partial\mathcal{L}/\partial p = -\lambda x$, $\partial\mathcal{L}/\partial q = -\lambda y$, $\partial\mathcal{L}/\partial m = \lambda$, and $\partial\mathcal{L}/\partial a = \ln y$. When we evaluate these four partials at (x^*, y^*), we see that the envelope theorem is confirmed.

2. The minimum point is $(x^*, y^*, z^*) = (a, 2a, 9a)$, where $a = -\sqrt{b}/6$, with $\lambda = -3/\sqrt{b}$.

The value of the objective function is $f^*(b) = x^* + 4y^* + 3z^* = -6\sqrt{b}$, and $df^*(b)/db = -3/\sqrt{b} = \lambda$.

3. (a) $x = aM/\alpha$, $y = bM/\beta$, $z = cM/\gamma$, $\lambda = 1/2M$, where $M = \sqrt{L}/\sqrt{a^2/\alpha + b^2/\beta + c^2/\gamma}$. (The first-order conditions give $x = a/2\lambda\alpha$, $y = b/2\lambda\beta$, $z = c/2\lambda\gamma$. Substituting in the constraint and solving for λ gives the solution.)

(b) We find that $M = \sqrt{L}/5$, and the given values of x, y, and z follow.

(c) For $L = 100$ one has $M = 2$ and $\lambda = 1/4$. The increase in the maximal value as L increases from 100 to 101, is approximately $\lambda \cdot 1 = 0.25$. The actual increase is $5(\sqrt{101} - \sqrt{100}) \approx 0.249378$.

4. (a) $\left(\frac{1}{4}\sqrt{15}, 0, \frac{1}{8}\right)$ and $\left(-\frac{1}{4}\sqrt{15}, 0, \frac{1}{8}\right)$ (with $\lambda = 1$) both solve the maximization problem, while $\left(0, 0, -\frac{1}{2}\right)$ solves the minimization problem. (b) $\Delta f^* \approx \lambda \Delta c = 1 \cdot 0.02 = 0.02$

5. $K^* = 2^{1/3} r^{-1/3} w^{1/3} Q^{4/3}$, $L^* = 2^{-2/3} r^{2/3} w^{-2/3} Q^{4/3}$, $C^* = 3 \cdot 2^{-2/3} r^{2/3} w^{1/3} Q^{4/3}$, $\lambda = 2^{4/3} r^{2/3} w^{1/3} Q^{1/3}$. The equalities $(*)$ are easily verified.

6. $\dfrac{\partial K^*}{\partial w} = \dfrac{\partial}{\partial w}\left(\dfrac{\partial C^*}{\partial r}\right) = \dfrac{\partial}{\partial r}\left(\dfrac{\partial C^*}{\partial w}\right) = \dfrac{\partial L^*}{\partial r}$, using the first and second equalities in $(*)$ in Example 14.7.3.

7. (a) With $\mathcal{L} = \sqrt{x} + ay - \lambda(px + qy - m)$, the first-order conditions for (x^*, y^*) to solve the problem are:
(i) $\mathcal{L}'_x = 1/2\sqrt{x^*} - \lambda p = 0$; (ii) $\mathcal{L}'_y = a - \lambda q = 0$.
Thus $\lambda = a/q$, and $x^*(p, q, a, m) = q^2/4a^2p^2$, $y^*(p, q, a, m) = m/q - q/4a^2p$. The Lagrangian is concave in (x, y), so this is the solution. The indirect utility function is $U^*(p, q, a, m) = \sqrt{x^*} + ay^* = q/4ap + am/q$.

(b) The partial derivatives of U^* w.r.t. the parameters are:

$\partial U^*/\partial p = -q/4ap^2$, $\partial U^*/\partial q = 1/4ap - am/q^2$, $\partial U^*/\partial m = a/q$, and $\partial U^*/\partial a = -q/4a^2p + m/q$.

On the other hand, with $\mathcal{L}(x, y, p, q, m, a) = \sqrt{x} + ay - \lambda(px + qy - m)$, when evaluated at (x^*, y^*), the partial derivatives of $\mathcal{L}$ with respect to the four parameters are:

$\partial \mathcal{L}^*/\partial p = -\lambda x^* = -(a/q)(q^2/4a^2p^2) = -q/4ap^2$, $\partial \mathcal{L}^*/\partial q = -\lambda y^* = -(a/q)(m/q - q/4a^2p) = 1/4ap - am/q^2$, $\partial \mathcal{L}^*/\partial m = \lambda$, and $\partial \mathcal{L}^*/\partial a = y^* = m/q - q/4a^2p$.

The envelope theorem is confirmed in all cases.

14.8

1. (a) With $\mathcal{L} = -x^2 - y^2 - \lambda(x - 3y + 10)$, Eqs (14.8.2) and (14.8.3) yield:

(i) $\mathcal{L}'_x = -2x - \lambda = 0$; (ii) $\mathcal{L}'_y = -2y + 3\lambda = 0$; (iii) $\lambda \geq 0$ with $\lambda = 0$ if $x - 3y < -10$.

Suppose $\lambda = 0$. Then (i) and (ii) imply $x = y = 0$, contradicting $x - 3y \leq -10$. Therefore $\lambda > 0$ and from (iii), $x - 3y = -10$. Furthermore, (i) and (ii) imply $\lambda = -2x = \frac{2}{3}y$, so $y = -3x$. Inserting this into $x - 3y = -10$ yields $x = -1$, and then $y = 3$. Since the Lagrangian is easily seen to be concave, $(x, y) = (-1, 3)$ is the solution.

(b) See Fig. A14.8.1. The solution is the point on the line $x - 3y = -10$ that is closest to the origin.

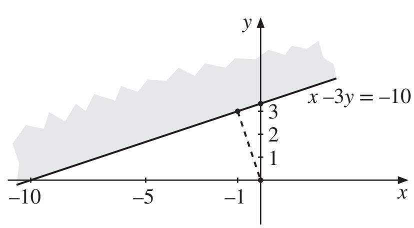


Figure A14.8.1

2. (a) The Kuhn–Tucker conditions yield:
(i) $1/(2\sqrt{x}) - \lambda p = 0$, (ii) $1/(2\sqrt{y}) - \lambda q = 0$, (iii) $\lambda \geq 0$, and $\lambda = 0$ if $px + qy < m$.
Clearing fractions in (i) and (ii) gives $1 = 2\lambda p\sqrt{x} = 2\lambda q\sqrt{y}$, from which we infer that x, y, λ are all positive, and also that $y = p^2x/q^2$. Because $\lambda > 0$, the budget equation $px + qy = m$ holds, implying that $x = mq/(pq + p^2)$. The corresponding value for y is easily found, and the demand functions are

$$x = x(p, q, m) = \frac{mq}{p(p+q)}, \qquad y = y(p, q, m) = \frac{mp}{q(p+q)}$$

These demand functions solve the problem because $\mathcal{L}(x, y)$ is easily seen to be concave.
(b) It is easy to see that the demand functions are homogeneous of degree 0, as expected.

3. (a) With $\mathcal{L} = 4 - \frac{1}{2}x^2 - 4y - \lambda(6x - 4y - a)$, the Kuhn–Tucker conditions are:

(i) $\partial\mathcal{L}/\partial x = -x - 6\lambda = 0$; (ii) $\partial\mathcal{L}/\partial y = -4 + 4\lambda = 0$; (iii) $\lambda \geq 0$ (with $\lambda = 0$ if $6x - 4y < a$).

(b) From (ii), $\lambda = 1$, so (i) gives $x = -6$. From (iii) and the given constraint, $y = -9 - \frac{1}{4}a$. The Lagrangian is concave, so we have found the solution.

(c) $V(a) = a + 22$, so $V'(a) = 1 = \lambda$.

4. (a) $\mathcal{L}(x, y) = x^2 + 2y^2 - x - \lambda(x^2 + y^2 - 1)$. The Kuhn–Tucker conditions are:

(i) $2x - 1 - 2\lambda x = 0$; (ii) $4y - 2\lambda y = 0$; (iii) $\lambda \geq 0$, with $\lambda = 0$ if $x^2 + y^2 < 1$.

(b) From (ii), $y(2 - \lambda) = 0$, so either (I) $y = 0$ or (II) $\lambda = 2$.

(I) $y = 0$. First, if $\lambda = 0$, then from (i), $x = 1/2$ and $(x, y) = (1/2, 0)$ is a candidate for an optimum (since it satisfies all the Kuhn–Tucker conditions).

Second, if $y = 0$ and $\lambda > 0$ then from (iii) and $x^2 + y^2 \leq 1$, one has $x^2 + y^2 = 1$. But then $x = \pm 1$, so $(x, y) = (\pm 1, 0)$ are candidates, with $\lambda = 1/2$ and 3/2, respectively.

(II) $\lambda = 2$. Then from (i) $x = -1/2$, and (iii) gives $y^2 = 3/4$, so $y = \pm\sqrt{3}/2$. Hence $(-1/2, \pm\sqrt{3}/2)$ are the two remaining candidates with $\lambda = 2$.

(c) Since f is continuous and the admissible set is closed and bounded, the extreme value theorem guarantees a maximum. The maximum point or points must be among the five points that satisfy the necessary conditions. Evaluating $x^2 + 2y^2 - x$ at each of those points shows that the maximum value is 9/4, attained at $(-1/2, \sqrt{3}/2)$ and at $(-1/2, -\sqrt{3}/2)$.

5. (a) For $0 < a < 1$, the solution is $x = \sqrt{a}$, $y = 0$, and $\lambda = a^{-1/2} - 1$; for $a \geq 1$, it is $x = 1$, $y = 0$, and $\lambda = 0$.

(b) Because the Lagrangian is concave (note that $\lambda \geq 0$), these give the respective maxima.

(c) For $a \in (0, 1)$, one has $f^*(a) = 2\sqrt{a} - a$, and $\mathrm{d}f^*(a)/\mathrm{d}a = \lambda$. If $a \geq 1$, then $f^*(a) = 1$, so $\mathrm{d}f^*(a)/\mathrm{d}a = 0 = \lambda$.

6. With $\mathcal{L} = aQ - bQ^2 - \alpha Q - \beta Q^2 + \lambda Q$, the Kuhn–Tucker conditions for Q^* to solve the problem are:

(i) $\mathrm{d}\mathcal{L}/\mathrm{d}Q = a - \alpha - 2(b + \beta)Q^* + \lambda = 0$; (ii) $\lambda \geq 0$, with $\lambda = 0$ if $Q^* > 0$.

By Theorem 14.8.1, these conditions are also sufficient because the Lagrangian is concave. We find that $Q^* = (a - \alpha)/2(b + \beta)$ and $\lambda = 0$ if $a > \alpha$, whereas $Q^* = 0$ and $\lambda = \alpha - a$ if $a \leq \alpha$. (See also Example 4.6.2.)

14.9

1. (a) Writing the constraints as $g_1(x, y) = x + e^{-x} - y \leq 0$ and $g_2(x, y) = -x \leq 0$, the Lagrangian is

$\mathcal{L} = \frac{1}{2}x - y - \lambda_1(x + e^{-x} - y) - \lambda_2(-x)$. The Kuhn–Tucker conditions are then:

(i) $\frac{1}{2} - \lambda_1(1 - e^{-x}) + \lambda_2 = 0$; (ii) $-1 + \lambda_1 = 0$;

(iii) $\lambda_1 \geq 0$, with $\lambda_1 = 0$ if $x + e^{-x} < y$; (iv) $\lambda_2 \geq 0$, with $\lambda_2 = 0$ if $x > 0$.

(b) From (ii), $\lambda_1 = 1$, so from (iii), $x + e^{-x} = y$. Either $x = 0$ or $x > 0$.

If $x > 0$, then (iv) implies that $\lambda_2 = 0$. Then (i) implies $\frac{1}{2} - (1 - e^{-x}) = 0$, or $e^{-x} = \frac{1}{2}$. Hence $x = \ln 2$, and so $y = x + e^{-x} = \ln 2 + \frac{1}{2}$.

If $x = 0$, then (i) implies $\lambda_2 = -\frac{1}{2}$, which contradicts $\lambda_2 \geq 0$.

We conclude that $(x, y) = (\ln 2, \ln 2 + \frac{1}{2})$ is the only point satisfying the Kuhn–Tucker conditions, with $(\lambda_1, \lambda_2) = (1, 0)$. (By sketching the constraint set and studying the level curves $\frac{1}{2}x - y = c$, it is easy to see that the point we found solves the problem.)

2. If $m \leq p\bar{x}/\alpha$, then $x^* = m\alpha/p$ and $y^* = (1 - \alpha)m/q$, with $\lambda = 1/m$ and $\mu = 0$.

If $m > p\bar{x}/\alpha$, then $x^* = \bar{x}$ and $y^* = (m - p\bar{x})/q$, with $\lambda = (1 - \alpha)/(m - p\bar{x})$ and $\mu = (\alpha m - p\bar{x})/\bar{x}(m - p\bar{x})$.

3. (a) The admissible set is the shaded region in Fig. A14.9.3. (b) See SM. (c) $(x^*, y^*) = (-1, 5)$.

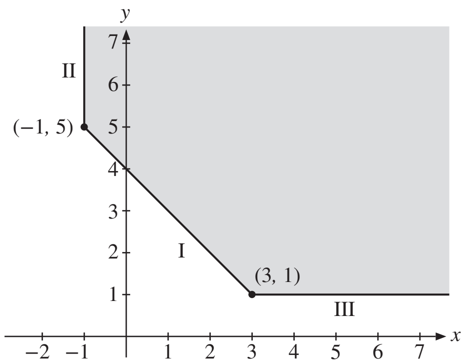


Figure A14.9.3

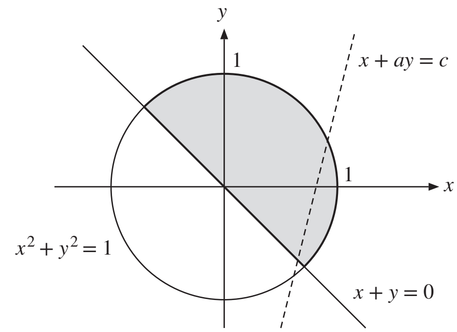


Figure A14.9.4

4. (a) The admissible set and one of the level curves for $x + ay$ are shown in Fig. A14.9.4. The requested necessary conditions, with $\mathcal{L} = x + ay - \lambda(x^2 + y^2 - 1) + \mu(x + y)$, are:
(i) $\mathcal{L}'_x = 1 - 2\lambda x + \mu = 0$; (ii) $\mathcal{L}'_y = a - 2\lambda y + \mu = 0$;
(iii) $\lambda \geq 0$, with $\lambda = 0$ if $x^2 + y^2 < 1$;
(iv) $\mu \geq 0$, with $\mu = 0$ if $x + y > 0$; (v) $x^2 + y^2 \leq 1$; (vi) $x + y \geq 0$.
(b) The solution is $(x^*, y^*) = \left(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right)$ in case $a \leq -1$, but $\left(x^*, y^*\right) = (1/\sqrt{1 + a^2}, a/\sqrt{1 + a^2}\right)$ in case $a > -1$.

5. $(x, y) = (4^{-2/3}, 4^{-1/3})$, with shadow prices $\lambda = 0$, $\mu = 0$, and $\nu = 1/2y = 4^{-1/6}$. See SM.

6. (a) See Fig. A14.9.6.
(b) With $\mathcal{L} = -(x + \frac{1}{2})^2 - \frac{1}{2}y^2 - \lambda(e^{-x} - y) - \mu(y - \frac{2}{3})$, the requested Kuhn–Tucker conditions are:
(i) $\mathcal{L}'_x = -2(x + \frac{1}{2}) + \lambda e^{-x} = 0$; (ii) $\mathcal{L}'_y = -y + \lambda - \mu = 0$; (iii) $\lambda \geq 0$, with $\lambda = 0$ if $e^{-x} - y < 0$;
(iv) $\mu \geq 0$, with $\mu = 0$ if $y < \frac{2}{3}$; (v) $e^{-x} - y \leq 0$; (vi) $y \leq \frac{2}{3}$.
The solution is $(x^*, y^*) = (\ln(3/2), 2/3)$, with $\lambda = 3[\ln(3/2) + 1/2]$ and $\mu = 3\ln(3/2) + 5/6$. See SM.

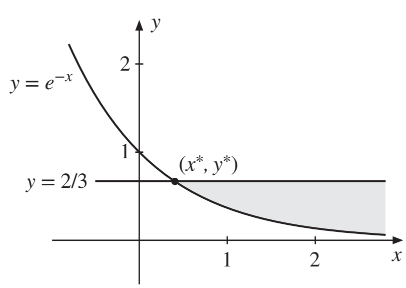


Figure A14.9.6

7. (a) With $\mathcal{L} = xz + yz - \lambda(x^2 + y^2 + z^2 - 1)$, the Kuhn–Tucker conditions are:

(i) $z - 2\lambda x = 0$; (ii) $z - 2\lambda y = 0$; (iii) $x + y - 2\lambda z = 0$; (iv) $\lambda \geq 0$, with $\lambda = 0$ if $x^2 + y^2 + z^2 < 1$.

(b) If $\lambda = 0$, then every point $(x, y, 0)$ with $x + y = 0$ with $x^2 + y^2 \leq 1$ satisfies the Kuhn–Tucker conditions. But the value of $xz + yz$ at these points is 0, and this is obviously not the maximum value.

Alternatively, in case $\lambda > 0$ and so $x^2 + y^2 + z^2 = 1$, then (i) and (ii) imply that $x = y = z/2\lambda$.

Next, it follows that $(z^2/4\lambda^2) + (z^2/4\lambda^2) + z^2 = 1$, so $z^2 = 4\lambda^2/(4\lambda^2 + 2)$. But then (iii) implies that $z/\lambda = 2\lambda z$ and so, because $z \neq 0$, that $2\lambda^2 = 1$. Therefore $z^2 = \frac{1}{2}$. The maximum points are at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2})$ and $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\sqrt{2})$, with $\lambda = \frac{1}{2}\sqrt{2}$. The extreme value theorem guarantees the existence of a maximum.

14.10

1. (a) With $\mathcal{L}(x, y) = x + \ln(1 + y) - \lambda(16x + y - 495)$, the K–T conditions for (x^*, y^*) to be a solution are:

(i) $\mathcal{L}_1'(x^*, y^*) = 1 - 16\lambda \leq 0 \;\; (= 0 \text{ if } x^* > 0)$; (ii) $\mathcal{L}_2'(x^*, y^*) = \dfrac{1}{1 + y^*} - \lambda \leq 0 \;\; (= 0 \text{ if } y^* > 0)$;

(iii) $\lambda \geq 0$, with $\lambda = 0$ if $16x^* + y^* < 495$; (iv) $x^* \geq 0$, $y^* \geq 0$; (v) $16x^* + y^* \leq 495$.

(b) Note that the Lagrangian is concave, so a point that satisfies the K–T conditions will be a maximum point. By condition (i), $\lambda \geq 1/16 > 0$, so (iii) and (v) imply (vi) $16x^* + y^* = 495$.

Suppose $x^* = 0$. Then (v) gives $y^* = 495$, and from (ii), $\lambda = 1/496$, contradicting $\lambda \geq 1/16$. Hence, $x^* > 0$, and so by (i), $\lambda = 1/16$.

Suppose $y^* = 0$. Then (ii) implies $\lambda \geq 1$, contradicting $\lambda = 1/16$. Thus $y^* > 0$, and so from (ii), $y^* = 15$ and then (v) yields $x^* = 30$.

So the only solution to the K–T conditions is $(x^*, y^*) = (30, 15)$, with $\lambda = 1/16$.

(c) Utility will increase by approximately $\lambda \cdot 5 = 5/16$. (Actually, the new solution is $(30\frac{5}{16}, 15)$, and the increase in utility is exactly 5/16. This is because the utility function has a special "quasi-linear" form.)

2. $(x, y) = (1, 0)$ is the only point satisfying all the conditions.

3. The only possible solution is $(x_1^*, x_2^*, k^*) = (1/2,\ 3/4,\ 3/4)$, with $\lambda = 0$ and $\mu = 3/2$.

Review exercises for Chapter 14

1. (a) With Lagrange multiplier λ, the first-order conditions imply $3 - 2\lambda x = 0$ and $4 - 2\lambda y = 0$, so $3y = 4x$. Inserting these into the constraint yields $x^2 = 81$, so $x = \pm 9$. Since the Lagrangian is concave, the solution is at $x = 9$, $y = 12$, with $\lambda = 1/6$.

(b) Using (14.2.3), $f^*(225 - 1) - f^*(225) \approx \lambda \cdot (-1) = -1/6$.

2. (a) $x = 2m/5p$, $y = 3m/5q$ (b) $x = m/3p$, $y = 2m/3q$ (c) $x = 3m/5p$, $y = 2m/5q$

3. (a) $\pi = xp(x) + yq(y) - C(x, y)$. The first-order conditions are (i) $p(x^*) = C_1'(x^*, y^*) - x^*p'(x^*)$ and (ii) $q(y^*) = C_2'(x^*, y^*) - y^*q'(y^*)$. See SM for the economic interpretations.

(b) With $\mathcal{L} = xp(x) + yq(y) - C(x, y) - \lambda(x + y - m)$, the first-order conditions for $(\hat{x}, \hat{y})$ to solve the problem are $\mathcal{L}_1' = p(\hat{x}) + \hat{x}p'(\hat{x}) - C_1'(\hat{x}, \hat{y}) - \lambda = 0$ and $\mathcal{L}_2' = q(\hat{y}) + \hat{y}q'(\hat{y}) - C_2'(\hat{x}, \hat{y}) - \lambda = 0$.

4. (a) The Lagrangian is $\mathcal{L}(x, y) = U(x, y) - \lambda[py - w(24 - x)]$. The first-order conditions imply that $pU_1' = wU_2' = \lambda wp$, which immediately yields (∗∗).

(b) Differentiating (∗) and (∗∗) w.r.t. w gives $py_w' = 24 - x - wx_w'$ and $p(U_{11}''x_w' + U_{12}''y_w') = U_2' + w(U_{21}''x_w' + U_{22}''y_w')$. Solving these equations yields the given formula for $x_w' = \partial x/\partial w$.

5. (a) $x = -2\sqrt{b}$, $y = 0$ solves the max problem, whereas $x = 4/3$, $y = \pm\sqrt{b - 4/9}$ solves the min problem.

(b) For $x = -2\sqrt{b}$, $y = 0$, $f^*(b) = 4b + 4\sqrt{b} + 1$. Since $\lambda = 4 + 2/\sqrt{b}$, the suggested equality is easily verified.

6. (a) With $\mathcal{L}(x,y) = v(x) + w(y) - \lambda(px + qy - m)$, the first-order conditions yield $v'(x) = \lambda p$ and $w'(y) = \lambda q$. Thus $v'(x)/w'(y) = p/q$.

(b) Since $\mathcal{L}''_{xx} = v''(x)$, $\mathcal{L}''_{yy} = w''(y)$, and $\mathcal{L}''_{xy} = 0$, we see that the Lagrangian is concave.

7. (a) The first-order conditions imply that $2x - 2 = 2y - 2$, so $x = y$. Inserting this into the constraint equation and squaring, then simplifying, one obtains the second equation in $(*)$.

(b) $\partial x/\partial a = 1/2x(3x + b)$, $\partial^2 x/\partial a^2 = -\frac{1}{4}(6x + b)[x(3x + b)]^{-3}$, and $\partial x/\partial b = -x/2(3x + b)$.

8. For $a \geq 5$, $(x, y) = (2, 1)$ with $\lambda = 0$. For $a < 5$, $(x, y) = (2\sqrt{a/5}, \sqrt{a/5})$, with $\lambda = \sqrt{5/a} - 1$.

9. (a) With $\mathcal{L} = xy - \lambda_1(x^2 + ry^2 - m) - \lambda_2(-x + 1)$, the Kuhn–Tucker conditions for (x^*, y^*) to solve the problem are:

(i) $\mathcal{L}'_1 = y^* - 2\lambda_1 x^* + \lambda_2 = 0$; (ii) $\mathcal{L}'_2 = x^* - 2r\lambda_1 y^* = 0$; (iii) $\lambda_1 \geq 0$, with $\lambda_1 = 0$ if $(x^*)^2 + r(y^*)^2 < m$;

(iv) $\lambda_2 \geq 0$, with $\lambda_2 = 0$ if $x^* > 1$; (v) $(x^*)^2 + r(y^*)^2 \leq m$; (vi) $x^* \geq 1$.

(b) Solution: For $m \geq 2$, it is $x^* = \sqrt{m/2}$ and $y^* = \sqrt{m/2r}$, with $\lambda_1 = 1/2\sqrt{r}$ and $\lambda_2 = 0$.

For $1 < m < 2$, it is $x^* = 1$, $y^* = \sqrt{(m-1)/r}$, with $\lambda_1 = 1/2\sqrt{r(m-1)}$ and $\lambda_2 = (2 - m)/\sqrt{r(m-1)}$.

(c) and (d): See SM.

10. With the Lagrangian $\mathcal{L} = R(Q) - C(Q) - \lambda(-Q)$, the first-order conditions for Q^* to be a solution are:

(i) $R'(Q^*) - C'(Q^*) + \lambda = 0$; (ii) $\lambda \geq 0$, with $\lambda = 0$ if $Q^* > 0$.

These conditions are also sufficient for optimality because the Lagrangian is concave in Q. A sufficient (and necessary) condition for $Q^* = 0$ to be optimal is that $\pi'(0) \leq 0$, or equivalently, $R'(0) \leq C'(0)$. (Draw a figure.)

11. (a) The maximization problem is: $\max(-rK - wL)$ subject to $-\sqrt{KL} \leq -Q$. With the Lagrangian $\mathcal{L} = -rK - wL - \lambda(-\sqrt{KL} + Q)$, the Kuhn–Tucker conditions for (K^*, L^*) to solve the problem are:

(i) $\mathcal{L}'_K = -r + \lambda(\sqrt{L^*}/2\sqrt{K^*}) = 0$; (ii) $\mathcal{L}'_L = -w + \lambda(\sqrt{K^*}/2\sqrt{L^*}) = 0$; (iii) $\lambda \geq 0$ ($\lambda = 0$ if $\sqrt{K^*L^*} > Q$). Obviously $\lambda = 0$ would contradict (i) and (ii), so $\lambda > 0$ and (iv) $\sqrt{K^*L^*} = Q$. Eliminating λ from (i) and (ii), we find $L^* = rK^*/w$. Then (iv) yields $K^* = Q\sqrt{w/r}$ and $L^* = Q\sqrt{r/w}$.

(b) $c^*(r, w, Q) = rK^* + wL^* = 2Q\sqrt{rw}$, so $\partial c^*/\partial r = Q\sqrt{w/r} = K^*$. If the price of capital r increases by 1, then the minimum cost will increase by about K^*, the optimal choice of capital input. The equation $\partial c^*/\partial w = Q\sqrt{r/w} = L^*$ has a similar interpretation.

Chapter 15

15.1

1. Equations (a), (c), (d), and (f) are linear, whereas (b) and (e) are nonlinear.

2. Yes, the system is linear in a, b, c, and d.

3. The three rows are $2x_1 + 4x_2 + 6x_3 + 8x_4 = 2$, $5x_1 + 7x_2 + 9x_3 + 11x_4 = 4$, and $4x_1 + 6x_2 + 8x_3 + 10x_4 = 8$.

4. The system is $\begin{cases} x_2 + x_3 + x_4 = b_1 \\ x_1 + x_3 + x_4 = b_2 \\ x_1 + x_2 + x_4 = b_3 \\ x_1 + x_2 + x_3 = b_4 \end{cases}$ with solution $\begin{cases} x_1 = -\frac{2}{3}b_1 + \frac{1}{3}(b_2 + b_3 + b_4) \\ x_2 = -\frac{2}{3}b_2 + \frac{1}{3}(b_1 + b_3 + b_4) \\ x_3 = -\frac{2}{3}b_3 + \frac{1}{3}(b_1 + b_2 + b_4) \\ x_4 = -\frac{2}{3}b_4 + \frac{1}{3}(b_1 + b_2 + b_3) \end{cases}$

(Adding the 4 equations, then dividing by 3, gives $x_1 + x_2 + x_3 + x_4 = \frac{1}{3}(b_1 + b_2 + b_3 + b_4)$.)

Subtracting each of the original equations in turn from this new equation gives $x_1, \dots, x_4$.
An alternative solution method is to eliminate the variables systematically, starting with (say) x_4.)

5. (a) The commodity bundle owned by individual j. (b) $a_{i1} + a_{i2} + \cdots + a_{in}$ is the total stock of commodity i. The first case is when $i = 1$. (c) $p_1 a_{1j} + p_2 a_{2j} + \cdots + p_m a_{mj}$

6. The solution is $x \approx 93.53$, $y \approx 482.11$, $s \approx 49.73$, and $c \approx 438.31$.

15.2

1. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}$, $3\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 6 & 9 \end{pmatrix}$

3. $u = 3$ and $v = -2$. (Equating the elements in row 1 and column 3 gives $u = 3$.
Then, equating those in row 2 and column 3 gives $u - v = 5$ and so $v = -2$.
The other elements then need to be checked, but this is obvious.)

4. $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 4 & 16 \end{pmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 & 2 & -6 \\ 2 & 2 & -2 \end{pmatrix}$, and $5\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} -3 & 8 & -20 \\ 10 & 12 & 8 \end{pmatrix}$

15.3

1. (a) $\mathbf{AB} = \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 \cdot (-1) + (-2) \cdot 1 & 0 \cdot 4 + (-2) \cdot 5 \\ 3 \cdot (-1) + \quad 1 \cdot 1 & 3 \cdot 4 + 1 \cdot 5 \end{pmatrix} = \begin{pmatrix} -2 & -10 \\ -2 & 17 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 12 & 6 \\ 15 & 3 \end{pmatrix}$.

(b) $\mathbf{AB} = \begin{pmatrix} 26 & 3 \\ 6 & -22 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 14 & 6 & -12 \\ 35 & 12 & 4 \\ 3 & 3 & -22 \end{pmatrix}$

(c) $\mathbf{AB}$ is not defined, whereas $\mathbf{BA} = \begin{pmatrix} -1 & 4 \\ 3 & 4 \\ 4 & 8 \end{pmatrix}$

(d) $\mathbf{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -6 \\ 0 & -8 & 12 \end{pmatrix}$ and $\mathbf{BA} = (16)$, a 1×1 matrix.

2. $\begin{pmatrix} -1 & 15 \\ -6 & -13 \end{pmatrix}$; $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\mathbf{C}(\mathbf{AB}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

3. $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{pmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 & 3 & -5 \\ 1 & -2 & -3 \\ -1 & -1 & -2 \end{pmatrix}$, $\mathbf{AB} = \begin{pmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{pmatrix}$,

$\mathbf{BA} = \begin{pmatrix} 0 & 4 & -9 \\ 19 & 3 & -3 \\ 5 & 1 & -3 \end{pmatrix}$, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 23 & 8 & 25 \\ 92 & -28 & 76 \\ 4 & -8 & -4 \end{pmatrix}$

4. (a) $\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5. (a) $\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$. The matrix $\mathbf{C}$ must be 2×2.

With $\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, we need $\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 2c_{21} & 2c_{22} \\ c_{11} + 3c_{21} & c_{12} + 3c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It follows that $c_{11} = -3/2$, $c_{12} = 1$, $c_{21} = 1/2$, and $c_{22} = 0$.

(b) $\mathbf{B} - 2\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$, so the first row of any product matrix $(\mathbf{B} - 2\mathbf{I})\mathbf{D}$ must be $(0, 0)$.

Hence, no such matrix $\mathbf{D}$ can possibly exist.

6. The product $\mathbf{AB}$ is defined only if $\mathbf{B}$ has n rows. And $\mathbf{BA}$ is defined only if $\mathbf{B}$ has m columns. So $\mathbf{B}$ must be an $n \times m$ matrix.

7. $\mathbf{B} = \begin{pmatrix} w - y & y \\ y & w \end{pmatrix}$, for arbitrary y, w.

8. $\mathbf{T}(\mathbf{Ts}) = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix}\begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix} = \begin{pmatrix} 0.2875 \\ 0.2250 \\ 0.4875 \end{pmatrix}$

15.4

1. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} = \begin{pmatrix} 3 & 2 & 6 & 2 \\ 7 & 4 & 14 & 6 \end{pmatrix}$

2. $(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz)$ (a 1×1 matrix)

3. It is straightforward to show that $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are both equal to the 2×2 matrix $\mathbf{D} = (d_{ij})_{2\times 2}$ whose four elements are $d_{ij} = a_{i1}b_{11}c_{1j} + a_{i1}b_{12}c_{2j} + a_{i2}b_{21}c_{1j} + a_{i2}b_{22}c_{2j}$ for $i = 1, 2$ and $j = 1, 2$.

4. (a) $\begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 9 \\ 1 & 3 & 3 \end{pmatrix}$ (b) $(1, 2, -3)$

5. (a) and (b) Equality occurs in (i) as well as in (ii) if and only if $\mathbf{AB} = \mathbf{BA}$.

(Note that $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 \neq \mathbf{A}^2 - \mathbf{B}^2$ unless $\mathbf{AB} = \mathbf{BA}$. The other case is similar.)

6. (a) Direct verification by matrix multiplication.
(b) $\mathbf{AA} = (\mathbf{AB})\mathbf{A} = \mathbf{A}(\mathbf{BA}) = \mathbf{AB} = \mathbf{A}$, so $\mathbf{A}$ is idempotent.

Then just interchange $\mathbf{A}$ and $\mathbf{B}$ to show that $\mathbf{B}$ is idempotent.

(c) As the induction hypothesis, suppose that $\mathbf{A}^k = \mathbf{A}$, which is true for $k = 1$.
Then $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \mathbf{AA} = \mathbf{A}$, which completes the proof by induction.

7. If $\mathbf{P}^3\mathbf{Q} = \mathbf{PQ}$, then $\mathbf{P}^5\mathbf{Q} = \mathbf{P}^2(\mathbf{P}^3\mathbf{Q}) = \mathbf{P}^2(\mathbf{PQ}) = \mathbf{P}^3\mathbf{Q} = \mathbf{PQ}$.

8. (a) Direct verification. (b) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ (c) Follow the hints. See SM for details.

15.5

1. $\mathbf{A}' = \begin{pmatrix} 3 & -1 \\ 5 & 2 \\ 8 & 6 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B}' = (0, 1, -1, 2)$, $\mathbf{C}' = \begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix}$

2. (a) $\mathbf{A}' = \begin{pmatrix}3 & -1\\ 2 & 5\end{pmatrix}$, $\mathbf{B}' = \begin{pmatrix}0 & 2\\ 2 & 2\end{pmatrix}$, $(\mathbf{A}+\mathbf{B})' = \begin{pmatrix}3 & 1\\ 4 & 7\end{pmatrix}$, $(\alpha\mathbf{A})' = \begin{pmatrix}-6 & 2\\ -4 & -10\end{pmatrix}$, $\mathbf{AB} = \begin{pmatrix}4 & 10\\ 10 & 8\end{pmatrix}$,
$(\mathbf{AB})' = \begin{pmatrix}4 & 10\\ 10 & 8\end{pmatrix} = \mathbf{B}'\mathbf{A}'$, and $\mathbf{A}'\mathbf{B}' = \begin{pmatrix}-2 & 4\\ 10 & 14\end{pmatrix}$.

(b) Verifying the rules (15.5.2) to (15.5.5) is now very easy.

3. Direct verification shows that for each of the two matrices the element in position ij equals the element in position ji, for $i = 1, 2, 3$ and $j = 1, 2, 3$.

4. Symmetry requires $a^2 - 1 = a + 1$ and $a^2 + 4 = 4a$. The second equation has the unique root $a = 2$, which also satisfies the first equation.

5. No! For example: $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$.

6. $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)' = (\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3))' = (\mathbf{A}_2\mathbf{A}_3)'\mathbf{A}_1' = (\mathbf{A}_3'\mathbf{A}_2')\mathbf{A}_1' = \mathbf{A}_3'\mathbf{A}_2'\mathbf{A}_1'$. For the general case use induction.

7. (a) Direct verification. (b) $\begin{pmatrix}p & q\\ -q & p\end{pmatrix}\begin{pmatrix}p & -q\\ q & p\end{pmatrix} = \begin{pmatrix}p^2+q^2 & 0\\ 0 & p^2+q^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \iff p^2+q^2 = 1$.

(c) If $\mathbf{P}'\mathbf{P} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$, then $(\mathbf{PQ})'(\mathbf{PQ}) = (\mathbf{Q}'\mathbf{P}')(\mathbf{PQ}) = \mathbf{Q}'(\mathbf{P}'\mathbf{P})\mathbf{Q} = \mathbf{Q}'\mathbf{I}_n\mathbf{Q} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$.

8. (a) $\mathbf{TS} = \begin{pmatrix} p^3+p^2q & 2p^2q+2pq^2 & pq^2+q^3\\ \frac{1}{2}p^3+\frac{1}{2}p^2+\frac{1}{2}p^2q & p^2q+pq+pq^2 & \frac{1}{2}pq^2+\frac{1}{2}q^2+\frac{1}{2}q^3\\ p^3+p^2q & 2p^2q+2pq^2 & pq^2+q^3\end{pmatrix} = \mathbf{S}$ because $p+q=1$. A similar argument shows that $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$. To derive the formula for $\mathbf{T}^3$, multiply each side of the last equation on the left by $\mathbf{T}$.

(b) The appropriate formula is $\mathbf{T}^n = 2^{1-n}\mathbf{T} + (1-2^{1-n})\mathbf{S}$.

15.6

1. (a) Gaussian elimination yields

$$\begin{pmatrix}1 & 1 & 3\\ 3 & 5 & 5\end{pmatrix}\begin{array}{c}-3\\ \end{array} \sim \begin{pmatrix}1 & 1 & 3\\ 0 & 2 & -4\end{pmatrix}\begin{array}{c}\\ 1/2\end{array} \sim \begin{pmatrix}1 & 1 & 3\\ 0 & 1 & -2\end{pmatrix}\begin{array}{c}\\ -1\end{array} \sim \begin{pmatrix}1 & 0 & 5\\ 0 & 1 & -2\end{pmatrix}$$

The solution is therefore $x_1 = 5$, $x_2 = -2$.
(b) Gaussian elimination yields

$$\begin{pmatrix}1 & 2 & 1 & 4\\ 1 & -1 & 1 & 5\\ 2 & 3 & -1 & 1\end{pmatrix}\begin{array}{cc}-1 & -2\\ & \\ & \end{array} \sim \begin{pmatrix}1 & 2 & 1 & 4\\ 0 & -3 & 0 & 1\\ 0 & -1 & -3 & -7\end{pmatrix}\begin{array}{c}\\ -1/3\\ \end{array} \sim \begin{pmatrix}1 & 2 & 1 & 4\\ 0 & 1 & 0 & -1/3\\ 0 & -1 & -3 & -7\end{pmatrix}\begin{array}{cc} & \\ 1 & -2\\ & \end{array}$$

$$\sim \begin{pmatrix}1 & 0 & 1 & 14/3\\ 0 & 1 & 0 & -1/3\\ 0 & 0 & -3 & -22/3\end{pmatrix}\begin{array}{c}\\ \\ -1/3\end{array} \sim \begin{pmatrix}1 & 0 & 1 & 14/3\\ 0 & 1 & 0 & -1/3\\ 0 & 0 & 1 & 22/9\end{pmatrix}\begin{array}{c}\\ \\ -1\end{array} \sim \begin{pmatrix}1 & 0 & 0 & 20/9\\ 0 & 1 & 0 & -1/3\\ 0 & 0 & 1 & 22/9\end{pmatrix}$$

The solution is therefore: $x_1 = 20/9$, $x_2 = -1/3$, $x_3 = 22/9$.

(c) The general solution is $x_1 = (2/5)s$, $x_2 = (3/5)s$, $x_3 = s$, where s is an arbitrary real number.

2. Using Gaussian elimination to eliminate x from the second and third equations, and then y from the third equation, we arrive at the following augmented matrix: $\begin{pmatrix}1 & 1 & -1 & 1\\ 0 & 1 & -3/2 & -1/2\\ 0 & 0 & a+5/2 & b-1/2\end{pmatrix}$.

For any z, the first two equations imply that $y = -\frac{1}{2} + \frac{3}{2}z$ and $x = 1 - y + z = \frac{3}{2} - \frac{1}{2}z$.

From the last equation we see that for $a \neq -\frac{5}{2}$, there is a unique solution with $z = (b - \frac{1}{2})\big/(a + \frac{5}{2})$.

For $a = -\frac{5}{2}$, there are no solutions if $b \neq \frac{1}{2}$, but there is one degree of freedom if $b = \frac{1}{2}$ (with z arbitrary).

3. For $c = 1$ and for $c = -2/5$ the solution is $x = 2c^2 - 1 + t$, $y = s$, $z = t$, $w = 1 - c^2 - 2s - 2t$, for arbitrary s and t.

For other values of c there are no solutions.

4. Move the first row down to row number three and use Gaussian elimination. There is a unique solution if and only if $a \neq 3/4$.

5. If $b_1 \neq \frac{1}{4}b_3$, there is no solution. If $b_1 = \frac{1}{4}b_3$, there is an infinite set of solutions that take the form $x = -2b_2 + b_3 - 5t$, $y = \frac{3}{2}b_2 - \frac{1}{2}b_3 + 2t$, $z = t$, with $t \in \mathbb{R}$.

15.7

1. $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mathbf{a} - \mathbf{b} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, $2\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 13 \\ 10 \end{pmatrix}$, and $-5\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} -4 \\ 13 \end{pmatrix}$

2. $\mathbf{a} + \mathbf{b} + \mathbf{c} = (-1, 6, -4)$, $\mathbf{a} - 2\mathbf{b} + 2\mathbf{c} = (-3, 10, 2)$, $3\mathbf{a} + 2\mathbf{b} - 3\mathbf{c} = (9, -6, 9)$

3. By definition of vector addition and scalar multiplication, the left-hand side of the equation must equal the vector $3(x, y, z) + 5(-1, 2, 3) = (3x - 5, 3y + 10, 3z + 15)$. For this to equal the vector $(4, 1, 3)$, corresponding components must be equal. So the vector equation is equivalent to the equation system $3x - 5 = 4$, $3y + 10 = 1$, and $3z + 15 = 3$, with the obvious solution $x = 3$, $y = -3$, $z = -4$.

4. $x_i = 0$ for all i.

5. Nothing, because $0 \cdot \mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$.

6. We need to find numbers t and s such that $t(2, -1) + s(1, 4) = (4, -11)$. This vector equation is equivalent to $(2t + s, -t + 4s) = (4, -11)$. Equating the two components gives the system (i) $2t + s = 4$; (ii) $-t + 4s = -11$.

This system has the solution $t = 3$, $s = -2$, so $(4, -11) = 3(2, -1) - 2(1, 4)$.

7. $4\mathbf{x} - 2\mathbf{x} = 7\mathbf{a} + 8\mathbf{b} - \mathbf{a}$, so $2\mathbf{x} = 6\mathbf{a} + 8\mathbf{b}$, and $\mathbf{x} = 3\mathbf{a} + 4\mathbf{b}$.

8. $\mathbf{a} \cdot \mathbf{a} = 5$, $\mathbf{a} \cdot \mathbf{b} = 2$, and $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = 7$. We see that $\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$.

9. The inner product of the two vectors is $x^2 + (x - 1)x + 3 \cdot 3x = x^2 + x^2 - x + 9x = 2x^2 + 8x = 2x(x + 4)$, which is 0 for $x = 0$ and for $x = -4$.

10. $\mathbf{x} = (5, 7, 12)$, $\mathbf{u} = (20, 18, 25)$, $\mathbf{u} \cdot \mathbf{x} = 526$.

11. (a) The firm's revenue is $\mathbf{p} \cdot \mathbf{z}$. Its costs are $\mathbf{p} \cdot \mathbf{x}$. (b) Profit = revenue − costs, which equals $\mathbf{p} \cdot \mathbf{z} - \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot (\mathbf{z} - \mathbf{x}) = \mathbf{p} \cdot \mathbf{y}$. If $\mathbf{p} \cdot \mathbf{y} < 0$, the firm makes a loss equal to $-\mathbf{p} \cdot \mathbf{y}$.

12. (a) Input vector = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (b) Output vector = $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (c) Cost = $(1, 3)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3$ (d) Revenue = $(1, 3)\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2$

(e) Value of net output = $(1, 3)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2 - 3 = -1$. (f) Loss = cost − revenue = $3 - 2 = 1$, so profit = -1.

15.8

1. $\mathbf{a} + \mathbf{b} = (3, 3)$ and $-\frac{1}{2}\mathbf{a} = (-2.5, 0.5)$. See Fig. A15.8.1.

2. (a) (i) $\lambda = 0$ gives $\mathbf{x} = (-1, 2) = \mathbf{b}$, (ii) $\lambda = 1/4$ gives $\mathbf{x} = (0, 7/4)$, (iii) $\lambda = 1/2$ gives $\mathbf{x} = (1, 3/2)$,

(iv) $\lambda = 3/4$ gives $\mathbf{x} = (2, 5/4)$, (v) $\lambda = 1$ gives $\mathbf{x} = (3, 1) = \mathbf{a}$. See Fig. A15.8.2.

(b) As λ runs through $[0,1]$, the vector $\mathbf{x}$ traces out the line segment joining $\mathbf{b}$ to $\mathbf{a}$ in Fig. A15.8.2. (c) See SM.

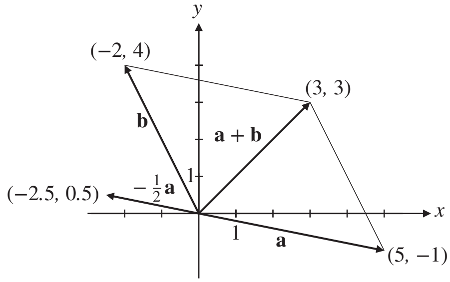


Figure A15.8.1

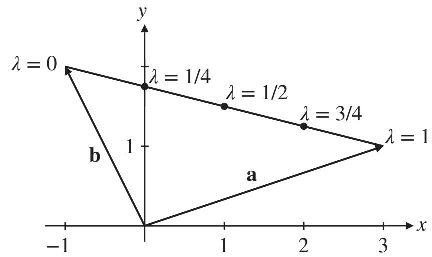


Figure A15.8.2

3. $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 3$, $\|\mathbf{c}\| = \sqrt{29}$. Also, $|\mathbf{a}\cdot\mathbf{b}| = 6 \le \|\mathbf{a}\|\cdot\|\mathbf{b}\| = 9$

4. (a) $x_1(1,2,1) + x_2(-3,0,-2) = (x_1 - 3x_2, 2x_1, x_1 - 2x_2) = (5,4,4)$ when $x_1 = 2$ and $x_2 = -1$.

(b) x_1 and x_2 would have to satisfy $x_1(1,2,1) + x_2(-3,0,-2) = (-3,6,1)$. Then $x_1 - 3x_2 = -3$, $2x_1 = 6$, and $x_1 - 2x_2 = 1$. The first two equations imply that $x_1 = 3$ and $x_2 = 2$, which contradicts the last equation.

5. The pairs of vectors in (a) and (c) are both orthogonal; the pair in (b) is not.

6. The vectors are orthogonal if and only if their inner product is 0. This is true if and only if

$x^2 - x - 8 - 2x + x = x^2 - 2x - 8 = 0$, which is the case for $x = -2$ and $x = 4$.

7. If $\mathbf{P}$ is orthogonal and $\mathbf{c}_i$ and $\mathbf{c}_j$ are two different columns of $\mathbf{P}$, then $\mathbf{c}_i'\mathbf{c}_j$ is the element in row i and column j of the matrix $\mathbf{P}'\mathbf{P} = \mathbf{I}$, so $\mathbf{c}_i'\mathbf{c}_j = 0$. If $\mathbf{r}_i$ and $\mathbf{r}_j$ are two different rows of $\mathbf{P}$, then $\mathbf{r}_i\mathbf{r}_j'$ is the element in row i and column j of $\mathbf{P}\mathbf{P}' = \mathbf{I}$, so again $\mathbf{r}_i\mathbf{r}_j' = 0$.

8. $(\|\mathbf{a}\| + \|\mathbf{b}\|)^2 = \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\cdot\|\mathbf{b}\| + \|\mathbf{b}\|^2$, whereas $\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2$.

Then $(\|\mathbf{a}\| + \|\mathbf{b}\|)^2 - \|\mathbf{a}+\mathbf{b}\|^2 = 2(\|\mathbf{a}\|\cdot\|\mathbf{b}\| - \mathbf{a}\cdot\mathbf{b}) \ge 0$ by the Cauchy–Schwarz inequality (15.8.2).

15.9

1. (a) $x_1 = 3t + 10(1-t) = 10 - 7t$, $x_2 = (-2)t + 2(1-t) = 2 - 4t$, and $x_3 = 2t + (1-t) = 1 + t$

(b) $x_1 = 1$, $x_2 = 3 - t$, and $x_3 = 2 + t$

2. (a) To show that $\mathbf{a}$ lies on L, put $t = 0$. (b) The direction of L is given by $(-1,2,1)$, and the equation of $\mathcal{P}$ is $(-1)(x_1 - 2) + 2(x_2 - (-1)) + 1\cdot(x_3 - 3) = 0$, or $-x_1 + 2x_2 + x_3 = -1$.

(c) We must have $3(-t+2) + 5(2t-1) - (t+3) = 6$, and so $t = 4/3$. Thus $P = (2/3, 5/3, 13/3)$.

3. $x_1 - 3x_2 - 2x_3 = -3$

4. $2x + 3y + 5z \le m$, with $m \ge 75$.

5. (a) Direct verification. (b) $(x_1, x_2, x_3) = (-2,1,-1) + t(-1,2,3) = (-2-t, 1+2t, -1+3t)$

Review exercises for Chapter 15

1. (a) $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

2. (a) $\mathbf{A}-\mathbf{B}=\begin{pmatrix}3&-2\\-2&2\end{pmatrix}$ (b) $\mathbf{A}+\mathbf{B}-2\mathbf{C}=\begin{pmatrix}-3&-4\\-2&-8\end{pmatrix}$ (c) $\mathbf{AB}=\begin{pmatrix}-2&4\\2&-3\end{pmatrix}$ (d) $\mathbf{C}(\mathbf{AB})=\begin{pmatrix}2&-1\\6&-8\end{pmatrix}$

(e) $\mathbf{AD}=\begin{pmatrix}2&2&2\\0&2&3\end{pmatrix}$ (f) **DC** is not defined. (g) $2\mathbf{A}-3\mathbf{B}=\begin{pmatrix}7&-6\\-5&5\end{pmatrix}$ (h) $(\mathbf{A}-\mathbf{B})'=\begin{pmatrix}3&-2\\-2&2\end{pmatrix}$

(i) and (j): $(\mathbf{C}'\mathbf{A}')\mathbf{B}'=\mathbf{C}'(\mathbf{A}'\mathbf{B}')=\begin{pmatrix}-6&5\\-4&5\end{pmatrix}$ (k) Not defined. (l) $\mathbf{D}'\mathbf{D}=\begin{pmatrix}2&4&5\\4&10&13\\5&13&17\end{pmatrix}$.

3. (a) $\begin{pmatrix}2&-5\\5&8\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$ (b) $\begin{pmatrix}1&1&1&1\\1&3&2&4\\1&4&8&0\\2&0&1&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$ (c) $\begin{pmatrix}a-1&3&-2\\a&2&-1\\1&-2&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\2\\1\end{pmatrix}$

4. $\mathbf{A}+\mathbf{B}=\begin{pmatrix}0&-4&1\\8&6&4\\-10&9&15\end{pmatrix}$, $\mathbf{A}-\mathbf{B}=\begin{pmatrix}0&6&-5\\-2&2&6\\-2&5&15\end{pmatrix}$, $\mathbf{AB}=\begin{pmatrix}13&-2&-1\\0&3&5\\-25&74&-25\end{pmatrix}$,

$\mathbf{BA}=\begin{pmatrix}-33&1&20\\12&6&-15\\6&4&18\end{pmatrix}$, $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})=\begin{pmatrix}74&-31&-48\\6&25&38\\-2&-75&-26\end{pmatrix}$

5. The matrix products on the left-hand side are $\begin{pmatrix}2a+b&a+b\\2x&x\end{pmatrix}$ and $\begin{pmatrix}a&b\\2a+x&2b\end{pmatrix}$, whose difference is $\begin{pmatrix}a+b&a\\x-2a&x-2b\end{pmatrix}$. Equating this to the matrix $\begin{pmatrix}2&1\\4&4\end{pmatrix}$ on the right-hand side yields the four equalities $a+b=2$, $a=1$, $x-2a=4$, and $x-2=4$. It follows that $a=b=1$, $x=6$.

6. (a) $\mathbf{A}^2=\begin{pmatrix}a^2-b^2&2ab&b^2\\-2ab&a^2-2b^2&2ab\\b^2&-2ab&a^2-b^2\end{pmatrix}$

(b) $(\mathbf{C}'\mathbf{BC})'=\mathbf{C}'\mathbf{B}'(\mathbf{C}')'=\mathbf{C}'(-\mathbf{B})\mathbf{C}=-\mathbf{C}'\mathbf{BC}$. So **A** is skew-symmetric if and only if $a=0$.

(c) $\mathbf{A}_1'=\frac{1}{2}(\mathbf{A}'+\mathbf{A}'')=\frac{1}{2}(\mathbf{A}'+\mathbf{A})=\mathbf{A}_1$, so $\mathbf{A}_1$ is symmetric. It is equally easy to prove that $\mathbf{A}_2$ is skew-symmetric, as well as that any square matrix **A** is therefore the sum $\mathbf{A}_1+\mathbf{A}_2$ of a symmetric matrix $\mathbf{A}_1$ and a skew-symmetric matrix $\mathbf{A}_2$.

7. (a) $\begin{pmatrix}1&4&1\\2&2&8\end{pmatrix}\begin{matrix}-2\\ \hookleftarrow\end{matrix}\sim\begin{pmatrix}1&4&1\\0&-6&6\end{pmatrix}\begin{matrix}\\ -1/6\end{matrix}\sim\begin{pmatrix}1&4&1\\0&1&-1\end{pmatrix}\begin{matrix}\hookleftarrow\\ -4\end{matrix}\sim\begin{pmatrix}1&0&5\\0&1&-1\end{pmatrix}$

The solution is $x_1=5$, $x_2=-1$. (b) The solution is $x_1=3/7$, $x_2=-5/7$, $x_3=-18/7$.

(c) The solution is $x_1=(1/14)x_3$, $x_2=-(19/14)x_3$, where x_3 is arbitrary. (One degree of freedom.)

8. We use the Gaussian method:

$$\begin{pmatrix}1&a&2&0\\-2&-a&1&4\\2a&3a^2&9&4\end{pmatrix}\begin{matrix}2\;\;-2a\\ \hookleftarrow\\ \longleftarrow\end{matrix}\sim\begin{pmatrix}1&a&2&0\\0&a&5&4\\0&a^2&9-4a&4\end{pmatrix}\begin{matrix}\\ -a\\ \hookleftarrow\end{matrix}\sim\begin{pmatrix}1&a&2&0\\0&a&5&4\\0&0&9-9a&4-4a\end{pmatrix}$$

For $a=1$, the last equation is superfluous; the solution is $x=3t-4$, $y=-5t+4$, $z=t$, with t arbitrary. If $a\neq1$, we have $(9-9a)z=4-4a$, so $z=4/9$. The two other equations then become $x+ay=-8/9$ and $ay=16/9$. If $a=0$, there is no solution. If $a\neq0$, the solution is $x=-8/3$, $y=16/9a$, and $z=4/9$.

9. $\|\mathbf{a}\| = \sqrt{35}$, $\|\mathbf{b}\| = \sqrt{11}$, and $\|\mathbf{c}\| = \sqrt{69}$. Also, $|\mathbf{a}\cdot\mathbf{b}| = |(-1)\cdot 1 + 5\cdot 1 + 3\cdot(-3)| = |-5| = 5$, and $\sqrt{35}\sqrt{11} = \sqrt{385}$ is obviously greater than 5, so the Cauchy–Schwarz inequality is satisfied.

10. (a) To produce **a**, put $\lambda = 1/2$. To produce **b** would require $6\lambda + 2 = 7$, $-2\lambda + 6 = 5$, and $-6\lambda + 10 = 5$, but these equations have no solution. For (b) and (c) see SM.

11. Because $\mathbf{PQ} = \mathbf{QP} + \mathbf{P}$, multiplying on the left by $\mathbf{P}$ gives $\mathbf{P}^2\mathbf{Q} = (\mathbf{PQ})\mathbf{P} + \mathbf{P}^2 = (\mathbf{QP} + \mathbf{P})\mathbf{P} + \mathbf{P}^2 = \mathbf{QP}^2 + 2\mathbf{P}^2$. See SM for details of how to repeat this argument for higher powers of $\mathbf{P}$.

Chapter 16

16.1

1. (a) $3\cdot 6 - 2\cdot 0 = 18$ (b) $ab - ba = 0$ (c) $(a+b)^2 - (a-b)^2 = 4ab$ (d) $3^t2^{t-1} - 3^{t-1}2^t = 3^{t-1}2^{t-1}(3-2) = 6^{t-1}$

2. See Fig. A16.1.2. The shaded parallelogram has area $3\cdot 6 = 18 = \begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix}$.

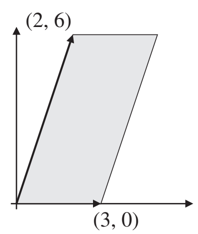


Figure A16.1.2

3. (a) Cramer's rule gives $x = \dfrac{\begin{vmatrix} 8 & -1 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \dfrac{-16+5}{-6+1} = \dfrac{11}{5}$ and $y = \dfrac{\begin{vmatrix} 3 & 8 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \dfrac{15-8}{-6+1} = \dfrac{7}{-5} = -\dfrac{7}{5}$.

(b) $x = 4$ and $y = -1$ (c) $x = \dfrac{a+2b}{a^2+b^2}$, $y = \dfrac{2a-b}{a^2+b^2}$ provided that $a^2 + b^2 \neq 0$

4. The numbers a and b must satisfy $a + 1 = 0$ and $a - 3b = -10$, so $a = -1$ and $b = 3$.

5. Expanding the determinant, $(2-x)(-x) - 8 = 0$, that is $x^2 - 2x - 8 = 0$, so $x = -2$ or $x = 4$.

6. The matrix product is $\mathbf{AB} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$,

implying that $|\mathbf{AB}| = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$.

On the other hand, $|\mathbf{A}||\mathbf{B}| = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$. A tedious process of expanding each expression, then cancelling four terms in the expression for $|\mathbf{AB}|$, reveals that the two expressions are equal.

7. If $\mathbf{A} = \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $|\mathbf{A} + \mathbf{B}| = 4$, whereas $|\mathbf{A}| + |\mathbf{B}| = 2$. (One has $|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$ for almost any choice of the matrices $\mathbf{A}$ and $\mathbf{B}$.)

8. Begin by writing the system as $\begin{cases} Y - C = I_0 + G_0 \\ -bY + C = a \end{cases}$. Then Cramer's rule yields

$$Y = \frac{\begin{vmatrix} I_0 + G_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + I_0 + G_0}{1 - b}, \qquad C = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1 - b}$$

Instead of using Cramer's rule, the expression for Y is most easily found by:

(i) solving the second equation to obtain $C = a + bY$; (ii) substituting this expression for C into the first equation;

(iii) solving the resulting equation for Y. Finally, use $C = a + bY$ again to find C.

9. (a) The equation $X_1 = M_2$ says that nation 1's exports equal nation 2's imports. Similarly, $X_2 = M_1$.

(b) Substituting for X_1, X_2, M_1, M_2, C_1, and C_2 gives:

(i) $(1 - c_1 + m_1)Y_1 - m_2Y_2 = A_1$; (ii) $(1 - c_2 + m_2)Y_2 - m_1Y_1 = A_2$.

Using Cramer's rule with $D = (1 - c_1 + m_1)(1 - c_2 + m_2) - m_1m_2$ yields

$$Y_1 = [A_2m_2 + A_1(1 - c_2 + m_2)]/D, \qquad Y_2 = [A_1m_1 + A_2(1 - c_1 + m_1)]/D$$

(c) Y_2 increases when A_1 increases.

16.2

1. (a) -2 (b) -2 (c) adf (d) $e(ad - bc)$

2. $\mathbf{AB} = \begin{pmatrix} -1 & -1 & -1 \\ 7 & 13 & 13 \\ 5 & 9 & 10 \end{pmatrix}$, $|\mathbf{A}| = -2$, $|\mathbf{B}| = 3$, and $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}| = -6$

3. (a) $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$ (b) $x_1 = x_2 = x_3 = 0$ (c) $x = 1$, $y = 2$, and $z = 3$

4. By Sarrus's rule the determinant is $(1 + a)(1 + b)(1 + c) + 1 + 1 - (1 + b) - (1 + a) - (1 + c)$, which simplifies to the given expression.

5. $\text{tr}(\mathbf{A}) = a + b - 1 = 0$ and thus $b = 1 - a$. Also, $|\mathbf{A}| = -2ab = 12$, and so $-2a(1 - a) = 12$, or $a^2 - a - 6 = 0$. The roots of this equation are $a = 3$ and $a = -2$. Thus the solutions are $(a, b) = (3, -2)$ and $(a, b) = (-2, 3)$.

6. By Sarrus's rule, the determinant is $p(x) = (1 - x)^3 + 8 + 8 - 4(1 - x) - 4(1 - x) - 4(1 - x) = -x^3 + 3x^2 + 9x + 5$. The equation we want to solve is therefore the cubic equation $-x^3 + 3x^2 + 9x + 5 = 0$. We have no simple general formula available for solving such equations, but since this is a polynomial equation with integer coefficients, it follows from (4.7.6) that every integer root of the equation (if there are any) must divide the constant term 5. The only candidates are therefore ± 5 and ± 1. It is easily seen that $p(5) = 0$ and $p(-1) = 0$, and so both $x - 5$ and $x + 1$ must be factors in $p(x)$. Thus $p(x) = (x - 5)(x + 1)q(x)$, and polynomial division yields $q(x) = x + 1$. So the determinant is 0 if and only if $x = -1$ or $x = 5$.

7. (a) $|\mathbf{A}_t| = 2t^2 - 2t + 1 = t^2 + (t - 1)^2 > 0$ for all t. (Alternatively, show that the quadratic polynomial has no real zeros.) (b) $\mathbf{A}_t^3 = \begin{pmatrix} 1 & 2t - 2t^2 & t - t^2 \\ 4t - 4 & 5t - 4 & -t^2 + 4t - 3 \\ 2 - 2t & t^2 - 4t + 3 & t^3 - 2t + 2 \end{pmatrix}$. We find that $\mathbf{A}_t^3 = \mathbf{I}_3$ for $t = 1$.

8. $Y = (a - bd + A_0)/[1 - b(1 - t)]$, $C = (a - bd + A_0b(1 - t))/[1 - b(1 - t)]$,
$T = [t(a + A_0) + (1 - b)d]/\ [1 - b(1 - t)]$.

16.3

1. (a) $1 \cdot 2 \cdot 3 \cdot 4 = 24$ (b) $d - a$ (c) $1 \cdot 1 \cdot 1 \cdot 11 - 1 \cdot 1 \cdot 4 \cdot 4 - 1 \cdot (-3) \cdot 1 \cdot 3 - 2 \cdot 1 \cdot 1 \cdot 2 = 0$

2. With $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{nn} \end{pmatrix}$, the product $\mathbf{AB}$ is easily seen to be upper triangular, with the elements $a_{11}b_{11}, a_{22}b_{22}, \dots, a_{nn}b_{nn}$ on the main diagonal. The determinant $|\mathbf{AB}|$ is, according to (16.3.3), the product of the n numbers $a_{ii}b_{ii}$. On the other hand, $|\mathbf{A}| = a_{11}a_{22}\cdots a_{nn}$, and $|\mathbf{B}| = b_{11}b_{22}\cdots b_{nn}$, so the required equality follows immediately.

3. $+a_{12}a_{23}a_{35}a_{41}a_{54}$. (Four lines between pairs of boxed elements rise as one goes to the right.)

4. $-a_{15}a_{24}a_{32}a_{43}a_{51}$. (There are nine lines that rise to the right.)

5. Carefully examining the determinant reveals that its only nonzero term is the product of its diagonal elements. So the equation is $(2-x)^4 = 0$, whose only solution is $x = 2$.

16.4

1. (a) $\mathbf{AB} = \begin{pmatrix} 13 & 16 \\ 29 & 36 \end{pmatrix}$, $\mathbf{BA} = \begin{pmatrix} 15 & 22 \\ 23 & 34 \end{pmatrix}$, $\mathbf{A}'\mathbf{B}' = \begin{pmatrix} 15 & 23 \\ 22 & 34 \end{pmatrix}$, $\mathbf{B}'\mathbf{A}' = \begin{pmatrix} 13 & 29 \\ 16 & 36 \end{pmatrix}$

(b) $|\mathbf{A}| = |\mathbf{A}'| = -2$ and $|\mathbf{B}| = |\mathbf{B}'| = -2$. So $|\mathbf{AB}| = 4 = |\mathbf{A}| \cdot |\mathbf{B}|$.

(c) $|\mathbf{A}'\mathbf{B}'| = 4$ and $|\mathbf{A}'| \cdot |\mathbf{B}'| = (-2) \cdot (-2) = 4$.

2. $\mathbf{A}' = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 5 \end{pmatrix}$ and $|\mathbf{A}| = |\mathbf{A}'| = -2$

3. (a) 0 (one column has only zeros). (b) 0 (rows 1 and 4 are proportional).

(c) $(a_1 - x)(-x)^3 = x^4 - a_1x^3$. (Use the definition of a determinant and observe that at most one term is nonzero.)

4. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = -12$, $3|\mathbf{A}| = 9$, $|-2\mathbf{B}| = (-2)^3(-4) = 32$, $|4\mathbf{A}| = 4^3|\mathbf{A}| = 4^3 \cdot 3 = 192$, and $|\mathbf{A}| + |\mathbf{B}| = -1$, whereas $|\mathbf{A} + \mathbf{B}|$ is not determined.

5. $\mathbf{A}^2 = \begin{pmatrix} a^2+6 & a+1 & a^2+4a-12 \\ a^2+2a+2 & 3 & 8-2a^2 \\ a-3 & 1 & 13 \end{pmatrix}$ and $|\mathbf{A}| = a^2 - 3a + 2$.

6. (a) The first and the second columns are proportional, so by rule (v) of Theorem 16.4.1, the determinant is 0.

(b) Add the second column to the third. This makes the first and third columns proportional.

(c) The first row is $x - y$ times the second row, so the first two rows are proportional.

7. $\mathbf{X}'\mathbf{X} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ and $|\mathbf{X}'\mathbf{X}| = 10$

8. By Sarrus's rule, for example, $|\mathbf{A}_a| = a(a^2+1) + 4 + 4 - 4(a^2+1) - a - 4 = a^2(a-4)$, so $|\mathbf{A}_1| = -3$ and $|\mathbf{A}_1^6| = |\mathbf{A}_1|^6 = (-3)^6 = 729$. (Note how much easier this is than first finding $\mathbf{A}_1^6$ and only then evaluating its determinant.)

9. Because $\mathbf{P}'\mathbf{P} = \mathbf{I}_n$, it follows from (16.4.1) and (16.3.4) that $|\mathbf{P}'||\mathbf{P}| = |\mathbf{I}_n| = 1$. But $|\mathbf{P}'| = |\mathbf{P}|$ by rule (ii) in Theorem 16.4.1, so $|\mathbf{P}|^2 = 1$. Hence, $|\mathbf{P}| = \pm 1$.

10. (a) Because $\mathbf{A}^2 = \mathbf{I}_n$ it follows from (16.4.1) that $|\mathbf{A}|^2 = |\mathbf{I}_n| = 1$, and so $|\mathbf{A}| = \pm 1$.

(b) Direct verification by matrix multiplication.

(c) We have $(\mathbf{I}_n - \mathbf{A})(\mathbf{I}_n + \mathbf{A}) = \mathbf{I}_n \cdot \mathbf{I}_n - \mathbf{A}\mathbf{I}_n + \mathbf{I}_n\mathbf{A} - \mathbf{A}\mathbf{A} = \mathbf{I}_n - \mathbf{A} + \mathbf{A} - \mathbf{A}^2 = \mathbf{I}_n - \mathbf{A}^2$, and this expression equals $\mathbf{0}$ if and only if $\mathbf{A}^2 = \mathbf{I}_n$.

11. (a) The first equality is true, the second is false. (The second equality becomes true if the factor 2 is replaced by 4.)

(b) Generally false. (Both determinants on the right are 0, even if $ad - bc \neq 0$.) (c) Both equalities are true.

(d) True. (The second determinant is the result of subtracting 2 times row 1 of the first determinant from its row 2.)

12. We want to show that $\mathbf{B}(\mathbf{PQ}) = (\mathbf{PQ})\mathbf{B}$. Using the associative law for matrix multiplication, we get

$$\mathbf{B}(\mathbf{PQ}) = (\mathbf{BP})\mathbf{Q} \overset{(1)}{=} (\mathbf{PB})\mathbf{Q} = \mathbf{P}(\mathbf{BQ}) \overset{(2)}{=} \mathbf{P}(\mathbf{QB}) = (\mathbf{PQ})\mathbf{B}.$$

This shows that $\mathbf{PQ}$ does indeed commute with $\mathbf{B}$. (At (1) we used $\mathbf{BP} = \mathbf{PB}$, and at (2) we used $\mathbf{BQ} = \mathbf{QB}$.)

13. Let $\mathbf{A} = \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}$. Then compute $\mathbf{A}^2$ and recall (16.4.1).

14. Start by adding each of the last $n-1$ rows to the first row. Each element in the first row then becomes $na + b$. Factor this out of the determinant. Next, add the first row multiplied by $-a$ to all the other $n-1$ rows. The result is an upper triangular matrix whose diagonal elements are $1, b, b, \ldots, b$, with product equal to b^{n-1}. The conclusion follows easily.

16.5

1. (a) 2. (Subtract row 1 from both row 2 and row 3 to get a determinant whose first column has elements $1, 0, 0$. Then expand by the first column.) (b) 30 (c) 0. (Columns 2 and 4 are proportional.)

2. In each of these cases we keep expanding by the last (remaining) column. The answers are:

(a) $-abc$ (b) $abcd$ (c) $1 \cdot 5 \cdot 3 \cdot 4 \cdot 6 = 360$

16.6

1. (a) Using (16.6.4) one has $\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. (b) Use (16.6.4).

2. $\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ a+b & 2a+1/4+3b & 4a+3/2+2b \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$ if and only if both $a + b = 4a + 3/2 + 2b = 0$ and $2a + 1/4 + 3b = 1$. This is true if and only if $a = -3/4$ and $b = 3/4$.

3. (a) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (c) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

4. From $\mathbf{A}^3 = \mathbf{I}$, it follows that $\mathbf{A}^2\mathbf{A} = \mathbf{I}$, so $\mathbf{A}^{-1} = \mathbf{A}^2 = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$.

5. (a) $|\mathbf{A}| = 1$, $\mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$, and so $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} - \mathbf{I}_3 = \mathbf{0}$.

(b) The last equality in (a) is equivalent to $\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}_3) = \mathbf{A}(\mathbf{A} - \mathbf{I}_3)^2 = \mathbf{I}_3$, so $\mathbf{A}^{-1} = (\mathbf{A} - \mathbf{I}_3)^2$.

(c) Choose $\mathbf{P} = (\mathbf{A} - \mathbf{I}_3)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, so that $\mathbf{A} = [(\mathbf{A} - \mathbf{I}_3)^2]^{-1} = \mathbf{P}^2$. The matrix $-\mathbf{P}$ also works.

6. (a) $\mathbf{AA}' = \begin{pmatrix} 21 & 11 \\ 11 & 10 \end{pmatrix}$, $|\mathbf{AA}'| = 89$, and $(\mathbf{AA}')^{-1} = \dfrac{1}{89}\begin{pmatrix} 10 & -11 \\ -11 & 21 \end{pmatrix}$.

(b) No, $\mathbf{AA}'$ is always symmetric by Example 15.5.3. Then $(\mathbf{AA}')^{-1}$ is symmetric by the note below Theorem 16.6.1.

7. (a) $\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} = \mathbf{PDIDP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$.

(b) The formula holds for $m = 1$. Suppose the formula is valid for $m = k$. Then

$$\mathbf{A}^{k+1} = \mathbf{AA}^k = \mathbf{PDP}^{-1}(\mathbf{PD}^k\mathbf{P}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}^k\mathbf{P}^{-1} = \mathbf{PDID}^k\mathbf{P}^{-1} = \mathbf{PDD}^k\mathbf{P}^{-1} = \mathbf{PD}^{k+1}\mathbf{P}^{-1}$$

so the formula is also valid for $m = k + 1$. By induction, it is valid for all natural numbers m.

8. $\mathbf{B}^2 + \mathbf{B} = \mathbf{I}$, $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I} = \mathbf{0}$, and $\mathbf{B}^{-1} = \mathbf{B} + \mathbf{I} = \begin{pmatrix} 1/2 & 5 \\ 1/4 & 1/2 \end{pmatrix}$.

9. Let $\mathbf{B} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Then $\mathbf{A}^2 = (\mathbf{I}_m - \mathbf{B})(\mathbf{I}_m - \mathbf{B}) = \mathbf{I}_m - \mathbf{B} - \mathbf{B} + \mathbf{B}^2$. Here

$$\mathbf{B}^2 = (\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{B}$$

Thus, $\mathbf{A}^2 = \mathbf{I}_m - \mathbf{B} - \mathbf{B} + \mathbf{B} = \mathbf{I}_m - \mathbf{B} = \mathbf{A}$.

10. $\mathbf{AB} = \begin{pmatrix} -7 & 0 \\ -2 & 10 \end{pmatrix}$, so $\mathbf{CX} = \mathbf{D} - \mathbf{AB} = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}$. But $\mathbf{C}^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$, so $\mathbf{X} = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}$.

11. (a) If $\mathbf{C}^2 + \mathbf{C} = \mathbf{I}$, then $\mathbf{C}(\mathbf{C} + \mathbf{I}) = \mathbf{I}$, and so $\mathbf{C}^{-1} = \mathbf{C} + \mathbf{I} = \mathbf{I} + \mathbf{C}$.

(b) Because $\mathbf{C}^2 = \mathbf{I} - \mathbf{C}$, it follows that $\mathbf{C}^3 = \mathbf{C}^2\mathbf{C} = (\mathbf{I} - \mathbf{C})\mathbf{C} = \mathbf{C} - \mathbf{C}^2 = \mathbf{C} - (\mathbf{I} - \mathbf{C}) = -\mathbf{I} + 2\mathbf{C}$.

Moreover, $\mathbf{C}^4 = \mathbf{C}^3\mathbf{C} = (-\mathbf{I} + 2\mathbf{C})\mathbf{C} = -\mathbf{C} + 2\mathbf{C}^2 = -\mathbf{C} + 2(\mathbf{I} - \mathbf{C}) = 2\mathbf{I} - 3\mathbf{C}$.

16.7

1. (a) $\begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$ (b) $\dfrac{1}{9}\begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}$ (c) $|\mathbf{C}| = 0$, so the matrix $\mathbf{C}$ has no inverse.

2. The inverse is $\dfrac{1}{|\mathbf{A}|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \dfrac{1}{72}\begin{pmatrix} -3 & 5 & 9 \\ 18 & -6 & 18 \\ 6 & 14 & -18 \end{pmatrix}$.

3. $(\mathbf{I} - \mathbf{A})^{-1} = \dfrac{5}{62}\begin{pmatrix} 18 & 16 & 10 \\ 2 & 19 & 8 \\ 4 & 7 & 16 \end{pmatrix}$

4. When $k = r$, the solution to the system is $x_1 = b^*_{1r}$, $x_2 = b^*_{2r}$, $\dots$, $x_n = b^*_{nr}$.

5. (a) $\mathbf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ (c) There is no inverse.

16.8

1. (a) $x = 1$, $y = -2$, and $z = 2$ (b) $x = -3$, $y = 6$, $z = 5$, and $u = -5$

2. The determinant of the system is equal to -10, so the solution is unique. The determinants in (16.8.2) are

$$D_1 = \begin{vmatrix} b_1 & 1 & 0 \\ b_2 & -1 & 2 \\ b_3 & 3 & -1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 3 & b_1 & 0 \\ 1 & b_2 & 2 \\ 2 & b_3 & -1 \end{vmatrix}, \quad D_3 = \begin{vmatrix} 3 & 1 & b_1 \\ 1 & -1 & b_2 \\ 2 & 3 & b_3 \end{vmatrix}$$

Now expand each of these determinants by the column (b_1, b_2, b_3).

The result is $D_1 = -5b_1 + b_2 + 2b_3$, $D_2 = 5b_1 - 3b_2 - 6b_3$, $D_3 = 5b_1 - 7b_2 - 4b_3$.

Hence, $x_1 = \frac{1}{2}b_1 - \frac{1}{10}b_2 - \frac{1}{5}b_3$, $x_2 = -\frac{1}{2}b_1 + \frac{3}{10}b_2 + \frac{3}{5}b_3$, $x_3 = -\frac{1}{2}b_1 + \frac{7}{10}b_2 + \frac{2}{5}b_3$.

3. Show that the coefficient matrix has determinant equal to $-(a^3 + b^3 + c^3 - 3abc)$, then use Theorem 16.8.2.

16.9

1. $x_1 = \frac{1}{4}x_2 + 100$, $x_2 = 2x_3 + 80$, $x_3 = \frac{1}{2}x_1$. The solution is $x_1 = 160$, $x_2 = 240$, $x_3 = 80$.

2. (a) Let x and y denote total production in industries A and I, respectively. Then $x = \frac{1}{6}x + \frac{1}{4}y + 60$ and $y = \frac{1}{4}x + \frac{1}{4}y + 60$. So $\frac{5}{6}x - \frac{1}{4}y = 60$ and $-\frac{1}{4}x + \frac{3}{4}y = 60$. (b) The solution is $x = 320/3$ and $y = 1040/9$.

3. (a) No sector delivers to itself. (b) The total amount of good i needed to produce one unit of each good.

(c) This column vector gives the number of units of each good needed to produce one unit of good j.

(d) No meaningful economic interpretation. (The goods are usually measured in different units, so it is meaningless to add them together. As the saying goes: "You can't add apples and oranges!")

4. $0.8x_1 - 0.3x_2 = 120$ and $-0.4x_1 + 0.9x_2 = 90$, with solution $x_1 = 225$ and $x_2 = 200$.

5. The Leontief system for this three-sector model is

$$\begin{aligned} 0.9x_1 - 0.2x_2 - 0.1x_3 &= 85 \\ -0.3x_1 + 0.8x_2 - 0.2x_3 &= 95 \\ -0.2x_1 - 0.2x_2 + 0.9x_3 &= 20 \end{aligned}$$

which has the claimed solution.

6. The input matrix is $\mathbf{A} = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & \gamma \\ \alpha & 0 & 0 \end{pmatrix}$. The sums of the elements in each column are less than 1 provided that $\alpha < 1$, $\beta < 1$, and $\gamma < 1$, respectively. Then, in particular, the product $\alpha\beta\gamma < 1$.

7. The quantity vector $\mathbf{x}_0$ must satisfy $(*)$ $(\mathbf{I}_n - \mathbf{A})\mathbf{x}_0 = \mathbf{b}$, and the price vector $\mathbf{p}_0'$ must satisfy $(**)$ $\mathbf{p}_0'(\mathbf{I}_n - \mathbf{A}) = \mathbf{v}'$. Multiplying $(**)$ from the right by $\mathbf{x}_0$ yields $\mathbf{v}'\mathbf{x}_0 = [\mathbf{p}_0'(\mathbf{I}_n - \mathbf{A})]\mathbf{x}_0 = \mathbf{p}_0'[(\mathbf{I}_n - \mathbf{A})\mathbf{x}_0] = \mathbf{p}_0'\mathbf{b}$.

Review exercises for Chapter 16

1. (a) $5(-2) - (-2)3 = -4$ (b) $1 - a^2$ (c) $6a^2b + 2b^3$ (d) $\lambda^2 - 5\lambda$

2. (a) -4 (b) 1. (Subtract row 1 from rows 2 and 3. Then subtract twice row 2 from row 3. The resulting determinant has only one nonzero term in its third row.) (c) 1. (Use exactly the same row operations as in (b).)

3. Transposing each side yields $\mathbf{A}^{-1} - 2\mathbf{I}_2 = -2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, so $\mathbf{A}^{-1} = 2\mathbf{I}_2 - 2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix}$. Hence, using (16.6.3), $\mathbf{A} = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix}^{-1} = -\frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 \\ -1/2 & 0 \end{pmatrix}$.

4. (a) $|\mathbf{A}_t| = t + 1$, so $\mathbf{A}_t$ has an inverse if and only if $t \neq -1$.

(b) Multiplying the given equation from the right by $\mathbf{A}_1$ yields $\mathbf{B}\mathbf{A}_1 + \mathbf{X} = \mathbf{I}_3$.

Hence $\mathbf{X} = \mathbf{I}_3 - \mathbf{B}\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}$.

5. $|\mathbf{A}| = (p+1)(q-2)$, $|\mathbf{A} + \mathbf{E}| = 2(p-1)(q-2)$. So $\mathbf{A} + \mathbf{E}$ has an inverse for $p \neq 1$ and $q \neq 2$.

Obviously, $|\mathbf{E}| = 0$. Hence $|\mathbf{B}\mathbf{E}| = |\mathbf{B}||\mathbf{E}| = 0$, so $\mathbf{B}\mathbf{E}$ has no inverse.

6. The determinant of the coefficient matrix is $\begin{vmatrix} -2 & 4 & -t \\ -3 & 1 & t \\ t-2 & -7 & 4 \end{vmatrix} = 5t^2 - 45t + 40 = 5(t-1)(t-8)$.

So by Cramer's rule, there is a unique solution if and only if $t \neq 1$ and $t \neq 8$.

7. $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3) = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 - \mathbf{A} - \mathbf{A}^2 - \mathbf{A}^3 - \mathbf{A}^4 = \mathbf{I} - \mathbf{A}^4 = \mathbf{I}$. Then use (16.6.4).

8. (a) $(\mathbf{I}_n + a\mathbf{U})(\mathbf{I}_n + b\mathbf{U}) = \mathbf{I}_n^2 + b\mathbf{U} + a\mathbf{U} + ab\mathbf{U}^2 = \mathbf{I}_n + (a + b + nab)\mathbf{U}$, because $\mathbf{U}^2 = n\mathbf{U}$, as is easily verified.

(b) $\mathbf{A}^{-1} = \dfrac{1}{10}\begin{pmatrix} 7 & -3 & -3 \\ -3 & 7 & -3 \\ -3 & -3 & 7 \end{pmatrix}$. See SM for details.

9. From the first equation, $\mathbf{Y} = \mathbf{B} - \mathbf{A}\mathbf{X}$. Inserting this into the second equation gives $\mathbf{X} = \mathbf{C} - 2\mathbf{A}^{-1}\mathbf{Y} = \mathbf{C} - 2\mathbf{A}^{-1}\mathbf{B} + 2\mathbf{X}$.

Solving for $\mathbf{X}$, one obtains $\mathbf{X} = 2\mathbf{A}^{-1}\mathbf{B} - \mathbf{C}$. Moreover, $\mathbf{Y} = \mathbf{A}\mathbf{C} - \mathbf{B}$.

10. (a) For $a \neq 1$ and $a \neq 2$, there is a unique solution. If $a = 1$, there is no solution.

If $a = 2$, there are infinitely many solutions.

(b) When $a = 1$ and $b_1 - b_2 + b_3 = 0$, or when $a = 2$ and $b_1 = b_2$, there are infinitely many solutions.

11. (a) $|\mathbf{A}| = -2$. $\mathbf{A}^2 - 2\mathbf{I}_2 = \begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix} = \mathbf{A}$, so $\mathbf{A}^2 + c\mathbf{A} = 2\mathbf{I}_2$ if $c = -1$.

(b) If $\mathbf{B}^2 = \mathbf{A}$, then $|\mathbf{B}|^2 = |\mathbf{A}| = -2$, which is impossible.

12. Note first that if $\mathbf{A}'\mathbf{A} = \mathbf{I}_n$, then rule (16.6.5) implies that $\mathbf{A}' = \mathbf{A}^{-1}$, so $\mathbf{A}\mathbf{A}' = \mathbf{I}_n$.
But then $(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})(\mathbf{A}'\mathbf{B}\mathbf{A}) = \mathbf{A}'\mathbf{B}^{-1}(\mathbf{A}\mathbf{A}')\mathbf{B}\mathbf{A} = \mathbf{A}'\mathbf{B}^{-1}\mathbf{I}_n\mathbf{B}\mathbf{A} = \mathbf{A}'(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = \mathbf{A}'\mathbf{I}_n\mathbf{A} = \mathbf{A}'\mathbf{A} = \mathbf{I}_n$.

By rule (16.6.5) again, it follows that $(\mathbf{A}'\mathbf{B}\mathbf{A})^{-1} = \mathbf{A}'\mathbf{B}^{-1}\mathbf{A}$.

13. For once we use "unsystematic elimination". Solve the first equation to get $y = 3 - ax$, then the second to get $z = 2 - x$, and the fourth to get $u = 1 - y$. Substituting for all these in the third equation gives the result $3 - ax + a(2 - x) + b(1 - 3 + ax) = 6$ or $a(b-2)x = -2a + 2b + 3$. There is a unique solution provided that $a(b-2) \neq 0$. The solution is:

$$x = \frac{2b - 2a + 3}{a(b-2)}, \qquad y = \frac{2a + b - 9}{b - 2}, \qquad z = \frac{2ab - 2a - 2b - 3}{a(b-2)}, \qquad u = \frac{7 - 2a}{b - 2}$$

14. $|\mathbf{B}^3| = |\mathbf{B}|^3$. Because $\mathbf{B}$ is a 3×3-matrix, we have $|-\mathbf{B}| = (-1)^3|\mathbf{B}| = -|\mathbf{B}|$. Since $\mathbf{B}^3 = -\mathbf{B}$, it follows that $|\mathbf{B}|^3 = -|\mathbf{B}|$, and so $|\mathbf{B}|(|\mathbf{B}|^2 + 1) = 0$. The last equation implies $|\mathbf{B}| = 0$, and thus $\mathbf{B}$ can have no inverse.

15. The determinant on the left is equal to $(a + x)d - c(b + y) = (ad - bc) + (dx - cy)$, and this is the sum of the determinants on the right.

16. For simplicity look at the case $r = 1$, and consider Eq. (16.5.1). See SM for details.

17. For $a \neq b$ the solutions are $x_1 = \frac{1}{2}(a + b)$ and $x_2 = -\frac{1}{2}(a + b)$. If $a = b$, the determinant is 0 for all values of x.

Chapter 17

17.1

1. (a) Figure A17.1.1a shows that the solution is at the intersection P of the two lines $3x_1 + 2x_2 = 6$ and $x_1 + 4x_2 = 4$. Solution: max $= 36/5$ for $(x_1, x_2) = (8/5, 3/5)$.

(b) Figure A17.1b shows we see that the solution is at the intersection P of the two lines $u_1 + 3u_2 = 11$ and $2u_1 + 5u_2 = 20$. Solution: min $= 104$ for $(u_1, u_2) = (5, 2)$.

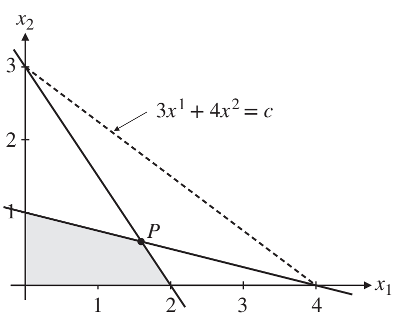


Figure A17.1.1a

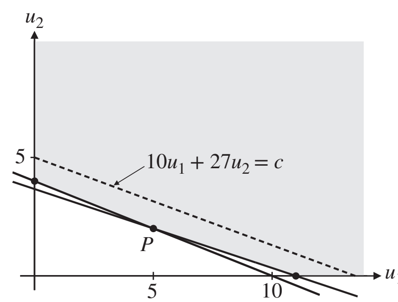


Figure A17.1.1b

2. (a) A graph shows that the solution is at the intersection of the lines $-2x_1 + 3x_2 = 6$ and $x_1 + x_2 = 5$.

Hence max $= 98/5$ for $(x_1, x_2) = (9/5, 16/5)$.

(b) The solution satisfies $2x_1 + 3x_2 = 13$ and $x_1 + x_2 = 6$. Hence max $= 49$ for $(x_1, x_2) = (5, 1)$.

(c) The solution satisfies $x_1 - 3x_2 = 0$ and $x_1 = 2$. Hence max $= -10/3$ for $(x_1, x_2) = (2, 2/3)$.

3. (a) max $= 18/5$ for $(x_1, x_2) = (4/5, 18/5)$. (b) max $= 8$ for $(x_1, x_2) = (8, 0)$. (c) max $= 24$ for $(x_1, x_2) = (8, 0)$.

(d) min $= -28/5$ for $(x_1, x_2) = (4/5, 18/5)$.

(e) max $= 16$ for all (x_1, x_2) of the form $(x_1, 4 - \frac{1}{2}x_1)$ where $x_1 \in [4/5, 8]$.

(f) min $= -24$ for $(x_1, x_2) = (8, 0)$ (follows from the answer to (c)).

4. (a) No maximum exists. Consider Fig. A17.1.4. As c becomes arbitrarily large, the dashed level curve $x_1 + x_2 = c$ moves to the north-east and still has the point $(c, 0)$ in common with the shaded set.

(b) Maximum at $(1, 0)$. The level curves are as in (a), but the direction of increase is reversed.

5. The slope of the line $20x_1 + tx_2 = c$ must lie between $-1/2$ (the slope of the flour border) and -1 (the slope of the butter border). For $t = 0$, the line is vertical and the solution is the point D in Fig. 17.1.2 in the text. For $t \neq 0$, the slope of the line is $-20/t$. Thus, $-1 \leq -20/t \leq -1/2$, which implies that $t \in [20, 40]$.

6. The LP problem is: max $700x + 1000y$ subject to $\begin{cases} 3x + 5y \leq 3900 \\ x + 3y \leq 2100 \\ 2x + 2y \leq 2200 \end{cases}$, $x \geq 0,\ y \geq 0$.

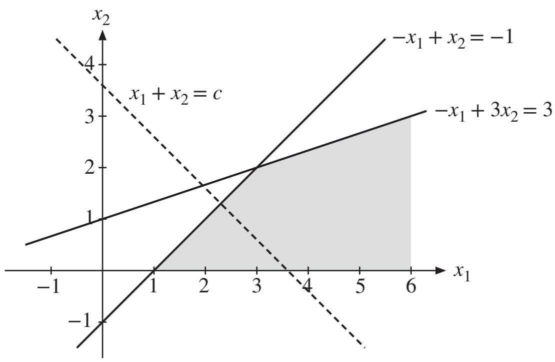


Figure A17.1.4

A figure showing the admissible set and an appropriate level line for the objective function will show that the solution is at the intersection of the two lines $3x + 5y = 3900$ and $2x + 2y = 2200$. Solving these equations yields $x = 800$ and $y = 300$. The firm should produce 800 sets of type A and 300 of type B.

17.2

1. (a) $(x_1, x_2) = (2, 1/2)$ and $u_1^* = 4/5$. (b) $(x_1, x_2) = (7/5, 9/10)$ and $u_2^* = 3/5$.

(c) Multiplying the two $\leq$ constraints by 4/5 and 3/5, respectively, then adding,

we obtain $(4/5)(3x_1 + 2x_2) + (3/5)(x_1 + 4x_2) \leq 6 \cdot (4/5) + 4 \cdot (3/5)$, which reduces to $3x_1 + 4x_2 \leq 36/5$.

2. $\min 8u_1 + 13u_2 + 6u_3$ subject to $\begin{cases} u_1 + 2u_2 + u_3 \geq 8 \\ 2u_1 + 3u_2 + u_3 \geq 9 \end{cases}$, $u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0$

3. (a) $\min\ 6u_1 + 4u_2$ subject to $\begin{cases} 3u_1 + \ \ u_2 \geq 3 \\ 2u_1 + 4u_2 \geq 4 \end{cases}$, $u_1 \geq 0,\ u_2 \geq 0$

(b) $\max\ 11x_1 + 20x_2$ subject to $\begin{cases} x_1 + 2x_2 \leq 10 \\ 3x_1 + 5x_2 \leq 2 \end{cases}$, $u_1 \geq 0,\ u_2 \geq 0$

4. (a) A graph shows that the solution is at the intersection of the lines $x_1 + 2x_2 = 14$ and $2x_1 + x_2 = 13$.

Hence $\max = 9$ for $(x_1^*, x_2^*) = (4, 5)$.

(b) The dual is $\min\ 14u_1 + 13u_2$ subject to $\begin{cases} u_1 + 2u_2 \geq 1 \\ 2u_1 + \ \ u_2 \geq 1 \end{cases}$, $u_1 \geq 0,\ u_2 \geq 0$.

A graph shows that the solution is at the intersection of the lines $u_1 + 2u_2 = 1$ and $2u_1 + u_2 = 1$.

Hence $\min = 9$ for $(u_1^*, u_2^*) = (1/3, 1/3)$.

17.3

1. (a) $x = 0$ and $y = 3$ gives $\max = 21$. See Fig. A17.3.1a, where the optimum is at P.

(b) The dual problem is $\min 20u_1 + 21u_2$ subject to $\begin{cases} 4u_1 + 3u_2 \geq 2 \\ 5u_1 + 7u_2 \geq 7 \end{cases}$, $u_1 \geq 0,\ u_2 \geq 0$.

The solution is $u_1 = 0$ and $u_2 = 1$, which gives $\min = 21$. See Fig. A17.3.1b. (c) Yes.

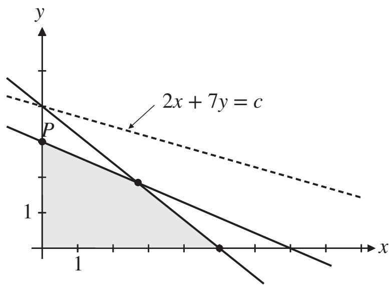


Figure A17.3.1a

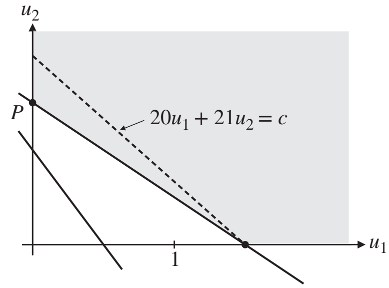


Figure A17.3.1b

2. max $300x_1 + 500x_2$ subject to $\begin{cases} 10x_1 + 25x_2 \leq 10\,000 \\ 20x_1 + 25x_2 \leq 8\,000 \end{cases}$, $\quad x_1 \geq 0,\ x_2 \geq 0.$

The solution can be found graphically. It is $x_1^* = 0$, $x_2^* = 320$, and the maximum value of the objective function is 160 000, the same as that found in Example 17.1.2 for the primal problem.

3. (a) The profit from selling x_1 small and x_2 medium television sets is $400x_1 + 500x_2$.

The first constraint, $2x_1 + x_2 \leq 16$, says that we cannot use more hours on assembly line 1 than its capacity allows.

The other constraints have similar interpretations.

(b) max $= 3800$ for $x_1 = 7$ and $x_2 = 2$. (c) Assembly line 1 should have its capacity increased.

17.4

1. According to formula (17.4.1), $\triangle z^* = u_1^* \triangle b_1 + u_2^* \triangle b_2 = 0 \cdot 0.1 + 1 \cdot (-0.2) = -0.2$.

2. (a) max $300x_1 + 200x_2$ subject to $\begin{cases} 6x_1 + 3x_2 \leq 54 \\ 4x_1 + 6x_2 \leq 48 \\ 5x_1 + 5x_2 \leq 50 \end{cases}$, $\quad x_1 \geq 0,\ x_2 \geq 0$

where x_1 and x_2 are the number of units produced of A and B, respectively. Solution: $(x_1, x_2) = (8, 2)$.
(b) Dual solution: $(u_1, u_2, u_3) = (100/3, 0, 20)$. (c) Increase in optimal profit: $\Delta\pi^* = u_1^* \cdot 2 + u_3^* \cdot 1 = 260/3$.

17.5

1. $4u_1^* + 3u_2^* = 3 > 2$ and $x^* = 0$; $5u_1^* + 7u_2^* = 7$ and $y^* = 3 > 0$.
Also $4x^* + 5y^* = 15 < 20$ and $u_1^* = 0$; $3x^* + 7y^* = 21$ and $u_2^* = 1 > 0$. So (17.5.1) and (17.5.2) are satisfied.

2. (a) See Figure A17.5.2. The minimum is attained at $(y_1^*, y_2^*) = (3, 2)$.

(b) The dual is max $15x_1 + 5x_2 - 5x_3 - 20x_4$ s.t. $\begin{cases} x_1 + x_2 - x_3 + x_4 \leq 1 \\ 6x_1 + x_2 + x_3 - 2x_4 \leq 2 \end{cases}$, $\quad x_j \geq 0\ (j = 1, \dots, 4)$.

The maximum is at $(x_1^*, x_2^*, x_3^*, x_4^*) = (1/5, 4/5, 0, 0)$.
(c) If the first constraint is changed to $y_1 + 6y_2 \geq 15.1$, the solution of the primal is still at the intersection of the lines (1) and (2) in Fig. A17.5.2, but with (1) shifted up slightly. The solution of the dual is completely unchanged. In both problems the optimal value increases by $(15.1 - 15) \cdot x_1^* = 0.02$.

3. (a) min $10\,000\, y_1 + 8\,000\, y_2 + 11\,000\, y_3$ s.t. $\begin{cases} 10y_1 + 20y_2 + 20y_3 \geq 300 \\ 20y_1 + 10y_2 + 20y_3 \geq 500 \end{cases}$, $\quad y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$

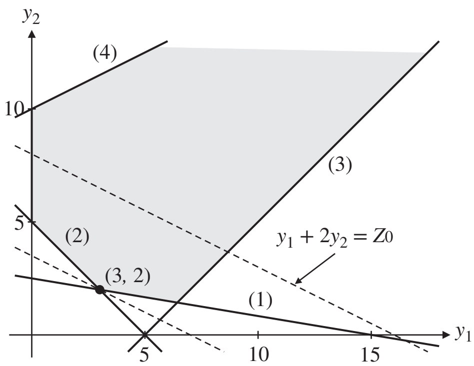


Figure A17.5.2

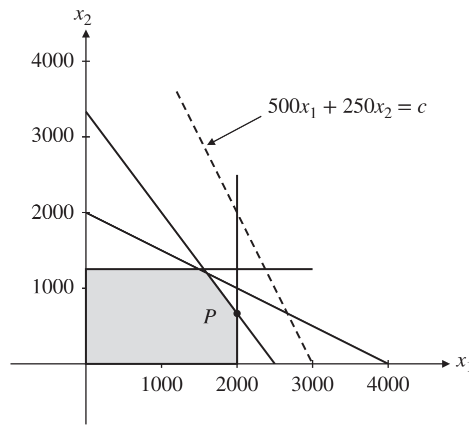


Figure A17.R.4

(b) The dual is: $\max\ 300x_1 + 500x_2$ subject to $\begin{cases} 10x_1 + 20x_2 \le 10\,000 \\ 20x_1 + 10x_2 \le \ \ 8\,000 \\ 20x_1 + 20x_2 \le 11\,000 \end{cases}$, $x_1 \ge 0,\ x_2 \ge 0$

Solution: max $= 255\,000$ for $x_1 = 100$ and $x_2 = 450$.
Solution of the primal: min $= 255\,000$ for $(y_1, y_2, y_3) = (20, 0, 5)$. (c) The minimum cost will increase by 2000.

4. (a) For $x_3 = 0$, the solution is $x_1 = x_2 = 1/3$. For $x_3 = 3$, the solution is $x_1 = 1$ and $x_2 = 2$.

(b) Let $z_{\max}$ denote the maximum value of the objective function.

If $0 \le x_3 \le 7/3$, then $z_{\max}(x_3) = 2x_3 + 5/3$ for $x_1 = 1/3$ and $x_2 = x_3 + 1/3$.

If $7/3 < x_3 \le 5$, then $z_{\max}(x_3) = x_3 + 4$ for $x_1 = x_3 - 2$ and $x_2 = 5 - x_3$.

If $x_3 > 5$, then $z_{\max}(x_3) = 9$ for $x_1 = 3$ and $x_2 = 0$. Because $z_{\max}(x_3)$ is increasing, the maximum is 9 for $x_3 \ge 5$.

(c) The solution to the original problem is $x_1 = 3$ and $x_2 = 0$, with x_3 as an arbitrary number ≥ 5.

Review exercises for Chapter 17

1. (a) $x^* = 3/2$, $y^* = 5/2$. (A diagram shows that the solution is at the intersection of $x + y = 4$ and $-x + y = 1$.)

(b) The dual is $\min\ 4u_1 + u_2 + 3u_3$ subject to $\begin{cases} u_1 - u_2 + 2u_3 \ge 1 \\ u_1 + u_2 - \ \ u_3 \ge 2 \end{cases}$, $u_1 \ge 0, u_2 \ge 0, u_3 \ge 0$.

Using complementary slackness, the solution of the dual is: $u_1^* = 3/2$, $u_2^* = 1/2$, and $u_3^* = 0$.

2. (a) $\max\ -x_1 + x_2$ subject to $\begin{cases} -x_1 + 2x_2 \le \ \ 16 \\ \ \ x_1 - 2x_2 \le \ \ \ 6 \\ -2x_1 - \ \ x_2 \le \ -8 \\ -4x_1 - 5x_2 \le -15 \end{cases}$, $x_1 \ge 0,\ x_2 \ge 0$. Solution: $(x_1, x_2) = (0, 8)$.

(b) $(y_1, y_2, y_3, y_4) = (\frac{1}{2}(b+1), 0, b, 0)$ for any b satisfying $0 \le b \le 1/5$.

(c) The maximand for the dual becomes $kx_1 + x_2$. The solution is unchanged provided that $k \le -1/2$.

3. (a) $x^* = 0$, $y^* = 4$. (A diagram shows that the solution is at the intersection of $x = 0$ and $4x + y = 4$.)

(b) The dual problem is

$$\max\ 4u_1 + 3u_2 + 2u_3 - 2u_4 \quad \text{subject to} \quad \begin{cases} 4u_1 + 2u_2 + 3u_3 - \ \ u_4 \leq 5 \\ \ \ u_1 + \ \ u_2 + 2u_3 + 2u_4 \leq 1 \end{cases}, \quad u_1, u_2, u_3, u_4 \geq 0$$

By complementary slackness, its solution is: $u_1^* = 1$, $u_2^* = u_3^* = u_4^* = 0$.

4. (a) See Fig. A17.R.4. The solution is at P, where $(x_1, x_2) = (2000, 2000/3)$; (b) See SM. (c) $a \leq 1/24$

5. (a) If the numbers of units produced of the three goods are x_1, x_2, and x_3, the profit is $6x_1 + 3x_2 + 4x_3$, and the times spent on the two machines are $3x_1 + x_2 + 4x_3$ and $2x_1 + 2x_2 + x_3$, respectively. The LP problem is therefore

$$\max\ 6x_1 + 3x_2 + 4x_3 \quad \text{subject to} \quad \begin{cases} 3x_1 + \ \ x_2 + 4x_3 \leq b_1 \\ 2x_1 + 2x_2 + \ \ x_3 \leq b_2 \end{cases}, \quad x_1, x_2, x_3 \geq 0$$

(b) The dual problem is obviously as given. Optimum at $P = (y_1^*, y_2^*) = (3/2, 3/4)$.
(c) $x_1^* = x_2^* = 25$. For (d) and (e) see SM.

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