Regular and inverse matrices

1. Let matrix A have odd numbers on the diagonal and even numbers off the diagonal. Can A be singular?

We can safely say, that our matrix A is a square matrix. That is because we call matrices singular (or regular) only when they are square.

To find our answer we can look at it from a different perspective. And that is:

Can A not be regular?

So let's set A to $2 \cdot 2$. And calculate for that. Maybe we can find a clue to finding our answer.

Let matrix A be (for now) defined as:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

As we can see, A is regular so we need to look elsewhere.

We can look at two attributes of regular matrices:

- Rows are not linearly dependant
- Columns are not linearly dependant

That means that i can't nullify one row or column completely with linear operations.

If:

$$k \in \mathbb{Z}, \quad 2|k$$

(I know it's not right but let's say that k is not a number but a list of numbers)

$$A = \begin{bmatrix} k+1 & k \\ k & k+1 \end{bmatrix}$$

We can say that these rows and columns are not dependant, because of the elements $k+1$. We can also say that if we keep increasing the size of A, this will stay the same, because in every new row and column, there will always be only one element equal to $k+1$. That means A will always be regular, which means **it can never be singular**.

(I have no idea how to prove this properly. I know this is not enough.)

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2. Let matrix $A \in \mathbb{R}^{n \times n}$ (where $n > 2$) have elements $a \in \mathbb{R}$ on the diagonal and elements $b \in \mathbb{R}$ off the diagonal. Determine for which values of a and b the matrix A is regular.

Lowest value of n is 3.

$$A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$$

Matrix is regular when:

- Rows are not linearly dependant
- Columns are not linearly dependant

That means that any linear operation within the matrix cannot create a zero vector, which means that the matrix was linearly dependant in some way.

If we subtract any row from the matrix from any row (linear operation) we get this row:

$$[0, \ldots, 0, a-b, 0, \ldots, 0, b-a, 0 \ldots]$$

Which means that if $a = b$: The whole row would only 0 (zero vector). So:

$$a \neq b$$

Every combination of $n-1$ rows will be containing a combination of these 3 elements:

$$0, a-b, b-a$$

But if we add every row (let's say to the first row) we get a row looking like this:

$$[a + b \cdot (n-1), a + b \cdot (n-1), \ldots, a + b \cdot (n-1)]$$

The whole row would be made up of a one element $a + b \cdot (n-1)$. That means for A to be regular:

$$a \neq b \cdot (n-1)$$

So A is regular when:

$$a \neq b \quad \text{and} \quad a \neq b \cdot (n-1)$$

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3. Determine for which values of n the matrix $A \in \mathbb{R}^{n \times n}$ is regular, given that its elements are defined as follows:
   - (a) $a_{ij} = i \cdot j$,
   - (b) $a_{ij} = i + j$.

a)

$$A = \begin{bmatrix} 1 \cdot 1 & 1 \cdot 2 & 1 \cdot 3 & \cdots & 1 \cdot n \\ 2 \cdot 1 & 2 \cdot 2 & 2 \cdot 3 & \cdots & 2 \cdot n \\ 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 & \cdots & 3 \cdot n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n \cdot 1 & n \cdot 2 & n \cdot 3 & \cdots & n \cdot n \end{bmatrix}$$

Every k-th row can be written as:

$$Row_k = Row_1 \cdot k$$

Which means every row and column after the first one are linearly dependant on the first. So matrix A is regular only when:

$$n = 1$$

b)

$$A = \begin{bmatrix} 1+1 & 1+2 & 1+3 & \cdots & 1+n \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ 3+1 & 3+2 & 3+3 & \cdots & 3+n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & n+3 & \cdots & n+n \end{bmatrix}.$$

If we subtract Row_{k-1} from Row_k then we get a row containing only number 1.
Since $k-1 = i, l-1 = j : a_{ij} - 1 = a_{kl}$. We can subtract the row containing only number 1 from the row i to get the same elements that are in row k. We need two rows to create the row containing only number 1 and another row to subtract from. So A can be regular only when:

$$n \leq 2$$

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4. Prove that for matrices $A, B \in \mathbb{R}^{n \times n}$, the following holds:

$$(ABA^{-1})^n = AB^nA^{-1}.$$

We can rewrite the left side of the equation:

$$ABA^{-1}ABA^{-1}ABA^{-1} \ldots ABA^{-1} = AB^nA^{-1}$$

From definition of inverse matrices:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

We can add brackets:

$$AB(A^{-1}A) \cdot B(A^{-1}A) \cdot B(A^{-1}A) \cdot \cdots \cdot BA^{-1} = AB^nA^{-1}$$

$$AB \cdot B^{n-2} \cdot BA^{-1} = AB^nA^{-1}$$

$$AB^nA^{-1} = AB^nA^{-1}$$

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5. Find a matrix $A \in \mathbb{R}^{n \times n}$ that has no zero elements and satisfies $A = A^{-1}$.

We can adjust our equation $A = A^{-1}$ to $A \cdot A = I$ if we multiply each side by A.
Identity matrix to power of n is still identity matrix so we can again rewrite the equation as:

$$A^2 = I^2$$

This doesn't mean that $A = I$, because square rooting both sides isn't an equivalent operation.
We can divide our problem into a set of problems. First let's look when:

$$A = I$$

$I^{n \times n}, n > 1$ contains number 0. So the only matrix that satisfies the assignment is $A \in \mathbb{R}^{1 \times 1}$, which is a scalar.
Now let's look when $A \neq I$.

Let's find a matrix for $n = 2$:

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}$$

$$a^2 + bc = 1$$

$$ab + bd = 0$$

$$ac + cd = 0$$

$$bc + d^2 = 1$$

There is not existing solution for these equations, when none of these parameters can be 0. I don't know how to prove it, but I would guess that it is same for larger matrices. So my answer: Matrix A doesn't exist.

6. Prove that $\mathrm{trace}(A) = \mathrm{trace}(BAB^{-1})$, where $\mathrm{trace}(M)$ is the sum of the diagonal elements of matrix M.

Trace is invariant to cyclic permutations of matrices.(Found on wikipedia) Here is the proof.

We will show that

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$$

for matrices A, B, and C.
By the definition of the trace, we have:

$$\mathrm{tr}(ABC) = \sum_{i=1}^{n} (ABC)_{ii}$$

- $$\mathrm{tr}(ABC) = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} (BC)_{ki}$$

- $$(BC)_{ki} = \sum_{j=1}^{n} B_{kj} C_{ji}$$

$$\mathrm{tr}(ABC) = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} \sum_{j=1}^{n} B_{kj} C_{ji}$$

Now, let's compute the trace of CAB:

- $$\mathrm{tr}(CAB) = \sum_{i=1}^{n} (CAB)_{ii}$$

- $$tr(CAB) = \sum_{i=1}^{n} \sum_{k=1}^{n} C_{ik}(AB)_{ki}$$
- $$(AB)_{ki} = \sum_{j=1}^{n} A_{kj}B_{ji}$$

Thus, the trace of CAB becomes:

$$\mathrm{tr}(CAB) = \sum_{i=1}^{n} \sum_{k=1}^{n} C_{ik} \left(\sum_{j=1}^{n} A_{kj}B_{ji} \right)$$

Now, compare the two sums for $\mathrm{tr}(ABC)$ and $\mathrm{tr}(CAB)$:

We can notice that these two expressions are similar, except for the names of the indices. We can manipulate the indices to show they are equal.
Thus, we conclude that:

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$$