Cauchy-Schwarz, Gram-Schmidt a ortogonalita (34 bodů)

1. Dokažte, že pro každé $a_1, a_2, a_3, a_4 \in \mathbb{R}$ platí:

$$5a_1 + a_2 + 3a_3 + a_4 \leq 6\sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}.$$

Podle Cauchy-Schwarzovy nerovnosti platí:

$$|\langle x, y\rangle| \leq ||x|| \cdot ||y||$$

Na levo tedy mame vlastne skalarni soucin vektoru a a vektoru $v = (5, 1, 6, 1)^t$.

$$x = a; y = v$$

Pod CS nerovnosti ziskavame:

$$|\langle a, v\rangle| \leq ||a|| \cdot ||v|| = \sum_{i=1}^{4} a_i v_i \leq \sqrt{\sum_{i=0}^{4} a_i^2} \cdot \sqrt{\sum_{i=0}^{4} v_i^2}$$

$$5a_1 + a_2 + 3a_3 + a_4 \leq \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} \cdot \sqrt{5^2 + 1 + 3^2 + 1}$$

$$5a_1 + a_2 + 3a_3 + a_4 \leq \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} \cdot 6$$

Mame dokazano.

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2. **(4 body)** Dokažte vztah mezi aritmetickým a kvadratickým průměrem.

Aritmeticky prumer A:

$$A = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Kvadraticky prumer K:

$$B = \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2}$$

A vztah mezi nimi je definovan jako:

$$A \leq B$$

Opet vyuzijeme CS nerovnost.

A muzeme prepsat jako:

$$A = \sum_{i}^{n} \frac{1}{n} x_i$$

coz je skalarani soucin mezi vektorem x a y, kde y ma na vsech pozicich $\frac{1}{n}$.

Dosadme do CS:

$$|\langle x, y \rangle| \leq ||x|| \cdot ||y||$$

$$\sum_{i}^{n} \frac{1}{n} x_i \leq \sqrt{\sum_{i}^{n} x_i^2} \cdot \sqrt{\sum_{i}^{n} \frac{1}{n^2}}$$

$$\sum_{i}^{n} \frac{1}{n} x_i \leq \sqrt{\sum_{i}^{n} x_i} \cdot \sqrt{n \cdot \frac{1}{n^2}} = \sqrt{\sum_{i}^{n} x_i} \cdot \sqrt{\frac{1}{n}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2}$$

$$A \leq B$$

Pravou stranu jsme upravili tak, aby se rovnala B. Tvrzeni je dokazano.

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3. **(12 bodů)** Stopa čtvercové matice je definována jako:

$$\mathrm{trace}(A) = \sum_{i} a_{ii}.$$

Ukažte, že platí:

- **(4 body)** $\mathrm{trace}(A)^2 \leq n \cdot \mathrm{trace}(A^T A)$,

$B = A^T A$

$$\left(\sum_{i} a_{ii} \right)^2 \leq n \cdot \sum_{i} b_{ii}$$

Nejdrive rozepiseme hodnoty pro prvky v B:

$$b_{ij} = \sum_{k=1} a_{ik}^T a_{kj} = \sum_{k=1} a_{ki} a_{kj}$$

$$b_{ii} = \sum_k a_{ki}^2$$

Dosadime zpatky:

$$\sum_i a_{ii} \cdot \sum_j a_{jj} \leq n \cdot \sum_i \sum_k a_{ki}^2$$

$$(\mathrm{trace}(A))^2 \leq n \sum_i \sum_k a_{ki}^2$$

Dosadime do CS kde $a = \{a_{11}, \ldots, a_{nn}\}$ a $y = \{1, 1, \ldots, 1\}$.

$$|\langle a, y\rangle| \leq ||a|| \cdot ||y||$$

$$|\mathrm{trace}(A)| \leq \sqrt{\sum_{i=1}^n a_{ii}^2} \cdot \sqrt{n}$$

$$|\mathrm{trace}(A)| \leq \sqrt{n \sum_{i=1}^n a_{ii}^2}$$

Protoze CS muzeme upravit na:

$$|\langle x, y\rangle|^2 \leq ||x||^2 \cdot ||y||^2$$

Plati:

$$(\mathrm{trace}(A))^2 \leq n \sum_{i=1}^n a_{ii}^2$$

Ted musime dokazat ze:

$$n \sum_{i=1}^n a_{ii}^2 \leq n \sum_{i=1}^n \sum_{k=1}^n a_{ki}^2$$

$$\sum_{i=1}^n a_{ii}^2 \leq \sum_{i=1}^n a_{ii}^2 + \sum_{j=1}^n \sum_{k \neq j}^n a_{ki}^2$$

$$0 \leq \sum_{j=1}^n \sum_{k \neq j}^n a_{ki}^2$$

Tohle vždy bude platit, protože sčítáme druhé mocniny reálných čísel, které jsou vždy větší rovny nule.

Tedy:

$$(\text{trace}(A))^2 \leq n \sum_{i=1}^{n} a_{ii}^2 \leq n \cdot \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2 = n \cdot \text{trace}(A^T A)$$

- **(4 body)** $\text{trace}(A^2) \leq \text{trace}(A^T A)$,

$$B = A^2; b_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

$$b_{ii} = \sum_{k=1}^{n} a_{ik} a_{ki}$$

Ziskavame:

$$\text{trace}(A^2) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} a_{ki}$$

$$\text{trace}(A^T A) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} a_{ki} \leq \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2$$

Aplikujeme CS na skalarni soucin radkoveho a sloupcoveho prostoru :

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} a_{ki} \leq \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2} \cdot \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2}$$

Prohodime indexy na leve strane. Timto nic nezmenime:

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} a_{ki} \leq \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2} \cdot \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2}$$

$$\text{trace}(A^2) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} a_{ki} \leq \sqrt{\left(\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2\right)^2} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2 = \text{trace}(A^T A)$$

- **(4 body)** $\text{trace}(A^T B) \leq \frac{1}{2}(\text{trace}(A^T A) + \text{trace}(B^T B))$.

Vzorce pro vsechny stopy v prikladu:

$$\operatorname{trace}(A^T A) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2$$

$$\operatorname{trace}(A^T B) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ik}$$

$$\operatorname{trace}(B^T B) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}^2$$

Víme že:

$$(a-b)^2 \geq 0$$

$$a^2 - 2ab + b^2 \geq 0$$

$$\frac{a^2 + b^2}{2} \geq ab$$

A to platí pro $\forall a, b \in \mathbb{R}$.

Tedy pokud si rozepiseme nerovnici, kterou se snazime dokazat:

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ik} \leq \frac{1}{2} (\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2 + \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}^2)$$

Muzeme nahlednout ze pro kazdy par $i, k \text{ kde } i, k \in [n]$ plati tato nerovnost.
Tedy

$$a_{ik} b_{ik} \leq \frac{1}{2} (a_{ik}^2 + b_{ik}^2)$$

Jelikoz to plati pro vsechny i, k, muzeme napsat:

$$\operatorname{trace}(A^T B) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ik} \leq \frac{1}{2} (\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2 + \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}^2) = \frac{1}{2} (\operatorname{trace}(A^T A) + \operatorname{trace}(B^T B))$$

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4. **(10 bodů)** Buď

$$v_1 = (1, 1, 0)^T, \quad v_2 = (1, 1, 1)^T.$$

- **(3 body)** Ortonormalizujte vektory v_1, v_2.

Vytvorime ortonormalni vektor a_1 z vektoru v_1:

$$a_1 = \frac{v_1}{||v_1||} = v_1 \cdot \frac{1}{\sqrt{2}}$$

K ziskani a_{2*} potrebujeme:

$$a_{2*} = v_2 - proj_{a_1}(v_2)$$
$$proj_{a_1}(v_2) = \frac{\langle v_2, a_1 \rangle}{||a_1||} \cdot a_1$$

Skalarni soucin:

$$\langle v_2, a_1 \rangle = \frac{1+1}{\sqrt{2}} = \sqrt{2}$$
$$proj_{u_1}(v_2) = \frac{\langle v_2, a_1 \rangle}{||a_1||} = \frac{\sqrt{2}}{1} \cdot \frac{1}{\sqrt{2}} \cdot v_1$$

Tedy:

$$a_{2*} = v_2 - v_1 = (0,0,1)^T$$

Normalizujeme a_2* :

$$a_2 = \frac{a_{2*}}{||a_{2*}||} = a_{2*} \cdot \frac{1}{1} = (0,0,1)^T$$

Vysledne vektory.

$$a_1 = \frac{1}{\sqrt{2}} u_1; a_2 = (0,0,1)^T$$

- **(3 body)** Proveďte ortonormalizaci v opačném pořadí vektorů.

Vytvorime ortonormalni vektor a_2 z vektoru v_2:

$$a_2 = \frac{v_2}{||v_2||} = \frac{1}{\sqrt{3}} v_2$$

K ziskani a_{1*} potrebujeme:

$$a_{1*} = v_1 - proj_{a_2}(v_1)$$

$$proj_{a_2}(v_1) = \frac{\langle v_1, a_2 \rangle}{||a_2||} \cdot a_2$$

Skalarni soucin:

$$\langle v_1, a_2 \rangle = \frac{2}{\sqrt{3}}$$

$$proj_{a_2}(v_1) = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} v_2 = \frac{2}{3} v_2$$

Tedy:

$$a_{1*} = v_1 - \frac{2}{3} v_2 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right)$$

Normalizujeme a_{1*} :

$$a_1 = \frac{a_{1*}}{||a_{1*}||} = \frac{a_{1*}}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}}} = a_{1*} \cdot \frac{3}{\sqrt{6}}$$

Vysledne vektory:

$$a_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right)^T; \quad a_2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)^T$$

- **(4 body)** Najděte projekci $x = (0,1,1)^T$ do podprostoru $U = \text{span}\{v_1, v_2\}$. Jaká je vzdálenost x od U?

$$a_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right); a_2 = (0,0,1)^T$$

$$proj_U(x) = \langle x, a_1 \rangle \cdot a_1 + \langle x, a_2 \rangle a_2$$

$$= \frac{1}{\sqrt{2}} \cdot a_1 + a_2$$

$$proj_U(x) = \left(\frac{1}{2}, \frac{1}{2}, 1 \right)^T$$

Vzdalenost je dana normou rozdilu x a projekce x do U.

$$||x - proj_U(x)|| = \sqrt{\frac{1}{4} + \frac{1}{4} + 0} = \sqrt{\frac{1}{2}}$$

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5. **(4 body)** Zortonormalizujte bázi podprostoru $\mathbb{R}^4$ popsaného soustavou:

$$x - y + u + v = 0, \quad x + u = 0.$$

Ziskavame:

$$x = -u; \text{ A tedy: } y = v$$

Soustavu muzeme zapsat pomoci vektoru:

$$(x, y, u, v) = (-u, y, u, y) = u(-1, 0, 1, 0)^T + y(0, 1, 0, 1)^T$$

To jsou dva vektory, ktere generuji tento podprostor.

Normalizujeme vektor $u = (-1, 0, 1, 0)^T$:

$$a_1 = \frac{u}{||u||} = u \cdot \frac{1}{\sqrt{2}}$$

K ziskani a_{2*} z vektoru $y = (0, 1, 0, 1)^T$potrebujeme:

$$a_{2*} = y - proj_{a_1}(y)$$

$$proj_{a_1}(y) = \frac{\langle y, a_1 \rangle}{||a_1||} \cdot a_1$$

$$\langle y, a_1 \rangle = 0$$

To znamena, ze je jiz ortogonalni protoze y je kolme na a_1,
Tedy $a_{2*} = y$

Normalizace y :

$$a_2 = \frac{y}{||y||} = y \cdot \frac{1}{\sqrt{2}}$$

Vysledne vektory:

$$a_1 = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right); a_2 = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$