Introductory Analysis

Proof Techniques

Proof of uniquesness

Assume two seperate elements with the requiered property and than proove that they must be equal to eachother.

Proof by contradiction

Assume that a false stament is true and show that this results in a contradiction

Reals 1

Axioms of the reals

The real numbers are a field which also has ordered and diedekin complete

Field A set F of with two opperators $+$ and $\cdot$ such that $\forall f_1, f_2, f_3 \in F$

- Addition
  - **(A1)** $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$ Addition is associative
  - **(A2)** $f_1 + f_2 = f_1 + f_2$ Addition is comutitive
  - **(A3)** $\exists 0 | f_1 + 0 = f_1$ Indentity of addition
  - **(A3)** $\exists - f_1 | f_1 + (-f_1) = 0$
- Multiplication
  - **(M1)** $(f_1 f_2) f_3 = f_1 (f_2 f_3)$ Multiplication is associative
  - **(M2)** $f_1 f_2 = f_1 f_2$ Multiplication is comutitive
  - **(M3)** $\exists 1 \neq 0 | f_1 \dot{1} = f_1$ Indentity of addition
  - **(M4)** $(\forall f_1 \in F | f_1 \neq 0) \exists f_1^{-1} | f_1 (f_1^{-1}) = 1$
- Distrabution
  - **(D1)** $f_1 (f_2 + f_3) = f_1 f_2 + f_1 f_2$ multiplication is distributive over addition

  Axioms of comparison

- Order
  - **(O1)** The trichotomy property of reals.
    - $(\forall r \in \mathbb{R})$ exactly one is true
    - $a > 0 \quad a = 0 \quad a < 0$

Let $r_1 r_2 \in \mathbb{R}$ then r_1 is called larger than r_2 written $r_1 > r_2$ if $r_1 - r_2 > 0$

- **(O2)** $a > 0 \wedge b > 0 \quad \rightarrow \quad a + b > 0$
- **(O3)** $a > 0 \wedge b > 0 \rightarrow ab > 0$

Diedekin completeness axiom

- Completeness
  - **(C)** All bounded sets have superium and infamium

Find the defintion of a monotonic transformation

Collararies

b) $a < b \wedge b < c \rightarrow a < c$ transitivty

sort of

- $r_1 \geq b \Leftrightarrow a > b \wedge a = b$
- $a \leq b \Leftrightarrow a = b \wedge a < b$

Proof

$b > a \Leftrightarrow b - a \geq 0$ **(i)**
$b - c \geq 0$ **(ii)**
combining i and ii with previous theorem $(b - a) + c - b \geq 0$ $a - c \geq 0$
(addition axioms) $c \geq 0$

Proof by contradiction $a > 0 \Leftrightarrow a^{-1} > 0 \wedge a < 0 \Leftrightarrow a^{-1} < 0$

Case $a > 0$
(i) $a \neq 0$ **Trichotomy**
(ii) $\exists a^{-1}$ **(M3)** $\wedge$ **(i)**
Case $a^{-1} < 0$

1. $-a^{-1} > 0$ **(a)**
2. $-a^{-1} \cdot a > 0$ **(O3)**
3. $-1 > 0$ **Contradiction**

Case $a^{-1} = 0 > \#.\ 1 = a \cdot a^{-1}$ **(M3)** $> \#. = 0$ **(Multiplication by zero) Contradiction**
Conculsion $a > 0 \Leftrightarrow a^-1 > 0$
Case $a < 0$
(i) $a \neq 0$ **Trichotomy**
(ii) $\exists a^{-1}$ **(M3)** $\wedge$ **(i)**
Case $a^{-1} = 0 > \#.\ 1 = a \cdot a^{-1} = 0$ **Contradiciton (Multiplication by zero) (M3)**
Case $a^{-1} > 0 > \#.\ -1 = -aa^{-1} > 0$

$1 > 0$ $a \neq 0 \rightarrow a^2 > 0$ $usethistoshow 1 > 0$

Absolute value

Absolute value

$$|a|\forall a \in \mathbb{R} := \begin{cases} a \geq 0 & = a \\ a < 0 & - a \end{cases}$$

PROPERTIES

- $|a| = 0 \Leftrightarrow a = 0$
  - if $a = 0 \rightarrow |a| = 0$ deffinition
  - if $|a| = 0 \rightarrow a = 0$ contrapositive
- $b| - a| = | - a|$
- use each three cases of dichotomy
- try d and e tutorial questions

Trainagle inequality

$||a| + |b|| \geq |a| + |b|$

- $|a + b| \leq |a| + |b| tut$
- $||a| - |b| \leq |a + b|$

proof

- $a = a - b + b \rightarrow |a| = |a - b + b| = |a - b| + |b|$
- $b = b - a + a \rightarrow |b| = |b - a + a| = |b - a| + |a|$
- by c $|b| \leq |a - b| + |a| \rightarrow |b| \leq |b - a|$
- combining
- multiply ** -1 **
- $|a - b| \leq |a| - |b| - |a - b| \leq |a| - |b| \leq a - b \mid$ by d
- $\Leftrightarrow$ triangle inequility to proove with minus just sub in $-b$

Posittive Square Root

$$\forall (r \in \mathbb{R} | r \geq 0) \quad \exists \left(r^{\frac{1}{2}} \in \mathbb{R} | r^{\frac{1}{2}} \geq 0 \wedge \left(a^{\frac{1}{2}}\right)^2 = a \right) \text{ and } r^{\frac{1}{2}} \text{ is unique}$$

PROOF

- **Case** $r = 0$

  $0^2 = 0 \quad \wedge \quad 0 \geq 0 \quad \wedge \quad 0 \text{ is unique} \quad \Rightarrow \quad r^{\frac{1}{2}} = 0$

- **Case** $r > 0$
  - $\exists r^{\frac{1}{2}}$
    - $\exists S_s \mid S_s = \sup S$ **(lemma 1, lemma 2, C)**
    - $\sup S^2 = r$ **(lemma3)**

$S = \{x \in \mathbb{R} \mid 0 < x, x^2 < 0\}$

- $r^{\frac{1}{2}}$ is unique**(lemma 4)**

lemma 1 $S \neq \emptyset$

- **Case** $a < 1$
  - $0 < a, a^2 < a \Rightarrow a \in S$
- **Case** $a \geq 1$
  - $0 < \frac{1}{2}, \frac{1}{2}^2 < a \Rightarrow \frac{1}{2} \in S$

lemma 2 S is bounded

- $a + 1$ is a bound
  - $(a+1)^2$ is an upperbound of S
    - $x > a + 1 \Rightarrow x^2 > a^2 + 2a + 1 > 2a > a \Rightarrow xS$

lemma 3 $S_s^2 = r$

Assume $S_s^2 \neq r$

- **Case** $S_s^2 < a$
  - **Contradiction** $S_s + \epsilon \in S$ for some $\epsilon > 0$ **(lemma 3.1)**
- **Case** $S_s^2 < 0$
  - **Contradiction**

    $S_s - \epsilon$ is an upperbound of S for some $\epsilon > 0$ **(lemma 3.2)**

lemma 3.1

- Let $\epsilon = \min\left\{\frac{r - S_s}{4S_s}, S_s\right\}$

$$\begin{aligned}(S_s + \epsilon)^2 - a &= S_s^2 - a + 2(S_s\epsilon + \epsilon)\epsilon \\ &\leq S_s - a + (2S_s + S_s)\frac{r - S_s^2}{4S_s} \\ &\leq \frac{1}{4}(S_s^2 - r) < 0\end{aligned}$$

- $(S_s + \epsilon)^2 < a$

lemma 3.2

- Let $\epsilon = \dfrac{S_s^2 - r}{2S_s}$

$$\begin{aligned}
(S_s - \epsilon)^2 - a &= S_s^2 - 2S_s\epsilon - \epsilon^2 - a \\
&> S_s^2 - 2S_s\epsilon \\
&> 0 \textbf{ (Sub in } \epsilon \textbf{ and solve)} \\
(S_s - \epsilon)^2 &\$> \text{a}
\end{aligned}$$

- **Contradiction** $S_s - \epsilon$ is an upperbound less than S_s

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lemma 4
(i) $x^2 = r \quad y^2 = r \quad \wedge \quad r_1, r_2 > 0$
(ii) $0 = r - r = x^2 - y^2 = (x + y)(x - y)$
(iii) $x + y > 0$ **(i)**
(iv)$x - y = 0$ **(iii, ii)**
Conclusion $x = y$ So the posisitve square root is unique **(iv)**

Glossary

Bound The bound of a set is a number such that all numbers in the set are less than or greater than the set. A bound is not unique.
Supremum The supereme or infamium of a set the least upper bound or the greatest lower bound and is unique
Maximum and miniumium If the superium or infaium are elements of the set than they are the max and miuminium otherwise there is no maxium or mimiumum

Absolute value

$$|a| \forall a \in \mathbb{R} := \begin{cases} a \geq 0 & = a \\ a < 0 & - a \end{cases}$$