

Multi Variable Culculus

Vector function¹

¹ Row and column vectors are treated interchangeably in the coarse

$$\vec{F} :: R^n \rightarrow R^m$$

$$\text{Domain} \subseteq R^n$$

$$\text{Range} \subseteq R^m$$

Continuity²

² All functions in the coarse are continuous

\vec{F} is continuous $\forall \vec{a}$ if the limit exists and is equal to the function for all \vec{a}

Limit:

The limit of a vector function $\vec{F}(\vec{X})$ is defined as

$$\lim_{\vec{x} \rightarrow \vec{a}} \vec{F}(\vec{x}) = \vec{l}$$

if for each $\epsilon > 0 \exists \delta > 0$

such that $0 < \|\vec{x} - \vec{a}\| < \delta$

$$\Leftrightarrow \|\vec{F}(\vec{l})\| < \epsilon$$

Scalar triple product

$$\vec{x} \cdot (\vec{y} \times \vec{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Projection

$$\text{proj}_a b = \frac{a \cdot b}{|a|^2} a$$

Components

$$\frac{a \cdot b}{|a|}$$

Do show and triangle inequality

Parametric form:

An equation is in parametric form if it is of the type $\vec{F} :: \mathbb{R} \rightarrow \mathbb{R}^n$
for a circle:

$$f(t) = \begin{pmatrix} r \cos(t) \\ r \sin(t) \end{pmatrix}$$

Continuity:

$$\vec{F} :: \mathbb{R} \rightarrow \mathbb{R}^n = \{f_i(t)\}_{i=1}^n \quad |\forall i \in [1, n] \ f_i :: \mathbb{R} \rightarrow \mathbb{R}$$

$$\vec{F} \text{ is continuous} \iff \forall i \in [1, n] \ f_i \text{ is continuous}$$

Differentiates of vector functions³

³ **Definition** := (is defined as)

Differentiates of $\vec{F} : \mathbb{R} \rightarrow \mathbb{R}^n$

$$\begin{aligned} \text{RHS} \quad \vec{F}' &= \lim_{a \rightarrow 0} \left(\frac{1}{a} \left(\vec{F}(x+a) - \vec{F}(x) \right) \right) \\ &= \lim_{a \rightarrow 0} \left(\frac{1}{a} \begin{pmatrix} f_1(x+a) - f_1(x) \\ \vdots \\ f_n(x+a) - f_n(x) \end{pmatrix} \right) \\ &= \lim_{a \rightarrow 0} \begin{pmatrix} \frac{f_1(x+a) - f_1(x)}{a} \\ \vdots \\ \frac{f_n(x+a) - f_n(x)}{a} \end{pmatrix} \\ &= \begin{pmatrix} \lim_{a \rightarrow 0} \left(\frac{f_1(x+a) - f_1(x)}{a} \right) \\ \vdots \\ \lim_{a \rightarrow 0} \left(\frac{f_n(x+a) - f_n(x)}{a} \right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{df_1(x)}{dx} \\ \vdots \\ \frac{df_n(x)}{dx} \end{pmatrix} \end{aligned}$$

Differentiates ⁴ of $f : \mathbb{R}^n \rightarrow \mathbb{R}$

⁴ functions resulting a scalar are not written as vectors!

The differentiate of f where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called the gradient of f and is written ∇f

$$\nabla f := (f_a, \dots, f_n)$$

Where f_x is $\frac{\partial f(a, \dots, n)}{\partial x}$ and $(a \dots n)$ are the arguments of f

Differentiates of $\vec{F} : \mathbb{R}^m \rightarrow \mathbb{R}^n$

$$\vec{F}(x_1 \dots x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$$

$$\Rightarrow \vec{F}'(x_1 \dots x_n) = \begin{pmatrix} \nabla f_1(x_1, \dots, x_n) \\ \vdots \\ \nabla f_m(x_1, \dots, x_n) \end{pmatrix}$$

Product rules

$\vec{F}, \vec{G} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$

- (i) $(\vec{F} \cdot \vec{G})' = \vec{F}'^T \vec{G}' + \vec{G}'^T \vec{F}'$
- (ii) $a(\vec{F} + b\vec{G})' = \vec{F}' + b\vec{G}' \forall a, b \in \mathbb{R}$
- (iii) $(f\vec{F})' = f\vec{F}' + \vec{F}f'$

$$\vec{F}, \vec{G} : \mathbb{R} \rightarrow \mathbb{R}^3$$

- (iv) $(F \times G)' = F \times G' + F' \times G$

The proofs are all simple evaluation of the individual components of each part.

Properties of the derivatives

- ∇ is linear, distributive over addition and commutative with multiplication $\nabla(fg) = g\nabla f + f\nabla g$ **Chain Rule**
- $\nabla \cdot (gF) = (\nabla g) \cdot F + g\nabla F$
- Same is true for cross product for \mathbb{R}_3 to \mathbb{R}_3
- $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - (\nabla \times G) \cdot F$ Partial derivatives are commutative
- for $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
 - $\nabla f \equiv 0$
 - $\nabla \cdot (\nabla F) = 0$

Tangents

To FIND the tangent of a equation.

- Take the vector equation of a parametric equation $\vec{F}(t)$ at t_0
- The point of the line is given by $\vec{F}(t_0)$
- The Direction is given by the derivative of the function times some vector or parameter u $T(u) = \vec{F}(t_0) + d\vec{F}(t_0; u) = \vec{F}(t) + \vec{F}' \cdot u$

get better wording

Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\vec{f}(\vec{x} + \vec{h}) = f(x) \text{ Try find another explanation or consult}$$

Full Notation

Definition

$:=$ (is defined as)

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

$$\nabla \vec{F} := \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \text{ This is only defined for } \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\text{div} \vec{F} := \nabla \cdot \vec{F} \text{ This is defined } (a)_4$$

$$\text{Laplacian } \nabla^2 f = \nabla \cdot \nabla f = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}$$

$$\text{Curl for } \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ curl } \vec{X} = \nabla \times \vec{F}$$