Multi Veriable Culculus

Vector function[1]

[1] Row and column vectors are treated interchangably in the coarse

$\vec{F} :: R^n \to R^m$

Domain $\subseteq R^n$

Range $\subseteq R^m$

Continuity[2]

[2] All functions in the coarse are contin-ues

$\vec{F}$ is continues $\forall \vec{a}$ if the limit exists and and is equal to the function for all a$

Limit:

The limit of of a vector function $\vec{F}(\vec{X})$ is defined as

$\lim_{\vec{x} \to \vec{a}} \vec{F}(\vec{x}) = \vec{l}$

if for each $\epsilon > 0 \exists \delta > 0$

such that $0 < ||\vec{x} - \vec{a}|| < \delta$

$\Leftrightarrow ||\vec{F}(\vec{l})|| < \epsilon$

Scalar triple product

$$\vec{x} \cdot (\vec{y} \times \vec{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Projection

$$\text{proj}_a b = \frac{a \cdot b}{|a|^2} a$$

Componenets

$$\frac{a \cdot b}{|a|}$$

Do shwats and traingle inequilty

Parametric form:

An equation is in parametric form if it is of the type $\vec{F} :: \mathbb{R} \to \mathbb{R}^n$

for a circle:

$$f(t) = \begin{pmatrix} r\cos(t) \\ r\sin(t) \end{pmatrix}$$

Continuity:

$\vec{F} :: \mathbb{R} \to \mathbb{R}^n = \{f_i(t)\}_{i=1}^{n} \quad |\forall i \in [1, n]\ f_i :: \mathbb{R} \to \mathbb{R}$

$\vec{F}$ is continious $\iff \forall i \in [1, n]\ f_i$ is continious

Differentiates of vector functions[3]

[3] **Definition** := (is defined as)

Differentiates of $\vec{F} : \mathbb{R} \to \mathbb{R}^n$

$$\begin{aligned}
&\text{RHS} && \text{LHS} \\
&\vec{F}' && = \lim_{a \to 0} \left(\frac{1}{a} \left(\vec{F}(x+a) - \vec{F}(x) \right) \right) \\
& && = \lim_{a \to 0} \left(\frac{1}{a} \begin{pmatrix} f_1(x+a) - f_1(x) \\ \vdots \\ f_n(x+a) - f_n(x) \end{pmatrix} \right) \\
& && = \lim_{a \to 0} \begin{pmatrix} \dfrac{f_1(x+a) - f_1(x)}{a} \\ \vdots \\ \dfrac{f_n(x+a) - f_n(x+a)}{a} \end{pmatrix} \\
& && = \begin{pmatrix} \lim_{a \to 0} \left(\dfrac{f_1(x+a) - f_1(x)}{x} \right) \\ \vdots \\ \lim_{a \to 0} \left(\dfrac{f_n(x+a) - f_n(x)}{x} \right) \end{pmatrix} \\
& && = \begin{pmatrix} \dfrac{df_1(x)}{dx} \\ \vdots \\ \dfrac{df_n(x)}{dx} \end{pmatrix}
\end{aligned}$$

Differenetiates [4]of $f : \mathbb{R}^n \to \mathbb{R}$

[4] functions resulting a scaler are not written as vectors!

The differentiate of f where $f : \mathbb{R} \to \mathbb{R}^n$ is called the gradient of f and is written ∇f

$\nabla f := \left(f_a, \cdots, f_n \right)$

Where f_x is $\dfrac{\partial f(a,\cdots n)}{\partial x}$ and $(a\cdots n)$ are the arguments of f

Differentaites of $\vec{F}:\mathbb{R}^m \to \mathbb{R}^n$

$$\vec{F}(x_1\cdots x_n) = \begin{pmatrix} f_1(x_1,\cdots x_n) \\ \vdots \\ f_m(x_1\cdots x_n)\end{pmatrix}$$

$$\Rightarrow \vec{F}'(x_1\cdots x_n) = \begin{pmatrix} \nabla f_1(x_1,\cdots x_n) \\ \vdots \\ \nabla f_m(x_1\cdots x_n)\end{pmatrix}$$

Product rules

$\vec{F},\vec{G}:\mathbb{R}^n \to \mathbb{R}^m$ and $f:\mathbb{R}^n \to \mathbb{R}$

- **(i)** $(\vec{F}\cdot\vec{G})' = \vec{F}^T\vec{G}' + \vec{G}^T\vec{F}'$
- **(ii)** $a(\vec{F}+b\vec{G}' = \vec{F})' + b\vec{G}'\forall a,b\in\mathbb{R}$
- **(iii)** $(f\vec{F})' = f\vec{F}' + \vec{F}f'$

  $\vec{F},\vec{G}:\mathbb{R}\to\mathbb{R}^3$

- **(iv)** $(F\times G)' = F\times G' + F'\times G$

  The proofs are all simple evalution of the indivdiual components of each part.

Properties of the derivatives

- $ ∇ $islineariedistrabutiveoveradditionandcommutitivewithmultiplication\nabla(fg) = g\nabla f + f\nabla g$ **Chain Rule**
- $ $\nabla\cdot$ $islinearoverRntoRmfunctions$\nabla\cdot(gF) = (\nabla g)\cdot F + g\nabla F$$
- Same is true for cross product for R3 to R3
- $\nabla\cdot(\mathbf{FG} = (\nabla\times\mathbf{F})\cdot\mathbf{G} - (\nabla\times\mathbf{G}\cdot F))Partialderivativesarecomutitive$
- for $f :: \mathbf{R^3} \to mathbb[R]$ and {F :: Rˆ3 → ℝ3}
  - $\nabla f \equiv 0$
  - $\nabla\cdot(\nabla F) = 0$

Tangents

To FIND the tangent of a equation.

- Take the vector equation of a parametric equation $\vec{F}(t)$ at t_0
- The point of the line is given by $\vec{F}(t_0)$
- The Direction is given by the derivative of the function times some vector or perameter u $T(u) = \vec{F}(t_0) + d\vec{F}(t_0; u) = \vec{F}(t) + \vec{F}' \cdot u$

get better wording

Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\vec{f}(\vec{x} + \vec{h}) = f(x)$ Try find another explaination or consult

Chain rule

Vector chain rule

For $f :: \mathbb{R}^n \to R$ and $\vec{G} :: \mathbb{R} \to \mathbb{R}^n$ $(f \circ \vec{G}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}_{\vec{G}(t)} \frac{\partial g_i}{\partial dt}_t$ **General Chain Rule**

For $\vec{R} :: \vec{R}^m \to \vec{R}^n$ and $\vec{G} :: \mathbb{R}^l \to \mathbb{R}^m$ (F $\circ G)x = (F(G(x))) \times Gx$

Directional deravitives

The directional derivative is defined as

$$D_u f(x) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} | \forall f :: \mathbb{R}^n \ to \mathbb{R}$$

This is equivelant to

$$D_u f(x) = u \cdot \nabla f(x).$$

Results

The direction of the maxiumium rate of increase of f is ∇f and the rate of increase in this direction is $||\nabla f(x)||$.
The direction of the miniumium rate of increase of f is $-\nabla f$ and the rate of increase in this direction is ||-nabla f(x)|| # Normals

Maxima and minima

Full Notation

Definition

:= (is defined as)

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

$$\nabla \vec{F} := \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \textbf{ This is only defined for } \mathbb{R}^n \to \mathbb{R}$$

$\text{div}\vec{F} := \nabla \cdot \vec{F}$ **This is definied** $(a)_4$

Laplacian $\nabla^2 f = \nabla \cdot \nabla f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$

Curl for $: R^3 $\to$ **ℝ3curl**$\vec{X} = \nabla \times \vec{F}$