MAT9004 Assignment 1

Due at the **end** of week 4.

1. A beekeeper estimates that the number of bees in a particular hive at the start of 2023 is 500, and that at the start of any subsequent day the number of bees in the hive has increased by 1% compared to what it was exactly one day earlier.

   (a) Find a function $b : \mathbb{R} \to (0, \infty)$ such that $b(t)$ models the population of the hive in the sense that $b(t)$ agrees with the beekeeper's estimates whenever $t \in \mathbb{N}$. Here $t \in \mathbb{R}$ measures the number of days since the start of the year (e.g. $t = 1/2$ is noon on New Year's day). Similarly, you should consider the population of bees to be a gradually growing real number (ignore the fact that it is actually restricted to being an integer). **[2]**

   (b) Explain why the inverse function b^{-1} exists. **[2]**

   (c) Calculate a formula for b^{-1}. **[2]**

   (d) The function b can be described as taking the time since the start of this year as input and outputting the population of bees in the hive at that time. Provide a similar description for what the function b^{-1} does. **[2]**

   (e) Find the derivative $b'(t)$. Explain what $b'(t)$ is measuring by giving a description of the same type that you gave in part (d). **[2]**

   (f) On which date does the population of bees in the hive reach 10000? Suppose that at the end of that day all but 1000 of the bees leave the hive to start a new colony elsewhere. **[2]**

   (g) Write a function $c : [0, 365] \to (0, \infty)$ which gives the population of bees in the hive throughout 2023, given the migration described in (f). Assume that the population continues to increase at the same rate (1% per day) after the migration. **[2]**

   (h) What is the population of bees in the hive at the end of the year? **[2]**

2. READIT is a vendor of e-books. They provide recommendations to customers based on ratings by other customers. Each customer rates each book that they read, by giving a star rating from 1 star to 5 stars (there are no "half" stars). Suppose READIT wants to recommend books to a customer called Wendy. For each customer X other than Wendy, they first calculate

$$w(X) = \begin{cases} \frac{n}{10} & \text{if } n < 10, \\ 1 & \text{if } n \geqslant 10, \end{cases}$$

   where n is the number of books that have been read (and rated) by both X and Wendy. Now for a book B that Wendy has not read they calculate a score $s(B)$ by the formula

$$s(B) = \frac{\sum_{X \in C} w(X) r(X)}{\sum_{X \in C} w(X)} \qquad (*)$$

   where C is the set of customers that have read B, and $r(X)$ is the rating that X gave to B when they read it. READIT recommends the books with the highest score to Wendy.

(a) Under some circumstances it will not be possible to calculate the score $s(B)$ using the formula $(*)$. In order for $(*)$ to produce a real number as an answer, a particular set D of customers must be non-empty. What is D? **[2]**

(b) Suppose that there is only one customer in the set D. What are the possible values of $s(B)$ in this situation? **[2]**

(c) Assuming that D is not empty, what are the minimum possible value and the maximum possible value for $s(B)$? **[2]**

(d) Explain why READIT uses the function $w(x)$ rather than just averaging the values of $r(X)$. **[2]**

3. The Laberal party are closely tracking their popularity through a 3 week election campaign. They find that on day d of the campaign the percentage of people who would vote for them is $v(d) = \frac{1}{200}(d^4 - 40d^3 + 594d^2 - 3888d + 13480)$. Consider v as a function of real values, $v : [0, 21] \to \mathbb{R}$.

(a) What are the domain and codomain of v? **[1]**

(b) Find all stationary points of v. **[2]**

(c) Classify each stationary point of v as a local minimum, local maximum or neither. **[2]**

(d) What is the range of v? **[2]**

(e) Over which intervals in its domain is v a convex function? **[2]**