# MAT9004 Assignment 2 — Solutions

**1. Energy consumption.**

The typical daily energy use of a household is described by a function $f : [-12, 12] \to [0, \infty)$. The value of $f(x)$ is the consumption rate at given time $x$. Both points $x = -12$ and $x = 12$ of the domain correspond to 3am so $f(12) = f(-12)$. It is also known that 9am and 9pm ($x = -6$ and $x = 6$, respectively) are local maxima of the consumption rate. In this question we model the rate by a polynomial of degree 4 that is $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and assume that the total energy consumption during the day equals 24.

(a) Write a linear system of equations describing the following properties: $f(12) = f(-12)$, $x = -6, 6$ are stationary points of $f$ and the total energy consumption is 24. **[4]**

**ANS:** Equation $f(12) = f(-12)$ is equivalent to $12^2a_3 + a_1 = 0$. Next, the condition $f'(-6) = f'(6) = 0$ gives

$$4 \cdot 6^3a_4 - 3 \cdot 6^2a_3 + 2 \cdot 6a_2 - a_1 = 4 \cdot 6^3a_4 + 3 \cdot 6^2a_3 + 2 \cdot 6a_2 + a_1 = 0.$$

Finally, the total energy consumption equals $\int_{-12}^{12} f(x)dx = \left[a_4\frac{x^5}{5} + a_2\frac{x^3}{3} + a_0x\right]_{-12}^{12}$. Thus, we get $\frac{12^4}{5}a_4 + \frac{12^2}{3}a_2 + a_0 = 1$. Combining all together, we get the following system:

$$\begin{aligned} \tfrac{12^4}{5}a_4 + \tfrac{12^2}{3}a_2 + a_0 &= 1, \\ 12^2a_3 + a_1 &= 0, \\ 4 \cdot 6^3a_4 - 3 \cdot 6^2a_3 + 2 \cdot 6a_2 - a_1 &= 0, \\ 4 \cdot 6^3a_4 + 3 \cdot 6^2a_3 + 2 \cdot 6a_2 + a_1 &= 0. \end{aligned}$$

(b) Using Gaussian elimination, find all solutions of the linear system obtained in (a). **[4]**

**ANS:** First, we rewrite the system from (a) as a matrix equation

$$\begin{pmatrix} 1 & 0 & \frac{12^2}{3} & 0 & \frac{12^4}{5} \\ 0 & 1 & 0 & 12^2 & 0 \\ 0 & -1 & 2 \cdot 6 & -3 \cdot 6^2 & 4 \cdot 6^3 \\ 0 & 1 & 2 \cdot 6 & 3 \cdot 6^2 & 4 \cdot 6^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Adding the second row to the third row and then subtracting the second row from the fourth, we get that

$$\begin{pmatrix} 1 & 0 & \frac{12^2}{3} & 0 & \frac{12^4}{5} \\ 0 & 1 & 0 & 12^2 & 0 \\ 0 & 0 & 2 \cdot 6 & 6^2 & 4 \cdot 6^3 \\ 0 & 0 & 2 \cdot 6 & -6^2 & 4 \cdot 6^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Next, we subtract the forth row from the third.

$$\begin{pmatrix} 1 & 0 & \frac{12^2}{3} & 0 & \frac{12^4}{5} \\ 0 & 1 & 0 & 12^2 & 0 \\ 0 & 0 & 2\cdot 6 & 6^2 & 4\cdot 6^3 \\ 0 & 0 & 0 & -2\cdot 6^2 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From the fourth row we find that $a_3 = 0$. Then from the second row we get $a_1 = 0$. For the remaining coefficients we get a reduced system

$$\begin{pmatrix} 1 & \frac{12^2}{3} & \frac{12^4}{5} \\ 0 & 2\cdot 6 & 4\cdot 6^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_2 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Take $a_4$ to be a free variable and put $a_4 = t$. From the second row we get $a_2 = -2\cdot 6^2 t$. Then, the first row gives

$$a_0 = 1 - \tfrac{12^2}{3}a_2 - \tfrac{12^4}{5}a_4 = 1 + 12^4 t(\tfrac{1}{6} - \tfrac{1}{5}) = 1 - \tfrac{\cdot 12^4}{30}t.$$

We conclude that the set of solutions $(a_0, a_1, a_2, a_3, a_4)$ of the linear system from (a) is

$$\left\{ \left(1 - \tfrac{\cdot 12^4}{30}t, 0, -2\cdot 6^2 t, 0, t\right) \,:\, t \in \mathbb{R} \right\}.$$

(c) Find all functions $f$ that satisfy all of the data given in the description of the question. **[7]**

**ANS:** Let $f_t(x) = tx^4 - 2\cdot 6^2 tx^2 + 1 - \frac{\cdot 12^4}{30}t$. From (b), we know that any function $f$ should be of this form. Note that $t \neq 0$ since $f$ should be a polynomial of degree 4. We compute $f_t(x)'' = 12tx^2 - 4\cdot 6^2 t$. From the second derivative test we conclude that **[3]** $x = 6$ and $x = -6$ are local maxima if and only if $t < 0$.

Since the codomain of $f$ is $[0, \infty)$ we also need to check $f_t(x) \geq 0$ for all $x \in [-12, 12]$. Note that $f_t'(x) = 4tx^3 - 4\cdot 6^2 tx = 0$ has three solutions $x = -6, 0, 6$. Recalling $t < 0$, we get from the second derivative that $x = 0$ is a local minimum. The minimal value of function $f$ can only occur at points $x = -12$, $x = 0$, $x = 12$. Therefore, it remains to check that $f(12) = f(-12) \geq 0$ and $f(0) \geq 0$. The second inequality always holds for $t < 0$, while the first gives

$$t\,12^4\left(1 - \tfrac{1}{2} - \tfrac{1}{30}\right) + 1 \geq 0.$$

This is equivalent to $t \geq -\frac{15}{7\cdot 12^4}$. Combining the above, we conclude that the set of **[4]** functions $f$ satisfying all the condition is $\left\{ f_t \,:\, t \in \left[-\frac{15}{7\cdot 12^4}, 0\right) \right\}$.

(d) Is it possible that the total energy consumed during the night hours from 1am to 5am **[3]** is smaller than 2 (that is the average over these four hours is less than half of the daily average)?

*If your answer is "yes" then you should give an example of such function $f$ from (c), otherwise justify why it is not possible.*

**ANS:** Yes. For example, take $f(x) = -\frac{2}{12^4}x^4 + \frac{1}{12^2}x^2 + \frac{16}{15}$. Observe that $f = f_t$ for $t = -\frac{2}{12^4} \in \left[-\frac{15}{7\cdot 12^4}, 0\right)$. We have that $\int f(x)dx = -\frac{2}{5\cdot 12^4}x^5 + \frac{1}{3\cdot 12^2}x^3 + \frac{16}{15}x + c$. Then, the energy consumed from 1am to 5am equals

$$\int_{-12}^{-10} f(x)dx + \int_{10}^{12} f(x)dx = -\frac{4}{5\cdot 12^4}(12^5 - 10^5) + \frac{2}{3\cdot 12^2}(12^3 - 10^3) + \frac{64}{15} \approx 1.9 < 2.$$

**2. Home building.**

When you are building a new home it is important to have an idea from the very start how much it may cost. The price varies dramatically depending on the size and the design of the property. Your task is to analyse the data from past construction works in order to estimate the contribution of different components. The table below show a part of construction costs breakdown (in thousand dollars) for three houses: the first has four bedrooms and three bathrooms, the second has two bedrooms and two bathrooms, and the third has two bedrooms and one bathroom.

| House | Plumbing | Electricity | Flooring |
|---|---|---|---|
| 4beds & 3baths | 14 | 15 | 10 |
| 2beds & 2baths | 11 | 10 | 7 |
| 2beds & 1baths | 8 | 9 | 6 |

(a) Find real constants $p_{\text{bed}}$, $p_{\text{bath}}$, $p_0$ such that $P(x, y) = p_{\text{bed}} \cdot x + p_{\text{bath}} \cdot y + p_0$ equals the plumbing costs given in the table, where $x$ is the number of bedrooms and $y$ is the number of bathrooms. Similarly, find functions $E(x, y) = e_{\text{bed}} \cdot x + e_{\text{bath}} \cdot y + e_0$ and $F(x, y) = f_{\text{bed}} \cdot x + f_{\text{bath}} \cdot y + f_0$ describing the costs for electricity and flooring, respectively. **[3]**

**ANS:** Take $p_{\text{bed}} = 0$, $p_{\text{bath}} = 3$ and $p_0 = 5$ so $P(x, y) = 3y + 5$. Then $P(2, 1) = 8$, $P(2, 2) = 11$, $P(4, 3) = 14$. For electricity and flooring we can take $E(x, y) = 2x + y + 4$ and $F(x, y) = x + y + 3$.

(b) In general, for a certain construction component, suppose the costs equal $w_1$ for the first house, $w_2$ for the second house, and $w_3$ for the third house. Find formulas for $w_{\text{bed}}$, $w_{\text{bath}}$ and $w_0$ such that $W(x, y) = w_{\text{bed}} \cdot x + w_{\text{bath}} \cdot y + w_0$ gives values $w_1$, $w_2$, $w_3$ for these three houses. **[3]**

**ANS:** Define $w_{\text{bed}} = \frac{1}{2}w_1 - w_2 + \frac{1}{2}w_3$, $w_{\text{bath}} = w_2 - w_3$ and $w_0 = w_2 + w_3 - w_1$. Then,

$$\begin{aligned}
W(4, 3) &= (2w_1 - 4w_2 + 2w_3) + (3w_2 - 3w_3) + (w_2 + w_3 - w_1) = w_1, \\
W(2, 2) &= (w_1 - 2w_2 + w_3) + (2w_2 - 2w_3) + (w_2 + w_3 - w_1) = w_2, \\
W(2, 1) &= (w_1 - 2w_2 + w_3) + (w_2 - w_3) + (w_2 + w_3 - w_1) = w_3.
\end{aligned}$$

This proves that the formula $W(x,y)$ is correct.

Alternatively one can observe that $W(4,3) = w_1$, $W(2,2) = w_2$ and $W(2,1) = w_3$ is equivalent to the linear system

$$\begin{pmatrix} 4 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} w_{\text{bed}} \\ w_{\text{bath}} \\ w_0 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}. \tag{1}$$

Then, solve it by Gaussian elimination.

(c) Find a matrix $A$ that $W(x,y) = A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, where "$\cdot$" is the dot product of vectors. **[2]**

**ANS:** By definition, $W(x,y) = \begin{pmatrix} w_{\text{bed}} \\ w_{\text{bath}} \\ w_0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$. Applying the formulas from (b), we can substitute $\begin{pmatrix} w_{\text{bed}} \\ w_{\text{bath}} \\ w_0 \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, where $A = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$.

(d) The matrix $A$ from part (c) should be the inverse of $B = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. Explain why one could expect this relation even without computing entries of $A$. **[2]**

**ANS:** Matrix $A$ is defined by $\begin{pmatrix} w_{\text{bed}} \\ w_{\text{bath}} \\ w_0 \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$. Substituting it into system (**??**), we find that

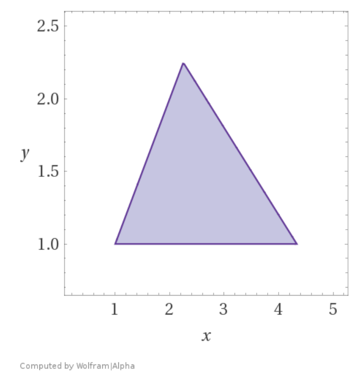
$$BA \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

for any $w_1, w_2, w_3$. Thus, $BA$ is the indentity matrix (to see this, take $(w_1, w_2, w_3)$ to be $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$).

(e) Sketch the region of all ordered pairs of real numbers $(x,y)$ such that $1 \leq y \leq x$ and $P(x,y) + E(x,y) + F(x,y) \leq 30$. List all points with integer coordinates lying in this region. **[2]**

**ANS:** We have $P(x,y) + E(x,y) + F(x,y) = 3x + 5y + 12$.

The region is located below the straight lines $y = x$, $3x + 5y = 18$ and above the line $y = 1$. The integer points in this region are $(1,1), (2,1), (3,1), (4,1), (2.2)$.




**3. Greedy monopolist**

The market for a certain product is governed by the equation

$$d_{n+1} = d_n + \alpha(d_n - s_n) + \beta(s_n - s_{n+1}),$$

where $d_n$ and $s_n$ represent the market demand and supply levels at year $n \in \mathbb{N}$, respectively. The parameter $\alpha$ corresponds to the rate of change of the demand with respect to the excess demand for the previous year and the parameter $\beta$ measures the market reaction to the supply fluctuations. A corporation took over control of all supply resources and decided to implement a greedy approach: to set the level of supply every year such that the revenue is maximised. Assume that the price is proportional to the demand and the production costs remain unchanged so the revenue at year $n$ is given by the equation

$$r_n = s_n(\gamma d_n - \delta).$$

In this question you find out whether the greedy approach performs well in a long term for this model. In the following set $s_0 = d_0 = 10$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\gamma = 2$, $\delta = 1$.

(a) Given fixed $d_n$, $s_n$ show that $f(x) = x\left(\gamma(d_n + \alpha(d_n - s_n) + \beta(s_n - x)) - \delta\right)$ has unique global maximum. This point is the level of supply $s_{n+1}$ at year $n+1$. **[1]**

**ANS:** Note that $f'(x) = -2x + 2(d_n + \frac{1}{4}(d_n - s_n) + \frac{1}{2}s_n) - 1$ has unique stationary point $x = \frac{5}{4}d_n + \frac{1}{4}s_n - \frac{1}{2}$. Since $f''(x) = -2 < 0$ we conclude that $f$ is concave. This implies that the stationary point is the global maximum.

(b) Find a matrix $A$ and a vector $\boldsymbol{b}$ such that $\begin{pmatrix} s_{n+1} \\ d_{n+1} \end{pmatrix} = A \begin{pmatrix} s_n \\ d_n \end{pmatrix} + \boldsymbol{b}$. **[2]**

**ANS:** From (a), we know that $s_{n+1} = \frac{5}{4}d_n + \frac{1}{4}s_n - \frac{1}{2}$. Therefore,

$$d_{n+1} = d_n + \tfrac{1}{4}(d_n - s_n) + \tfrac{1}{2}(s_n - s_{n+1}) = \tfrac{5}{8}d_n + \tfrac{1}{8}s_n + \tfrac{1}{4}.$$

Then, we take $A = \begin{pmatrix} \frac{1}{4} & \frac{5}{4} \\ \frac{1}{8} & \frac{5}{8} \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$.

(c) Find a vector $\boldsymbol{u}$ that $\boldsymbol{u} = A\boldsymbol{u} + \boldsymbol{b}$. Show that $\begin{pmatrix} s_{n+1} \\ d_{n+1} \end{pmatrix} - \boldsymbol{u} = A^{n+1}\left(\begin{pmatrix} s_0 \\ d_0 \end{pmatrix} - \boldsymbol{u}\right)$. **[4]**

**ANS:** The equation $\boldsymbol{u} = A\boldsymbol{u} + \boldsymbol{b}$ is equivalent to the linear system **[1]**

$$\begin{pmatrix} 1 - \frac{1}{4} & -\frac{5}{4} \\ -\frac{1}{2} & 1 - \frac{5}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

Solving this system gives $u_1 = u_2 = 1$. Next, note that **[1]**

**[1]**

$$\begin{pmatrix} s_{n+1} \\ d_{n+1} \end{pmatrix} - \boldsymbol{u} = A\begin{pmatrix} s_n \\ d_n \end{pmatrix} + \boldsymbol{b} - A\boldsymbol{u} - \boldsymbol{b} = A\left(\begin{pmatrix} s_n \\ d_n \end{pmatrix} - \boldsymbol{u}\right).$$

Applying this relation $n+1$ times completes the proof of part (c). **[1]**

(d) Write $A^2$ and $A^3$ as scalars times $A$ . Guess a general formula for $A^n$ for all $n \geq 1$ and prove that your formula is correct. Find expressions for $s_n$ and $d_n$. **[3]**

**ANS:** We have **[1]**

$$A^2 = \frac{1}{64}\begin{pmatrix} 2 & 10 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 10 \\ 1 & 5 \end{pmatrix} = \frac{1}{64}\begin{pmatrix} 14 & 70 \\ 7 & 35 \end{pmatrix} = \frac{7}{8}A,$$
$$A^3 = A^2 \cdot A = \frac{7}{8}A \cdot A = \left(\frac{7}{8}\right)^2 A.$$

Similarly, for $n \geq 1$, we get **[1]**

$$A^n = \left(\frac{7}{8}\right)A^{n-1} = \cdots = \left(\frac{7}{8}\right)^{n-1} A = \frac{7^{n-1}}{8^n}\begin{pmatrix} 2 & 10 \\ 1 & 5 \end{pmatrix}.$$

Then, using (c), we obtain that **[1]**

$$\begin{pmatrix} s_n \\ d_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A^n\begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 + \frac{3^3}{2}\left(\frac{7}{8}\right)^{n-1} \\ 1 + \frac{3^3}{2^2}\left(\frac{7}{8}\right)^{n-1} \end{pmatrix}.$$

(e) Using the formula $\sum_{i=0}^{k} \lambda^i = \frac{1-\lambda^{k+1}}{1-\lambda}$ write an exact explicit expression for the total revenue of the corporation over the first ten years $R_{\text{greed}} = \sum_{n=1}^{10} s_n(\gamma d_n - \delta)$. **[3]**

**ANS:** Using (d), we find that

$$\sum_{n=1}^{10} s_n(\gamma d_n - \delta) = \sum_{n=1}^{10}\left(1 + \frac{3^3}{2}\left(\frac{7}{8}\right)^{n-1}\right)^2 = \sum_{n=1}^{10}\left(1 + 3^3\left(\frac{7}{8}\right)^{n-1} + \frac{3^6}{4}\left(\frac{7}{8}\right)^{2(n-1)}\right)$$
$$= 10 + 3^3\sum_{i=0}^{9}\left(\frac{7}{8}\right)^i + \frac{3^6}{4}\sum_{i=0}^{9}\left(\frac{7^2}{8^2}\right)^i$$
$$= 10 + 3^3\frac{1-\left(\frac{7}{8}\right)^{10}}{1-\frac{7}{8}} + \frac{3^6}{4}\frac{1-\left(\frac{7}{8}\right)^{20}}{1-\frac{7^2}{8^2}}.$$

(f) Compute the revenue $R_{\text{equil}}$ that the corporation can get (over the same period) by the equilibrium approach: setting $s_n = s_0$ for all $n = 1, \ldots, 10$. Approximate $R_{\text{greed}}/R_{\text{equil}}$ to two decimal places. **[2]**

**ANS:** From the formulas for the demand at year $n+1$, we find that if $s_n = d_n$ and $s_n = s_{n+1}$ then $d_{n+1} = d_n$. Therefore, if $s_n = s_0 = 10$ for all $n = 1, \ldots, 10$ then $d_n = d_0 = 10$ for this period. Thus, $R_{\text{equil}} = 10 s_0(2d_0 - 1) = 1900$. Using the formula from (e), we compute $R_{\text{greed}}/R_{\text{equil}} \approx 0.47$.

(g) How can the corporation outperform the equilibrium approach over the same period? **[1]**

**ANS:** Set $s_n = s_0$ for all $n = 1, \ldots, 9$. Then use the greedy appoach for $n = 10$ that is $s_n = 14.5$. The revenue for the first nine years is the same as for the equilibrium approach, but the revenue for the last year is bigger. Alternatively, use the strategy from (h).

(h) Design a strategy that would bring more than $6R_{\text{equil}}$ over the first ten years. **[2]**

**ANS:** Set $s_n = 0$ for all $n = 1, \ldots, 9$. This will inflate the demand so $d_n = 15(\frac{5}{4})^{n-1}$ for all $n = 1, \ldots, 9$. Then, at year 10 use the greedy approach and set $s_{10} = 15(\frac{5}{4})^9 - \frac{1}{2}$. Then, $d_{10} = 7.5(\frac{5}{4})^9 + \frac{1}{4}$. The total revenue is just the revenue for the last year which is

$$R = s_{10}(2d_{10} - 1) = \left(15(\tfrac{5}{4})^9 - \tfrac{1}{2}\right)^2 \approx 12378.5 > 6R_{\text{equil}}.$$

## 4. Gradient descent.

Gradient descent is one of the most popular algorithms in data science and by far the most common way to optimise neural networks. A function is minimised by iteratively moving a little bit in the direction of negative gradient. For the two-dimensional case, the step of iteration is given by the formula

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \varepsilon \, \nabla f(x_n, y_n).$$

In general, $\varepsilon$ does not have to be a constant, but in this question, for demonstrative purposes, we set $\varepsilon = 0.1$. Let $f(x, y) = 3.5x^2 - 4xy + 6.5y^2$ and $x_0$ and $y_0$ be any real numbers.

(a) For all $x, y \in \mathbb{R}$ compute $\nabla f(x, y)$ and find a matrix $A$ such that **[3]**

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \varepsilon \, \nabla f(x, y).$$

Write an expression for $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in terms of $x_0$ and $y_0$ and powers of $A$.

**ANS:** $\nabla f(x,y) = \begin{pmatrix} 7x-4y \\ -4x+13y \end{pmatrix}$. Then, we take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \varepsilon \begin{pmatrix} 7 & -4 \\ -4 & 13 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & -0.3 \end{pmatrix}.$$

By the definition of the iteration step, we find that $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

(b) Find the eigenvalues of $A$. **[1]**

**ANS:** First, we write the characteristic equation

$$\det(A-\lambda I) = (0.3-\lambda)(-0.3-\lambda) - 0.4^2 = \lambda^2 - 0.5^2 = 0.$$

Thus, $\lambda = 0.5$ and $\lambda = -0.5$ are the eigenvalues of $A$.

(c) Find one eigenvector corresponding to each eigenvalue. **[2]**

**ANS:** We have $(A-0.5I)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{0}$ and $(A+0.5I)\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \mathbf{0}$. Therefore, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding $\lambda = 0.5$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector corresponding $\lambda = -0.5$.

(d) Find matrices $P$ and $D$ such that $D$ is diagonal and $A = PDP^{-1}$. **[1]**

**ANS:** Using eigenvectors found (c) as columns, we define $P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ and $D = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It was given on lectures that $A = PDP^{-1}$.

(e) Find matrices $D^n$, $P^{-1}$ and $A^n$. Find formulas for $x_n$ and $y_n$. **[4]**

**ANS:** We have $D^n = \dfrac{1}{2^n}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$ and $P^{-1} = \dfrac{1}{\det P}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$. Then, we get $A^n = PD^nP^{-1} = \frac{1}{5\cdot 2^n}\begin{pmatrix} 4+(-1)^n & 2-2(-1)^n \\ 2-2(-1)^n & 1+4(-1)^n \end{pmatrix}$. Using (a), we find that $x_n = \frac{4+(-1)^n}{5\cdot 2^n}x_0 + \frac{2-2(-1)^n}{5\cdot 2^n}y_0$ and $y_n = \frac{2-2(-1)^n}{5\cdot 2^n}x_0 + \frac{1+4(-1)^n}{5\cdot 2^n}y_0$.

(f) Suppose $x_0 = y_0 = 1$. Find the smallest $N \in \mathbb{N}$ such that $\left\| \begin{pmatrix} x_N \\ y_N \end{pmatrix} \right\| \le 0.05$. **[3]**

**ANS:** Observe that, for any $n \in \mathbb{N}$,

$$(4+(-1)^n)^2 + (2-2(-1)^n)^2 = 16 + 8(-1)^n + 1 + 4 - 8(-1)^n + 4 = 25.$$

Similarly, $(1+4(-1)^n)^2 + (2-2(-1)^n)^2 = 25$. We also have that

$$(4+(-1)^n)(2-2(-1)^n) + (1+4(-1)^n)(2-2(-1)^n) = 0.$$

Indeed, if $n$ is even then $2-2(-1)^n=0$, otherwise $4+(-1)^n+1+(-4)^n=0$. Using the formulas for $x_n$ and $y_n$ from (e), we get that

$$\left\|\begin{pmatrix}x_n\\y_n\end{pmatrix}\right\|=\sqrt{x_n^2+y_n^2}=\tfrac{1}{5\cdot 2^n}\sqrt{25x_0^2+25y_0^2}=\tfrac{1}{2^n}\left\|\begin{pmatrix}x_0\\y_0\end{pmatrix}\right\|.$$

Recalling $x_0=y_0=1$ and observing $\sqrt{2}/2^4>0.05>\sqrt{2}/2^5$, we get that $N=5$.

(g) Sketch the region $R$ consisting of those $(x_0,y_0)$ such that $x_N\geq 0$, $y_N\geq 0$ and **[4]**

$$\left\|\begin{pmatrix}x_N\\y_N\end{pmatrix}\right\|\leq 0.05, \qquad \left\|\begin{pmatrix}x_{N-1}\\y_{N-1}\end{pmatrix}\right\|>0.05,$$

where $N$ is the number found in part (f). Write an equation for the boundary of $R$. Which points of the boundary belongs to $R$ and which do not?

**ANS:** Using the formulas obtained in (f), we get that

$$0.8=0.05\cdot 2^4<\left\|\begin{pmatrix}x_0\\y_0\end{pmatrix}\right\|=\sqrt{x_0^2+y_0^2}\leq 0.05\cdot 2^5=1.6.$$

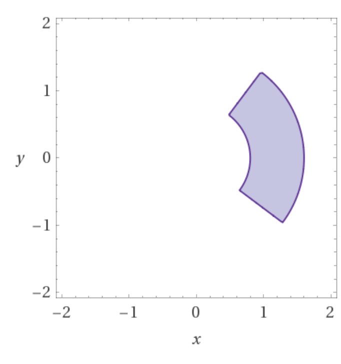
We also have the condition $x_5,y_5\geq 0$ which is equivalent to the inequalities

$$3x_0+4y_0\geq 0 \qquad \text{and} \qquad 4x_0-3y_0\geq 0.$$

Note that $3(3x_0+4y_0)+4(4x_0-3y_0)=25x_0\geq 0$.

Thus, $R$ is the part of annulus between circles $x^2+y^2=1.6^2$ and $x^2+y^2=0.8^2$, located above the straight line $y=-\frac{3}{4}x$ and below the straight line $y=\frac{4}{3}x$. For example, the boundary of $R$ is given by the following equation:

$$\begin{aligned}&\{(x,y)\in\mathbb{R}^2\,:\,x^2+y^2=0.8^2, 3x+4y\geq 0, 4x-3y\geq 0\}\cup\\&\{(x,y)\in\mathbb{R}^2\,:\,x^2+y^2=1.6^2, 3x+4y\geq 0, 4x-3y\geq 0\}\cup\\&\{(x,y)\in\mathbb{R}^2\,:\,0.8\leq x^2+y^2\leq 1.6^2, y=-\tfrac{3}{4}x\leq 0\}\cup\\&\{(x,y)\in\mathbb{R}^2\,:\,0.8\leq x^2+y^2\leq 1.6^2, y=\tfrac{4}{3}x\geq 0\}.\end{aligned}$$




The boundary points that do not belong to $R$ form a part of the circle $x^2+y^2=0.8^2$ between corners $\frac{4}{25}(4,-3)$ and $\frac{3}{25}(3,4)$, which are intersections of the circle with lines $y=-\frac{3}{4}x$ and $y=\frac{4}{3}x$ (with $x\geq 0$).