MAT9004 Assignment 2

Due at the end of Week 8.

1. Energy consumption.

The typical daily energy use of a household is described by a function $f : [-12, 12] \to [0, \infty)$. The value of $f(x)$ is the consumption rate at given time x. Both points $x = -12$ and $x = 12$ of the domain correspond to 3am so $f(12) = f(-12)$. It is also known that 9am and 9pm ($x = -6$ and $x = 6$, respectively) are local maxima of the consumption rate. In this question we model the rate by a polynomial of degree 4 that is $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and assume that the total energy consumption during the day equals 24.

(a) Write a linear system of equations describing the following properties: $f(12) = f(-12)$, $x = -6, 6$ are stationary points of f and the total energy consumption is 24. **[4]**

(b) Using Gaussian elimination, find all solutions of the linear system obtained in (a). **[4]**

(c) Find all functions f that satisfy all of the data given in the description of the question. **[7]**

(d) Is it possible that the total energy consumed during the night hours from 1am to 5am is smaller than 2 (that is the average over these four hours is less than half of the daily average)? **[3]**

If your answer is "yes" then you should give an example of such function f from (c), otherwise justify why it is not possible.

2. Home building.

When you are building a new home it is important to have an idea from the very start how much it may cost. The price varies dramatically depending on the size and the design of the property. Your task is to analyse the data from past construction works in order to estimate the contribution of different components. The table below show a part of construction costs breakdown (in thousand dollars) for three houses: the first has four bedrooms and three bathrooms, the second has two bedrooms and two bathrooms, and the third has two bedrooms and one bathroom.

House	Plumbing	Electricity	Flooring
4beds & 3baths	14	15	10
2beds & 2baths	11	10	7
2beds & 1baths	8	9	6

(a) Find real constants p_{bed}, p_{bath}, p_0 such that $P(x, y) = p_{\text{bed}} \cdot x + p_{\text{bath}} \cdot y + p_0$ equals the plumbing costs given in the table, where x is the number of bedrooms and y is the number of bathrooms. Similarly, find functions $E(x, y) = e_{\text{bed}} \cdot x + e_{\text{bath}} \cdot y + e_0$ and $F(x, y) = f_{\text{bed}} \cdot x + f_{\text{bath}} \cdot y + f_0$ describing the costs for electricity and flooring, respectively. **[3]**

(b) In general, for a certain construction component, suppose the costs equal w_1 for the first house, w_2 for the second house, and w_3 for the third house. Find formulas for w_{bed}, w_{bath} and w_0 such that $W(x,y) = w_{\text{bed}} \cdot x + w_{\text{bath}} \cdot y + w_0$ gives values w_1, w_2, w_3 for these three houses. **[3]**

(c) Find a matrix A that $W(x,y) = A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, where "$\cdot$" is the dot product of vectors. **[2]**

(d) The matrix A from part (c) should be the inverse of $B = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. Explain why one could expect this relation even without computing entries of A. **[2]**

(e) Sketch the region of all ordered pairs of real numbers (x,y) such that $1 \le y \le x$ and $P(x,y) + E(x,y) + F(x,y) \le 30$. List all points with integer coordinates lying in this region. **[2]**

3. Greedy monopolist

The market for a certain product is governed by the equation

$$d_{n+1} = d_n + \alpha(d_n - s_n) + \beta(s_n - s_{n+1}),$$

where d_n and s_n represent the market demand and supply levels at year $n \in \mathbb{N}$, respectively. The parameter α corresponds to the rate of change of the demand with respect to the excess demand for the previous year and the parameter β measures the market reaction to the supply fluctuations. A corporation took over control of all supply resources and decided to implement a greedy approach: to set the level of supply every year such that the revenue is maximised. Assume that the price is proportional to the demand and the production costs remain unchanged so the revenue at year n is given by the equation

$$r_n = s_n(\gamma d_n - \delta).$$

In this question you find out whether the greedy approach performs well in a long term for this model. In the following set $s_0 = d_0 = 10$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\gamma = 2$, $\delta = 1$.

(a) Given fixed d_n, s_n show that $f(x) = x\left(\gamma(d_n + \alpha(d_n - s_n) + \beta(s_n - x)) - \delta\right)$ has unique global maximum. This point is the level of supply s_{n+1} at year $n+1$. **[1]**

(b) Find a matrix A and a vector $\boldsymbol{b}$ such that $\begin{pmatrix} s_{n+1} \\ d_{n+1} \end{pmatrix} = A \begin{pmatrix} s_n \\ d_n \end{pmatrix} + \boldsymbol{b}$. **[2]**

(c) Find a vector $\boldsymbol{u}$ that $\boldsymbol{u} = A\boldsymbol{u} + \boldsymbol{b}$. Show that $\begin{pmatrix} s_{n+1} \\ d_{n+1} \end{pmatrix} - \boldsymbol{u} = A^{n+1} \left(\begin{pmatrix} s_0 \\ d_0 \end{pmatrix} - \boldsymbol{u} \right)$. **[4]**

(d) Write A^2 and A^3 as scalars times A . Guess a general formula for A^n for all $n \geq 1$ and prove that your formula is correct. Find expressions for s_n and d_n. **[3]**

(e) Using the formula $\sum_{i=0}^{k} \lambda^i = \frac{1-\lambda^{k+1}}{1-\lambda}$ write an exact explicit expression for the total revenue of the corporation over the first ten years $R_{\text{greed}} = \sum_{n=1}^{10} s_n(\gamma d_n - \delta)$. **[3]**

(f) Compute the revenue R_{equil} that the corporation can get (over the same period) by the equilibrium approach: setting $s_n = s_0$ for all $n = 1, \ldots, 10$. Approximate $R_{\text{greed}}/R_{\text{equil}}$ to two decimal places. **[2]**

(g) How can the corporation outperform the equilibrium approach over the same period? **[1]**

(h) Design a strategy that would bring more than $6R_{\text{equil}}$ over the first ten years. **[2]**

4. Gradient descent.

Gradient descent is one of the most popular algorithms in data science and by far the most common way to optimise neural networks. A function is minimised by iteratively moving a little bit in the direction of negative gradient. For the two-dimensional case, the step of iteration is given by the formula

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \varepsilon \, \nabla f(x_n, y_n).$$

In general, ε does not have to be a constant, but in this question, for demonstrative purposes, we set $\varepsilon = 0.1$. Let $f(x, y) = 3.5x^2 - 4xy + 6.5y^2$ and x_0 and y_0 be any real numbers.

(a) For all $x, y \in \mathbb{R}$ compute $\nabla f(x, y)$ and find a matrix A such that **[3]**

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \varepsilon \, \nabla f(x, y).$$

Write an expression for $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in terms of x_0 and y_0 and powers of A.

(b) Find the eigenvalues of A. **[1]**

(c) Find one eigenvector corresponding to each eigenvalue. **[2]**

(d) Find matrices P and D such that D is diagonal and $A = PDP^{-1}$. **[1]**

(e) Find matrices D^n, P^{-1} and A^n. Find formulas for x_n and y_n. **[4]**

(f) Suppose $x_0 = y_0 = 1$. Find the smallest $N \in \mathbb{N}$ such that $\left\| \begin{pmatrix} x_N \\ y_N \end{pmatrix} \right\| \leq 0.05$. **[3]**

(g) Sketch the region R consisting of those (x_0, y_0) such that $x_N \geq 0$, $y_N \geq 0$ and **[4]**

$$\left\| \begin{pmatrix} x_N \\ y_N \end{pmatrix} \right\| \leq 0.05, \qquad \left\| \begin{pmatrix} x_{N-1} \\ y_{N-1} \end{pmatrix} \right\| > 0.05,$$

where N is the number found in part (f). Write an equation for the boundary of R. Which points of the boundary belongs to R and which do not?