1. The Fourier series coefficients are given by

$$c_k = \frac{e^{-jk\pi/20}}{1+jk^3}$$

To find the magnitude of this we use the rule "magnitude of a ratio is the ratio of the magnitudes"! That gives

$$|c_k| = \frac{|e^{-jk\pi/20}|}{|1+jk^3|}$$

The numerator of c_k is in polar form so the magnitude is the (non-negative!) number out front… so in this case it is just 1. The denominator is in rectangular form so we can use the "square-root of sum of squares" rule. That gives

$$|c_k| = \frac{|e^{-jk\pi/20}|}{|1+jk^3|} = \frac{1}{\sqrt{1^2+(k^3)^2}} = \frac{1}{\sqrt{1+k^6}}$$

Now we can just plot this.

Now to find the angle we use the rule "angle of a ratio if the difference of the angles"! That gives

$$\angle c_k = \angle\{e^{-jk\pi/20}\} - \angle\{1+jk^3\}$$

The numerator of c_k is in polar form so the angle is the exponent (without the j!). The denominator is in rectangular form so we can use the atan formula:

$$\begin{aligned}\angle c_k &= \angle\{e^{-jk\pi/20}\} - \angle\{1+jk^3\} \\ &= -k\pi/20 - \tan^{-1}\left\{\frac{k^3}{1}\right\} \\ &= -k\pi/20 - \tan^{-1}\{k^3\}\end{aligned}$$

Now we can just plot this.

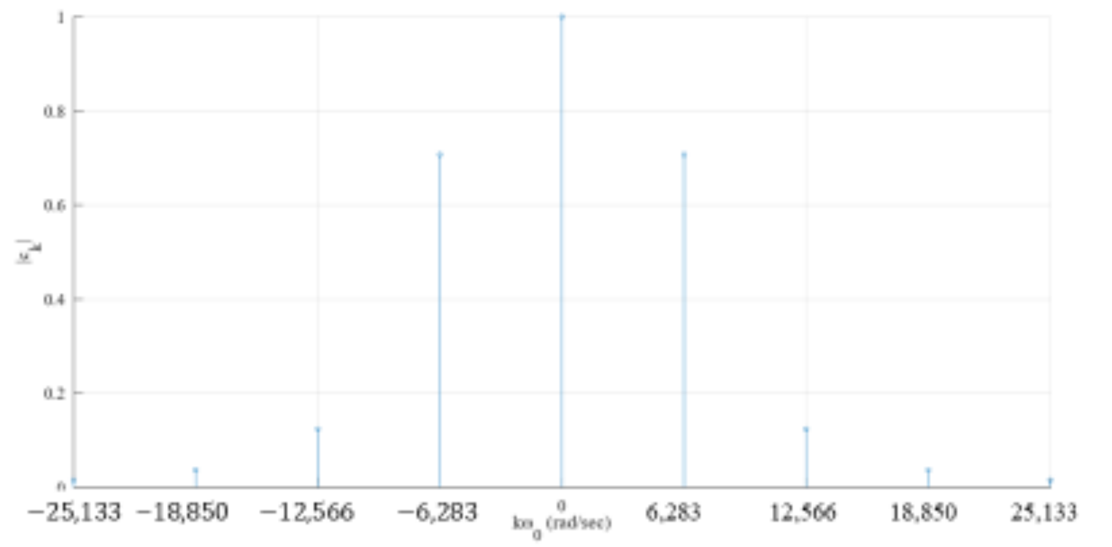
We are to do the plots vs $k\omega_o$ (rather than just k – which is sometimes done but it is better to plot vs $k\omega_o$ so we can see the actual *frequency* that each term is at. So we need the value of ω_o – which given that the period is given as $T = 1msec$ we know that $\omega_0 = \frac{2\pi}{T}$ so we get that

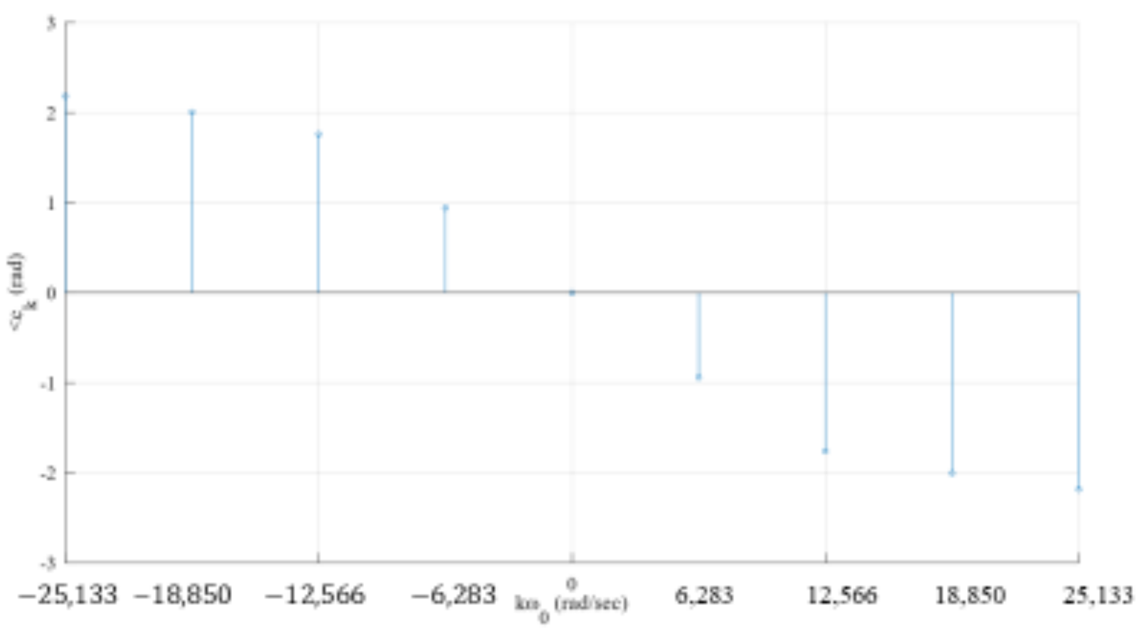
$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{10^{-3}} = 2000\pi \approx 6,283 \text{ rad/sec}$$

Since we are told that our plot only needs to cover the range of $\pm 30{,}000$ rad/sec we only need to plot using k values that yield $k\omega_0$ values in that range. So that means only $-4 \le k \le 4$. Also, we know that $|c_k|$ will have even symmetry and $\angle c_k$ will have odd symmetry, we only need to compute values for $k = 0,1,2,3,4$:

k	$\|c_k\|$	$\angle c_k$(rad)
0	1	0
1	0.707	-0.942
2	0.124	-1.761
3	0.037	-2.005
4	0.016	-2.183

These values can now be plotted by hand on gridded paper… or can be plotted using software (like MATLAB) as I did here:





2. For this circuit we have R_1 and C in series and R_2 and L in parallel… and then those two impedances form a voltage divider. So first we find those two series and parallel combos, called here Z_s and Z_p:

$$Z_s = R_1 + \frac{1}{j\omega C} = \frac{R_1(j\omega C)}{j\omega C} + \frac{1}{j\omega C} = \frac{1 + j\omega R_1 C}{j\omega C}$$

$$Z_p = \frac{R_2(j\omega L)}{R_2 + j\omega L} = \frac{j\omega R_2 L}{R_2 + j\omega L}$$

For the series combo, found a common denominator and then combined into the final form shown. For the parallel combo I used the alternative form of "product over sum" (only works with two things in parallel!) and then re-wrote (using the "reciprocal of sum of reciprocals" form works but almost always requires more work to get it into a simple form. Note: I always work to get any impedance combos into the form of rectangular-over-rectangular BEFORE subbing them into the voltage divider form!!! (Yes, for the series here I could have used $\frac{1}{j} = -j$ to get a rectangular form $Z_s = R_1 - j(\frac{1}{\omega C})$… but my experience shows that is not as helpful as what I did here!)

Now just use the formula for the voltage divider (after replacing the two voltages by their phasor symbols!!!) to get:

$$\vec{Y} = \underbrace{\left[\frac{Z_p}{Z_s + Z_p}\right]}_{=H(\omega)} \vec{X}$$

and then I can identify as the frequency the thing that multiplies the input phasor to give the output phasor. So we have

$$H(\omega) = \frac{Z_p}{Z_s + Z_p} = \frac{\left[\dfrac{j\omega R_2 L}{R_2 + j\omega L}\right]}{\left[\dfrac{1 + j\omega R_1 C}{j\omega C}\right] + \left[\dfrac{j\omega R_2 L}{R_2 + j\omega L}\right]}$$

Now we have a bunch of "over-over-over" stuff that we need to clear out by multiplying top and bottom by the things we want to get of. Starting with $j\omega C$ and using the fact that $j^2 = -1$ we get

$$H(\omega) = \frac{\left[\dfrac{j\omega R_2 L}{R_2 + j\omega L}\right]}{\left[\dfrac{1 + j\omega R_1 C}{j\omega C}\right] + \left[\dfrac{j\omega R_2 L}{R_2 + j\omega L}\right]} \frac{j\omega C}{j\omega C} = \frac{\left[\dfrac{-\omega^2 C R_2 L}{R_2 + j\omega L}\right]}{\left[1 + j\omega R_1 C\right] + \left[\dfrac{-\omega^2 C R_2 L}{R_2 + j\omega L}\right]}$$

Now to clear the $R_2 + j\omega L$ we get:

$$H(\omega) = \frac{\left[\dfrac{-\omega^2 C R_2 L}{R_2 + j\omega L}\right]}{\left[1 + j\omega R_1 C\right] + \left[\dfrac{-\omega^2 C R_2 L}{R_2 + j\omega L}\right]} \frac{R_2 + j\omega L}{R_2 + j\omega L} = \frac{-\omega^2 C R_2 L}{\left[1 + j\omega R_1 C\right]\left[R_2 + j\omega L\right] - \omega^2 C R_2 L}$$

And then we need to expand out the first term in the denominator to get

$$\begin{aligned} H(\omega) &= \frac{-\omega^2 C R_2 L}{\left[1 + j\omega R_1 C\right]\left[R_2 + j\omega L\right] - \omega^2 C R_2 L} \\ &= \frac{-\omega^2 C R_2 L}{\left(R_2 + j\omega R_1 R_2 C + j\omega L - \omega^2 R_1 L C\right) - \omega^2 C R_2 L} \\ &= \frac{-\omega^2 C R_2 L}{\left[R_2 - \omega^2 \left(R_1 + R_2\right) L C\right] + j\omega(R_1 R_2 C + L)} \end{aligned}$$

I've put this into the specified form: rectangular form over rectangular form $(A + jB)/(D + jE)$ where

$$\begin{aligned} A &= -\omega^2 C R_2 L \\ B &= 0 \\ D &= R_2 - \omega^2 \left(R_1 + R_2\right) L C \\ E &= \omega(R_1 R_2 C + L) \end{aligned}$$

In practice, we almost always want to end up with our frequency response $H(\omega)$ in this form!!

3. Although you *can* plug in the numerical values right away… it is easier to avoid that until the end – work with the symbols until the end! In practice, there is often a change in component value and if you've kept things in terms of the symbols until the end it is VERY easy to make the change to the new values!

We know that $d_k = H(k\omega_o)c_k$ so we can first work with that form and (this is important… it helps reduce LOTS of work) we can exploit the rules that the "magnitude of a product is product of magnitudes" and the "angle of a product is sum of the angles"! So we have that

$$|d_k| = |H(k\omega_o)c_k| = |H(k\omega_o)||c_k|$$

$$\angle d_k = \angle\{H(k\omega_o)c_k\} = \angle\{H(k\omega_o)\} + \angle\{c_k\}$$

We already have the forms for $|c_k|$ and $\angle c_k$ (from problem #1) so we focus here on finding magnitude and phase of the frequency response. For simplicity we can ignore the argument being $k\omega_o$ and just let it be ω for now and sub in later.

$$\begin{aligned}
|H(\omega)| &= \left|\frac{-\omega^2 CR_2L}{\left[R_2 - \omega^2(R_1+R_2)LC\right] + j\omega(R_1R_2C+L)}\right| \\
&= \frac{\left|-\omega^2 CR_2L\right|}{\left|\left[R_2 - \omega^2(R_1+R_2)LC\right] + j\omega(R_1R_2C+L)\right|} \\
&= \frac{\omega^2 CR_2L}{\sqrt{\left[R_2 - \omega^2(R_1+R_2)LC\right]^2 + \omega^2(R_1R_2C+L)^2}}
\end{aligned}$$

And

$$\angle H(\omega) = \underbrace{\angle\{-\omega^2 CR_2L\}}_{\substack{\angle \text{ neg real } \# = \pm\pi \\ \text{Except when } \omega=0 \text{ it is } 0}} - \underbrace{\angle\left\{\left[R_2 - \omega^2(R_1+R_2)LC\right] + j\omega(R_1R_2C+L)\right\}}_{\substack{=\tan^{-1}\left\{\frac{\omega(R_1R_2C+L)}{R_2-\omega^2(R_1+R_2)LC}\right\} \\ \text{Note: this is 0 when } \omega=0}}$$

Remember, we really only need to compute these at the same $k\omega_o$ values we did for Problem #1 – and we again have symmetry to exploit. And **NOW** we sub in the numerical values for the components! I used MATLAB/Octave to do this.

```
R1=1000;R2=500;C=0.1e-6;L=100e-3;
wo=2000*pi;
k=0:4;
w=k*wo;
H_mag=(w.^2)*C*R2*L./(sqrt( (R2 - (w.^2)*(R1+R2)*L*C).^2 + (w.^2)*(R1*R2*C+L)^2));
% For the phase we have to handle w=0 case separately
k=0
% We know the angel is 0 when w = 0
H_ang(k+1)=0
k=1:4;
w=k*wo;
H_ang(k+1)= pi - atan2(w*(R1*R2*C+L),(R2 - (w.^2)*(R1+R2)*L*C));
% atan2 is a better form of atan... it handles all four quadrants!!
```

k	$\lvert H(k\omega_o)\rvert$	$\angle H(k\omega_o)$(rad)
0	0	0
1	0.208	1.473
2	0.297	0.790
3	0.317	0.530
4	0.324	0.398

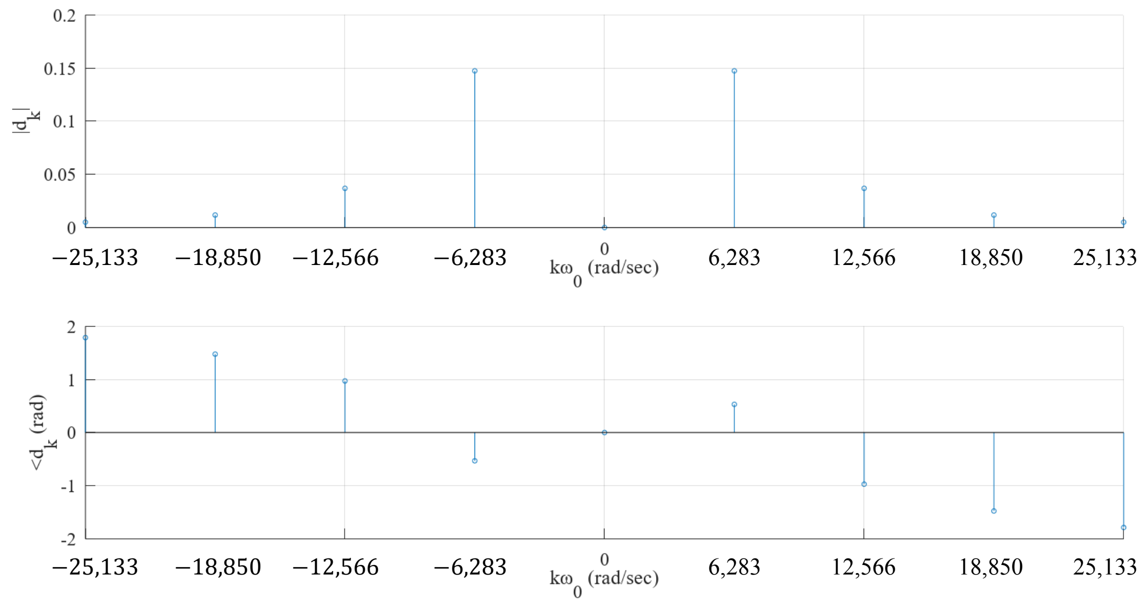
Now we can use these values with the values we have for c_k from problem #1 and use these formulas:

$$|d_k| = |H(k\omega_o)c_k| = |H(k\omega_o)||c_k|$$

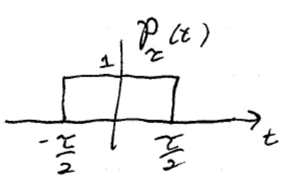
$$\angle d_k = \angle\{H(k\omega_o)c_k\} = \angle\{H(k\omega_o)\} + \angle\{c_k\}$$

k	$\lvert c_k\rvert$	$\lvert H(k\omega_o)\rvert$	$\lvert d_k\rvert = \lvert c_k\rvert\lvert H(k\omega_o)\rvert$
0	1	0	0
1	0.707	0.208	0.147
2	0.124	0.297	0.037
3	0.037	0.317	0.012
4	0.016	0.324	0.005

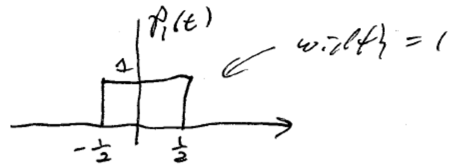
k	$\angle c_k$(rad)	$\angle H(k\omega_o)$(rad)	$\angle d_k = \angle c_k + \angle H(k\omega_o)$
0	0	0	0
1	-0.942	1.473	0.531
2	-1.761	0.790	-0.971
3	-2.005	0.530	-1.475
4	-2.183	0.398	-1.785


0.2
0.15
0.1
0.05
0
|d_k|
−25,133
−18,850
−12,566
−6,283
0
6,283
12,566
18,850
25,133
kω_0 (rad/sec)
2
1
0
-1
-2
<d_k (rad)
−25,133
−18,850
−12,566
−6,283
0
6,283
12,566
18,850
25,133
kω_0 (rad/sec)

4. When we look at the given signal we see that it is made out of two rectangular pulses, each with width of 1. Recall that



Set $\tau = 1$



From this we can build the signal by shifting them to the right places and weighting them appropriately:

$$x(t) = 2p_1(t+1.5) - 2p_1(t-1.5)$$

To get the FT of $x(t)$ we'll need the FT of $p_1(t)$ and then can apply the time shift property to account for the shifts (and the linearity property to account for the weighting and summation of two pulses).
So from the FT table we have

$$\mathcal{F}\left\{p_\tau(t)\right\} = \tau\,\mathrm{sinc}\left(\frac{\tau\omega}{2\pi}\right)$$

So then

$$\mathcal{F}\left\{p_1(t)\right\} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

Now using the time shift property (and the linearity property):

$$X(\omega) = 2\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)e^{j1.5\omega} - 2\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)e^{-j1.5\omega}$$

$$= 2\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)\left[e^{j1.5\omega} - e^{-j1.5\omega}\right]$$

$$= 2\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)2j\left[\frac{e^{j1.5\omega} - e^{-j1.5\omega}}{2j}\right]$$

$$= 4j\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)\sin(1.5\omega)$$

where Euler's formula has been used.

Note that this FT is purely imaginary so its phase can only be $\pm\pi/2$ rad – it will be $+\pi/2$ when the sinc(…)sin(…) product is positive and it will be $-\pi/2$ the sinc(…)sin(…) product is negative (and one can say that it will be 0 when that product is 0… although technically it can be any angle then!). Thus we can say that

$$\angle X(\omega) = \frac{\pi}{2} sign\left\{ \text{sinc}\left(\frac{\omega}{2\pi}\right) \sin(1.5\omega) \right\}$$

The magnitude – using the rule that the magnitude of a product is a product of the magnitudes – is:

$$|X(\omega)| = \left| 4j \,\text{sinc}\left(\frac{\omega}{2\pi}\right) \sin(1.5\omega) \right| = |4j| \left| \text{sinc}\left(\frac{\omega}{2\pi}\right) \right| \left| \sin(1.5\omega) \right|$$

$$= 4 \left| \text{sinc}\left(\frac{\omega}{2\pi}\right) \right| \left| \sin(1.5\omega) \right|$$

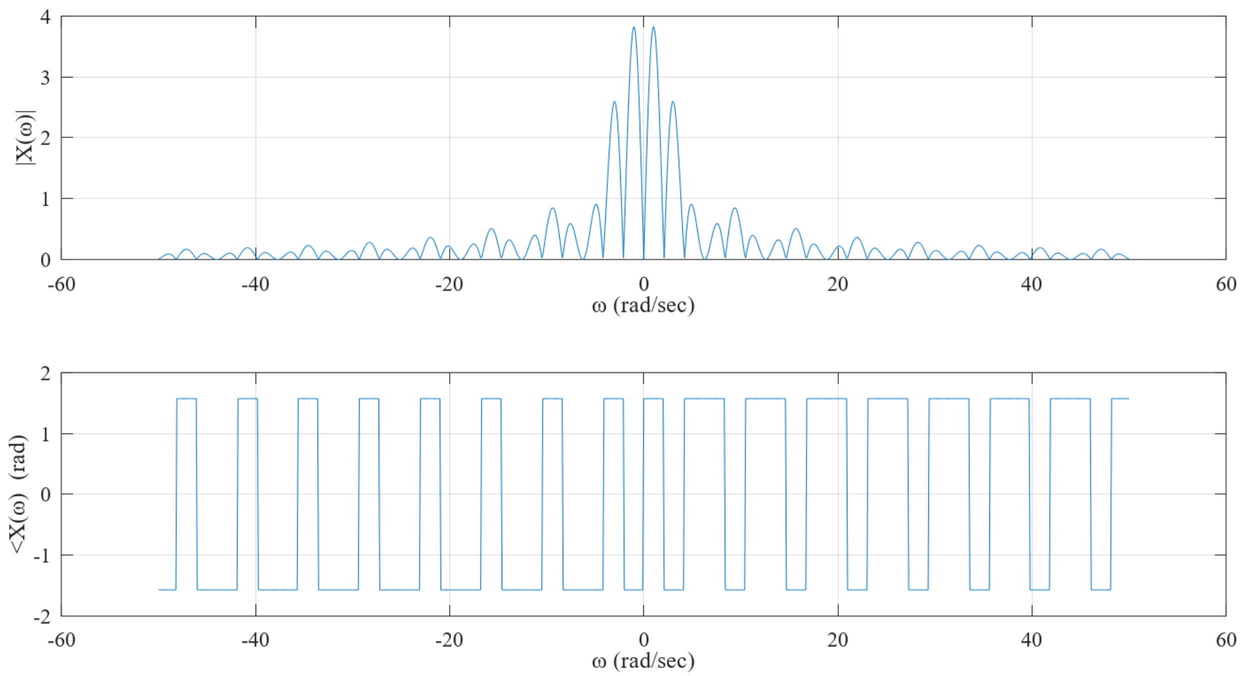
Now we can use the characteristics in MATLAB (or Octave) to make the plots:

```
w=-50:0.1:50;

X_mag = 4*abs(sinc(w/(2*pi))).*abs(sin(1.5*w));  % use abs function in MATLAB

X_ang = (pi/2)*sign(sinc(w/(2*pi)).*sin(1.5*w));  % use “sign” function in MATLAB

subplot(2,1,1)
plot(w,X_mag)
grid
xlabel('\omega (rad/sec)')
ylabel('|X(\omega)|')
subplot(2,1,2)
plot(w,X_ang)
grid
xlabel('\omega (rad/sec)')
ylabel('<X(\omega)  (rad)')
```



|X(ω)|
ω (rad/sec)
<X(ω) (rad)
ω (rad/sec)

5. Given FT is $X(\omega) = e^{-jc\omega}\,\mathrm{sinc}^2(0.1\omega), \quad c \in \mathbb{R}$

a. If $x_a(t) = x(4t)$ then this is a time-scaling property. From the FT properties table we have that

$$X_a(\omega) = \frac{1}{4}X(\omega/4) = \frac{1}{4}e^{-jc\omega/4}\,\mathrm{sinc}^2(0.1\omega/4), \quad c \in \mathbb{R}$$

b. If $x_b(t) = x(4t-5)$, then this uses a combination of time-scaling and time shift…. And it matters which order:

$$x(t) \underset{\substack{shift \\ t \to t-5}}{\rightarrow} x(t-5) \underset{\substack{scale \\ t \to 4t}}{\rightarrow} x(4t-5)$$

So now we have to apply those FT properties in the same order:

$$e^{-jc\omega}\,\mathrm{sinc}^2(0.1\omega) \underset{\text{shift by 5}}{\rightarrow} \left[e^{-jc\omega}\,\mathrm{sinc}^2(0.1\omega)\right]e^{-j5\omega} \underset{\text{scale by 4}}{\rightarrow} \frac{1}{4}\left[e^{-jc\omega/4}\,\mathrm{sinc}^2(0.1\omega/4)\right]e^{-j5\omega/4}$$

So that yields

$$X_b(\omega) = \frac{1}{4}\left[e^{-jc\omega/4}\,\mathrm{sinc}^2(0.1\omega/4)\right]e^{-j5\omega/4}$$

c. If $x_c(t) = x(t)e^{-j200t}$, then this is frequency shift (also called modulation):

$$X_c(\omega) = X(\omega-200) = e^{-jc(\omega-200)}\,\mathrm{sinc}^2(0.1(\omega-200)), \quad c \in \mathbb{R}$$

d. If $x_d(t) = x(t)\cos(200t)$, then this is a frequency shift (also called modulation):

$$X_d(\omega) = \frac{1}{2}\left[X(\omega-200) + X(\omega+200)\right]$$

$$= \frac{1}{2}\left[e^{-jc(\omega-200)}\,\mathrm{sinc}^2(0.1(\omega-200)) + e^{-jc(\omega+200)}\,\mathrm{sinc}^2(0.1(\omega+200))\right]$$

6. (a) This is a Path #1 problem! So all we need to is find $H(\omega_o)$ where $\omega_o = 100$ and then use that.

$$H(100) = e^{-j0.1\times 100}\,\text{sinc}\left(100/500\right) = e^{-j10}\,\text{sinc}\left(1/5\right) = 0.935e^{-j10}$$

where evaluating the sinc gives $sinc\left(\frac{1}{5}\right) = 0.935$.

So now the output will be $y(t) = |H(100)| \times 5\cos(100t - \pi/4 + \angle H(100))$

So that becomes:

$$y(t) = 0.935 \times 5\cos(100t - \pi/4 - 10)$$

Or

$$y(t) = 4.675\cos(100t - 10.785)$$

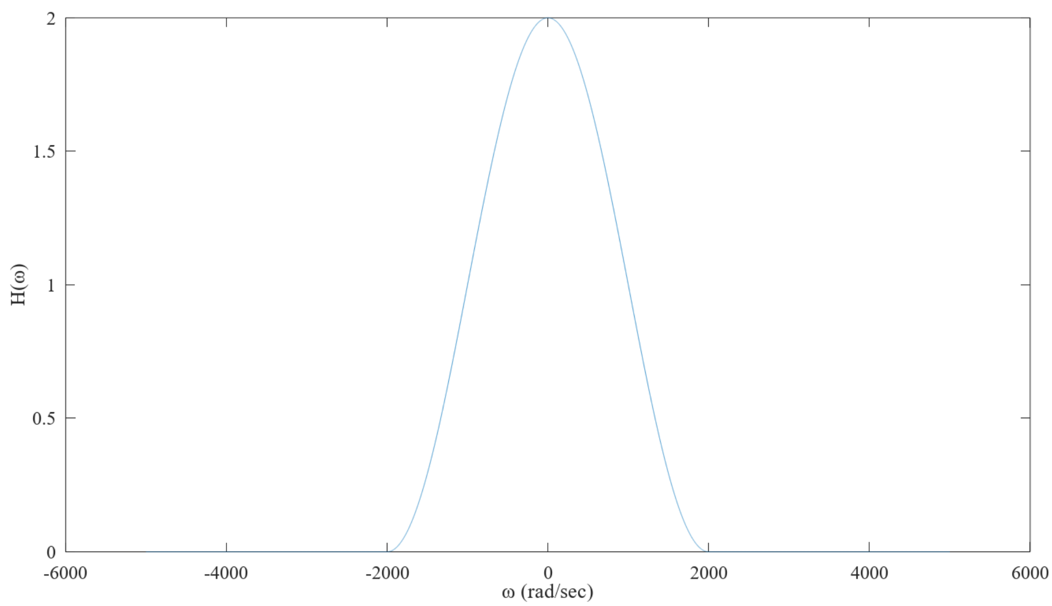
Of course, that -10.785 rad angle can be adjusted by adding or subtracting any multiple of 2π – in particular so the result lies between $\pm\pi$.

(b) Any value of ω_o such that $|H(\omega_o)| = 0$. We have that

$$|H(\omega)| = \left|e^{-j0.1\omega}\,\text{sinc}\left(\omega/500\right)\right| = \left|e^{-j0.1\omega}\right|\left|\text{sinc}\left(\omega/500\right)\right| = \left|\text{sinc}\left(\omega/500\right)\right|$$

So, we need to find all ω_o such that $\left|\text{sinc}\left(\omega_o/500\right)\right| = 0$. We know that $sinc(x) = \sin(\pi x)\,/\pi x$ so $sinc(x) = 0$ whenever x is an integer. Thus, $\left|\text{sinc}\left(\omega_o/500\right)\right| = 0$ whenever $\omega_o/500 = k$ for k an integer. Thus, the values of ω_o we seek are integer multiples of 500 rad/sec.

7. In the given function for $H(\omega)$ the $p_{4000}(\omega)$ is 0 outside $\omega \in [-2000,2000]$. Within that range, the $\cos(2\pi\omega/4000)$ goes through one complete cycle… and adding 1 to it makes the values go no lower than 0 and no higher than 2. Plotting this in MATLAB/Octave gives



From this we can see that
(a) This is NOT an ideal filter because the passband is not constant.
(b) This is a lowpass filter – frequencies near 0 are passed (and boosted by close to 2) and frequencies above 2000 rad/sec are completely stopped.

(c) For an input of $x(t) = 5sinc\left(\frac{2000t}{\pi}\right)$ this is a Path #3 problem. So we need to find the FT $X(\omega)$. From the FT table

$\tau\,\mathrm{sinc}[\tau t/2\pi]$	$2\pi p_\tau(\omega)$

So we have that $\frac{\tau}{2\pi} = \frac{2000}{\pi}$ or that $\tau = 4000$ so we have that

$$x(t) = \frac{5}{4000}\left[4000\,\mathrm{sinc}\left(\frac{4000t}{2\pi}\right)\right] \leftrightarrow X(\omega) = \frac{5}{4000}2\pi p_{4000}(\omega)$$

Now the FT of the output via the Path #3 view is

$$Y(\omega) = X(\omega)H(\omega) = \left[\frac{5}{4000}2\pi p_{4000}(\omega)\right]\left[1+\cos\left(\frac{2\pi\omega}{4000}\right)\right]p_{4000}(\omega)$$

Now, graphically, it is easy to see that $p_{4000}(\omega) \times p_{4000}(\omega) = p_{4000}(\omega)$ so we have that

$$Y(\omega)=\frac{\pi}{400}\left[1+\cos\left(\frac{2\pi\omega}{4000}\right)\right]p_{4000}(\omega)=\frac{\pi}{400}p_{4000}(\omega)+\frac{\pi}{400}p_{4000}(\omega)\cos\left(\frac{2\pi\omega}{4000}\right)$$

Now deal with each of those two terms individually and find the IFT:

$$y_1(t)=\mathscr{F}^{-1}\left\{\frac{\pi}{400}p_{4000}(\omega)\right\}=\mathscr{F}^{-1}\left\{\frac{2\pi}{800}p_{4000}(\omega)\right\}=\frac{1}{800}\mathscr{F}^{-1}\left\{2\pi p_{4000}(\omega)\right\}$$

$$=\frac{4000}{800}\operatorname{sinc}\left(\frac{4000t}{2\pi}\right)=5\operatorname{sinc}\left(\frac{4000t}{2\pi}\right)$$

Now for the second term we use Euler to expand the cosine:

$$\frac{\pi}{400}p_{4000}(\omega)\cos\left(\frac{2\pi\omega}{4000}\right)=\frac{\pi}{400}p_{4000}(\omega)\left[\frac{e^{j\frac{2\pi\omega}{4000}}+e^{-j\frac{2\pi\omega}{4000}}}{2}\right]=\frac{1}{2}\frac{\pi}{400}p_{4000}(\omega)\left[e^{j\frac{2\pi\omega}{4000}}+e^{-j\frac{2\pi\omega}{4000}}\right]$$

Now using the result for the first term together with the time-shift property we get

$$y_2(t)=\mathscr{F}^{-1}\left\{\frac{\pi}{400}p_{4000}(\omega)\cos\left(\frac{2\pi\omega}{4000}\right)\right\}=\mathscr{F}^{-1}\left\{\frac{1}{2}\frac{\pi}{400}p_{4000}(\omega)\left[e^{j\frac{2\pi\omega}{4000}}+e^{-j\frac{2\pi\omega}{4000}}\right]\right\}$$

$$=\frac{1}{2}\left[\mathscr{F}^{-1}\left\{\frac{\pi}{400}p_{4000}(\omega)e^{j\frac{2\pi\omega}{4000}}\right\}+\mathscr{F}^{-1}\left\{\frac{\pi}{400}p_{4000}(\omega)e^{-j\frac{2\pi\omega}{4000}}\right\}\right]$$

$$=\frac{1}{2}\left[5\operatorname{sinc}\left(\frac{4000\left(t+\frac{2\pi}{4000}\right)}{2\pi}\right)+5\operatorname{sinc}\left(\frac{4000\left(t-\frac{2\pi}{4000}\right)}{2\pi}\right)\right]$$

So then the output is the sum of these two terms so

$$y(t)=y_1(t)+y_2(t)=5\operatorname{sinc}\left(\frac{4000t}{2\pi}\right)+\frac{1}{2}\left[5\operatorname{sinc}\left(\frac{4000\left(t+\frac{2\pi}{4000}\right)}{2\pi}\right)+5\operatorname{sinc}\left(\frac{4000\left(t-\frac{2\pi}{4000}\right)}{2\pi}\right)\right]$$