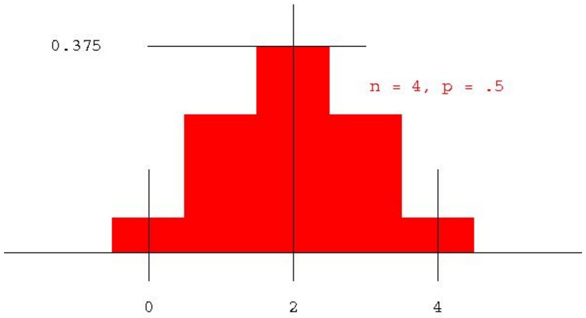
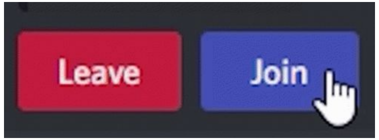
1

Discrete Random Variables, PMF, CDF, Expectation



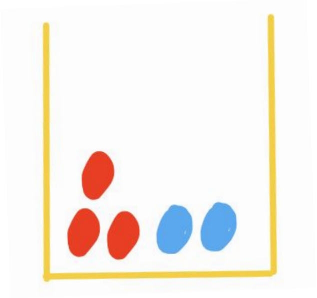


Challenge - **Decision Making**

Newton School of Technology

Game stall in a college fest - **3R**, **2B** balls in a bag

Rules - *Pick a ball → Put it back → Repeat **4 times***



Play

RRRR → Win: **+150₹**

Anything else → Lose: **-10₹**

Exercise - Do this activity, pass the bag to your neighbour

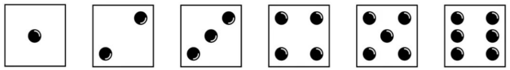
Ques 1. Will you play this game*? Will others play this?*

Ques 2. *If not, for **what reward** will you be ok to play? 200₹, 500₹?*

Recap - **Experiment, SS, Events**

Tossing a fair coin; SS = {H, T}
Events - Getting heads, tails

Tossing a fair coin two times; SS = {HH, HT, TH, TT}
Events - At least one heads; no heads; both heads, Same on both tosses



Rolling a fair dice; SS = {1,2, 3, 4, 5, 6}
Events - Getting a number (say 6), Getting even number

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Rolling 2 die; SS = 36 possibilities
Events - Getting sum *k*, same no. on both dice, one of them is 6

From **Events** to **Random Variable**

Newton School of Technology

Tossing a fair coin
SS = {H, T}
X= No. of Heads

Outcome	X (Number of Heads)
H	1
T	0

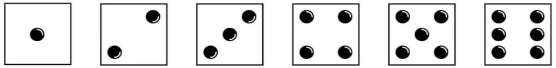
Tossing a fair coin two times; SS = {HH, HT, TH, TT}
X= Both toss result in same face

Outcome(s)	X (Same or Not)
HH, TT	1 (Same)
HT, TH	0 (Not Same)

From **Events** to **Random Variable**

Tossing a fair coin two times;
SS = {HH, HT, TH, TT}, X = No of heads

Outcome(s)	X (Number of Heads)
HH	2
HT, TH	1
TT	0

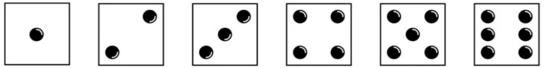


Rolling a fair dice; SS = {1,2, 3, 4, 5, 6}
X = Getting 6 on the die

Outcome(s)	X (Getting 6 or Not)
6	1 (Got 6)
1, 2, 3, 4, 5	0 (Not 6)

From **Events** to **Random Variable**

Newton School of Technology



Rolling a fair dice; SS = {1,2, 3, 4, 5, 6}

Outcome(s)	X (Getting Even or Not)
2, 4, 6	1 (Even)
1, 3, 5	0 (Not Even)

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Rolling 2 die; SS = 36 possibilities

Event - Getting same number on both the die
RV - Same number on two die

Can you tabulate possible values of this RV?
Left as an exercise

From **Events** to **Random Variable**

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Rolling 2 die; SS = 36 possibilities

Outcomes (Dice Rolls)	RV Value (Same Number on Both Dice)
(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)	1 (Same)
All other cases	0 (Different)

From **Events** to **Random Variable**

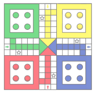
Newton School of Technology

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Rolling 2 die

Event - One of them is six
RV - No. of sixes

Can you tabulate possible values of number of sixes?



Event - Getting sum k
RV - Sum of numbers on die

Can you tabulate possible values of sum?

From Events to Random Variable

Newton School of Technology

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Rolling 2 die; SS = 36 possibilities

Outcomes	X (No. of sixes)
(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)	0
(1,6), (2,6), (3,6), (4,6), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5)	1
(6,6)	2

Outcomes	Sum (k)
(1,1)	2
(1,2), (2,1)	3
(1,3), (2,2), (3,1)	4
(1,4), (2,3), (3,2), (4,1)	5
(1,5), (2,4), (3,3), (4,2), (5,1)	6
(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)	7
(2,6), (3,5), (4,4), (5,3), (6,2)	8
(3,6), (4,5), (5,4), (6,3)	9
(4,6), (5,5), (6,4)	10
(5,6), (6,5)	11
(6,6)	12

Why do we need **Random Variable?**

Newton School of Technology

E: *A subset of outcomes*

RV: *Assign numbers to outcomes*

Die Roll **E**: Getting Even

X=0 {1, 3, 5} X=1 {2, 4, 6}

Coin Toss E: Getting Head

X=0 (No Head) X=1 (1 Head)

3R, 2B **E**: Getting 4R

X=0 (0R) X=1 (1R) X=2 (2R) X=3 (3R) X=4 (4R)

Outcome/Events → Number Why??

E: Descriptive (define specific outcomes)

RV: Analytical & computational (allows mathematical modeling)

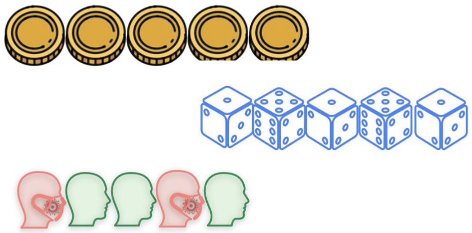
'X = Number of Red Balls'
instead of
'X = 4 Red Balls or Not'

Why? Answer coming up

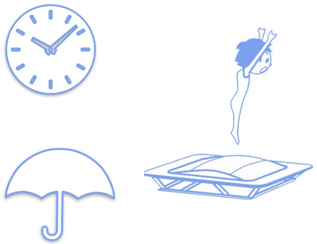
Discrete Random Variable

Discrete Random Variable (DRV) is a type of random variable that can take on a **countable number of distinct values**.
Each value has an associated probability, and the sum of all probabilities equals 1.

- Flipping *k* coins **– Number of heads** (0, 1, 2, 3, 4, 5, ..,k)
- Rolling *k* dice **– Number of 1s** (0, 1, 2, 3, 4, 5, ,..*k*)
- People in Hospital **– Number of sick patients** arriving in an hour (0, 1, 2, ...)



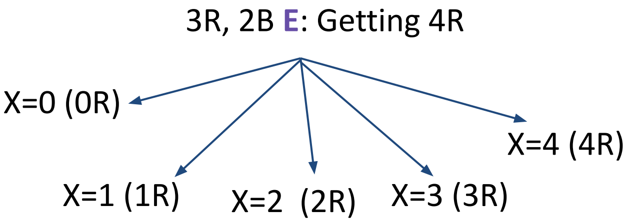
Discrete Random Variable

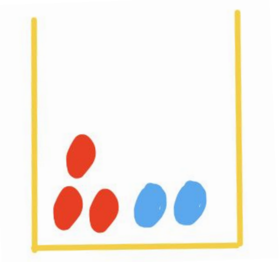


Continuous Random Variable

Probability Mass Function (PMF)

Probability Mass function



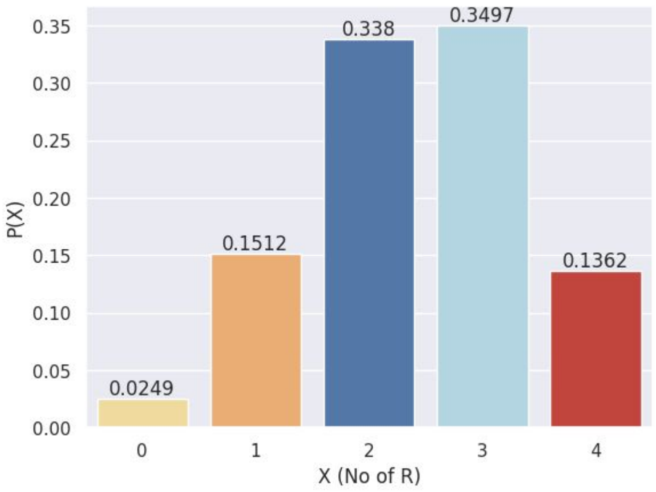


What's the probability of getting 4R balls?

$$P(X=4) = \frac{\text{Number of students who got 4R balls}}{\text{Total number of students who played the game}}$$

P(*X*=0),P(X=1), P(*X*=2), P(*X*=3) - Calculate!

Probability Mass function

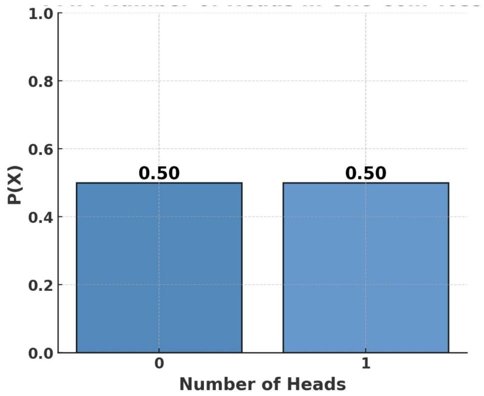


PMF - Function describing probability of a **discrete** random variable takes a specific value.

Maps each possible value of a random variable to its corresponding probability of occurrence.

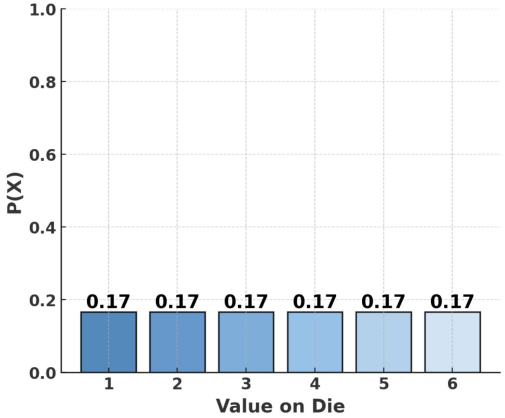
Question: Probability Mass function

Tossing a coin;
E - Getting Heads
X = No. of heads



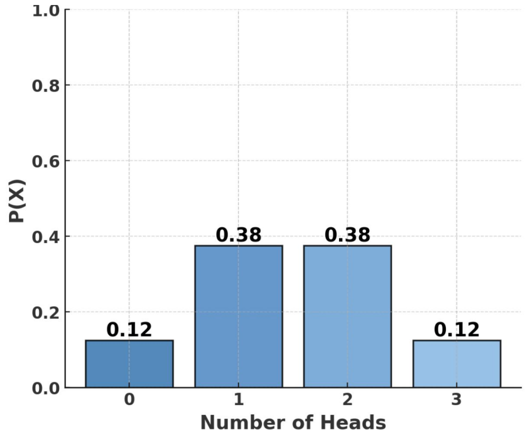
*Outcomes **equally likely***

Rolling a die;
E - Getting 1 to 6
X = Value on the die



*Outcomes **equally likely***

Tossing 3 coins;
E - Getting all heads
X = No. of heads



*Outcomes **Not equally likely***

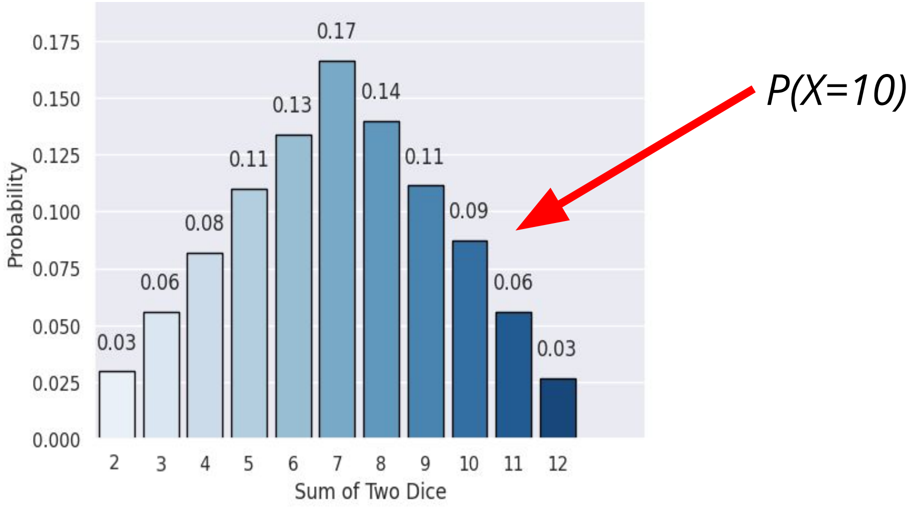
Question: Probability Mass function

Experiment - Roll a die two times

What is your sum of values of two dice?

All outcomes are ***not equally likely***

What's P(*X*=*10*)?



Cumulative Distribution Function (CDF)

Cumulative Distribution Function

3R, 2B **E**: Getting 4R

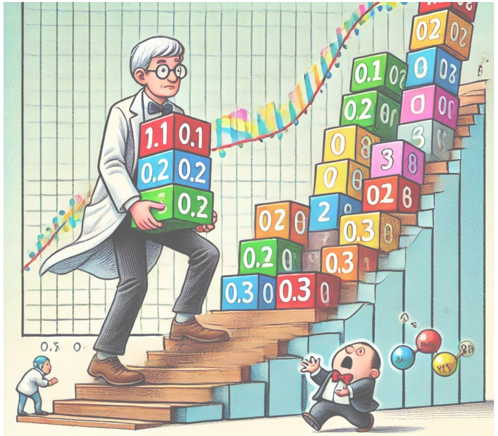
X=0 (0R)

X=1 (1R)

X=2 (2R)

X=3 (3R)

X=4 (4R)



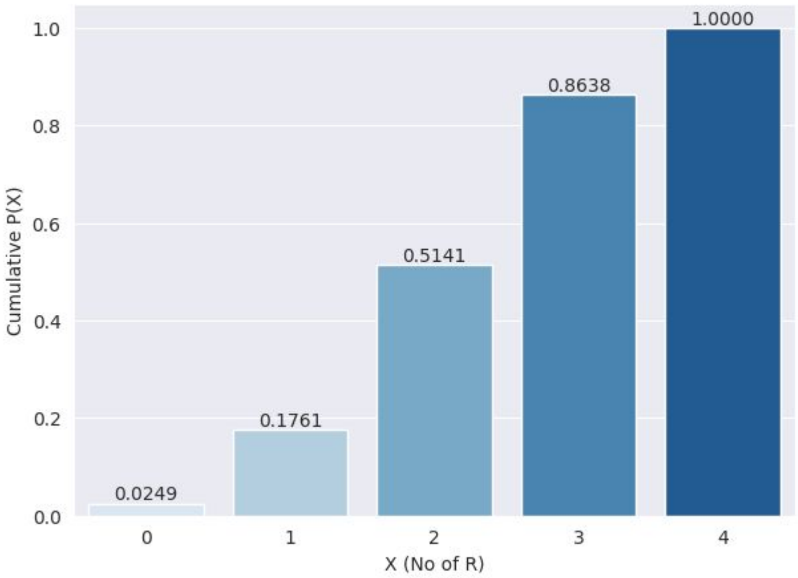
What's the probability of getting less than 2R balls?

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

What's the probability of getting upto 4 red balls?

$$P(X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

Cumulative Distribution Function



CDF gives the probability that the random variable is less than or equal to a certain value:

$$F(x) = P(X \leq x)$$

Cumulative Distribution Function

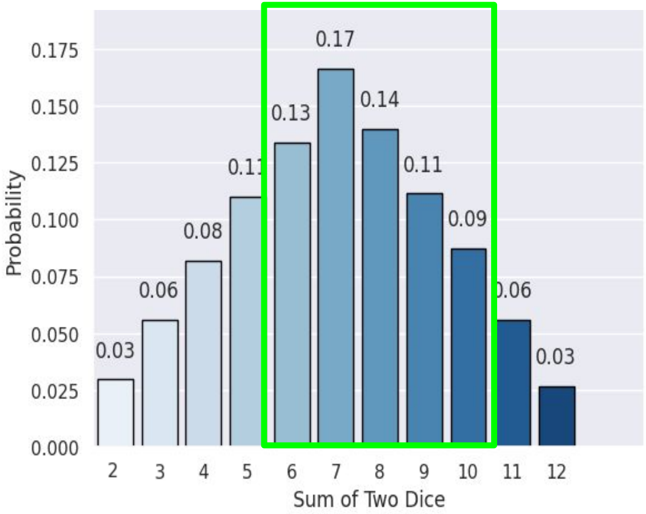
Why CDF? Helps us find **probabilities for range of values efficiently**

Experiment - **Rolling a die two times**

X = Sum of values on die in both throw

What's the probability of winning?

Game Rules - If sum is **between 6 and 10 - you win**, else you lose!!



Calculation using PMF

$$P(6 \leq X \leq 10) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= 0.13 + 0.17 + 0.14 + 0.11 + 0.09$$

$$= 0.64$$

Cumulative Distribution Function

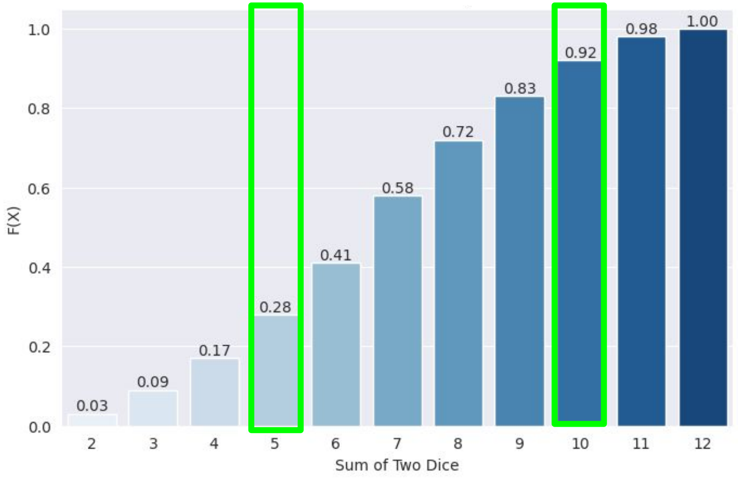
Why CDF? Helps us find **probabilities for range of values efficiently**

Experiment - **Rolling a die two times**

X = Sum of values on die in both throw

Game Rules - If sum is **between 6 and 10 - you win**, else you lose!!

What's the probability of winning?



Calculation using CDF

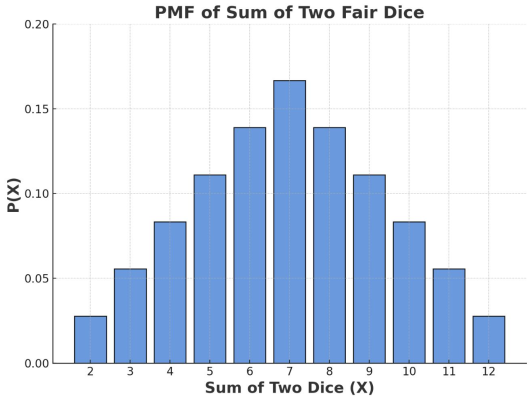
$$F(10) = 0.92, \quad F(5) = 0.28$$

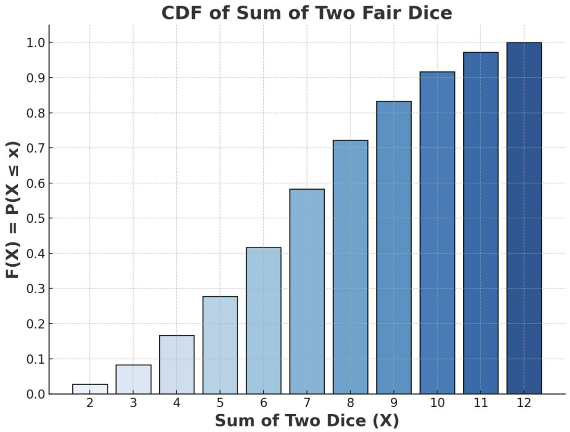
$$P(6 \leq X \leq 10) = 0.92 - 0.28 = 0.64$$

Same answer!!!

Summary - Discrete RV, PMF, CDF

- **Random Variable:** A numerical representation of events for analysis.
- **Discrete Random Variable (DRV):** A random variable with a **countable** set of distinct values.
- **Probability Mass Function (PMF):** A function $P(X)$ that assigns probabilities to each DRV value.
- **Cumulative Distribution Function (CDF):** A function $F(x) = P(X \leq x)$ giving the probability of values ≤ **x**.





Expectation, Expectation Value, Decision Making

Challenge – Decision Making

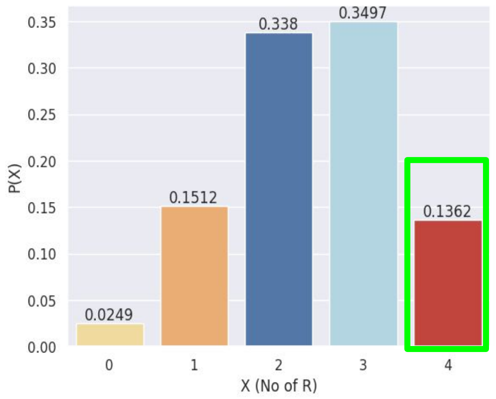
Newton School of Technology

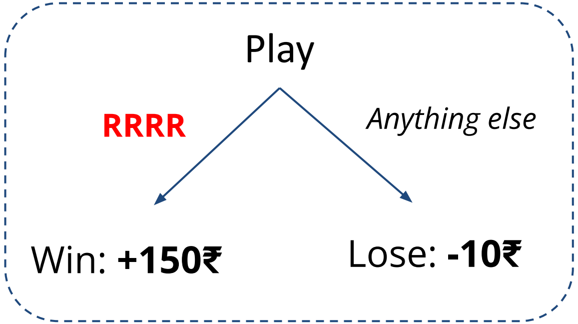
Ques 1. Will you play this game*? Will others play this?*

What's the probability of winning?

$$P(\text{RRRR}) = \text{use PMF } P(X = 4)$$

P(Win) is 0.13 (very low) but stakes are also high!

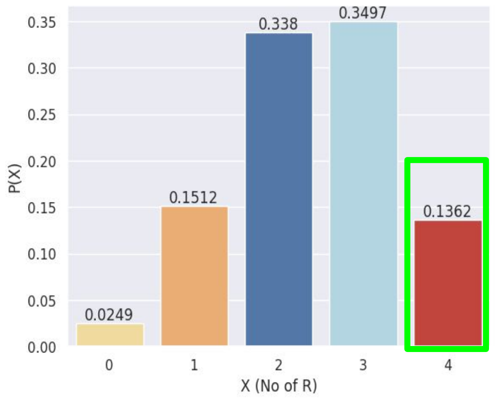




Challenge - Decision Making

Ques 1. Will you play this game*? Will others play this?*

P(Win) is **0.13** (very low) but **stakes are also high!**



Exercise - How many R {0, 1, 2, 3, 4} balls will you get **at average** if you perform multiple trials**?**

$$0 \times 0.0249 \; + \; 1 \times 0.1512 \; + \; 2 \times 0.338 \; + \; 3 \times 0.3497 \; + \; 4 \times 0.1362$$

$$= \; 0 \; + \; 0.1512 \; + \; 0.676 \; + \; 1.0491 \; + \; 0.5448$$

$$= \; 2.4211 \; \approx \; 2.42$$

We will get about **2.42 R balls per trial** if you repeat the game multiple times. Why?

Expectation, Expected Value

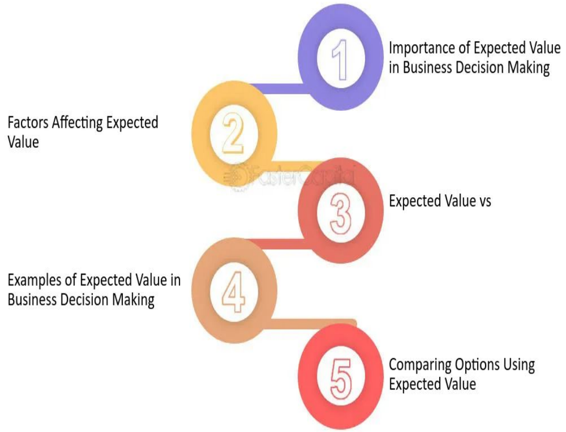
Newton School of Technology

*We will get about **2.42 R balls per trial** if you repeat the game multiple times*

Conceptually - Long-run average outcome of a random variable if it were to be repeated many times under identical conditions.

Mathematically - Sum the products of each possible outcome of the variable and its corresponding probabilit

$$E(X) = \sum_i x_i \cdot p_i$$



Crucial in decision-making processes. Calculating the expected outcomes for different actions, decision-makers can **choose the option with the most beneficial average result**.

Challenge – Decision Making

Newton School of Technology

Lets use concept of **Expectation** and Expected Value in calculating **Expected Earning**

Lets define *X* as amount earned/lost in one play

Event	X	P(X)
4R	150	0.1362
0R, 1R, 2R, 3R	-10	0.8638

$$E(X) = (150 \times 0.1362) + (-10 \times 0.8638)$$
$$= 20.43 - 8.64$$
$$= 11.79$$

Since, expected earning over multiple trials > 0, it should be okay to play the game?

But does **+11.79 ₹** sounds rewarding to attempt multiple trials?

What about organisers PoV?
Fair for them? Take home exercise

Additional References

Newton School of Technology

Reference Book - "A First Course in Probability" by Sheldon Ross

1. Chapter 4 - Random Variables
    a. In Section 4.1 (Random Variable), 4.2 (Discrete Random Variable), 4.3 (Expected Values
2. **Videos**
    a. Random Variable lecture Khan Academy Random Variable
    b. Discrete Random Variables
    c. Intuitive Understanding of Expectation Value
    d. Expected Value Examples
    e. The Most controversial Problem in Philosophy
    f. St. Petersburg Paradox
3. **Blogs and articles**
    a. MIT Lectures on Discrete Random Variable
    b. Paradoxes for Probabilities : Expectation