

Universal entrywise eigenvector fluctuations in delocalized spiked matrix models and asymptotics of rounded spectral algorithms

Shujing Chen* and Dmitriy Kunisky†

Department of Applied Mathematics & Statistics, Johns Hopkins University

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Abstract

We consider the distribution of the top eigenvector \hat{v} of a spiked matrix model of the form $H = \theta vv^* + W$, in the supercritical regime where H has an outlier eigenvalue of comparable magnitude to $\|W\|$. We show that, if v is sufficiently delocalized, then the distribution of the individual entries of \hat{v} (not, we emphasize, merely the inner product $\langle \hat{v}, v \rangle$) is universal over a large class of generalized Wigner matrices W having independent entries, depending only on the first two moments of the distributions of the entries of W . This complements the observation of Capitaine and Donati-Martin (2018) that these distributions are *not* universal when v is instead sufficiently localized. Further, for W having entrywise variances close to constant and thus resembling a Wigner matrix, we show by comparing to the case of W drawn from the Gaussian orthogonal or unitary ensembles that averages of entrywise functions of \hat{v} behave as they would if \hat{v} had Gaussian fluctuations around a suitable multiple of v . We apply these results to study spectral algorithms followed by rounding procedures in dense stochastic block models and synchronization problems over the cyclic and circle groups, obtaining the first precise asymptotic characterizations of the error rates of such algorithms.

*Email: schen344@jhu.edu

†Email: kunisky@jhu.edu.

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1 Introduction

We will study a universality phenomenon in *spiked matrix models*, of which we take the simple rank-one case

$$H = \theta vv^* + W,$$

where $\theta \in \mathbb{R}$, $v \in \mathbb{S}^{n-1}(\mathbb{C}) \subset \mathbb{C}^n$ (the unit sphere of \mathbb{C}^n), and $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ is a Hermitian “noise” matrix normalized such that $\|W\|$ is of constant order. From a statistical point of view, H can be seen as an “observation” of the hidden “signal” v , from which one seeks to estimate v . A natural choice of estimator is $\hat{v} = v_1(H)$, the unit eigenvector of H corresponding to the largest eigenvalue of H , which we denote $\hat{\lambda} = \lambda_1(H)$. Our general goal will be to understand in a fine-grained way the quality of estimate of v that we obtain from \hat{v} .

Let us explain some of what is known about such models under the following assumption on W , which we will also use in the first part of our main results.

Definition 1.1 (Generalized Wigner matrix). *A generalized Wigner matrix with $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ with parameters (γ_W, ξ_W) is a random matrix having the following properties:*

1. *The entries on and above the diagonal $(W_{ij})_{1 \leq i \leq j \leq n}$ are independent.*
2. *$\mathbb{E}W_{ij} = 0$ for all $i, j \in [n]$.*
3. *The magnitudes $|W_{ij}|$ admit a uniform tail bound of the form*

$$\mathbb{P}[\sqrt{n} \cdot |W_{ij}| \geq t] \leq \xi_W^{-1} \exp(-t^{\xi_W}) \quad (1.1)$$

We will call this condition the entries W_{ij} being ξ_W -power-subexponential (Definition 2.1).

4. *The numbers $\sigma_{ij}^2 := \mathbb{E}|W_{ij}|^2$ satisfy*

$$\gamma_W^{-1} \leq n \cdot \sigma_{ij}^2 \leq \gamma_W \text{ for all } i, j \in [n], \quad (1.2)$$

as well as

$$\sum_{j=1}^n \sigma_{ij}^2 = 1 \text{ for all } i \in [n]. \quad (1.3)$$

This class of matrices is a well-studied *universality class*, sharing many basic behaviors. For instance, as a useful point of reference for the results we discuss next, under these assumptions the empirical distribution of W almost surely converges weakly to the semicircle distribution supported on $[-2, 2]$, and $\lambda_1(W)$ and $\|W\|$ both converge in probability to 2 (see, e.g., [AGZ10] for an exposition of these classical results, or our Theorem 2.13).

We mostly take v to be deterministic, and later in our results we will mention the case of v random, in which case it is random independently of W (in this situation results for v deterministic can be applied by conditioning on the value of v).

The following describes the main features of the *phase transition* that the behavior of the spectrum of H undergoes as θ varies, a version for this setting of the *Baik–Ben Arous–Péché* transition. We sketch how the proof of this version follows from the *isotropic local law* proved by [BEK⁺14] in Theorem 2.21 below; this technique has been used by several works such as [KY13, KY14] in the past and we merely adapt the particular local law on which it is based.

Theorem 1.2. Let $W = W^{(n)} \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a sequence of generalized Wigner matrices with parameters (γ_W, ξ_W) not depending on n and let $v = v^{(n)} \in \mathbb{S}^{n-1}(\mathbb{C})$. Write $H = H^{(n)} = \theta vv^* + W$, $\hat{\lambda} = \hat{\lambda}^{(n)} = \lambda_1(H^{(n)})$ for the largest eigenvalues of these matrices, and $\hat{v} = \hat{v}^{(n)}$ for the associated eigenvectors of H . Then, the following hold, with all convergences in probability as $n \rightarrow \infty$:

$$\hat{\lambda} \rightarrow \lambda(\theta) := \begin{cases} 2 & \text{if } \theta \leq 1 \\ \theta + \theta^{-1} & \text{if } \theta > 1 \end{cases}, \quad (1.4)$$

$$\langle \hat{v}, v \rangle^2 \rightarrow \rho(\theta)^2 := \begin{cases} 0 & \text{if } \theta \leq 1 \\ 1 - \theta^{-2} & \text{if } \theta > 1 \end{cases}. \quad (1.5)$$

It will also be useful for us to have a specific notation for

$$\tau(\theta)^2 := 1 - \rho(\theta)^2 = \frac{1}{\theta^2}, \quad (1.6)$$

the typical value of $1 - \langle \hat{v}, v \rangle^2$, the magnitude of the component of \hat{v} orthogonal to v .

This gives a precise understanding of the top eigenvalue $\hat{\lambda}$ and eigenvector \hat{v} of H to leading order, though for the eigenvector only in terms of its correlation with its noiseless counterpart v . For both quantities, it is reasonable to ask about the next-order fluctuations: how do a rescaling of $\hat{\lambda} - \lambda(\theta)$ and $\hat{v} - \rho(\theta)v$ behave? (The latter is not quite well-defined for the reason in Remark 1.5, but let us ignore this issue for the moment.)

While it would be natural to conjecture that both of these quantities should also behave universally over a generalized Wigner matrices or some other such large class, in fact this is not the case. In our setting, it is known that *non-universality* of the fluctuations of both eigenvalues and eigenvectors—that is, a sensitivity to the specific distribution of entries of W —arises when v is *localized*, having some large entries. See, for instance, [CDMF09, CDMF12, RS12, BGGM11, KY13, KY14] for such results on eigenvalues, [CDM18, BDW21], and [MY22, FFHL22] for similar results about eigenvectors but in different settings (where the signal is of macroscopic rank and the noise is Gaussian in the first case, and where $\theta = \theta(n) \rightarrow \infty$ in the second case). In the former works, it is also shown that the fluctuations of eigenvalues are universal when v is sufficiently delocalized.

Summary of contributions To the best of our knowledge, until now a gap has remained in the above line of work concerning a situation that is particularly important in applications: it is not known that the *eigenvector* fluctuations—those of \hat{v} —are universal provided that v is *delocalized*, in any stronger sense beyond just the behavior of the specific inner product $\langle \hat{v}, v \rangle$. This is the question that we take up in this paper: we will show such a universality result in a rather strong entrywise sense, in the style of the results for eigenvectors of purely random matrices with no low-rank perturbation of [KY11]. Our final goal will be to treat quantities of the form

$$\frac{1}{n^2} \sum_{i,j=1}^n \psi(n \cdot v_i \bar{v}_j, n \cdot \hat{v}_i \bar{\hat{v}}_j).$$

If we view ψ as a measurement of proximity of two complex numbers (where we scale by n since the typical scale of each of these numbers is $\Theta(1/n)$), this gives us access to various ways of viewing the amount of error made by a spectral algorithm by measuring various notions of entrywise distance between vv^* and $\hat{v}\hat{v}^*$.

In particular, when we have prior information that v belongs to some special subset of \mathbb{C}^n , we may include *rounding* functions in the definition of ψ , that encode a procedure estimating vv^* by first computing $\hat{v}\hat{v}^*$ and then “projecting” its value in some suitable entrywise sense to the set in

which the entries $v_i \overline{v_j}$ of vv^* can possibly lie. As a simple example, if we know in advance that $v \in \{\pm 1/\sqrt{n}\}^n$ and W is real-valued, then it is sensible to take the entrywise sign $\text{sgn}(\widehat{v}_i \widehat{v}_j)/n$ as an estimate of $v_i v_j$. We can then measure the natural notion of error of what fraction of these rounded signs disagree with those of vv^* , i.e., the quantity:

$$\frac{1}{n^2} \sum_{i,j=1}^n \mathbf{1}\{\text{sgn}(\widehat{v}_i \widehat{v}_j) \neq \text{sgn}(v_i v_j)\},$$

arguably a more natural and meaningful notion of error for this discrete setting than continuous quantities like $\|\widehat{v}\widehat{v}^* - vv^*\|_F^2$. To the best of our knowledge, ours are the first precise asymptotics for the amount of entrywise error made by such algorithms under general non-Gaussian models of the noise W , and we will see that we can give the same precise analysis of even fully discrete models such as dense stochastic block models and synchronization over cyclic groups.

1.1 Theoretical results

Our first main result is that, provided that v is sufficiently delocalized and that W is sufficiently unstructured, statistics of the noisy top eigenvector \widehat{v} do not depend on the entry distribution of W . The specific delocalization assumption we make on v is the following:

Assumption 1.3. *We assume that $v \in \mathbb{C}^n$ has $\|v\| = 1$, and that*

$$\|v\|_\infty \leq C_v n^{-1/2+\varepsilon_v}$$

for some $\varepsilon_v \in (0, 1/20)$ and $C_v > 0$. We call (ε_v, C_v) the parameters of this assumption on v .

Our first main result is then as follows.

Theorem 1.4 (Universality of entry statistics). *Let $\theta > 1$ and $v \in \mathbb{S}^{n-1}(\mathbb{C})$ satisfy Assumption 1.3. Let W and X be two generalized Wigner matrices. Suppose that the second moments of the real and imaginary parts of X match (recalling that their first moments are fixed to be zero by the assumption of being generalized Wigner matrices): for all $i, j \in [n]$, we have*

$$\begin{aligned} \mathbb{E}\text{Re}(W_{ij})^2 &= \mathbb{E}\text{Re}(X_{ij})^2, \\ \mathbb{E}\text{Im}(W_{ij})^2 &= \mathbb{E}\text{Im}(X_{ij})^2, \\ \mathbb{E}\text{Re}(W_{ij})\text{Im}(W_{ij}) &= \mathbb{E}\text{Re}(X_{ij})\text{Im}(X_{ij}). \end{aligned}$$

Let $\widehat{v}(W)$ be the eigenvector associated to the largest eigenvalue of $\theta vv^* + W$ and $\widehat{v}(X)$ be that associated to the largest eigenvalue of $\theta vv^* + X$. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a \mathcal{C}^5 function such that ϕ and its first five derivatives are bounded. Then, for any $\varepsilon > 0$,

$$\max_{1 \leq i, j \leq n} \left| \mathbb{E}\phi\left(n \cdot \widehat{v}(W)_i \overline{\widehat{v}(W)_j}\right) - \mathbb{E}\phi\left(n \cdot \widehat{v}(X)_i \overline{\widehat{v}(X)_j}\right) \right| \leq C n^{-1/2+10\varepsilon_v+\varepsilon},$$

for a constant C depending only on the parameters $\varepsilon, \theta, \phi$ in the statement of this result, the parameters (ε_v, C_v) of Assumption 1.3 on v , and the generalized Wigner matrix parameters (γ_W, ξ_W) and (γ_X, ξ_X) .

Remark 1.5. *The reason for taking products of pairs of entries of the various \widehat{v} above is that \widehat{v} itself is only defined up to sign, while $\widehat{v}\widehat{v}^*$, the orthogonal projection to the \widehat{v} direction, is a well-defined geometric object; above we formulate our result in terms of its entries.*

This is a conceptually interesting result, but one that does not allow concrete calculations of the actual values of entrywise quantities like the above. By comparing to special Gaussian distributions of random matrices, however, we can allow for that as well under more stringent assumptions on W . We consider which W can be compared by Theorem 1.4 to the following classical distributions of random matrices:

Definition 1.6 (Gaussian orthogonal and unitary ensembles). *Let $n \geq 1$. We then define the following distributions of random matrices in $\mathbb{C}_{\text{herm}}^{n \times n}$.*

- The Gaussian orthogonal ensemble, denoted $\text{GOE}(n)$ for given dimension n , is the law of $W \in \mathbb{R}_{\text{sym}}^{n \times n}$ with W_{ij} independent for $1 \leq i \leq j \leq n$ drawn as

$$\begin{aligned} W_{ij} &\sim \mathcal{N}\left(0, \frac{1}{n}\right) \text{ for } 1 \leq i < j \leq n, \\ W_{ii} &\sim \mathcal{N}\left(0, \frac{2}{n}\right) \text{ for } 1 \leq i \leq n. \end{aligned}$$

Equivalently, $\{\sqrt{n} \cdot W_{ij} : 1 \leq i < j \leq n\} \cup \{\sqrt{n/2} \cdot W_{ii} : 1 \leq i \leq n\}$ are i.i.d. with law $\mathcal{N}(0, 1)$.

- The Gaussian unitary ensemble, denoted $\text{GUE}(n)$ for given dimension n , is the law of $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ with W_{ij} independent for $1 \leq i \leq j \leq n$ drawn as

$$\begin{aligned} W_{ij} &\sim \mathcal{N}_{\mathbb{C}}\left(0, \frac{1}{n}\right) \text{ for } 1 \leq i < j \leq n, \\ W_{ii} &\sim \mathcal{N}_{\mathbb{R}}\left(0, \frac{1}{n}\right) \text{ for } 1 \leq i \leq n. \end{aligned}$$

Here $\mathcal{N}_{\mathbb{R}}(0, \sigma^2) = \mathcal{N}(0, \sigma^2)$ is the ordinary Gaussian measure, while $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ is the complex Gaussian measure, the law of $Z = X + iY$ with $X, Y \sim \mathcal{N}_{\mathbb{R}}(0, \sigma^2/2)$ independently, so that $\mathbb{E}|Z|^2 = \sigma^2$. Equivalently, a GUE matrix W has $\{\sqrt{2n} \cdot \text{Re}(W_{ij}) : 1 \leq i < j \leq n\} \cup \{\sqrt{2n} \cdot \text{Im}(W_{ij}) : 1 \leq i < j \leq n\} \cup \{\sqrt{n} \cdot W_{ii} : 1 \leq i \leq n\}$ are i.i.d. with law $\mathcal{N}(0, 1)$.

Here and throughout we denote the imaginary unit by i to avoid conflict with indices named i . It is easy to check that GOE and GUE matrices are both generalized Wigner matrices, up to a negligible renormalization in the GOE case.

When our matrices are drawn from these distributions, the behavior of \hat{v} can be analyzed quite precisely. That is because $X \sim \text{GOE}(n)$ and $X \sim \text{GUE}(n)$ are respectively *orthogonally* and *unitarily invariant*, satisfying the property that QXQ^* has the same law as X for any orthogonal or any unitary Q , respectively. Because of this, the component of \hat{v} that is orthogonal to v is itself a likewise invariant random vector in this subspace, whose norm by Theorem 1.2 is close to $\tau(\theta) = \theta^{-1}$. Thus, setting aside the sign ambiguity in \hat{v} mentioned in Remark 1.5, approximating this component by a Gaussian random vector (which is also orthogonally invariant) of comparable norm, we expect

$$\hat{v}(X) \xrightarrow{(d)} \mathcal{N}_{\mathbb{F}}\left(\rho(\theta)v, \frac{\tau(\theta)^2}{n}(I_n - vv^*)\right).$$

Here, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is the field we are working over depending on whether X is a real or complex Wigner matrix. Further, since the mean is a constant multiple of v , we should be able to neglect the contribution of $\frac{1}{\theta^2 n} vv^*$ in the covariance. Similar ideas have appeared before in the literature in [CH12, LCC24], but we give a full derivation of a precise version of such an approximation specific to our setting in Section 4.1.

Through Theorem 1.4, together with such analysis, we expect to obtain more concrete predictions for generalized Wigner matrices whose first two moments match those of either a GOE or GUE matrix. That leads to the following related definition of a more restrictive class of random matrices, in which we also allow some slack in the assumption of exact moment matching that we will see we may deal with in the proof of our next result, and which makes these results more useful for applications.

Definition 1.7 (Weakly Wigner matrices). *A weakly Wigner matrix $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ with parameters $(\xi_W, \varepsilon_W, C_W)$ is a matrix satisfying the following properties. The first three are the same as the first three properties of a generalized Wigner matrix (Definition 1.1):*

1. *The entries on and above the diagonal $(W_{ij})_{1 \leq i \leq j \leq n}$ are independent.*
2. *$\mathbb{E}W_{ij} = 0$ for all $i, j \in [n]$.*
3. *The $|W_{ij}|$ are ξ_W -power-subexponential.*

Further, we require one of the following two additional properties:

4. *We say that W is weakly \mathbb{R} -Wigner if for some constants $\varepsilon_W, C_W > 0$ we have*

$$\begin{aligned} \mathbb{E}W_{ii}^2 &\leq \frac{C_W}{n} \text{ for all } 1 \leq i \leq n, \\ \left| \mathbb{E}W_{ij}^2 - \frac{1}{n} \right| &\leq \frac{C_W}{n^{1+\varepsilon_W}} \text{ for all } 1 \leq i < j \leq n. \end{aligned}$$

- 4'. *We say that W is weakly \mathbb{C} -Wigner if*

$$\begin{aligned} \mathbb{E}W_{ii}^2 &\leq \frac{C_W}{n} \text{ for all } 1 \leq i \leq n, \\ \left| \mathbb{E}\text{Re}(W_{ij})^2 - \frac{1}{2n} \right| &\leq \frac{C_W}{n^{1+\varepsilon_W}} \text{ for all } 1 \leq i < j \leq n, \\ \left| \mathbb{E}\text{Im}(W_{ij})^2 - \frac{1}{2n} \right| &\leq \frac{C_W}{n^{1+\varepsilon_W}} \text{ for all } 1 \leq i < j \leq n, \\ |\mathbb{E}\text{Re}(W_{ij})\text{Im}(W_{ij})| &\leq \frac{C_W}{n^{1+\varepsilon_W}} \text{ for all } 1 \leq i < j \leq n. \end{aligned}$$

We will often use the variable $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ specifying whether the matrices we are working with are real or complex weakly Wigner matrices.

We note that it is actually not quite true that all weakly Wigner matrices are generalized Wigner matrices, because the exact condition (1.3) need not hold. However, the weakly Wigner matrix assumptions imply that it holds up to an $O(n^{-\varepsilon_W})$ error, which we will see in our arguments suffices for our purposes below.

Our second main result formalizes the result of the above sketch of an argument combined with Theorem 1.4. In particular, we obtain a precise prediction for averages of entrywise functions of \widehat{vv}^* for $\widehat{v} = \widehat{v}(W)$ defined with weakly \mathbb{F} -Wigner matrices W for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ matching what we predicted for GOE and GUE matrices, respectively, based on their special invariance properties. As will be useful in applications, we may also allow these functions to depend on the corresponding entries of vv^* , allowing us to analyze various notions of the quality of entrywise approximation that \widehat{vv}^* gives to vv^* .

Theorem 1.8. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\theta > 1$, suppose that $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ is a weakly \mathbb{F} -Wigner matrix, let $v \in \mathbb{S}^{n-1}(\mathbb{F})$ satisfy Assumption 1.3, and let \hat{v} be the eigenvector associated to the largest eigenvalue of $\theta vv^* + W$. Let $\psi : \mathbb{C}^2 \rightarrow \mathbb{R}$ be a \mathcal{C}^5 function such that the values of ψ and its first five derivatives are bounded. Define the associated function $\Psi : \mathbb{C}_{\text{herm}}^{n \times n} \times \mathbb{C}_{\text{herm}}^{n \times n} \rightarrow \mathbb{R}$ by

$$\Psi(A, B) := \frac{1}{n^2} \sum_{i,j=1}^n \psi(A_{ij}, B_{ij}).$$

Define a standard Gaussian random vector over the same field, $g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)$, i.e., having entries $g_i \sim \mathcal{N}_{\mathbb{F}}(0, 1)$ drawn i.i.d. Then, for any $\varepsilon > 0$,

$$\begin{aligned} & |\mathbb{E}\Psi(n \cdot vv^*, n \cdot \hat{v}\hat{v}^*) - \mathbb{E}\Psi(n \cdot vv^*, (\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)(\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)^*)| \\ & \leq C(n^{-1/2+10\varepsilon_v+\varepsilon} + n^{-\varepsilon_W/2+\varepsilon}), \end{aligned}$$

where C is a constant depending only on the parameters $\mathbb{F}, \varepsilon, \theta, \psi$ in the statement of this result, the parameters (ε_v, C_v) of Assumption 1.3 on v , the generalized Wigner matrix parameters (γ_W, ξ_W) , and the additional weakly Wigner matrix parameters (ε_W, C_W) .

Suppose further that v is also drawn at random and independently of W such that $\sqrt{n} \cdot v_i$ are i.i.d. according to some bounded probability measure μ on \mathbb{F} . Then, for any $\varepsilon > 0$,

$$\begin{aligned} & \left| \mathbb{E}\Psi(n \cdot vv^*, n \cdot \hat{v}\hat{v}^*) - \mathbb{E}_{\substack{v, w \sim \mu \\ g, h \sim \mathcal{N}_{\mathbb{F}}(0, 1)}} \psi(v\bar{w}, (\rho(\theta)v + \tau(\theta)g)(\rho(\theta)\bar{w} + \tau(\theta)h)) \right| \\ & \leq C(n^{-1/2+\varepsilon} + n^{-\varepsilon_W/2+\varepsilon}), \end{aligned} \tag{1.7}$$

where C is a constant depending only on the parameters $\mathbb{F}, \varepsilon, \theta, \psi$ in the statement of this result, the distribution μ of the entries of v , and the weakly Wigner matrix parameters $(\xi_W, \varepsilon_W, C_W)$.

Remark 1.9. We emphasize the important assumption that v has entries in the same field as W ; in particular, we exclude the case of $v \in \mathbb{C}^n$ while W is a weakly \mathbb{R} -Wigner matrix.

We think of $\psi(a, b)$ as an *entrywise loss* function for the estimation of v by \hat{v} , measuring some notion of distance between $a, b \in \mathbb{C}$. $\Psi(A, B)$ then averages this loss over the entries of a pair of matrices. The final bound (1.7) then describes a “single-letter formula” for the expectation of any such loss. In particular, if we have an asymptotic sequence of $v = v^{(n)} \in \mathbb{S}^{n-1}(\mathbb{F})$ drawn as above for some fixed entrywise distribution μ not depending on n , a sequence of weakly \mathbb{F} -Wigner $W = W^{(n)}$ satisfying the definition with any parameters $(\xi_W, \varepsilon_W, C_W)$ not depending on n , and $\hat{v} = \hat{v}^{(n)}$ the top eigenvectors of $\theta v^{(n)} v^{(n)*} + W^{(n)}$, then the above implies the exact asymptotic result

$$\lim_{n \rightarrow \infty} \mathbb{E}\Psi(n \cdot vv^*, n \cdot \hat{v}\hat{v}^*) = \mathbb{E}_{\substack{v, w \sim \mu \\ g, h \sim \mathcal{N}_{\mathbb{F}}(0, 1)}} \psi(v\bar{w}, (\rho(\theta)v + \tau(\theta)g)(\rho(\theta)\bar{w} + \tau(\theta)h)). \tag{1.8}$$

Thus, we characterize the asymptotic expectation of any entrywise measurement of the loss achieved by such a spectral algorithm as $n \rightarrow \infty$ by an expectation of fixed dimension on the right-hand side, over just the four scalar random variables v, w, g , and h , which is complicated to evaluate in closed form for most ψ but can easily be estimated numerically by straightforward Monte Carlo integration methods.

1.2 Applications

We give applications to the following setting where spectral algorithms have been used with entrywise rounding in the literature [Sin11, MAC12, EM19, GZ19, MH21]. Let G be a compact Lie group, and write $\text{Haar}(G)$ for its Haar measure. We suppose that we draw $x_1, \dots, x_n \sim \text{Haar}(G)$, from which we can form the matrix of pairwise differences:

$$M_{ij} := x_i x_j^{-1},$$

a matrix $M \in G^{n \times n}$ that is G -Hermitian in the sense that $M_{ji} = M_{ij}^{-1}$. We observe a noisy version of this matrix,

$$Y_{ij} \sim \begin{cases} M_{ij} & \text{with probability } p, \\ \text{Haar}(G) & \text{with probability } 1-p \end{cases} \quad (1.9)$$

for each $1 \leq i < j \leq n$, and set $Y_{ii} = e$ the group identity¹ and $Y_{ji} = Y_{ij}^{-1}$ to preserve the G -Hermitian property. Here $p = p(n) \in (0, 1)$ is some parameter governing the amount of information available in the observation Y . Our goal is to produce an estimator \widehat{M} of M . (Note that, for the same reason as in Remark 1.5, it is not sensible to try to recover the x_i themselves.) Such problems in general are referred to as *group synchronization*, and this particular noise model is called the *truth-or-Haar model*, so named by [PWBM16].

While quite general G have been considered in the literature, we consider two particular examples that are well-suited to our results: the finite cyclic groups $G = \mathbb{Z}/L$ for some $L \geq 2$, and the infinite circle group $G = U(1) \cong \text{SO}(2)$, which we identify with $\mathbb{R}/2\pi$. Note that the cyclic groups are subgroups of $U(1)$; all of these cases are sometimes referred to as *angular synchronization* problems. In these cases, Singer in [Sin11] observed that a reasonable spectral algorithm is as follows. (See Section 1.3 below for further references.)

First, from $Y \in G^{n \times n}$, we build $H \in \mathbb{C}_{\text{herm}}^{n \times n}$ by applying an injective group character to the entries of Y . To be concrete, we take:

$$\chi(x) := \begin{cases} \exp(2\pi ix/L) & \text{if } G = \mathbb{Z}/L, \\ \exp(ix) & \text{if } G = U(1) \end{cases}, \quad (1.10)$$

and set

$$H_{ij} := \frac{1}{\sqrt{n}} \chi(Y_{ij}).$$

Note that this H is Hermitian since we have chosen Y to be G -Hermitian, and as we will see it satisfies the same normalization as our generalized Wigner and weakly \mathbb{F} -Wigner matrices above. Next, let $\widehat{v} = v_1(H)$. Finally, let $\text{Round} : \mathbb{C} \rightarrow G$ be some function that rounds complex numbers to G . We think of Round as an inverse of χ , extended to an “approximate inverse” on all of \mathbb{C} rather than just $\chi(G) \subset \mathbb{C}$. Using this rounding function, we define the estimator

$$\widehat{M}_{ij} := \text{Round}(n \cdot \widehat{v}_i \overline{\widehat{v}_j}).$$

For instance, for $G = U(1)$ it is natural to take $R(z) := z/|z|$, while for $G = \mathbb{Z}/L$ to round by considering the nearest L th root of unity.

Theorem 1.10. *Suppose in the above setting that*

$$p = p(n) = \frac{\theta}{\sqrt{n}}$$

¹The diagonal entries will not contribute to the statistics we consider, so this choice is inconsequential.

for some $\theta > 1$. Let $\text{Round} : \mathbb{C} \rightarrow G$ be a piecewise constant function with a rectifiable set of discontinuity points if $G = \mathbb{Z}/L$, or a piecewise smooth function with the same property if $G = U(1)$. Let $\ell : G \times G \rightarrow \mathbb{R}$ be arbitrary if $G = \mathbb{Z}/L$, or a smooth function if $G = U(1)$. Let $\mathbb{F} = \mathbb{R}$ if $G = \mathbb{Z}/2$ and $\mathbb{F} = \mathbb{C}$ otherwise. Then, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i,j=1}^n \ell(M_{ij}, \widehat{M}_{ij}) = \mathbb{E}_{\substack{x,y \sim \text{Haar}(G) \\ g,h \sim \mathcal{N}_{\mathbb{F}}(0,1)}} \ell\left(xy^{-1}, \text{Round}\left((\rho(\theta)\chi(x) + \tau(\theta)h)(\rho(\theta)\overline{\chi(y)} + \tau(\theta)h)\right)\right).$$

As in the abstract case earlier, this gives a “nearly closed form” for the error rate of such estimators under various notions of error, provided we can estimate the integral of fixed dimension on the right-hand side. Along with the proof of this result, in Section 5 we present numerical experiments showing that, even for moderately large finite n , the right-hand side gives an excellent approximation to the error rate on the left-hand side.

As a simple concrete example, for $G = \mathbb{Z}/L$ we may take simply

$$\ell(x, y) := \mathbf{1}\{x \neq y\},$$

in which case the left-hand side is

$$\frac{1}{n^2} \sum_{i,j=1}^n \ell(M_{ij}, \widehat{M}_{ij}) = \frac{\#\{(i, j) \in [n]^2 : M_{ij} \neq \widehat{M}_{ij}\}}{n^2},$$

the fraction of incorrectly estimated entries of M .

The case $L = 2$ has another important interpretation: as we discuss in Section 5, this is equivalent to a dense version of the *stochastic block model*, a model of community detection in random graphs. In this case, the x_i are Boolean and may be identified with a label given to each vertex i in a graph, and M_{ij} describes whether vertices i and j have the same label or different labels. In this case, our results describe the asymptotic rate of mislabeling for essentially arbitrary rounded spectral estimators of these pairwise community membership relations.

1.3 Related work

Spiked matrix models The spiked matrix model originates in statistics in the work of [Joh01], who proposed a version where the signal is applied to a covariance matrix of vector observations, though a special case of a model similar to ours was discussed much earlier in the foundational work [FK81]. Those predictions were proved in the seminal work of [BBAP05]. Subsequently, many variants of such results appeared; the first results to the effect of our Theorem 1.2 for models with additive noise appear to have been shown soon after by [Péc06, FP07]. Other more general versions are shown, for instance, in [CDMF09, BGN10], and an overview of this line of work from the mathematical point of view may be found in [Cap17], and from the statistical point of view in [PA14, JP18]. The general approach we take to analyzing spiked matrix models using the resolvent appears in [BGN10, BGGM11] and was used more systematically together with *isotropic local laws* by [KY13, KY14, KY16], whose techniques we will follow closely. Another relevant work for studying eigenvectors by this method is the earlier one of [KY11], though this concerns matrices like our W with no low-rank perturbation.

Universality and non-universality As we discussed briefly above, the general phenomenon of non-universality of fluctuations in spiked matrix models depending on the structure of the signal

was gradually uncovered by works including [CDMF09, CDMF12, RS12, BGGM11, KY13, KY14]. For eigenvectors, non-universality for localized signals was shown by [CDM18], with regard to a quantity similar to $\langle \hat{v}, v \rangle$. We note that this is a special projection of \hat{v} to study because, for $\theta > 1$, per Theorem 1.2 it is of magnitude $\Theta(1)$; in contrast, for v delocalized the entries \hat{v}_i are of magnitude $o(1)$ and therefore considerably more delicate to understand. Distributions of similar quantities in the case of asymmetric and rectangular matrices are considered by [BDW21]. The results perhaps most similar to ours are those of [FFHL22], but those have the important difference of considering, in our notation, the case $\theta = \theta(n) \rightarrow \infty$, in which case \hat{v} is very close to v and the Gaussian fluctuations they identify are of vanishing magnitude.

Approximate message passing and state evolution One important other branch of the literature where similar statements of Gaussian fluctuations appear is in the characterization of *approximate message passing (AMP)* algorithms. See, e.g., [FVRS21] for a general survey of this area. These algorithms perform a general kind of *nonlinear power method*, alternating multiplying a vector by a matrix like our H and applying an entrywise nonlinearity. Various expressions of the Gaussian distribution of the result of such algorithms are known as *state evolution* descriptions. In principle, the power method itself for computing \hat{v} is a special case of AMP where the nonlinearities are omitted, and thus various averages of nonlinear functions of \hat{v} can be described as measurements of the state of a suitable AMP algorithm. However, there is an important caveat that much of the work on AMP concerns only asymptotic results as $n \rightarrow \infty$ for a finite number of iterations $t = O(1)$ of such an algorithm, from a random initialization. Such an algorithm essentially would compute $H^t x$ for a random x independent of H , which does not faithfully approximate \hat{v} , and thus such analysis cannot be used in our setting directly.

In the AMP literature there have been essentially two workarounds considered to this kind of issue (variants of which are also relevant to other, more sophisticated instances of AMP). First, one may perform non-asymptotic analysis of a number of iterations $t = t(n)$ depending on n , as pioneered by [RV18] and developed further by [LW22, LFW23, CR23]. However, such analysis requires strong assumptions on W such as Gaussianity or orthogonal invariance. The recent work [Han25] is in a similar spirit to ours, but only can treat $t(n) \leq (\log n)^{1/3}$ iterations of an AMP algorithm, while $t(n) = \Theta(\log n)$ are required to faithfully estimate \hat{v} in spiked matrix models.

Second, a line of work of [MV21b, MV21a, MTV22], also discussed in Section 3.2 of [FVRS21], has sought to analyze AMP initialized not from a random vector x independent of H , but from precisely our \hat{v} , a so-called *spectral initialization* from the top eigenvector. Of course, if this were possible, then we could study \hat{v} very directly using such analysis. However, these works involve various tricks that rely deeply on the Gaussian structure of W , and in their current form do not seem to imply any analysis of \hat{v} comparable to ours for non-Gaussian W , or indeed for any W besides very symmetric Gaussian models like the GOE and GUE—the information about \hat{v} that these approaches use stems from precisely the same invariance properties of the GOE and GUE that we take advantage of in our proof of Theorem 1.8.

1.4 Notation

We use the standard notation $[n] = \{1, \dots, n\}$. We occasionally use the special function

$$\varphi(n) := (\log n)^{\log \log n}.$$

We write $i = \sqrt{-1}$ for the imaginary unit, not to be confused with the indices $i \in [n]$. For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we denote the associated unit sphere in dimension n by $\mathbb{S}^{n-1}(\mathbb{F}) = \{x \in \mathbb{F}^n : \|x\| = 1\}$.

For a proposition P about various variables defined in our arguments, we write $\mathbf{1}\{P\}$ to equal 1 if P is true and 0 if P is false. Reusing this notation, in probabilistic arguments we also use $\mathbf{1}_{\mathcal{E}}$ for the indicator random variable of an event \mathcal{E} .

We write $\mathbb{C}_{\text{herm}}^{n \times n}$ for the set of Hermitian matrices and $\mathbb{R}_{\text{sym}}^{n \times n}$ for the set of real symmetric matrices. $I = I_n$ denotes the $n \times n$ identity matrix; we omit the subscript if it is clear from context. For $X \in \mathbb{C}_{\text{herm}}^{n \times n}$, we write $\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X)$ for its ordered real eigenvalues and $v_1(X), \dots, v_n(X)$ for the associated eigenvectors, provided the corresponding eigenvalues are simple.

The asymptotic notations $\lesssim, \gtrsim, O(\cdot), \Theta(\cdot), o(\cdot), \omega(\cdot), \Omega(\cdot)$ have their usual meanings, always with respect to the limit $n \rightarrow \infty$. We sometimes write subscripts on these notations for parameters that the implicit constants depend on, but when proving a result whose statement gives this dependence explicitly, we omit those parameters for the sake of brevity. In general, all of the quantities we have called “parameters” in the statements, with subscripts of “ v ” or “ W ”, are viewed as constants for the purposes of our proofs.

2 Preliminaries

2.1 Probability

Let us give a name to the tail bound assumption (1.1) that we make on the magnitudes of the entries of our generalized Wigner matrices.

Definition 2.1. *We say that a random variable X is ξ -power-subexponential if, for all $t > 0$,*

$$\mathbb{P}[|X| \geq t] \leq \xi^{-1} \exp(-t^\xi).$$

Proposition 2.2. *If X is ξ -power-subexponential and $p \geq 1$, then there exists $C(\xi, p)$ such that $\mathbb{E}|X|^p \leq C(\xi, p)$.*

Proof. We have

$$\begin{aligned} \mathbb{E}|X|^p &= \int_{t=0}^{\infty} \mathbb{P}[|X|^p \geq t] dt \\ &= \int_{t=0}^{\infty} \mathbb{P}[|X| \geq t^{1/p}] dt \\ &\leq \xi^{-1} \int_{t=0}^{\infty} \exp(-t^{\xi/p}) dt, \end{aligned}$$

where the remaining integral is finite and depends only on ξ and p . \square

The following notation for tail bounds up to a polynomial amount of “slack” will be useful throughout.

Definition 2.3 (Polynomial stochastic domination). *For sequences of non-negative random variables $X = X_n$ and $Y = Y_n$, we say X is polynomially stochastically dominated by Y , written as $X \prec Y$ or $(X_n) \prec (Y_n)$, if for every $\varepsilon > 0$ and for all $D > 0$, there exists $n_0 = n_0(\varepsilon, D)$ such that, for all $n \geq n_0$,*

$$\mathbb{P}[X_n > n^\varepsilon Y_n] \leq n^{-D}.$$

Also, if \mathcal{E}_n is a sequence of events, we say that \mathcal{E}_n occurs with polynomially high probability if, for every $D > 0$, there exists $n_0 = n_0(D)$ such that, for all $n \geq n_0$,

$$\mathbb{P}[\mathcal{E}_n] \leq n^{-D}.$$

The following property of polynomial stochastic domination is elementary to verify by the union bound.

Proposition 2.4. *Suppose that $X_{n,1}, \dots, X_{n,k}, Y_{n,1}, \dots, Y_{n,k}$ are non-negative random variables such that $(X_{n,i}) \prec (Y_{n,i})$ for each fixed $i \in [k]$. Then,*

$$\begin{aligned} \sum_{i=1}^k X_{n,i} &\prec \sum_{i=1}^k Y_{n,i}, \\ \prod_{i=1}^k X_{n,i} &\prec \prod_{i=1}^k Y_{n,i}. \end{aligned}$$

The following gives a convenient condition under which polynomial stochastic domination results can be converted to control of expectations.

Proposition 2.5. *Let $(X_n)_{n \geq 1}$ be a sequence of non-negative random variables and $(Y_n)_{n \geq 1}$ be a sequence of deterministic non-negative numbers. Suppose that the following hold:*

1. $X_n \prec Y_n$.
2. $\sup_{n \rightarrow \infty} \mathbb{E}X_n^2 < \infty$.
3. There exists $C > 0$ such that $Y_n \geq n^{-C}$ for all n .

Then, for any $\varepsilon > 0$,

$$\mathbb{E}X_n \lesssim_\varepsilon n^\varepsilon Y_n.$$

Proof. Let $K = \sup_{n \rightarrow \infty} \mathbb{E}X_n^2$. We have

$$\mathbb{E}X_n = \mathbb{E}X_n \mathbf{1}_{X_n \leq n^\varepsilon Y_n} + \mathbb{E}X_n \mathbf{1}_{X_n > n^\varepsilon Y_n}$$

and using the bound from the indicator on the first term and the Cauchy-Schwarz inequality on the second,

$$\leq n^\varepsilon Y_n + (\mathbb{E}X_n^2)^{1/2} (\mathbb{P}[X_n > n^\varepsilon Y_n])^{1/2}$$

Now, choosing some $D \geq 2C$, if $n \geq n_0(\varepsilon, D)$, then we have

$$\begin{aligned} &\leq n^\varepsilon Y_n + K^{1/2} n^{-D/2} \\ &\leq n^\varepsilon Y_n + K^{1/2} Y_n \\ &\leq (1 + K^{1/2}) n^\varepsilon Y_n, \end{aligned}$$

completing the proof. \square

2.2 Matrix resolvents and low-rank perturbations

We will work extensively with the resolvents of random matrices, whose basic properties we enumerate below.

Definition 2.6. *The resolvent of a matrix X at the point $z \in \mathbb{C}$ is*

$$\text{Res}_z(X) = (zI - X)^{-1},$$

defined at any z that is not an eigenvalue of X . In particular, if $X \in \mathbb{C}_{\text{herm}}^{n \times n}$, then $\text{Res}_z(X)$ is defined away from a compact subset of \mathbb{R} .

Proposition 2.7 (Riesz projection formula). *Let Γ be a counterclockwise simple contour in \mathbb{C} that does not pass through any eigenvalues of a matrix $X \in \mathbb{C}_{\text{herm}}^{n \times n}$ and encircles a single eigenvalue λ of X . Let P be the orthogonal projection to the eigenspace of X associated to λ . Then,*

$$P = \frac{1}{2\pi i} \oint_{\Gamma} \text{Res}_z(X) dz.$$

Proposition 2.8 (Resolvent identities). *Let $X, Y \in \mathbb{C}_{\text{herm}}^{n \times n}$ and $w, z \in \mathbb{C} \setminus (\text{spec}(X) \cup \text{spec}(Y))$. Then, the following hold:*

1. (First resolvent identity) $R_X(z) - R_X(w) = (w - z)R_X(z)R_X(w)$.
2. (Second resolvent identity) $R_X(z) - R_Y(z) = R_X(z)(X - Y)R_Y(z)$.
3. (Resolvent expansion) $R_X(z) = \sum_{s=0}^t R_Y(z)((X - Y)R_Y(z))^s + R_X(z)((X - Y)R_Y(z))^{t+1}$ for any $t \geq 0$.

The two resolvent identities are standard, while the resolvent expansion follows from repeatedly applying the second resolvent identity.

One important use of resolvents is in characterizing the eigenvalues and eigenvectors of a low-rank perturbation of a matrix. We will use these results on random spiked matrix models, but they are true deterministically as well. We state the results for the general case

$$H = \sum_{i=1}^k \theta_i v_i v_i^* + W = V \text{Diag}(\theta) V^* + W.$$

To study the eigenvalues, we use the following device, used in derivations of the main phase transition phenomena of spiked matrix models by, e.g., [KY13, KY14]; see also Section 5 of [BGN10]. This characterization of the eigenvalues of a low-rank perturbation is sometimes called the *secular equation*.

Proposition 2.9. *Suppose that $\lambda \notin \text{spec}(W)$. Then, $\lambda \in \text{spec}(H)$ if and only if*

$$\det(\text{Diag}(\theta)^{-1} - V^* R_W(\lambda) V) = 0.$$

If $k = 1$, $\theta = \theta_1$, and $v = v_1$, then this reduces to

$$v^* R_W(\lambda) v = \frac{1}{\theta}.$$

Proof. We use the determinantal description of eigenvalues: we have $\lambda \in \text{spec}(Y)$ if and only if

$$\begin{aligned} 0 &= \det(\lambda I - Y) \\ &= \det(\lambda I - W - X \text{Diag}(\theta) X^*) \end{aligned}$$

and, using that $\lambda \notin \text{spec}(W)$, we have that $R_W(\lambda)^{-1} = (\lambda I - W)^{-1}$ is well-defined and non-singular, whereby we can factor out

$$\begin{aligned} &= \det(R_W(\lambda)^{-1}) \cdot \det(I - R_W(\lambda) X \text{Diag}(\theta) X^*) \\ &= \det(R_W(\lambda)^{-1}) \cdot \det(I - \text{Diag}(\theta) X^* R_W(\lambda) X), \end{aligned}$$

and finally we use that the leading factor is non-zero. \square

We also use the following calculation of the resolvent of a spiked matrix model as a low-rank perturbation of the resolvent of the underlying noise matrix.

Proposition 2.10. *If $z \notin \text{spec}(W) \cup \text{spec}(H)$, then*

$$R_H(z) = R_W(z) + R_W(z)V(\text{Diag}(\theta)^{-1} - V^*R_W(z)V)^{-1}V^*R_W(z).$$

Proof. We calculate directly:

$$\begin{aligned} R_{W+V\text{Diag}(\theta)V^*}(z) &= (zI - W - V\text{Diag}(\theta)V^*)^{-1} \\ &= (I - R_W(z)V\text{Diag}(\theta)V^*)^{-1}R_W(z) \\ &= (I + R_W(z)V(\text{Diag}(\theta)^{-1} + V^*V)^{-1}V^*)R_W(z) \\ &= R_W(z) + R_W(z)V(\text{Diag}(\theta)^{-1} - V^*R_W(z)V)^{-1}V^*R_W(z), \end{aligned}$$

where we have used the Woodbury matrix inverse identity in the middle. \square

Using this, we may compute the eigenvectors of H .

Proposition 2.11. *Suppose that $\lambda \in \text{spec}(H) \setminus \text{spec}(W)$ is an eigenvalue with some multiplicity ℓ , and let P be the orthogonal projection to the associated (ℓ -dimensional) eigenspace. Then, the dimension of $\ker(\text{Diag}(\theta)^{-1} - V^*R_W(\lambda)V)$ is ℓ . Letting $U \in \mathbb{C}^{k \times \ell}$ have as its columns an orthonormal basis for this kernel, we have*

$$P = R_W(\lambda)VU(U^*V^*R_W(\lambda)^2VU)^{-1}U^*V^*R_W(\lambda).$$

Equivalently, the eigenspace of H associated to λ is the column space of $R_W(\lambda)VU$, spanned by $R_W(\lambda)Vu$ over all u such that $(\text{Diag}(\theta)^{-1} - V^*R_W(\lambda)V)u = 0$.

If $k = 1$, $\theta = \theta_1$, $v = v_1$, and λ is a simple eigenvalue with eigenvector \hat{v} (matching our earlier notation in the case of λ the largest eigenvalue of H), then this reduces to

$$\hat{v}\hat{v}^* = \frac{1}{v^*R_W(\lambda)^2v}R_W(\lambda)vv^*R_W(\lambda),$$

and \hat{v} up to rescaling is $R_W(\lambda)v$.

Proof. Let Γ be a closed contour encircling λ but no other eigenvalues of W or H . We have, by Propositions 2.7 and 2.10,

$$\begin{aligned} P &= \frac{1}{2\pi i} \oint_{\Gamma} R_H(z) dz \\ &= \frac{1}{2\pi i} \oint_{\Gamma} (R_W(z) + R_W(z)V(\text{Diag}(\theta)^{-1} - V^*R_W(z)V)^{-1}V^*R_W(z)) dz \end{aligned}$$

and, since Γ avoids the eigenvalues of W , $R_W(z)$ is analytic on an open neighborhood of the interior of Γ , and thus by Cauchy's integral theorem the first term integrates to zero and we have

$$= \frac{1}{2\pi i} \oint_{\Gamma} R_W(z)V(\text{Diag}(\theta)^{-1} - V^*R_W(z)V)^{-1}V^*R_W(z) dz$$

Now, by Proposition 2.9, the integrand has poles precisely z equaling the eigenvalues of H . Since the only one of these that Γ encircles is λ , the above equals the (matrix-valued) residue at this pole. Again by our assumption, $R_W(z)$ is analytic in an open neighborhood of the interior of Γ , and so we may take the (matrix-valued) residue of the inner term $(\text{Diag}(\theta)^{-1} - V^* R_W(z)V)^{-1}$ at $z = \lambda$. Let $U \in \mathbb{C}^{k \times \ell}$ have as its columns an orthonormal basis of $\ker(\text{Diag}(\theta)^{-1} - V^* R_W(\lambda)V)$. Since $\frac{d}{dz} R_W(z) = -R_W(z)^2$, this is given by $U(U^* V^* R_W(\lambda)^2 V U)^{-1} U^*$. Thus we may evaluate and we find

$$= R_W(\lambda) V U (U^* V^* R_W(\lambda)^2 V U)^{-1} U^* V^* R_W(\lambda).$$

Note that, since $\lambda \in \mathbb{R}$, $R_W(\lambda)$ is Hermitian, and so this is just the formula for the orthogonal projection to the span of the vectors $R_W(\lambda) V U$. \square

2.3 Random matrix theory

2.3.1 Semicircle limit theorems

Let us review some of the properties of generalized Wigner random matrices we have mentioned above.

Definition 2.12. *The semicircle distribution, denoted μ_{sc} , is the probability measure with density*

$$\frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}\{x \in [-2, 2]\} dx$$

with respect to Lebesgue measure.

The following general limit theorem is a direct consequence of the much more precise results of [EYY12] on generalized Wigner matrices.

Theorem 2.13. *Let $W^{(n)} \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a sequence of generalized Wigner matrices (Definition 1.1) with parameters (γ_W, ξ_W) not depending on n . Then, the following convergences hold in probability:*

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n f(\lambda_i(W^{(n)})) &\rightarrow \int f d\mu_{\text{sc}}, \\ \lambda_1(W^{(n)}) &\rightarrow 2, \\ \lambda_n(W^{(n)}) &\rightarrow -2, \\ \|W^{(n)}\| &\rightarrow 2. \end{aligned}$$

In the first claim, $f : \mathbb{R} \rightarrow \mathbb{R}$ is any bounded continuous function or any polynomial.

Many of the calculations to follow will also involve the following object evaluated with the semicircle measure, a few of whose properties we establish now.

Definition 2.14. *For a probability measure μ on \mathbb{R} supported on some $K \subseteq \mathbb{R}$, its Cauchy transform is the function $G_\mu : \mathbb{C} \setminus K \rightarrow \mathbb{C}$ defined by*

$$G_\mu(z) = \int \frac{1}{z - x} d\mu(x).$$

Proposition 2.15. *The Cauchy transform $G_{\mu_{\text{sc}}}(z)$ of the semicircle distribution is defined on $\mathbb{C} \setminus [-2, 2]$ and satisfies:*

$$G_{\mu_{\text{sc}}}(z) = \frac{1}{2}(z - \sqrt{z^2 - 4}),$$

$$G'_{\mu_{\text{sc}}}(z) = \frac{1}{2} \left(1 - \frac{z}{\sqrt{z^2 - 4}} \right).$$

In particular, on $(2, \infty) \subset \mathbb{R}$, $G'_{\mu_{\text{sc}}}$ increases monotonically from $-\infty$ to 0.

2.3.2 Tail bounds

We follow works like [KY13] in adopting the notation

$$\varphi(n) := (\log n)^{\log \log n}$$

for a nuisance factor that appears in the following results. For intuition, it is useful to keep in mind that $\varphi(n)$ grows slower than any polynomial of n but faster than any polynomial of $\log n$: for any $C > 0$,

$$(\log n)^C \ll \varphi(n) \ll n^{1/C}.$$

The following concrete norm bounds will be useful.

Proposition 2.16 (Wigner matrix norm deviation bounds [EYY12]). *Let $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a generalized Wigner matrix. Then,*

$$\mathbb{P}[\|W\| \geq 2 + \varphi(n)n^{-2/3}] \leq C \exp(-\varphi(n))$$

for a constant C depending only on the parameters (γ_W, ξ_W) of the generalized Wigner matrix.

Proposition 2.17 (Resolvent norm bounds). *Let $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a generalized Wigner matrix, and let $z \in \mathbb{C}$ have $\text{Im}(z) > n^{-C}$ and $\text{Re}(z) > 2 + c$ for some $c, C > 0$. Then,*

$$\mathbb{E}\|R_W(z)\|^k \leq K$$

for some constant K depending only on c, C, k , and the parameters (γ_W, ξ_W) of the generalized Wigner matrix. The same also holds for W replaced by the matrix Q formed by setting entries (a, b) and (b, a) to zero in W , for any $1 \leq a \leq b \leq n$.

Proof. We have

$$\begin{aligned} \|R_W(z)\| &= \|(zI - W)^{-1}\| \\ &\leq \frac{1}{d(z, \text{spec}(W))} \\ &\leq \frac{1}{\text{Im}(z)}. \end{aligned}$$

Fix some $\delta < c$. Define the event

$$\mathcal{E} = \{\|W\| \leq 2 + \delta\} = \{\text{spec}(W) \subseteq [-2 - \delta, 2 + \delta]\}.$$

Then, we have using Proposition 2.16 that

$$\begin{aligned}\mathbb{E}\|R_W(z)\|^k &= \mathbb{E}\mathbf{1}_{\mathcal{E}}\|R_W(z)\|^k + \mathbb{E}\mathbf{1}_{\mathcal{E}^c}\|R_W(z)\|^k \\ &\leq \left(\frac{1}{c-\delta}\right)^k + \text{Im}(z)^{-k}\mathbb{P}[\mathcal{E}^c] \\ &\leq O_{c,k}(1) + n^{Ck}\exp(-\varphi(n)),\end{aligned}$$

and the result for W follows since $\varphi(n)$ grows faster than any power of $\log n$. The result for Q follows by the same argument since $\|W - Q\| = 2|W_{ab}|$ which satisfies a power-subexponential tail bound. \square

2.3.3 Local laws

We will use the following powerful result about deterministic quadratic forms with the resolvent of a generalized Wigner random matrix.

Theorem 2.18 (Isotropic local law, Theorem 2.15 and Remark 2.6 of [BEK⁺14]). *Fix $c, C > 0$. Let $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a generalized Wigner matrix and define*

$$\Omega = \Omega(c, C) = \{z \in \mathbb{C} : \text{Re}(z) > 2 + c, \text{Im}(z) > 0, |z| \leq C\}.$$

Then,

$$\sup_{z \in \Omega} |x^* R_W(z)y - G_{\mu_{\text{sc}}}(z) \cdot x^* y| \prec n^{-1/2}, \quad (2.1)$$

with the constants $n_0(\varepsilon, D)$ in the polynomial stochastic domination depending only on c, C , and the parameters (γ_W, ξ_W) of the generalized Wigner matrix.

Corollary 2.19. *Let $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a generalized Wigner matrix. For some $i, j \in [n]$, let Q be formed by setting entries (i, j) and (j, i) in W to zero. Then, the conclusion (2.1) of Theorem 2.18 holds with W replaced by Q .*

Proof. It suffices to show that

$$\sup_{z \in \Omega} \|R_W(z) - R_Q(z)\| \prec n^{-1/2}.$$

By the second resolvent identity, we have

$$\begin{aligned}\|R_W(z) - R_Q(z)\| &= \|R_W(z)(W - Q)R_Q(z)\| \\ &\leq |W_{ij}| \cdot \|R_W(z)\| \cdot \|R_Q(z)\| \\ &\leq |W_{ij}| \cdot \|R_W(z)\| \cdot (\|R_W(z)\| + \|R_W(z) - R_Q(z)\|),\end{aligned}$$

and rearranging this gives

$$\leq \frac{|W_{ij}| \cdot \|R_W(z)\|^2}{(1 - |W_{ij}| \cdot \|R_W(z)\|)}$$

provided that $|W_{ij}| \cdot \|R_W(z)\| < 1$. We have $|W_{ij}| \prec n^{-1/2}$ since by assumption it is power-subexponential, while it follows from Proposition 2.16 that $\sup_z \|R_W(z)\| \prec 1$ over the range of z in the statement, and the result follows by a suitable union bound. \square

We will also use the following corollary, which essentially says that one may differentiate with respect to z inside the expression appearing in the local law and still obtain the same quality of guarantee.

Corollary 2.20 (Derivative of isotropic local law). *In the setting of Theorem 2.18, we also have*

$$\sup_{z \in \Omega} |x^* R_W(z)^2 y + G'_{\mu_{sc}}(z) x^* y| \prec n^{-1/2},$$

with the same dependence of constants.

Proof. Define the function

$$f(z) := x^* R_W(z) y - G_{\mu_{sc}}(z) x^* y.$$

This function is analytic on $\mathbb{C} \setminus \text{spec}(W)$. Define two domains similar to the one in Theorem 2.18:

$$\begin{aligned} \Omega &= \{z \in \mathbb{C} : \operatorname{Re}(z) > 2 + c, \operatorname{Im}(z) > 0, |z| \leq C\}, \\ \Omega' &= \{z \in \mathbb{C} : \operatorname{Re}(z) > 2 + c/2, \operatorname{Im}(z) > 0, |z| \leq C + c/2\} \end{aligned}$$

We have $\Omega \subset \Omega'$. By the local law of Theorem 2.18, we have, in the notation from its statement,

$$\sup_{z \in \Omega'} |f(z)| \prec n^{-1/2}.$$

For any $w \in \Omega$, let C_w be the circle of radius $c/4$ around w . Note that C_w is contained in Ω' . Thus, by Cauchy's integral formula, we have

$$\sup_{w \in \Omega} |f'(w)| = \sup_{w \in \Omega} \left| \frac{1}{2\pi i} \oint_{C_w} \frac{f(z)}{(z-w)^2} dz \right| \prec n^{-1/2}$$

by the above observation, absorbing constants depending on c into the polynomial stochastic domination notation. But, since $\frac{d}{dz} R_W(z) = -R_W(z)^2$, we have

$$f'(z) = -x^* R_W(z)^2 y - G'_{\mu_{sc}}(z) x^* y,$$

so the result follows. \square

2.3.4 Spiked matrix model estimates

Combining the above results, we may prove estimates on the largest eigenvalue and associated eigenvector of spiked matrix models with generalized Wigner matrix noise that will be useful later. These arguments for deducing such bounds from isotropic local laws are standard; we outline them omitting some details for the sake of completeness.

Theorem 2.21. *Let W be a generalized Wigner matrix, $\theta > 1$, and $v \in \mathbb{S}^{n-1}(\mathbb{C})$. Write $\hat{\lambda}$ for the largest eigenvalue of $H = \theta vv^* + W$ and \hat{v} for the associated eigenvector. Then, we have*

$$\begin{aligned} |\hat{\lambda} - \lambda(\theta)| &\prec n^{-1/2}, \\ |\langle \hat{v}, v \rangle|^2 - \rho(\theta)^2 &\prec n^{-1/2}, \end{aligned}$$

with the constants in the polynomial stochastic domination depending only on θ and the parameters (γ_W, ξ_W) of the generalized Wigner matrix. Further, for any $\varepsilon > 0$, with polynomially high probability H has at most one eigenvalue in the interval $[2 + \varepsilon, \infty)$.

Proof. For the statement about $\widehat{\lambda}$, by Proposition 2.9 we have that this quantity is the largest real solution z of the equation

$$v^* R_W(z) v = \frac{1}{\theta}.$$

By Proposition 2.16, there is a constant $C > \lambda(\theta)$ such that there is no such $z > C$ with polynomially high probability. For $c \in (0, \lambda(\theta) - 2)$, let

$$\Omega = \{z \in \mathbb{C} : \operatorname{Re}(z) > 2 + c, \operatorname{Im}(z) > 0, |z| \leq C\}.$$

By the local law of Theorem 2.18, we have

$$\sup_{z \in \Omega} |v^* R_W(z) v - G_{\mu_{sc}}(z)| \prec n^{-1/2}.$$

By the formula for $G_{\mu_{sc}}(z)$ given in Proposition 2.15, one may calculate that, on $z \in [2 + c, C]$, there is a unique z solving $G_{\mu_{sc}}(z) = 1/\theta$, which is precisely $z = \lambda(\theta) = \theta + 1/\theta$. Since $G_{\mu_{sc}}(z)$ is smooth and monotone on this interval, the result follows.

For the statement about \widehat{v} , by Proposition 2.11 we have

$$|\langle \widehat{v}, v \rangle|^2 = \frac{(v^* R_W(\widehat{\lambda}) v)^2}{v^* R_W(\lambda)^2 v^*}.$$

The result then follows from applying the local law of Theorem 2.18 and its derivative in Corollary 2.20 on the numerator and denominator, uniformly over Ω in which $\widehat{\lambda}$ lies with polynomially high probability. \square

3 Universality of entrywise statistics: Proof of Theorem 1.4

Recall the setting: we have two generalized Wigner matrices, denoted $W, X \in \mathbb{C}_{\text{herm}}^{n \times n}$, and a deterministic vector $v \in \mathbb{S}^{n-1}(\mathbb{C})$. On v we have only assumed that $\|v\| = 1$ and that

$$\|v\|_\infty \leq n^{-1/2+\varepsilon_v},$$

for some $\varepsilon_v \in (0, 1/20)$. We define

$$\begin{aligned} H &= H(W) = \theta v v^* + W, \\ \widehat{\lambda} &= \widehat{\lambda}(W) = \lambda_1(H(W)), \\ \widehat{v} &= \widehat{v}(W) = v_1(H(W)), \\ F(W) &= \phi(n \cdot \widehat{v}(H(W))_i \cdot \widehat{v}(H(W))_j), \end{aligned}$$

for some fixed $1 \leq i \leq j \leq n$ and ϕ five times continuously differentiable and with bounded values and derivatives. We will use the notation defined earlier

$$\lambda = \lambda(\theta) := \theta + \theta^{-1} > 2.$$

Our goal is to show that $\mathbb{E}F(W) \approx \mathbb{E}F(X)$. We proceed in two steps. First, we make some initial simplifications using our calculations related to the eigenspaces of H from Proposition 2.11. In particular, we use our explicit formula for P to reduce $F(W)$ to a simpler function of W . Then, we complete the remaining proof by the Lindeberg method.

3.1 Initial simplifications

Let $\varepsilon \in (0, \varepsilon_v)$ be a constant for the purposes of the proof to be chosen later, and fix $0 < \delta_1 < \delta_2$ such that $\lambda \in (2 + \delta_1, 2 + \delta_2)$. We view $\varepsilon, \delta_1, \delta_2$ as well as the function ϕ and the generalized Wigner matrix parameters $(\gamma_W, \xi_W, \gamma_X, \xi_X)$, all as constants for the purposes of this proof, and the dependence of various asymptotic notations on these parameters is not mentioned from now on.

Let \mathcal{E} be the event that the following conditions hold for W :

1. There is a single eigenvalue of $H(W)$ in $[2 + \delta_1, 2 + \delta_2]$, of multiplicity 1, which is also the top eigenvalue $\hat{\lambda} = \lambda_1(H(W))$.
2. $|\hat{\lambda} - \lambda| \leq n^{-1/2+\varepsilon}$.
3. $\|W\| < 2 + \delta_1$, and in particular $\hat{\lambda} \notin \text{spec}(W)$.
4. The following bounds hold:

$$\begin{aligned} |e_i^* R_W(\hat{\lambda})v - v_i| &\leq n^{-1/2+\varepsilon}, \\ |e_j^* R_W(\hat{\lambda})v - v_j| &\leq n^{-1/2+\varepsilon}, \\ |v^* R_W(\hat{\lambda})^2 v + G'_{\mu_{\text{sc}}}(\hat{\lambda})| &\leq n^{-1/2+\varepsilon}. \end{aligned}$$

Proposition 3.1. \mathcal{E} holds with polynomially high probability.

Proof. The result follows by combining the isotropic local law (Theorem 2.18), its derivative (Theorem 2.20), and Theorem 2.21 on the eigenvalues of spiked matrix models. \square

Define

$$\begin{aligned} \Lambda_1 &:= -G'_{\mu_{\text{sc}}}(2 + \delta_2), \\ \Lambda_2 &:= -G'_{\mu_{\text{sc}}}(2 + \delta_1). \end{aligned}$$

By Proposition 2.15, we have $0 < \Lambda_1 < \Lambda_2$ and $-G'_{\mu_{\text{sc}}}(\lambda) \in [\Lambda_1, \Lambda_2]$ since $-G_{\mu_{\text{sc}}}(z)$ is monotonically decreasing on the interval $(2, \infty)$.

Since ϕ is bounded, we have by the bound on the probability that \mathcal{E} does not happen that, giving a concrete bound,

$$|\mathbb{E}F(W) - \mathbb{E}\mathbf{1}_{\mathcal{E}}F(W)| \leq \|\phi\|_{L^\infty} \mathbb{P}[\mathcal{E}^c] \lesssim n^{-1}. \quad (3.1)$$

By Proposition 2.11, we have

$$\begin{aligned} \mathbb{E}\mathbf{1}_{\mathcal{E}}F(W) &= \mathbb{E}\mathbf{1}_{\mathcal{E}}\phi(n \cdot \widehat{v}_i \overline{\widehat{v}_j}) \\ &= \mathbb{E}\mathbf{1}_{\mathcal{E}}\phi\left(n \cdot \frac{(e_i^* R_W(\hat{\lambda})v)(v^* R_W(\hat{\lambda})e_j)}{v^* R_W(\hat{\lambda})^2 v}\right). \end{aligned}$$

We will now show that $\mathbb{E}F(W)$ is close to what we get when we perform two operations on the above expression: (1) we replace the random eigenvalue $\hat{\lambda}$ by its deterministic typical location λ , and (2) we apply the isotropic local law to the denominator. There is a small but important caveat that we must attend to:

Remark 3.2 (Resolvent for discrete models). *When the entries of W have discrete distributions, which our Definition 1.1 does not rule out, it is possible that $\lambda \in \text{spec}(W)$ with small but positive probability. On this event, $R_W(\lambda)$ is undefined, and in particular expectations involving $R_W(\lambda)$ are undefined.*

To deal with this, consider the slight further perturbation

$$\tilde{\lambda} = \tilde{\lambda}(\theta) = \lambda(\theta) + in^{-1/2}.$$

For the above technical reason, we instead consider replacing $\hat{\lambda}$ by $\tilde{\lambda}$.

It is convenient to state this reduction in terms of the following auxiliary function:

$$\tilde{\phi}(t) := \phi\left(\frac{-1}{G'_{\mu_{sc}}(\tilde{\lambda})}t\right).$$

This is just a constant rescaling of ϕ and thus also satisfies all the same smoothness and boundedness assumptions as ϕ did. Let us then define

$$\tilde{F}(W) := \tilde{\phi}\left(n \cdot (e_i^* R_W(\tilde{\lambda}) v)(v^* R_W(\tilde{\lambda}) e_j)\right). \quad (3.2)$$

Lemma 3.3. *The following bounds hold:*

$$\begin{aligned} |\mathbb{E}F(W) - \mathbb{E}\tilde{F}(W)| &\lesssim n^{-1/2+3\varepsilon_v}, \\ |\mathbb{E}F(X) - \mathbb{E}\tilde{F}(X)| &\lesssim n^{-1/2+3\varepsilon_v}, \end{aligned}$$

and therefore also

$$|\mathbb{E}F(W) - \mathbb{E}F(X)| \leq |\mathbb{E}\tilde{F}(W) - \mathbb{E}\tilde{F}(X)| + O(n^{-1/2+3\varepsilon_v}).$$

Proof. We will show the first bound for W ; the same argument will apply symmetrically for X , and the two bounds combined will give the third bound.

By the conditions included in \mathcal{E} , we have that, on this event,

$$\begin{aligned} \left| \frac{1}{v^* R_W(\hat{\lambda})^2 v} + \frac{1}{G'_{\mu_{sc}}(\hat{\lambda})} \right| &\leq \left| \frac{1}{v^* R_W(\hat{\lambda})^2 v} + \frac{1}{G'_{\mu_{sc}}(\hat{\lambda})} \right| + \left| \frac{1}{G'_{\mu_{sc}}(\hat{\lambda})} - \frac{1}{G'_{\mu_{sc}}(\tilde{\lambda})} \right| \\ &= \frac{|v^* R_W(\hat{\lambda})^2 v + G'_{\mu_{sc}}(\hat{\lambda})|}{|v^* R_W(\hat{\lambda})^2 v| \cdot |G'_{\mu_{sc}}(\hat{\lambda})|} + O(n^{-1/2+\varepsilon}) \\ &\leq \frac{n^{-1/2+\varepsilon}}{(\Lambda_1 - n^{-1/2+\varepsilon})\Lambda_1} + O(n^{-1/2+\varepsilon}) \\ &= O(n^{-1/2+\varepsilon}). \end{aligned}$$

Therefore, we also have that on \mathcal{E} ,

$$\left| n \cdot \frac{(e_i^* R_W(\hat{\lambda}) v)(v^* R_W(\hat{\lambda}) e_j)}{v^* R_W(\hat{\lambda})^2 v} + n \cdot \frac{(e_i^* R_W(\hat{\lambda}) v)(v^* R_W(\hat{\lambda}) e_j)}{G'_{\mu_{sc}}(\tilde{\lambda})} \right| = O(n^{-1/2+\varepsilon+2\varepsilon_v}).$$

We also have that on \mathcal{E} , using the first resolvent identity,

$$\begin{aligned} |e_i^* R_W(\tilde{\lambda})v - e_i^* R_W(\hat{\lambda})v| &\leq \|R_W(\tilde{\lambda}) - R_W(\hat{\lambda})\| \\ &= \|(\hat{\lambda} - \lambda)R_W(\lambda)R_W(\hat{\lambda})\| \\ &\leq \frac{|\hat{\lambda} - \tilde{\lambda}|}{|\tilde{\lambda} - \|W\|| \cdot |\hat{\lambda} - \|W\||} \\ &= O(n^{-1/2+\varepsilon}). \end{aligned}$$

Also applying this symmetrically to the $v^* R_W(\hat{\lambda})e_j$ term, we finally find that, since ϕ is smooth and ϕ' is bounded, and combining with (3.1),

$$\mathbb{E}F(W) = \mathbb{E}\mathbf{1}_{\mathcal{E}}\phi\left(n \cdot \frac{-1}{G'_{\mu_{\text{sc}}}(\tilde{\lambda})} \cdot (e_i^* R_W(\tilde{\lambda})v)(v^* R_W(\tilde{\lambda})e_j)\right) + O(n^{-1/2+\varepsilon+2\varepsilon_v})$$

and, undoing our first truncation step where we introduced the indicator of \mathcal{E} , we have

$$\begin{aligned} &= \mathbb{E}\phi\left(n \cdot \frac{-1}{G'_{\mu_{\text{sc}}}(\tilde{\lambda})} \cdot (e_i^* R_W(\tilde{\lambda})v)(v^* R_W(\tilde{\lambda})e_j)\right) + O(n^{-1/2+\varepsilon+2\varepsilon_v}) \\ &= \mathbb{E}\tilde{\phi}\left(n \cdot (e_i^* R_W(\tilde{\lambda})v)(v^* R_W(\tilde{\lambda})e_j)\right) + O(n^{-1/2+\varepsilon+2\varepsilon_v}), \end{aligned} \quad (3.3)$$

completing the proof, where we note that the stated error bound follows since we have taken $\varepsilon < \varepsilon_v$ by assumption. \square

3.2 Lindeberg method

Now, we consider comparing $\mathbb{E}\tilde{F}(W)$ and $\mathbb{E}\tilde{F}(X)$ by the Lindeberg method. Our final result will be the following:

Lemma 3.4. *For \tilde{F} as defined above in (3.2), for any $\varepsilon > 0$,*

$$|\mathbb{E}\tilde{F}(X) - \mathbb{E}\tilde{F}(W)| \lesssim_{\varepsilon} n^{-1/2+10\varepsilon_v+\varepsilon}.$$

The proof of Theorem 1.4 is then completed by combining Lemma 3.3 with Lemma 3.4.

Following the prescription of the Lindeberg method (our use of the method is also quite similar to the specific one in [KY11]), let us construct an interpolating path from W to X . Let $L = \frac{n(n+1)}{2}$ be the number of entries on and above the diagonal in a Hermitian matrix. Fix some enumeration $(a_1, b_1), \dots, (a_L, b_L)$ of all $(a, b) \in [n]^2$ with $1 \leq a \leq b \leq n$. Let $W^{(0)} = W$, $W^{(L)} = X$, and $W^{(\ell)}$ denote W with $W_{a_1 b_1}, \dots, W_{a_\ell b_\ell}$ replaced by the corresponding entries of X , and by their conjugates symmetrically below the diagonal.

We follow the usual prescription of the Lindeberg method. First, we bound by a telescoping sum,

$$\begin{aligned} |\mathbb{E}\tilde{F}(X) - \mathbb{E}\tilde{F}(W)| &= \left| \sum_{\ell=1}^L \left[\mathbb{E}\tilde{F}(W^{(\ell)}) - \mathbb{E}\tilde{F}(W^{(\ell-1)}) \right] \right| \\ &\leq \sum_{\ell=1}^L |\mathbb{E}\tilde{F}(W^{(\ell)}) - \mathbb{E}\tilde{F}(W^{(\ell-1)})|. \end{aligned} \quad (3.4)$$

We now consider bounding the effect of each individual swap, i.e., the size of

$$|\mathbb{E}\tilde{F}(W^{(\ell)}) - \mathbb{E}\tilde{F}(W^{(\ell-1)})| \quad (3.5)$$

for each fixed ℓ . For the sake of conciseness, let us write $a = a_\ell$ and $b = b_\ell$ for the moment. Let us define

$$E_{ab}^W = \begin{cases} W_{ab}e_a e_b^* + \overline{W_{ab}}e_b e_a^* & \text{if } a \neq b, \\ W_{aa}e_a e_a^* & \text{if } a = b. \end{cases},$$

and similarly E_{ab}^X . Let

$$Q = W^{(\ell)} - E_{ab}^X = W^{(\ell-1)} - E_{ab}^W,$$

the matrix after $\ell - 1$ swaps have been performed but where the entries about to be swapped at step ℓ (in positions $(a = a_\ell, b = b_\ell)$ and (b, a)) have been set to zero.

It is useful to pause to understand the distribution of Q . If we placed the corresponding entries of X or W into positions (a, b) and (b, a) in Q , then we would obtain a generalized Wigner matrix in the sense of Definition 1.1. Thus, Q is such a generalized Wigner matrix with two entries set to zero, which is precisely the situation covered by our Corollary 2.19, which states that the isotropic local law for generalized Wigner matrices (Theorem 2.18) applies just as well to such Q . We will use this fact many times below.

Rewriting our expression in (3.5) in terms of this Q , we have

$$|\mathbb{E}\tilde{F}(W^{(\ell)}) - \mathbb{E}\tilde{F}(W^{(\ell-1)})| = |\mathbb{E}\tilde{F}(Q + E_{ab}^X) - \mathbb{E}\tilde{F}(Q + E_{ab}^W)|.$$

We would like to understand the leading order effect of this difference. We do this by expanding each term separately at first. Recall that $\tilde{F}(W)$ is a function of W only through $R_W(\tilde{\lambda})$, where $\tilde{\lambda} = \theta + \theta^{-1} + i n^{-1/2}$ is a complex constant. Let us note in passing that this constant falls in the regions treated in the local laws (Theorem 2.18 and Corollary 2.19) for suitable choices of the parameters there. All resolvents below will be evaluated at this $\tilde{\lambda}$ only, so for W and all other matrices involved, let us set

$$R_W := R_W(\tilde{\lambda}).$$

We first compute the effect on the resolvent of the above perturbations of Q . By the resolvent expansion of Proposition 2.8, we have

$$R_{W^{(\ell-1)}} = R_{Q+E_{ab}^W} = \sum_{s=0}^4 R_Q(E_{ab}^W R_Q)^s + R_{W^{(\ell-1)}}(E_{ab}^W R_Q)^5. \quad (3.6)$$

We will see momentarily why taking the expansion to order 4 is the correct choice here.

To work with these expressions, let us set up some notation for what happens when we expand the E_{ab}^W .

Definition 3.5 (Conjugation words). *Write $\text{id} : \mathbb{C} \rightarrow \mathbb{C}$ for the identity map $\text{id}(z) = z$, and $\text{conj} : \mathbb{C} \rightarrow \mathbb{C}$ for the conjugation map $\text{conj}(z) = \bar{z}$. For a given $1 \leq a \leq b \leq n$, define*

$$\mathcal{S}_{a,b,s} = \begin{cases} \{\text{id}, \text{conj}\}^s & \text{if } a \neq b, \\ \{\text{id}\}^s & \text{if } a = b \end{cases}.$$

For $S = (f_1, \dots, f_s) \in \mathcal{S}_{a,b,s}$, viewed as a string of functions of length s , let

$$z^S := f_1(z) \cdots f_s(z).$$

Also, associate to such S indices $\alpha_{S,1}, \beta_{S,1}, \dots, \alpha_{S,s}, \beta_{S,s} \in \{a, b\}$, so that $\alpha_{S,i} = a$ and $\beta_{S,i} = b$ if $f_i = \text{id}$, while $\alpha_{S,i} = b$ and $\beta_{S,i} = a$ if $f_i = \text{conj}$.

Then, we may expand

$$(E_{ab}^W R_Q)^s = \sum_{S \in \mathcal{S}_{a,b,s}} W_{ab}^S e_{\alpha_{S,1}}(e_{\beta_{S,1}}^* R_Q e_{\alpha_{S,2}}) \cdots (e_{\beta_{S,s-1}}^* R_Q e_{\alpha_{S,s}}) e_{\beta_{S,s}}^* R_Q,$$

where we note that the quantities in parentheses are just scalars giving certain entries of R_Q . Since such expressions will often come up, let us write $e_a^* R_Y e_b = (R_Y)_{ab} =: R_Y(a, b)$ for matrices $Y \in \{W^{(\ell-1)}, W^{(\ell)}, Q\}$. Plugging this into (3.6), we have

$$\begin{aligned} R_{W^{(\ell-1)}} &= \sum_{s=0}^4 \sum_{S \in \mathcal{S}_{a,b,s}} W_{ab}^S \cdot R_Q(\beta_{S,1}, \alpha_{S,2}) \cdots R_Q(\beta_{S,s-1}, \alpha_{S,s}) \cdot R_Q e_{\alpha_{S,1}} e_{\beta_{S,s}}^* R_Q \\ &\quad + \sum_{S \in \mathcal{S}_{a,b,5}} W_{ab}^S \cdot R_Q(\beta_{S,1}, \alpha_{S,2}) \cdots R_Q(\beta_{S,t}, \alpha_{S,t+1}) \cdot R_{W^{(\ell-1)}} e_{\alpha_{S,1}} e_{\beta_{S,t+1}}^* R_Q. \end{aligned}$$

The expressions we will finally be interested in are $e_i^* R_Y v$ and $v_* R_Y e_j$. Let us extend the previous notation and write $R_Y(i, v)$ and $R_Y(v, j)$ for these, respectively, and extending in the same way to other mixed quadratic forms of this kind, for all Y mentioned above.

Then, firstly we may rewrite

$$\tilde{F}(W^{(\ell-1)}) = \tilde{\phi}(n \cdot R_{W^{(\ell-1)}}(i, v) R_{W^{(\ell-1)}}(v, j)), \quad (3.7)$$

and similarly for $\ell - 1$ replaced by ℓ , and for the above expansion of the inner terms, we have for instance

$$\begin{aligned} R_{W^{(\ell-1)}}(i, v) &= M^{(\ell)}(i, v, W) + \Delta^{(\ell)}(i, v, W), \\ M^{(\ell)}(i, v, Y) &= \sum_{s=0}^t \sum_{S \in \mathcal{S}_{a,b,s}} Y_{ab}^S \cdot R_Q(i, \alpha_{S,1}) R_Q(\beta_{S,1}, \alpha_{S,2}) \cdots R_Q(\beta_{S,s-1}, \alpha_{S,s}) R_Q(\beta_{S,s}, v) \\ &= R_Q(i, v) + \sum_{s=1}^4 \sum_{S \in \mathcal{S}_{a,b,s}} Y_{ab}^S \cdot R_Q(i, \alpha_{S,1}) R_Q(\beta_{S,1}, \alpha_{S,2}) \cdots R_Q(\beta_{S,s-1}, \alpha_{S,s}) R_Q(\beta_{S,s}, v), \\ \Delta^{(\ell)}(i, v, Y) &= \sum_{S \in \mathcal{S}_{a,b,5}} Y_{ab}^S \cdot R_{Q+E_{ab}^Y}(i, \alpha_{S,1}) R_Q(\beta_{S,1}, \alpha_{S,2}) \cdots R_Q(\beta_{S,t}, \alpha_{S,5}) R_Q(\beta_{S,t+1}, v). \end{aligned}$$

We define these expressions taking $a = a_\ell$ and $b = b_\ell$ under our previous ordering, and allow for a general matrix input Y , so that we also have

$$R_{W^{(\ell)}}(i, v) = M^{(\ell)}(i, v, X) + \Delta^{(\ell)}(i, v, X),$$

reusing the same definitions. Also, as a convention, we take the term $s = 0$ in this expression to contribute $R_Q(i, v)$, compatible with our previous calculation.

We will now show the following estimates, which imply that the various Δ terms in the input to $\tilde{\phi}$ in (3.7) may be ignored.

Lemma 3.6. *In the above setting, for each $\ell \in [L]$, we have*

$$|\mathbb{E}\tilde{F}(W^{(\ell-1)}) - \mathbb{E}\tilde{\phi}(n \cdot M^{(\ell)}(i, v, W) M^{(\ell)}(v, j, W))| \lesssim n^{-5/2+3\varepsilon_v}, \quad (3.8)$$

$$|\mathbb{E}\tilde{F}(W^{(\ell)}) - \mathbb{E}\tilde{\phi}(n \cdot M^{(\ell)}(i, v, X) M^{(\ell)}(v, j, X))| \lesssim n^{-5/2+3\varepsilon_v}, \quad (3.9)$$

and thus by (3.4)

$$\begin{aligned}
& |\mathbb{E}\tilde{F}(W) - \mathbb{E}\tilde{F}(X)| \\
& \lesssim \sum_{\ell=1}^L |\mathbb{E}\tilde{\phi}(n \cdot M^{(\ell)}(i, v, W)M^{(\ell)}(v, j, W)) - \mathbb{E}\tilde{\phi}(n \cdot M^{(\ell)}(i, v, X)M^{(\ell)}(v, j, X))| \\
& \quad + O(n^{-1/2+3\varepsilon_v}). \tag{3.10}
\end{aligned}$$

Having shown this, we will have again made useful progress, since the inputs into $\tilde{\phi}$ in term ℓ here are polynomials in W_{ab} , X_{ab} , their conjugates, and the entries of R_Q which, crucially, are independent of those quantities by definition.

To do this, it will be useful to establish some estimates on the sizes of resolvent quadratic forms and associated expectations.

Proposition 3.7 (Resolvent quadratic form bounds). *In the above setting, for any $\alpha, \beta \in [n]$ distinct and any $Y \in \{W^{(\ell)}, W^{(\ell-1)}, Q\}$, we have*

$$\begin{aligned}
|Y(\alpha, \alpha)| &\prec n^{-1/2}, \\
|Y(\alpha, \beta)| &\prec n^{-1/2}, \\
|R_Y(\alpha, \beta)| &\prec n^{-1/2}, \\
|R_Y(\alpha, \alpha)| &\prec 1, \\
|R_Y(\alpha, v)| &\prec n^{-1/2+\varepsilon_v}, \\
|R_Y(v, v)| &\prec 1.
\end{aligned}$$

Further, consider a product of terms

$$\Pi = R_{Y_1}(\zeta_{11}, \zeta_{12}) \cdots R_{Y_s}(\zeta_{s1}, \zeta_{s2}) \cdot Y_{s+1}(\alpha_{11}, \alpha_{12}) \cdots Y_{s+t}(\alpha_{t1}, \alpha_{t2}),$$

where $\alpha_{xy} \in \{1, \dots, n\}$ and $\zeta_{xy} \in \{1, \dots, n, v\}$ for each $x \in [s]$ and $y \in [2]$, where in this notation “ v ” is viewed just as a formal symbol, and $Y_x \in \{W^{(\ell)}, W^{(\ell-1)}, Q\}$. Let s_v be the number of pairs (ζ_{x1}, ζ_{x2}) that contain v exactly once and s_{offdiag} be the number of pairs (ζ_{x1}, ζ_{x2}) for which $\zeta_{x1} \neq \zeta_{x2}$. Then, we have

$$|\Pi| \prec n^{-t/2-s_{\text{offdiag}}/2+s_v\varepsilon_v} \tag{3.11}$$

and, for any $\varepsilon > 0$,

$$\mathbb{E}|\Pi| \lesssim_\varepsilon n^{-t/2-s_{\text{offdiag}}/2+s_v\varepsilon_v+\varepsilon}. \tag{3.12}$$

Proof. The polynomial stochastic domination bounds on the entries of Y (which are all zero or entries of W or X) follow from the ξ -power-subexponential tail bound on these entries in (1.1). The next set of polynomial stochastic domination bounds on resolvent quadratic forms follow from the local laws of Theorem 2.18 (for $Y \in \{W^{(\ell)}, W^{(\ell-1)}\}$) and Corollary 2.19 (for $Y = Q$), since applying the triangle inequality to those results it suffices to observe that $|G_{\mu_{\text{sc}}}(\tilde{\lambda})|$ is just a constant, while $|v_i| \leq \|v\|_\infty \leq n^{-1/2+\varepsilon_v}$ by assumption.

The bound of (3.11) then follows by Proposition 2.4, since this is just a product of several polynomial stochastic dominations from the first set of results. Then, (3.12) says that we may take expectations on either side of this polynomial stochastic domination, provided we insert a factor of n^ε on the right-hand side. By Proposition 2.5, to show this it suffices to check that $\mathbb{E}|\Pi|^2$ is bounded independently of n . But, we have

$$|\Pi| \leq \|R_{Y_1}\| \cdots \|R_{Y_s}\| \cdot |Y_{s+1}(\alpha_{11}, \alpha_{12})| \cdots |Y_{s+t}(\alpha_{t1}, \alpha_{t2})|,$$

and so by Hölder's inequality,

$$\begin{aligned}\mathbb{E}|\Pi|^2 &\leq (\mathbb{E}\|R_{Y_1}\|^{2s+2t})^{\frac{1}{s+t}} \cdots (\mathbb{E}\|R_{Y_s}\|^{2s+2t})^{\frac{1}{s+t}} \\ &\quad (\mathbb{E}|Y_{s+1}(\alpha_{11}, \alpha_{12})|^{2s+2t})^{\frac{1}{s+t}} \cdots (\mathbb{E}|Y_{s+t}(\alpha_{t1}, \alpha_{t2})|^{2s+2t})^{\frac{1}{s+t}}\end{aligned}$$

and we then indeed find that $\mathbb{E}|\Pi|^2 = O(1)$ by Proposition 2.17 and Proposition 2.2 which show that each factor here is $O(1)$.

We note that a condition of that Proposition is that $\text{Im}(z) > n^{-\Omega(1)}$, so we are using that we are evaluating our resolvents at $\tilde{\lambda}$ rather than $\lambda \in \mathbb{R}$, and this cannot be avoided for working with such expectations (without introducing other technical devices like restricting to particular events) for the reason discussed in Remark 3.2. \square

Proof of Lemma 3.6. It suffices to show (3.8), then (3.9) follows by a symmetric argument, and (3.10) follows from the two taken together and the triangle inequality since $L = O(n^2)$.

By Proposition 3.7, we have that

$$M^{(\ell)}(i, v, W) \prec n^{-1/2+\varepsilon_v},$$

with the dominant contribution coming from the $s = 0$ term, and

$$\Delta^{(\ell)}(i, v, W) \prec n^{-3+\varepsilon_v},$$

where in the $\Delta^{(\ell)}$ bounds a factor of $n^{-5/2}$ comes from 5 factors of entries of Y , and another factor of $n^{-1/2+\varepsilon_v}$ comes from one factor of the form $R_Y(\alpha, v)$ for some $\alpha \in [n]$. (We use here that the $M^{(\ell)}$ and $\Delta^{(\ell)}$ are finite sums of the form treated by Proposition 3.7, which may be combined over finite sums by Proposition 2.4.) The same bounds hold for (i, v) replaced by (v, j) as well. Then, we may expand

$$\begin{aligned}\widetilde{F}(W^{(\ell-1)}) &= \widetilde{\phi}(n \cdot R_{W^{(\ell-1)}}(i, v) R_{W^{(\ell-1)}}(v, j)) \\ &= \widetilde{\phi}(n \cdot M^{(\ell)}(i, v, W) M^{(\ell)}(v, j, W) + n \cdot M^{(\ell)}(i, v, W) \Delta^{(\ell)}(v, j, W) \\ &\quad + n \cdot \Delta^{(\ell)}(i, v, W) M^{(\ell)}(v, j, W) + n \cdot \Delta^{(\ell)}(i, v, W) \Delta^{(\ell)}(v, j, W)) \\ &=: \widetilde{\phi}(n \cdot M^{(\ell)}(i, v, W) M^{(\ell)}(v, j, W) + \Xi),\end{aligned}$$

where the above bounds imply that

$$|\Xi| \prec n^{-5/2+2\varepsilon_v}.$$

We then have

$$|\widetilde{F}(W^{(\ell-1)}) - \widetilde{\phi}(n \cdot M^{(\ell)}(i, v, W) M^{(\ell)}(v, j, W))| \leq \|\widetilde{\phi}'\|_{L^\infty} |\Xi|,$$

and the result follows upon taking expectations and using the triangle inequality, since $\mathbb{E}|\Xi|$ may be bounded by another triangle inequality as a sum of terms each controlled by Proposition 3.7. Applying the Proposition costs another factor of n^ε for an arbitrarily small $\varepsilon > 0$, and we take $\varepsilon = \varepsilon_v$ to obtain the result as stated. \square

Remark 3.8 (Order of resolvent expansion). *We see in the above bounds why our choice of fourth order in the resolvent expansion of (3.6) was correct: in order for n^2 terms with the above error bound to not contribute in total to our bound on $|\mathbb{E}F(W) - \mathbb{E}F(X)|$, we must have each term to be bounded by $n^{-2-\delta}$ for some $\delta > 0$, which will no longer hold by the above argument if we take a third order (or shorter) expansion.²*

²It seems that with slightly more care in the combinatorics of how many terms of what “types” in terms of the intersection among the $\{a_\ell, b_\ell\}$ and $\{i, j\}$ index pairs one could take an expansion to order 3, but we take the longer expansion that is more transparently correct for the sake of exposition.

We now continue to finish the proof of the main result of this section, modulo one technical result we will encounter at the end of our calculation that we defer to the following section.

Proof of Lemma 3.4. Starting from the result of Lemma 3.6, we must control summands of the form

$$|\mathbb{E}\tilde{\phi}(n \cdot M^{(\ell)}(i, v, W)M^{(\ell)}(v, j, W)) - \mathbb{E}\tilde{\phi}(n \cdot M^{(\ell)}(i, v, X)M^{(\ell)}(v, j, X))|.$$

We now take a Taylor expansion of $\tilde{\phi}$ in each term in such a difference. We define

$$\Gamma(Y) = \Gamma^{(\ell, i, j)}(Y) = M^{(\ell)}(i, v, Y)M^{(\ell)}(v, j, Y) - R_Q(i, v)R_Q(v, j).$$

By the same application of Proposition 3.7 as above in the proof of Lemma 3.6, we have

$$|\Gamma(Y)| \prec n^{-3/2+2\varepsilon_v}$$

for each $Y \in \{W, X\}$. Taking a Taylor expansion to fourth order around $n \cdot R_Q(i, v)R_Q(v, j)$ in each expectation, we find

$$\begin{aligned} & \tilde{\phi}(n \cdot M^{(\ell)}(i, v, Y)M^{(\ell)}(v, j, Y)) \\ &= \tilde{\phi}(n \cdot R_Q(i, v)R_Q(v, j) + n \cdot \Gamma(Y)) \\ &= \sum_{u=0}^4 \frac{1}{u!} \tilde{\phi}^{(u)}(n \cdot R_Q(i, v)R_Q(v, j)) \cdot n^u \cdot \Gamma(Y)^u + \Xi', \end{aligned}$$

where Ξ' is a random error term (the prime mark distinguishing it from the one in the proof of Lemma 3.4 above) that is bounded by

$$|\Xi'| \leq \|\tilde{\phi}^{(5)}\|_{L^\infty} \cdot n^5 \cdot |\Gamma(Y)|^5 \prec n^{-5/2+10\varepsilon_v}.$$

Thus, again using the expectation bounds in Proposition 3.7, we have

$$\begin{aligned} & \left| \mathbb{E}\tilde{\phi}(n \cdot M^{(\ell)}(i, v, Y)M^{(\ell)}(v, j, Y)) - \sum_{u=0}^4 \frac{1}{u!} \tilde{\phi}^{(u)}(n \cdot R_Q(i, v)R_Q(v, j)) \cdot n^u \cdot \Gamma(Y)^u \right| \\ &\leq \mathbb{E}|\Xi'| \\ &\lesssim_\varepsilon n^{-5/2+10\varepsilon_v+\varepsilon} \end{aligned}$$

for $\varepsilon > 0$ arbitrarily small.

Combining this with Lemma 3.6, we find

$$\begin{aligned} & |\mathbb{E}\tilde{F}(W) - \mathbb{E}\tilde{F}(X)| \\ &\lesssim \sum_{\ell=1}^L \left| \mathbb{E} \sum_{u=0}^4 \frac{1}{u!} \tilde{\phi}^{(u)}(n \cdot R_Q(i, v)R_Q(v, j)) \cdot n^u \cdot (\Gamma^{(\ell, i, j)}(W)^u - \Gamma^{(\ell, i, j)}(X)^u) \right| + O(n^{-1/2+10\varepsilon_v+\varepsilon}). \end{aligned}$$

(In the same way as discussed in Remark 3.8, we see from this calculation that fourth order was the correct order of Taylor expansion for this argument to work.) Let us write

$$\varepsilon' := 10\varepsilon_v + \varepsilon$$

for this remaining error exponent, which we will carry through to the end of the proof and which gives rise to the dependence on ε_v in Theorem 1.4.

Let us define $\Phi_{u,i,j}(Q) = \frac{1}{u!} \tilde{\phi}^{(u)}(n \cdot R_Q(i, v) R_Q(v, j))$. The only properties of this quantity we will need is that it is $O(1)$ almost surely (by the boundedness of $\tilde{\phi}$) and its derivatives and that it is independent of W_{ab} and X_{ab} (since it is a function only of Q , where this entry is set to zero). Also, let us develop notation for the quantities appearing in powers of Γ . Expanding Γ itself from its definition, we have

$$\begin{aligned} \Gamma^{(\ell, i, j)}(Y) &= \sum_{\substack{(s_1, s_2) \in \{0, 1, 2, 3, 4\}^2 \\ (s_1, s_2) \neq (0, 0)}} \sum_{\substack{S_1 \in \mathcal{S}_{a,b,s_1} \\ S_2 \in \mathcal{S}_{a,b,s_2}}} Y_{ab}^{S_1} Y_{ab}^{S_2} \\ &\quad \cdot R_Q(i, \alpha_{S_1, 1}) R_Q(\beta_{S_1, 1}, \alpha_{S_1, 2}) \cdots R_Q(\beta_{S_1, s_1-1}, \alpha_{S_1, s_1}) R_Q(\beta_{S_1, s_1}, v) \\ &\quad \cdot R_Q(v, \alpha_{S_2, 1}) R_Q(\beta_{S_2, 1}, \alpha_{S_2, 2}) \cdots R_Q(\beta_{S_2, s_2-1}, \alpha_{S_2, s_2}) R_Q(\beta_{S_2, s_2}, j) \\ &=: \sum_{\substack{(s_1, s_2) \in \{0, 1, 2, 3, 4\}^2 \\ (s_1, s_2) \neq (0, 0)}} \sum_{\substack{S_1 \in \mathcal{S}_{a,b,s_1} \\ S_2 \in \mathcal{S}_{a,b,s_2}}} Y_{ab}^{S_1} Y_{ab}^{S_2} J_{i,j,S_1}(Q) J_{i,j,S_2}(Q). \end{aligned}$$

It will be useful to rearrange a little bit more, by defining the union $\mathcal{S}_{a,b} := \mathcal{S}_{a,b,0} \sqcup \cdots \sqcup \mathcal{S}_{a,b,4}$. For $S \in \mathcal{S}_{a,b}$, is a string of indeterminate length now, write $s(S)$ for its length, so that $S \in \mathcal{S}_{a,b,s(S)}$. Then, writing \emptyset for the empty string that is the only element of $\mathcal{S}_{a,b,0}$, we may write the above as a single sum,

$$= \sum_{(S_1, S_2) \in \mathcal{S}_{a,b}^2 \setminus \{(\emptyset, \emptyset)\}} Y_{ab}^{S_1} Y_{ab}^{S_2} J_{i,j,S_1}(Q) J_{i,j,S_2}(Q).$$

We may then write the result of taking a power of this expression in terms of matrices with entries taking values in $\mathcal{S}_{a,b}$:

$$\Gamma^{(\ell, i, j)}(Y)^u = \sum_{\substack{S \in \mathcal{S}_{a,b}^{u \times 2} \\ \text{no row of } S \text{ equals } (\emptyset, \emptyset)}} \prod_{e=1}^u \prod_{f=1}^2 Y_{ab}^{S_{ef}} J_{i,j,S_{ef}}(Q).$$

Lastly, for such a matrix S , define

$$\begin{aligned} Y_{ab}^S &:= \prod_{e,f} Y_{ab}^{S_{ef}}, \\ J_{i,j,S}(Q) &:= \prod_{e,f} J_{i,j,S_{ef}}(Q), \\ |S| &:= \sum_{e,f} s(S_{ef}), \\ |S|_0 &:= \#\{(e, f) : S_{ef} \neq \emptyset\}. \end{aligned}$$

Using this, we may rewrite our bound concisely as

$$\begin{aligned} &|\mathbb{E}F(W) - \mathbb{E}F(X)| \\ &\lesssim \sum_{\ell=1}^L \left| \mathbb{E} \sum_{u=0}^4 \sum_{\substack{S \in \mathcal{S}_{a,b}^{u \times 2} \\ \text{no row of } S \text{ equals } (\emptyset, \emptyset)}} \Phi_{u,i,j}(Q) J_{i,j,S}(Q) \cdot n^u \cdot (W_{ab}^S - X_{ab}^S) \right| + O(n^{-1/2+\varepsilon'}) \end{aligned}$$

where we may use independence of Q and (W_{ab}, X_{ab}) , giving when combined with a triangle inequality

$$\leq \sum_{\ell=1}^L \sum_{u=0}^4 \sum_{\substack{S \in \mathcal{S}_{a,b}^{u \times 2} \\ \text{no row of } S \text{ equals } (\emptyset, \emptyset)}} |\mathbb{E}[\Phi_{u,i,j}(Q) J_{i,j,S}(Q)]| \cdot n^u \cdot |\mathbb{E}[W_{ab}^S] - \mathbb{E}[X_{ab}^S]| + O(n^{-1/2+\varepsilon'})$$

Here, since the expressions in the second expectation are monomials of degree $|S|$ in W_{ab}, X_{ab} , and their conjugates, they will be exactly zero unless $|S| \geq 3$, since by assumption the joint moments of the real and imaginary parts of W and X match up to order two. On the other hand, we have $|J_{i,j,S}(Q)| \prec n^{-u+2u\varepsilon_v} \prec n^{-u+8\varepsilon_v}$ for all $S \in \mathcal{S}_{a,b}$ and $|Y_{ab}| \prec n^{-1/2}$ for $Y \in \{W, X\}$ by Proposition 3.7. Thus, also applying the expectation bounds from Proposition 3.7, each term above has a simple *a priori* bound of $O(n^{-u-|S|/2+u+8\varepsilon_v+\varepsilon}) = O(n^{-|S|/2+9\varepsilon_v})$. (Here and below we let $\varepsilon \in (0, \varepsilon_v)$ be a temporary parameter as needed.) In particular, if $|S| \geq 5$ then this is at most $O(n^{-5/2+9\varepsilon_v})$, and since $9\varepsilon_v < \varepsilon'$ this may be subsumed into the error term we already have, giving

$$\leq \sum_{\ell=1}^L \sum_{u=0}^4 \sum_{\substack{S \in \mathcal{S}_{a,b}^{u \times 2} \\ \text{no row of } S \text{ equals } (\emptyset, \emptyset) \\ |S| \in \{3, 4\}}} n^u \cdot |\mathbb{E}[\Phi_{u,i,j}(Q) J_{i,j,S}(Q)]| \cdot |\mathbb{E}[W_{ab}^S] - \mathbb{E}[X_{ab}^S]| + O(n^{-1/2+\varepsilon'}),$$

Now, for the remaining terms we will not have any particular control of the difference of expectations involving W and X , so a direct bound on these using Proposition 2.2 reduces this to

$$\lesssim \sum_{\ell=1}^L \sum_{u=0}^4 \sum_{\substack{S \in \mathcal{S}_{a,b}^{u \times 2} \\ \text{no row of } S \text{ equals } (\emptyset, \emptyset) \\ |S| \in \{3, 4\}}} n^{u-|S|/2} \cdot |\mathbb{E}[\Phi_{u,i,j}(Q) J_{i,j,S}(Q)]| + O(n^{-1/2+\varepsilon'}).$$

Now, consider grouping the terms in the outer sum over ℓ according to, firstly, whether $a_\ell = b_\ell$ or not, and second according to the size of $\{a_\ell, b_\ell\} \cap \{i, j\}$. The total number of terms where either $a_\ell = b_\ell$ or $\{a_\ell, b_\ell\} \cap \{i, j\} \geq 1$ is $O(n)$. On the other hand, note first that, since $|J_{i,j,S}(Q)| \prec n^{-u+8\varepsilon_v}$ as we derived earlier, by Proposition 3.7 (and the Cauchy-Schwarz inequality to extract the $\Phi_{u,i,j}$ factor) every term in the sum above is $O(n^{-|S|/2+8\varepsilon_v+\varepsilon})$. Further, whenever $S \in \mathcal{S}_{a,b}^{u \times 2}$ and no row of S equals (\emptyset, \emptyset) , then in particular every row contributes at least 1 to $|S|$, so $|S| \geq u \geq 3$. Therefore, we have the simpler bound that every term in the sum above is $O(n^{-|S|/2+8\varepsilon_v+\varepsilon}) \leq O(n^{-3/2+9\varepsilon_v})$. So, the sum of $O(n)$ such terms is always $O(n^{-1/2+9\varepsilon_v})$, and up to such error, which is again subsumed in our current error term, we may restrict our attention to the case $a_\ell \neq b_\ell$ and $\{a_\ell, b_\ell\} \cap \{i, j\} = \emptyset$. Since there are $O(n^2)$ such terms, we have:

$$|\mathbb{E}\tilde{F}(W) - \mathbb{E}\tilde{F}(X)| \lesssim \max_{\substack{a, b \in [n] \text{ distinct} \\ \{a, b\} \cap \{i, j\} = \emptyset \\ u \leq 4 \\ S \in \mathcal{S}_{a,b}^{u \times 2} \\ \text{no row of } S \text{ equals } (\emptyset, \emptyset) \\ |S| \in \{3, 4\}}} n^{2+u-|S|/2} \cdot |\mathbb{E}[\Phi_{u,i,j}(Q) J_{i,j,S}(Q)]| + O(n^{-1/2+\varepsilon'}).$$

Now, we note that when $\{a, b\} \cap \{i, j\} = \emptyset$, then we can improve our above bound to $|J_{i,j,S}(Q)| \prec n^{-u-|S|_0/2+2|S|\varepsilon_v}$. Thus, every expression in the maximum above, for a given value of u , is bounded by $O(n^{2-|S|/2-|S|_0/2+8\varepsilon_v+\varepsilon})$. All S in the maximum have $|S|_0 \geq 1$ (since otherwise they would have

a row identically zero), so if $|S| = 4$ then any such term in the maximum has value $O(n^{-1/2+9\varepsilon_v})$, again smaller than our current error term, and so such terms can effectively be ignored.

So, we may restrict our attention to terms with $|S| = 3$. In this case, if $|S|_0 \geq 2$, then again such a term has value $O(n^{-1/2+9\varepsilon_v})$ since $|S|/2 + |S|_0/2 \geq 5/2$, and such terms can likewise be ignored. So, we may finally restrict our attention to the case $|S| = 3$ and $|S|_0 = 1$. In this case, we must have $u = 1$, and $S = [T \ \emptyset]$ or $S = [\emptyset \ T]$ for some $T \in \mathcal{S}_{a,b,3}$. Expanding the definitions back out, we find

$$\begin{aligned} & |\mathbb{E}\tilde{F}(W) - \mathbb{E}\tilde{F}(X)| \\ & \lesssim \max_{\substack{i,j \in [n] \\ a,b \in [n] \text{ distinct} \\ \{a,b\} \cap \{i,j\} = \emptyset \\ T \in \mathcal{S}_{a,b,3}}} n^{3/2} \cdot |\mathbb{E}[\Phi_{1,i,j}(Q) R_Q(i, \alpha_{T,1}) R_Q(\beta_{T,1}, \alpha_{T,2}) R_Q(\beta_{T,2}, \alpha_{T,3}) R_Q(\beta_{T,3}, v) R_Q(v, j)]| \\ & \quad + O(n^{-1/2+\varepsilon'}), \end{aligned}$$

where we use that the cases of $S = [T \ \emptyset]$ and $S = [\emptyset \ T]$ take the same form up to exchanging the role of i and j . Now, recall that given $T \in \mathcal{S}_{a,b,3}$, we have $\alpha_{T,c}, \beta_{T,d} \in \{a, b\}$, and $\alpha_{T,c}$ and $\beta_{T,c}$ are distinct (so one is a and the other is b). Thus, whenever $\beta_{T,1} \neq \alpha_{T,2}$ or $\beta_{T,2} \neq \alpha_{T,3}$, by Proposition 3.7 the expression in the maximum is $O(n^{3/2} \cdot n^{-2+\varepsilon}) \leq O(n^{-1/2+\varepsilon})$, and thus may be absorbed into the current error term. So, we may further reduce to a specific combination of resolvent quadratic forms, removing the dependence on T and arriving at an entirely concrete expression:

$$\lesssim \max_{\substack{i,j \in [n] \\ a,b \in [n] \text{ distinct} \\ \{a,b\} \cap \{i,j\} = \emptyset}} n^{3/2} \cdot |\mathbb{E}[\Phi_{1,i,j}(Q) R_Q(i, a) R_Q(a, a) R_Q(b, b) R_Q(b, v) R_Q(v, j)]| + O(n^{-1/2+\varepsilon'}).$$

Expanding out the definition of $\Phi_{1,i,j}(Q)$, this proof is then completed upon using the result of Lemma 3.9 below. \square

The last technical ingredient we will need is the following. We remove some of the specific details of the form of the $\Phi_{1,i,j}(Q)$ factor for the sake of brevity in the proof.

Lemma 3.9. *In the above setting, suppose $i, j \in [n]$ and that $a, b \in [n] \setminus \{i, j\}$ are distinct. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a bounded L -Lipschitz function. Then, for any $\varepsilon > 0$,*

$$|\mathbb{E}[f(R_Q(i, v) R_Q(v, j)) \cdot R_Q(i, a) R_Q(a, a) R_Q(b, b) R_Q(b, v) R_Q(v, j)]| \lesssim_{\varepsilon, L} n^{-2+2\varepsilon_v+\varepsilon}.$$

We note that this will conclude the proof of Theorem 1.4: Lemma 3.9 completes the proof of Lemma 3.4, and combining this with Lemma 3.3 in turn completes the proof of Theorem 1.4, as described earlier.

3.3 Resolvent monomial expectation via decoupling: Proof of Lemma 3.9

Let $Q^{(i)} \in \mathbb{C}_{\text{herm}}^{(n-1) \times (n-1)}$ be obtained from Q by removing the i -th row and column. Also, denote by $\mathcal{F}^{(i)} := \sigma\{Q_{pq} : i \notin \{p, q\}\}$ the σ -algebra generated by all entries of Q except row and column i and write $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | \mathcal{F}^{(i)}]$ for the operation of taking conditional expectation with respect to this σ -algebra (in simpler language, the operation of averaging over the i th row and column of Q).

Since in this section we will only work with resolvents of matrices related to Q , let us slightly change notation and define

$$R := R_Q(\tilde{\lambda}),$$

and write $R^{(i)}$ for $R_{Q^{(i)}}(\tilde{\lambda})$ extended to have dimension $n \times n$ by adding an i th row and column equal to zero. Here $\tilde{\lambda} = \theta + \theta^{-1} + i\eta^{-1/2}$ is the same complex value at which we have been evaluating all resolvents in the previous section as well.

We borrow a proof technique appearing in [KY11] for the analysis of the eigenvectors of Wigner matrices themselves, which is based on the observation that $\mathbb{E}_i R_Q(i, a)$ is much smaller than the typical value of $R_Q(i, a)$ with no averaging. The following are the main technical devices used to make this argument, identities relating R and $R^{(i)}$.

Proposition 3.10. *For $k \notin \{i, j\}$, we have*

$$R(i, j) = R^{(k)}(i, j) + \frac{R(i, k)R(k, j)}{R(k, k)}.$$

Proof. This is a special case of the formula for the inverse of the entries of a block matrix, where we view the (k, k) entry as a 1×1 block. \square

Proposition 3.11. *For $i, j \in [n]$ distinct, we have*

$$\begin{aligned} R(i, j) &= R(i, i) \sum_{k \neq i} Q_{ik} R^{(i)}(k, j), \\ R(i, j) &= R(j, j) \sum_{k \neq j} R^{(j)}(i, k) Q_{kj}. \end{aligned}$$

Proof. This is a special case of the same formula referenced above, but now where we view the (i, i) or (j, j) entry as the 1×1 block with respect to which we expand the inverse. \square

The following is then the main estimate that we will use. Note that Proposition 3.7 implies that $\mathbb{E}|R_Q(i, a)|^2 \lesssim_\varepsilon n^{-1+\varepsilon}$, on which this result improves considerably.

Lemma 3.12. *For any $i, a \in [n]$ distinct, for any $\varepsilon > 0$,*

$$\mathbb{E}|\mathbb{E}_i R_Q(i, a)|^2 \lesssim_\varepsilon n^{-2+\varepsilon}.$$

Proof. By Proposition 3.11, for indices $a \neq i$,

$$R(i, a) = R(i, i) \sum_{k \neq i} Q_{ik} R^{(i)}(k, a).$$

Decompose $R(i, i) = G_{\mu_{sc}}(\tilde{\lambda}) + (R(i, i) - G_{\mu_{sc}}(\tilde{\lambda}))$, where $G_{\mu_{sc}}$ is the Cauchy transform of the semicircle law. Then

$$R(i, a) = G_{\mu_{sc}}(\tilde{\lambda}) \cdot \sum_{k \neq i} Q_{ik} R^{(i)}(k, a) + (R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})) \cdot \sum_{k \neq i} Q_{ik} R^{(i)}(k, a).$$

Now we consider taking the conditional expectation \mathbb{E}_i (conditional on $\mathcal{F}^{(i)}$). Since $R^{(i)}$ is $\mathcal{F}^{(i)}$ -measurable (i.e., independent of row and column i of Q) and by the definition of generalized Wigner matrices we have $\mathbb{E}_i[Q_{ik}] = 0$, we have

$$\mathbb{E}_i \left[G_{\mu_{sc}}(\tilde{\lambda}) \cdot \sum_{k \neq i} Q_{ik} R^{(i)}(k, a) \right] = G_{\mu_{sc}}(\tilde{\lambda}) \cdot \sum_{k \neq i} R^{(i)}(k, a) \mathbb{E}_i[Q_{ik}] = 0.$$

So, we are left with just

$$\mathbb{E}_i[R(i, a)] = \mathbb{E}_i \left[(R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})) \cdot \sum_{k \neq i} Q_{ik} R^{(i)}(k, a) \right].$$

Applying the Cauchy-Schwarz inequality on this conditional expectation, we have

$$\left| \mathbb{E}_i[R(i, a)] \right|^2 \leq \left(\mathbb{E}_i |R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})|^2 \right) \left(\mathbb{E}_i \left| \sum_{k \neq i} Q_{ik} R^{(i)}(k, a) \right|^2 \right), \quad (3.13)$$

and taking the unconditional expectation,

$$\mathbb{E} \left| \mathbb{E}_i[R(i, a)] \right|^2 \leq \mathbb{E} \left[\mathbb{E}_i |R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})|^2 \cdot \mathbb{E}_i \left| \sum_{k \neq i} Q_{ik} R^{(i)}(k, a) \right|^2 \right]. \quad (3.14)$$

Expanding the second factor, we see that we may simplify it to

$$\begin{aligned} \mathbb{E}_i \left| \sum_{k \neq i} Q_{ik} R^{(i)}(k, a) \right|^2 &= \mathbb{E}_i \left[\sum_{k, l \neq i} Q_{ik} \overline{Q_{il}} \cdot R^{(i)}(k, a) \overline{R^{(i)}(l, a)} \right] \\ &= \sum_{k, l \neq i} R^{(i)}(k, a) \overline{R^{(i)}(l, a)} \cdot \mathbb{E}_i [Q_{ik} \overline{Q_{il}}] \\ &= \sum_{k \neq i} \mathbb{E}_i [|Q_{ik}|^2] \cdot |R^{(i)}(k, a)|^2. \end{aligned}$$

We have $\mathbb{E}_i [|Q_{ik}|^2] = \mathbb{E} [|Q_{ik}|^2] \leq Cn^{-1}$ for some $C > 0$ using our definition of generalized Wigner matrices and that Q is formed from a matrix of this kind by setting some entries to zero. So, we have

$$\mathbb{E}_i \left| \sum_{k \neq i} Q_{ik} R^{(i)}(k, a) \right|^2 \leq n^{-1} \cdot \sum_k |R^{(i)}(k, a)|^2,$$

which is an $\mathcal{F}^{(i)}$ -measurable random variable (not depending on row and column i of Q).

Substituting back, we may then rewrite as a full expectation,

$$\begin{aligned} \mathbb{E} \left| \mathbb{E}_i[R(i, a)] \right|^2 &\leq n^{-1} \cdot \mathbb{E} \left[\mathbb{E}_i [|R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})|^2] \cdot \sum_k |R^{(i)}(k, a)|^2 \right] \\ &= n^{-1} \cdot \mathbb{E} \left[|R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})|^2 \cdot \sum_k |R^{(i)}(k, a)|^2 \right] \end{aligned}$$

using that $\sum_k |R^{(i)}(k, a)|^2 = \|R^{(i)}e_a\|^2 \leq \|R^{(i)}\|^2$, we have

$$\begin{aligned} &\leq n^{-1} \cdot \mathbb{E} \left[|R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})|^2 \cdot \|R^{(i)}\|^2 \right] \\ &\leq n^{-1} \cdot \left(\mathbb{E} |R(i, i) - G_{\mu_{sc}}(\tilde{\lambda})|^4 \right)^{1/2} \cdot \left(\mathbb{E} \|R^{(i)}\|^4 \right)^{1/2}. \end{aligned}$$

By Proposition 2.17, the third factor is $O(1)$. By Corollary 2.19 (the isotropic local law for Q), Proposition 2.5, and Proposition 2.17, the second factor is $O_\varepsilon(n^{-1+\varepsilon})$ for any $\varepsilon > 0$. The result then follows from combining these bounds. \square

Now we use this to give the proof of the main result of this section.

Proof of Lemma 3.9. We note before continuing that the bounds on resolvent entries and quadratic forms with e_α and v proved in Proposition 3.7 all apply equally well to $R^{(i)}$, which is zero outside of a principle $(n - 1) \times (n - 1)$ submatrix that equals the resolvent of $Q^{(i)}$, where $Q^{(i)}$ is either a Wigner matrix of dimension $n - 1$ or such a matrix with one or two entries set to zero, to which Proposition 3.7 applies directly.

We would like to move towards applying Lemma 3.12. To that end, define

$$\begin{aligned}\mathcal{A} &:= f(R(i, v)R(v, j)) \cdot R(a, a)R(b, b)R(b, v)R(v, j), \\ \mathcal{A}^{(i)} &:= f(R^{(i)}(i, v)R^{(i)}(v, j)) \cdot R^{(i)}(a, a)R^{(i)}(b, b)R^{(i)}(b, v)R^{(i)}(v, j).\end{aligned}$$

Recall that we are trying to prove a bound on $\mathbb{E}\mathcal{A} \cdot R(i, a)$. Since $\mathcal{A}^{(i)}$ is $\mathcal{F}^{(i)}$ -measurable, we will try to replace \mathcal{A} by $\mathcal{A}^{(i)}$. We have:

$$\begin{aligned}|\mathbb{E}[\mathcal{A} \cdot R(i, a)]| &= |\mathbb{E}[(\mathcal{A} - \mathcal{A}^{(i)}) \cdot R(i, a)] + \mathbb{E}[\mathcal{A}^{(i)} \cdot R(i, a)]| \\ &\leq |\mathbb{E}[(\mathcal{A} - \mathcal{A}^{(i)}) \cdot R(i, a)]| + |\mathbb{E}[\mathcal{A}^{(i)} \cdot \mathbb{E}_i[R(i, a)]]|,\end{aligned}\tag{3.15}$$

where in the second term we have used that $\mathcal{A}^{(i)}$ is $\mathcal{F}^{(i)}$ -measurable. We now bound each term individually.

Consider the second term of (3.15) first. Using Cauchy–Schwarz,

$$|\mathbb{E}[\mathcal{A}^{(i)} \cdot \mathbb{E}_i[R(i, a)]]| \leq \left(\mathbb{E}|\mathcal{A}^{(i)}|^2\right)^{1/2} \left(\mathbb{E}|\mathbb{E}_i[R(i, a)]|^2\right)^{1/2}$$

We already have

$$\mathbb{E}|\mathbb{E}_i[R(i, a)]|^2 \lesssim n^{-2+\varepsilon}$$

for any $\varepsilon > 0$. It remains to bound $\mathbb{E}[|\mathcal{A}^{(i)}|^2]$. Since f is bounded, we have by Proposition 3.7

$$|\mathcal{A}^{(i)}| \leq \|f\|_\infty \cdot |R^{(i)}(a, a)| \cdot |R^{(i)}(b, b)| \cdot |R^{(i)}(b, v)| \cdot |R^{(i)}(v, j)| \prec n^{-1+2\varepsilon_v},$$

and by the expectation bounds in Proposition 3.7 we have

$$\mathbb{E}[|\mathcal{A}^{(i)}|^2] \lesssim n^{-2+4\varepsilon_v+\varepsilon}.$$

for any $\varepsilon > 0$. Thus, the entire second term of (3.15) is bounded by

$$|\mathbb{E}[\mathcal{A}^{(i)} \cdot \mathbb{E}_i[R(i, a)]]| \lesssim n^{-1+2\varepsilon_v+\varepsilon/2} \cdot n^{-1+\varepsilon/2} = n^{-2+2\varepsilon_v+\varepsilon}.\tag{3.16}$$

Now we consider the first term of (3.15). We first reorganize the expression $\mathcal{A} - \mathcal{A}^{(i)}$. Define

$$\begin{aligned}\mathcal{B} &:= R(a, a)R(b, b)R(b, v)R(v, j), \\ \mathcal{B}^{(i)} &:= R^{(i)}(a, a)R^{(i)}(b, b)R^{(i)}(b, v)R^{(i)}(v, j).\end{aligned}$$

Then, we have

$$\mathcal{A} - \mathcal{A}^{(i)} = \left(f(R(i, v)R(v, j)) - f(R^{(i)}(i, v)R^{(i)}(v, j))\right) \cdot \mathcal{B} + f(R^{(i)}(i, v)R^{(i)}(v, j)) \cdot (\mathcal{B} - \mathcal{B}^{(i)}).\tag{3.17}$$

Since f is L -Lipschitz,

$$\begin{aligned} & \left| f(R(i, v)R(v, j)) - f(R^{(i)}(i, v)R^{(i)}(v, j)) \right| \\ & \leq L \cdot \left| R(i, v)R(v, j) - R^{(i)}(i, v)R^{(i)}(v, j) \right| \\ & \leq L \cdot \left(|R(i, v)| \cdot |R(v, j)| + |R^{(i)}(i, v)| \cdot |R^{(i)}(v, j)| \right) \end{aligned}$$

and by Proposition 3.7, we therefore have

$$\prec n^{-1+2\varepsilon_v}.$$

And, again by Proposition 3.7, we have

$$|\mathcal{B}| \prec n^{-1+2\varepsilon_v},$$

so by Proposition 2.4, the first term of (3.17) is polynomially stochastically dominated as

$$\left| \left(f(R(i, v)R(v, j)) - f(R^{(i)}(i, v)R^{(i)}(v, j)) \right) \cdot \mathcal{B} \right| \prec n^{-2+4\varepsilon_v}. \quad (3.18)$$

Meanwhile, for the second term of (3.17), we may apply a telescoping expansion to $\mathcal{B} - \mathcal{B}^{(i)}$, obtaining

$$\begin{aligned} \mathcal{B} - \mathcal{B}^{(i)} &= R(a, a)R(b, b)R(b, v)R(v, j) - R^{(i)}(a, a)R^{(i)}(b, b)R^{(i)}(b, v)R^{(i)}(v, j) \\ &= (R(a, a) - R^{(i)}(a, a))R(b, b)R(b, v)R(v, j) \\ &\quad + R^{(i)}(a, a)(R(b, b) - R^{(i)}(b, b))R(b, v)R(v, j) \\ &\quad + R^{(i)}(a, a)R^{(i)}(b, b)(R(b, v) - R^{(i)}(b, v))R(v, j) \\ &\quad + R^{(i)}(a, a)R^{(i)}(b, b)R^{(i)}(b, v)(R(v, j) - R^{(i)}(v, j)). \end{aligned}$$

We now control the individual differences that appear here. Using Proposition 3.10, since $\{a, b\} \cap \{i, j\} = \emptyset$, we have

$$\begin{aligned} R(a, a) - R^{(i)}(a, a) &= \frac{R(a, i)R(i, a)}{R(i, i)}, \\ R(b, b) - R^{(i)}(b, b) &= \frac{R(b, i)R(i, b)}{R(i, i)}, \end{aligned}$$

and thus by Proposition 3.7 we have

$$\begin{aligned} |R(a, a) - R^{(i)}(a, a)| &\prec n^{-1}, \\ |R(b, b) - R^{(i)}(b, b)| &\prec n^{-1}. \end{aligned}$$

For the other differences, we must be slightly more careful, but using Proposition 3.10 again, we have

$$\begin{aligned} R(b, v) - R^{(i)}(b, v) &= \sum_{k=1}^n (R(b, k) - R^{(i)}(b, k))v_k \\ &= \sum_{k \in [n] \setminus \{i\}} \frac{R(b, i)R(i, k)}{R(i, i)}v_k + (R(b, i) - R^{(i)}(b, i))v_i \\ &= \frac{R(b, i)R(i, v)}{R(i, i)} - R^{(i)}(b, i)v_i \end{aligned}$$

By Proposition 3.7 and our assumption on v we have

$$|R(b, v) - R^{(i)}(b, v)| \prec n^{-1+\varepsilon_v},$$

and by the same token for the other similar term we have

$$|R(v, j) - R^{(i)}(v, j)| \prec n^{-1+\varepsilon_v}$$

as well.

Putting everything together, we find

$$|\mathcal{B} - \mathcal{B}^{(i)}| \prec n^{-3/2+2\varepsilon_v},$$

and therefore the whole second term in (3.17) is bounded as

$$\left| f(R^{(i)}(i, v)R^{(i)}(v, j)) \cdot (\mathcal{B} - \mathcal{B}^{(i)}) \right| \prec n^{-3/2+2\varepsilon_v}$$

as well, since f is bounded. Combining this with (3.18) and plugging both bounds into (3.17), we have

$$|\mathcal{A} - \mathcal{A}^{(i)}| \prec n^{-3/2+2\varepsilon_v},$$

so

$$|(\mathcal{A} - \mathcal{A}^{(i)}) \cdot R(i, a)| \prec n^{-2+2\varepsilon_v},$$

and thus

$$|\mathbb{E}[(\mathcal{A} - \mathcal{A}^{(i)}) \cdot R(i, a)]| \lesssim_\varepsilon n^{-2+2\varepsilon_v+\varepsilon}.$$

Finally, applying our bounds to either term of (3.15), we get

$$|\mathbb{E}[\mathcal{A} \cdot R(i, a)]| \lesssim_\varepsilon n^{-2+2\varepsilon_v+\varepsilon},$$

completing the proof. \square

4 Gaussian formula for entrywise averages: Proof of Theorem 1.8

Recall the setting of the Theorem: unlike the above proof of Theorem 1.4, the statement of this result depends on the field to which the entries of W belong. So, as in the Introduction, let us write

$$\mathbb{F} = \begin{cases} \mathbb{R} & \text{if } W \text{ is a weakly real Wigner matrix,} \\ \mathbb{C} & \text{if } W \text{ is a weakly complex Wigner matrix} \end{cases}.$$

Recall that we write $\mathcal{N}_{\mathbb{F}}(0, 1)$ for the standard Gaussian scalar distribution associated to each field (normalized so that $\mathbb{E}|Z|^2 = 1$ for $Z \in \mathcal{N}_{\mathbb{F}}(0, 1)$ in either case), and we write $\mathcal{N}_{\mathbb{F}}(0, I_n)$ for the law of a vector of n i.i.d. such Gaussians.

In Theorem 1.8, we have $\psi : \mathbb{C}^2 \rightarrow \mathbb{R}$ a C^5 function with bounded values and first five derivatives, and an associated function $\Psi : \mathbb{C}_{\text{herm}}^{n \times n} \times \mathbb{C}_{\text{herm}}^{n \times n} \rightarrow \mathbb{R}$ defined by

$$\Psi(A, B) := \frac{1}{n^2} \sum_{i,j=1}^n \psi(A_{ij}, B_{ij}).$$

Theorem 1.8 then bounds, for $\widehat{v}(W) = v_1(\theta vv^* + W)$ and $g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)$, the quantity

$$|\mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(W)\widehat{v}(W)^*) - \mathbb{E}\Psi(n \cdot vv^*, (\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)(\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)^*)|.$$

Before proceeding, let us establish a few tools for working with such functions Ψ . First, it follows immediately from our assumptions that Ψ is $O(n^{-1})$ -Lipschitz:

Proposition 4.1. *For any $A, B, A', B' \in \mathbb{C}_{\text{herm}}^{n \times n}$, we have*

$$|\Psi(A, B) - \Psi(A', B')| \leq \frac{L}{n} (\|A - A'\|_F + \|B - B'\|_F),$$

for a constant L depending only on the bounds on ψ and its derivatives.

Proof. The result follows since we have entrywise

$$\begin{aligned} |\psi(A_{ij}, B_{ij}) - \psi(A'_{ij}, B'_{ij})| &\leq |\psi(A_{ij}, B_{ij}) - \psi(A'_{ij}, B_{ij})| + |\psi(A'_{ij}, B_{ij}) - \psi(A'_{ij}, B'_{ij})| \\ &\leq L (|A_{ij} - A'_{ij}| + |B_{ij} - B'_{ij}|) \end{aligned}$$

for a suitable L . Thus, we have

$$|\Psi(A, B) - \Psi(A', B')| \leq \frac{L}{n^2} \sum_{i,j=1}^n |A_{ij} - A'_{ij}| + \frac{L}{n^2} \sum_{i,j=1}^n |B_{ij} - B'_{ij}| \leq \frac{L}{n} \|A - A'\|_F + \frac{L}{n} \|B - B'\|_F$$

using the Cauchy-Schwarz inequality on either term. \square

Next, we give a standard bound on the change in eigenvector projections under small perturbations of a matrix; the result we give is a simple reformulation of the Davis-Kahan inequality; see [DK70, YWS15].

Proposition 4.2. *Let $H, \Delta \in \mathbb{C}_{\text{herm}}^{n \times n}$. Let $\lambda_1(H)$ be a simple eigenvalue and suppose $\|\Delta\| < \lambda_1(H) - \lambda_2(H)$. Then,*

$$\|v_1(H)v_1(H)^* - v_1(H + \Delta)v_1(H + \Delta)^*\|_F \leq \sqrt{2} \cdot \frac{\|\Delta\|}{\lambda_1(H) - \lambda_2(H)}.$$

Combining the two results and rescaling per our setting gives the following useful corollary.

Corollary 4.3. *Let $H = \theta vv^* + W$ and $H' = \theta vv^* + W'$, and write $\widehat{v}(W) = v_1(H)$ and $\widehat{v}(W') = v_1(H')$ for the associated top eigenvectors. If $\|W - W'\| < \lambda_1(H) - \lambda_2(H)$, then*

$$|\Psi(n \cdot vv^*, n \cdot \widehat{v}(W)\widehat{v}(W)^*) - \Psi(n \cdot vv^*, n \cdot \widehat{v}(W')\widehat{v}(W')^*)| \leq \sqrt{2} \cdot L \cdot \frac{\|W - W'\|}{\lambda_1(H) - \lambda_2(H)}.$$

By Theorem 2.21, we have that $\lambda_1(H) - \lambda_2(H) = \Omega(1)$ with polynomially high probability. Therefore, in effect, perturbations of W with operator norm $o(1)$ have a small effect on the value of Ψ we are interested in. We will use this observation several times below.

Returning to the main proof, let us define

$$\text{GFE}(n) := \begin{cases} \text{GOE}(n) & \text{if } \mathbb{F} = \mathbb{R}, \\ \text{GUE}(n) & \text{if } \mathbb{F} = \mathbb{C} \end{cases}.$$

We prove the Theorem in several steps. First, we show using the above argument that we may, at the cost of a small error, replace W a weakly Wigner matrix with W that is also normalized to be a generalized Wigner matrix, satisfying the exact condition (1.3).

Lemma 4.4. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\theta > 1$, and $v \in \mathbb{S}^{n-1}(\mathbb{F})$ deterministic. Let $W \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a weakly \mathbb{F} -Wigner matrix. Define

$$\begin{aligned}\sigma_{ij}^2 &:= \mathbb{E}|W_{ij}|^2, \\ r_i &:= \sum_{j=1}^n \sigma_{ij}^2, \\ D &:= \text{Diag}(r_1, \dots, r_n).\end{aligned}$$

Write

$$W^{(\text{reg})} := D^{-1/2}WD^{-1/2}.$$

Then, $W^{(\text{reg})}$ is also a weakly \mathbb{F} -Wigner matrix, with parameters depending only on the parameters of W , and further is a generalized Wigner matrix satisfying (1.3). Also, we have, for any $\varepsilon > 0$,

$$\left| \mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(W)\widehat{v}(W)^*) - \mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(W^{(\text{reg})})\widehat{v}(W^{(\text{reg})})^*) \right| \lesssim n^{-\varepsilon_W + \varepsilon}. \quad (4.1)$$

Proof. That $W^{(\text{reg})}$ satisfies (1.3) follows immediately from the definition. From the definition of weakly Wigner matrices, we have $|1 - r_i| \leq C_W n^{-\varepsilon_W}$, and it then follows from elementary bounds that $W^{(\text{reg})}$ remains a weakly Wigner matrix as well. For the final bound, note that we have by the above observation $|I - D| \leq C_W n^{-\varepsilon_W}$, and thus $|I - D^{-1/2}| \lesssim n^{-\varepsilon_W}$ as well. So, we may bound

$$\begin{aligned}\|W - W^{(\text{reg})}\| &= \|W - D^{-1/2}WD^{-1/2}\| \\ &\leq \|W(I - D^{-1/2}) + (I - D^{-1/2})WD^{-1/2}\| \\ &\leq \|W\| \cdot \|I - D^{-1/2}\| \cdot (1 + \|D^{-1/2}\| \\ &\prec n^{-\varepsilon_W}.\end{aligned}$$

The result then follows from Corollary 4.3, with Theorem 2.21 used as mentioned after the Corollary. \square

Next, we show that we may, at the cost of a small error, replace $W^{(\text{reg})}$ in the above quantity with a Gaussian matrix.

Lemma 4.5. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\theta > 1$, and $v \in \mathbb{S}^{n-1}(\mathbb{F})$ deterministic with $\|v\|_\infty \leq n^{-1/2+\varepsilon_v}$ for some $\varepsilon_v \in (0, 1/20)$. Let $W^{(\text{reg})} \in \mathbb{C}_{\text{herm}}^{n \times n}$ be a weakly \mathbb{F} -Wigner matrix that is also a generalized Wigner matrix, and let $X^{(0)}$ be an \mathbb{F} -valued Gaussian random matrix with the same first two moments as $W^{(\text{reg})}$. Write $H(Y) = \theta vv^* + Y$ for $Y \in \{W, X^{(0)}\}$ and $\widehat{v}(Y) = v_1(H(Y))$. Let ψ, Ψ be as above. Then, for any $\varepsilon > 0$,

$$\left| \mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(W^{(\text{reg})})\widehat{v}(W^{(\text{reg})})^*) - \mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^*) \right| \lesssim n^{-1/2+10\varepsilon_v+\varepsilon}. \quad (4.2)$$

Proof. This is a simple consequence of Theorem 1.4: note that W and $X^{(0)}$ satisfy the assumptions of that result, and we may bound

$$\begin{aligned}&\left| \mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(W^{(\text{reg})})\widehat{v}(W^{(\text{reg})})^*) - \mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^*) \right| \\ &\leq \frac{1}{n^2} \sum_{i,j=1}^n \left| \mathbb{E}\psi \left(n \cdot v_i \overline{v_j}, n \cdot \widehat{v}(W^{(\text{reg})})_i \overline{\widehat{v}(W^{(\text{reg})})_j} \right) - \mathbb{E}\psi \left(n \cdot v_i \overline{v_j}, n \cdot \widehat{v}(W^{(\text{reg})})_i \overline{\widehat{v}(W^{(\text{reg})})_j} \right) \right| \\ &\lesssim \frac{1}{n^2} \sum_{i,j=1}^n n^{-1/2+10\varepsilon_v+\varepsilon} \\ &\leq n^{-1/2+10\varepsilon_v+\varepsilon},\end{aligned}$$

as claimed. \square

Note that above $X^{(0)}$ is itself a weakly \mathbb{F} -Wigner matrix, since the condition of being weakly \mathbb{F} -Wigner for a generalized Wigner matrix only depends on the first two moments. We next show that the top eigenvector of any such $X^{(0)}$ is close to that of a coupled GFE matrix. The simple idea of the proof is that the weakly \mathbb{F} -Wigner assumption implies that there is an $X \sim \text{GFE}(n)$ coupled to our $X^{(0)}$ such that $\|X - X^{(0)}\| \prec 1$, whereby the top eigenvectors of outlier eigenvalues of rank-one perturbations of these matrices are close by standard eigenvector perturbation inequalities.

Lemma 4.6. *In the setting of Lemma 4.5, let $X \sim \text{GFE}(n)$. Then, for any $\varepsilon, K > 0$ there exists $C = C(\varepsilon, K) > 0$ and a coupling between $X^{(0)}$ (the weakly Wigner Gaussian random matrix from Lemma 4.5) and X such that*

$$\left\| \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^* - \widehat{v}(X)\widehat{v}(X)^* \right\|_F \prec n^{-\varepsilon_w/2} \quad (4.3)$$

Proof. From the condition of being weakly Wigner, we have that, given such a Gaussian matrix X , there exists a coupled $Y \sim \text{GFE}(n)$ such that

$$X = (1 - \delta)Y + \Delta^{(1)} + \Delta^{(2)} = Y + \underbrace{(-\delta Y + \Delta^{(1)} + \Delta^{(2)})}_{=: \Delta},$$

where $\delta = O(n^{-\varepsilon_w/2})$, $\Delta^{(1)}$ is diagonal with independent Gaussian diagonal entries having variance $O(n^{-1})$, and $\Delta^{(2)}$ is Hermitian with independent centered complex Gaussian entries on and above the diagonal with $\mathbb{E}|\Delta_{ij}^{(2)}|^2 = O(n^{-1-\varepsilon_w})$.

To construct this decomposition, we may for instance first take $\Delta^{(1,1)}$ to be the matrix of diagonal entries of X and $\Delta^{(1,2)}$ to be the matrix of diagonal entries of a $\text{GFE}(n)$ matrix independent of X . Then, by the weakly Wigner assumption, there exists $\delta = O(n^{-\varepsilon_w/2})$ such that $\text{Cov}(\text{vec}(X - \Delta^{(1,1)} + \Delta^{(1,2)})) \succeq \text{Cov}(\text{vec}((1 - \delta)Y))$ in positive semidefinite order. Then, we may choose $\Delta^{(2)}$ jointly Gaussian, independent of Y and satisfying the above assumptions such that $X - \Delta^{(1,1)} + \Delta^{(1,2)} \xrightarrow{\text{(law)}} (1 - \delta)Y + \Delta^{(2)}$, and the claim follows by taking $\Delta^{(1)} := \Delta^{(1,1)} - \Delta^{(1,2)}$.

We then immediately find that $\|\delta Y\| \prec n^{-1/2-\varepsilon_w/2}$ and $\|\Delta^{(1)}\| \prec n^{-1+\varepsilon}$ for any $\varepsilon > 0$. From standard bounds such as those of Theorem 1.1 and Corollary 3.9 of [BVH16], it also follows that $\|\Delta^{(2)}\| \prec n^{-\varepsilon_w/2}$. Thus we also have $\|\Delta\| \leq \|\delta Y\| + \|\Delta^{(1)}\| + \|\Delta^{(2)}\| \prec n^{-\varepsilon_w/2}$.

By Theorem 2.21, with polynomially high probability, the gap between the top two eigenvalues of $\theta vv^* + X$ is of size $\Omega(1)$. The result then follows by Proposition 4.2. \square

The third and more substantial part of the proof is a direct analysis of such expressions for GFE matrices. For the sake of brevity below, let us define

$$y = y(\theta, v, g) := \rho(\theta)v + \frac{\tau(\theta)}{\sqrt{n}}g.$$

Lemma 4.7. *In the setting of Lemma 4.6, let $g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)$. Then, for any $\varepsilon, K > 0$ there exists $C = C(\varepsilon, K) > 0$ and a coupling between X and g such that*

$$\left\| \widehat{v}(X)\widehat{v}(X)^* - yy^* \right\|_F \prec n^{-1/2}. \quad (4.4)$$

This is our main technical tool, which we prove in the following section. For now, let us see how this together with Lemma 4.5 completes the proof of Theorem 1.8.

Corollary 4.8. *In the setting of Lemma 4.7, we have that, for any $\varepsilon > 0$,*

$$\left| \mathbb{E}\Psi(n \cdot vv^*, n \cdot \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^*) - \mathbb{E}\Psi(n \cdot vv^*, n \cdot yy^*) \right| \lesssim n^{-1/2+\varepsilon} + n^{-\varepsilon_W/2+\varepsilon}.$$

Proof. By our assumptions, $|\Psi|$ is bounded, say by some constant $C > 0$, and Ψ is Lipschitz as a function of a pair of matrices endowed with the distance $d((A, B), (A', B')) = \|A - A'\|_F + \|B - B'\|_F$, say with some Lipschitz constant $L > 0$. Also, combining Lemma 4.6 and Lemma 4.7, we find that there is a coupling between $X^{(0)}$ and g such that

$$\left\| \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^* - yy^* \right\|_F \prec n^{-1/2} + n^{-\varepsilon_W/2}.$$

Let us view $X^{(0)}$ and g as being realized on the same probability space to achieve this. Fix $\varepsilon > 0$, and define the event

$$\mathcal{E} = \left\{ \left\| \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^* - yy^* \right\|_F \leq n^{-1/2+\varepsilon} + n^{-\varepsilon_W/2+\varepsilon} \right\}.$$

Then, by the definition of polynomial stochastic domination, there is a constant $C > 0$ such that $\mathbb{P}[\mathcal{E}^c] \leq Cn^{-1/2}$. We then have

$$\begin{aligned} & \left| \mathbb{E} \left[\Psi(n \cdot vv^*, n \cdot \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^*) - \Psi(n \cdot vv^*, n \cdot yy^*) \right] \right| \\ & \leq 2C\mathbb{P}[\mathcal{E}^c] + \mathbb{E} \mathbf{1}_{\mathcal{E}} \left| \Psi(n \cdot vv^*, n \cdot \widehat{v}(X^{(0)})\widehat{v}(X^{(0)})^*) - \Psi(n \cdot vv^*, n \cdot yy^*) \right| \\ & \lesssim n^{-1/2} + n^{-1/2+\varepsilon} + n^{-\varepsilon_W/2+\varepsilon}, \end{aligned}$$

giving the result. \square

Theorem 1.8 then follows from combining Corollary 4.8 with Lemma 4.4 and Lemma 4.5. It remains to prove Lemma 4.7, which we do below.

4.1 Analysis of Gaussian noise models: Proof of Lemma 4.7

It will also be useful to define

$$\mathcal{U}_{\mathbb{F}}(n) = \left\{ \begin{array}{ll} \mathcal{O}(n), & \text{the } n \times n \text{ orthogonal group} \\ \mathcal{U}(n), & \text{the } n \times n \text{ unitary group} \end{array} \begin{array}{l} \text{if } \mathbb{F} = \mathbb{R}, \\ \text{if } \mathbb{F} = \mathbb{C} \end{array} \right\}.$$

The following is the main property of GFE matrices that we will use.

Proposition 4.9. *For each $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the following hold:*

- $g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)$ has a $\mathcal{U}_{\mathbb{F}}(n)$ invariant law as a vector; that is, for all $U \in \mathcal{U}_{\mathbb{F}}(n)$, $Ug \stackrel{(\text{law})}{=} g$.
- $X \sim \text{GFE}(n)$ has a $\mathcal{U}_{\mathbb{F}}(n)$ -invariant law as Hermitian matrix; that is, for all $U \in \mathcal{U}_{\mathbb{F}}(n)$, $UXU^* \stackrel{(\text{law})}{=} X$.

Proof of Lemma 4.7. Let us write $\widehat{v}(X) := v_1(\theta vv^* + X)$. For $U \in \mathcal{U}_{\mathbb{F}}(n)$, we have by Proposition 4.9 the equality of distributions

$$\widehat{v}(X) \stackrel{(\text{law})}{=} \widehat{v}(UXU^*) = v_1(\theta vv^* + UXU^*) = Uv_1(\theta(U^*v)(U^*v)^* + X).$$

We may choose such U such that $U^*v = e_1$ (note here the delicate point that we are using that $v \in \mathbb{F}^n$, avoiding the case where $\mathbb{F} = \mathbb{R}$ while v has complex values), so we find that there is some such (deterministic) U for which

$$\hat{v}(X) \stackrel{\text{(law)}}{=} U v_1 (\theta e_1 e_1^* + X) U^*.$$

Define

$$\tilde{y} := \rho(\theta)e_1 + \frac{\tau(\theta)}{\sqrt{n}}g.$$

Note that, by Proposition 4.9 again, we have

$$\tilde{y} \stackrel{\text{(law)}}{=} U^*y.$$

Let us couple the two sides of this equality so that, on the same probability space, we have $\tilde{y} = U^*y$. Then, we have

$$\begin{aligned} \|v_1(\theta e_1 e_1^* + X)v_1(\theta e_1 e_1^* + X)^* - \tilde{y}\tilde{y}^*\|_F &= \|U v_1(\theta e_1 e_1^* + X)v_1(\theta e_1 e_1^* + X)^* U^* - U\tilde{y}\tilde{y}^* U^*\|_F \\ &\stackrel{\text{(law)}}{=} \|\hat{v}(X)\hat{v}(X)^* - yy^*\|_F \end{aligned}$$

for a corresponding coupling of X and g . Thus, in effect we may take $v = e_1$ without loss of generality, so let us make this assumption going forward, in which case we may also take $y = \tilde{y}$.

We will only consider a single distribution of X now, so let us take $X \sim \text{GFE}(n)$ and $\hat{v} := \hat{v}(X) = v_1(\theta e_1 e_1^* + X)$. We remove the sign (for $\mathbb{F} = \mathbb{R}$) or phase (for $\mathbb{F} = \mathbb{C}$) ambiguity by choosing \hat{v} to have its first coordinate real and non-negative. Write $\hat{v}^{(1)}$ for \hat{v} with the first entry set to zero, so that we have

$$\hat{v} = |\langle \hat{v}, e_1 \rangle|e_1 + \hat{v}^{(1)}. \quad (4.5)$$

Let us write $\mathcal{U}_{\mathbb{F}}^{(1)}(n)$ for the set of matrices in $\mathcal{U}_{\mathbb{F}}(n)$ that are zero in the first row and column, which equivalently are the matrices of $\mathcal{U}_{\mathbb{F}}(n-1)$ padded by a row and column equal to zero. Then, $\theta e_1 e_1^* + X$ is $\mathcal{U}_{\mathbb{F}}^{(1)}(n)$ -invariant, in the same sense as in the second part of Proposition 4.9, since the first summand is fixed by the action of these matrices and the distribution of the second summand is unchanged. Thus, its top eigenvector \hat{v} is $\mathcal{U}_{\mathbb{F}}^{(1)}(n)$ -invariant, in the same sense as in the first part of Proposition 4.9. Since the two summands in (4.5) each belong to fixed subspaces of this group action, they are each individually invariant as well. In particular, $\hat{v}^{(1)}/\|\hat{v}^{(1)}\|$ has the law of an $(n-1)$ -dimensional random vector drawn uniformly at random from $\mathbb{S}^{n-2}(\mathbb{F})$, with a zero entry prepended to it.

Let us introduce $g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)$ and write $g^{(1)}$ for g with the first entry set to zero. By the above observations, we have the equality of laws

$$\frac{\hat{v}^{(1)}}{\|\hat{v}^{(1)}\|} \stackrel{\text{(law)}}{=} \frac{g^{(1)}}{\|g^{(1)}\|}.$$

Let us couple X and g such that this equality holds. From this g , we set

$$y = \rho(\theta)e_1 + \frac{\tau(\theta)}{\sqrt{n}}g.$$

Then, we have

$$\begin{aligned}
& \|\widehat{v}\widehat{v}^* - yy^*\|_F \\
&= \|(\widehat{v} - y)\widehat{v}^* + y(\widehat{v} - y)^*\|_F \\
&\leq (1 + \|y\|)\|\widehat{v} - y\| \\
&\leq \left(1 + \rho(\theta) + \tau(\theta)\frac{\|g\|}{\sqrt{n}}\right) \left(|\langle \widehat{v}, e_1 \rangle| - \rho(\theta) + \left| \frac{\tau(\theta)}{\sqrt{n}} g - \frac{\|\widehat{v}^{(1)}\|}{\|g^{(1)}\|} g^{(1)} \right| \right) \\
&\leq \left(1 + \rho(\theta) + \tau(\theta)\frac{\|g\|}{\sqrt{n}}\right) \left(|\langle \widehat{v}, e_1 \rangle| - \rho(\theta) + \frac{|g_1|}{\|g^{(1)}\|} + \left| \frac{\tau(\theta)}{\sqrt{n}} - \frac{\|\widehat{v}^{(1)}\|}{\|g^{(1)}\|} \right| \|g^{(1)}\| \right) \\
&= \left(1 + \rho(\theta) + \tau(\theta)\frac{\|g\|}{\sqrt{n}}\right) \left(|\langle \widehat{v}, e_1 \rangle| - \rho(\theta) + \frac{|g_1|}{\|g^{(1)}\|} + \left| \tau(\theta) \frac{\|g^{(1)}\|}{\sqrt{n}} - \|\widehat{v}^{(1)}\| \right| \right) \\
&\leq \left(1 + \rho(\theta) + \tau(\theta)\frac{\|g\|}{\sqrt{n}}\right) \left(|\langle \widehat{v}, e_1 \rangle| - \rho(\theta) + \frac{|g_1|}{\|g^{(1)}\|} + \left| \tau(\theta) - \|\widehat{v}^{(1)}\| \right| \frac{\|g^{(1)}\|}{\sqrt{n}} + \left| \frac{\|g^{(1)}\|}{\sqrt{n}} - 1 \right| \right).
\end{aligned}$$

We recall here that $\tau(\theta) = \sqrt{1 - \rho(\theta)^2}$ and by definition $\|\widehat{v}^{(1)}\| = \sqrt{1 - |\langle \widehat{v}, e_1 \rangle|^2}$, so quantities in the third term of the second factor are determined by those in the first term.

To complete the proof, it suffices to show that the above expression is small with high probability. To that end, fix $\varepsilon > 0$ and define the events

$$\begin{aligned}
\mathcal{E}_1 &:= \left\{ 1 - n^{-1/4} \leq \frac{\|g\|}{\sqrt{n}} \leq 1 + n^{-1/4} \right\}, \\
\mathcal{E}_2 &:= \{|g_1| \leq \log n\}, \\
\mathcal{E}_3 &:= \left\{ |\langle \widehat{v}, e_1 \rangle| - \rho(\theta) \leq n^{-1/2+\varepsilon} \right\}, \\
\mathcal{E} &:= \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3.
\end{aligned}$$

We see from elementary manipulations that, on the event \mathcal{E} , we have

$$\|\widehat{v}\widehat{v}^* - yy^*\|_F \lesssim n^{-1/2+\varepsilon}.$$

Fix some $K > 0$. We have by the tail bounds proved in [LM00] that

$$\mathbb{P}[\mathcal{E}_1^c] \lesssim n^{-K},$$

and by well-known Gaussian tail bounds we also have

$$\mathbb{P}[\mathcal{E}_2^c] \lesssim n^{-K}.$$

Finally, by Theorem 2.21, we also have $|\langle \widehat{v}, e_1 \rangle|^2 - \rho(\theta)| \prec n^{-1/2}$, and thus

$$\mathbb{P}[\mathcal{E}_3^c] \lesssim n^{-K}.$$

Thus $\mathbb{P}[\mathcal{E}^c] \lesssim n^{-K}$, concluding the proof. \square

5 Applications

5.1 Angular synchronization: Proof of Theorem 1.10

We recall the construction of the random matrix H that this result concerns: we have an underlying family of group elements $x_1, \dots, x_n \sim \text{Haar}(G)$, and define the group-valued matrix $M_{ij} = x_i x_j^{-1}$. From this, we define Y a noisy version of M by (1.9), and H the entrywise image of Y under the character χ given in (1.10). Let us write $v_i := \chi(x_i)/\sqrt{n}$. We have $|v_i| = 1/\sqrt{n}$, so $v \in \mathbb{S}^{n-1}(\mathbb{F})$ (recall that $\mathbb{F} = \mathbb{R}$ if $G = \mathbb{Z}/2$ and $\mathbb{F} = \mathbb{C}$ for all other groups we consider, $G = \mathbb{Z}/L$ for $L \geq 3$ and $G = U(1)$) and this v has $\|v\|_\infty = n^{-1/2}$, thus satisfying Assumption 1.3 for any $\varepsilon_v > 0$.

In total, then, the entries of H are distributed as, independently for $1 \leq i < j \leq n$,

$$H_{ij} \sim \left\{ \begin{array}{ll} \sqrt{n} \cdot v_i \bar{v}_j & \text{with probability } p, \\ \chi(\text{Haar}(G))/\sqrt{n} & \text{with probability } 1-p \end{array} \right\}, \quad (5.1)$$

where $p = \theta/\sqrt{n}$. Since we construct Y to be G -Hermitian, we also have $H = H^*$. Under our specific definition from the Introduction we will always have $H_{ii} = 1/\sqrt{n} = \chi(e)/\sqrt{n}$ for e the identity of G , but this detail will be inconsequential as we have shown above.

We then have, conditional on the underlying randomness of the x_i ,

$$\mathbb{E}[H \mid x] = p\sqrt{n} \cdot vv^* = \theta \cdot vv^*.$$

We then define

$$W := H - \theta vv^*,$$

which will be centered conditional on x . Specifically, its entries conditional on x are distributed as

$$W_{ij} \sim \left\{ \begin{array}{ll} (\sqrt{n} - \theta) \cdot v_i \bar{v}_j & \text{with probability } p, \\ \chi(\text{Haar}(G))/\sqrt{n} - \theta v_i \bar{v}_j & \text{with probability } 1-p \end{array} \right\}. \quad (5.2)$$

This is a centered Hermitian matrix with independent entries on and above the diagonal. We have $|v_i \bar{v}_j| = 1/n$, and thus the joint second moments of the real and imaginary parts of W are the same as those of $\chi(\text{Haar}(G))/\sqrt{n}$ up to an additive error $O(n^{-3/2})$:

$$\begin{aligned} \mathbb{E}[\text{Re}(W_{ij})^2 \mid x] &= n^{-1} \mathbb{E}_{y \sim \text{Haar}(G)} \mathbb{E}[\text{Re}(\chi(y))^2] + O(n^{-3/2}), \\ \mathbb{E}[\text{Im}(W_{ij})^2 \mid x] &= n^{-1} \mathbb{E}_{y \sim \text{Haar}(G)} \mathbb{E}[\text{Im}(\chi(y))^2] + O(n^{-3/2}), \\ \mathbb{E}[\text{Re}(W_{ij})\text{Im}(W_{ij}) \mid x] &= n^{-1} \mathbb{E}_{y \sim \text{Haar}(G)} \mathbb{E}[\text{Re}(\chi(y))\text{Im}(\chi(y))] + O(n^{-3/2}). \end{aligned}$$

For $y \sim \text{Haar}(G)$, we have $|\chi(y)|^2 = \text{Re}(y)^2 + \text{Im}(y)^2 = 1$, while $\chi(y)^2 = (\text{Re}(y)^2 - \text{Im}(y)^2) + 2\text{Re}(y)\text{Im}(y)\mathbf{i}$. On the other hand, when $\mathbb{F} = \mathbb{C}$ then $\mathbb{E}\chi(y)^2 = \mathbb{E}\chi(y^2) = 0$ as this averages a non-trivial character over the subgroup of squares in G . So, in this case we have that W is a weakly \mathbb{C} -Wigner matrix, and a direct calculation shows that when $\mathbb{F} = \mathbb{R}$ (i.e., when $G = \mathbb{Z}/2$), then W is a weakly \mathbb{R} -Wigner matrix, in which case one only needs to consider $\mathbb{E}[W_{ij}^2 \mid x]$. Thus, in all cases, W is a weakly \mathbb{F} -Wigner matrix, with the parameter C_W an absolute constant and $\varepsilon_W = 1/2$.

Now, let \hat{v} be the top eigenvector of this H . By Theorem 1.8, we then obtain that, provided that ψ is \mathcal{C}^5 with bounded values and derivatives, conditionally on x (whereby v is not random), we have

$$\left| \Psi(n \cdot vv^*, n \cdot \hat{v}\hat{v}^*) - \mathbb{E}_{g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)} \Psi(n \cdot vv^*, (\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)(\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)^*) \right| \lesssim n^{-1/5}.$$

Next, we show that the expectation over x (and thus over the randomness in v) of the second expression converges to a single-letter formula. Indeed, expanding the definition of Ψ , we have

$$\begin{aligned}
& \mathbb{E}_{x_1, \dots, x_n \sim \text{Haar}(G)} \mathbb{E}_{g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)} \Psi(n \cdot vv^*, (\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)(\rho(\theta)\sqrt{n} \cdot v + \tau(\theta)g)^*) \\
&= \mathbb{E}_{x_1, \dots, x_n \sim \text{Haar}(G)} \mathbb{E}_{g \sim \mathcal{N}_{\mathbb{F}}(0, I_n)} \frac{1}{n^2} \sum_{i,j=1}^n \psi(n \cdot \chi(x_i) \overline{\chi(x_j)}, (\rho(\theta) \cdot \chi(x_i) + \tau(\theta)g_i)(\rho(\theta) \cdot \overline{\chi(x_j)} + \tau(\theta)\overline{g_j})) \\
&= \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E}_{\substack{x_i, x_j \sim \text{Haar}(G) \\ g_i, g_j \sim \mathcal{N}_{\mathbb{F}}(0, 1)}} \psi(n \cdot \chi(x_i) \overline{\chi(x_j)}, (\rho(\theta) \cdot \chi(x_i) + \tau(\theta)g_i)(\rho(\theta) \cdot \overline{\chi(x_j)} + \tau(\theta)\overline{g_j})) \\
&= \mathbb{E}_{\substack{x, y \sim \text{Haar}(G) \\ g, h \sim \mathcal{N}_{\mathbb{F}}(0, 1)}} \psi(n \cdot \chi(x) \overline{\chi(y)}, (\rho(\theta) \cdot \chi(x) + \tau(\theta)g)(\rho(\theta) \cdot \overline{\chi(y)} + \tau(\theta)h)) + O(n^{-1}),
\end{aligned}$$

since the diagonal terms contribute negligibly and all other expectations are equal.

To derive the actual statement of Theorem 1.10, we choose ψ such that

$$\psi(\chi(x) \overline{\chi(y)}, z) = \ell(xy^{-1}, \text{Round}(z))$$

for $x, y \in G$ and $z \in \mathbb{C}$. Since χ is an injective character, this is always possible. The resulting ψ may not be smooth, but we may extend to the choices of ℓ and Round specified in the theorem with rectifiable points of discontinuity by a standard mollification argument, approximating this ψ above and below, $\psi^- \leq \psi \leq \psi^+$, by smooth functions that differ from ψ by some parameter $\varepsilon > 0$ on sets of Gaussian measure tending to zero, applying Theorem 1.8 on these ψ , and taking $\varepsilon \rightarrow 0$, completing the proof.

5.2 Discussion and numerical experiments

Let us give some additional discussion and present numerical experiments verifying these results.

For the loss function, we mentioned in the Introduction, for discrete groups $G = \mathbb{Z}/L$, perhaps the most sensible discrete loss function merely counts errors,

$$\ell(x, y) = \mathbf{1}\{x \neq y\}.$$

For $G = U(1)$, we follow [Sin11] in taking the continuous loss

$$\ell(x, y) = 1 - \cos(x - y),$$

where we recall that we view $x, y \in \mathbb{R}/2\pi$, whereby they may be viewed as angles (here we take the difference $x - y$ modulo 2π).

For the rounding function, we use the natural choice

$$\text{Round}(z) := \arg \min_{x \in G} |\chi(x) - z|.$$

For the case $G = U(1)$, if we identify G with the unit circle in \mathbb{C} , then this is simply

$$\text{Round}(z) = \frac{z}{|z|}.$$

This is not defined at $z = 0$, but a routine approximation argument lets us use a well-defined rounding function that with high probability agrees with the above. For $G = \mathbb{Z}/L$, it rounds z

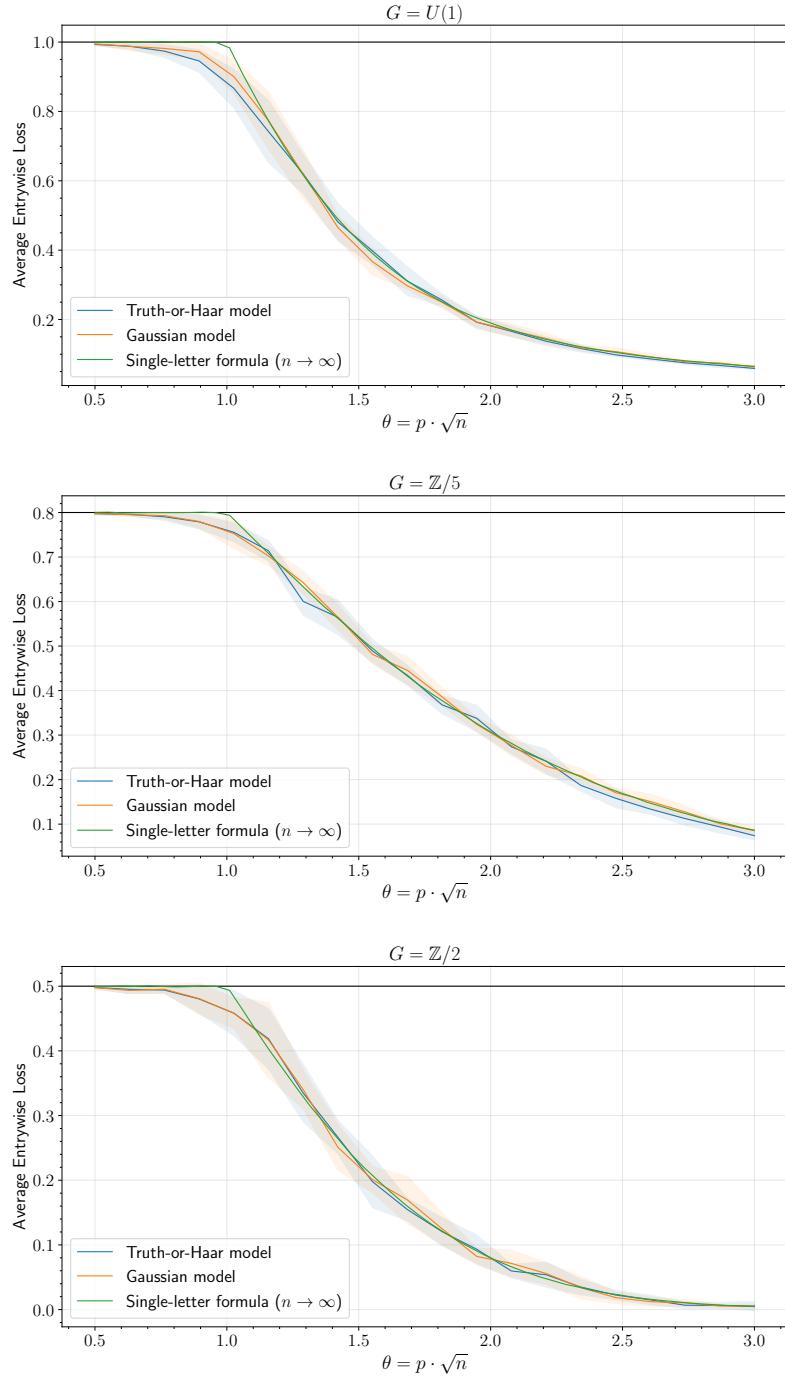


Figure 1: We show the performance of the rounded spectral algorithm with the rounding and loss functions described in Section 5.2 on the groups $G = U(1)$, $G = \mathbb{Z}/5$, and $G = \mathbb{Z}/2$, the latter recovering the setting of the dense stochastic block model. We consider dimension $n = 500$ in all cases, and plot the associated average of the entrywise loss $\ell : G \times G \rightarrow \mathbb{R}$ over all pairs $[n]^2$ of estimated group elements. The green curve shows an estimate of the single-letter formula given in Theorem 1.10 found by Monte Carlo estimation of the associated finite-dimensional expectation. For the Truth-or-Haar and Gaussian noise models, for several choices of the signal strength θ , we perform 10 trials of the algorithm and show the mean and one standard deviation of the average loss over these trials.

to the nearest root of unity. In particular, for $G = \mathbb{Z}/2$, which gives the stochastic block model setting, we have on $z \in \mathbb{R}$ that

$$\text{Round}(z) = \text{sgn}(z),$$

the natural sign rounding function for $\{\pm 1/\sqrt{n}\}$ -valued signals.

By way of comparison, we also consider an analogous model where Gaussian noise is directly applied to our observation, i.e., where we take

$$H = \theta vv^* + X$$

for v as above and $X \sim \text{GFE}(n)$. This has been studied in works such as [PWBM18, KBK24] as a simplified proxy for the truth-or-Haar model, but our arguments apply equally well to this and show that the result of Theorem 1.10 also applies under this model.

In Figure 1, we present a comparison of empirical average losses for the groups $G = U(1)$, $G = \mathbb{Z}/5$, and $G = \mathbb{Z}/2$ under both of these noise models with the prediction of Theorem 1.10. We find excellent agreement, especially away from the critical value $\theta = 1$. Around this critical value, similar experiments with different values of n show convergence to the single-letter formula prediction, albeit slowly the closer one is to this critical value.

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