# A Unifying Convexification Framework for Chance-Constrained Programs via Bilinear Extended Formulations over a Simplex

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#### Abstract

Chance-constrained programming is a widely used framework for decision-making under uncertainty, yet its mixed-integer reformulations involve nonconvex mixing sets with a knapsack constraint, leading to weak relaxations and computational challenges. Most existing approaches for strengthening the relaxations of these sets rely primarily on extensions of a specific class of valid inequalities, limiting both convex hull coverage and the discovery of fundamentally new structures. In this paper, we develop a novel convexification framework that reformulates chance-constrained sets as bilinear sets over a simplex in a lifted space and employs a step-bystep aggregation procedure to derive facet-defining inequalities in the original space of variables. Our approach generalizes and unifies established families of valid inequalities in the literature while introducing new ones that capture substantially larger portions of the convex hull. Main contributions include: (i) the development of a new aggregation-based convexification technique for bilinear sets over a simplex in a lower-dimensional space; (ii) the introduction of a novel bilinear reformulation of mixing sets with a knapsack constraint—arising from single-row relaxations of chance constraints—over a simplex, which enables the systematic derivation of strong inequalities in the original variable space; and (iii) the characterization of facet-defining inequalities within a unified framework that contains both existing and new families. Preliminary computational experiments demonstrate that our inequalities describe over 90% of the facet-defining inequalities of the convex hull of benchmark instances, significantly strengthening existing relaxations and advancing the polyhedral understanding of chance-constrained programs.

**Keywords:** Chance constraints, Convex hull, Bilinear formulation, Cutting planes, Lift-and-project.

## 1 Introduction

Many decision-making problems in uncertain environments, arising in engineering and operation, must account for reliability requirements, service levels, or risk measures [2, 27]. Chance-constrained

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programming (CCP), first introduced by [8], has become a widely used modeling framework to address such needs. This research area continues to attract significant interest, with recent advances in both theory and computation [28, 30, 35]; see also Küçükyavuz and Jiang [19] for a comprehensive survey.

In this paper, we study the polyhedral structure of an important substructure that arises in mixed-integer programming (MIP) reformulations of CCPs over a finite support, known as the mixing set with a knapsack constraint. The nonconvexity of these sets has motivated extensive research aimed at understanding their polyhedral properties. A fundamental direction in this line of work is the development of explicit valid inequalities for their convex hull. Such inequalities can be embedded directly into mathematical models to strengthen relaxations or incorporated into branch-and-cut algorithms [21, 23, 27], thereby improving both computational efficiency and solution quality.

An important class of valid inequalities for CCPs is the *strengthened mixing* inequalities. In their seminal work, Luedtke et al. [24] studied a *single-row* relaxation of the chance constraint, modeled as a mixing set with a 0-1 knapsack constraint, and derived a family of valid (and sometimes facet-defining) inequalities that generalize the classical *star* or *mixing* inequalities [3, 15]. Building on this, Küçükyavuz [18] developed compact extended formulations for joint chance constraints, thereby unifying and extending earlier inequalities in [24]. Later, Abdi and Fukasawa [1] characterized a broader class of valid inequalities for the general-probability case, subsuming those in Küçükyavuz [18], Luedtke et al. [24], while Zhao et al. [34] derived an even more general family encompassing all known explicit inequalities for the single-row relaxation.

Although the literature provides several families of valid inequalities for mixing sets, two main limitations remain. First, computational evidence shows that existing families capture only a limited portion of the convex hull, yielding relatively weak relaxations that often fail to deliver notable bound improvements when added to the initial relaxation of CCPs. Second, the prevailing approach to deriving new inequalities has largely focused on a specific class, typically by starting from a known family and expanding it through the addition of new elements. This constrains the discovery of fundamentally different classes of inequalities with distinct forms that fall outside previously established families. As a result, much of the existing work presents extended variants of known inequalities and relies on case-by-case analysis to prove validity. This approach not only lacks an intuitive or constructive methodology to inspire the derivation of new families with different structures, but also limits coverage of the convex hull—contributing to the relatively weak relaxation bounds noted earlier. These limitations highlight the need for continued efforts in designing systematic procedures to identify new and richer classes of valid inequalities, with the potential to improve optimality gap closure more meaningfully during the solution process.

To address these limitations, we propose a novel convexification approach through a different lens: reformulating the original mixing set with a knapsack constraint as a bilinear set over a simplex in a lifted space, which allows for employing a systematic aggregation method to characterize important families of facet-defining inequalities for the convex hull of the set in the original space of variables. The motivation for this work stems from [11] and [17], which introduced step-by-step aggregation methods for bilinear sets to obtain a closed-form representation of their convex hull in the original space of variables. Computational experiments across various application areas, from network interdiction to transportation, have demonstrated two key advantages of this convexification approach compared to classical methods such as disjunctive programming [4, 5] and the reformulation-linearization technique [31, 32]: (i) it enables the rapid generation of valid inequalities for bilinear sets to tighten convex relaxations, and (ii) it provides a systematic process to derive explicit facet-defining inequalities of the convex hull in the original space of variables. We refer the reader to [7, 10, 12, 13, 16, 22, 26, 29, 33] for a collection of works on the polyhedral study and convexification approaches for bilinear problems under various settings.

Building on this foundation, the present paper extends the existing literature by introducing a new convexification method for a bilinear set with a different structure from those previously studied. The main differences between the structures we study here and those in [11, 17] are as follows: (i) we focus on sets that contain disjoint bilinear constraints, whereas previous works examined the graph of disjoint bilinear functions over linear constraints; (ii) referring to the disjoint bilinear terms in the constraints as  $x_iy_j$ , we derive the convex hull in the space of the x variables only, while previous studies obtained the convex hull in the space of both x and y variables; and (iii) in our case, the y variables are binary, whereas in the prior works, the y variables can be continuous. These differences necessitate the design of a new aggregation method with components that differ from those proposed in prior works.

This extension enables us to apply our convexification framework to mixing sets with a knap-sack constraint, through a novel reformulation of the problem as a bilinear set over a simplex in a lifted space. The advantages of this convexification framework are two-fold. First, it provides a step-by-step aggregation procedure for deriving valid inequalities for the mixing set with a knapsack constraint (see Theorem 3). Unlike existing approaches, this yields a constructive recipe for generating multiple classes of valid inequalities in different forms. Second, it leads to the closed-form characterization of a broad family of valid inequalities that not only encompasses many existing families from the literature but also introduces new ones (see Theorem 4). Together, these inequalities collectively capture a significantly larger portion of the convex hull, as confirmed by our computational experiments on benchmark instances.

In summary, the contributions of this work are as follows. First, we develop a step-by-step aggregation technique to derive facet-defining inequalities for the convex hull of bilinear sets containing disjoint bilinear terms of the form  $x_iy_j$  in the x-variable space, where the y variables are binary and constrained to lie in a simplex. We believe this aggregation-based lift-and-project technique may be of independent interest beyond the class of CCPs studied in this paper. Second, we propose a novel reformulation of the mixing set with a knapsack constraint as a bilinear set over a simplex in a lifted space, allowing for the application of our convexification procedure to obtain valid inequalities. Third, our convexification procedure provides a systematic framework to characterize important families of valid inequalities that not only subsume several well-known

families in the literature but also unify them under a common structure. Fourth, we identify new classes of valid inequalities that capture large portions of the convex hull not attainable by existing approaches. Fifth, preliminary computational experiments on benchmark instances show that our new inequalities describe over 90% of the facet-defining inequalities of the convex hulls, representing a substantial improvement over prior results. Collectively, these contributions advance our understanding of the polyhedral structure of CCPs.

The remainder of this paper is organized as follows. Section 2 provides background on the MIP reformulation of CCPs as a mixing set with a knapsack constraint, along with a review of explicit families of valid inequalities derived in the literature. Section 3 introduces our aggregation-based convexification procedure for a general bilinear set containing disjoint bilinear terms in the constraints, which may also be of independent interest beyond CCPs. In Section 4, we present a reformulation of the mixing set with a knapsack constraint as a bilinear set in a lifted space, and derive families of valid inequalities for this set in the original variable space via our convexification procedure. Section 5 reports preliminary computational experiments on benchmark instances from the literature, demonstrating the extent to which our families of inequalities capture the convex hull compared to existing approaches. We end with conclusions and directions of future research in Section 6.

**Notation.** We denote vectors in boldface. For a set  $S \subseteq \{(\boldsymbol{x}; \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m\}$ , we refer to the convex hull and projection of S on to the  $\boldsymbol{x}$ -space by  $\operatorname{conv}(S)$  and  $\operatorname{proj}_{\boldsymbol{x}}(S)$ , respectively. Given  $x \in \mathbb{R}$ , we define  $(x)^+ = \max\{x, 0\}$ . Further, we define the indicator function  $\mathbb{I}(x) = 0$  if x = 0, and  $\mathbb{I}(x) = 1$  otherwise.

## 2 Background on Chance-Constrained Programming

We consider a probabilistically-constrained linear program with a random right-hand side:

$$\min_{\boldsymbol{u}} \quad \boldsymbol{c}^{\mathsf{T}} \boldsymbol{u} \tag{1a}$$

s.t. 
$$\mathbb{P}(W\boldsymbol{u} \ge \boldsymbol{\xi}) \ge 1 - \varepsilon,$$
 (1b)

$$u \in U$$
, (1c)

where  $U \subseteq \mathbb{R}_+^{d_u}$  is a polyhedron,  $W \in \mathbb{R}^{d_{\xi} \times d_u}$  is a matrix,  $\boldsymbol{\xi} \in \mathbb{R}^{d_{\xi}}$  is a random vector,  $\varepsilon \in [0,1]$  is a risk parameter, and  $\boldsymbol{c} \in \mathbb{R}^{d_u}$ . Assume that  $\boldsymbol{\xi}$  has a finite support on m scenarios, i.e.,  $\boldsymbol{\xi} \in \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m\}$  with  $\mathbb{P}(\boldsymbol{\xi} = \boldsymbol{\xi}_i) = \pi_i > 0$  for each  $i \in M := \{1, \dots, m\}$  and  $\sum_{i=1}^m \pi_i = 1$ . We assume without loss of generality that  $\boldsymbol{\xi}_i \geq \boldsymbol{0}$  for each  $i \in M$ . Moreover, we impose that  $\pi_i \leq \varepsilon$  for each i, since if  $\pi_i > \varepsilon$ , then  $W\boldsymbol{u} \geq \boldsymbol{\xi}_i$  must hold for any feasible  $\boldsymbol{u}$ . In such cases, we can incorporate these inequalities directly into the definition of U and exclude the corresponding scenario i from further consideration. We also note that (1) can represent a sampling-based approximation of a CCP, where the samples are given by  $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_m\}$  and  $\pi_i = 1/m$  for all  $i \in M$ .

#### 2.1 MIP Reformulation of CCP

In this section, we present the widely-used formulation of the CCP, defined in (1), as a MIP. For each  $i \in M$ , we introduce a binary variable  $x_i \in \{0,1\}$ , where  $x_i = 0$  enforces the constraint  $Wu \geq \xi_i$ . Since  $\xi_i \geq 0$ , this also implies that  $Wu \geq 0$  for every feasible u. Defining v = Wu, we can then reformulate (1) as

$$\min_{\boldsymbol{v},\boldsymbol{u},\boldsymbol{x}} \quad \boldsymbol{c}^{\mathsf{T}}\boldsymbol{u} \tag{2a}$$

s.t. 
$$\boldsymbol{u} \in U$$
, (2b)

$$\boldsymbol{v} - W\boldsymbol{u} = \boldsymbol{0},\tag{2c}$$

$$x_i = 0 \Rightarrow \mathbf{v} \ge \boldsymbol{\xi}_i, \quad i \in M,$$
 (2d)

$$\boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{x} \leq \varepsilon,$$
 (2e)

$$\boldsymbol{x} \in \{0,1\}^m, \ \boldsymbol{v} \ge \boldsymbol{0}. \tag{2f}$$

In this formulation, the inequality (2e) is sometimes referred to as either a *chance constraint* or a *knapsack constraint*; in this paper, we adopt the latter terminology. It is common in the literature to study the polyhedral structure of a subset of the feasible region of (2) in the space of variables (v; x), defined as

$$\mathcal{G} = \left\{ (\boldsymbol{v}, \boldsymbol{x}) \in \mathbb{R}_+^{d_{\boldsymbol{\xi}}} \times \{0, 1\}^m \left| \begin{array}{c} x_i = 0 \Rightarrow \boldsymbol{v} \geq \boldsymbol{\xi}_i, \ i \in M \\ \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{x} \leq \varepsilon \end{array} \right. \right\}.$$

We can write  $\mathcal{G}$  as

$$\mathcal{G} = \bigcap_{j=1}^{d_{\xi}} \Big\{ (oldsymbol{v}, oldsymbol{x}) : (v_j, oldsymbol{x}) \in \mathcal{F}_j \Big\},$$

where, for  $j \in \{1, \ldots, d_{\xi}\}$ , we define

$$\mathcal{F}_j = \left\{ (v_j, \boldsymbol{x}) \in \mathbb{R}_+ \times \{0, 1\}^m \middle| \begin{array}{c} x_i = 0 \Rightarrow v_j \ge \xi_{ij}, \ i \in M \\ \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{x} \le \varepsilon \end{array} \right\}.$$

Therefore, a natural first step in developing a strengthened formulation for  $\mathcal{G}$  is to develop a strengthened formulation for each  $\mathcal{F}_j$ . In particular, we have that any facet-defining inequality for  $\operatorname{conv}(\mathcal{F}_j)$  is also facet-defining for  $\operatorname{conv}(\mathcal{G})$ . This follows because the set of m+1 affinely independent points in  $\mathcal{F}_j$  that serves as the support of a facet-defining inequality of  $\operatorname{conv}(\mathcal{F}_j)$  can be trivially extended to a set of  $m+d_{\xi}$  affinely independent supporting points of this inequality in  $\mathcal{G} \subseteq \mathbb{R}^{m+d_{\xi}}$  by appropriately assigning values to the  $v_i$  variables for each  $i \neq j$ ; see [18, 24] for a detailed account. Thus, we consider a generic set

$$\mathcal{F}_c = \left\{ (z, \boldsymbol{x}) \in \mathbb{R}_+ \times \{0, 1\}^m \middle| \begin{array}{c} x_i = 0 \Rightarrow z \ge h_i, \ i \in M \\ \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{x} \le \varepsilon \end{array} \right\}, \tag{3}$$

by dropping the index j, replacing  $v_j$  with z, and substituting  $\xi_{ij}$  with  $h_j$  for each  $i \in M$ . In what follows, we refer to  $\mathcal{F}_c$  as the mixing set with a knapsack constraint. To perform our polyhedral analysis of this set, we begin by introducing notation and terminology. Without loss of generality, we assume that the right-hand-side values are ordered, i.e.,  $h_1 \geq h_2 \geq \cdots \geq h_m$ . Define  $p = \max\{k : \sum_{i=1}^k \pi_i \leq \varepsilon\}$ . In addition, let  $\{\langle 1 \rangle, \ldots, \langle m \rangle\}$  be a permutation of  $\{1, \ldots, m\}$  such that  $\pi_{\langle 1 \rangle} \leq \ldots \leq \pi_{\langle m \rangle}$ . Define  $\vartheta = \max\{k : \sum_{i=1}^k \pi_{\langle i \rangle} \leq \varepsilon\}$ . By definition, it is clear that  $p, \vartheta \in M$ .

An important special case of  $\mathcal{F}_c$  arises when the underlying probability distribution is uniform. In this case, we have  $\pi_i = 1/m$  for all  $i \in M$ , which implies  $p = \vartheta = \lfloor m\varepsilon \rfloor$ . Consequently, the knapsack constraint  $\pi^{\intercal}x \leq \varepsilon$  reduces to the cardinality constraint  $\mathbf{1}^{\intercal}x \leq p$ , where  $\mathbf{1}$  is the vector of all ones with matching dimension. Due to its numerous practical applications, this special case has received significant attention in the literature. For ease of reference, we define it separately here and refer to it throughout the paper as the *mixing set with a cardinality constraint*, formally given by

$$\mathcal{F}_c^{=} = \left\{ (z, \boldsymbol{x}) \in \mathbb{R}_+ \times \{0, 1\}^m \middle| \begin{array}{c} x_i = 0 \Rightarrow z \ge h_i, \ i \in M \\ \mathbf{1}^{\mathsf{T}} \boldsymbol{x} \le p \end{array} \right\}. \tag{4}$$

Next, in Section 2.2, we review existing results on the polyhedral structure of  $\mathcal{F}_c$  that provide closed-form expression of valid inequalities for this set.

#### 2.2 Existing Families of Valid Inequalities in Closed Form

Set  $\mathcal{F}_c$ , as defined in (3), has been extensively studied in the literature. When  $\varepsilon = 1$  (i.e., the chance constraint is trivially satisfied),  $\mathcal{F}_c$  reduces to a mixing set defined as

$$\mathcal{F}_c^1 = \left\{ (z, \boldsymbol{x}) \in \mathbb{R}_+ \times \{0, 1\}^m \,\middle|\, x_i = 0 \Rightarrow z \ge h_i, \ i \in M \right\},\tag{5}$$

and is studied in [3, 14, 15, 25]. The following result has served as the foundation for deriving valid inequalities for the CCP over the past several decades.

**Proposition 1.** [3, 14, 15] Consider  $\{t_1, \dots, t_l\} \subseteq M$  with  $t_1 < \dots < t_l$ . Then, the star inequalities

$$z + \sum_{\iota=1}^{l} (h_{t_{\iota}} - h_{t_{\iota+1}}) x_{t_{\iota}} \ge h_{t_{1}}, \tag{6}$$

where  $h_{t_{l+1}} := 0$ , are valid for the mixing set (5).

Later, [24] extends Proposition 1 to the case where  $\varepsilon \leq 1$ , as defined in  $\mathcal{F}_c$  given by (3).

**Proposition 2.** [24, Theorem 2] Consider  $\{t_1, \dots, t_l\} \subseteq \{1, \dots, p\}$  with  $t_1 < \dots < t_l$ . Then, the strengthened star inequalities

$$z + \sum_{\iota=1}^{l} (h_{t_{\iota}} - h_{t_{\iota+1}}) x_{t_{\iota}} \ge h_{t_{1}}, \tag{7}$$

where  $h_{t_{l+1}} := h_{p+1}$ , are valid for  $\mathcal{F}_c$ .

The following result extends the strengthened star inequalities for the special case of a uniform probability distribution by adding new elements that allow variables with negative coefficients to be included in the inequality.

**Proposition 3.** [24, Theorem 4] Consider the mixing set with a cardinality constraint  $\mathcal{F}_c^=$ . Let  $r \in \{1, \ldots, p\}, \{t_1, \ldots, t_l\} \subseteq \{1, \ldots, r\}, \text{ and } \{q_1, \ldots, q_{p-r}\} \subseteq \{p+1, \ldots, m\}, \text{ where } t_1 < \cdots < t_l \text{ and } q_1 < \cdots < q_{p-r}. Define <math>\phi_{q_1} = h_{r+1} - h_{r+2}$  and

$$\phi_{q_{\iota}} = \max \left\{ \phi_{q_{\iota-1}}, h_{r+1} - h_{r+\iota+1} - \sum_{k=1}^{\iota-1} \phi_{q_k} \right\}, \quad \iota = 2, \dots, p-r.$$

Then, the inequalities

$$z + \sum_{\iota=1}^{l} (h_{t_{\iota}} - h_{t_{\iota+1}}) x_{t_{\iota}} + \sum_{\iota=1}^{p-r} \phi_{q_{\iota}} (1 - x_{q_{\iota}}) \ge h_{t_{1}}, \tag{8}$$

where  $h_{t_{l+1}} := h_{r+1}$ , are valid for  $\mathcal{F}_c^=$ .

The following result extends the family of inequalities (8) by allowing the permutation of variables with negative coefficients and expanding their index range.

**Proposition 4.** [18, Theorem 3] Consider the mixing set with a cardinality constraint  $\mathcal{F}_c^=$ . Let  $\{t_1,\ldots,t_l\}\subseteq\{1,\ldots,r\}$  where  $t_1<\cdots< t_l$ , and  $\{q_1,\ldots,q_{p-r}\}$  be such that  $q_\iota\geq r+\iota+1$  for  $\iota\in\{1,\ldots,p-r\}$ . Define  $\phi_{q_1}=h_{r+1}-h_{r+2}$  and

$$\phi_{q_{\iota}} = \max \left\{ \phi_{q_{\iota-1}}, h_{r+1} - h_{r+\iota+1} - \sum_{\substack{k=1\\q_k > r+\iota+1}}^{\iota-1} \phi_{q_k} \right\}, \quad \iota = 2, \dots, p-r.$$

Then, the inequalities

$$z + \sum_{\iota=1}^{l} (h_{t_{\iota}} - h_{t_{\iota+1}}) x_{t_{\iota}} + \sum_{\iota=1}^{p-r} \phi_{q_{\iota}} (1 - x_{q_{\iota}}) \ge h_{t_{1}}, \tag{9}$$

where  $h_{t_{l+1}} := h_{r+1}$ , are valid for  $\mathcal{F}_c^=$ .

Subsequently, [1, 34] extended the result in Proposition 4 to the general-probability setting for  $\mathcal{F}_c$ . The next result presents the family of inequalities introduced in [34, Theorem 1] under Assumption 1 of that paper, which subsumes those derived in [1, Theorem 14].

**Proposition 5.** [34, Theorem 1] Consider  $r \in \{1, ..., p\}$ ,  $l \in \{1, ..., r\}$ ,  $v \in \{1, ..., \vartheta - r\}$ . Consider a sequence of integers  $\{s_0, s_1, ..., s_{v+1}\}$  such that  $1 \le s_1 \le ... \le s_v \le s_{v+1} = p - r + 1$ . Let  $P := \{t_1, ..., t_l\} \subseteq \{1, ..., r\}$ , where  $t_1 < ... < t_l$ . Moreover, let  $Q := \{q_1, ..., q_v\} \subseteq \{r + s_1 + s_1 + s_2 + s_1 + s_2 + s_2 + s_3 + s_4 + s_$ 

 $1, \ldots, m$ , where  $q_{\iota} \ge r + \min\{1 + s_{\iota}, s_{\iota+1}\}$  and

$$\sum_{k=1}^{r+s_{\iota}} \pi_{k} + \sum_{k=\iota}^{v} \pi_{w_{k}} > \varepsilon, \quad \sum_{k=1}^{r+s_{\iota}-1} \pi_{k} + \sum_{k=\iota}^{v} \pi_{w_{k}} \le \varepsilon, \tag{10}$$

for  $\iota \in \{1, \ldots, v\}$ , where  $\{w_1, \ldots, w_v\}$  is a permutation of Q such that  $\pi_{w_1} \geq \ldots \geq \pi_{w_v}$ . Define  $\phi_{q_1} = h_{r+s_1} - h_{r+s_2}$  and

$$\phi_{q_{\iota}} = \max \left\{ \phi_{q_{\iota-1}}, h_{r+s_1} - h_{r+s_{\iota}+1} - \sum_{\substack{k=1\\q_k \ge r + \min\{1+s_{\iota}, s_{\iota+1}\}}}^{\iota-1} \phi_{q_k} \right\}, \quad \iota = 2, \dots, v.$$

Then, the inequalities

$$z + \sum_{\iota=1}^{l} (h_{t_{\iota}} - h_{t_{\iota+1}}) x_{t_{\iota}} + \sum_{\iota=1}^{v} \phi_{q_{\iota}} (1 - x_{q_{\iota}}) \ge h_{t_{1}}, \tag{11}$$

with  $h_{t_{l+1}} := h_{r+s_1}$ , are valid for  $\mathcal{F}_c$ .

The results presented in Propositions 1–5 illustrate the progression of closed-form valid inequalities for  $\mathcal{F}_c$  over the years, with each subsequent result subsuming the preceding ones as special cases. As seen from these results, the fundamental structure of the inequalities has remained largely unchanged, originating from the star inequality (6), with gradual extensions achieved by adding new elements to broaden the family. Although this process has yielded an extensive family of inequalities—whose most general form is presented in Proposition 5—their structural foundation remains constrained by the core form of the star inequality. Consequently, several important classes of valid inequalities for  $\mathcal{F}_c$  still cannot be represented within the family of (11), as we illustrate next.

Example 1. Consider an instance of  $\mathcal{F}_c^=$  with m = 10,  $\{h_1, \ldots, h_m\} = \{20, 18, 14, 11, 6, 5, 4, 3, 2, 1\}$ , and  $p = \vartheta = 4$ . Then, it is easy to verify that the constraint

$$z + 6x_1 + 2x_4 + 3(1 - x_5) + 3(1 - x_6) \ge 20 \tag{12}$$

is facet-defining for  $conv(\mathcal{F}_c^=)$ . However, this inequality does not belong to the family of inequalities described in (11), as outlined next. Based on the structure of the above inequality, the setup in Proposition 5 would require setting r = 4,  $P = \{1,4\}$ , and  $Q = \{5,6\}$ . However, since  $p = r = \vartheta$ , the conditions in that proposition enforce  $Q = \emptyset$ , which conflicts with the requirement  $Q = \{5,6\}$  for this inequality. Moreover, the coefficients of variables in P, namely  $x_1$  and  $x_4$ , do not match the values prescribed in Proposition 5. In particular, the coefficient of  $x_1$ , which is 6, does not match  $(h_1 - h_4) = 9$ , and the coefficient of  $x_4$ , which is 2, does not match  $(h_4 - h_5) = 5$  as  $s_1 = 1$ . As a result, the above inequality exhibits a fundamentally different structure from those in (11), both in the segment corresponding to variables with positive coefficients and in that corresponding to

variables with negative coefficients.

Motivated by Example 1, in Section 4, we develop a unifying convexification framework that not only subsumes many existing families of valid inequalities for  $\mathcal{F}_c$  from the literature but also generates a rich collection of new families of valid inequalities that could not be captured by previous results, such as the inequality in Example 1. These new inequalities can constitute a substantial portion of the convex hull, as demonstrated by our computational experiments in Section 5.

## 3 A Convexification Procedure for Bilinear Sets over a Simplex

In this section, we develop a step-by-step aggregation procedure to derive facet-defining inequalities for the convex hull of a bilinear set over a simplex, which contains disjoint bilinear terms in its constraints. In Section 4, we then show how  $\mathcal{F}_c$  in (3) can be reformulated as a bilinear set of similar structure, making it amenable to our convexification procedure for generating valid inequalities for its convex hull.

For 
$$N := \{1, \dots, n\}$$
,  $M := \{1, \dots, m\}$ ,  $K := \{1, \dots, \kappa\}$ , and  $T := \{1, \dots, \tau\}$ , we consider 
$$S = \left\{ (\boldsymbol{x}; \boldsymbol{y}) \in \Xi \times \Delta_m \,\middle|\, \boldsymbol{y}^{\mathsf{T}} A^k \boldsymbol{x} + (\boldsymbol{b}^k)^{\mathsf{T}} \boldsymbol{x} + (\boldsymbol{c}^k)^{\mathsf{T}} \boldsymbol{y} \ge d_k, \,\,\forall k \in K \right\}, \tag{13}$$

where  $\Xi = \{ \boldsymbol{x} \in \mathbb{R}^n_+ \mid E\boldsymbol{x} \geq \boldsymbol{f} \}$ , and  $\Delta_m = \{ \boldsymbol{y} \in \mathbb{Z}^m_+ \mid \mathbf{1}^\intercal \boldsymbol{y} \leq 1 \}$  is an integral simplex. In this definition,  $A^k \in \mathbb{R}^{m \times n}$ ,  $\boldsymbol{b}^k \in \mathbb{R}^n$ ,  $\boldsymbol{c}^k \in \mathbb{R}^m$ ,  $d_k \in \mathbb{R}$ ,  $E \in \mathbb{R}^{\tau \times n}$ ,  $\boldsymbol{f} \in \mathbb{R}^{\tau}$  are parameters of appropriate dimension, K is the index set of bilinear constraints in  $\mathcal{S}$ , and T is the index set of linear constraints besides the non-negativity constraints in  $\Xi$ .

Define  $S(\bar{y})$  to be the restriction of S at point  $y = \bar{y}$ , i.e.,

$$\mathcal{S}(ar{oldsymbol{y}}) = \left\{ (oldsymbol{x}; oldsymbol{y}) \in \mathbb{R}^{n+m} \left| egin{array}{c} oldsymbol{y} = ar{oldsymbol{y}} \ ar{oldsymbol{y}}^{\intercal} A^k oldsymbol{x} + (oldsymbol{b}^k)^\intercal oldsymbol{x} \geq d_k - (oldsymbol{c}^k)^\intercal ar{oldsymbol{y}}, \ \ orall oldsymbol{x} \in K \ E oldsymbol{x} \geq oldsymbol{f} \ oldsymbol{x} \geq oldsymbol{0} \end{array} 
ight. 
ight.$$

Since y is integral in  $\Delta_m$ , we can write  $S = \bigcup_{j=0}^m S(e^j)$ , where  $e^j$  for  $j \in M$  is the j-th unit vector in  $\mathbb{R}^m$ , and  $e^0$  is the origin in  $\mathbb{R}^m$ . We impose the following assumption for the remainder of this section.

#### **Assumption 1.** We have that

- (i)  $S(e^j) \neq \emptyset$  for all  $j \in M \cup \{0\}$ ,
- (ii) All sets  $S(e^j)$ , for  $j \in M \cup \{0\}$ , share the same recession cone.

Since S is represented as a disjunctive union of  $S(e^j)$  for  $j \in M \cup \{0\}$ , Assumption 1 implies that conv(S) is a polyhedron. We refer to the procedure for finding the convex hull of S in the space of variables x as bilinear lift-and-project (BL&P). We aim to obtain valid inequalities for

 $\operatorname{proj}_x \operatorname{conv}(\mathcal{S})$  through a weighted aggregation of the constraints in the description of  $\mathcal{S}$  according to the following procedure.

- (A1) We select  $\hat{k} \in K$  to be the index of a bilinear constraint in K used in the aggregation. We also select  $\hat{j} \in M \cup \{0\}$  to indicate that the previously selected constraint  $\hat{k}$  has a weight of  $y_{\hat{j}}$  if  $\hat{j} \in M$ , or  $1 \mathbf{1}^{\intercal} \mathbf{y}$  if  $\hat{j} = 0$  in the aggregation. In the resulting weighted inequality, we replace  $y_i^2$  with  $y_i$  for each  $i \in M$ , and we replace  $y_i y_l$  with 0 for each  $i, l \in M$  and  $i \neq l$ . We refer to this weighted constraint as the base constraint.
- (A2) We select  $K^j$  for  $j \in M \cup \{0\}$  to be subsets of K. For each  $j \in M$  (resp. j = 0) and for each  $k \in K^j$  such that  $(k, j) \neq (\hat{k}, \hat{j})$ , we multiply the bilinear constraint  $\mathbf{y}^{\mathsf{T}} A^k \mathbf{x} + (\mathbf{b}^k)^{\mathsf{T}} \mathbf{x} + (\mathbf{c}^k)^{\mathsf{T}} \mathbf{y} \geq d_k$  with  $\alpha_k^j y_j$  where  $\alpha_k^j \geq 0$  (resp. with  $\alpha_k^0 (1 \mathbf{1}^{\mathsf{T}} \mathbf{y})$  where  $\alpha_k^0 \geq 0$ ). In the resulting weighted inequality, we replace  $y_i^2$  with  $y_i$  for each  $i \in M$ , and we replace  $y_i y_l$  with 0 for each  $i, l \in M$  and  $i \neq l$ .
- (A3) We select  $\mathcal{T}^j$  for  $j \in M \cup \{0\}$  to be subsets of T whose intersection is empty. For each  $j \in M$  (resp. j = 0) and for each  $t \in \mathcal{T}^j$ , we multiply the constraint  $E_t \cdot x \geq f_t$  with  $\beta_t^j y_j$  where  $\beta_t^j \geq 0$  (resp. with  $\beta_t^0 (1 \mathbf{1}^{\mathsf{T}} y)$  where  $\beta_t^0 \geq 0$ ).

The above sets are compactly represented as  $[\mathcal{K}^0, \mathcal{K}^1, \dots, \mathcal{K}^m | \mathcal{T}^0, \mathcal{T}^1, \dots, \mathcal{T}^m]$ , which is called a  $BL\mathcal{E}P$  assignment. Each BL&P assignment is identified by the pair  $(\hat{k}, \hat{j})$  which represents the base constraint selected in step (A1) of the procedure. We next aggregate all aforementioned weighted constraints. In the resulting aggregated inequality, the remaining bilinear terms and y-variables are replaced as follows:

- (R1) For each  $i \in N$ , let  $p_i$  be the most positive coefficient (i.e., the largest coefficient that is also positive) among all remaining terms  $x_i y_j$  for  $j \in M$ , and let  $q_i$  be the number of these terms with that coefficient. Set  $p_i = q_i = 0$  if such coefficient does not exist. Then, replace the entire remaining terms  $x_i y_j$  for  $j \in M$  with a single term  $p_i x_i$ .
- (R2) Let  $p_0$  be the most positive coefficient (i.e., the largest coefficient that is also positive) among all remaining variables  $y_j$  for  $j \in M$ , and let  $q_0$  be the number of these variables with that coefficient. Set  $p_0 = q_0 = 0$  if such coefficient does not exist. Then, replace the entire remaining variables  $y_j$  for  $j \in M$  with the single constant  $p_0$ .

During this procedure, we require that weights  $\alpha$  and  $\beta$  be chosen such that:

(C1) the coefficients of at least  $\sum_{j=0}^{m} (|\mathcal{K}^{j}| + |\mathcal{T}^{j}|) + \sum_{i=0}^{n} (\mathbb{I}(q_{i}) - q_{i})$  bilinear terms, together with those of the y-variables, become zero during the aggregation—excluding the replacements made in (R1) and (R2).

The resulting linear inequality after performing the above steps is referred to as a class- $(\hat{k}, \hat{j})$   $BL \mathcal{C}P$  inequality.

Claim 1. The BLEP inequalities are valid for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ , and the collection of all BLEP inequalities is sufficient to describe  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ .

Our goal in this section is to prove this claim, which is provided in Theorem 2. We begin with finding the convex hull description of S using disjunctive programming [4].

**Proposition 6.** Define  $Q = \{(x, y, u^0, u^1, ..., u^m) | (14a) - (14h) \}$ , where

$$(\boldsymbol{b}^k)^{\intercal} \left( \boldsymbol{x} - \sum_{j=1}^m \boldsymbol{u}^j \right) \ge d_k (1 - \mathbf{1}^{\intercal} \boldsymbol{y})$$
  $\forall k \in K$  (14a)

$$\left(A_{j.}^{k} + (\boldsymbol{b}^{k})^{\mathsf{T}}\right)\boldsymbol{u}^{j} \ge \left(d_{k} - \boldsymbol{c}_{j}^{k}\right)y_{j} \qquad \forall k \in K, \ j \in M$$
(14b)

$$E\left(\boldsymbol{x} - \sum_{j=1}^{m} \boldsymbol{u}^{j}\right) \ge \boldsymbol{f}(1 - \mathbf{1}^{\mathsf{T}}\boldsymbol{y}) \tag{14c}$$

$$E\boldsymbol{u}^{j} \ge \boldsymbol{f}y_{j} \tag{14d}$$

$$x - \sum_{i=1}^{m} u^j \ge 0 \tag{14e}$$

$$\mathbf{u}^j \ge \mathbf{0} \tag{14f}$$

$$1 - \mathbf{1}^{\mathsf{T}} \boldsymbol{y} \ge 0 \tag{14g}$$

$$y \ge 0 \tag{14h}$$

Then,  $\operatorname{proj}_{\boldsymbol{x}} \operatorname{conv}(\mathcal{S}) = \operatorname{proj}_{\boldsymbol{x}}(\mathcal{Q}).$ 

*Proof.* The disjunctive programming formulation of  $S = \bigcup_{j=0}^{m} S(e^{j})$  using  $u^{j} \in \mathbb{R}^{n}$  and  $v^{j} \in \mathbb{R}^{m}$  as replicas of x and y in disjunct  $j \in M \cup \{0\}$  with convex multiplier  $\xi_{j}$  is as follows:

$$\mathbf{v}^j = \mathbf{e}^j \xi_j \qquad \forall j \in M \cup \{0\}$$

$$(\mathbf{e}^{j})^{\mathsf{T}} A^{k} \mathbf{u}^{j} + (\mathbf{b}^{k})^{\mathsf{T}} \mathbf{u}^{j} \ge \left( d_{k} - (\mathbf{c}^{k})^{\mathsf{T}} \mathbf{e}^{j} \right) \xi_{j} \qquad \forall k \in K, j \in M \cup \{0\}$$

$$(15b)$$

$$Eu^{j} \ge f\xi_{j} \qquad \forall j \in M \cup \{0\}$$
 (15c)

$$\mathbf{u}^j \ge \mathbf{0} \tag{15d}$$

$$x = \sum_{j=0}^{m} u^j \tag{15e}$$

$$y = \sum_{j=0}^{m} v^j \tag{15f}$$

$$\sum_{j=0}^{m} \xi_j = 1 \tag{15g}$$

$$\xi \ge 0.$$
 (15h)

Let  $\mathcal{P}$  be the feasible region described by the above constraints. Under Assumption 1, we have  $\operatorname{conv}(\mathcal{S}) = \operatorname{proj}_{(\boldsymbol{x},\boldsymbol{y})}(\mathcal{P})$ ; see [9]. Using equations (15a), (15e) – (15g), we may substitute  $\xi_j$  with  $y_j$  for  $j \in M$ , substitute  $\xi_0$  with  $1 - \mathbf{1}^{\mathsf{T}}\boldsymbol{y}$ , and replace  $\boldsymbol{u}^0$  with  $\boldsymbol{x} - \sum_{j=1}^m \boldsymbol{u}^j$  in the rest of the equations to obtain formulation (14a) – (14h).

Following the interpretation of the new variables added in the disjunctive programming formulation, we may view variables  $u_i^j$  in (14a)–(14h) as  $x_i y_j$  for each  $i \in N$  and  $j \in M$ . To obtain the projection of (14a)–(14h) in the space of  $\boldsymbol{x}$  variables, we define the vector of dual multipliers  $\boldsymbol{\pi} = (\boldsymbol{\alpha}^0, \dots, \boldsymbol{\alpha}^m; \boldsymbol{\beta}^0, \dots, \boldsymbol{\beta}^m; \boldsymbol{\gamma}^0, \dots, \boldsymbol{\gamma}^m; \theta_0, \dots, \theta_m)$ , where  $\boldsymbol{\alpha}^0 \in \mathbb{R}_+^{\kappa}$  (resp.  $\boldsymbol{\alpha}^j \in \mathbb{R}_+^{\kappa}$ ) is the vector of dual multipliers associated with constraints (14a) (resp. (14b)),  $\boldsymbol{\beta}^0 \in \mathbb{R}_+^{\kappa}$  (resp.  $\boldsymbol{\beta}^j \in \mathbb{R}_+^{\kappa}$ ) includes the dual multipliers associated with (14c) (resp. (14d)),  $\boldsymbol{\gamma}^0 \in \mathbb{R}_+^{\kappa}$  (resp.  $\boldsymbol{\gamma}^j \in \mathbb{R}_+^{\kappa}$ ) contains the dual multipliers corresponding to (14e) (resp. (14f)), and  $\boldsymbol{\theta}_0 \in \mathbb{R}_+$  (resp.  $\boldsymbol{\theta}_j \in \mathbb{R}_+$ ) is the dual multiplier for (14g) (resp. (14h)).

**Proposition 7.** Define  $C = \left\{ \pi \in \mathbb{R}_+^{(m+1)(\kappa+\tau+n+1)} \, \middle| \, (16a), (16b) \right\}$ , where

$$\sum_{k=1}^{\kappa} \left( (A_{j,i}^{k} + b_{i}^{k}) \alpha_{k}^{j} - b_{i}^{k} \alpha_{k}^{0} \right) + \sum_{t=1}^{\tau} \left( E_{t,i} \beta_{t}^{j} - E_{t,i} \beta_{t}^{0} \right) + \gamma_{i}^{j} - \gamma_{i}^{0} = 0 \qquad \forall i \in \mathbb{N}, \ j \in \mathbb{M}$$
 (16a)

$$\sum_{k=1}^{\kappa} \left( (c_k^j - d_k) \alpha_k^j + d_k \alpha_k^0 \right) + \sum_{t=1}^{\tau} \left( -f_t \beta_t^j + f_t \beta_t^0 \right) + \theta_j - \theta_0 = 0.$$
  $\forall j \in M$  (16b)

For any point  $\bar{\pi} \in \mathcal{C}$ , the inequality

$$\left(\sum_{k=1}^{\kappa} \bar{\alpha}_k^0(\boldsymbol{b}^k) + E^{\mathsf{T}} \bar{\boldsymbol{\beta}}^0 + \bar{\boldsymbol{\gamma}}^0\right)^{\mathsf{T}} \boldsymbol{x} \ge \sum_{k=1}^{\kappa} \bar{\alpha}_k^0 d_k + (\bar{\boldsymbol{\beta}}^0)^{\mathsf{T}} \boldsymbol{f} - \bar{\theta}_0$$
(17)

is valid for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ . Further, (17) is facet-defining for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$  only if  $\bar{\boldsymbol{\pi}}$  is an extreme ray of  $\mathcal{C}$ .

Proof. It follows from Proposition 6 that  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S}) = \operatorname{proj}_{\boldsymbol{x}}(\mathcal{Q})$ , where  $\mathcal{Q}$  is the set defined in that proposition. Therefore, we use polyhedral projection to project out variables  $\boldsymbol{u}$  and  $\boldsymbol{y}$  in the description of  $\mathcal{Q}$ . Considering the non-negative dual weights  $\boldsymbol{\pi}$  as defined previously, we obtain the projection constraint (16a) for each variable  $u_i^j$  for  $i \in N$  and  $j \in M$ . Similarly, for each variable  $y_j$  for  $j \in M$ , the projection constraint is (16b). These constraints form the projection cone  $\mathcal{C}$  defined in the proposition statement. As a result, the weighted combination of the constraints in  $\mathcal{Q}$  with dual weights in  $\mathcal{C}$  yields the valid inequality (17) for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ . It is well-known that a facet-defining inequality of the form (17) corresponds to an extreme ray of the projection cone  $\mathcal{C}$ ; see [9].

We refer to valid inequalities of  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$  that are implied by the constraints describing  $\Xi$  as  $\operatorname{vertical}$  inequalities. As a result, we are interested in obtaining non-vertical valid inequalities for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ .

**Proposition 8.** Let  $\bar{\pi} \in \mathcal{C}$  be the dual weights that yield a non-vertical valid inequality of the form (17) for  $\operatorname{proj}_{x} \operatorname{conv}(\mathcal{S})$ . Then,  $\bar{\alpha}_{k}^{j} > 0$  for some  $j \in M \cup \{0\}$  and  $k \in K$ .

Proof. Assume by contradiction that there exists a non-vertical valid inequality of the form (17) for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$  obtained from the dual weights  $\bar{\boldsymbol{\pi}}\in\mathcal{C}$  such that  $\bar{\alpha}_k^j=0$  for all  $j\in M\cup\{0\}$  and  $k\in K$ . Then, the resulting inequality reduces to  $(E^{\mathsf{T}}\bar{\boldsymbol{\beta}}^0+\bar{\boldsymbol{\gamma}}^0)^{\mathsf{T}}\boldsymbol{x}\geq(\bar{\boldsymbol{\beta}}^0)^{\mathsf{T}}\boldsymbol{f}$ . This inequality can be obtained from combining inequalities  $E\boldsymbol{x}\geq\boldsymbol{f}$  with weights  $\bar{\boldsymbol{\beta}}^0$  and inequalities  $\boldsymbol{x}\geq\boldsymbol{0}$  with weights  $\bar{\boldsymbol{\gamma}}^0$ . Since all these inequalities are in the description of  $\Xi$ , we conclude that  $(E^{\mathsf{T}}\bar{\boldsymbol{\beta}}^0+\bar{\boldsymbol{\gamma}}^0)^{\mathsf{T}}\boldsymbol{x}\geq(\bar{\boldsymbol{\beta}}^0)^{\mathsf{T}}\boldsymbol{f}$  is implied by the constraints in  $\Xi$ , making it a vertical inequality by definition. This is a contradiction to the initial assumption imposed above.

It follows from Propositions (7) and 8 that each non-vertical facet-defining inequality in  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$  is of the form (17) for some extreme ray  $\bar{\boldsymbol{\pi}} \in \mathcal{C}$ , where  $\bar{\alpha}_{\hat{k}}^{\hat{j}} > 0$  for some  $\hat{j} \in M \cup \{0\}$  and  $\hat{k} \in K$ . Since  $\bar{\boldsymbol{\pi}}$  is a ray, we may scale it by fixing  $\bar{\alpha}_{\hat{k}}^{\hat{j}} = 1$ . As a result, we can view  $\bar{\boldsymbol{\pi}}$  as an extreme point of the set  $\mathcal{C}_{(\hat{k},\hat{j})} = \left\{ \boldsymbol{\pi} \in \mathbb{R}_{+}^{(m+1)(\kappa+\tau+n+1)} \,\middle|\, \alpha_{\hat{k}}^{\hat{j}} = 1, \, (16a), (16b) \right\}$ .

Using the results of Propositions 7 and 8 together with the definition of  $C_{(\hat{k},\hat{j})}$ , we directly obtain the following result.

**Proposition 9.** The collection of inequalities (17) obtained from the dual weights  $\bar{\pi}$  corresponding to the extreme points of  $C_{(\hat{k},\hat{j})}$  for all  $\hat{k} \in K$  and  $\hat{j} \in M \cup \{0\}$  includes all non-vertical facet-defining inequalities in the description of  $\operatorname{proj}_{x} \operatorname{conv}(\mathcal{S})$ .

Conversely, any inequality (17) obtained from the dual weight  $\bar{\pi}$  corresponding to a feasible solution of  $C_{(\hat{k},\hat{j})}$  for some  $\hat{k} \in K$  and  $\hat{j} \in M \cup \{0\}$  is valid for  $\operatorname{proj}_{\boldsymbol{x}} \operatorname{conv}(\mathcal{S})$ .

Next, we identify structural properties of the extreme points of  $C_{(\hat{k},\hat{j})}$  that can facilitate the derivation of useful valid inequalities for  $\operatorname{proj}_{x}\operatorname{conv}(\mathcal{S})$ .

**Proposition 10.** Let  $\bar{\pi}$  be an extreme point of  $C_{(\hat{k},\hat{j})}$  for some  $\hat{k} \in K$  and  $\hat{j} \in M \cup \{0\}$ . Then there exist  $K^j \subseteq K$  and  $T^j \subseteq T$  for all  $j \in M \cup \{0\}$ , and  $U_1 \subseteq N \times M$  and  $U_2 \subseteq M$  such that:

- (i) for each  $j \in M \cup \{0\}$ ,  $\bar{\alpha}_k^j > 0$  for  $k \in \mathcal{K}^j$ , and  $\bar{\alpha}_k^j = 0$  otherwise, except for  $\bar{\alpha}_{\hat{k}}^{\hat{j}}$  which is 1.
- (ii)  $\hat{k} \notin \mathcal{K}^{\hat{j}}$ ,
- (iii) for each  $j \in M \cup \{0\}$ ,  $\bar{\beta}_t^j > 0$  for  $t \in \mathcal{T}^j$ , and  $\bar{\beta}_t^j = 0$  otherwise,
- (iv)  $\bigcap_{j=0}^m \mathcal{T}^j = \emptyset$ ,
- (v)  $\Phi_{i,j} = 0$  for all  $(i,j) \in U_1$ , where

$$\Phi_{i,j} = \sum_{k \in K: k = \hat{k}, j = \hat{j}} (A_{j,i}^{k} + b_{i}^{k}) - \sum_{k \in K: k = \hat{k}, \hat{j} = 0} b_{i}^{k} + \sum_{k \in \mathcal{K}^{j}} (A_{j,i}^{k} + b_{i}^{k}) \bar{\alpha}_{k}^{j} - \sum_{k \in \mathcal{K}^{0}} b_{i}^{k} \bar{\alpha}_{k}^{0} + \sum_{t \in \mathcal{T}^{j}} E_{t,i} \bar{\beta}_{t}^{j} - \sum_{t \in \mathcal{T}^{0}} E_{t,i} \bar{\beta}_{t}^{0}$$

(vi)  $\Psi_j = 0$  for all  $j \in U_2$ , where

$$\Psi_j = \sum_{k \in K: k = \hat{k}, j = \hat{j}} (c_k^j - d_k) + \sum_{k \in K: k = \hat{k}, \hat{j} = 0} d_k + \sum_{k \in \mathcal{K}^j} (c_k^j - d_k) \bar{\alpha}_k^j + \sum_{k \in \mathcal{K}^0} d_k \bar{\alpha}_k^0 - \sum_{t \in \mathcal{T}^j} f_t \bar{\beta}_t^j + \sum_{t \in \mathcal{T}^0} f_t \bar{\beta}_t^0$$

- (vii) for each  $i \in N$ ,  $\bar{\gamma}_i^0 = \max_{j \in M} \{(\Phi_{i,j})^+\}$ ,
- (viii) for each  $i \in N$  and  $j \in M$ ,  $\bar{\gamma}_i^j = \bar{\gamma}_i^0 \Phi_{i,j}$ ,
- (ix)  $\bar{\theta}_0 = \max_{j \in M} \{ (\Psi_j)^+ \},$
- (x) for each  $j \in M$ ,  $\bar{\theta}_j = \bar{\theta}_0 \Psi_j$ ,
- (xi) the valid inequality of  $\operatorname{proj}_{x} \operatorname{conv}(\mathcal{S})$  produced by  $\bar{\pi}$  is of the following form

$$\sum_{i=1}^{n} \left( \sum_{k \in K: k = \hat{k}, \hat{j} = 0} b_i^k + \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 b_i^k + \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 E_{t,i} + \bar{\gamma}_i^0 \right) x_i \ge \sum_{k \in K: k = \hat{k}, \hat{j} = 0} d_k + \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 d_k + \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 f_t - \bar{\theta}_0$$

$$(xii) |U_1| + |U_2| \ge \sum_{j=0}^m |\mathcal{K}^j| + |\mathcal{T}^j| + \sum_{i=1}^n \mathbb{I}(\bar{\gamma}_i^0) - \sum_{i=1}^n \sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0) (1 - \mathbb{I}(\bar{\gamma}_i^j)) + \mathbb{I}(\bar{\theta}_0) - \sum_{j=1}^m \mathbb{I}(\bar{\theta}_0) (1 - \mathbb{I}(\bar{\theta}_0))$$

Conversely, assume that  $\bar{\pi} \in \mathbb{R}^{(m+1)(\kappa+\tau+n+1)}$  satisfies conditions (i)-(xii) above for for some  $\hat{k} \in K$ ,  $\hat{j} \in M \cup \{0\}$ ,  $K^j \subseteq K$ ,  $T^j \subseteq T$  for all  $j \in M \cup \{0\}$ ,  $U_1 \subseteq N \times M$  and  $U_2 \subseteq M$ . Then,  $\bar{\pi} \in \mathcal{C}_{(\hat{k},\hat{j})}$ .

*Proof.* The coefficient matrix for the system of equations describing  $C_{(\hat{k},\hat{j})}$  corresponding to the basic solution  $\bar{\pi}$  after appropriate rearrangement of rows and columns would be as follows:

1	0	0	0	0	0	0	0		
$X_{(2,1)}$	$X_{(2,2)}$	$X_{(2,3)}$	$X_{(2,4)}$	0	0	0	$X_{(2,8)}$	(10)	
$X_{(3,1)}$	$X_{(3,2)}$	$X_{(3,3)}$	0	$X_{(3,5)}$	0	0	$X_{(3,8)}$		(19)
$X_{(4,1)}$	$X_{(4,2)}$	$X_{(4,3)}$	$X_{(4,4)}$	0	I	0	$X_{(4,8)}$	,	(18)
$X_{(5,1)}$	$X_{(5,2)}$	$X_{(5,3)}$	0	$X_{(5,5)}$	0	I	$X_{(5,8)}$		
0	0	0	0	0	0	0	$X_{(6,8)}$		

where  $X_{(p,q)}$  is the submatrix of appropriate dimension in row block p and column block q, and 0 and I are the zero and identity submatrices of appropriate dimension.

In the matrix (18), the first column corresponds to variable  $\bar{\alpha}_{\hat{k}}^j$ . The second and third column blocks are respectively associated with variables  $\bar{\alpha}_k^j$  for  $(k,j) \in K \times M \cup \{0\} \setminus \left\{(\hat{k},\hat{j})\right\}$  and  $\bar{\beta}_t^j$  for  $(t,j) \in T \times M \cup \{0\}$  that have positive values in the basic solution. Similarly, the fourth and fifth column blocks are respectively associated with variables  $\bar{\gamma}_i^0$  for  $i \in N$  and  $\bar{\theta}_0$  that have positive values in the basic solution. The next two column blocks respectively correspond to variables  $\bar{\gamma}_i^j$  for  $(i,j) \in N \times M$  and  $\bar{\theta}_j$  for  $j \in M$  with positive values in the basic solution. The last column block represents the remaining basic variables that appear with value zero in the basic solution.

In the basis matrix (18), the first row shows the coefficients of the first constraint in the description of  $C_{(\hat{k},\hat{j})}$ , i.e.,  $\bar{\alpha}_{\hat{k}}^{\hat{j}} = 1$ . The second row block corresponds to constraints (16a) for all  $(i,j) \in N \times M$  that have a non-zero coefficient for at least one positive variable among  $\bar{\alpha}_k^0$ ,  $\bar{\alpha}_k^j$  for  $k \in K$ ,  $\bar{\beta}_t^0$ ,  $\bar{\beta}_t^j$  for  $t \in T$ , and  $\bar{\gamma}_i^0$ . This definition implies that for each row in the second row block, there is a non-zero element in the same row of at least one of the submatrices  $X_{(2,2)}$ ,  $X_{(2,3)}$ , and  $X_{(2,4)}$ . Similarly, the third row block corresponds to constraints (16b) for all  $j \in M$  that have a non-zero coefficient for at least one positive variable among  $\bar{\alpha}_k^0$ ,  $\bar{\alpha}_k^j$  for  $k \in K$ ,  $\bar{\beta}_t^0$ ,  $\bar{\beta}_t^j$  for  $t \in T$ , and  $\bar{\theta}_0$ . This definition implies that for each row in the third row block, there is a non-zero element in the same row of the at least one of the submatrices  $X_{(3,2)}$ ,  $X_{(3,3)}$ , and  $X_{(3,5)}$ . The fourth row block contains the coefficients of constraints (16a) for all  $(i,j) \in N \times M$  such that  $\bar{\gamma}_i^j > 0$ . As a result, the submatrix representing the coefficients of  $\bar{\gamma}_i^j$  in these constraints forms an identity matrix as shown in (18). Similarly, the fifth row block contains the coefficients of constraints (16b) for all

 $j \in M$  such that  $\bar{\theta}_j > 0$ . As a result, the submatrix representing the coefficients of  $\bar{\theta}_j$  in these constraints forms an identity matrix as shown in (18). The last row block corresponds to all the remaining constraints in (16a) and (16b) that are not included in the previous row blocks.

To prove part (i), for each  $j \in M \cup \{0\}$ , define  $\mathcal{K}^j$  to include constraints  $k \in K$  whose aggregation weights  $\bar{\alpha}_k^j$  correspond to a column in the second column block. By definition, these variables are positive, and the remaining variables that are not associated with columns of the second column block are zero. Because of the structure of the matrix (18), the column corresponding to  $\bar{\alpha}_{\hat{k}}^j$  is not among those in the second column block, because it is represented in the first column. As a result,  $\hat{k} \notin \mathcal{K}^{\hat{j}}$ , which verifies part (ii). To prove part (iii), for each  $j \in M \cup \{0\}$ , define  $\mathcal{T}^j$  to include constraints  $t \in T$  whose aggregation weights  $\bar{\beta}_t^j$  correspond to a column in the third column block. By definition, these variables are positive, and the remaining variables that are not associated with columns of the third column block are zero. To show part (iv), assume by contradiction that there exists  $t^* \in \bigcap_{j=0}^m \mathcal{T}^j$ . Therefore, the columns corresponding to the coefficients of variables  $\bar{\beta}_{t^*}^j$  for all  $j \in M \cup \{0\}$  are included in the third column block of (18). It follows from the description of the constraints (16a) and (16b) in  $\mathcal{C}_{(\hat{k},\hat{j})}$  that these columns linearly dependent as their summation yields the zero vector. This is a contradiction to the fact that (18) is a basis matrix.

For part (v), we define  $U_1$  to be the set of  $(i,j) \in N \times M$  corresponding to constraints (16a) in rows of the second row block of (18) such that  $\bar{\gamma}_i^0 = 0$ . As a result, (16a) implies that  $\sum_{k=1}^{\kappa} \left( (A_{j,i}^k + b_i^k) \bar{\alpha}_k^j - b_i^k \bar{\alpha}_k^0 \right) + \sum_{t=1}^{\tau} \left( E_{t,i} \bar{\beta}_t^j - E_{t,i} \bar{\beta}_t^0 \right) = 0$  for all  $(i,j) \in U_1$ . Using the results of parts (i)–(iii), we may rewrite the left-hand-side of this equation as  $\sum_{k \in K: k=\hat{k}, j=\hat{j}} (A_{j,i}^k + b_i^k) - \sum_{k \in K: k=\hat{k}, \hat{j}=0} b_i^k + \sum_{k \in \mathcal{K}^j} (A_{j,i}^k + b_i^k) \bar{\alpha}_k^j - \sum_{k \in \mathcal{K}^0} b_i^k \bar{\alpha}_k^0 + \sum_{t \in \mathcal{T}^j} E_{t,i} \bar{\beta}_t^j - \sum_{t \in \mathcal{T}^0} E_{t,i} \bar{\beta}_0^1$ , which is equal to  $\Phi_{i,j}$  defined in part (v). Similarly for part (vi), we define  $U_2$  to be the set of  $j \in M$  corresponding to constraints (16b) in rows of the third row block of (18) if  $\bar{\theta}_0 = 0$ , otherwise  $U_2 = \emptyset$ . As a result, (16b) implies that  $\sum_{k=1}^{\kappa} \left( (c_k^j - d_k) \bar{\alpha}_k^j + d_k \bar{\alpha}_k^0 \right) + \sum_{t=1}^{\tau} \left( -f_t \bar{\beta}_t^j + f_t \bar{\beta}_t^0 \right) = 0$  for all  $j \in U_2$ . Using the results of parts (i)–(iii), we may rewrite the left-hand-side of this equation as  $\sum_{k \in K: k=\hat{k}, j=\hat{j}} (c_k^j - d_k) + \sum_{k \in K: k=\hat{k}, \hat{j}=0} d_k + \sum_{k \in \mathcal{K}} (c_k^j - d_k) \bar{\alpha}_k^j + \sum_{k \in \mathcal{K}^0} d_k \bar{\alpha}_k^0 - \sum_{t \in \mathcal{T}^j} f_t \bar{\beta}_t^j + \sum_{t \in \mathcal{T}^0} f_t \bar{\beta}_t^0$ , which is equal to  $\Psi_j$  defined in part (vi).

For part (vii), consider  $i \in N$ . On the one hand, it follows from (16a) that the vectors containing the coefficient of  $\gamma_i^j$  for all  $j \in M \cup \{0\}$  are linearly dependent as their summation yields the zero vector. As a result, all these vectors cannot be included in the basis matrix (18). Therefore,  $\bar{\gamma}_i^j = 0$  for some  $j \in M \cup \{0\}$ . On the other hand, the constraints (16a) for fixed  $i \in N$  imply that  $\bar{\gamma}_i^0 \geq \sum_{k=1}^{\kappa} (A_{j,i}^k + b_i^k) \bar{\alpha}_k^j - b_i^k \bar{\alpha}_k^0 + \sum_{t=1}^{\tau} E_{t,i} \bar{\beta}_j^j - E_{t,i} \bar{\beta}_t^0$  for each  $j \in M$  because  $\bar{\gamma}_i^j \geq 0$  by definition of  $\mathcal{C}_{(\hat{k},\hat{j})}$ . Using the results of parts (i)–(iii), we may rewrite the right-hand-side of this inequality as  $\Phi_{i,j}$  given previously. Also, since  $\bar{\pi}$  is a feasible solution of  $\mathcal{C}_{(\hat{k},\hat{j})}$ , we have that  $\bar{\gamma}_i^0 \geq 0$ . Thus, we can combine the above inequalities as  $\bar{\gamma}_i^0 \geq \max_{j \in M} \{(\Phi_{i,j})^+\}$ . Because of the previous argument, this inequality must be satisfied at equality since otherwise  $\bar{\gamma}_i^j$  would be strictly positive for all  $j \in M \cup \{0\}$ . Once  $\bar{\gamma}_i^0$  is calculated, we may use the equations in (16a) to calculate  $\bar{\gamma}_i^j = \bar{\gamma}_i^0 - \sum_{k=1}^{\kappa} \left( (A_{j,i}^k + b_i^k) \bar{\alpha}_k^j - b_i^k \bar{\alpha}_k^0 \right) - \sum_{t=1}^{\tau} \left( E_{t,i} \bar{\beta}_t^j - E_{t,i} \bar{\beta}_t^0 \right) = \bar{\gamma}_i^0 - \Phi_{i,j}$  for all  $j \in M$ , verifying the equation in part (viii). The proof of parts (ix) and (x) follow from similar arguments to those

of parts (vii) and (viii) when applied to (16b) and variables  $\theta_j$  for  $j \in M \cup \{0\}$ . The inequality form in part (xi) is obtained by using the results of parts (i)–(iii) to properly separate the terms in the inequality (17).

For part (xii), since the submatrices in position (4,6) and (5,7) of matrix (18) are identity matrices, we can use basic column operations to make all elements of the submatrices  $X_{(4,2)}$ ,  $X_{(4,3)}$ ,  $X_{(4,4)}$ ,  $X_{(5,2)}$ ,  $X_{(5,3)}$ , and  $X_{(5,5)}$  zero. As a result, the nonzero elements of column blocks 2–5 are in rows included in the row blocks 2 and 3, forming the following submatrix:

$$\begin{bmatrix}
X_{(2,2)} & X_{(2,3)} & X_{(2,4)} & 0 \\
X_{(3,2)} & X_{(3,3)} & 0 & X_{(3,5)}
\end{bmatrix}.$$
(19)

Since the column vectors in this submatrix must be linearly independent due to being included in the basis matrix, the number of its columns cannot exceed the number of its rows. We calculate the number of columns as follows. By definition of sets  $\mathcal{K}^j$ , the number of columns in the first column block of (19) is equal to  $\sum_{j=0}^{m} |\mathcal{K}^{j}|$ . Similarly, the definition of  $\mathcal{T}^{j}$  implies that the number of columns in the second column block of (19) is equal to  $\sum_{j=0}^{m} |\mathcal{T}^{j}|$ . It follows from the definition of the columns included in the third column block that the number of these columns is equal to the number of variables  $\bar{\gamma}_i^0$  for  $i \in N$  that are strictly positive, which can be written as  $\sum_{i=1}^n \mathbb{I}(\bar{\gamma}_i^0)$ . Using a similar argument, we can calculate the number of columns in the fourth column block as  $\mathbb{I}(\bar{\theta}_0)$ . Next, we calculate the number of rows in (19). It follows from the structure of (18) that the rows associated with pair  $(i,j) \in N \times M$  in the second row block must have  $\bar{\gamma}_i^j = 0$ . By definition of  $U_1$  given above, these rows are partitioned into those with  $(i,j) \in U_1$  where  $\bar{\gamma}_i^0 = 0$ , and those with  $(i,j) \notin U_1$  where  $\bar{\gamma}_i^0 > 0$ . The latter partition can be described as  $\{(i,j) \in N \times M | \bar{\gamma}_i^j = 0, \; \bar{\gamma}_i^0 > 0 \}$ . We may calculate the cadinality of this set as  $\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{I}(\bar{\gamma}_{i}^{0})(1 - \mathbb{I}(\bar{\gamma}_{i}^{j}))$ . Therefore, the number of rows in the first row block of (19) is equal to  $|U_1| + \sum_{i=1}^n \sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0)(1 - \mathbb{I}(\bar{\gamma}_i^j))$ . Using a similar argument for the second row block of (19), we can calculate the number of its rows as  $|U_2| + \sum_{j=1}^m \mathbb{I}(\bar{\theta}_0)(1 - \mathbb{I}(\bar{\theta}_j))$ . Consequently, using the argument above, the number of rows in (19) must be no less than the number of its columns, i.e.,  $|U_1| + \sum_{i=1}^n \sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0)(1 - \mathbb{I}(\bar{\gamma}_i^j)) + |U_2| + \sum_{i=1}^n \sum_{j=1}^n \mathbb{I}(\bar{\gamma}_i^0)(1 - \mathbb{I}(\bar{\gamma}_i^0)) + |U_2| + \sum_{i=1}^n \sum_{j=1}^n \mathbb{I}(\bar{\gamma}_i^$  $\sum_{j=1}^{m} \mathbb{I}(\bar{\theta}_0)(1 - \mathbb{I}(\bar{\theta}_j)) \ge \sum_{j=0}^{m} |\mathcal{K}^j| + |\mathcal{T}^j| + \sum_{i=1}^{n} \mathbb{I}(\bar{\gamma}_i^0) + \mathbb{I}(\bar{\theta}_0), \text{ yielding the result in part (xi)}.$ 

Next, we prove the converse statement. It follows from conditions (i)–(iv), and (vii)–(x) that all components of  $\bar{\pi}$  are non-negative and  $\bar{\alpha}_{\hat{k}}^{\hat{j}} = 1$ . As established earlier in this proof, conditions (vii)–(x) together with the equations in (v) and (vi) imply that  $\bar{\pi}$  satisfies equations (16a) and (16b). We conclude that  $\bar{\pi} \in \mathcal{C}_{(\hat{k},\hat{j})}$ .

We are now ready to prove the main result of this section outlined in Claim 1. In particular, we show that the structural properties of the extreme points of  $\mathcal{C}_{(\hat{k},\hat{j})}$  obtained in Proposition 10 trans-

late into the steps of the BL&P procedure to derive the facet-defining inequalities for  $\operatorname{proj}_{x} \operatorname{conv}(\mathcal{S})$ . We further show that any BL&P inequality is valid for  $\operatorname{proj}_{x} \operatorname{conv}(\mathcal{S})$ . The combination of these statements lead to our main result, as presented next.

**Theorem 2.** The collection of all BL&P inequalities together with  $\Xi$  and  $\Delta_m$  describes  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ .

Proof. To prove the result, we first show that any non-vertical facet-defining inequality of  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$  can be obtained as a BL&P inequality. Let  $g(\boldsymbol{x}) \geq g_0$  be a non-vertical facet-defining inequality of  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ . It follows from Proposition 9 that  $g(\boldsymbol{x}) \geq g_0$  is of the form (17) for some extreme point  $\bar{\boldsymbol{\pi}}$  of  $\mathcal{C}_{(\hat{k},\hat{j})}$  for some  $\hat{k} \in K$  and  $\hat{j} \in M \cup \{0\}$ . Proposition 10 implies that there exist  $\mathcal{K}^j \subseteq K$  and  $\mathcal{T}^j \subseteq T$  for all  $j \in M \cup \{0\}$ , and  $U_1 \subseteq N \times M$  and  $U_2 \subseteq M$  that satisfy properties (i)–(xii). We use these sets and properties to define the proper parameters for the BL&P procedure that produces  $g(\boldsymbol{x}) \geq g_0$ . We begin with step (A1), where we choose  $\hat{k}$  to be the index of the bilinear constraint and  $\hat{j}$  to represent the weight of this constraint in the aggregation. The pair  $(\hat{k}, \hat{j})$  defines the class of the desired BL&P inequality. According to this step, the weighted constraint will be of the form

$$\sum_{i=1}^{n} b_i^{\hat{k}} x_i - \sum_{i=1}^{n} \sum_{j=1}^{m} b_i^{\hat{k}} x_i y_j \ge d_{\hat{k}} - \sum_{j=1}^{m} d_{\hat{k}} y_j, \qquad \text{if } \hat{j} = 0$$

$$\sum_{i=1}^{n} A_{\hat{j},i}^{\hat{k}} x_i y_{\hat{j}} + \sum_{i=1}^{n} b_i^{\hat{k}} x_i y_{\hat{j}} + c_{\hat{j}}^{\hat{k}} y_{\hat{j}} \ge d_{\hat{k}} y_{\hat{j}}, \qquad \text{if } \hat{j} \in M.$$

For step (A2), we choose sets  $\mathcal{K}^j$  for  $j \in M \cup \{0\}$  to be those defined in Proposition 10 with aggregation weights  $\bar{\alpha}_k^j$  for each  $k \in \mathcal{K}^j$ . These sets satisfy the conditions of (A2) because of properties (i) and (ii) of Proposition 10. This yields the weighted constraints of the form

$$\begin{split} &\sum_{i=1}^n \alpha_k^0 b_i^k x_i - \sum_{i=1}^n \sum_{j=1}^m \alpha_k^0 b_i^k x_i y_j \geq \alpha_k^0 d_k - \sum_{j=1}^m \alpha_k^0 d_k y_j, \\ &\sum_{i=1}^n \alpha_k^j A_{j,i}^k x_i y_j + \sum_{i=1}^n \alpha_k^j b_i^k x_i y_j + \alpha_k^j c_j^k y_j \geq \alpha_k^j d_k y_j, \end{split} \qquad \forall k \in \mathcal{K}^j, \forall j \in M. \end{split}$$

For step (A3), we choose sets  $\mathcal{T}^j$  for  $j \in M \cup \{0\}$  to be those defined in Proposition 10 with aggregation weights  $\bar{\beta}_t^j$  for each  $t \in \mathcal{T}^j$ . These sets satisfy the conditions of (A3) because of properties (iii) and (iv) of Proposition 10. This yields the weighted constraints of the form

$$\sum_{i=1}^{n} \beta_t^0 E_{t,i} x_i - \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_t^0 E_{t,i} x_i y_j \ge \beta_t^0 f_t - \sum_{j=1}^{m} \beta_t^0 f_t y_j, \qquad \forall t \in \mathcal{T}^0$$

$$\sum_{i=1}^{n} \beta_t^0 E_{t,i} x_i y_j \ge \beta_t^0 f_t y_j, \qquad \forall t \in \mathcal{T}^j, \forall j \in M.$$

According to the BL&P procedure, we aggregate the above inequalities to obtain

$$\sum_{i \in N} \sum_{j \in M} \left( -\sum_{k \in K: k = \hat{k}, \hat{j} = 0} b_i^k + \sum_{k \in K: k = \hat{k}, j = \hat{j}} (A_{j,i}^k + b_i^k) - \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 b_i^k + \sum_{k \in \mathcal{K}^j} \bar{\alpha}_k^j (A_{j,i}^k + b_i^k) \right) \\
- \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 E_{t,i} + \sum_{t \in \mathcal{T}^j} \bar{\beta}_t^j E_{t,i} x_i y_j \\
+ \sum_{j \in M} \left( \sum_{k \in K: k = \hat{k}, \hat{j} = 0} d_k + \sum_{k \in K: k = \hat{k}, j = \hat{j}} (c_k^j - d_k) + \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 d_k + \sum_{k \in \mathcal{K}^j} \bar{\alpha}_k^j (c_k^j - d_k) \right) \\
+ \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 f_t - \sum_{t \in \mathcal{T}^j} \bar{\beta}_t^j f_t y_j \\
+ \sum_{i \in N} \left( \sum_{k \in K: k = \hat{k}, \hat{j} = 0} b_i^k + \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 b_i^k + \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 E_{t,i} \right) x_i \ge \sum_{k \in K: k = \hat{k}, \hat{j} = 0} d_k + \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 d_k + \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 f_t. \tag{20}$$

It follows from property (v) of Proposition 10 that the coefficient of  $x_i y_j$  in the above aggregated inequality is equal to  $\Phi_{i,j}$  defined therein. Among these terms, those with  $(i,j) \in U_1$  will have coefficient zero according to the BL&P procedure. The remaining bilinear terms  $x_i y_j$  with  $(i,j) \notin U_1$  are replaced following the step (R1) in the BL&P procedure. In particular, for each  $i \in N$ , we calculate the most positive coefficient of the remaining terms  $x_i y_j$  as  $p_i = \max_{j \in M: (i,j) \notin U_1} \{\Phi_{i,j}\}$ . If  $p_i < 0$ , we set it equal to zero. Since  $\Phi_{i,j} = 0$  for all  $(i,j) \in U_1$ , we may rewrite this equation as  $p_i = \max_{j \in M} \{(\Phi_{i,j})^+\}$ . This value is equal to  $\bar{\gamma}_i^0$  according to the property (vii) of Proposition 10. Following the step (R1) of the BL&P procedure, all these remaining bilinear terms are replaced with  $p_i x_i = \bar{\gamma}_i^0$ .

Similarly, the coefficient of  $y_j$  in the aggregated inequality (20) is equal to  $\Psi_j$  according to the property (vi) of Proposition 10. The coefficient of variables with indices  $j \in U_2$  become zero. The remaining variables  $y_j$  with  $j \notin U_2$  are replaced following the step (R2) in the BL&P procedure. In particular, we calculate the most positive coefficient of the remaining variables  $y_j$  as  $p_0 = \max_{j \notin U_2} \{\Psi_j\}$ . If  $p_0 < 0$ , we set it equal to zero. Since  $\Psi_j = 0$  for all  $j \in U_2$ , we may rewrite this equation as  $p_0 = \max_{j \in M} \{(\Psi_j)^+\}$ . This value is equal to  $\bar{\theta}_0$  according to the property (viii) of Proposition 10. Following the step (R2) of the BL&P procedure, all these remaining variables are replaced with  $p_0 = \bar{\theta}_0$ .

Considering the above simplifications, the aggregated inequality (20) reduces to

$$\sum_{i \in N} \left( \sum_{k \in K: k = \hat{k}, \hat{j} = 0} b_i^k + \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 b_i^k + \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 E_{t,i} + \bar{\gamma}_i^0 \right) x_i \ge \sum_{k \in K: k = \hat{k}, \hat{j} = 0} d_k + \sum_{k \in \mathcal{K}^0} \bar{\alpha}_k^0 d_k + \sum_{t \in \mathcal{T}^0} \bar{\beta}_t^0 f_t - \bar{\theta}_0,$$

which is the inequality  $g(x) \geq g_0$  produced by the extreme point  $\bar{\pi}$  of  $C_{(\hat{k},\hat{j})}$  according to the property (xi) of Proposition 10.

We next show that the condition (C1) of the BL&P procedure is satisfied for the derived inequality. It follows from the previous arguments that the coefficients of bilinear terms  $x_i y_i$  for  $(i,j) \in U_1$  and the variables  $y_j$  for  $j \in U_2$  become zero during the aggregation process. According to the condition (C1),  $|U_1| + |U_2|$  must be no less than  $\sum_{j=0}^{m} (|\mathcal{K}^j| + |\mathcal{T}^j|) + \sum_{i=0}^{n} (\mathbb{I}(q_i) - q_i)$ , where  $q_i$  for  $i \in N$  (resp.  $q_0$ ) is the number of the remaining bilinear terms  $x_i y_j$  (resp. variables  $y_j$ ) with coefficient  $p_i$  (resp.  $p_0$ ) in the aggregated inequality (20). On the other hand, according to the property (xii) of Proposition 10,  $|U_1| + |U_2| \ge \sum_{j=0}^m |\mathcal{K}^j| + |\mathcal{T}^j| + \sum_{i=1}^n \mathbb{I}(\bar{\gamma}_i^0) - \sum_{i=1}^n \sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0) (1 - \sum_{i=1}^n \sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0)) = \sum_{j=1}^n \mathbb{I}(\bar{\gamma}_i^0) = \sum_{j=1}^n \mathbb{I}(\bar{\gamma}_i$  $\mathbb{I}(\bar{\gamma}_i^j)) + \mathbb{I}(\bar{\theta}_0) - \sum_{j=1}^m \mathbb{I}(\bar{\theta}_0)(1 - \mathbb{I}(\bar{\theta}_j)).$  Therefore, it suffices to show that (I)  $\mathbb{I}(\bar{\gamma}_i^0) = \mathbb{I}(q_i)$  for  $i \in N$ , (II)  $\mathbb{I}(\bar{\theta}_0) = q_0$ , (III)  $\sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0)(1 - \mathbb{I}(\bar{\gamma}_i^j)) = q_i$  for  $i \in N$ , and (IV)  $\sum_{j=1}^m \mathbb{I}(\bar{\theta}_0)(1 - \mathbb{I}(\bar{\theta}_j)) = q_0$ . To prove (I), we have  $q_i > 0$  if and only if  $p_i > 0$  by definition. It was shown in the previous sections of this proof that  $p_i = \bar{\gamma}_i^0$ , which implies that  $\mathbb{I}(\bar{\gamma}_i^0) = \mathbb{I}(q_i)$ . The proof of (II) follows from a similar argument. To prove (III), for  $i \in N$ , it is easy to verify that  $\sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0)(1 - \mathbb{I}(\bar{\gamma}_i^j))$  counts the number of  $j \in M$  such that  $\mathbb{I}(\bar{\gamma}_i^0) = 1$  and  $\mathbb{I}(\bar{\gamma}_i^j) = 0$ . Equivalently, we count the number of  $j \in M$  such that  $\bar{\gamma}_i^0 > 0$  and  $\bar{\gamma}_i^j = 0$ . It follows from the property (vii) of Proposition 10 that  $\bar{\gamma}_i^j = \bar{\gamma}_i^0 - \Phi_{i,j}$ . Therefore,  $\bar{\gamma}_i^j = 0$  implies that  $\bar{\gamma}_i^0 = \Phi_{i,j}$ . We previously established that  $\Phi_{i,j}$  is the coefficient of the bilinear term  $x_i y_j$  in the aggregated inequality (20). As a result,  $\bar{\gamma}_i^j = 0$  implies that the coefficient of the bilinear term  $x_i y_j$  in the aggregated inequality (20) is equal to  $\bar{\gamma}_i^0$ , which is equal to  $p_i$  according to the above discussion. In summary,  $\sum_{j=1}^m \mathbb{I}(\bar{\gamma}_i^0)(1-\mathbb{I}(\bar{\gamma}_i^j))$  is equal to the number of the remaining bilinear term  $x_i y_j$  in the aggregated inequality (20) with coefficients equal to  $p_i$ , which is  $q_i$  according to the condition (C1) of the BL&P procedure. The proof of (IV) follows from a similar argument.

To complete the proof, it remains to show that any BL&P inequality is valid for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ . The above arguments in this proof have established that the steps of the BL&P procedure correspond to the conditions (i)–(xii) of Proposition 10 for a point  $\bar{\pi}$ . It follows from the converse statement in Proposition 10 that any point  $\bar{\pi}$  that satisfies conditions (i)–(xii) is feasible to  $\mathcal{C}_{(\hat{k},\hat{j})}$  for some  $\hat{k} \in K$  and  $\hat{j} \in M \cup \{0\}$ . The converse statement in Proposition 9 implies that any inequality obtained from a feasible solution to  $\mathcal{C}_{(\hat{k},\hat{j})}$  is valid for  $\operatorname{proj}_{\boldsymbol{x}}\operatorname{conv}(\mathcal{S})$ , proving the result.

In view of Theorem 2, it follows from its proof and the connections to Proposition 9 that condition (C1) in the BL&P procedure is a necessary requirement for the resulting inequality to be facet-defining for conv(S). However, this condition is not required for the inequality to be valid. In other words, a linear inequality produced by the BL&P procedure that does not satisfy (C1) remains valid for conv(S), but it will not be facet-defining.

We conclude this section by providing several remarks that help adjust and simply the steps of the BL&P procedure for special structures that commonly appear in various application domains. The first remark shows that for sparse structures of the bilinear set  $\mathcal{S}$ , the BL&P procedure can be substantially simplified. These structures appear in numerous applications, including in CCP (see Section 4).

Remark 1. Consider a bilinear constraint  $k^* \in K$  in the description of S that contains a single y variable, and is of the form  $(\mathbf{a}^{\intercal}\mathbf{x})y_{j^*} + c_{j^*}y_{j^*} \geq 0$  for some  $j^* \in M$ . Following step (A1) of the

BL&P procedure, if this inequality is used for the base constraint, i.e.,  $\hat{k} = k^*$ , the only choice for  $\hat{j}$  will be  $\hat{j} = j^*$ . This leads to the weighted inequality  $(\boldsymbol{a}^{\intercal}\boldsymbol{x})y_{j^*} + c_{j^*}y_{j^*} \geq 0$ , which means using the same bilinear constraint with constant aggregation weight 1 as the base constraint. This is due to the fact that using  $\hat{j} \neq j^*$  would lead to zero terms on the left-hand-side of the aggregated inequality. Similarly, following step (A2) of the BL&P procedure, if this inequality is used in the aggregation, the only choice will be to select it in the set  $\mathcal{K}^{j^*}$ . This yields the weighted inequality  $\alpha_{k^*}^{j^*}(\boldsymbol{a}^{\intercal}\boldsymbol{x})y_{j^*} + \alpha_{k^*}^{j^*}c_{j^*}y_{j^*} \geq 0$ , which is using the same bilinear constraint with a constant aggregation weight  $\alpha_{k^*}^{j^*}$ . We can use similar arguments for the case where a bilinear constraint  $k^* \in K$  in the description of S is of the form  $(\boldsymbol{a}^{\mathsf{T}}\boldsymbol{x})(\mathbf{1}-\mathbf{1}^{\mathsf{T}}\boldsymbol{y})+c_0(\mathbf{1}-\mathbf{1}^{\mathsf{T}}\boldsymbol{y})\geq 0$ . Following step (A1) of the BL&P procedure, if this inequality is selected for the base constraint, i.e.,  $\hat{k} = k^*$ , the only choice for  $\hat{j}$ will be  $\hat{j} = 0$ . This leads to using this inequality in the aggregation with a constant weight of 1 as the base constraint. Similarly, following step (A2) of the BL&P procedure, if this inequality is used in the aggregation, the only choice will be to select it in the set  $\mathcal{K}^0$ . This yields using this inequality with a constant aggregation weight  $\alpha_{k^*}^0$ . These simplifications can significantly reduce the number of possible options for BL&P assignments, which could in turn lead to more efficient cut-generation procedures.

In many applications, some of the variables x in the set  $\Xi$  may have an upper bound. Although this bound can be treated as a regular constraint in step (A3) of the BL&P framework, the procedure can be further streamlined to reduce the size of the BL&P assignment, as detailed in the following remark.

Remark 2. Assume that  $\Xi$  contains the constraints of the form  $-x_{i^*} \geq -1$ , representing scaled upper bounds for certain variables  $x_{i^*}$  for  $i^* \in I^*$ , where  $I^* \subseteq N$ . In step (A3) of the BL&P procedure, these constraints can be excluded from the assignment sets  $\mathcal{T}^j$  for  $j \in M$ . Instead, we add the following step (R0) before the steps (R1) and (R2).

(R0) For each  $i^* \in I^*$ , select a subset  $\mathcal{U}^{i^*}$  of the remaining bilinear terms  $u_{i^*j}x_{i^*}y_j$  with positive coefficient  $u_{i^*j} > 0$  for  $j \in M$  in the aggregated inequality, and then replace them with  $u_{i^*j}y_j$ . Similarly, select a subset  $\mathcal{V}^{i^*}$  of the remaining terms  $v_jy_j$  with negative coefficient  $v_j < 0$  for  $j \in M$  in the aggregated inequality, and then replace them with  $v_jx_{i^*}y_j$ .

The next remarks discuss a simplifying step in the BL&P process when S contains bilinear complementarity constraints.

Remark 3. Assume that S contains the bilinear constraint  $-x_{i^*}y_{j^*} \geq 0$  for some  $i^* \in N$  and  $j^* \in M$ . This constraint is imposed when there is a complementarity relationship between  $x_{i^*}$  and  $y_{j^*}$ , since it will imply that  $x_{i^*}y_{j^*} = 0$  as  $x_{i^*} \geq 0$  and  $y_{j^*} \geq 0$ . Within the BL&P procedure, this bilinear constraint can be omitted from S and excluded from the aggregation process. Instead, in the aggregated inequality, we can substitute the remaining term of the form  $ux_{i^*}y_{j^*}$  for any  $u \in \mathbb{R}$  with 0 prior to applying step (R1).

Remark 4. Assume that  $\Xi$  contains the constraints of the form  $-x_{i^*} \geq -1$ , for some  $i^* \in N$ . In addition, assume that S contains the bilinear constraint  $-(1-x_{i^*})y_{j^*} \geq 0$  for some  $i^* \in N$  and

 $j^* \in M$ . This constraint is imposed when there is a complementarity relationship between the complement of  $x_{i^*} \in [0,1]$  and  $y_{j^*}$ , since it will imply that  $(1-x_{i^*})y_{j^*}=0$  as  $x_{i^*} \leq 1$  and  $y_{j^*} \geq 0$ . In the BL&P procedure, we can omit this bilinear constraint from  $\mathcal{S}$ , and thereby exclude it from the aggregation process. Instead, in the aggregated inequality, we can perform one of the following two substitution options after step (R0) discussed in Remark 2 and before step (R1) in the BL&P procedure: (i) we may replace any remaining term of the form  $u_{i^*,j^*}x_{i^*}y_{j^*}$  with  $u_{i^*,j^*}y_{j^*}$  when  $u_{i^*,j^*} < 0$ ; (ii) we may replace any remaining term of the form  $u_{i^*,j^*}y_{j^*}$  with  $u_{i^*,j^*}x_{i^*}y_{j^*}$  when  $u_{i^*,j^*} > 0$ .

## 4 Convexification of CCP through Bilinear Extended Reformulation

In this section, we introduce a novel technique for deriving valid inequalities for the convex hull of the mixing set with a knapsack constraint,  $\mathcal{F}_c$ , as defined in (3). Our approach begins by reformulating  $\mathcal{F}_c$  as a bilinear set over a simplex in a higher-dimensional space. We then apply the BL&P procedure to this bilinear set to generate families of valid inequalities for conv( $\mathcal{F}_c$ ), covering both the general case with an arbitrary probability distribution and the special case with a uniform probability distribution.

#### 4.1 Bilinear Reformulation

Consider the bilinear set

$$S_c = \{(z, \boldsymbol{x}; \boldsymbol{y}) \in \Xi_c \times \Delta_m | (22a) - (22e) \},$$
(21)

where

$$(1 - x_i)(1 - \mathbf{1}^{\mathsf{T}} \mathbf{y}) \ge 0, \qquad i \in M \tag{22a}$$

$$-(1-x_i)(1-\mathbf{1}^{\mathsf{T}}\boldsymbol{y}) \ge 0, \qquad i \in M$$
 (22b)

$$zy_i - h_i y_i \ge 0, i \in M (22c)$$

$$-x_i y_i \ge 0, i \in M (22d)$$

$$-(1 - x_i)y_j \ge 0,$$
  $i, j \in M : i < j,$  (22e)

and where  $M = \{1, \dots, m\}$ ,  $\Xi_c = \{(z, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}_+^m \mid -\boldsymbol{x} \ge -1, -\boldsymbol{\pi}^{\intercal} \boldsymbol{x} \ge -\varepsilon\}$ , and  $\Delta_m = \{\boldsymbol{y} \in \mathbb{Z}_+^m \mid \mathbf{1}^{\intercal} \boldsymbol{y} \le 1\}$ .

The bilinear set  $S_c$  conforms to the structure of the general set S introduced in (13), where the variables  $(z, \boldsymbol{x})$  in  $S_c$  correspond to the  $\boldsymbol{x}$  variables in S. As a result, we can apply the BL&P procedure to obtain valid inequalities for the projection of the convex hull of  $S_c$  onto the  $(z, \boldsymbol{x})$ space. To this end, we fist need to show that Assumption 1 is satisfied.

#### **Lemma 1.** Assumption 1 holds for $S_c$ .

Proof. Let  $S_c(e^j)$  be the restriction of  $S_c$  at point  $\mathbf{y} = e^j$  for  $j \in M \cup \{0\}$ . For part (i) of Assumption 1, we show that  $S_c(e^j) \neq 0$  for all  $j \in M \cup \{0\}$ . Consider j = 0. Then, the point  $(\bar{z}, \bar{x}; \bar{y}) = (0, \mathbf{1}; \mathbf{0})$  is feasible to  $S_c(e^0)$ . Consider  $j \in M$ . Then, the point  $(\bar{z}, \bar{x}; \bar{y}) = (h_j, \mathbf{1} - e^j; e^j)$  is feasible to  $S_c(e^j)$ . For part (ii) of Assumption 1, it is easy to verify that all sets  $S_c(e^j)$  for  $j \in M \cup \{0\}$  share the same recession cone described as  $\{(z, x; y) \mid z \geq 0, x = 0, y = 0\}$ .

The next proposition shows the relationship between the convex hulls of  $\mathcal{F}_c$  and  $\mathcal{S}_c$ .

## **Proposition 11.** It holds that $\mathcal{F}_c \subseteq \operatorname{proj}_{(z,x)} \mathcal{S}_c$ .

Proof. Consider a point  $(\bar{z}, \bar{x}) \in \mathcal{F}_c$ . We show that there exists  $\bar{y}$  such that  $(\bar{z}, \bar{x}; \bar{y}) \in \mathcal{S}_c$ . There are two cases. For the first case, assume that  $\bar{x}_i = 1$  for all  $i \in M$ . As a result, (3) implies that  $\bar{z} \geq 0$ . Let  $\bar{y}_j = 0$  for all  $j \in M$ . It is easy to verify that the point  $(\bar{z}, \bar{x}; \bar{y})$  satisfies all constraints in  $\mathcal{S}_c$ , and thus  $(\bar{z}, \bar{x}; \bar{y}) \in \mathcal{S}_c$ . For the second case, assume that  $\bar{x}_i = 0$  for some  $i \in M$ . Let  $i^* \in M$  be the smallest index of the variables with  $\bar{x}_i = 0$ . As a result, (3) implies that  $\bar{z} \geq h_{i^*}$ . Let  $\bar{y}_{i^*} = 1$  and  $\bar{y}_j = 0$  for  $j \in M \setminus \{i^*\}$ . It is easy to verify that the point  $(\bar{z}, \bar{x}; \bar{y})$  satisfies all constraints in  $\mathcal{S}_c$ , and thus  $(\bar{z}, \bar{x}; \bar{y}) \in \mathcal{S}$ .

Lemma 1 and Proposition 11 imply that the BL&P method can be applied to the bilinear set  $\mathcal{S}_c$  to obtain valid inequalities for the convex hull of the original set  $\mathcal{F}_c$ . Although the results presented in this section apply to the general form of  $\mathcal{F}_c$ —where the knapsack constraint involves an arbitrary probability distribution  $\pi$ —many existing studies on deriving explicit valid inequalities focus on the special case of a uniform distribution, as defined by  $\mathcal{F}_c^-$  in (4); see, for instance, [18, 24]. In this special case, where the chance constraint reduces to a cardinality constraint, the relationship between  $\mathcal{F}_c$  and  $\mathcal{S}_c$ , as described in Proposition 11, becomes stronger. In fact, we can show that under this setting, both sets share the same convex hull in the (z, x)-space.

## **Proposition 12.** It holds that $\operatorname{conv}(\mathcal{F}_c^{-}) = \operatorname{proj}_{(z,x)} \operatorname{conv}(\mathcal{S}_c^{-})$ .

Proof. Since  $\mathcal{F}_c^=$  and  $\mathcal{S}_c^=$  are special case  $\mathcal{F}_c$  and  $\mathcal{S}_c$ , where  $\pi_i = 1/m$  for all  $i \in M$  and  $p = \lfloor m\varepsilon \rfloor$ . Therefore, it follows from Proposition 11 that  $\mathcal{F}_c^= \subseteq \operatorname{proj}_{(z,x)} \mathcal{S}_c^=$ . Taking the convex hull from both sides and using the commutative property of the convex hull and projection operators, we obtain  $\operatorname{conv}(\mathcal{F}_c^=) \subseteq \operatorname{proj}_{(z,x)} \operatorname{conv}(\mathcal{S}_c^=)$ .

Next, we show the reverse inclusion  $\operatorname{conv}(\mathcal{F}_c^=) \supseteq \operatorname{proj}_{(z,\boldsymbol{x})} \operatorname{conv}(\mathcal{S}_c^=)$ . Consider a point  $(\bar{z},\bar{\boldsymbol{x}}) \in \operatorname{proj}_{(z,\boldsymbol{x})} \operatorname{conv}(\mathcal{S}_c^=)$ . There exists  $\bar{\boldsymbol{y}} \in \mathbb{R}^m$  such that  $(\bar{z},\bar{\boldsymbol{x}};\bar{\boldsymbol{y}}) \in \operatorname{conv}(\mathcal{S}_c^=)$ . As a result, there exists a collection of points  $(\hat{z}^k,\hat{\boldsymbol{x}}^k;\hat{\boldsymbol{y}}^k) \in \mathcal{S}_c^=$  for some  $k \in \{1,\ldots,\kappa\}$  such that  $(\bar{z},\bar{\boldsymbol{x}};\bar{\boldsymbol{y}}) = \sum_{k=1}^{\kappa} \lambda_k (\hat{z}^k,\hat{\boldsymbol{x}}^k;\hat{\boldsymbol{y}}^k)$  with  $\sum_{k=1}^{\kappa} \lambda_k = 1$  and  $\lambda_k \geq 0$  for all  $k \in \{1,\ldots,\kappa\}$ . We next show that  $(\hat{z}^k,\hat{\boldsymbol{x}}^k) \in \operatorname{conv}(\mathcal{F}_c^=)$  for all  $k \in \{1,\ldots,\kappa\}$ . Consider  $(\hat{z}^k,\hat{\boldsymbol{x}}^k;\hat{\boldsymbol{y}}^k)$  for some  $k \in \{1,\ldots,\kappa\}$ . There are two cases. For the first case, assume that  $\hat{y}^k_j = 0$  for all  $j \in M$ . It follows from (22a) and (22b) that  $\hat{x}^k_i = 1$  for all  $i \in M$ . Further, (22c) implies that  $\hat{z}^k \geq 0$ . Hence,  $(\hat{z}^k,\hat{\boldsymbol{x}}^k) \in \mathcal{F}_c^- \subseteq \operatorname{conv}(\mathcal{F}_c^-)$ . For the second case, assume that  $\hat{y}^k_{j^*} = 1$  for some  $j^* \in M$  and  $\hat{y}^k_j = 0$  all  $j \in M \setminus \{j^*\}$ . It

follows from (22c) that  $\hat{z}^k \geq h_{j^*}$ . Furthermore, (22d) implies that  $\hat{x}^k_{j^*} = 0$ , while (22e) implies that  $\hat{x}^k_i = 1$  for all  $i < j^*$ . Since the set  $\Xi_c$  in the description of  $\mathcal{S}^{=}_c$  is a unit cube intersected with a cardinality constraint, all of its extreme points are integral. Therefore, there exists a collection of integral points  $\tilde{x}^l$  for  $l \in \{1, \dots, \ell\}$  of  $\Xi_c$  such that  $\hat{x}^k = \sum_{l=1}^{\ell} \mu_l \tilde{x}^l$  with  $\sum_{l=1}^{\ell} \mu_l = 1$  and  $\mu_l > 0$  for all  $l \in \{1, \dots, \ell\}$ . According to the previous argument, since  $\hat{x}^k_i \in \{0, 1\}$  for all  $i \leq j^*$ , we must have  $\tilde{x}^l_i = \hat{x}^k_i$  for all  $i \leq j^*$  and all  $l \in \{1, \dots, \ell\}$ . In particular, for each  $l \in \{1, \dots, \ell\}$ , we have  $\tilde{x}^l_{j^*} = 0$  and  $\tilde{x}^l_i = 1$  for  $i < j^*$ , which implies that the point  $(\hat{z}^k, \hat{x}^l)$  is feasible to  $\mathcal{F}^{=}_c$ . Hence, we can write that  $(\hat{z}^k, \hat{x}^k) = \sum_{l=1}^{\ell} \mu_l(\hat{z}^k, \hat{x}^l)$ , which shows that  $(\hat{z}^k, \hat{x}^k) \in \text{conv}(\mathcal{F}^{=}_c)$ . In sum, we have shown that  $(\bar{z}, \bar{x})$  is represented as a convex combination of points  $(\hat{z}^k, \hat{x}^k) \in \text{conv}(\mathcal{F}^{=}_c)$  for all  $k \in \{1, \dots, \kappa\}$ , proving that  $(\bar{z}, \bar{x}) \in \text{conv}(\mathcal{F}^{=}_c)$ .

#### 4.2 Valid Inequalities for the Mixing Set with a Knapsack Constraint

In this section, we demonstrate how the BL&P procedure yields the explicit form of an important family of valid inequalities for  $conv(\mathcal{F}_c)$ , which we refer to as BL&P inequalities. This family subsumes several known results from the literature while also introducing a broad class of new inequalities with distinct structural forms that lie beyond the scope of existing results.

**Theorem 3.** Let  $r \in \{1, \ldots, p\}$ ,  $l \in \{1, \ldots, r\}$ , and define  $L := \{1, \ldots, l\}$ . Select  $\delta_{t_{\iota}} \in \mathbb{R}$  for  $\iota \in L$  such that  $\delta_{t_{\iota}} \geq h_{t_{\iota+1}} - h_{t_{\iota}}$  and  $\sum_{\iota \in L} \delta_{t_{\iota}} \leq h_{r+1}$ , where  $t_{l+1} := r+1$ . Consider  $v \in \{1, \ldots, p-r+l\}$ , and define  $V := \{1, \ldots, v\}$ . Let  $P := \{t_{\iota} : \iota \in L\} \subseteq \{1, \ldots, r\}$  and  $Q := \{q_{\iota} : \iota \in V\} \subseteq \{r+1, \ldots, m\}$ , where  $t_1 < \cdots < t_l$  and  $q_1 < \cdots < q_v$ , and define  $a_j := |\{\iota \in L : \iota_{\iota} < j\}|$ . Moreover, select  $A_j \subseteq V$  for  $j \in M$ .

Suppose that for some  $(\phi_{q_1}, \ldots, \phi_{q_v}) \in \mathbb{R}^v_+$  there exists  $\beta_j \in \mathbb{R}_+$  for each  $j \in M$  that satisfies

$$\max\left\{\frac{\phi_{q_{\iota}}}{m\pi_{q_{\iota}}}: \iota \in A_{j}, \ q_{\iota} > j\right\} \leq \beta_{j} \leq \min\left\{\frac{\phi_{q_{\iota}}}{m\pi_{q_{\iota}}}: \iota \in V \setminus A_{j}, \ q_{\iota} > j\right\},\tag{23a}$$

$$\beta_{j}\left(F_{j}-\pi_{j}-\varepsilon+\sum_{\substack{\iota\in V\\q_{\iota}>j}}\pi_{q_{\iota}}-\sum_{\substack{\iota\in A_{j}\\q_{\iota}>j}}\pi_{q_{\iota}}\right)\geq\left(h_{t_{a_{j}+1}}-h_{j}-\sum_{\substack{\iota\in L\\t_{\iota}< j}}\delta_{t_{\iota}}-\mathbb{I}(j\in Q)\phi_{j}-\sum_{\substack{\iota\in A_{j}\\q_{\iota}> j}}\phi_{q_{\iota}}\right)/m,\quad(23b)$$

where  $F_j := \sum_{i=1}^j \pi_j$ . Then, the BL&P inequality

$$z + \sum_{\iota=1}^{l} (h_{t_{\iota}} - h_{t_{\iota+1}} + \delta_{t_{\iota}}) x_{t_{\iota}} + \sum_{\iota=1}^{v} \phi_{q_{\iota}} (1 - x_{q_{\iota}}) \ge h_{t_{1}}, \tag{24}$$

is valid for  $\operatorname{proj}_{(z,x)}\operatorname{conv}(\mathcal{S}_c)$ ; hence, is valid for  $\operatorname{conv}(\mathcal{F}_c)$ .

Proof. To comply with the indexing rules of the BL&P procedure, we refer to the bilinear inequalities (22a)–(22e) as  $type \ k=1,\ldots,5$ , respectively. Since each bilinear inequality of type  $k=1,\ldots,4$  entails multiple constraints with indices  $j\in M$ , we distinguish each individual constraint by using the notation  $j^k$  in the BL&P procedure. We also refer to the index of the knapsack constraint  $-\pi^{\intercal}x \geq -\varepsilon$  in  $\Xi$  with c.

Suppose that for some  $(\phi_{q_1}, \ldots, \phi_{q_v}) \in \mathbb{R}^v_+$  and  $\beta_j \in \mathbb{R}_+$ , for each  $j \in M$ , (23) holds. We show that (24) is valid for  $\operatorname{proj}_{(z,x)} \operatorname{conv}(\mathcal{S}_c)$ , which implies its validity for  $\operatorname{conv}(\mathcal{F}_c)$  by Proposition 11. To show the former statement, according to Theorem 2, it suffices to prove that (24) can be obtained as a BL&P inequality. In particular, we show that this BL&P inequality is produced by a class- $(t_1^3, t_1)$  BL&P assignment, where  $t_1^3$  indicates the bilinear constraint of type 3 with constraint index  $t_1$  as defined in set P. The choice of  $t_1$  for the second component of the class follows from Remark 1 since the bilinear constraints in  $\mathcal{S}_c$  contain a single y variable. The aggregation sets for this BL&P assignment are selected as follows. Define  $\beta_0 = \frac{h_{r+1} - \sum_{i \in L} \delta_{t_i}}{m(1-\varepsilon)}$ . It is clear that  $\beta_0 \geq 0$  for any  $\varepsilon \in [0,1)$  as  $h_{r+1} \geq \sum_{i \in L} \delta_{t_i}$  by assumption. Let  $\mathcal{K}^0 = \{j^1 : j \in Q\} \cup \{j^2 : j \in M\}$ ,  $\mathcal{K}^j = \{j^3\}$  for  $j \in M \setminus \{t_1\}$ ,  $\mathcal{T}^0 = \{c\}$ , and  $\mathcal{T}^j = \{c\}$  for  $j \in M$ . All other sets  $\mathcal{K}^j$  and  $\mathcal{T}^j$  are empty.

Next, we assign aggregation weights for the constraints in the BL&P assignment as described in steps (A1)–(A3) of the BL&P procedure. Because of the special structure of the bilinear constraints in  $\mathcal{S}_c$ , determining the aggregation weights of these constraints can be substantially simplified as discussed in Remark 1. In particular, the base constraint has a constant weight of 1. The bilinear constraint  $t_\iota^2 \in \mathcal{K}^0$ , for  $\iota \in L$ , has a constant weight of  $(h_{t_\iota} - h_{t_{\iota+1}} + \delta_{t_\iota} + m\pi_{t_\iota}\beta_0)$ . This weight is non-negative because  $h_{t_\iota} - h_{t_{\iota+1}} + \delta_{t_\iota} \geq 0$  by assumption, and  $\beta_0 \geq 0$  as discussed above. The constraints  $q_\iota^1, q_\iota^2 \in \mathcal{K}^0$ , for  $\iota \in V$ , have constant weights of  $(\phi_{q_\iota} - m\pi_{q_\iota}\beta_0)^+$  and  $(m\pi_{q_\iota}\beta_0 - \phi_{q_\iota})^+$ , respectively. The bilinear constraint  $j^2 \in \mathcal{K}^0$ , for  $j \in M \setminus (P \cup Q)$ , has a constant weight of  $m\pi_j\beta_0$ . The bilinear constraint  $j^3 \in \mathcal{K}^j$ , for  $j \in M \setminus \{t_1\}$ , has a constant weight of 1. The knapsack constraint  $c \in \mathcal{T}^0$  has a weight of  $m\beta_0(1-\mathbf{1}^{\intercal}y)$ . Furthermore, the knapsack constraint  $c \in \mathcal{T}^j$ , for  $j \in M$ , has a weight of  $m\beta_j y_j$ . Note that all these weights are non-negative. All other weights are zero.

The next step in the BL&P procedure is to aggregate the above weighted constraints to obtain the aggregated inequality

$$\sum_{j \in M} z y_j + \sum_{\iota \in L} (h_{t_\iota} - h_{t_{\iota+1}} + \delta_{t_\iota}) x_{t_\iota} + \sum_{\iota \in V} \phi_{q_\iota} (1 - x_{q_\iota}) + \sum_{j \in M} (h_{t_1} - h_j - \sum_{\iota \in V} \phi_{q_\iota} + m\varepsilon\beta_j) y_j + \sum_{i \in M} \sum_{j \in M} \sigma_{ij} x_i y_j \ge h_{t_1},$$
(25)

where  $\sigma_{ij}$ , for  $i, j \in M$ , is defined as

$$\sigma_{ij} = \begin{cases} -h_{t_{\iota}} + h_{t_{\iota+1}} - \delta_{t_{\iota}} - m\pi_{t_{\iota}}\beta_{j}, & i = t_{\iota}, \ \iota \in L, \ j \in M, \\ \phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j}, & i = q_{\iota}, \ \iota \in V, \ j \in M, \\ -m\pi_{i}\beta_{j}, & i \in M \setminus (P \cup Q), \ j \in M. \end{cases}$$

$$(26)$$

As the next step in the BL&P procedure, we apply the rules (R1) and (R2) to substitute the bilinear terms and y variables in (25). Note that  $\sigma_{ij} \leq 0$  for  $i \in P$  (i.e.,  $i = t_{\iota}, \iota \in L$ ) and  $j \in M$  because  $\delta_{t_{\iota}} \geq -h_{t_{\iota}} + h_{t_{\iota+1}}$  and  $\beta_{j} \geq 0$  by definition. We also have  $\sigma_{ij} \leq 0$  for  $i \in M \setminus (P \cup Q)$  and  $j \in M$ . However, for  $i \in Q$  (i.e.,  $i = q_{\iota}, \iota \in V$ ) and  $j \in M$ ,  $\sigma_{ij}$  can be positive or non-positive. Using

the above observations about the sign of  $\sigma_{ij}$ , we can separate the terms in  $\sum_{j \in M} \sum_{i \in M} \sigma_{ij} x_i y_j$  in (25) as follows:

$$\sum_{j \in M} \sum_{\iota \in L} (-h_{t_{\iota}} + h_{t_{\iota+1}} - \delta_{t_{\iota}} - m\pi_{t_{\iota}}\beta_{j}) x_{t_{\iota}} y_{j} + \sum_{j \in M} \sum_{i \in M \setminus (P \cup Q)} (-m\pi_{i}\beta_{j}) x_{i} y_{j}$$

$$+ \sum_{j \in M} \sum_{\iota \in V} (\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j}) x_{q_{\iota}} y_{j}.$$

$$(27)$$

Since (22d) matches the constraint form described in Remark 3, we can apply the substitution rule in that remark to all  $x_iy_j$  terms with i=j to eliminate them. Similarly, since (22e) matches the constraint form described in Remark 4, we can apply the substitution rule (i) in that remark to all  $x_iy_j$  terms with i < j that have non-positive coefficients. This includes terms  $x_{t_i}y_j$  for  $j \in M$  and  $\iota \in L$  with  $t_\iota < j$  since their coefficient  $-h_{t_\iota} + h_{t_{\iota+1}} - \delta_{t_\iota} - m\pi_{t_\iota}\beta_j$  is non-positive as established above, as well as terms  $x_iy_j$  for  $j \in M$  and  $i \in M \setminus (P \cup Q)$  with i < j since their coefficient  $-m\pi_i\beta_j$  is non-positive. For term  $x_{q_\iota}y_j$  for  $j \in M$  and  $\iota \in V$  with  $q_\iota < j$ , there are two cases. If the coefficient  $\phi_{q_\iota} - m\pi_{q_\iota}\beta_j$  of this term is non-negative, then we may apply rule (R0) as described in Remark 2 to replace it with  $(\phi_{q_\iota} - m\pi_{q_\iota}\beta_j)y_j$ . If this coefficient is non-positive, we may apply the substitution rule (i) in Remark 4 to replace it with  $(\phi_{q_\iota} - m\pi_{q_\iota}\beta_j)y_j$ ; both cases leading to the same substitution term. Using these substitutions while further separating the terms in (27) based on the precedence of i and j indices in the bilinear terms, we obtain

$$\sum_{j \in M} \sum_{\iota \in L, t_{\iota} > j} (-h_{t_{\iota}} + h_{t_{\iota+1}} - \delta_{t_{\iota}} - m\pi_{t_{\iota}}\beta_{j}) x_{t_{\iota}} y_{j} + \sum_{j \in M} y_{j} \sum_{\iota \in L, t_{\iota} < j} (-h_{t_{\iota}} + h_{t_{\iota+1}} - \delta_{t_{\iota}} - m\pi_{t_{\iota}}\beta_{j}) 
+ \sum_{j \in M} \sum_{i \in M \setminus (P \cup Q), i > j} (-m\pi_{i}\beta_{j}) x_{i} y_{j} - \sum_{j \in M} y_{j} \sum_{i \in M \setminus (P \cup Q), i < j} m\pi_{i}\beta_{j} 
+ \sum_{j \in M} \sum_{\iota \in V \setminus A_{j}, q_{\iota} > j} (\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j}) x_{q_{\iota}} y_{j} + \sum_{j \in M} y_{j} \sum_{\iota \in V \setminus A_{j}, q_{\iota} < j} (\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j}) 
+ \sum_{j \in M} \sum_{\iota \in A_{j}, q_{\iota} > j} (\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j}) x_{q_{\iota}} y_{j} + \sum_{j \in M} y_{j} \sum_{\iota \in A_{j}, q_{\iota} < j} (\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j}).$$
(28)

Note that in (28), the coefficient of  $x_{q_{\iota}}y_{j}$  term, for  $j \in M$  and  $\iota \in V \setminus A_{j}$  with  $q_{\iota} > j$ , is non-negative by (23a). Therefore, we may apply rule (R0) as described in Remark 2 to replace this term with  $(\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j})y_{j}$ . Moreover, the coefficient of  $x_{q_{\iota}}y_{j}$  term, for  $j \in M$  and  $\iota \in A_{j}$  with  $q_{\iota} > j$ , is non-positive by (23a). All remaining product terms  $x_{i}y_{j}$  have a non-positive coefficient as discussed earlier. Thus, we can apply rule (R1) in the BL&P procedure to eliminate these terms with non-positive coefficients, which reduces (28) to

$$\sum_{j \in M} y_{j} \sum_{\iota \in L, t_{\iota} < j} (-h_{t_{\iota}} + h_{t_{\iota+1}} - \delta_{t_{\iota}} - m\pi_{t_{\iota}}\beta_{j}) - \sum_{j \in M} y_{j} \sum_{i \in M \setminus (P \cup Q), i < j} m\pi_{i}\beta_{j}$$

$$+ \sum_{j \in M} y_{j} \sum_{\iota \in V \setminus A_{j}, q_{\iota} > j} (\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j}) + \sum_{j \in M} y_{j} \sum_{\iota \in V \setminus A_{j}, q_{\iota} < j} (\phi_{q_{\iota}} - m\pi_{q_{\iota}}\beta_{j})$$

$$+\sum_{j\in M} y_j \sum_{\iota\in A_j, q_\iota < j} (\phi_{q_\iota} - m\pi_{q_\iota}\beta_j). \tag{29}$$

Combining all  $y_j$  terms, for  $j \in M$ , in (29) with the  $y_j$  terms on the left-hand-side of (25) yields

$$\begin{split} &\sum_{j\in M}\left(h_{t_1}-h_j-\sum_{\iota\in V}\phi_{q_\iota}+m\varepsilon\beta_j\right)y_j+\sum_{j\in M}y_j\sum_{\iota\in L,t_\iota< j}\left(-h_{t_\iota}+h_{t_{\iota+1}}-\delta_{t_\iota}-m\pi_{t_\iota}\beta_j\right)\\ &-\sum_{j\in M}y_j\sum_{\iota\in M\setminus (P\cup Q),i< j}m\pi_i\beta_j+\sum_{j\in M}y_j\sum_{\iota\in V\setminus A_j,q_\iota\neq j}\left(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j\right)\\ &+\sum_{j\in M}y_j\sum_{\iota\in A_j,q_\iota< j}\left(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j\right)\\ &=\sum_{j\in M}\left(h_{t_{a_j+1}}-h_j-\sum_{\iota\in L,t_\iota< j}\delta_{t_\iota}-\sum_{\iota\in V}\phi_{q_\iota}+m\left(\varepsilon-\sum_{\iota\in L,t_\iota< j}\pi_{t_\iota}-\sum_{i\in M\setminus (P\cup Q),i< j}\pi_i\right)\beta_j\right)y_j\\ &+\sum_{j\in M}y_j\left(\sum_{\iota\in V\setminus A_j,q_\iota\neq j}\left(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j\right)+\sum_{\iota\in A_j,q_\iota< j}\left(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j\right)\right)\\ &=\sum_{j\in M}\left(h_{t_{a_j+1}}-h_j-\sum_{\iota\in L,t_\iota< j}\delta_{t_\iota}-\sum_{\iota\in V}\phi_{q_\iota}+m\left(\varepsilon-F_j+\pi_j+\sum_{\iota\in V,q_\iota< j}\pi_{q_\iota}\right)\beta_j\right)y_j\\ &+\sum_{j\in M}y_j\left(\sum_{\iota\in V\setminus A_j,q_\iota\neq j}\left(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j\right)+\sum_{\iota\in A_j,q_\iota\neq j}\left(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j\right)-\sum_{\iota\in A_j,q_\iota> j}\left(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j\right)\right)\\ &=\sum_{j\in M}\left(h_{t_{a_j+1}}-h_j-\sum_{\iota\in L}\delta_{t_\iota}-\mathbb{I}(j\in Q)\phi_j+m\left(\varepsilon-F_j+\pi_j+\sum_{\iota\in V}\pi_{q_\iota}-\sum_{\iota\in V}\pi_{q_\iota}+\mathbb{I}(j\in Q)\pi_j\right)\beta_j\right)y_j\\ &-\sum_{j\in M}\sum_{\iota\in A_j}y_j(\phi_{q_\iota}-m\pi_{q_\iota}\beta_j), \end{split}$$

where the second equality follows from the definition of  $F_j$  and the fact that  $\{\iota \in A_j : q_\iota \neq j\} = \{\iota \in A_j : q_\iota < j\} \cup \{\iota \in A_j : q_\iota > j\}$ , while the third equality is due to  $\{\iota \in V \setminus A_j : q_\iota \neq j\} \cup \{\iota \in A_j : q_\iota \neq j\} = \{\iota \in V : q_\iota \neq j\}$  for  $j \in M$ , and the fact that  $\sum_{\iota \in V} \phi_{q_\iota} - \sum_{\iota \in V, q_\iota \neq j} \phi_{q_\iota} = \mathbb{I}(j \in Q)\phi_j$ . For each  $j \in M$ , the coefficient of  $y_j$  in (30) can be simplified to

$$m\beta_{j}\left(\varepsilon - F_{j} + \pi_{j} - \sum_{\substack{\iota \in V \\ q_{\iota} > j}} \pi_{q_{\iota}} + \sum_{\substack{\iota \in A_{j} \\ q_{\iota} > j}} \pi_{q_{\iota}}\right) - h_{j} + h_{t_{a_{j}+1}} - \sum_{\substack{\iota \in L \\ t_{\iota} < j}} \delta_{t_{\iota}} - \mathbb{I}(j \in Q)\phi_{j} - \sum_{\substack{\iota \in A_{j} \\ q_{\iota} > j}} \phi_{q_{\iota}}, \tag{31}$$

which is enforced to be non-positive by (23b). This allows us to apply rule (R2) in the BL&P procedure to eliminate these terms from the left-hand-side of (25). Finally, we apply rule (R1) to  $\sum_{j\in M} zy_j$  on the left-hand-side of (25) to replace it with z. As a result, this inequality reduces to (24). The remark following Theorem 2 implies that (24) is valid for  $conv(S_c)$ .

The next result provides a necessary condition for the BL&P inequalities derived in Theorem 3 to be facet-defining for the convex hull of the underlying bilinear set.

**Proposition 13.** The BL&P inequality (24) is facet-defining for  $\operatorname{proj}_{(z,x)} \operatorname{conv}(S_c)$  only if at least 2m+1 of the inequalities in (23a) and (23b) for  $j \in M$ , and  $\pi_i\beta_j \geq 0$  for  $j \in M$ ,  $i \in M \setminus (P \cup Q)$ , i > j, hold at equality.

*Proof.* The remark following Theorem 2 implies that, if the BL&P inequality (24) is facet-defining for  $\operatorname{proj}_{(z,\boldsymbol{x})}\operatorname{conv}(\mathcal{S}_c)$ , then it satisfies condition (C1). In the BL&P procedure outlined in the proof of Theorem 3, the total number of constraints in the sets  $\mathcal{K}^j$  and  $\mathcal{T}^j$  for  $j \in M$  that participate in the aggregation with nonzero weights is at most 3m, since by definition, at most one of the constraints  $q_{\iota}^1$  and  $q_{\iota}^2$  in  $\mathcal{K}^0$  can have a nonzero weight for each  $\iota \in V$ . According to the aggregation procedure described in the proof of Theorem 3, rule (R1) is applied to the terms with positive coefficients, represented by  $\sum_{i \in M} zy_i$ , where the maximum coefficient equals one and is achieved by m terms. Thus, it follows from the condition (C1) of the BL&P procedure that the number of bilinear and y terms whose coefficients become zero must be at least 3m + (1-m) = 2m + 1. In particular, 2m+1 of the coefficients of the bilinear terms in (28), together with the coefficients of the y-variables in (31), must become zero. Note that for the coefficient of  $x_t, y_j$  with  $j \in M$  and  $\iota \in L$ , where  $t_{\iota} > j$ , to vanish in (28), it must hold that  $m\pi_{t_{\iota}}\beta_{j} = -h_{t_{\iota}} + h_{t_{\iota+1}} - \delta_{t_{\iota}}$ . Since the left-hand side of this equality is non-negative and the right-hand side is non-positive by assumption, we must have  $\beta_j = 0$  and  $\delta_{t_i} = -h_{t_i} + h_{t_{i+1}}$ . This implies that the coefficient of  $x_{t_i}$  becomes zero in (24), meaning that index  $t_{\iota}$  can be removed from set P, which in turn corresponds to a different parameter configuration for obtaining the inequality. Consequently, we may exclude the coefficients of  $x_{t_{\iota}}y_{j}$  for  $j \in M$  and  $\iota \in L$  with  $t_{\iota} > j$  from the set of coefficients selected to be zero. Thus, the remaining coefficients to be zero correspond to the conditions represented by constraints (23a), (23b) for all  $j \in M$ , and  $\pi_i \beta_j \geq 0$  for all  $j \in M$  and  $i \in M \setminus (P \cup Q)$  with i > j, among which at least 2m + 1 must hold at equality. 

The family of BL&P inequalities described in Theorem 3 subsumes several existing results in the literature, including those in Propositions 1–4, as well as Proposition 5 in the case of a uniform probability distribution, as will be demonstrated in Section 4.3. At the same time, this family encompasses a substantially broader variety of valid inequalities for  $conv(\mathcal{F}_c)$  than previously available results, as evidenced by the computational experiments presented in Section 5. To further illustrate, we next present an example of facet-defining BL&P inequalities derived from Theorem 3 that cannot be obtained using Proposition 5, which provides the most general form of known inequalities for  $\mathcal{F}_c$  in the literature, as discussed in Section 2.

Example 2. Consider the instance of  $\mathcal{F}_c^=$  given in Example 1, where  $m=10, \{h_1, \ldots, h_m\} = \{20, 18, 14, 11, 6, 5, 4, 3, 2, 1\}$ , and  $p=\vartheta=4$ . We established in Example 1 that the inequality (12) cannot be represented by the family of valid inequalities in Proposition 5. In contrast, we show here that this inequality belongs to the family of BL&P inequalities (24) derived in Theorem 3. For this problem instance, we have r=4,  $P=\{1,4\}$ , and  $Q=\{5,6\}$ . Following the settings in that theorem, we may choose  $\delta_1=\delta_4=-3$ , and select  $A_5=\{5\}$ ,  $A_6=\{5,6\}$ , and  $A_j=\emptyset$  for all  $j\in M\setminus\{5,6\}$ . Then, we can write the conditions in (23) together with the non-negativity

requirements for  $\beta$  as follows:

$$0 \leq \beta_{1} \leq \min\{\phi_{5}, \phi_{6}\},\$$

$$0 \leq \beta_{2} \leq \min\{\phi_{5}, \phi_{6}, 4\},\$$

$$0 \leq \beta_{3} \leq \min\{\phi_{5}, \phi_{6}\},\$$

$$3 \leq \beta_{4} \leq \min\{\phi_{5}, \phi_{6}\},\$$

$$\max\{0, 6 - \phi_{5}\} \leq \beta_{5} \leq \phi_{6},\$$

$$\beta_{6} \geq \max\{0, 7 - \phi_{6}\},\$$

$$\beta_{7} \geq 4, \ \beta_{8} \geq 3, \ \beta_{9} \geq 2.5, \ \beta_{10} \geq 2.2.$$

It is clear that, with the choice of  $\phi_5 = \phi_6 = 3$  to attain the lower bound value of  $\beta_4$ , there exist values  $\beta_1 = \beta_2 = \beta_3 = 0$ ,  $\beta_4 = \beta_5 = 3$ ,  $\beta_6 = \beta_7 = 4$ ,  $\beta_8 = 3$ ,  $\beta_9 = 2.5$ , and  $\beta_{10} = 2.2$  that satisfy the above set of conditions. Hence, the resulting BL&P inequality

$$z + 6x_1 + 2x_4 + 3(1 - x_5) + 3(1 - x_6) \ge 20$$

which matches (12), is valid for  $conv(\mathcal{F}_c^=)$ .

In light of the above example, we note that although the family of BL&P inequalities in Theorem 3 includes inequalities that do not belong to the family presented in Proposition 5, it does not fully subsume that family under the general probability setting, as illustrated in Example 3. Nevertheless, the computational results in Section 5 demonstrate that the family of BL&P inequalities provides a substantially larger coverage of the convex hull for the studied instances compared to the inequalities in Proposition 5.

Example 3. [34, Example 1] Consider an instance of  $\mathcal{F}_c$ , where m = 10,  $\varepsilon = 0.5$ ,  $\{h_1, \ldots, h_m\} = \{40, 38, 34, 31, 26, 16, 8, 4, 2, 1\}$ . Let  $\pi_1 = \ldots = \pi_4 = \varepsilon/4$  and  $\pi_5 = \ldots = \pi_{10} = \varepsilon/6$ . For this problem instance, we have p = 6 and  $\vartheta = 4$ . It is shown in [34, Example 1] that

$$z + 2x_1 + 4(1 - x_4) + 4(1 - x_7) + 8(1 - x_8) \ge 40$$

is facet-defining for  $conv(\mathcal{F}_c)$ , where  $r=1, P=\{1\}$ , and  $Q=\{4,7,8\}$ . Following the setting in Theorem 3, we may choose  $\delta_1=0$ . However, it can be shown that for the choice  $\phi_4=\phi_7=4$  and  $\phi_8=8$ , which is required to produce this inequality form, there does not exist any  $\beta_j \in \mathbb{R}_+$  for some  $j \in M$  that satisfies (23) for any choice of  $A_j$ . Thus, this inequality is not captured by the family of BL&P inequalities in Theorem 3.

As noted earlier, beyond encompassing new classes of valid inequalities that cannot be captured by existing results (as illustrated in Example 2), Theorem 3 also subsumes many known results on valid inequalities for  $\mathcal{F}_c$ . A summary of the relationship between the families of inequalities developed in the literature (Propositions 1–5) and the BL&P inequalities established in Theorems 3 in covering  $\operatorname{conv}(\mathcal{F}_c)$  under a general probability distribution is illustrated in Figure 1. More

specifically, for the widely studied special case where the underlying probability distribution is uniform, we show in the next section that a subfamily of the BL&P inequalities in (24) strictly subsumes all existing results in the literature, as presented in Section 2.2.

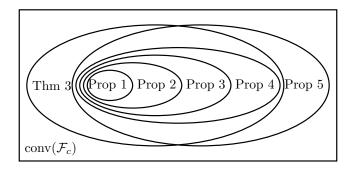


Figure 1: Relationship between existing inequalities (Propositions 1–5) and BL&P inequalities (Theorems 3) in characterizing  $conv(\mathcal{F}_c)$  under a general probability distribution.

#### 4.3 Valid Inequalities for the Mixing Set with a Cardinality Constraint

In this section, we present a special subfamily of the BL&P inequalities in (24) corresponding to the case of a uniform probability distribution. In this setting, the explicit parameter values can be readily determined and directly incorporated into the BL&P inequality, eliminating the need to verify the existence of  $\beta$  satisfying the conditions in (23). This leads to an efficient approach for generating closed-form valid inequalities for CCPs. We show in Corollary 1 that this subfamily encompasses all families of valid inequalities previously derived in the literature, as presented in Propositions 1–5. Moreover, it also includes valid inequalities that cannot be captured by Proposition 5 (see Example 4).

**Theorem 4.** Consider the mixing set with a cardinality constraint  $\mathcal{F}_c^=$ , as defined in (4). Let  $r \in \{1,\ldots,p\}$ ,  $l \in \{1,\ldots,r\}$ , and  $v \in \{1,\ldots,p-r\}$ . Define  $V := \{1,\ldots,v\}$  and let  $s_\iota = p-r-v+\iota$  for  $\iota \in V$ . Define  $L := \{1,\ldots,l\}$  and let  $P = \{t_\iota : \iota \in L\} \subseteq \{1,\ldots,r\}$ , where  $t_1 < \cdots < t_l$ . Select  $\delta_{t_\iota} \in \mathbb{R}$  for  $\iota \in L$  such that  $\delta_{t_\iota} \geq h_{t_{\iota+1}} - h_{t_\iota}$ ,  $\sum_{\iota=1}^{k-1} \delta_{t_\iota} \geq 0$  for  $k = 2,\ldots,l$ , and  $\sum_{\iota \in L} \delta_{t_\iota} \leq h_{r+s_1} - h_{r+s_2}$ , with  $t_{l+1} := r + s_1$ . Let  $Q = \{q_1,\ldots,q_v\}$ , where  $r + s_\iota + 1 \leq q_\iota \leq m$  for  $\iota \in V$ . Define

$$\phi_{q_1} = h_{r+s_1} - h_{r+s_2} - \sum_{k \in L} \delta_{t_k}, \tag{32a}$$

$$\phi_{q_{\iota}} = \max \left\{ \phi_{q_{\iota-1}}, h_{r+s_1} - h_{r+s_{\iota}+1} - \sum_{k \in L} \delta_{t_k} - \sum_{\substack{k=1 \ q_k \ge r+s_{\iota}+1}}^{\iota-1} \phi_{q_k} \right\}, \quad \iota = 2, \dots, v.$$
 (32b)

Then, the BL&P inequality

$$z + \sum_{\iota=1}^{l} (h_{t_{\iota}} - h_{t_{\iota+1}} + \delta_{t_{\iota}}) x_{t_{\iota}} + \sum_{\iota=1}^{v} \phi_{q_{\iota}} (1 - x_{q_{\iota}}) \ge h_{t_{1}}, \tag{33}$$

is valid for  $\operatorname{proj}_{(z,x)}\operatorname{conv}(\mathcal{S}_c^{=})$ ; hence, is valid for  $\operatorname{conv}(\mathcal{F}_c^{=})$ .

Proof. By Theorem 3, it suffices to show that for each  $j \in M$ , there exists  $\beta_j \in \mathbb{R}_+$  that satisfies (23) for some  $A_j \subseteq V$ . First, note that  $\phi_{q_{\iota}} \geq 0$  for all  $\iota \in V$  because of (32a) and (32b), and the assumption that  $\sum_{\iota \in L} \delta_{t_{\iota}} \leq h_{r+s_1} - h_{r+s_2}$ . Following the notation in Theorem 3, for this special case, we have  $\varepsilon = p/m$ ,  $\pi_j = 1/m$ , and  $F_j = j/m$  for  $j \in M$ . We present the results for multiple cases of  $j \in M$  as follows.

(I) Assume that  $j \in M \setminus (P \cup Q)$  and  $j > \max\{p+1, \max\{q_{\iota} : \iota \in V\}\}$ . We choose  $A_j = \emptyset$ . In this case, constraint (23a) does not exist, and constraint (23b) can be written as

$$\beta_j \ge (h_{r+s_1} - h_j - \sum_{\iota \in L} \delta_{t_{\iota}})/(j - p - 1),$$

using the fact that j > p + 1. Therefore, we can simply pick  $\beta_j = \max \{0, (h_{r+s_1} - h_j - \sum_{i \in L} \delta_{t_i})/(j-p-1)\}$  to satisfy the conditions in Theorem 3.

(II) Assume that  $j \in M \setminus (P \cup Q)$  and  $j = r + s_1$ . We choose  $A_j = \emptyset$ . In this case,  $q_{\iota} > j$  for all  $\iota \in V$  as  $q_{\iota} \geq r + s_{\iota} + 1 \geq r + s_1 + 1 > r + s_1 = j$ , where the first inequality follows from the definition of  $q_{\iota}$  and the second inequality is implied by the definition of  $s_{\iota}$ . As a result, constraint (23a) can be written as

$$\beta_i \le \min\{\phi_{q_i} : \iota \in V\}.$$

Furthermore, constraint (23b) is trivially satisfied for this case since the right-hand-side will reduce to  $-\sum_{\iota\in L} \delta_{t_{\iota}}$ , and the coefficient of  $\beta_{j}$  on the left-hand-side is also zero as j-1-p+v=0 due to  $j=r+s_{1}$  and  $s_{1}=p-r-v+1$ . Therefore, we can simply select  $\beta_{j}=0$  to satisfy the conditions in Theorem 3.

(III) Assume that  $j \in M \setminus (P \cup Q)$  and  $j < r + s_1$ . We choose  $A_j = \emptyset$ . Similarly to the previous case, we have  $q_i > j$  for all  $i \in V$ . Therefore, constraints (23a) reduces to and

$$\beta_j \le \min\{\phi_{q_\iota} : \iota \in V\}. \tag{34}$$

Further, the coefficient of  $\beta_j$  in (23b) reduces to  $j-1-p+v=j-r-s_1<0$ , where the equality follows from the definition of  $s_1=p-r-v+1$ , and the inequality follows the

assumption of this case. Thus, (23b) can be written as

$$\beta_j \le (h_j - h_{t_{a_j+1}} + \sum_{\substack{\iota \in L \\ t_{\iota} < j}} \delta_{t_{\iota}}) / (r + s_1 - j),$$
(35)

where  $h_j - h_{t_{a_j+1}} \ge 0$  and  $\sum_{\substack{t \in L \\ t_\iota < j}} \delta_{t_\iota} \ge 0$  by definition. Because the right-hand-side values of both inequalities (34) and (35) are non-negative, we can pick  $\beta_j = 0$  to satisfy the conditions in Theorem 3.

(IV) Assume that  $j \in M \setminus (P \cup Q)$  and  $\max\{p+1, \min\{q_{\iota} : \iota \in V\}\} < j < \max\{p+1, \max\{q_{\iota} : \iota \in V\}\}$ . Clearly, if  $\max\{p+1, \min\{q_{\iota} : \iota \in V\}\} = \max\{p+1, \max\{q_{\iota} : \iota \in V\}\} = p+1$ , then this case does not occur. Otherwise, we choose  $A_j = \{\iota \in V : q_{\iota} > j\}$ . Hence, constraints (23a) and (23b) can be written as

$$\beta_j \ge \max\{\phi_{q_\iota} : \iota \in A_j\},$$
  
$$\beta_j \ge \left(h_{r+s_1} - h_j - \sum_{\iota \in L} \delta_{t_\iota} - \sum_{\iota \in A_j} \phi_{q_\iota}\right) / (j - p - 1),$$

where the second inequality follows from the fact that j > p + 1 by assumption. The above inequalities provide lower bounds for  $\beta_j$ . Therefore, we can pick

$$\beta_j = \max \left\{ (h_{r+s_1} - h_j - \sum_{\iota \in L} \delta_{t_\iota} - \sum_{\iota \in A_j} \phi_{q_\iota}) / (j - p - 1), \max\{\phi_{q_\iota} : \iota \in A_j\} \right\}$$

to satisfy the conditions in Theorem 3 as  $\phi_{q_{\iota}} \geq 0$  by definition.

(V) Assume that  $j \in M \setminus (P \cup Q)$  and  $r + s_1 + 1 \le j \le p + 1$ . We can write that  $j = r + s_t + 1$  for some  $1 \le \iota \le v$ . Choose  $A_j = \{k \in \{1, \ldots, \iota\} : q_k \ge j\}$ . It is clear that  $\iota \in A_j$  as  $q_\iota \ge r + s_\iota + 1 = j$  by assumption, where the equality across the chain is impossible since it would imply  $j \in Q$ . Thus,  $\iota \in A_j$  and  $q_\iota > j$ . As a result, constraints (23a) and (23b) can be written as

$$\phi_{q_{\iota}} = \max\{\phi_{q_{k}} : k \in A_{j}, \ q_{k} > j\} \le \beta_{j} \le \min\{\phi_{q_{k}} : k \in V \setminus A_{j}, \ q_{k} > j\} = \phi_{q_{\iota+1}},$$
 (36a)

$$\phi_{q_{\iota}} \ge h_{r+s_{1}} - h_{j} - \sum_{k \in L} \delta_{t_{k}} - \sum_{\substack{k=1 \ q_{k} > j}}^{\iota-1} \phi_{q_{k}} = h_{r+s_{1}} - h_{r+s_{\iota}+1} - \sum_{k \in L} \delta_{t_{k}} - \sum_{\substack{k=1 \ q_{k} \ge r+s_{\iota}+1}}^{\iota-1} \phi_{q_{k}}.$$
 (36b)

The left equality in (36a) follows from the definition of  $\phi_{q_k}$  for  $k \in V$  in (32), and the facts that  $\iota \in A_j$  and  $q_\iota > j$ . For the right equality in (36a), we claim that  $V \setminus A_j \cap \{k \in V : q_k > j\} = \{\iota + 1, \ldots, v\}$ . Note that  $V \setminus A_j = \{\iota + 1, \ldots, v\} \cup \{k \in \{1, \ldots, \iota - 1\} : q_k < j\}$ . Therefore, we can write  $V \setminus A_j \cap \{k \in V : q_k > j\} = \{k \in \{\iota + 1, \ldots, v\} : q_k > j\}$ . Consider  $k \in \{\iota + 1, \ldots, v\}$ . It follows from the definition of  $q_k$  that  $q_k \geq r + s_k + 1 > r + s_\iota + 1 = j$ . Thus,  $k \in V \setminus A_j \cap \{k \in V : q_k > j\}$ , proving our claim. Moreover, the inequality in (36b) is due to the

fact that the coefficient of  $\beta_j$  in (23b) is  $j-1-p+\left|\{k\in V\setminus A_j:q_k>j\}\right|=j-1-p+v-\iota=0$ , where the first equality follows from the earlier argument and the second equality holds because of the definitions  $j=r+s_t+1$  and  $s_t=p-r-v+\iota$ . The equality in (36b) follows from the assumption  $j=r+s_t+1$  and the fact that  $q_k\neq j$  for all  $k\in V$  by assumption of this case. Therefore, we can pick  $\beta_j=\phi_{q_\iota}$ , which satisfies (36a) because of (32b) as  $\phi_{q_\iota}\leq \phi_{q_{\iota+1}}$ . Furthermore, (36b) is directly implied by (32b).

(VI) Assume that  $j \in M \setminus (P \cup Q)$  and  $p+1 < j < \max\{p+1, \min\{q_{\iota} : \iota \in V\}\}$ . Clearly, for this case to occur, we must have  $\max\{p+1, \min\{q_{\iota} : \iota \in V\}\} = \min\{q_{\iota} : \iota \in V\} \neq p+1$ . As a result, we have  $j < q_{\iota}$  for all  $\iota \in V$ . Choose  $A_j = V$ . Therefore, constraints (23a) and (23b) can be written as

$$\beta_j \ge \max\{\phi_{q_\iota} : \iota \in V\} = \phi_{q_v},\tag{37a}$$

$$\beta_j \ge (h_{r+s_1} - h_j - \sum_{k \in L} \delta_{t_k} - \sum_{k \in V} \phi_{q_k})/(j - p - 1),$$
 (37b)

where the inequality in (37a) follows from the fact that  $A_j \cap \{k \in V : q_k > j\} = V$  because  $q_i > j$  for all  $i \in V$ , and the equality in this relation is due to the definition of  $\phi_{q_i}$  for  $i \in V$  in (32). Moreover, the inequality in (37b) holds because the coefficient of  $\beta_j$  in (23b) is j-p-1, which is positive by the assumption of this case. Since both (37a) and (37b) impose a lower bound on  $\beta_j$ , we can simply pick

$$\beta_j = \max \left\{ \phi_{q_v}, (h_{r+s_1} - h_j - \sum_{k \in L} \delta_{t_k} - \sum_{k \in V} \phi_{q_k}) / (j - p - 1) \right\}$$

to satisfy these conditions.

(VII) Assume that  $j \in Q$  and  $r + s_1 + 1 \le j \le p + 1$ . On the one hand, since  $j \in Q$ , we can write  $j = q_i$  for some  $1 \le i \le v$ . On the other hand, since  $r + s_1 + 1 \le j \le p + 1$ , we can write that  $j = r + s_t + 1$  for some  $1 \le \iota \le v$ . We claim that  $i \le \iota$ . Assume by contradiction that  $i > \iota$ . Then, we must have  $j = q_i \ge r + s_i + 1 > r + s_\iota + 1 = j$ , where the equalities follow from the previous arguments for this case, the first inequality is implied by the theorem requirement for  $q_i$ , and the second inequality is due to the fact that  $s_i > \iota$  as  $i > \iota$  by definition, yielding a contradiction. Next, we consider two sub-cases. For the first sub-case, assume that  $i < \iota$ . Choose  $A_j = \{k \in \{1, \ldots, \iota\} : q_k \ge j\}$ . It is clear that  $\iota \in A_j$  as  $q_\iota \ge r + s_\iota + 1 = j = q_i$ , where the equality across the chain is impossible since it would imply  $q_\iota = q_i$ . Thus,  $\iota \in A_j$  and  $q_\iota > j$ . As a result, constraints (23a) and (23b) can be written as

$$\phi_{q_{i}} = \max\{\phi_{q_{k}} : k \in A_{j}, \ q_{k} > j\} \le \beta_{j} \le \min\{\phi_{q_{k}} : k \in V \setminus A_{j}, \ q_{k} > j\} = \phi_{q_{i+1}}, \tag{38a}$$

$$\phi_{q_{\iota}} \ge h_{r+s_{1}} - h_{j} - \sum_{k \in L} \delta_{t_{k}} - \phi_{j} - \sum_{\substack{k=1 \ q_{k} > j}}^{\iota - 1} \phi_{q_{k}} = h_{r+s_{1}} - h_{r+s_{\iota} + 1} - \sum_{k \in L} \delta_{t_{k}} - \sum_{\substack{k=1 \ q_{k} \ge r + s_{\iota} + 1}}^{\iota - 1} \phi_{q_{k}}.$$

$$(38b)$$

The left equality in (38a) follows from the definition of  $\phi_{q_k}$  for  $k \in V$  in (32), and the facts that  $\iota \in A_j$  and  $q_\iota > j$ . For the right equality in (38a), we use the fact that  $V \setminus A_j \cap \{k \in V : q_k > j\} = \{\iota + 1, \ldots, v\}$ , which follows from an argument similar to that of case (V), discussed above. Moreover, the inequality in (38b) is due to the fact that the coefficient of  $\beta_j$  in (23b) is zero, which can be shown using a similar argument to that of case (V). The equality in this relation follows from the facts that  $j = r + s_\iota + 1$ , and  $\{k \in \{1, \ldots, \iota - 1\} : q_k > j\} \cup \{j\} = \{k \in \{1, \ldots, \iota - 1\} : q_k \geq j\}$  since  $j = q_i$  for some  $i \in \{1, \ldots, \iota - 1\}$  by the assumption of this case. Therefore, we can pick  $\beta_j = \phi_{q_\iota}$ , which satisfies (38a) because of (32b) as  $\phi_{q_\iota} \leq \phi_{q_{\iota+1}}$ . Furthermore, (38b) is implied by (32b).

For the second sub-case, assume that  $i = \iota$ . Choose  $A_j = \{k \in \{1, \ldots, \iota\} : q_k \geq j\}$ . It is clear that  $\iota \in A_j$  as  $q_{\iota} = j$ . Hence, constraints (23a) and (23b) can be written as

$$\phi_1^* = \max\{\phi_{q_k} : k \in A_j, \ q_k > j\} \le \beta_j \le \min\{\phi_{q_k} : k \in V \setminus A_j, \ q_k > j\} = \phi_{q_{i+1}}, \tag{39a}$$

$$\phi_{q_{\iota}} \ge h_{r+s_{1}} - h_{j} - \sum_{k \in L} \delta_{t_{k}} - \sum_{\substack{k=1 \ q_{k} > j}}^{\iota} \phi_{q_{k}} = h_{r+s_{1}} - h_{r+s_{\iota}+1} - \sum_{k \in L} \delta_{t_{k}} - \sum_{\substack{k=1 \ q_{k} \ge r+s_{\iota}+1}}^{\iota-1} \phi_{q_{k}}, \quad (39b)$$

where  $0 \le \phi_1^* \le \phi_{q_{\iota-1}}$  since  $\phi_{q_k} \le \phi_{q_{\iota-1}}$  for all  $k \in \{1, \ldots, \iota-1\}$ . The right equality in (39a) follows from an argument similar to that of the first sub-case above. Moreover, the inequality in (39b) is due to the fact that the coefficient of  $\beta_j$  in (23b) is zero, which can be computed similarly to case (V), discussed previously. The equality in (39b) follows from the facts that  $j = r + s_{\iota} + 1$  and  $\{k \in \{1, \ldots, \iota\} : q_k > j\} = \{k \in \{1, \ldots, \iota-1\} : q_k \ge j\}$  since  $j = q_i = q_{\iota}$  by the assumption of this case. Therefore, we can pick  $\beta_j = \phi_{q_{\iota+1}}$ , which satisfies (39a) because of (32b) as  $\phi_1^* \le \phi_{q_{\iota-1}} \le \phi_{q_{\iota+1}}$ . Furthermore, (39b) is directly implied by (32b).

(VIII) Assume that  $j \in Q$  and j > p + 1. We can write  $j = q_{\iota}$  for some  $\iota = 1, \ldots, v$ . We choose  $A_j = \{k \in V : q_k > j\}$ . Hence, constraints (23a) and (23b) for j can be written as

$$\beta_j \ge \max\{\phi_{q_k} : k \in A_j\} := \phi_2^*,$$
  
 $\beta_j \ge (h_{r+s_1} - h_j - \sum_{k \in L} \delta_{t_k} - \phi_{q_i} - \sum_{k \in A_j} \phi_{q_k})/(j - p - 1),$ 

where  $\phi_2^* \ge \phi_{q_1} \ge 0$  and j - p - 1 > 0 by assumption. Therefore, we can select  $\beta_j = \max \left\{ \phi_2^*, (h_{r+s_1} - h_j - \sum_{k \in L} \delta_{t_k} - \phi_{q_i} - \sum_{k \in A_j} \phi_{q_k})/(j - p - 1) \right\}$  to satisfy the above inequalities.

(IX) Assume that  $j \in P$ . We choose  $A_j = \emptyset$ . In this case, constraints (23a) and (23b) reduce to

$$\beta_j \le \min\{\phi_{q_\iota} : \iota \in V\},\$$
 $\beta_j(j-1-p+v) \ge (h_{t_{a_j+1}} - h_j - \sum_{\substack{\iota \in L \\ t_\iota < j}} \delta_{t_\iota}).$ 

We can write that  $j-1-p+v=j-1-r-(p-r-v) \leq j-1-r < 0$  where the first inequality follows from  $p-r-v \geq 0$  and the second inequality holds by assumption. Using this result together with the fact that  $h_j = h_{t_{a_j+1}}$  as  $j \in P$ , we can simplify the above inequalities as

$$\beta_j \le \min\{\phi_{q_\iota} : \iota \in V\},$$
  
$$\beta_j \le \sum_{\substack{\iota \in L \\ t, < j}} \delta_{t_\iota} / (p - v + 1 - j).$$

Because both right-hand-side values of these inequalities are non-negative, we can simply pick  $\beta_i = 0$  to satisfy the conditions in Theorem 3. This completes the proof.

Next, we show that the family of BL&P inequalities in Theorem 4 encompasses the existing results in the literature, represented by Proposition 5, as a special case under the uniform probability distribution.

Corollary 1. Consider the setting of Theorem 4 for the mixing set with a cardinality constraint  $\mathcal{F}_c^-$ . By setting  $\delta_{t_\iota} = 0$  for all  $\iota \in L$ , the BL&P inequality (33) reduces to the inequality (11) in Proposition 5.

Proof. We have previously established in Section 2.1 that, for  $\mathcal{F}_c^=$ , it holds that  $p = \tau = \lfloor m\varepsilon \rfloor$ . Consequently, the requirement (10) in Proposition 5 implies  $s_{\iota} = p - r - v + \iota$  for all  $\iota \in 1, \ldots, v$ . Thus, we obtain  $r + \min\{1 + s_{\iota}, s_{\iota+1}\} = r + s_{\iota} + 1$ . This ensures that the parameter definitions and assumptions in Theorem 4 and Proposition 5 coincide when  $\delta_{t_{\iota}} = 0$  for all  $\iota \in L$ . As a result, the definition of  $\phi_{q_{\iota}}$  is consistent across Theorem 4 and Proposition 5, leading to the equivalence between the BL&P inequality (33) and the inequality (11).

The following example illustrates the result of Corollary 1 by presenting a facet-defining inequality for  $conv(\mathcal{F}_c^{=})$  that can be derived from Theorem 4 but not from Proposition 5, thereby demonstrating that the family of BL&P inequalities (33) *strictly* subsumes the existing results.

Example 4. Consider an instance of  $\mathcal{F}_c^=$ , with m=10,  $\{h_1,\ldots,h_m\}=\{20,18,14,11,6,5,4,3,2,1\}$ , and p=4. We apply Theorem 4 by selecting r=1, l=1, v=3,  $P=\{1\}$ , and  $Q=\{6,8,7\}$ , i.e.,  $q_1=6$ ,  $q_2=8$ , and  $q_3=7$ . Moreover, we choose  $\delta_1=1$ . Following the equations (32a) and (32b), we can calculate  $\phi_6=3$ ,  $\phi_7=5$ , and  $\phi_8=3$ . As a result, (33) yields the BL&P inequality

$$z + 3x_1 + 3(1 - x_6) + 5(1 - x_7) + 3(1 - x_8) \ge 20,$$

which can be shown to be facet-defining for  $\operatorname{conv}(\mathcal{F}_c^=)$ . However, this inequality cannot be derived from Proposition 5, partly because the coefficient of  $x_1$ , which is 3, does not match  $(h_1 - h_2) = 2$  as prescribed therein.

As demonstrated in Corollary 1, when the values of  $\delta_{t_{\iota}}$  are fixed to zero, the structure of the BL&P inequality (33) coincides with that of Proposition 5, which in turn aligns with Proposition 4; see the proof of Corollary 1. Consequently, for any fixed values of  $\delta_{t_{\iota}}$  with  $\iota \in V$ , we can employ an argument similar to that in Section 3.1 of [18] to develop an efficient separation algorithm that identifies the parameters in Theorem 4 yielding the most violated inequality at a given point.

Although Theorem 4 generalizes existing results in the literature for mixing sets with cardinality constraints, it still constitutes a subfamily of the BL&P inequalities described in Theorem 3. This is because Theorem 4 imposes several restrictions on the choice of parameters in Theorem 3, which, in turn, enables the derivation of explicit valid inequalities without the need to verify the existence of a vector  $\boldsymbol{\beta}$  satisfying the conditions in (23). Some of these restrictions are as follows: (i) In Theorem 4, we have  $v = |Q| \leq p - r$ , whereas in Theorem 3,  $v = |Q| \leq p - r + |P|$ ; (ii) In Theorem 4, the range of Q is restricted to  $\{r + s_1 + 1, \ldots, m\}$ , with  $s_1 = p - r - v + 1 \geq 1$ , whereas in Theorem 3, the range of Q may be within  $\{r + 1, \ldots, m\}$ ; and (iii) Theorem 4 imposes additional restrictions on  $\delta_{t_k}$  compared to Theorem 3.

To illustrate, the following example presents a facet-defining inequality that can be derived from Theorem 3 but not from Theorem 4, indicating that the former theorem yields a richer family of valid inequalities—even under the special case of uniform probability distributions—as further demonstrated by the computational experiments in Section 5.

Example 5. Consider the instance of  $\mathcal{F}_c^-$ , given in Example 1, with m=10,  $\{h_1,\ldots,h_m\}=\{20,18,14,11,6,5,4,3,2,1\}$ , and p=4. We established in Example 2 that inequality (12) belongs to the family of BL&P inequalities (24) derived in Theorem 3. In contrast, we show here that this inequality cannot be represented by the family of valid inequalities in Theorem 4. Based on the structure of inequality (12), for this problem instance, we must have r=4,  $P=\{1,4\}$ , and  $Q=\{5,6\}$ . However, since p=r, the conditions in Theorem 4 enforce  $Q=\emptyset$ , which contradicts with the requirement  $Q=\{5,6\}$  for this inequality. Moreover, we must choose  $\delta_1=\delta_4=-3$  to ensure that the coefficient of variables in P, namely  $x_1$  and  $x_4$ , are  $(h_1-h_4+\delta_1)=6$  and  $(h_4-h_5+\delta_4)=5$ , respectively. While  $\delta_1\geq h_4-h_1=-9$  and  $\delta_4\geq h_5-h_4=-6$ , with  $\delta_1+\delta_2\leq h_5-h_6=1$  as  $s_1=1$  and  $s_2=2$ , the requirement that  $\delta_1\geq 0$  is not satisfied. As a result, inequality (12) cannot be represented by the family of inequalities in Theorem 4.

We conclude this section by summarizing the relationship between the families of inequalities developed in the literature (Propositions 1–5) and the BL&P inequalities established in this work (Theorems 3–4) in characterizing  $conv(\mathcal{F}_c^=)$  under a uniform probability distribution, as illustrated in Figure 2.

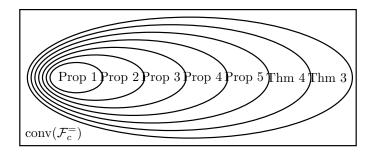


Figure 2: Relationship between existing inequalities (Propositions 1–5) and BL&P inequalities (Theorems 3–4) in characterizing  $conv(\mathcal{F}_c^=)$  under a uniform probability distribution.

## 5 Computational Experiments

In this section, we present preliminary computational results that demonstrate the effectiveness of the families of BL&P inequalities derived in Sections 4.2 and 4.3 in characterizing the convex hull of a chance-constrained set. Specifically, we consider two sets of benchmark instances for the mixing set with a cardinality constraint that have been studied in the literature, as described below:

Example 6. Instances of  $\mathcal{F}_c^=$  with  $m \in \{1, ..., 10\}$ , where the values of  $\{h_1, ..., h_m\}$  correspond to the first m elements of the sequence  $\{20, 18, 14, 11, 6, 5, 4, 3, 2, 1\}$ , and p is selected from the range  $\{1, 2, ..., m\}$ , as presented in [24].

Example 7. Instances of  $\mathcal{F}_c^=$  with  $m \in \{1, \ldots, 10\}$ , where the values of  $\{h_1, \ldots, h_m\}$  correspond to the first m elements of the sequence  $\{40, 38, 34, 31, 26, 16, 8, 4, 2, 1\}$ , and p is selected from the range  $\{1, 2, \ldots, m\}$ , as presented in [18].

To compute the facet-defining inequalities of the convex hull for each configuration, we used a Python wrapper for the Qhull library [6]. All experiments were conducted on a Linux Ubuntu 20.04 system running on a PC equipped with an Intel Core i7-9700 3.00 GHz processor and 32 GB of RAM. For m = 10 and  $p \ge 5$ , we were unable to compute the entire convex hull within 12 hours, which further highlights the complexity of these problems.

Tables 1 and 2 illustrate the strength of existing families of valid inequalities, as characterized by Proposition 5, as well as the strength of the BL&P inequalities proposed in Theorems 3 and 4, in describing the convex hull for Examples 6 and 7, respectively. In each table, we report the percentage of facet-defining inequalities of the convex hull that are captured by the inequalities from Proposition 5 and Theorems 3 and 4, under columns labeled with the corresponding results. The column "Imp.4" reports the improvement in convex hull coverage achieved by the subfamily of BL&P inequalities in Theorem 4 over the coverage provided by Proposition 5. Similarly, the column "Imp.3" shows the additional coverage achieved by the general BL&P inequalities in Theorem 3 over those in Theorem 4. Finally, the column "Total Imp." reports the overall improvement obtained by combining the BL&P inequalities from Theorems 3 and 4, relative to the baseline established by Proposition 5. We note that when p = m,  $\mathcal{F}^{=}$  reduces to a mixing set (5), for which the star inequalities in Proposition 1 are sufficient to describe the convex hull; the same holds for

Theorems 3 and 4. Similarly, when p = 1, there is *only* one facet-defining inequality of the form  $z + (h_1 - h_2)z_1 \ge h_m$ , which is captured by the strengthened star inequalities in Proposition 2 as well as by Theorems 3 and 4. Therefore, we exclude these trivial cases when p = 1 and p = m from Tables 1 and 2.

Observe that for Examples 6 and 7, the convex hull is fully characterized by existing results only when p=2 or p=m-1. This indicates that our inequalities are equally effective in these cases, as also stated in Corollary 1. In contrast, when the existing results do not fully characterize the convex hull, Theorems 3 and 4 complement them to achieve full characterization in many additional cases. Specifically, Theorem 4 provides additional characterization ranging from [7.69%, 25.54%] for Example 6 and [7.68%, 25.54%] for Example 7. Further, Theorem 3 contributes an additional [7.69%, 46.61%] and [7.69%, 42.32%] characterization for Examples 6 and 7, respectively. Notably, for Example 6, Theorem 3 fully characterizes the convex hull in all but one of the remaining 17 cases. In the exceptional case, when m=9 and p=7, it still achieves a coverage of 99.77% of the convex hull. In these instances, our BL&P inequalities provide a further improvement of [15.38%, 68.05%] over the existing results in Proposition 5. For Example 7, there are six out of 17 remaining cases where Theorem 3 captures at least 90.92% of the convex hull, offering an additional [15.38%, 62.77%] improvement over prior results. In summary, these computational results highlight that the family of BL&P inequalities introduced in this work advances our understanding of the polyhedral structure of chance-constrained models, offering substantial improvements over existing formulations in both coverage and completeness.

#### 6 Conclusion and Future Research

Chance-constrained programming offers a versatile framework for modeling optimization problems under uncertainty; however, its mixed-integer reformulations introduce nonconvex mixing sets with knapsack constraints, leading to weak continuous relaxations and increased computational complexity. Prior research has developed various families of valid inequalities, most notably mixing inequalities, to better describe these sets. However, existing approaches capture only a limited portion of the convex hull and rely on case-specific extensions of known inequalities, restricting the discovery of fundamentally new structures and systematic convexification methods. These gaps motivate the need for a unifying framework that can generate stronger and more comprehensive representations of chance-constrained sets.

To address these challenges, this paper introduces a new bilinear convexification framework that reformulates mixing sets with a knapsack constraint as bilinear sets over a simplex in a lifted space. The proposed aggregation-based procedure enables systematic derivation of facet-defining inequalities in the original space of variables, generalizing and unifying known families while revealing new classes of valid inequalities that capture much larger portions of the convex hull. Preliminary computational studies demonstrate that the new inequalities describe over 90% of the facets of benchmark instances, offering significant strengthening over existing relaxations.

Table 1: Comparison between existing and new results in characterizing the convex hull for Example 6.

$\overline{m}$	p	Proposition 5 (%)	Theorem 4 (%)	Imp. <sup>4</sup> (%)	Theorem 3 (%)	Imp. <sup>3</sup> (%)	Total Imp. (%)
3	2	100.0	100.0	-	100.0	-	-
4	2	100.0	100.0	_	100.0	_	-
4	3	100.0	100.0	-	100.0	-	-
5	2	100.0	100.0	-	100.0	-	-
5	3	84.62	92.31	7.69	100.0	7.69	15.38
5	4	100.0	100.0	-	100.0	-	-
6	2	100.0	100.0	-	100.0	-	-
6	3	72.73	86.36	13.63	100.0	13.64	27.27
6	4	76.32	86.84	10.52	100.0	13.16	23.68
6	5	100.0	100.0	-	100.0	-	-
7	2	100.0	100.0	-	100.0	-	_
7	3	64.71	82.35	17.64	100.0	17.65	35.29
7	4	59.3	76.74	17.44	100.0	23.26	40.7
7	5	63.0	75.0	12.0	100.0	25.0	37.0
7	6	100.0	100.0	-	100.0	-	_
8	2	100.0	100.0	-	100.0	-	-
8	3	59.18	79.59	20.41	100.0	20.41	40.82
8	4	48.81	70.24	21.43	100.0	29.76	51.19
8	5	41.95	60.4	18.45	100.0	39.6	58.05
8	6	61.43	71.43	10.0	100.0	28.57	38.57
8	7	100.0	100.0	-	100.0	-	-
9	2	100.0	100.0	-	100.0	-	_
9	3	55.22	77.61	22.39	100.0	22.39	44.78
9	4	41.98	65.87	23.89	100.0	34.13	58.02
9	5	31.95	53.39	21.44	100.0	46.61	68.05
9	6	37.78	54.07	16.29	100.0	45.93	62.22
9	7	60.81	70.95	10.14	99.77	28.82	38.96
9	8	100.0	100.0	-	100.0	-	-
10	2	100.0	100.0	-	100.0	-	
10	3	52.27	76.14	23.87	100.0	23.86	47.73
10	4	37.23	62.77	25.54	100.0	37.23	62.77

Table 2: Comparison between existing and new results in characterizing the convex hull for Example 7.

$\overline{m}$	p	Proposition 5	Theorem 4	Imp. <sup>4</sup>	Theorem 3	Imp. <sup>3</sup>	Total Imp.
	Г	(%)	(%)	(%)	(%)	(%)	(%)
3	2	100.0	100.0	-	100.0	-	-
4	2	100.0	100.0	-	100.0	-	_
4	3	100.0	100.0	-	100.0	-	-
5	2	100.0	100.0	-	100.0	-	-
5	3	84.62	92.31	7.69	100.0	7.69	15.38
5	4	100.0	100.0	-	100.0	-	-
6	2	100.0	100.0	-	100.0	-	_
6	3	72.73	86.36	13.63	100.0	13.64	27.27
6	4	76.32	86.84	10.52	100.0	13.16	23.68
6	5	100.0	100.0	-	100.0	-	-
7	2	100.0	100.0	-	100.0	-	_
7	3	64.71	82.35	17.64	100.0	17.65	35.29
7	4	59.3	76.74	17.44	100.0	23.26	40.7
7	5	70.87	80.58	9.71	97.09	16.51	26.22
7	6	100.0	100.0	-	100.0	-	-
8	2	100.0	100.0	-	100.0	-	_
8	3	59.18	79.59	20.41	100.0	20.41	40.82
8	4	48.81	70.24	21.43	100.0	29.76	51.19
8	5	48.97	64.01	15.04	92.92	28.91	43.95
8	6	62.8	70.4	7.6	96.0	25.6	33.2
8	7	100.0	100.0	-	100.0	-	-
9	2	100.0	100.0	-	100.0	-	-
9	3	55.22	77.61	22.39	100.0	22.39	44.78
9	4	41.98	65.87	23.89	100.0	34.13	58.02
9	5	37.51	54.83	17.32	91.34	36.51	53.83
9	6	35.88	48.6	12.72	90.92	42.32	55.04
9	7	53.59	61.27	7.68	95.66	34.39	42.07
9	8	100.0	100.0		100.0		-
10	2	100.0	100.0	-	100.0	-	-
10	3	52.27	76.14	23.87	100.0	23.86	47.73
10	4	37.23	62.77	25.54	100.0	37.23	62.77

The proposed aggregation-based convexification procedure advances the polyhedral understanding of CCPs and opens new avenues for research. The class of valid inequalities proposed in this paper is derived from a single-row relaxation of the chance constraint, which yields a single mixing set with a knapsack constraint. As previously discussed, any facet-defining inequality for such a set is also facet-defining for the joint CCPs, where the key substructure is the intersection of multiple mixing sets with a shared knapsack constraint. However, there also exist facet-defining inequalities for joint CCPs that cannot be obtained from single-row relaxations. A line of research has focused on extending the strength of single-row relaxations to multi-row intersections, see, e.g., [20, 24, 34]. An interesting direction for future research is to investigate the convexification of the intersection of multiple mixing sets with a knapsack constraint through the lens of the bilinear reformulation introduced in this work, thereby enabling the application of the proposed aggregation-based lift-and-project technique.

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