

Assignment 14

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Question

Referring to Example 15.8 , let e_1, e_2, \dots, e_N represent the states with two end reflecting barriers and transition matrix as in (15-22). Thus $p_{i,i+1}=p$, $p_{i,i-1}=q$ for $2 \leq i \leq N-1$, $p_{11}=q, p_{12}=p$, and $p_{N,N-1}=q, p_{NN}=p$.

Reference

Example 15.8: Suppose the two boundaries in Example 15-7 reflect the particle to the adjacent state instead of adsorbing it. With e_1, e_2, \dots, e_N representing the N states, the end reflection probabilities to the right and left are given by

$$p_{1,2}=p \text{ and } p_{N,N-1}=q$$

(15-22) and this gives the $N \times N$ transition matrix to be

$$p = \begin{pmatrix} q & p & 0 & 0 & . & . & . & 0 \\ q & 0 & p & 0 & 0 & . & . & . \\ 0 & q & 0 & p & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & q & 0 & p \\ 0 & 0 & 0 & . & . & 0 & q & p \end{pmatrix}$$

Solution

$$\sum_{j=1}^N p_{ij} x_j^{(k)} = \lambda_k x_i^{(k)}$$

Substituting in above equation we get

$$x_1 = s(qx_1 + px_2) \quad (2)$$

$$x_i = s(qx_{i-1} + px_{i+1}) \quad i = 2, 3, \dots, N-1 \quad (3)$$

$$x_N = s(qx_{N-1} + px_N) \quad (4)$$

Where we have used $s = 1/\lambda$. Clearly $\lambda=1$ corresponds to the specific solution $x_i = 1$. To find all other solutions, notice that $x_i = s(qx_{i-1} + px_{i+1})$ satisfies the particular solution

$$x_i = \xi^i$$

provided ξ is a root of the quadratic equation

$$\xi = qs + ps\xi^2 \quad (5)$$

The two roots of the quadratic equation are

$$\xi_1(s) = \frac{1 + \sqrt{1 - 4pqs^2}}{2ps}$$

$\xi_2(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}$ and the general solution to $x_i = s(qx_{i-1} + px_{i+1})$ is given by
for $i = 2, \dots, N-1$

$$x_i = a(s)\xi_1^i(s) + b(s)\xi_2^i(s) \quad (6)$$

where $a(s)$ and $b(s)$ are yet to be determined .For equation 2 to satisfy equation 6 , it must have the same form as equation 3 and hence we must have $x_1 = x_0$. Similarly for equation 4 to satisfy equation 6 , on comparing it with equation 3 , we must have $x_N = x_{N+1}$. But

$$x_1 = x_0 \Rightarrow a(s)[1-\xi_1(s)] = -b(s)[1-\xi_2(s)] \quad (7)$$

and

$$x_N = x_{N+1} \Rightarrow a(s)[1-\xi_1(s)]\xi_1^N(s) = -b(s)[1-\xi_2(s)]\xi_2^N(s) \quad (8)$$

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nd from equation 7 and 8 we must have

$$\xi_1^N(s) = \xi_2^N(s) \text{ with } \xi_1(s) \neq \xi_2(s)$$

But from equation 5, we get $\xi_1(s) \xi_2(s) = q/p$ so that equation $\xi_1^N(s) = \xi_2^N(s)$ reduces to

$\lambda_k = \frac{1}{s_k}$ For $\lambda_0 = 1$ we obtain directly $x_i^{(0)} = 1$ we get

$$x_i^{(k)} = \left(\frac{q}{p}\right)^{i/2} \sin \frac{\pi k i}{N} - \left(\frac{q}{p}\right)^{(i+1)/2} \sin \frac{\pi k (i-1)}{N} \text{ where } k = 1, 2, \dots, N-1$$