統計學（一）

第四章 離散型機率分佈
(Discrete Probability Distributions)

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本課程內容參考書目

- **教科書**
  - P. Newbold, W. L. Carlson and B. Thorne(2013). *Statistics for Business and the Economics*, 8^{th} Edition, Pearson.
- **參考書目**
  - Berenson, M. L., Levine, D. M., and Krehbiel, T. C. (2009). *Basic business statistics: Concepts and applications*, 11^{th} EditionPrentice Hall.
  - Larson, H. J. (1982). *Introduction to probability theory and statistical inference*, 3^{rd} Edition, New York: Wiley.
  - Miller, I., Freund, J. E., and Johnson, R. A. (2000). *Miller and Freund's Probability and statistics for engineers*, 6^{th} Edition, Prentice Hall.
  - Montgomery, D. C., and Runger, G. C. (2011). *Applied statistics and probability for engineers*, 5^{th} Edition, Wiley.
  - Watson, C. J. (1997). *Statistics for management and economics*, 5th Edition. Prentice Hall.
  - 唐麗英、王春和（2013），「從範例學MINITAB統計分析與應用」，博碩文化公司。
  - 唐麗英、王春和（2008），「SPSS 統計分析 」，儒林圖書公司。
  - 唐麗英、王春和（2007），「Excel 統計分析」，第二版，儒林圖書公司。
  - 唐麗英、王春和（2005），「STATISTICA與基礎統計分析」，儒林圖書公司。

Random Variables（隨機變數）

- **Random Variables (R.V.)**

  A **random variable** is a variable that takes on **numerical** values determined by the outcome of a random experiment.

  – **Note:**

  Use capital letter to denote a random variable: X

  Use corresponding lowercase letter to denote a possible value of a random variable: $x=1, 2, \ldots$ or $2 < x < 5$.

Random Variables（隨機變數）

- **Two types of Random Variable:**

  1) Discrete R.V. （離散型）

  A discrete R.V. is a R.V. that can take on only a **finite** or **at most** a **countable infinite** number of values. That is. X is a discrete R.V. if its range is a discrete set.

  2) Continuous R.V.（連續型）

  A **continuous R.V**. is a R.V. that can take any values in an interval.

Random Variables（隨機變數）

- **例 1:**

  Each of the following experiments results in one value of the random variable (one measurement).

  a) State whether the random variable is a discrete or continuous.

  b) Determine, at least in principle, all possible values of the random variable.

     1) The number of the leaves on a tree.
     2) The time required to read the book “How to lie with Statistics”
     3) The number of women in a jury of 12.
     4) The speed of a passing car.
     5) The numbers of heads observed when flip a coin two times.
     6) The sum of the two numbers that occur when roll a pair of fair dice one time.

Probability Distributions for Discrete R.V.

- **Probability Distribution Function for a Discrete Random Variable**
  - The probability distribution function, $P(\mathbf{X})$, of a **discrete** random variable **X** expresses the probability that **X** takes the values x, as a function of x. That is,

    $p(x_i) = P(\mathbf{X}=x_i)$ for all values of x.

  - **Remark:** The properties of a probability function for a discrete R.V.:

    i) $\sum_{\text{all } i} p(x_i) = 1$

    ii) $0 \le p(x_i) \le 1$

Probability Distributions for Discrete R.V.

- **How to find the probability function for a discrete R.V. X?**
  - Construct a table listing each value that the R.V. X can assume.
    (建立一表列出離散型隨機變數 X 的所有可能值(x))
  - Then calculate $p(x_i)$ for each value of X.
    (計算出所有可能值 X 之相對機率 p(x))

Probability Distributions for Discrete R.V.

- **例 1:**

  Toss a coin 3 times. Let X be the number of heads observed.

  a)What is the probability function for X?

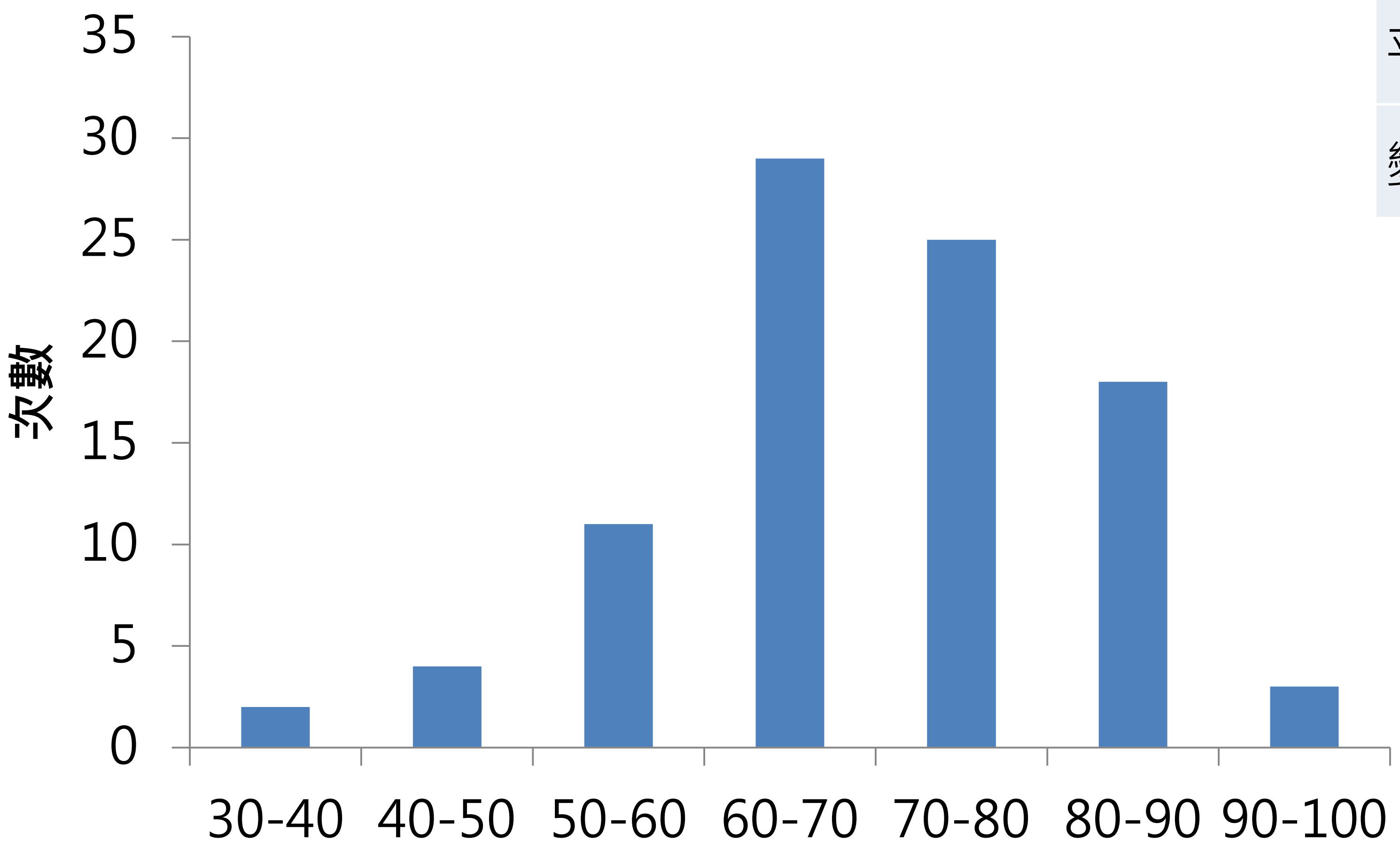
  b)Graph $P(X=x_i)$.

Probability Distributions for Discrete R.V.

- **例 2:**

  Roll two fair dice one time. Let X be the sum of the two numbers that occur. What is the probability function for X?

第一次期中考成績分布



平均數	70.30
變異數	12.81

30-40	2
40-50	4
50-60	11
60-70	29
70-80	25
80-90	18
90-100	3
Total	92

Probability Distributions for Discrete R.V.

- **例 3:**

  Assume that a basketball player has the same probability 0.7 of making every one of his 5 free throws and that his attempts are independent. Let Y be the number of shots he makes in the five attempts.

  a) What is the probability function for Y ?

  b) Find P(Y is odd)

  c) Find $P(Y \geq 3)$

  d) Find $P(Y \leq 1)$

Probability Distributions for Discrete R.V.

- **例 4:**

  The game of Chuck-a-Luck is played as follows. Three fair dice are rolled. You as the bettor are allowed to bet \$1 (or some other amount) on the occurrence of one of the integers 1, 2, 3, 4, 5, 6. If the number you bet occurs one time, you win \$1; If the number you bet occurs two times, you win \$2 and If the number you bet occurs three times, you win \$3. If the number you bet does not occur you lose your money. Suppose you bet \$1 on the occurrence of a 5. Let V be the net amount you win in one play of this game. Find probability function of V.

Probability Distributions for Discrete R.V.

- **Cumulative Probability Function, c.d.f.**
  (累加機率函數)

  – The Cumulative Probability Function, $F(x_0)$, for a random variable X is given as:
  $$F_X(t) = P(X \leq t) \quad \text{for} \quad -\infty \leq t \leq \infty$$

  – **Remark:** If X is a discrete R.V., then $F_X(t) = \sum_{x \leq t} P(X = t)$. (i.e. $F_X(t)$是一累積機率函數)

Probability Distributions for Discrete R.V.

- **例 5:**

  Suppose a hat contains four slips of paper; each slip bears the number 1, 2, 3 and 4. One slip is drawn from the hat without looking. Let **X** be the number on the slip that is drawn.

  a) What is the probability function of X?

  b) What is the distribution function of X?

  c) Graph the distribution function of X.

Probability Distributions for Discrete R.V.

- **例 6:**

  Show that $\mathrm{P}(\mathrm{a} < X \leq b) = \mathrm{F_X(b)} - \mathrm{F_X(a)}$.

  Proof:

  - **Note:** If X is a discrete R.V., then $(\mathrm{X} \leq \mathrm{b}) \neq \mathrm{P}(\mathrm{X} < b)$. Why ?

Properties of Discrete Random Variables

- **The Expected Value of a Discrete R.V. (期望值)**
  - If X is a discrete R.V. with the probability mass function p(x), the Expected Value of X, denote by E(X) or μ_X (Greek letter mu), is

    $$E(X) = \mu_X = \sum_{\text{all } x} x \cdot p(x)$$

  - provide $\sum |X| \cdot p(x) < \infty$. If the sum diverges, the expectation is undefined.

Properties of Discrete Random Variables

- **例 1:**

  Suppose that the probability function for the number of errors, X, on pages from business textbooks is as follows:

  $P(0) = 0.81$, $P(1) = 0.17$, $P(2) = 0.02$

  $E(X) = 0*0.81+1*0.17+2*0.02=0.21$,

  the expectation is **undefined**.

Properties of Discrete Random Variables

- **The Variance and Standard Deviation of a R.V. X**
  - Var(X)= σ_X^2 =E[$(X-\mu_X)^2$]= $E(X^2)-\mu_X^2$
  - St. D.(X)= σ_X=$\sqrt{\sigma_X^2}$

Theorem 1:

$$\boxed{\begin{array}{l}\text{If } X \text{ is any R.V. (discrete or continuous), then} \\ E(X)=\underline{\mathbf{C}}, \text{ where } \mathbf{C} \text{ is any constant.} \\ E[C \cdot H(X)]=C \cdot E[H(X)] \\ E[H(X)+J(X)]= E[H(X)]+ E[J(X)]\end{array}}$$

Properties of Discrete Random Variables

Theorem 2:

$$\boxed{\sigma_X^2 = E[(X-\mu_X)^2] = E(X^2) - \mu_X^2}$$

Theorem 3 :

If E(X) and Var(X) exist and Y=aX+b, where a and b are any constants, then $\mu_Y = a\mu_X + b$, $\sigma_Y^2 = a^2\sigma_X^2$, $\sigma_y = |a| \cdot \sigma_X$

Properties of Discrete Random Variables

- **例 2:**

  Suppose $E(X)=5$, $Var(X)=10$, Find

  (a) $E(3X-5)$

  (b) $Var(3X-5)$

Properties of Discrete Random Variables

- **例 3:**

  Flip a fair coin two times. Let X be the number of heads observed. Find

  a) E(X)

  b) Var(X)

  c) St.D.(X)

Properties of Discrete Random Variables

- **例 4:**

  Flip a fair die one time. Let Y be the number of dots facing up. Find

  a) E(Y)

  b) Var(Y)

  c) St.D.(Y)

Properties of Discrete Random Variables

- **例 5:**

  A roulette wheel has the number 1 through 36, as well as 0 and 00. If you bet $1 that an odd number comes up, you win or lose $1 according to whether or not that event occurs. (0 and 00 are considered as the even numbers in the game).

  a) What is the **expected** net gain?

  b) Is this a fair game? Will you play it?

Properties of Discrete Random Variables

- **例 6:**

  Recall the game of **Chuck-a-Luck** in 4.2例4. Let V be the net gain.

  Find the **expected net gain** of this game. b) Is this a fair game?

Remark: If X is a **discrete** R.V., then $E(X) = \mu_X$ is the **balance point** of the **probability function**.

- **Some Standard Probability Distributions**

The following are four useful discrete probability distributions:

– Binomial Probability Distribution（二項分佈）

– Bernoulli Probability Distribution（白努力分佈）

– Hypergeometric Probability Distribution （超幾何分佈）

– Poisson Probability Distribution （波瓦松分佈）

二項分佈
(Binomial Probability Distribution)

The Binomial Probability Distribution (二項分佈)

- **The Binomial Probability Distributions**
  - **Binomial Experiment**

An experiment is called a **binomial experiment** if it satisfies the following four conditions:

1) The experiment consists of **n** independent identical trials.
2) Each trial results in one of two outcomes: Success (S) or Failure (F)
3) The probability of success on a single trial is equal to **p** and remains the same from trial to trial. The probability of a **failure** is **(1-p)** or **q**.
4) We are interested in **X**, the **number of success** observed during the **n** trials.

一個實驗必須滿足以下四個條件，才能稱為二項實驗。

1) 某一實驗獨立、重複的試行n次。
2) 每一試行均產生兩結果：成功(Success)或失敗(Failure)。
3) 每一試行成功的機率均為p，失敗的機率為(1-p)或q。
4) 我們對試行n次中，成功X次之機率有興趣。

The Binomial Probability Distribution (二項分佈)

- **例 1:**

  Flip a coin 10 times, and we are interested in observing number of heads obtained Is this a binomial experiment?

The Binomial Probability Distribution (二項分佈)

- **Binomial Random Variable**
  - Let X be the **total number of successes** in a **binomial experiment** with n trials and probability of success on a single trial **p**. Then **X** is called the **binomial random variable** with parameters **n** and **p**. It can be denoted by $\mathbf{X \sim b(x; n, p)}$.

The Binomial Probability Distribution (二項分佈)

- **The Binomial Probability Distributions**

$p(x) = P(X=x) = C(n, x)p^x q^{n-x}$ for **$X = 0, 1, 2, \ldots, n$.**

where **n** = total number of trials

x = number of **successes** in **n** trials.

C(n, x) = number of arrangements of **x** successes during the **n** trials.

p = probability of **success** on a single trial

q = probability of **failure** on a single trial

在n次獨立的二項實驗試行中，出現x次成功的機率為

$$\mathbf{p(x) = C(n, x)p^x q^{n-x} \qquad x = 0, 1, 2, \ldots, n}$$

其中 n表全部的試行數

x表在n次試行中成功的次數；

C(n, x) 表n次試行中取x次成功次數的組合數；

p表每一試行成功的機率；

q=1-p表每一試行失敗的機率。

The Binomial Probability Distribution (二項分佈)

- **The Binomial Probability Distributions**

  - **Remark**:

    i) When n=1 then X ~ b(x;1, p), X is called a Bernoulli random variable.

    ii) A binomial random variable with parameter n and p is in fact the sum of n Bernoulli R.V.

    i) 當X 服從(n=1，p) 之二項分布，則X 稱為 白努力 (Bernoulli） 隨機變數。

    ii) 一個服從二項分布（n, p）之隨機變數Y是n個白努力隨機變數之和.

The Binomial Probability Distribution (二項分佈)

- **例 2:**

  Suppose a marksman has probability .3 of hitting the bull’s-eye. What is the probability that in 5 trials he will makes

  a) exactly 3 bull’s-eyes?

  b) one or fewer bull’s-eyes?

  c) at least one bull’s-eye?

The Binomial Probability Distribution (二項分佈)

- **例 3:**

  Suppose we have 20 multiple choices problems in final exam. You can pass the course if you answer at least 12 problems correctly. You must pick the single correct answer out of 5 choices for each problem. Assume that you come to the class completely unprepared and simply make a random guess at the correct answer to each problem. What is the probability that you would pass the course?

The Binomial Probability Distribution (二項分佈)

- **例 4:**

  If $X \sim b(x; n, p)$, find $E(X)$ and $Var(X)$.

The Binomial Probability Distribution (二項分佈)

• The Mean and Variance for a Binomial R.V.

The mean and variance of a binomial random variable, X, can be computed by the following formula:

$$E(X) = \mu_x = np$$

$$Var(X) = \sigma_x^2 = npq$$

– **Remark**: If X is a Bernoulli random variable, then E(X) = p , and Var(X) = pq , since n = 1 .

The Binomial Probability Distribution (二項分佈)

- **例 5:**

  The probability of conviction(有罪判決) by jury(陪審團) is .9. Let X be the number of acquittals(無罪判決) on the next three trials.

  a) Give the probability distribution of X.

  b) Find probability of at least two acquittals.

  c) Compute E(X) and Var(X).

The Binomial Probability Distribution (二項分佈)

- **The cumulative distribution function (c.d.f.) for a binomial random variable**

$$F_X(k) = \sum_{j=0}^{k} \binom{n}{j} p^j q^{n-j} ,$$

where k is a number between 0 and n

$$= B(\, x \,;\, n \,,\, p \,)$$

- **Table 3** on pages 744-748 displays the **cumulative Binomial distribution function.**

The Binomial Probability Distribution (二項分佈)

- **例 6:**

  利用Table 3 查出例2及例3之機率.

  例2:

  例3:

利用表求
P(one or fewer bull's-eyes?)=

Table 3 Cumulative Binomial Probabilities

The table shows the probability of x or fewer successes in n independent trials each with probability of success P. For example, the probability of two or less successes in four independent trials, each with probability of success, 0.35 is 0.874.

n	x	P .05	.10	.15	.20	.25	.30	.35	.40	.45	.500
2	0	.902	.81	.722	.64	.562	.49	.422	.36	.302	.25
	1	.998	.99	.978	.96	.937	.91	.877	.84	.797	.75
	2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3	0	.857	.729	.614	.512	.422	.343	.275	.216	.166	.125
	1	.993	.972	.939	.896	.844	.784	.718	.648	.575	.500
	2	1.00	.999	.997	.992	.984	.973	.957	.936	.909	.875
	3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.000
4	0	.815	.656	.522	.41	.316	.24	.179	.13	.092	.062
	1	.986	.948	.89	.819	.738	.652	.563	.475	.391	.312
	2	1.00	.996	.988	.973	.949	.916	.874	.821	.759	.687
	3	1.00	1.00	.999	.998	.996	.992	.985	.974	.959	.937
	4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.000
5	0	.774	.59	.444	.328	.237	.168	.116	.078	.05	.031
	1	.977	.919	.835	.737	.633	.528	.428	.337	.256	.187
	2	.999	.991	.973	.942	.896	.837	.765	.683	.593	.500
	3	1.00	1.00	.998	.993	.984	.969	.946	.913	.869	.812
	4	1.00	1.00	1.00	1.00	.999	.998	.995	.99	.982	.969
	5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.000
6	0	.735	.531	.377	.262	.178	.118	.075	.047	.028	.016
	1	.967	.886	.776	.655	.534	.42	.319	.233	.164	.109
	2	.998	.984	.953	.901	.831	.744	.647	.544	.442	.344
	3	1.00	.999	.994	.983	.962	.93	.883	.821	.745	.656
	4	1.00	1.00	1.00	.998	.995	.989	.978	.959	.931	.891
	5	1.00	1.00	1.00	1.00	1.00	.999	.998	.996	.992	.984
	6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.000
7	0	.698	.478	.321	.21	.133	.082	.049	.028	.015	.008
	1	.956	.85	.717	.577	.445	.329	.234	.159	.102	.062
	2	.996	.974	.926	.852	.756	.647	.532	.42	.316	.227
	3	1.00	.997	.988	.967	.929	.874	.80	.71	.608	.500
	4	1.00	1.00	.999	.995	.987	.971	.944	.904	.847	.773
	5	1.00	1.00	1.00	1.00	.999	.996	.991	.981	.964	.937
	6	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.996	.992
	7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.000

超幾何分佈
(Hypergeometric Probability Distribution)

The Hypergeometric Distribution（超幾何分佈）

- **The Hypergeometric Random Variable**
  1. The experiment consists of randomly drawing n elements **without replacement**（取後不放回）from a set of **N** elements, a of which are **S's** (for Success) and **(N-a)** of which are **F's (for Failure).**
  2. The **hypergeometric** random variable **X** is the **number of S's** in the draw of n elements.

The Hypergeometric Distribution（超幾何分佈）

- **The Hypergeometric Probability Function**

The **probability function** of a hypergeometric random variable **X** is

$$P(x) = P(X=x) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}} \quad \text{For } x = 0, 1, 2, \ldots, a$$

Where

N = total number of elements（群體總數）

a = Number of S's in the N elements（群體中成功的個數）

n = Number of elements drawn（從群體中抽取n個）

x = Number of S's drawn in the n elements（抽取n個中成功的個數）

The Hypergeometric Distribution（超幾何分佈）

- **The Mean and Variance for a Hypergeometric R.V. X:**

$$\mathbf{E(X)=\frac{na}{N}}\ ;$$

$$\mathbf{Var(X)=\frac{na(N-a)(N-n)}{N^2(N-1)}}$$

The Hypergeometric Distribution（超幾何分佈）

- **例 1:**

  Suppose an employer randomly selects three new employees from a total of ten applications, six men and four women. Let X be the number of women who are hired.

  a) Give the probability function of **X**.

  b) Find the probability that **no** women are hired.

  c) Find the probability that women **outnumber** men in the selection.

  d) Compute **E(X)** and **Var(X)**

The Hypergeometric Distribution（超幾何分佈）

- **例 2:**

  A shipment of 100 tape recorders contains 25 that are defective. If 10 of them are randomly chosen for inspection, find the probability that 2 of the 10 will be defective by using

  a) the formula for the hypergeometric distribution;

  b) the formula for the binomial distribution as an approximation.

The Hypergeometric Distribution（超幾何分佈）

– **Note:**

The difference between the two values is only 0.010 . In general, it can be shown that the **hypergeometric** distribution approaches binomial distribution with $\mathbf{p} = a/N$ when $N \rightarrow \infty$, and a good rule of thumb is to use the **binomial** distribution as an approximation to the hypergeometric distribution if $\mathbf{n} \leq \frac{N}{10}$.

波瓦松分佈
(Poisson Probability Distribution)

The Poisson Distribution（波瓦松分佈）

- **The Poisson distribution**
  - is used to describe the **number** of **rare events** which occur in a given unit of time or space.
  - **波瓦松分佈**是用來形容在某一特定時間或面積內稀有事件發生之機率

- **波瓦松隨機變數的一些例子：**
  1) 幾週內保險公司收到的要保信數
  2) 幾分鐘內經過剪票口的旅客數
  3) 一段短時間內經轉接的電話次數
  4) 一段時間內地震發生次數

The Poisson Distribution（波瓦松分佈）

- **Three Assumptions for a Poisson Process:**
  1) In a sufficiently short length of time, say of length $\Delta \mathbf{t}$, only 0 or 1 event can occur (i.e., it is impossible to have two or more simultaneous occurrences in a sufficiently short length of time).

  2) The probability of exactly 1 event occurring in $\Delta \mathbf{t}$ is equal to $\lambda \Delta \mathbf{t}$ (i.e., the probability of exactly 1 event occurring in $\Delta \mathbf{t}$ is proportional to the length of the interval).

  3) Any nonoverlapping intervals of length $\Delta \mathbf{t}$ are independent Bernoulli trials.

The Poisson Distribution（波瓦松分佈）

- **The Poisson Probability Distribution**

Assuming the events occur independent and randomly of one another, and **X** represents the **number of events** occurred during a period of time over which an **average** of **μ** such events can be expected to occur, the **Poisson probability function** is

$$p(x) = \frac{(\lambda t)^x e^{-\lambda t}}{X!} = \frac{\mu^x e^{-\mu}}{X!} \quad \text{for } x = 0, 1, 2, \ldots$$

where **μ** $= \lambda t$ is the mean number of Poisson-distributed events over sampling medium that is being examined and e $= 2.718$.

假設事件是隨機且彼此獨立的發生，單位時間的平均次數為μ，而X表示一段時間事件發生的次數，則波瓦松機率密度函數如下：

$$\mathrm{P(x)} = \frac{(\lambda t)^x e^{-\lambda t}}{x!} = \frac{\mu^x e^{-\mu}}{x!} \quad for \quad x = 0,1,2, \ldots$$

其中，μ＝波瓦松分佈事件在某一特定時間(或面積)內發生的平均數
λ＝單位時間(或面積)內發生的平均數
t ＝特定之時間(或面積)
e ＝2.718

The Poisson Distribution（波瓦松分佈）

- Table 4 in the Appendix (page 749) gives Exponentials for calculating the Poisson Probability.

- **Table of Individual Poisson Probabilities**: See Table 5 in the Appendix (pages 750-758).

- **Table of Cumulative Poisson Probabilities**: See Table 6 in the Appendix (pages 759-767).

The Poisson Distribution（波瓦松分佈）

- **The Mean and Standard Deviation of a Poisson R. V. X:**

$$\mathrm{E(X)} = \mu = \lambda t$$

$$\mathrm{Var(X)} = \sigma^2 = \lambda t$$

$$\mathrm{St.D.(X)} = \sigma = \sqrt{\lambda} \quad .$$

The Poisson Distribution（波瓦松分佈）

- **The Poisson Approximation to the Binomial:**
  (以波瓦松分佈近似二項分佈)
  - The Poisson Probability distribution provides good approximations to the Binomial Probability when n is large and p is small (preferable with $\mathbf{np \le 7}$).

The Poisson Distribution（波瓦松分佈）

- **例 1:**

  The number of customers arriving at a teller's window at Bay Bank is Poisson distributed with a mean rate of .75 person per minute.

  (a) What is the probability that two customers will arrive in the next 6 minutes?

  (b) Use table 5 in Appendix to find the answer in (a).

利用表求

P(that two customers will arrive in the next 6 minutes)=

Table 5 Individual Poisson Probabilities Continued

	Mean Arrival Rate λ									
	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
0	.0166	.0150	.0136	.0123	.0111	.0101	.0091	.0082	.0074	.0067
1	.0679	.0630	.0583	.0540	.0500	.0462	.0427	.0395	.0365	.0337
2	.1393	.1323	.1254	.1188	.1125	.1063	.1005	.0948	.0894	.0842
3	.1904	.1852	.1798	.1743	.1687	.1631	.1574	.1517	.1460	.1404
4	.1951	.1944	.1933	.1917	.1898	.1875	.1849	.1820	.1789	.1755
5	.1600	.1633	.1662	.1687	.1708	.1725	.1738	.1747	.1753	.1755
6	.1093	.1143	.1191	.1237	.1281	.1323	.1362	.1398	.1432	.1462
7	.0640	.0686	.0732	.0778	.0824	.0869	.0914	.0959	.1002	.1044
8	.0328	.0360	.0393	.0428	.0463	.0500	.0537	.0575	.0614	.0653
9	.0150	.0168	.0188	.0209	.0232	.0255	.0281	.0307	.0334	.0363
10	.0061	.0071	.0081	.0092	.0104	.0118	.0132	.0147	.0164	.0181
11	.0023	.0027	.0032	.0037	.0043	.0049	.0056	.0064	.0073	.0082
12	.0008	.0009	.0011	.0013	.0016	.0019	.0022	.0026	.0030	.0034
13	.0002	.0003	.0004	.0005	.0006	.0007	.0008	.0009	.0011	.0013
14	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0003	.0004	.0005

	Mean Arrival Rate λ									
	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
0	.0061	.0055	.0050	.0045	.0041	.0037	.0033	.0030	.0027	.0025
1	.0311	.0287	.0265	.0244	.0225	.0207	.0191	.0176	.0162	.0149
2	.0793	.0746	.0701	.0659	.0618	.0580	.0544	.0509	.0477	.0446
3	.1348	.1293	.1239	.1185	.1133	.1082	.1033	.0985	.0938	.0892
4	.1719	.1681	.1641	.1600	.1558	.1515	.1472	.1428	.1383	.1339
5	.1753	.1748	.1740	.1728	.1714	.1697	.1678	.1656	.1632	.1606
6	.1490	.1515	.1537	.1555	.1571	.1584	.1594	.1601	.1605	.1606
7	.1086	.1125	.1163	.1200	.1234	.1267	.1298	.1326	.1353	.1377
8	.0692	.0731	.0771	.0810	.0849	.0887	.0925	.0962	.0998	.1033
9	.0392	.0423	.0454	.0486	.0519	.0552	.0586	.0620	.0654	.0688
10	.0200	.0220	.0241	.0262	.0285	.0309	.0334	.0359	.0386	.0413
11	.0093	.0104	.0116	.0129	.0143	.0157	.0173	.0190	.0207	.0225
12	.0039	.0045	.0051	.0058	.0065	.0073	.0082	.0092	.0102	.0113
13	.0015	.0018	.0021	.0024	.0028	.0032	.0036	.0041	.0046	.0052
14	.0006	.0007	.0008	.0009	.0011	.0013	.0015	.0017	.0019	.0022

The Poisson Distribution（波瓦松分佈）

- **例 2:**

  The number of bubbles found in place glass windows produced by a process at Glasser Industries is Poisson-distribution with a rate of .004 bubbles per square foot. A 20-by-5-foot plate glass window is about to be installed.

  a) Find the probability that it will have no bubbles in it.

  b) What is the probability that it will have no more than one bubbles in it?

The Poisson Distribution（波瓦松分佈）

- **例 3:**

  An analyst predicted that 3.5% of all small corporations would file for bankruptcy in the coming year. For a random sample of 100 small corporations, estimate the probability that at least 3 will file for bankruptcy in the next year, assuming that the analyst's prediction is right.

  a) Use the binomial distribution to compute the exact probability. (Use table 2 in the Appendix.)

  b) Compute the Poisson approximation.

Jointly Distributed Random Variables

Jointly Distributed Random Variables

- **Joint Probability (Mass) Function for Discrete Random Variables**
  - Suppose that X and Y are discrete random variables defined on the same probability space and that they take on values $x_1, x_2, \ldots$, and $y_1, y_2, \ldots$, respectly. Their joint probability distribution $\mathbf{P(x, y)}$ (X與 Y 之聯合機率分佈) is

$$\mathbf{P(x, y) = P(X = x, Y = y)}$$

Jointly Distributed Random Variables

- **The joint probability distribution must satisfy the following conditions:**

1) $P(x, y) \geq 0 . \forall (x, y) \in R$
2) $\sum_{\text{all } x} \sum_{\text{all } y} P(x, y) = 1 .$

Jointly Distributed Random Variables

- **How to find the joint probability mass function for the discrete X and Y?**
  - Construct a table listing each value that the R.V. **X** and **Y** can assume. Then find $\mathbf{p}(\mathbf{x},\mathbf{y})$ for each combination of $\mathbf{P}(\mathbf{x},\mathbf{y})$.

- **例 1**

  Toss a fair coin 3 times. Let **X** be the number of heads on the first toss and **Y** the total number of heads observed for the three tosses. What is the joint probability function of $(\mathbf{X},\mathbf{Y})$?

Jointly Distributed Random Variables

- **[Ans]**

		Y				P(X=x)
		0	1	2	3	
X	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
P(Y=y)		1/8	3/8	3/8	1/8	1

$$S = \{(HHH), (HHT), \ldots, (TTT)\}$$

Note:
Each entry represents a P(x, y). e.g.,
- P(0, 2) = P(X=0, Y=2) = 1/8
- P(1, 0) = P(X=1, Y=0) = 0

Jointly Distributed Random Variables

Sometimes we are interested in only the probability mass function for X or for Y.
i.e., $P_X(x) = P(X = x)$ or $P_Y(y) = P(Y = y)$

The **marginal functions** can be found, by

$$P_X(x) = P(X = x) = \sum_y P(X = x, Y = y)$$

$$P_Y(y) = P(Y = y) = \sum_x P(X = x, Y = y)$$

- **How to find the marginal probability function (邊際函數) from the joint probability table?**
  - To find $P_Y(y)$, sum down the appropriate column of the table.
  - To find $P_X(x)$, sum across the appropriate row of the table.
  - **Note:** Since $P_Y(y)$ and $P_X(x)$ are located in the row and column **"margins"**, these distributions are called **marginal probability functions.**

Jointly Distributed Random Variables

- **例 2 :**

In 例 1, find the marginal probability functions for X and Y.

		Y				P(X=x)
		0	1	2	3	
X	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
P(Y=y)		1/8	3/8	3/8	1/8	1

[Ans] $P_X(x) = \frac{4}{8}$, for $x = 0,1$

$$P_Y(0) = P(Y = 0) = \frac{1}{8}$$

$$P_Y(1) = P(Y = 1) = \frac{3}{8}$$

$$P_Y(2) = P(Y = 2) = \frac{3}{8}$$

$$P_Y(3) = P(Y = 3) = \frac{1}{8}$$

Jointly Distributed Random Variables

- **例 3:**

  Sally Peterson, a marketing analyst, has been asked to develop a probability model for the relationship between the sale of luxury cookware and age group. This model will be important for developing a marketing campaign for a new line of chef-grade cookware. She believes that purchasing patterns for luxury cookware are different for different age groups. find the marginal probability functions for X and Y.

Jointly Distributed Random Variables

- **[Ans]**

The marginal probability functions for X and Y is

Purchase Decision(Y)	Age Group (X)			
	1 (16 to 25)	2 (26 to 45)	3 (46 to 65)	P(Y)
1 (buy)	0.10	0.20	0.10	0.40
2(not buy)	0.25	0.25	0.10	0.60
P(x)	0.35	0.45	0.20	1.00

$P_X(1) = P(X = 1) = 0.35$ $\qquad$ $P_Y(1) = P(Y = 1) = 0.40$

$P_X(2) = P(X = 2) = 0.45$ $\qquad$ $P_Y(2) = P(Y = 2) = 0.60$

$P_X(3) = P(X = 3) = 0.20$

- **The Expected values of Function of Two Random Variables**
  - **Recall:** The *expected value* of a Random variable is

$$E(X) = \sum_{x} x \cdot P(x) \quad \text{if X is a discreate random variable.}$$

  - **Remark:** Let g(x) is a function of a random variable X. Then

$$E[g(x)] = \sum_{x} g(x) \cdot P(x) \quad \text{if X is a } \textbf{discreate} \text{ random variable.}$$

Jointly Distributed Random Variables

- **The Expected values of Function of Two Random Variables**
  - Let g(x, y) be a function of random variables X and Y. Then the expected value (or mean) of g(x, y) is defined to be

$$\boxed{E[g(x,y)]=\sum_y \sum_x g(x,y) \cdot P(x,y) \quad \text{if X and Y are } \textbf{discrete}}$$

- **The Variance of Function of Two Random Variables**
  - Let g(x, y) be a function of random variables X and Y. Then the variance of g(x, y) is defined to be

$$\mathrm{Var}[g(x,y)]=\sum_{y}\sum_{x}[g(x,y)]^{2}P(x,y)-(E[g(x,y)])^{2}$$

if X and Y are **discrete**

Jointly Distributed Random Variables

- **例 4:**

  Suppose that Charlotte King has two stocks, A and B. Let X and Y be random variables of possible percent returns (0%, 5%, 10%, and 15%) for each of these two stocks, with the joint probability distribution given in table.

	Y return			
X return	0%	5%	10%	15%
0%	0.0625	0.0625	0.0625	0.0625
5%	0.0625	0.0625	0.0625	0.0625
10%	0.0625	0.0625	0.0625	0.0625
15%	0.0625	0.0625	0.0625	0.0625

  a) Find the marginal probabilities.

  b) Find the means and variances of both X and Y.

Jointly Distributed Random Variables

- **[Ans]**

a)

	Y return			
X return	0%	5%	10%	15%
0%	0.0625	0.0625	0.0625	0.0625
5%	0.0625	0.0625	0.0625	0.0625
10%	0.0625	0.0625	0.0625	0.0625
15%	0.0625	0.0625	0.0625	0.0625

$P_X(1) = P(X = 0\%) = 0.25$ $\quad$ $P_Y(1) = P(Y = 0\%) = 0.25$

$P_X(2) = P(X = 5\%) = 0.25$ $\quad$ $P_Y(2) = P(Y = 5\%) = 0.25$

$P_X(3) = P(X = 10\%) = 0.25$ $\quad$ $P_Y(3) = P(Y = 10\%) = 0.25$

$P_X(4) = P(X = 15\%) = 0.25$ $\quad$ $P_Y(4) = P(Y = 15\%) = 0.25$

Jointly Distributed Random Variables

- **[Ans]**

  b) The **mean** for X is as follows:

$$\mu_x = \sum_x xP(x) = 0(0.25) + 0.5(0.25) + 0.1(0.25) + 0.15(0.25)$$
$$= 0.075$$

Similarly, the mean of Y is $\mu_y = 0.075$

The **variance** of X is

$$\sigma_x^2 = \sum_x (x - \mu_x)^2 P(x) = P(X) \sum_x (x - \mu_x)^2$$
$$= (0.25)[(0 - 0.075)^2 + \cdots + (0.15 - 0.075)^2]$$
$$= 0.003125$$
$$\sigma_X = \sqrt{0.003125} = 0.0559$$

Similarly, the variance of Y is $\sigma_y = 0.0559$

Independence and conditional Distributions

– **Recall**: Measures of Relationship between Variables

兩變數間之關聯性指標

– Two measures of association between two random variables衡量兩變數間關聯性之指標有二：

1. Covariance（共變異數）

$$\mathbf{Cov(X, Y)} = E[(X-\mu_X)(Y-\mu_Y)] = E(XY)-\mu_X\mu_Y$$
$$= \sum_x \sum_y xy\, P(x, y) - \mu_x\, \mu_y$$

2. Correlation (相關係數)

$$\boldsymbol{\rho} = \frac{\mathbf{Cov(X, Y)}}{\boldsymbol{\sigma_X \sigma_Y}}, \qquad \text{provide that } \sigma_X < \infty \text{ } and \text{ } \sigma_Y < \infty$$

Independence and conditional Distributions

- **Theorem**
  - If X and Y are **independent** , then **Cov(X,Y)=0**

Proof: If X and Y are independent ,then we know $E(XY)= E[X] \cdot E[Y]$

$Cov(X,Y)=E(XY)-E(X)E(Y)= E(XY)-E(XY)=0$

- **Question**
  - If Cov(X,Y)=0, X,Y are independent?? **NO!!**
- **Remark**
  - If Cov(X,Y)=0, X and Y may <u>**not**</u> be independent.

Jointly Distributed Random Variables

- **例 5:**

  Suppose that Charlotte King has two stocks, A and B. Let X and Y be random variables of possible percent returns (0%, 5%, 10%, and 15%) for each of these two stocks, with the joint probability distribution given in table.

	Y return			
X return	0%	5%	10%	15%
0%	0.0625	0.0625	0.0625	0.0625
5%	0.0625	0.0625	0.0625	0.0625
10%	0.0625	0.0625	0.0625	0.0625
15%	0.0625	0.0625	0.0625	0.0625

  Determine if X and Y are independent.

Jointly Distributed Random Variables

- **[Ans]**

  To test for independence, check $P(x, y) = P(x)P(y)$ for all possible pairs of values x and y.

  $P(x, y) = 0.0625$ for all possible value

  $P(x) = 0.25$, $P(y) = 0.25$ for all possible value

  $P(x, y) = 0.0625 = P(x)P(y)$

  Therefore, X and Y are independent.

Independence and conditional Distributions

- **Correlation (相關係數)**
  - **Note:**

$$-1 \leq \boldsymbol{\rho} \leq \mathbf{1}$$

X and Y are said to be positively linearly correlated if $\boldsymbol{\rho} > 0$.

X and Y are said to be negatively linearly correlated if $\boldsymbol{\rho} < 0$.

X and Y are said to have no linear correlation if $\boldsymbol{\rho} = \mathbf{0}$.

Independence and conditional Distributions

- **例 6：**
  - Let X,Y be discrete random variable with P(X=x, Y=y) as follows:

X \ Y	-1	0	1	$P_X(x)$
-1	1/8	1/8	1/8	3/8
0	1/8	0	1/8	2/8
1	1/8	1/8	1/8	3/8
$P_Y(y)$	3/8	2/8	3/8	1

a) Find Cov(X,Y).

b) Find the Correlation Coefficient of X and Y.

c) Are X and Y independent?

Independence and conditional Distributions

- **[Ans]**

a)
$$E(X) = \sum_{x} x \cdot P_X(x) = (-1)(3/8) + 0(2/8) + 1(3/8) = 0$$

$$E(Y) = \sum_{y} y \cdot P_Y(y) = (-1)(3/8) + 0(2/8) + 1(3/8) = 0$$

$$E(XY) = \sum_{x} \sum_{y} xy\, P(X = x, Y = y)$$

$=(-1)(-1)(1/8)+0(-1)(1/8)+(1)(-1)(1/8)+(-1)(0)(1/8)+0 \cdot 0 \cdot 0+1 \cdot 0(1/8)+ (-1)(1)(1/8)$

$+0(1)(1/8)+1(1)(1/8)=1/8-1/8-1/8+1/8=0$

$\therefore \mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=0-0 \cdot 0=0$

b) $$\rho = \frac{\mathrm{Cov}(x,y)}{\sigma_x \sigma_y} = 0$$

Independence and conditional Distributions

- **[Ans]**

c)

But $P(X=0, Y=0) \overset{?}{=} P(X=0)\,P(Y=0)$?

$$0 \overset{?}{=} 2/8 \cdot 2/8 \implies P(X=0, Y=0) \neq P(X=0)\,P(Y=0)$$

$\therefore$ NO! X,Y are <u>**not**</u> independent.

Independence and conditional Distributions

- **Conditional Probability Function for Discrete Random Variables**
  - The *conditional probability distributions for X given Y and Y given X* are denoted by :

$$\mathbf{P_1(x \mid y) = P(x, y)/P_2(y)}$$

$$\mathbf{P_2(y \mid x) = P(x, y)/P_1(x)}$$

Independence and conditional Distributions

- **例 7:**
  - Let X,Y be discrete random variable with P(X=x, Y=y) as follows, find the conditional probability function of X given Y=1.

X \ Y	0	1	2	3	$P_X(x)$
0	1/8	2/8	1/8	0	
1	0	1/8	2/8	1/8	
		3/8			

[Ans]

$$P(X=0|Y=1)=\frac{P(X=0,Y=1)}{P(Y=1)}=\frac{(\frac{2}{8})}{(\frac{3}{8})}=\frac{2}{3}$$

$$P(X=1|Y=1)=\frac{P(X=1,Y=1)}{P(Y=1)}=\frac{(\frac{1}{8})}{(\frac{3}{8})}=\frac{1}{3}$$

- **The Conditional Mean and Variance for Discrete Random Variables**

$$\mu_{Y|X} = E[Y|X] = \sum_Y (y|x)\, P(y|x)$$

$$\sigma^2{}_{Y|X} = E\left[(Y - \mu_{Y|X})^2 \middle| X\right] = \sum_Y \left(Y - \mu_{Y|X}\right)^2 |x)\, P(y|x)$$

$$= \sum_Y (Y^2|x)\; P(y|x) - \mu_{Y|X}{}^2$$

Independence and conditional Distributions

- **例 8:**

In 例 1 , find the expected value of Y given that x = 1

		Y				P(X=x)
		0	1	2	3	
X	0	1/8	2/8	1/8	0	4/8
	1	0	1/8	2/8	1/8	4/8
P(Y=y)		1/8	3/8	3/8	1/8	1

- **[Ans]**

$$E[Y|x=1] = \sum_{Y} (y|x=1)P(y|x=1)$$

$$= 0(0) + 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{4}\right) = 2$$

Jointly Continuous Random Variables

兩個隨機變數之和的平均數與變異數公式

$E(X_1+X_2) = \boldsymbol{\mu_1 + \mu_2}$

$Var(X_1+X_2) = \boldsymbol{\sigma_1^2 + \sigma_2^2 + 2\mathrm{Cov}(x_1, x_2)}$

兩個隨機變數之差的平均數與變異數公式:

$E(X_1-X_2) = \boldsymbol{\mu_1 - \mu_2}$

$Var(X_1-X_2) = \boldsymbol{\sigma_1^2 + \sigma_2^2 - 2\mathrm{Cov}(x_1, x_2)}$

Note: If X_1 and X_2 are uncorrelated or independent, $\boldsymbol{\mathrm{Cov}(x_1, x_2)} = 0$

Jointly Continuous Random Variables

- 兩個隨機變數X與 Y線性組合的平均數與變異數公式:

Let W= aX + bY, where a and b are constants, then

$$\mu_W = a\mu_X + b\mu_Y$$

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\ \mathrm{Cov}(X, Y)$$

$$= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\ \rho\ \sigma_X\ \sigma_Y$$

Let W=aX - bY, where a and b are constants, then

$$\mu_W = a\mu_X - b\mu_Y$$

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 - 2ab\ \mathrm{Cov}(X, Y)$$

$$= a^2\sigma_X^2 + b^2\sigma_Y^2 - 2ab\ \rho\ \sigma_X\ \sigma_Y$$

Linear Function of Random Variables

- **例 9:**

  An investor has \$1,000 to invest and two investment opportunities, each requiring a minimum of \$500. The profit per \$100 from the first can be represented by a random variable X, having the following probability functions:

  $$P(X = -5) = 0.4 \text{ and } P(X = 20) = 0.6$$

  The profit per \$100 from the second is given by the random variable Y, having the following probability functions:

  $$P(Y = 0) = 0.6 \text{ and } P(Y = 25) = 0.4$$

  X, Y are independent. The investor has the following possible strategies:

  a. \$1,000 in the first investment

  b. \$1,000 in the second investment

  c. \$500 in each investment

  Find the mean and variance of the profit from each strategy.

Linear Function of Random Variables

- **[Ans]**

a) $\mu_x = E(X) = (-5)(0.4) + (20)(0.6) = \$10$

profit mean= E(10X) = 10E(X) = $100

$$\sigma_x^2 = \sum_x (x - \mu_x)^2 P(x) = (-5 - 10)^2(0.4) + (20 - 10)^2(0.6) = 150$$

profit variance =Var(10X) = 100Var(X) =15,000

b) $\mu_y = E(Y) = (0)(0.6) + (25)(0.4) = \$10$

mean profit = E(10Y) = 10E(Y) = $100

$$\sigma_y^2 = \sum_y (y - \mu_y)^2 P(y) = (0 - 10)^2(0.6) + (25 - 10)^2(0.4) = 150$$

profit variance =Var(10Y) = 100Var(Y) =15,000

Linear Function of Random Variables

- **[Ans]**

c) profit mean

$$E(5X+5Y)=E(5X)+E(5Y)=5[E(X)+E(Y)]=\$100$$

profit variance =

$Var(5X+5Y) = Var(5X)+Var(5Y) = 25Var(X) + 25Var(Y) =7{,}500$

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