統計學（一）

第三章 機率
(Elements of Chance: Probability Methods)

授課教師：唐麗英教授

國立交通大學 工業工程與管理學系
聯絡電話：(03)5731896
e-mail：litong@cc.nctu.edu.tw

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本課程內容參考書目

- **教科書**
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- **參考書目**
  - Berenson, M. L., Levine, D. M., and Krehbiel, T. C. (2009). *Basic business statistics: Concepts and applications*, 11^{th} EditionPrentice Hall.
  - Larson, H. J. (1982). *Introduction to probability theory and statistical inference*, 3^{rd} Edition, New York: Wiley.
  - Miller, I., Freund, J. E., and Johnson, R. A. (2000). *Miller and Freund's Probability and statistics for engineers*, 6^{th} Edition, Prentice Hall.
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  - 唐麗英、王春和（2008），「SPSS 統計分析 」，儒林圖書公司。
  - 唐麗英、王春和（2007），「Excel 統計分析」，第二版，儒林圖書公司。
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Probability (機率)

- **Experiment (實驗)**
  - An **experiment** is the **process** by which an observation or measurement is recorded.
  - 實驗是指一個可記錄一些觀察體資料的**過程**

例 1:

- 1)
- 2)
- 3) Noting whether a switch is turned on or off.
- 4) Selecting a lot of 500 products and finding out how many of them are defective.

Probability (機率)

- **Sample Space (S) (樣本空間)**
  - The **collection** of ***all*** **possible outcomes** is called the ________ _______ of the experiment.
  - 一個實驗的所有可能出現的結果之集合稱為樣本空間。

例 2:

1) 擲一個骰子一次，請列出樣本空間。
2) 丟一個銅板兩次，請列出樣本空間。

- **Event (事件)**
  - An event is an outcome (結果) of the experiment.
  - **事件**為實驗的一個結果。

  - **例 3:**
    1) 擲一個骰子一次，令事件A表看到的點數為奇數，請列出事件A包含之元素。
    2) 丟一個銅板兩次，令事件B表看到兩個正面，請列出事件B包含之元素。

Probability (機率)

- **Simple Event (簡單事件)**
  - A simple event is an event that **cannot** be decomposed into two or more other events.

- **Compound Event (Joint event) (複合事件)**
  - A **compound event** is an event that **can** be decomposed into two or more other events.

Probability (機率)

- **Probability of an event A:**
  - $\mathbf{P(A)} = \frac{\#(\mathbf{A})}{\#(\mathbf{S})}$ where #(A) = # of element in event A

    #(S) = # of element in the sample space S

- **The Axioms of Probability**
  - ___≤ P(A) ≤ ___ for each event A in S.
  - P(∅) = ___, P(S) =___.
  - If A1, A2, …,Ak are mutually exclusive events, then
  - P(A1∪A2∪…∪Ak) = ______________________________

Probability (機率)

- **例 4:**

  Toss a die(骰子) once and observe the number appearing on the upper face.

  Some events would be:

  Event A: observe an odd number. Event B: observe an event number.

  Event E1: observe a number '1'. Event E4: observe a number '4'.

  Event E2: observe a number '2'. Event E5: observe a number '5'.

  Event E3: observe a number '3'. Event E6: observe a number '6'.

  – Experiment:
  – Sample Space:
  – Simple Event:
  – Compound event:
  – Probability of getting an odd number = P( ) =
  – Probability of getting a number '6' = P( ) =
  – Probability of getting a number < '3' = P( ) =

Probability (機率)

- **例 5:**

  Consider the experiment of tossing two coins. Assume both coins are fair. List all possible outcomes in the sample space. Consider the events

  A: { observe exactly one head }

  B: { observe at least one head }

  Find P(A), P(B)

- **例 6:**

  Drawing a card from a deck of 52 cards, what is the probability of getting

  a) a spade

  b) an ace

  c) a spade ace ?

Probability (機率)

- **例 7:**

  What is the probability of having exactly one girl in a two-child family? Assume equal chance of getting a boy or a girl.

Probability (機率)

- **例 8:**

  In a three-child family, what is the probability of having

  a)3 boys?

  b) 2 boys and a girl-in that order?

  c) 2 boys and a girl-in any order?

Probability (機率)

- **例 9:**

  (a) If a pair of dice are rolled, what is the probability that both dice show the same face ?

  (b) If three dice are rolled, what is the probability that all dice show different face ?

Counting Principles

• Counting Principles

Computing a probability by the formula, $P(A) = \frac{\#(A)}{\#(S)}$

has involved the counting of the number of outcome in event A and the total number of outcome in the sample space. However, in many cases, because of the large number of possible outcome that may occur, it is not feasible to list all the possible outcomes, for 例, drawing two cards from a deck of 52 cards, how would we determine the number of different possible outcome?

– For these type of cases, we may apply the following counting rules

Counting Principles

- **Counting Rules**
  - The Multiplication Principle (乘法原理)
  - Permutations (排列)
  - Combinations (組合)

The Multiplication Principle (乘法原理)

- **The Multiplication Principle**
  - Suppose that an act requires n steps to complete. If the steps can be performed successively in m_1, m_2, m_3, … , m_n ways, then the act can be performed in $(m_1 \cdot m_2 \cdot m_3 \cdot \ldots \cdot m_n)$ different ways.

The Multiplication Principle (乘法原理)

- **例 1 :**

  How many different two-letter “words” can be made up from the four letters A, B, C, D? The “word” may not make sense.

  a) If a letter may be repeated.

  b) If a letter may not be repeated.

  - ANS : 4*4=16
  - ANS: 4*3=12

The Multiplication Principle (乘法原理)

- **例 2：**

  How many different license plates are these consisting of

  a) five digits?

  b) three letters followed by three digits?

  - ANS: 10^5
  - ANS: $26^3 \cdot 10^3$

- **例 3 :**

  How many eight-digit telephone numbers are possible if zero cannot be used as the first digit?

  - ANS: $9 \cdot 10^7$

The Multiplication Principle (乘法原理)

- **例 4：**

  In how many different ways can a true-false test consisting of 9 questions be answered?

  - ANS: 2^9

Permutations (排列)

- **Factorial(階層)**
  - The product of n positive integers is called “ n factorial”, written n!.

例: a) 5!=120

b) 4!=24

c) 10!=3628800

  - **Note :** (1) $0! \equiv 1$ (2) $n!=n(n-1)!$

Permutations (排列)

- **Permutations**

1) The number of ways of selecting r objects from n distinct objects in order is

$$P(n,r) = \frac{n!}{(n-r)!} \qquad 1 \le r \le n$$

2) The number of ways of arranging n distinct objects in order is

$$P(n,n) = \frac{n!}{(n-n)!} = n!$$

– **Note :** For permutations: The order is important !!!

e.g. AB and BA are two distinct permutations

Permutations (排列)

- **例 5 :**

The number of permutations of the digits 0-9 is

ANS: 10!=3,628,800

Permutations (排列)

- **例 6 :**

  A woman getting dressed up for a night out is asked by her significant other to wear a red dress, high-heeled sneakers, and a wig. In how many different order can she put on these objects?

  ANS: P(3,3)=3!=6

Permutations (排列)

- **例 7：**

  If there are 7 horses in a race, in how many ways can three from among them finish first, second, and third?

  ANS: $P(7,3) = \frac{7!}{(7-3)!} = 7 \cdot 6 \cdot 5 = 210$

Permutations (排列)

- **例 8：**

  12 dancers take their bows at the end of their act. They are all standing in a row. How many total arrangements are possible?

  ANS: $12!=479{,}001{,}600$

Permutations (排列)

- **例 9 :**

  In how many ways can 5 different trees be planted in a circle?

  ANS: (5-1)!=4!=24

Theorem : The number of permutations of distinct objects arranged in a <u>circle</u> is (n-1)!

- **Combinations**
  - The number of ways of selecting r objects from n distinct objects, irrespective of order, is
  - $\binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$ $\qquad 1 \leq r \leq n$
  - **Note** : For combination: The order is not important!!!

Combinations (組合)

- **例 10：**

  What is the number of combinations of 52 cards taken two at a time?

ANS: $\binom{52}{2} = \frac{52!}{2! \cdot 50!} = \frac{52 \cdot 51}{1 \cdot 2} = 1{,}326$

Combinations (組合)

- **例 11：**

  How many <u>five-card</u> hands can be dealt from a deck of 52-cards?

$$\text{ANS: } \binom{52}{5} = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

Combinations (組合)

- **例 12 :**
  a) In how many ways can a chairperson and secretary be selected from a committee of 10 people?

  b) In how many ways can a subcommittee of two people be selected from a committee of 10 people?

ANS: $P(10,2) = \frac{10!}{8!} = 10 \cdot 9 = 90$

ANS: $C(10,2) = \frac{10 \cdot 9}{1 \cdot 2} = 45$

Combinations (組合)

- **例 13 :**

  In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

ANS: $\#(S)=\binom{52}{5}=2{,}598{,}960$

A=2 Aces, 3 jacks

$$\#(A) = \binom{4}{2}\binom{4}{3} = \frac{4\cdot 3}{1\cdot 2}\cdot 4 = 24$$

$$\therefore P(A) = \frac{24}{2{,}598{,}960} = 0.9\cdot 10^{-5}$$

Probability Laws

Probability Laws

- **Three possible relations between events A and B:**
  1. **Dependent(相依)** – A and B are *dependent* events if the occurrence of A dependents on the occurrence of B or vise versa.
  2. **Independent (獨立)** – The occurrence of A has no effect on the occurrence of B.
  3. **Mutually Exclusive(互斥)** – A and B are *M.E.* if there is no outcome that belongs to (M.E.) both of them.

Probability Laws

- **例 1:**

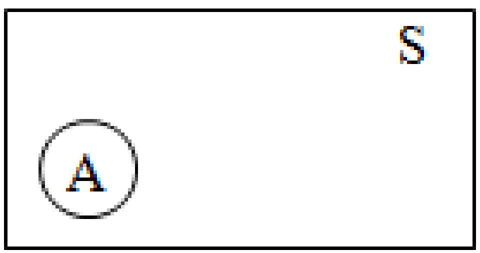
  Suppose that 兄弟隊 have a baseball game with 三商隊. After 7 innings(局), the possible outcomes are: { win, lose, tie }.

  Let A = { win }, B = { lose } and C = { tie }.

  What is the relationship among events A, B and C ?

Probability Laws

- **Venn Diagram:**
  - Venn Diagram is a convenient pictorial representation of sets and probabilities.



$$P(S) = 1$$

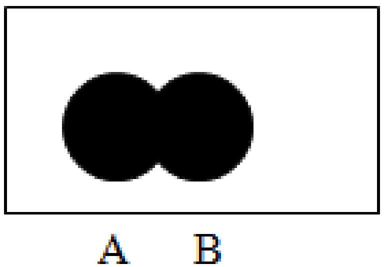
$$0 \le P(A) \le 1$$

Probability Laws

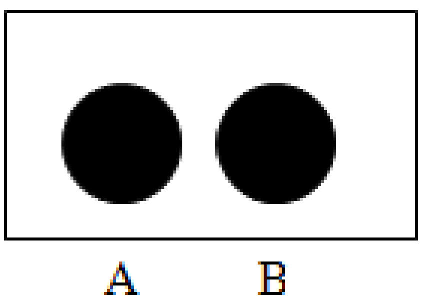
- **Union (聯集)**
  - The union of two events A and B is the event that occurs if either A or B or both occur on a single performance of the experiment. We denote the union of events A and B by the symbol “**A ∪ B**“.

Probability Laws

- **例 2:**
  - Venn Diagram for **A** ∪ **B**



A, B are independent or dependent.



A, B are mutually exclusive.

Probability Laws

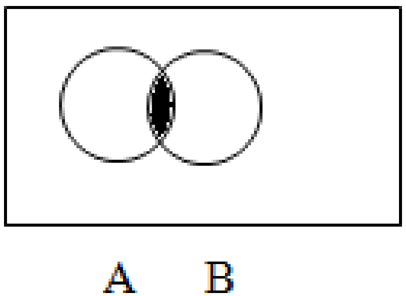
- **例 3:**

  Draw a card from a deck of 52. Event "Black <u>or</u> Ace" will include all the cards that are Black or Ace or were both.

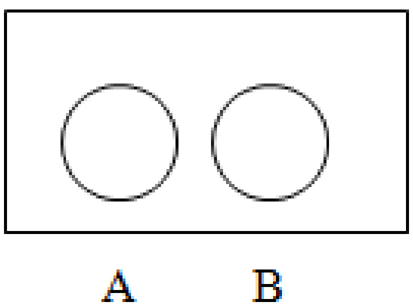
Probability Laws

- **Intersection (交集)**
  - The **intersection** of two events **A** and **B** is the event that occurs if **both A and B occur** on a single performance of the experiment. We denote the intersection of events A and B by the symbol "$\mathbf{A \cap B}$".

- **例 4:**
  - Venn Diagram for **A ∩ B**



A, B are independent or dependent.



A, B are mutually exclusive.

- **例 5:**

  Event “Black and Ace” will include the cards { Black, Ace }.

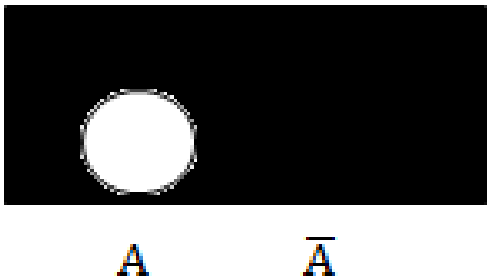
Probability Laws

- **Complementary Event (互補)**
  - The **complement** of any event A is the event that A **does not** occur. (That is, the complement of an event A is the collection of all sample points that are in the sample space but **not** in A). We denote the complement of A by the symbol "$\overline{A}$" or **A'** .

Probability Laws

- **例 6:**

  Venn Diagram for A



– Note: $\mathbf{P(A) + P(\bar{A}) = P(S) = 1}$

$$\mathbf{P(A) = 1 - P(\bar{A})}$$

$$\mathbf{P(\bar{A}) = 1 - P(A)}$$

Probability Laws

- **例 7:**

  If S = {1, 2, 3, 4, 5, 6, 7, 8}, A={1, 2, 3, 4}, B={3, 4, 5, 6}, then

  - $\mathrm{A} \cup \mathrm{B} = \mathrm{A\ or\ B} = \{1, 2, 3, 4, 5, 6\}$
  - $\mathrm{A} \cap \mathrm{B} = \mathrm{A\ andB} = \{3,4\}$
  - $\overline{\mathrm{A}} = \{5, 6, 7, 8\}$
  - $\overline{\mathrm{B}} = \{1, 2, 7, 8\}$

Probability Laws

- **例 8:**

  Toss a fair die one time.

  Define A = { Toss an even number } = { 2, 4, 6 }

  B = { Toss a number less than or equal to 3 }

  = { 1, 2, 3}

  Describe $A \cup B$ for this experiment. $A \cup B$ ={ 1, 2, 3 ,4, 6 }

  Describe $A \cap B$ for this experiment. $A \cap B$ = { 2 }

  Describe $\overline{A}$ for this experiment. $\overline{A} = \{1,3,5\}$

  Describe $\overline{B}$ for this experiment. $\overline{B} = \{4, 5, 6\}$

  Calculate $P(A \cup B)$, $P(A \cap B)$, $P(A)$, and $P(\overline{B})$.

Probability Laws

- **例 9:**

  If three dice are rolled one time, what is the probability that three numbers that turn up <u>different</u> ?

ANS: Experiment : toss three dice

$S = \{ (1,1,1), \ldots, (6,6,6) \}$ $\#(S) = 6^3 = 216$

$P(\text{three \#S all different}) = P(A) = 1 - P(\bar{A}) = 1 - \frac{210}{216} = 0.972$

$\bar{A}$ = three #S all the same = $\{ (1,1,1), (2,2,2), \ldots, (6,6,6) \}$

$P(\bar{A}) = \frac{6}{216}$

- **Conditional Probability (條件機率)**
  - The conditional probability of A occurring given that B has occurred, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \text{ and } B)}{P(B)} \qquad \text{where } P(B) > 0$$

  - **Note**: The equation of conditional probability can be written as

$$p(A \cap B) = P(B) * P(A|B)$$
$$p(A \cap B) = P(A) * P(B|A)$$

Probability Laws

- **例 10:**

  Drawing a card from a deck of 52 cards. What is the probability of getting an ace given that the card is black ?

ANS: $P(\text{Ace} \mid \text{Black}) = \dfrac{P(\text{Ace or Black})}{P(\text{Black})} = \dfrac{2/52}{26/52} = \dfrac{1}{13}$

Probability Laws

- **例 11:**

  Toss a fair die one time and the following events are defined:

  A = { observe an odd number }

  B = { observe a number less than 4}

  Find P(A|B) and P(B|A).

  ANS: $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{2/6}{3/6} = \dfrac{2}{3}$

  ANS: $P(B|A) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{2/6}{3/6} = \dfrac{2}{3}$

  – **Note**: Not always P(A|B) = P(B|A)

Probability Laws

- **Independent events**
  1) Events A and B are independent if $P(A|B) = P(A)$ or $P(B|A) = P(B)$
  2) Equivalently, Events A and B are independent *if and only if* $P(A \cap B)=P(A)P(B)$.

Probability Laws

- **例 12:**

  If P(A) = 0.40, P(B) =0.30 and P(A and B) = 0.20.

  Are A and B independent ? Why ?

Probability Laws

- **Mutually exclusive events**
  1) Events A and B are M.E. if $P(A|B) = 0$ or $P(B|A) = 0$
  2) Equivalently, Events A and B are M.E. *if and only if* $P(A \cap B) = 0$

Probability Laws

- **例 13:**

  Two dice are tossed and the following events are defined:

  A = {sum of the two numbers showing is an **odd** number }

  B = {sum of the two numbers showing is **8, 9** or **12** }.

  Are events A and B independent ? Why ? If not, what is the relation between A and B ?

Probability Laws

- **例 14:**

  Suppose A and B are independent events with P(A) = 0.4 and P(B) = 0.5. What is the probability that

  a) **both** occur?

  b) **at least one** occur?

  c) **exactly one** occur?

  d) A occurs **only**?

  e) B occurs **only**?

  f) neither occur?

- **例 15:**

  A survey of students at 交通大學 showed that 10% smoke cigarette, 30% drink tea, and 5% both smoke cigarette and drink tea. What percentage of these students

  a) **neither** smoke **nor** drink tea?

  b) smoke cigarette **only**?

  c) drink tea **only**?

  d) Are ‘smoking cigarette’ and ‘drinking tea’ **independent**? Why?

Probability Laws

- **Addition Rule:**

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  - **Note**:

    If A and B are M.E., then $P(A \cup B) = P(A) + P(B)$

    If A and B are independent, then

    $P(A \cup B) = P(A) + P(B) - P(A)*P(B)$

- **例 16:**

  Drawing a card from a deck of 52 cards.

  Let event $A = \{Ace\}$

  event $B = \{Black\}$

  event $H = \{Heart\}$

  event $S = \{Spade\}$

  Find a) $P(A \cap B)$ b) $P(A \cup B)$

  c) $P(H \cap S)$ d) $P(H \cup S)$

貝氏定理
(Bays' Theorem)

Bays' Theorem(貝氏定理)

- **Bays' Theorem:**
  - An application of the conditional probability.

$$P(A \mid B) = \frac{P(A \cap B)}{B(B)} = \frac{P(B \mid \mathbf{A}) \cdot P(A)}{P(B \mid \mathbf{A}) \cdot P(A) + P(B \mid \bar{A}) \cdot P(\bar{A})}$$

  - **Note**:

$P(B) = P(B \mid \mathbf{A}) \cdot P(A) + P(B \mid \bar{A}) \cdot P(\bar{A})$ --This is called "total probability"

Bays' Theorem(貝氏定理)

- **例 1:**

The chance a patient in a hospital has a certain **disease** is **0.05**. He is given a test to determine whether he has the disease. If (**given that**) he has the disease , the chance is **0.99** that the test will be **positive**; If (**given that**) he does **not** have the disease, the chance is **0.03** that the test will be positive. What is the probability that

a) the patient **has** the disease **given that** the test is positive?

b) the patient **dose not** have the disease **given that** the test is **negative**?

Bays' Theorem(貝氏定理)

- **例 2:**

  Machines I and II respectively produce **30%** and **70%** of a factory output. Machines **I** produces **3% defectives** and machine **II** produces **4% defectives**. If **(Given that)** an item is **defective**, what is the chance it was produced by machine I?

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