Note. General Ito's integral.

$E\left[\int_0^t X(s)\,dW(s)\right] = 0$

$\mathrm{var}\left[\int_0^t X(s)\,dW(s)\right] = \int_0^t E[X^2(s)]\,ds$

Lecture 4

Black-Scholes-Merton評價模型

戴天時

授課大綱

- 回顧: Ito's lemma
- 模型假設以及避險組合的建立
- 選擇權的評價偏微分方程式（The Black-Scholes PDE）
- 歐式選擇權的評價模型推導
- 避險參數:
- 考慮現金股息下的股票選擇權模型
  - 連續付息
  - 離散付息

回顧　Ito's Lemma

- f(t,X(t))的Ito's lemma 表示如下:

$$df(t,X(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f(X(t))}{\partial X(t)}dX(t) + \frac{1}{2}\frac{\partial^2 f(X(t))}{\partial X(t)^2}(dX(t))^2$$

- 範例:考慮標的物的遠期價格　$f(S,t) \equiv Se^{r(T-t)}$
  - 標的物的隨機過程: $dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$

$$df(S,t) = -rSe^{r(T-t)}dt + e^{r(T-t)}dS(t) + \frac{1}{2}\times 0(dS(t))^2$$

$$= -rf(S,t)dt + e^{r(T-t)}(\mu S(t)dt + \sigma S(t)dB(t))$$

$$= (\mu - r)f(S,t)dt + \sigma f(S,t)dB(t)$$

under risk-neutral measure, $\mu = r$ s.t.

Let $f(t) = f(0)\cdot e^{X(t)}$ $\Rightarrow$ $d\ln f(t) = \frac{1}{f}df + \frac{1}{2}\frac{-1}{f^2}df^2 = dX(t)$

$dX(t) = \sigma\cdot dB(t) - \frac{1}{2}\sigma^2 dt$ …… 報酬率 sp.

under neutral risk measure, $\mu = r \cdot a$

$X_t \cdot B(0,t)$ is a martingale $\Rightarrow$ $V(t)B(0,t) = \left[a(t)\cdot X_t + \frac{b(t)}{B(0,t)}\right]\cdot B(0,t)$ is martingale as well.

市場模型假設

- 假定標的物的隨機過程: $dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$
  - μ　代表報酬率,(在真實市場中不爲無風險利率) 在neutral risk measure下
  - σ　代表價格波動
- 衍生性商品的價格過程f(t,S(t)),可使用Ito's lemma表現如下: 才是

$$df(t,S(t)) = \frac{\partial f}{\partial t}dt + \frac{\partial f(S(t))}{\partial S(t)}dS(t) + \frac{1}{2}\frac{\partial^2 f(S(t))}{\partial S(t)^2}(dS(t))^2$$

certain　　uncertain

$$= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dB(t)$$

$df(t; X_1, X_2 \ldots) = f_t\,dt + \sum_{i=1}^{d}\left(f_{x_i}dx_i + \frac{1}{2}f_{x_i x_i}(dx_i)^2\right) + \sum_{i \neq j}\sum f_{x_i x_j}dx_i\,dx_j$

where $f \in C^2$

$df = f_t + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x \partial x}\cdot\frac{1}{2}(dx)^2 = (\mu - r) f\,dt + \sigma f\,dB(t)$

風險中立下為0

constant

$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t)$

stochastic

避險組合的建立

- 為了規避衍生性商品的價格風險 $\frac{\partial f}{\partial S}\sigma S dB(t)$ 消dB(t)，可透過購買(放空)標的物來達成
- 考慮一投資組合V,包含1單位衍生性商品和 $-\frac{\partial f}{\partial S}$ 單位的標的物

$df = f_t\,dt + f_s\,ds + \frac{1}{2} f_{ss}(ds)^2$

$f_s[\mu s\,dt + \sigma s\,dB]$

現貨部位

$$dV = \boxed{-\frac{\partial f}{\partial S}dS(t)} + df(t,S(t)) = -\frac{\partial f}{\partial S}(\mu S(t)dt + \sigma S(t)dB(t))$$

衍生部位

$$+\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S(t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2(t)\right)dt + \boxed{\frac{\partial f}{\partial S}\sigma S dB(t)}$$

ds

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2(t)\right)dt$$

$f(s,t) = s\cdot e^{-r(T-t)}$

$\therefore f_{ss} = 0$

★ f(X(t),t) is a martingale

Note. $ds = \mu\cdot s\cdot dt + \sigma s\,dB$

$df = f_t\,dt + f_s\,ds + \frac{f_{ss}}{2}(ds)^2$

$dV = -f_s\,ds + df = f_t\,dt + \frac{1}{2}f_{ss}\sigma^2 s^2\,dt = \left(-\frac{\partial f}{\partial S}\cdot S + f\right)\cdot r\,dt$

$\Rightarrow f_t\,dt + \frac{1}{2}f_{ss}\sigma^2 s^2\,dt + f_s\cdot s\cdot r\,dt = r f\,dt$

$E[S(t)] = S_0 e^{r(T-t)}$; $E[X(t)] = (r - \frac{1}{2}\sigma^2)\,dt$

Black Scholes偏微分方程式

- 投資組合V不會有任何價格風險
  - 其報酬率為無風險利率
  - $dV = (f - \frac{\partial f}{\partial S}S)rdt$

$\frac{dV}{dt} = [f - \frac{\partial f}{\partial S}\cdot S]\cdot r$

- 整合上述結果,可得Black Scholes偏微分方程式如下:

$$dV = (f - \frac{\partial f}{\partial S}S)rdt = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial t}} + \boxed{rS\frac{\partial f}{\partial S}} + \boxed{\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}} = rf$$

偏微分方程求解和期望值計算的關聯

- 求解偏微分方程的問題,在滿足某些條件下,可化簡成期望值求解的問題
- 假定f(x,t)滿足下列方程

$R(0,T) = 0$

$$\frac{\partial f}{\partial t} + r(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2} = \boxed{0}$$

- 令 $dX(t) = r(X(t),t)dt + \sigma(X(t),t)dB(t)$
  - 根據 Ito's lemma,可得

$$df(X(t),t) = \left(\frac{\partial f}{\partial t} + r(X(t),t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(X(t),t)\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma(X(t),t)\frac{\partial f}{\partial x}dB(t)$$

$$= \boxed{\frac{\partial f}{\partial x}\sigma(X(t),t)dB(t)}$$

f(X(t),t)為一 martingale process

$\mathcal{F}_\tau$

t

τ

T

偏微分方程求解和期望值計算的關聯

$ds = \mu\cdot s\,dt + \sigma s\,dB$

fundamental calculus theorem.

$$f(X(\tau),\tau) + \int_{t=\tau}^{T} df(X(t),t) = \boxed{f(X(T),T)}$$

在T時 measurable

取條件期望值

$f(X(\tau),\tau)$ is a martingale, $\forall T \geq 0$

$$E\left(f(X(\tau),\tau) + \int_{t=\tau}^{T} df(X(t),t) \mid F_\tau\right) = E\left(f(X(T),T) \mid F_\tau\right)$$

过去資訊对增量⊥

因為 $E\left(\int_{t=\tau}^{T} df(X(t),t) \mid F_\tau\right) = E\left(\int_{t=\tau}^{T}\frac{\partial f}{\partial x}\sigma(X(t),t)dB(t) \mid \boxed{F_\tau}\right) = 0$

有無沒差

mean=0

$$f(X(\tau),\tau) = E\left(f(X(T),T) \mid F_\tau\right)$$

建立在r=0下

偏微分方程求解和期望值計算的關聯

- 假定 f(x,T)=C(x) (邊界條件),因為f(X(t),t)是 martingale process:

$$f(X(\tau),\tau)=\mathrm{E}(f(X(T),T)\mid F_\tau)$$

Markov property

$$=\mathrm{E}(C(X(T))\mid F_\tau)=\mathrm{E}(C(X(T)\mid X(\tau))$$

base on EMH

- 考慮Black Scholes偏微分方程

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}=rf$$

可得 $$df(S(t),t)=\left(\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}\right)dt+\sigma S\frac{\partial f}{\partial S}dB(t)$$

rf

$$=rfdt+\sigma S\frac{\partial f}{\partial S}dB(t)$$

Feynman-Kac Formula

- 考慮下述式子微分

$\frac{d\,-(t-\tau)r}{dt}=-r$

$$d\left(e^{-\int_\tau^t rds}\boxed{f(S(t),t)}\right)=e^{-\int_\tau^t rds}\boxed{df(S(t),t)}-re^{-\int_\tau^t rds}\boxed{f(S(t),t)}dt$$

全微分

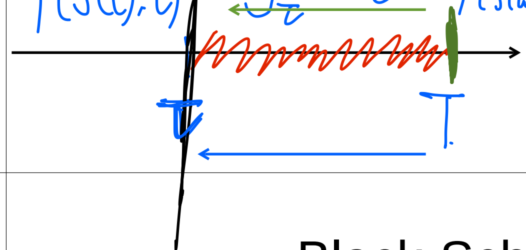
$$=\boxed{e^{-\int_\tau^t rds}}\left(rfdt+\sigma S\frac{\partial f}{\partial S}dB(t)\right)-\boxed{e^{-\int_\tau^t rds}rf(S(t),t)dt}=e^{-\int_\tau^t rds}\frac{\partial f}{\partial S}\sigma SdB(t)$$

- 兩邊積分得

$$e^{-\int_\tau^T rds}f(S(T),T)=f(S(\tau),\tau)+\int_\tau^T e^{-\int_\tau^t rds}\frac{\partial f}{\partial S}\sigma SdB(t)$$

$f(S(\tau),\tau)$

$\int_\tau^T d\,e^{-\int_\tau^a rds}f(S(a),a)\,da$



Martingale term,取期望值可消去

$f(X(T),T)=C(X(T))$

Feynman-Kac Formula

- 將f(x,T)=C(x) 帶入,取期望值

$E[f(X(T),T)/F_\tau]$

$$E\left(e^{-\int_\tau^T rds}C(S(T))\mid F_\tau\right)=e^{-\int_\tau^T rds}E(C(S(T))\mid F_\tau)=f(S(\tau),\tau)$$

假定利率為常數

- $f(S(\tau),\tau)$代表衍生性商品在時間 τ 的價格,從上式可知,$f(S(\tau),\tau)$的價格為到期日的payoff (C(x))的期望值再折現
  - 其中標的物價格滿足 $dS(t)=rSdt+\sigma SdB(t)$

風險中立機率下的標的物價格

Black-Scholes Formula
以歐式買權為例

- 標的物價格過程為 $S(t)=S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma B(t)}$
  - 假定到期日為T,履約價格為X,買權到期的payoff= $(S(T)-X)^+$
- 帶回上式得

期望值折現

$$價格=e^{-rT}E(S(T)-X)^+$$

$$=\boxed{e^{-rT}E(S(T)1_{\{S(T)\ge X\}})-e^{-rT}E(X1_{\{S(T)\ge X\}})}$$

分2种机率 case.

- 分別求解上述兩個期望值如下
  - 定義 N(t) 為標準常態 機率的分配函數

常態分配機率密度函數: $P_N(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$C_t = X_t \cdot B(t,T) \cdot N(D_1) + \boxed{k \cdot B(t,T)}\, N(D_2)$

Black-Scholes Formula

以歐式買權爲例

買权履約机会

$B(t,T)$　　$k \cdot N(D_2)$

證明 $e^{-rT}E(X1_{\{S(T)\ge X\}}) = e^{-rT}XN\left(\boxed{\dfrac{\ln\left(S(0)/X\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}}\right)$

$S(0)e^{(r-\sigma^2/2)T+\sigma B(T)} \ge X$

$\Rightarrow e^{(r-\sigma^2/2)T+\sigma B(T)} \ge \dfrac{X}{S(0)}$

$\Rightarrow (r-\sigma^2/2)T + \sigma B(T) \ge \ln\dfrac{X}{S(0)}$

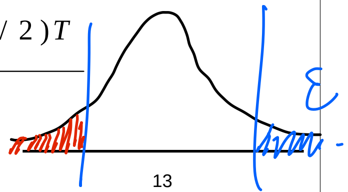
$\Rightarrow B(T) = \varepsilon\sqrt{T} \ge \dfrac{\ln\frac{X}{S(0)} - (r-\sigma^2/2)T}{\sigma}$

$\Rightarrow \varepsilon \ge \dfrac{\ln\frac{X}{S(0)} - (r-\sigma^2/2)T}{\sigma\sqrt{T}} \equiv -d_2$

$S(t) = S(0)\cdot e^{(r+\frac{1}{2}\sigma^2)\cdot t+\sigma B(t)}$

$E[I\{S_T \ge k\}]$

$= P\{S_T \ge k\}$



ε

$-\varepsilon$

$dS(t) = S(t)\cdot\mu\cdot dt + S(t)\cdot\sigma\cdot dB(t)$

S(t)和I是不独立的

Black-Scholes Formula

以歐式買權爲例

證明: $e^{-rT}E(S(T)1_{\{S(T)\ge X\}}) = S(0)N\left(\boxed{\dfrac{\ln\left(S(0)/X\right)+\left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}}}\right)$

$e^{-rT}E(S(T)1_{\{S(T)\ge X\}}) = e^{-rT}\int_{-d_2}^{\infty} \boxed{S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}}\,\boxed{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx}$ 求Expected

同P.13結論

$= S(0)\dfrac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\sigma^2T/2+\sigma\sqrt{T}x}e^{-x^2/2}dx = S(0)\dfrac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{\frac{x^2-2\sigma\sqrt{T}x+\sigma^2T}{-2}}dx$

$-\dfrac{(x-\sigma\sqrt{T})^2}{1}\cdot\dfrac{1}{2}$

$= S(0)\dfrac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{\frac{(x-\boxed{\sigma\sqrt{T}})^2}{-2}}dx$

將d2 Normalize

$-d_2-\sigma\sqrt{T} = \dfrac{\ln\frac{X}{S(0)}-(r-1/2\sigma^2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T} = \dfrac{\ln\frac{X}{S(0)}-(r+1/2\sigma^2)T}{\sigma\sqrt{T}} \equiv -d_1$

使用Put-Call Parity評價賣權

- 買權的價格可使用上述Black-Scholes Formula評價
- 考慮買權和賣權到期日的payoff:
  - 買權: $(S(T)-X)^+$　　賣權: $(X-S(T))^+$
- 如果買一個買權,賣一個賣權,到期日payoff
  - $(S(T)-X)^+ - (X-S(T))^+ = S(T)-X$
  - 其 payoff相當於期初借 Xe^{-rT} ,並購買標的物
- 在無套利的假定之下,假定C,P爲買權和賣權的價格,可得 $\boxed{C-P=S(0)-Xe^{-rT}}$

投組成本

使用BS-Formula和Put-Call Parity計算選擇權價格

- 使用Excel評價買權和賣權的價格(見BS.xls)
- 買權價格:
  - 使用Black-Scholes Formula

    $S(0)N\left(\dfrac{\ln\left(S(0)/X\right)+\left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}}\right) - e^{-rT}XN\left(\dfrac{\ln\left(S(0)/X\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$
  - 常態分配的累積密度函數: NORMSDIST(x)
  - 連續複利折現: exp(x)
- 賣權價格:
  - 使用put-call parity: $C-P=S(0)-Xe^{-rT}$

課堂演練

- BS formula的賣權價值: $Xe^{-rT}N(-d_2)-SN(-d_1)$
- 更改BS.xls,計算賣權價格
- 其結果應和使用put-call parity求出的答案相同
- 如何驗證評價程式的正確性?

BS Call formula + Put-Call Parity → Put Value

BS Put formula → $\boxed{\text{價格應該consistent}}$

不能說相同
因 float error.

避險參數(Greek Letters)

- 考慮市場價格發生變化時,選擇權價格的變化:
- 避險參數定義如下:
  - 假定f爲衍生性商品,S爲標的物
  - Delta: $\frac{\partial f}{\partial S}$
  - Gamma: $\frac{\partial^2 f}{\partial S^2}$
  - Theta: $\frac{\partial f}{\partial t}$
  - Vega: $\frac{\partial f}{\partial \sigma}$
  - Rho: $\frac{\partial f}{\partial r}$

詳細推導過程:見Y.K. Kwok, *Mathematical Models of Financial Derivatives*

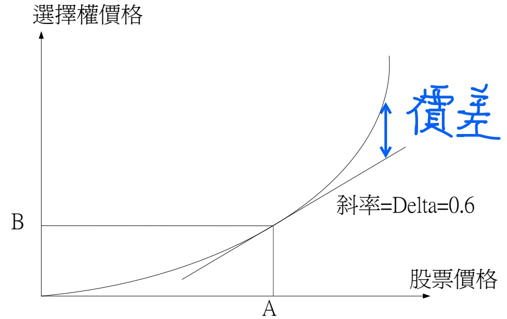
圖檔繪製:請參見GreekLetters.xls

$$\frac{\ln\frac{S_0}{K}-(r-\frac{1}{2}\sigma^2)\cdot T}{\sigma\sqrt{T}}$$

$C = S\cdot N(d_1) + K\cdot B(0,T)\cdot N(d_2)$

Delta

- 定義: $\frac{\partial f}{\partial S}$
- 衡量當標的物價格改變,選擇權價格的變化



- Delta決定避險時,所須購買/賣的標的物數量
  - 稱Delta neutral

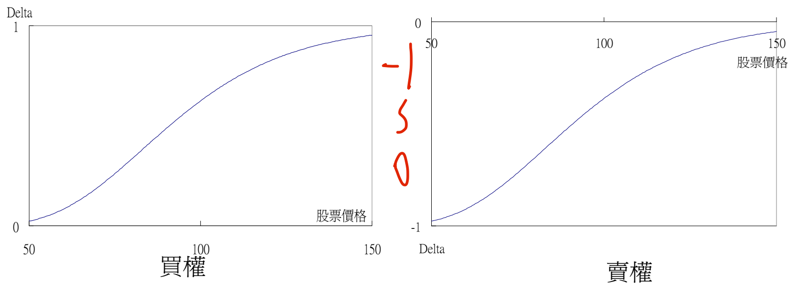
對cdf微分得pdf

Delta

- 買權 $\frac{\partial C}{\partial S}=N(d_1)$,賣權: $N(d_1)-1$

(履約價格爲100、到期日爲1年、無風險利率爲0.1、標的物股票的波動率爲30%。)

0 .5 1

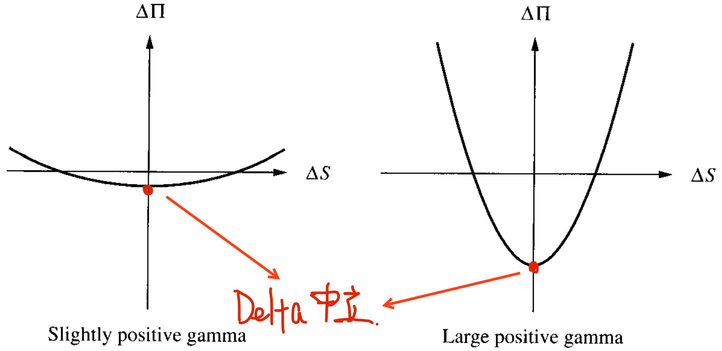


- 靠近到期日時,價平附近的選擇權Delta會有巨幅的變化,造成避險的困難

因若有transaction cost,會有巨額成本

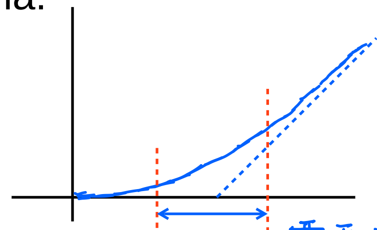
Gamma

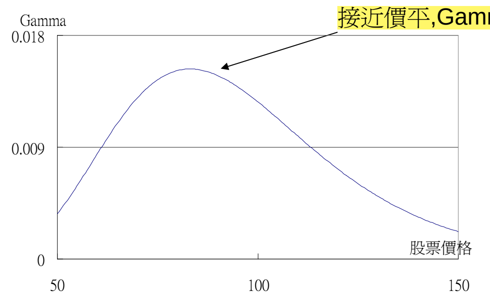
- 定義: $\frac{\partial^2 f}{\partial S^2}$
- 衡量當標的物價格變化時,避險部位的改變
  - 例如:高Gamma:部位改變大,較難避險



Gamma

- 歐式買權和賣權的Gamma:
  - $N'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{-2}}$
  - $\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T}}$





價平附近
fss最高

Gamma Neutral

- 先前討論的避險策略: Delta Neutral
  - 投資組合價值不受標的物價格波動影響
  - 必須不斷改變避險部位
    - 提高交易成本
- 如果存在一Delta-Gamma Neutral策略
  - 避險部位改變的頻率可下降

藉由不同option結合，消去Gamma和Delta.

Gamma Neutral投資組合的建立

$\Gamma_1 = -3000$
$\delta_2 = 0.6$
$\Gamma_2 = 1.5$

$\frac{-3000}{1.5} = -2000$

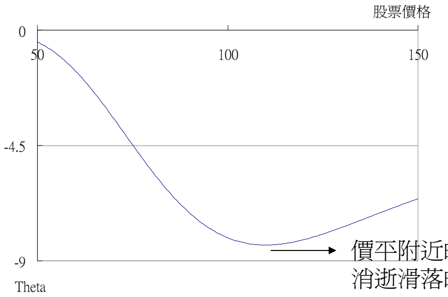
$2000 \times 0.6 = 1200$

- 假定有一個Delta-neutral的投資組合,其Gamma爲-3000,假定另有一個選擇權,其Delta和Gamma分別爲0.6和1.5
- 爲了確保投資組合爲Gamma Neutral,購買(3000/1.5)=2000單位的選擇權
- 爲了確保新投資組合爲Delta Neutral,須放空(2000*0.6)=1200單位的標的物
  - 標的物價格對自己二階微分=0,所以不會改變投資組合的Gamma

Theta

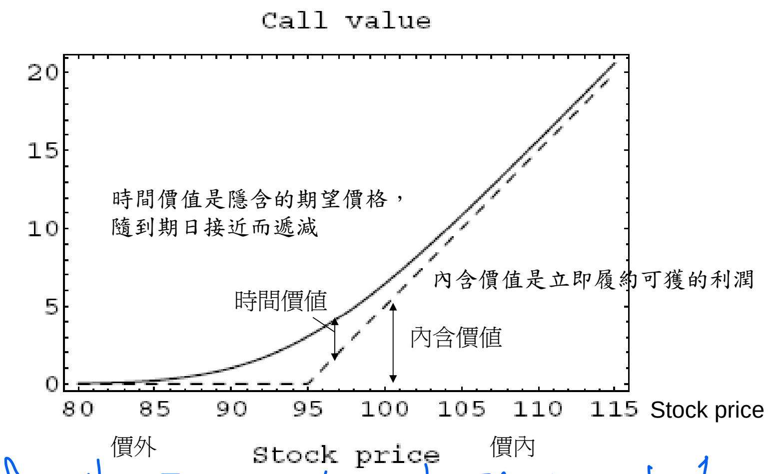
- 定義: $\frac{\partial f}{\partial t}$
- 衡量時間變化下,選擇權價格的變化:
- 買權: $\theta_c = -\frac{SN'(d_1)\sigma}{2\sqrt{T}} - rXe^{-rT}N(d_2)$ < 0
- 賣權: $\theta_p = -\frac{SN'(d_1)\sigma}{2\sqrt{T}} + \underline{rXe^{-rT}N(-d_2)}$ 未定

$\longrightarrow \boxed{\theta_c < \theta_p}$



價平附近的選擇權,其價格隨時間消逝滑落的速度最快

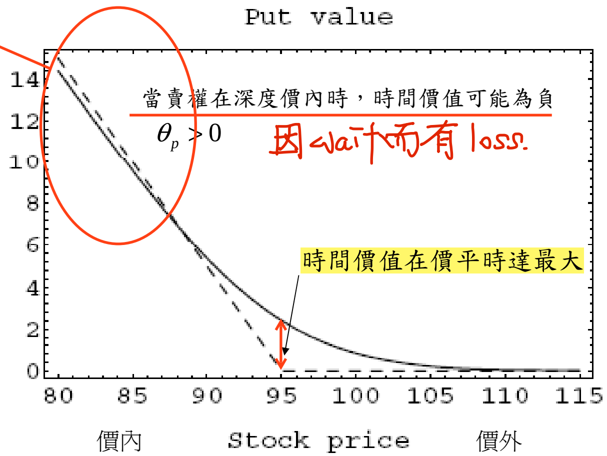
Call Value
(Intrinsic value + time value)



The slope of call price monotonically increase to 1.

Put Value
(Intrinsic value + time value)

立該 exercise
→ 因折現
而招致價損



Black-Scholes偏微分方程和Greek Letters的關係

- 前面推導出了Black Scholes方程如下:

$$\underbrace{\frac{\partial f}{\partial t}}_{\text{theta}} + rS\underbrace{\frac{\partial f}{\partial S}}_{\text{delta}} + \frac{1}{2}\sigma^2 S^2 \underbrace{\frac{\partial^2 f}{\partial S^2}}_{\text{gamma}} = rf$$

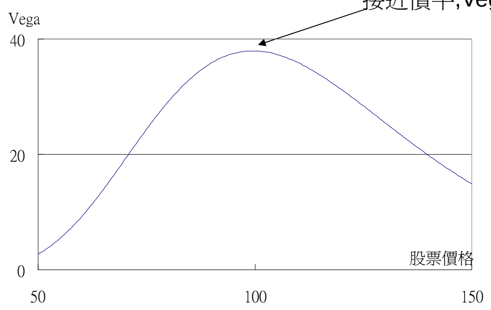
  – 包含Delta, Theta, Gamma
- 將Greek letters帶入,可得

$$\theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rf$$

☆真正架構Hedge,代回此式應成立。

0206 option event ← call & put · 皆漲跌板
終止保証金优化·

Vega

- 定義: $\frac{\partial f}{\partial \sigma} = S\sqrt{T}N'(d_1)$ > 0, 表不確定性
- 假定標的物的波動率改變,Vega衡量波動率變化時,選擇權價格改變率
- 

σ 上升 s.t C_t, P_t 上升.
σ 下降 s.t C_t, P_t 下降.

Delta-Gamma-Vega Neutral 投資組合的建立

- Vega neutral的策略,可減少因市場狀態巨幅改變,所造成的損失
- 假定有一Delta neutral投資組合,Gamma= -5000, Vega=-8000
- 假定有兩個選擇權
  - 1: Delta=0.6, Gamma=0.5, Vega=2
  - 2: Delta=0.5, Gamma=0.8, Vega=1.2
- 加入這兩個選擇權,可使投資組合 Delta-Gamma-Vega Neutral

Delta-Gamma-Vega Neutral 投資組合的建立

使Γ免疫

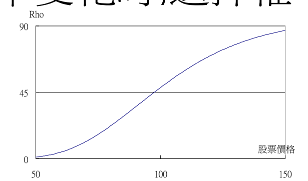
- 令兩個選擇權的購買量 u, v 使θ免疫
  - -5000+0.5u+0.8v=0 Gamma neutral
  - -8000+2.0u+1.2v=0 Vega neutral
- 解方程式得 u=400, v=6000
- 讓投資組合為Delta neutral,需放空
  400*0.6+6000*0.5=3240 單位標的物

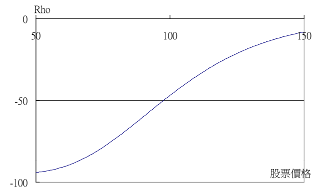
使δ免疫

Rho

- 定義: $\frac{\partial f}{\partial r}$
- 假定利率改變,Rho衡量利率變化時,選擇權價格改變率
- 買權 $\frac{\partial f}{\partial r} = XTe^{-rT}N(d_2) > 0$
- 賣權 $\frac{\partial f}{\partial r} = -XTe^{-rT}N(-d_2) < 0$

因折現





Delta-Gamma-Vega Neutral

- 假定市場利率=3%,標的物價格=6000,波動率=40%,有兩個在一個月後到期的買權,其履約價格分別為5900,6000
- 使用Excel計算這兩個選擇權的Delta, Gamma,Vega(見DeltaGammaVega.xls)

表示不同K之

$$\frac{\partial f}{\partial S} = N(d_1) \qquad \Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T}} \qquad \frac{\partial f}{\partial \sigma} = S\sqrt{T}N'(d_1)$$

不同 d_1

$$a\frac{N'(d_1)}{S\sigma\sqrt{T}} + b\frac{N'(d_1')}{S\sigma\sqrt{T}} = 0$$

$$aS\sqrt{T}N'(d_1) + bS\sqrt{T}N'(d_1') = 0$$

只需用兩個選擇權就可維持Gamma-Vega Neutral

Delta-Gamma-Vega Neutral

- (Delta Neutral)假定購買10000單位履約價格=5900的選擇權,計算需放空多少標的物
- 計算上述投資組合的Gamma,Vega
- (Gamma-Vega Neutral),計算需要買入(放空)另一種選擇權,使得整個投資組合Gamma=Vega=0
- (Delta rebalance)計算需要購買放空多少標的物,使得新的投資組合Delta=0

課堂演練 Delta-Gamma-Vega Neutral

- 將上述例子從買權換成賣權,重新建立一 Delta-Gamma-Vega Neutral portfolio.

填息、填权→股價回補

多index option.

現金股息下的股票$\boxed{\text{選擇權}}$模型

- 股票會因為配息而導致價格改變,因此也影響到股票選擇權的評價
- 配息模型
  - $\boxed{\text{連續付息}}$ (Merton's model)
    - 適用於股票指數選擇權的評價
    - 有精確的$\boxed{\text{解析解}}$

來逼近

  - 離散付息
    - 適合評價一般股票選擇權
    - 修改Black Scholes formula而得近似解

連續付息模型

- 假定股息率爲q

年化

每一瞬間皆以q去付div.

- 標的物價格波動: $dS(t)=(\mu-q)S(t)dt+\sigma S(t)dB(t)$
- 衍生性商品的價格波動:

$$df(t,S(t))=\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial S}(\mu-q)S+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2S^2\right)dt+\frac{\partial f}{\partial S}\sigma S dB(t)$$

- 購買一個單位的衍生品,放空 $\frac{\partial f}{\partial S}$ 單位的標的物(需考慮股息)

$$dV=-\frac{\partial f}{\partial S}dS(t)+df(t,S(t))=\left(\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2S^2-q\frac{\partial f}{\partial S}S\right)dt$$

Feynman Kac.

連續付息模型

- 上述投資組合無風險,報酬率爲無風險利率

$$dV=\left(\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2S^2-q\frac{\partial f}{\partial S}S\right)dt=r(f-\frac{\partial f}{\partial S}S)dt$$

$$\Rightarrow\frac{\partial f}{\partial t}+(r-q)S\frac{\partial f}{\partial S}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2S^2=rf$$

- 根據Feynman-Kac Formula,買權價格修正如下:

$$Se^{-qT}N\left(\frac{\ln\left(S/X\right)+\left(r-q+\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)-e^{-rT}XN\left(\frac{\ln\left(S/X\right)+\left(r-q-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$$

將 $r'=r-q$. 代入.

離散付息模型

- 假定股票在時間t付息D.則股息現值爲 De^{-rt}
- 將股票價值扣除股息得 $S'\equiv S-De^{-rt}$
  - 假定股息呈無風險利率成長
  - 扣除股息的股票價值服從對數常態分配
- 其結果相當於將Black Scholes Formula修正如下:

$$S'N\left(\frac{\ln\left(S'/X\right)+\left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)-e^{-rT}XN\left(\frac{\ln\left(S'/X\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$$

課堂演練

- 使用Excel評價股票買權的價格(修改BS.xls)
  - 連續付息:

修正R

$$SN\left(\frac{\ln\left(S/X\right)+\left(r-q+\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)-e^{-rT}XN\left(\frac{\ln\left(S/X\right)+\left(r-q-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$$

  - 離散付息

修正S.

$$S'N\left(\frac{\ln\left(S'/X\right)+\left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)-e^{-rT}XN\left(\frac{\ln\left(S'/X\right)+\left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}\right)$$