What is coordinate system, vectors and transformations?

Suppose, I've a manifold with Euclidean geometry (i.e. a manifold with Euclidean geometry laid on it). In short, a Euclidean space R^n. Let us take a 2d space for simplicity, i.e. it requires two values to define a unique point in the space. We might take single value to describe any point in space by serializing the 2d array of points, but it would make no sense as the set is uncountably infinite making offset of serializing undefined. Suppose, points in space are identified by independent variables x and y. Now, we define another pair of independent variables p and q for identifying points in the space. Then $x = x(p,q)$ and $y = y(p,q)$ and vice-versa. These, **we'll define as coordinates**.

Select a point P in space. Draw a straight line (path which extremizes the path length) originating from it in any direction. Let the line be denoted by $L = \{(x,y) \mid x(t) \ and \ y(t)\}$ in x, y coordinates or $L' = \{(p,q) \mid p(t) \ \ and \ q(t)\}$ in p, q coordinates, parametrized by t. Then a small element of the line at point P is given by,

$$\frac{dx}{dt}\Delta t = v_x \qquad in\ x - direction \qquad\qquad \frac{dp}{dt}\Delta t = v_p \qquad in\ p - direction$$
$$\frac{dy}{dt}\Delta t = v_y \qquad in\ y - direction \qquad\qquad \frac{dq}{dt}\Delta t = v_q \qquad in\ q - direction$$

$$\begin{aligned}\frac{dp}{dt}\Delta t = v_p = \frac{dp}{dx}\left(\frac{dx}{dt}\Delta t\right) + \frac{dp}{dy}\left(\frac{dy}{dt}\Delta t\right) = \frac{dp}{dx}v_x + \frac{dp}{dy}v_y \\ \frac{dq}{dt}\Delta t = v_q = \frac{dq}{dx}\left(\frac{dx}{dt}\Delta t\right) + \frac{dq}{dy}\left(\frac{dy}{dt}\Delta t\right) = \frac{dq}{dx}v_x + \frac{dq}{dy}v_y\end{aligned} \Rightarrow \begin{pmatrix} v_p \\ v_q \end{pmatrix} = \begin{pmatrix} \frac{dp}{dx} & \frac{dp}{dy} \\ \frac{dq}{dx} & \frac{dq}{dy} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\begin{pmatrix} v_p \\ v_q \end{pmatrix} = M \begin{pmatrix} v_x \\ v_y \end{pmatrix} \qquad where \qquad M = \begin{pmatrix} \frac{dp}{dx} & \frac{dp}{dy} \\ \frac{dq}{dx} & \frac{dq}{dy} \end{pmatrix}$$

Now let us define few things. Here, we represented a line in two coordinate systems, (x, y) and (p, q). Then we took a differential element of the line and represented it as (v_x, v_y) and (v_p, v_q) in coordinate systems (x, y) and (p, q) respectively. And derived a relationship between (v_x, v_y) and (v_p, v_q). Now, **we can define a vector** as any object which satisfies above relationship when represented in different coordinate systems. Let us rewrite (v_x, v_y) and (v_p, v_q) as,

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = v_x \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + v_y \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \qquad and \qquad \begin{pmatrix} v_p \\ v_q \end{pmatrix} = v_p \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + v_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2$$

Subscripts 1 and 2 denote coordinate systems (x, y) and (p, q) respectively. Let us give symbols to them.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 = \boldsymbol{e_x} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 = \boldsymbol{e_y} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \boldsymbol{e_p} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 = \boldsymbol{e_q}$$

Let ep be represented as (e_{px}, e_{py}) in (x, y) coordinate system. Then,

$$\boldsymbol{e_p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{in\ (p,q)\ coordinate\ system} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{in\ (p,q)\ coordinate\ system} = M \begin{pmatrix} e_{px} \\ e_{py} \end{pmatrix}_{in\ (x,y)\ coordinate\ system}$$
$$\Rightarrow \begin{pmatrix} e_{px} \\ e_{py} \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{e_p} = e_{px}\boldsymbol{e_x} + e_{py}\boldsymbol{e_y} = M^{-1}_{11}\boldsymbol{e_x} + M^{-1}_{21}\boldsymbol{e_y}$$

Similarly,

$$\boldsymbol{e_q} = e_{qx}\boldsymbol{e_x} + e_{qy}\boldsymbol{e_y} = M^{-1}_{12}\boldsymbol{e_x} + M^{-1}_{22}\boldsymbol{e_y}$$

In matrix notation

$$\begin{pmatrix} \boldsymbol{e_p} \\ \boldsymbol{e_q} \end{pmatrix} = M^{-1} \begin{pmatrix} \boldsymbol{e_x} \\ \boldsymbol{e_y} \end{pmatrix}$$

Here, we have expressed $\boldsymbol{e_p}$ and $\boldsymbol{e_q}$ in terms of $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$, same as we did to represent v_p and v_q in terms of v_x and v_x. Note, when we denote a vector as (v_x, v_y), we have to specify the coordinate system with respect to which it is represented. Instead, if we represent the vector as $v = v_x\boldsymbol{e_x} + v_y\boldsymbol{e_y}$, then it can be viewed as an entity independent of any coordinate system as we have implicitly mentioned the coordinate system using some object $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$. This object $(\boldsymbol{e_x},\ \boldsymbol{e_y})$ is called **basis for the coordinate system 1**, similarly $(\boldsymbol{e_p},\ \boldsymbol{e_q})$ **is basis for the coordinate system 2**.

Hence, we cannot write $(v_x, v_y) = (v_p, v_q)$ but we can write $v = v_x\boldsymbol{e_x} + v_y\boldsymbol{e_y} = v_p\boldsymbol{e_p} + v_q\boldsymbol{e_q}$. That is,

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} (\boldsymbol{e_x} \quad \boldsymbol{e_y}) = \begin{pmatrix} v_p \\ v_q \end{pmatrix} (\boldsymbol{e_p} \quad \boldsymbol{e_q})$$

Such that,

$$\begin{pmatrix} v_p \\ v_q \end{pmatrix} = M \begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad and \quad \begin{pmatrix} \boldsymbol{e_p} \\ \boldsymbol{e_q} \end{pmatrix} = M^{-1} \begin{pmatrix} \boldsymbol{e_x} \\ \boldsymbol{e_y} \end{pmatrix}$$

Note that while transforming between different bases or coordinate systems either transform components (v_x, v_y) or transform basis vectors $(\boldsymbol{e_x},\ \boldsymbol{e_y})$, but don't transform both together.

By observing similarity between above transformations, we can say that $(\boldsymbol{e_x},\ \boldsymbol{e_y})$ and $(\boldsymbol{e_p},\ \boldsymbol{e_q})$ are also a kind of vectors. Vectors which transform as $\begin{pmatrix} v_p \\ v_q \end{pmatrix} = M \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ are called contravariant vectors and vectors which transform as $\begin{pmatrix} \boldsymbol{e_p} \\ \boldsymbol{e_q} \end{pmatrix} = M^{-1} \begin{pmatrix} \boldsymbol{e_x} \\ \boldsymbol{e_y} \end{pmatrix}$ are called covariant vectors.

We can easily generalize this to more abstract notion of vectors, but that is a subject matter of Linear Algebra.