What is coordinate system, vectors and transformations?

Suppose, I've a manifold with Euclidean geometry (i.e. a manifold with Euclidean geometry laid on it). In short, a Euclidean space R^n. Let us take a 2D space for simplicity, i.e. it requires two values to define a unique point in the space. Suppose, points in space are identified by independent variables x and y. Now, we define another pair of independent variables p and q for identifying points in the space. Then $x = x(p,q)$ and $y = y(p,q)$ and vice-versa. These, **we'll define as coordinates**.

Select a point P in space. Draw a straight line (path which extremizes the path length) originating from it in any direction. Let the line be denoted by $L = \{(x,y) \mid x(t)\ and\ y(t)\}$ in x, y coordinates or $L' = \{(p,q) \mid p(t)\ \ and\ q(t)\}$ in p, q coordinates, parametrized by t. Then a small element of the line at point P is given by,

$$\frac{dx}{dt}\Delta t = v_x \qquad in\ x - direction \qquad\qquad \frac{dp}{dt}\Delta t = v_p \qquad in\ p - direction$$
$$\frac{dy}{dt}\Delta t = v_y \qquad in\ y - direction \qquad\qquad \frac{dq}{dt}\Delta t = v_q \qquad in\ q - direction$$

$$\begin{aligned}\frac{dp}{dt}\Delta t = v_p = \frac{dp}{dx}\left(\frac{dx}{dt}\Delta t\right) + \frac{dp}{dy}\left(\frac{dy}{dt}\Delta t\right) = \frac{dp}{dx}v_x + \frac{dp}{dy}v_y \\ \frac{dq}{dt}\Delta t = v_q = \frac{dq}{dx}\left(\frac{dx}{dt}\Delta t\right) + \frac{dq}{dy}\left(\frac{dy}{dt}\Delta t\right) = \frac{dq}{dx}v_x + \frac{dq}{dy}v_y\end{aligned} \Rightarrow \begin{pmatrix} v_p \\ v_q \end{pmatrix} = \begin{pmatrix} \frac{dp}{dx} & \frac{dp}{dy} \\ \frac{dq}{dx} & \frac{dq}{dy} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\begin{pmatrix} v_p \\ v_q \end{pmatrix} = M \begin{pmatrix} v_x \\ v_y \end{pmatrix} \qquad where \qquad M = \begin{pmatrix} \frac{dp}{dx} & \frac{dp}{dy} \\ \frac{dq}{dx} & \frac{dq}{dy} \end{pmatrix}$$

Now let us define few things. Here, we represented a line in two coordinate systems, (x, y) and (p, q). Then we took a differential element of the line and represented it as (v_x, v_y) and (v_p, v_q) in coordinate systems (x, y) and (p, q) respectively. And derived a relationship between (v_x, v_y) and (v_p, v_q). Now, **we can define a vector** as any object which satisfies above relationship (along with vector axioms) when represented in different coordinate systems. Let us rewrite (v_x, v_y) and (v_p, v_q) as,

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = v_x \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + v_y \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \qquad and \qquad \begin{pmatrix} v_p \\ v_q \end{pmatrix} = v_p \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + v_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2$$

Subscripts 1 and 2 denote coordinate systems (x, y) and (p, q) respectively. Let us give symbols to them.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 = \boldsymbol{e_x} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 = \boldsymbol{e_y} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \boldsymbol{e_p} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 = \boldsymbol{e_q}$$

Let $\boldsymbol{e_p}$ be represented as (e_{px}, e_{py}) in (x, y) coordinate system. Then,

$$\boldsymbol{e_p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{in\ (p,q)\ coordinate\ system} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{in\ (p,q)\ coordinate\ system} = M \begin{pmatrix} e_{px} \\ e_{py} \end{pmatrix}_{in\ (x,y)\ coordinate\ system}$$
$$\Rightarrow \begin{pmatrix} e_{px} \\ e_{py} \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{e_p} = e_{px}\boldsymbol{e_x} + e_{py}\boldsymbol{e_y} = M_{11}^{-1}\boldsymbol{e_x} + M_{21}^{-1}\boldsymbol{e_y}$$

Similarly,

$$\boldsymbol{e_q} = e_{qx}\boldsymbol{e_x} + e_{qy}\boldsymbol{e_y} = M_{12}^{-1}\boldsymbol{e_x} + M_{22}^{-1}\boldsymbol{e_y}$$

In matrix notation

$$\begin{pmatrix}\boldsymbol{e_p}\\ \boldsymbol{e_q}\end{pmatrix} = M^{-1}\begin{pmatrix}\boldsymbol{e_x}\\ \boldsymbol{e_y}\end{pmatrix}$$

Here, we have expressed $\boldsymbol{e_p}$ and $\boldsymbol{e_q}$ in terms of $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$, same as we did to represent v_p and v_q in terms of v_x and v_x. Note, when we denote a vector as (v_x, v_y), we have to specify the coordinate system with respect to which it is represented. Instead, if we represent the vector as $v = v_x\boldsymbol{e_x} + v_y\boldsymbol{e_y}$, then it can be viewed as an entity independent of any coordinate system as we have implicitly mentioned the coordinate system using some object $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$. This object $(\boldsymbol{e_x},\ \boldsymbol{e_y})$ is called **basis for the coordinate system 1**, similarly $(\boldsymbol{e_p},\ \boldsymbol{e_q})$ **is basis for the coordinate system 2**.

Hence, we cannot write $(v_x, v_y) = (v_p, v_q)$ but we can write $v = v_x\boldsymbol{e_x} + v_y\boldsymbol{e_y} = v_p\boldsymbol{e_p} + v_q\boldsymbol{e_q}$. That is,

$$v = \begin{pmatrix}v_x\\ v_y\end{pmatrix}(\boldsymbol{e_x} \quad \boldsymbol{e_y}) = \begin{pmatrix}v_p\\ v_q\end{pmatrix}(\boldsymbol{e_p} \quad \boldsymbol{e_q})$$

Such that,

$$\begin{pmatrix}v_p\\ v_q\end{pmatrix} = M\begin{pmatrix}v_x\\ v_y\end{pmatrix} \quad and \quad \begin{pmatrix}\boldsymbol{e_p}\\ \boldsymbol{e_q}\end{pmatrix} = M^{-1}\begin{pmatrix}\boldsymbol{e_x}\\ \boldsymbol{e_y}\end{pmatrix}$$

Note that while transforming between different bases or coordinate systems either transform components (v_x, v_y) or transform basis vectors $(\boldsymbol{e_x},\ \boldsymbol{e_y})$, but don't transform both together.

By observing similarity between above transformations, we can say that $(\boldsymbol{e_x},\ \boldsymbol{e_y})$ and $(\boldsymbol{e_p},\ \boldsymbol{e_q})$ are also a kind of vectors. Vectors which transform as $\begin{pmatrix}v_p\\ v_q\end{pmatrix} = M\begin{pmatrix}v_x\\ v_y\end{pmatrix}$ are called contravariant vectors and vectors which transform as $\begin{pmatrix}\boldsymbol{e_p}\\ \boldsymbol{e_q}\end{pmatrix} = M^{-1}\begin{pmatrix}\boldsymbol{e_x}\\ \boldsymbol{e_y}\end{pmatrix}$ are called covariant vectors.

We can easily generalize this to more abstract notion of vectors, but that is a subject matter of Linear Algebra.