Function fitting

Lecture 4; Reading: ISLR sections 2.1, 3.2.1, 3.5

IFT6758, Fall 2020

Leveraging Input → Output relationship in data

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- Person's characteristics → Whether they'll vote
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Mathematical expression

- $x_i = (x_{i_1}, \ldots, x_{i_p})$ ← **inputs**
- $\boldsymbol{y_i} = \boldsymbol{f}(\boldsymbol{x_i})$ ← **inputs to output relationship**

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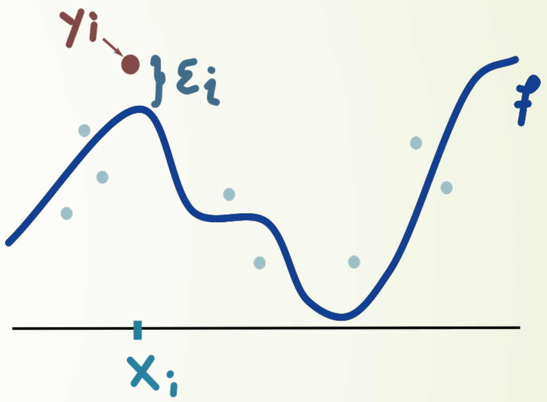

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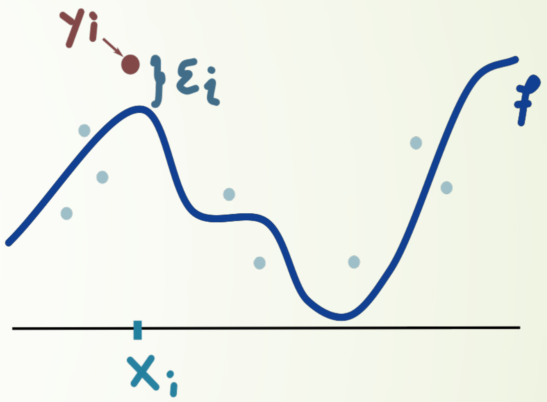

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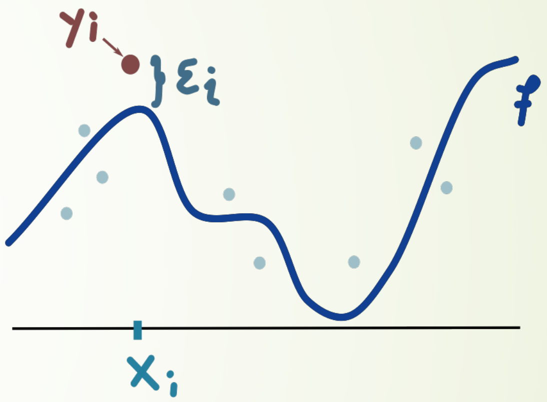

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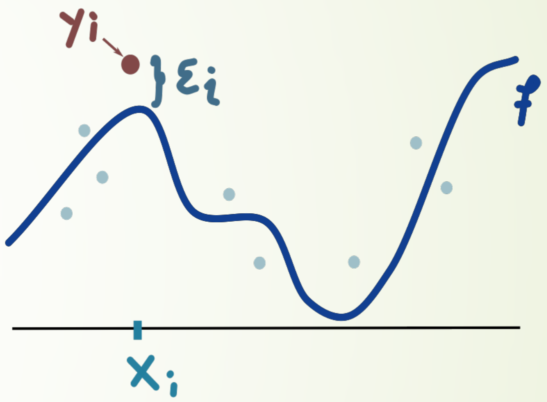

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- $\boldsymbol{f}$ describes **systematic variation** in $\boldsymbol{y_i}$
- $\boldsymbol{\epsilon_i}$ reflects variations whose source is unknown

to us. Consider coin tosses.

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- **Reason 1: Prediction**
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We want **quantitative estimates**, not visual summaries

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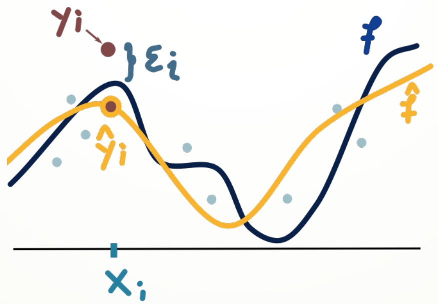

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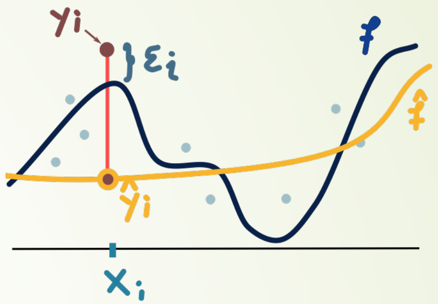

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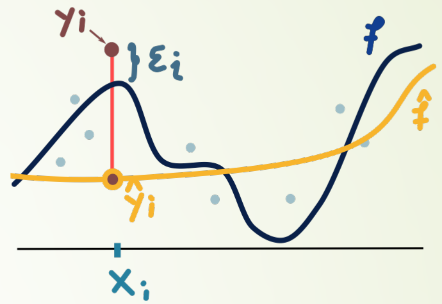

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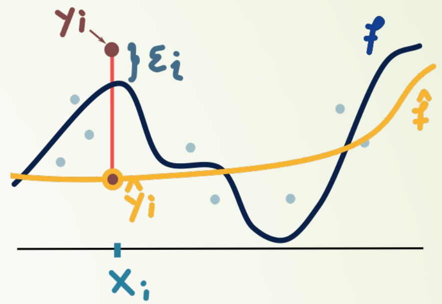

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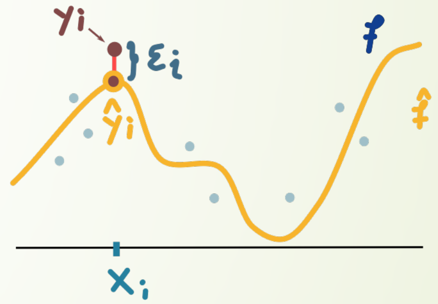

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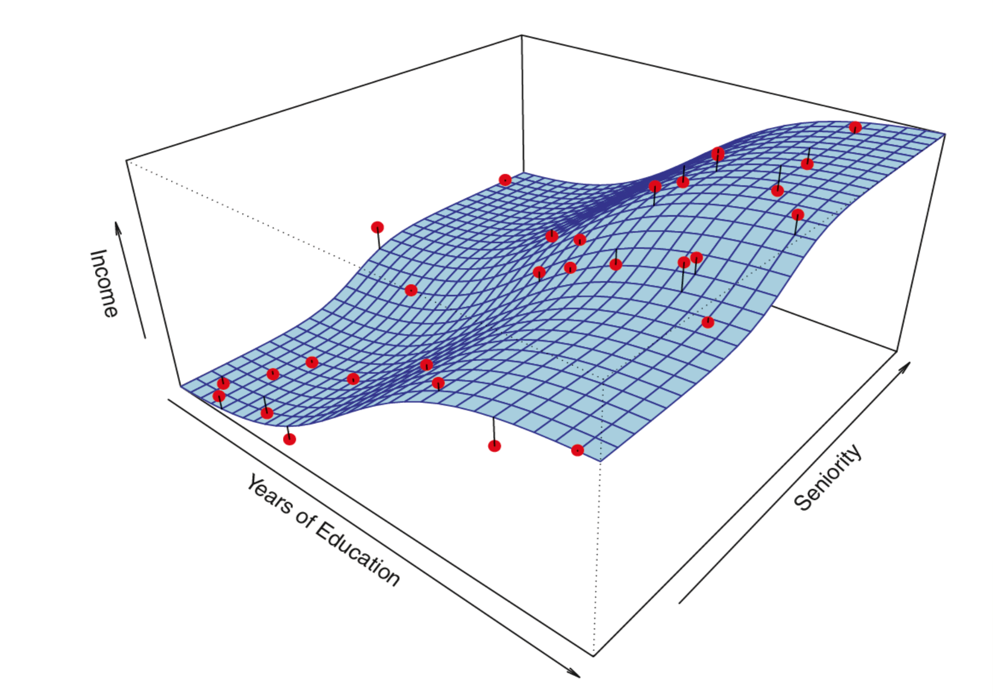

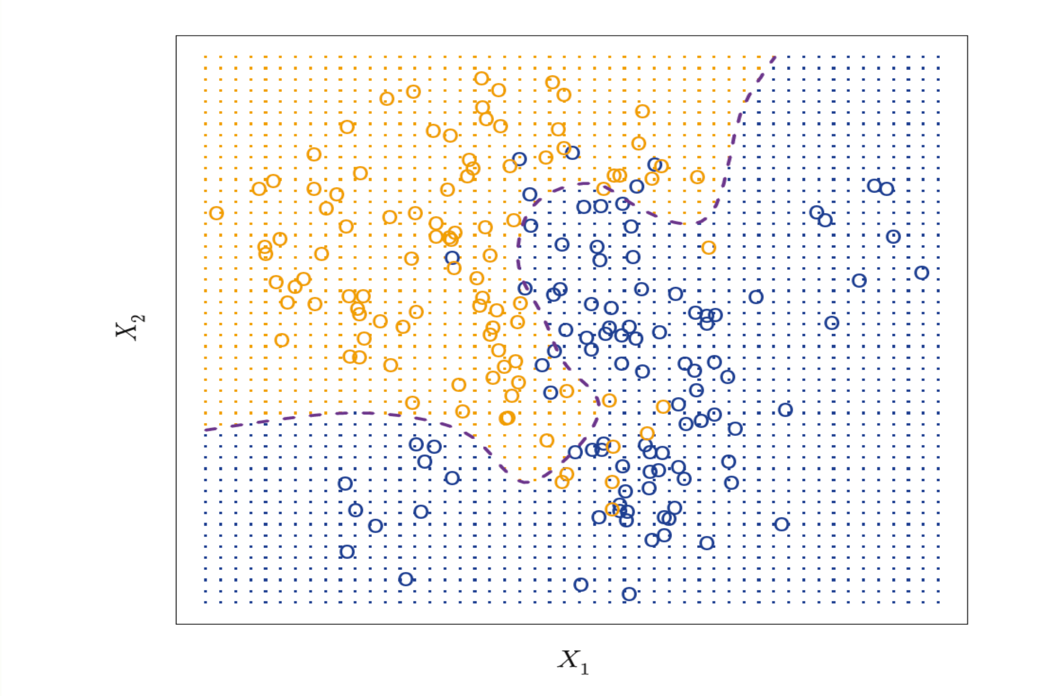

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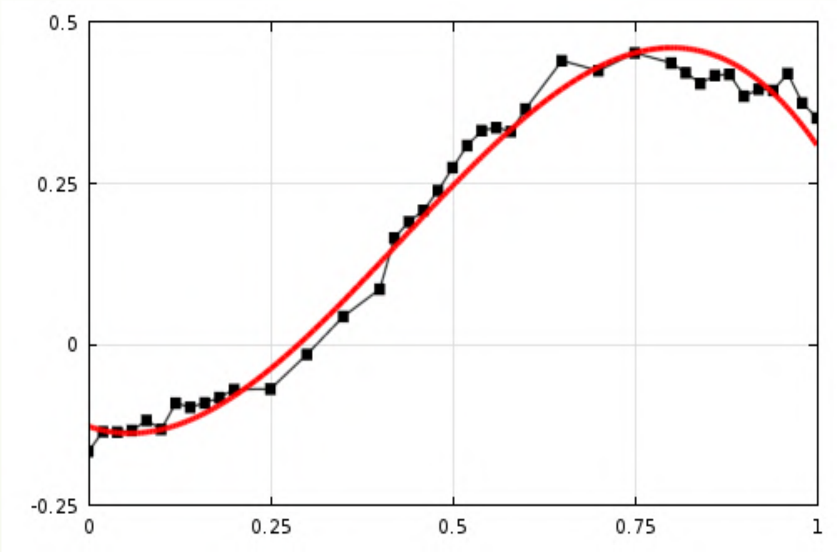

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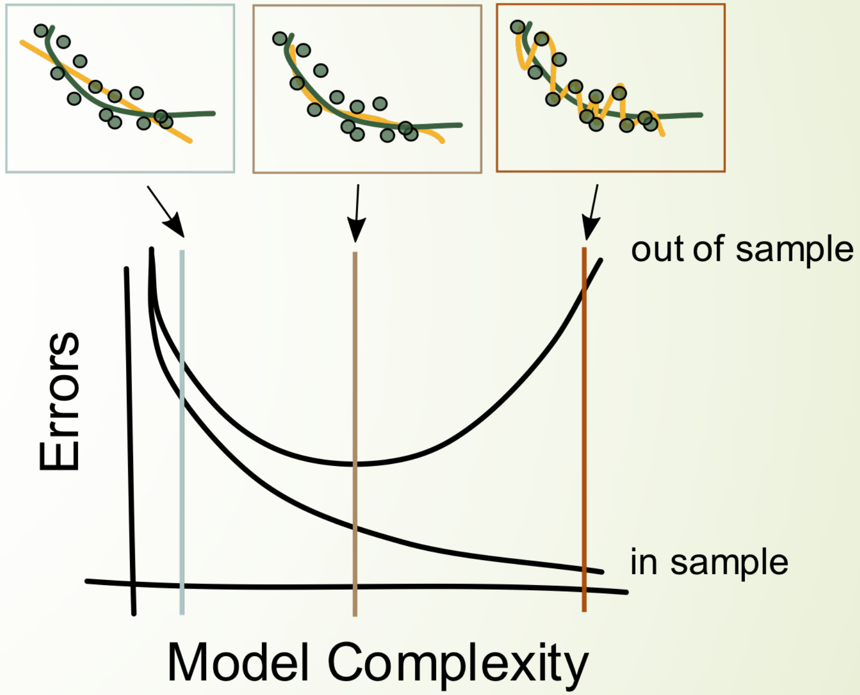

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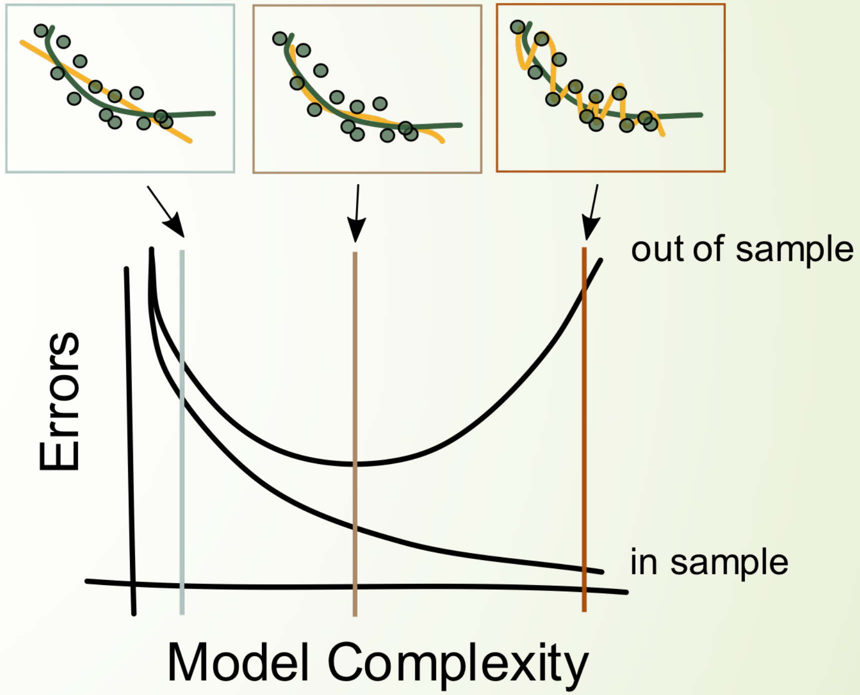

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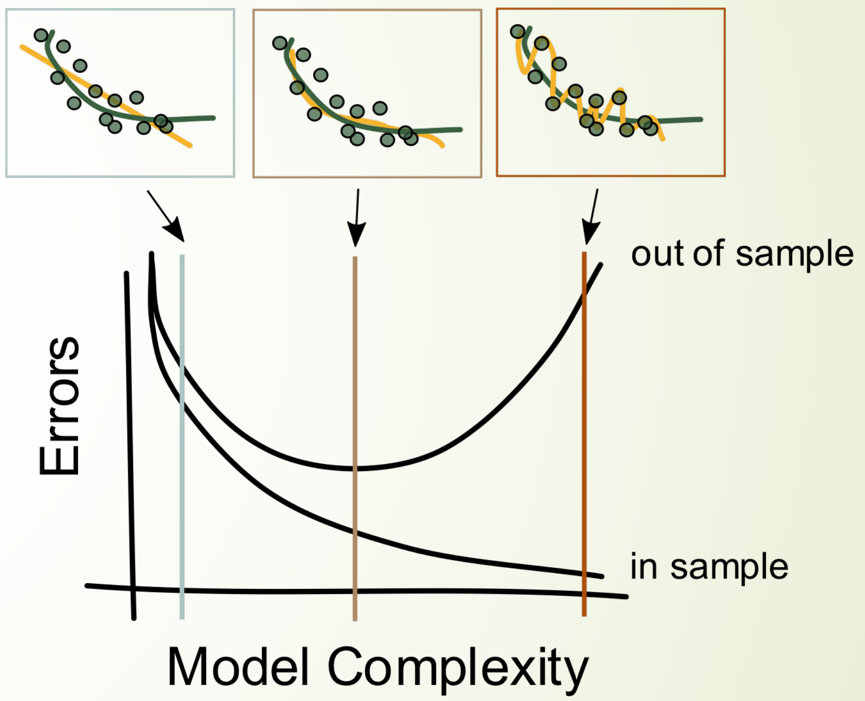
- **Incurring some bias can** lead to **better stability** and overall **better predictions**



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Algorithms: K-Nearest Neighbors (KNN) Regression

- K fixed and given
- **Samples:** $(x_i, \mathrm{y_i})_{\{\mathrm{i=1}\}}^{\mathrm{N}}$
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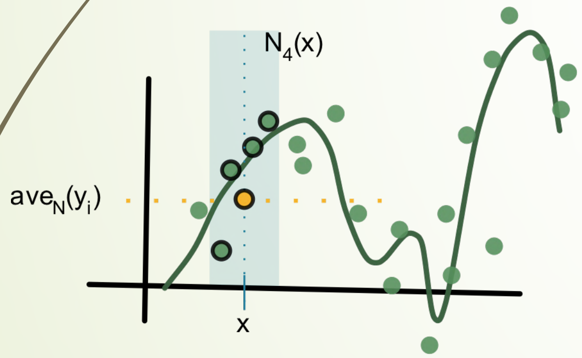

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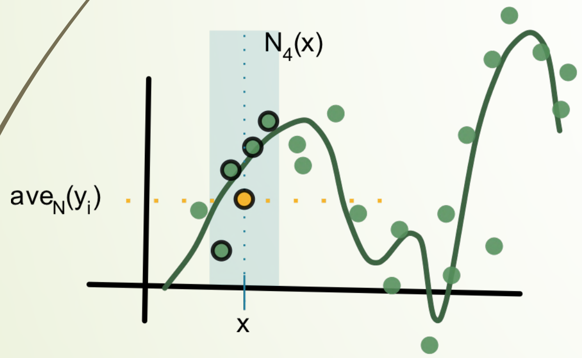

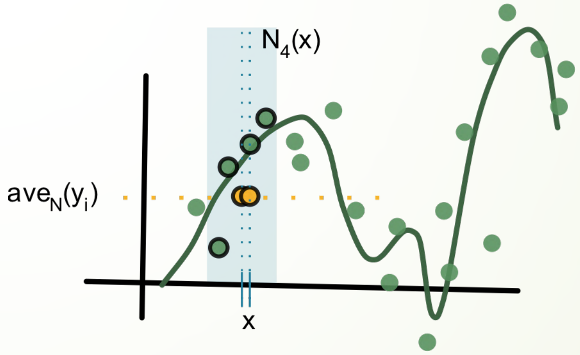

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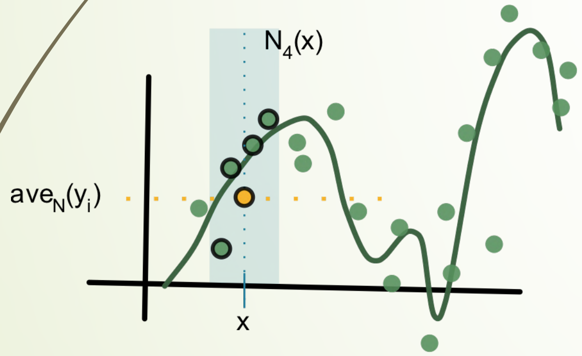

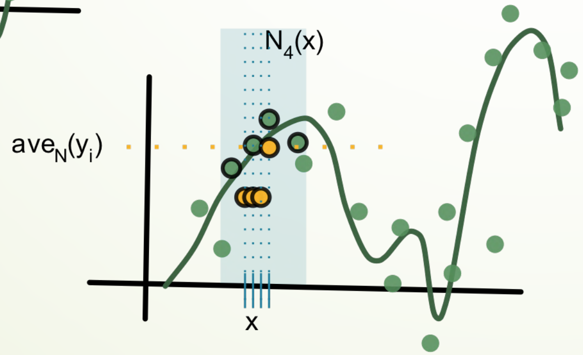

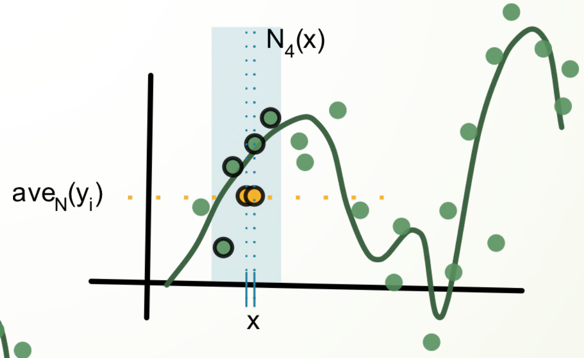

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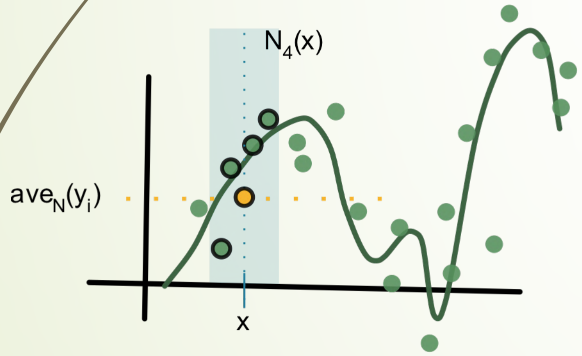

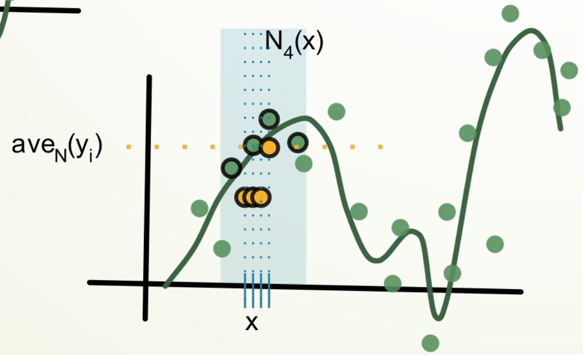

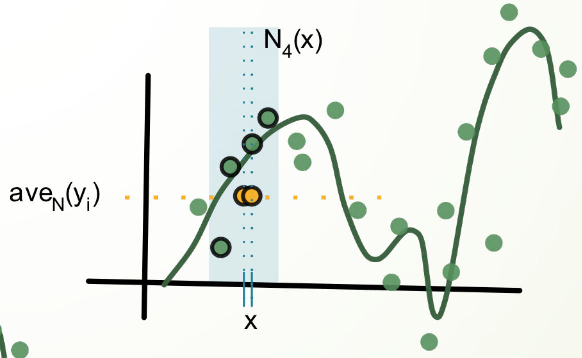

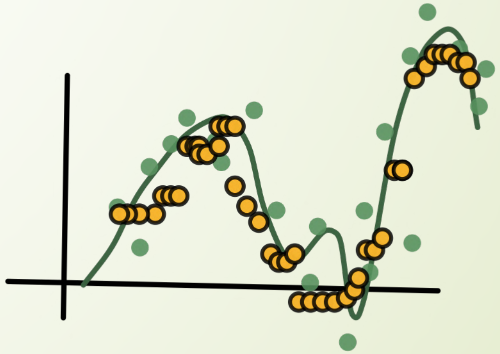

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  - Larger $\boldsymbol{K}$ learns smoother functions; smaller $\boldsymbol{K}$ can match more complex functions when the sampling density is high enough

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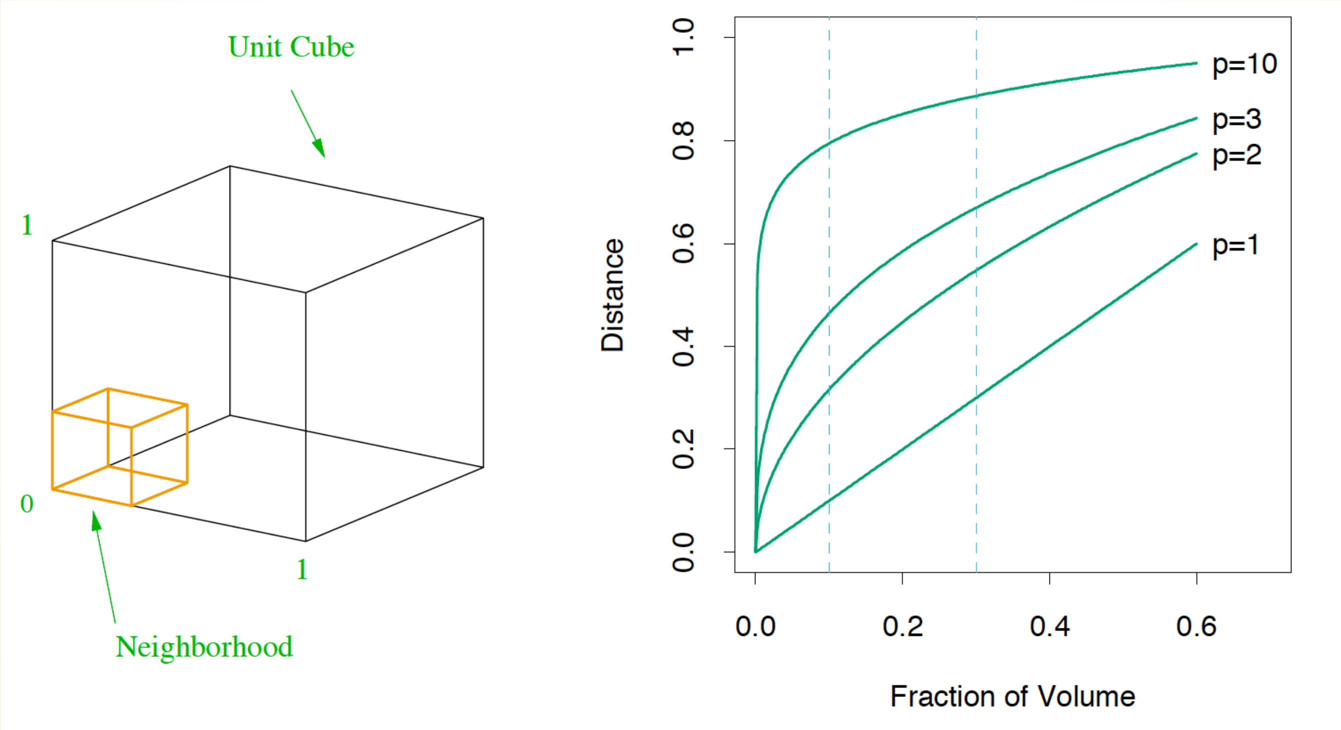

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- It means you average over points that are quite different

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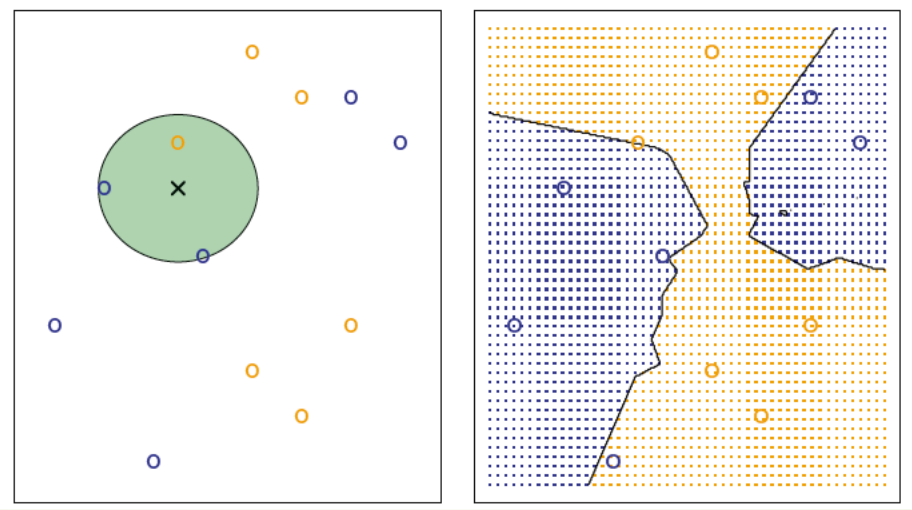

Next up: Linear regression, logistic regression, decision trees

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