A computational approach to diffusion coefficients.

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Introduction

During years of research, the study of the statistical properties of chaotic systems has been fundamental to progress in various areas. We can observe interesting relationships between concepts of probability and dynamical systems, which represent a harmonious advance for mathematics.

In this poster, we present a study of dynamic systems with noise, utilizing concepts from probability and dynamic systems, along with rigorous computational approaches, to obtain quantitative results on the statistical properties of our systems. Let's use [Nis23] to introduce the ideas of systems with noise and so we will get Random dynamical system. Now let $T : [-1,1] \to [-1,1]$ be a measurable map, in a first approach let's think of T Lipschitz to facilitate some estimates. Again we see the map T induces an operator on $L : \mathcal{M}_{\text{fin}}([-1,1]) \to \mathcal{M}_{\text{fin}}([-1,1])$ where $\mathcal{M}_{\text{fin}}([-1,1])$ is the space of finite measures on $[-1,1]$.

Definition (Boundary condition)**.** We will call boundary condition one of the two following maps:

- $\pi_P(x) = x \bmod 2$, called a periodic boundary conditions,
- $\pi_R(x) = (\min_{i\in\mathbb{Z}} |(x+1) - 4i|) - 1$, called a reflecting boundary conditions.

When the choice of the boundary condition is unimportant we will denote a boundary condition by π. We will denote by π_* the push-forward map acting on measures by

$$(\pi_*\nu)(A) = \nu\left(\pi^{-1}(A)\right).$$

Given a probability measure ν on $[-1,1]$ we define its extension $\hat{\nu}$ on $\mathbb{R}$ as the unique measure $\hat{\nu}$ on $\mathbb{R}$ such that $\hat{\nu}(A) = \nu(A \cap [-1,1])$ for all A measurable in $\mathbb{R}$.

Definition (Convolution)**.** Let ν be any probability measure in $[-1,1]$, and let ρ be a bounded variation function on $[-\xi,\xi]$ with $\int_{-\xi}^{\xi}\rho = 1$; their convolution is the unique probability measure $\hat{\rho} * \hat{\nu}$ on $\mathbb{R}$ such that

$$\hat{\rho} * \hat{\nu}(A) = \int_{[-\xi,\xi]} \hat{\rho}(y)\hat{\nu}(A-y)dm(y)$$

where $A - y$ to denote the set $\{x - y \mid x \in A\}$.

Definition (Mother noise kernel)**.** Let ρ a bounded variation function such that $\rho(x) \geq c > 0$ for all $x \in [-1,1]$, $\rho(x) = 0$ outside $[-1,1]$ and $\int_{-1}^{1}\rho(x)dm = 1$; we will call such a function a mother noise kernel.

In the following, define

$$\rho_\xi(x) := \frac{1}{\xi}\rho\left(\frac{x}{\xi}\right),$$

we will call ξ the amplitude of the noise.
Let $([-\xi,\xi], \mathcal{S}, p)$ be a probability space and $\Sigma = [-\xi,\xi]^{\mathbb{N}}$ endowed with the σ-algebra $\Omega = \mathcal{S}^{\mathbb{N}}$ and the probability measure $\mathbb{P} = p^{\mathbb{N}}$. We will use the standard notation for the shift map on Σ, $\sigma(\{\omega_i\}_{i\in\mathbb{N}}) = \{\omega_{i+1}\}_{i\in\mathbb{N}}$.

Definition. Let $([-1,1], \mathcal{B})$ be a measurable space and let consider the product σ-algebra on the space $[-1,1] \times \Sigma$. A random transformation over σ is a measurable function

$$F : \Sigma \times [-1,1] \to \Sigma \times [-1,1]$$

1. $F(\omega, x) = (\sigma(\omega), F_\omega(x))$, where $T : [-1,1] \to [-1,1]$ be a measurable non-singular function.
2. The map $\omega \to F_\omega(x)$ depends only from the 0-th coordinate of ω, therefore we shall use, with abuse, the notation $(\omega)_0$.

This skew product models the evolution of the stochastic process

$$X_0 = x_0, \quad X_{n+1} = \pi\left(T\left(X_n\right) + \omega_n\right)$$

where ω_n is a random variable with probability density ρ_ξ and π is either a periodic or reflecting boundary condition. In general we denoted the random dynamical system by $T_\xi = T(x) + \xi$.

Let's define the following Koopman operator for our system with noise, then let consider for some observable $\varphi \colon [-1,1] \to \mathbb{R}$ in $L^2(\nu)$

$$K\varphi(x) = \int \varphi(T(x) + \xi)\rho_\xi(\xi)d\xi,$$

making a change of variable we have to $v = T(x) + \xi$, so $\xi = v - T(x)$ and $d\xi = dv$, then

$$K\varphi(x) = \int \varphi(v)\rho_\xi(v - T(x))dv.$$

One of the equations that plays an important role is the Poisson equation given by $\varphi - c = g - Kg$, where $c = \int \varphi d\nu$ and g is the solution of the equation.

The Problem

Let's consider $\psi := \varphi - c$, and with that we have that our coefficient of diffusion is given by

$$\sigma^2 = \sum_{n=-\infty}^{\infty} \int \psi\left(\psi \circ T^n\right) d\nu,$$

by the formula of Green-Kubo we have that the diffusion coefficient can be rewritten as

$$\sigma^2 = \int \psi^2 \, d\nu + 2\sum_{n=1}^{\infty} \int \psi\left(\psi \circ T^n\right) d\nu.$$

Assuming that our system is mixing, we have that

$$g = \sum_{n=0}^{\infty} K^n \psi,$$

is a solution of the Poisson equation. By performing some calculations using the Poisson equation, we can obtain the following expression for the diffusion coefficient:

$$\sigma^2 = 2\int \psi \, g \, d\nu - \int \psi^2 \, d\nu,$$

where g is the solution of the Poisson equation. Then we will explore technics that allows us to transport quantitative information from finite dimensional objects to our infinite dimensional objects. The first step in our approach will be to discretize K and φ, and then approach c and g by strictly controlling the error.

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a uniform grid with step size $x_{i+1} - x_i = \frac{1}{N}$. The hat basis function $h_i(x)$ is defined by

$$h_i(x) = \begin{cases} N(x - x_{i-1}), & x \in [x_{i-1}, x_i], \\ N(x_{i+1} - x), & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

and define the projection operator $\Pi_\delta : Lip([-1,1]) \to C^\infty([-1,1])$ as

$$\Pi_\delta(f)(x) = \sum_{i=0}^{N} f(x_i)h_i(x),$$

where $\delta = \frac{1}{N}$. This basis of functions will help us to discretize our operator and observable, and thus starting the computational approach. We will use an adaptation of the approach found in [Bre+25], so that we do not need to estimate diffusion coefficients as [WBa+16].

Result

Using the discretization via the projection operator Π_δ, we can approximate K and φ by $K_\delta := \Pi_\delta K \Pi_\delta$ and $\varphi_\delta := \Pi_\delta(\varphi)$ respectively. With that, we can define an approximate solution of the Poisson equation $\tilde{g}$, and thus obtain an approximate diffusion coefficient

$$\tilde{\sigma}_\delta^2 = 2\int \psi_\delta \, \tilde{g} \, d\mu - \int \psi_\delta^2 \, d\mu.$$

Our concern is to be able to accurately estimate the error of

$$|\sigma^2 - \tilde{\sigma}_\delta^2|,$$

and for this we will use the estimates of errors $\tilde{g} = \sum_{n=0}^{N} K_\delta^n \psi_\delta$, $|c - \tilde{c}| \leq \int |r| \, d\nu$, where $r := (Id - K)\tilde{g} - (\varphi - \tilde{c})$. We also need to estimate the error the product error

$$\int (\phi - c) \, g \, d\nu,$$

we have that our functions by in $L^2(\nu)$, then by Cauchy-Schwarz inequality, we have

$$\int (\phi - c) \, g \, d\nu \leq \|(\phi - c)\|_{L^2(\nu)} \|g\|_{L^2(\nu)}.$$

As we are using the base of hats functions, we can notice that

$$\int_{x_i}^{x_{i+1}} \varphi_{i+1}(x)\varphi_i(x)d\,x = \frac{x_{i+1} - x_i}{6} \leq \frac{1}{6k}\|\rho_\xi\|_{BV},$$

then if we take a partition thinner than the noise performed, we can estimate the error of the product.

Example

Let consider $T : [0,1] \to [0,1]$, where $T(x) = 2x + 0.01\sin(2\pi x) \mod (1)$, and the observable $\varphi(x) = x^2$. Then we can use the package to estimate $\tilde{c}$, $\tilde{g}$, and finally estimate the error of the approximate diffusion coefficient. Scan a Qrcode:

References

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