Unit 2. Partial differential equations (PDEs) in physics

Lecture 204: PDE classification

Reference book:

“Introduction to Computational Astrophysical Hydrodynamics” by Zingale.
http://bender.astro.sunysb.edu/hydro_by_example/CompHydroTutorial.pdf

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Classification

Partial differential equations (PDEs) are usually grouped into one of three different classes:

Parabolic	*Heat equation*
Elliptic	*Poisson equation*
Hyperbolic	*Wave equation*

Parabolic equations

The canonical parabolic PDE is the heat equation:

$$\frac{\partial \phi}{\partial t} = k \frac{\partial^2 f}{\partial x^2}$$

This has aspects of both hyperbolic and elliptic PDEs.

The heat equation represents diffusion—an initially sharp feature will spread out into a smoother profile on a timescale that depends on the coefficient k. We'll encounter parabolic equations for thermal diffusion and other types of diffusion (like species, mass), and with viscosity.

Diffusion

Physically, a diffusive process obeys Fick's law—the quantity that is diffusing, ϕ, moves from higher to lower concentration at a rate proportional to the gradient,

$$F = -k\nabla\phi$$

If we think of this as a diffusive flux, then we can write the time-rate-of-change of ϕ as a conservation law:

$$\phi_t + \nabla \cdot F(\phi) = 0$$

This gives rise to the diffusion equation:

$$\phi_t = \nabla \cdot (k\nabla\phi)$$

Diffusion

In one-dimension, and assuming that the diffusion coefficient, k, is constant, we have

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial \phi}{\partial x}\right)$$

The diffusion equation can describe thermal diffusion (for example, as part of the energy equation in compressible flow), species/mass diffusion for multi-species flows, or the viscous terms in incompressible flows. In this form, the diffusion coefficient (or conductivity), k, can be a function of x, or even ϕ.

Diffusion

We will consider a constant diffusion coefficient as our model problem:

$$\frac{\partial \phi}{\partial t} = k\frac{\partial^2 \phi}{\partial x^2}$$

The diffusion equation is the prototypical parabolic PDE. The basic behavior of the diffusion equation is to take strongly peaked concentrations of ϕ and smooth them out with time.