

Reacting Mixing Layer

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Abstract

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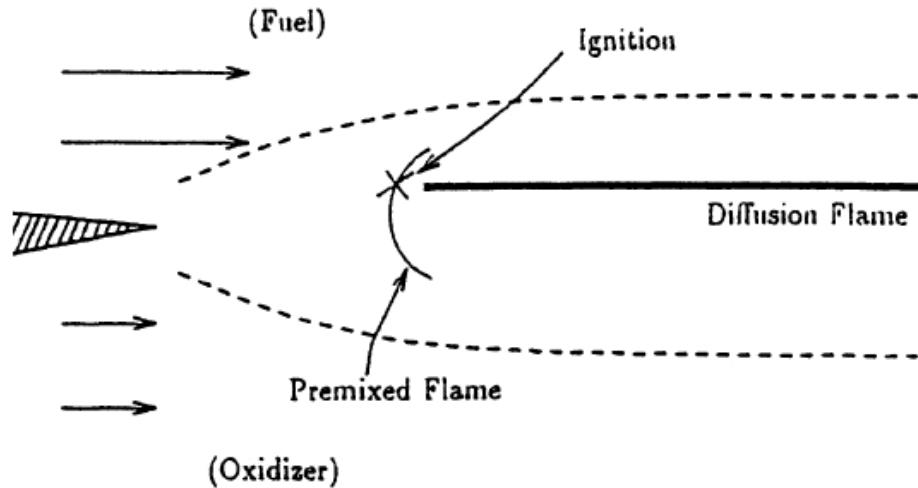


Figure 1: Reacting Mixing Layer

1 Reacting Mixing Layer

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4 1. REACTING MIXING LAYER EQUATIONS

5 1.1. CONSERVATION'S EQUATIONS

6 In a compressible, conservative form and in a perfect gas the conservation's
7 equations are:

8

9 **Mass conservation:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \quad \text{for } i = 1, 2, 3, \quad (1)$$

10

11

12 **Species conservation:**

$$\frac{\partial}{\partial t}(\rho Y^m) + \frac{\partial}{\partial x_i}(\rho Y^m(u_i + V^m)) = s^m \Omega \quad \text{for } i = 1, 2, 3 \quad \text{for } m = \text{Oxidizer, Fuel} \quad (2)$$

13

14

15 Using a constitutive equation for V^m

$$V^m = -\rho D^m \frac{\partial Y^m}{\partial x_i} \quad (3)$$

16

17 where D^m is the diffusion coefficient of specie respect to the most abundant
18 specie.

19 **who is Ω ?**

20

21 **Momemtum conservation:**

22

23

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j \quad \text{for } i, j = 1, 2, 3 \quad (4)$$

24

25

26 where

27

28

$$\tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (5)$$

29

30

31 and using the Stoke's relation:

32

33

$$\lambda = -\frac{2}{3} \mu \quad (6)$$

$$\tau_{ij} = \mu \left(-\frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \quad (7)$$

34

35

36 **Energy conservation:**

37

38

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x_i}(\rho u_i h) = \frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + \frac{\partial u_j}{\partial x_i} \tau_{ij} - \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + Q\Omega \quad (8)$$

39

40

41 **Perfect gas equation:**

42

43

$$p = \rho T R \quad (9)$$

44

45

$$h = c_p T \quad (10)$$

46

47

48

49 In the above equations ρ is the density, $u_i = (u, v, w)$ are the different velocities in the three different dimensions x , y and z . p is the pressure of the flow, 50 T is the temperature of the flow, f_j is the body force acting on the flow. Y^m 51 are the different mass fraction for the two different species, oxygen and fuel. 52 μ , D^m and k are the viscosity, diffusion coefficient for the different species 53 and the gas thermal conductivity. Q is the heat release for the chemical 54 reaction and Ω which is the mass reaction rate.

56

57 **Base Flow equations:**

58

59 We are interesting in the a two dimensional compressible mixing layer, be- 60 cause the tridimensional effect can be neglected for the study of instabilities 61 in the first stages, it states that for *the onset of linear instability in parallel* 62 *shear flows, the least stable (i.e., first to become unstable) disturbances are* 63 *two-dimensional* [2]. In the incompressible case, the process of transition to

64 turbulence in a mixing layer is dictated by: the growth of two dimensional
 65 coherent structures and the development of secondary instability, the merg-
 66 ing of the large-structures and finally the breakdown into small-scales three
 67 dimensional turbulence [1].

68 The mean flow for two dimensional mixing layer which separates two
 69 fluids, the oxidizer and the fuel, at different speeds and temperature with
 70 zero pressure gradient can be assumed that is governed by boundary layer
 71 equations. This mean that gradients for the different properties are in the y
 72 direction, in Cartesian coordinates.

73 To applied the boundary layer equations is necessary to defined a charac-
 74 teristic length of the flow. In this problem exist two characteristics lengths,
 75 one can be defined as the thickness of the mixing layer δ and other can be
 76 defined as the distance that the mixing layer needs to grow L_c .

77 Assume that δ is sufficient small compared with the x length where the
 78 mixing layer is development, it mean that $\delta/L_c \ll 1$. The mean velocity
 79 scale can be approximate, in the x direction is the velocity of order of U_c ,
 80 and $\partial/\partial x$ is of order of $1/L_c$ and assuming that ρ is of order of ρ_c then:

$$\frac{\partial}{\partial x}(\rho u) \sim \frac{\rho_c U_c}{L_c} \quad (11)$$

81 Using the conservation of mass equation 1 in two dimensions (x, y) , the
 82 order of

$$\frac{\partial}{\partial y}(\rho v) \sim \frac{\rho_c U_c}{L_c} \quad (12)$$

85
 86 We know that in the mixing layer the velocity v is smaller than u , then
 87 we can assume that

$$v \sim \frac{\delta}{L_c} U_c \quad \text{being} \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}. \quad (13)$$

91
 92 With this assumptions the terms in the conservation equations of order:
 93

95

$$\frac{\partial^2}{\partial x^2} \sim \frac{1}{L_c^2}, \quad (14)$$

96

97
98 are smaller than term of order

99

$$100 \quad \frac{\partial^2}{\partial y^2} \sim \frac{1}{\delta^2}. \quad (15)$$

101

102
103 Thereby, using the different length scales in the others conservations equa-
104 tions and assuming steady state to the mean flow we get:

105 **Mass conservation:**

107

$$108 \quad \frac{\partial}{\partial x}(\rho u) = \frac{\partial}{\partial y}(\rho v) \quad (16)$$

109

110
111
112 **Species conservation:**

114

$$115 \quad \frac{\partial}{\partial x}(\rho Y^o(u + V^o)) + \frac{\partial}{\partial y}(\rho Y^o(v + V^o)) = s^o \Omega \quad (17)$$

116

117

$$118 \quad \frac{\partial}{\partial x}(\rho Y^F(u + V^F)) + \frac{\partial}{\partial y}(\rho Y^F(v + V^F)) = s^F \Omega \quad (18)$$

119

120
121 Using a constitutive equation for V^m

122

$$123 \quad V^m = -\rho D^m \frac{\partial}{\partial x_i}(\ln(Y^m)) \quad \text{for } m = 1, 2 \quad (19)$$

124

125

126 Hence

127

128

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{\partial}{\partial x_i} \left(\rho D^m \frac{\partial Y^m}{\partial x_i} \right) \quad \text{for } m = 1, 2 \quad (20)$$

129

130

131

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{\partial}{\partial x} \left(\rho D^m \frac{\partial Y^m}{\partial x} \right) - \frac{\partial}{\partial y} \left(\rho D^m \frac{\partial Y^m}{\partial y} \right) \quad \text{for } m = 1, 2 \quad (21)$$

132

133

134 Using the characteristic length scales:

135

136

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{1}{L_c} \frac{\rho_c D_c^m Y_c^m}{L_c} - \frac{1}{\delta} \frac{\rho_c D_c^m Y_c^m}{\delta} \quad \text{for } m = 1, 2 \quad (22)$$

137

138

139

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{\rho_c D_c^m Y_c^m}{L_c^2} - \frac{\rho_c D_c^m Y_c^m}{\delta^2} \quad \text{for } m = 1, 2 \quad (23)$$

140

141

142

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\cancel{\frac{\rho_c D_c^m Y_c^m}{L_c^2}} - \overset{\approx 0}{\frac{\rho_c D_c^m Y_c^m}{\delta^2}} \quad \text{for } m = 1, 2 \quad (24)$$

143

144

145 The Species conservation can be wrote as:

146

147

$$\frac{\partial}{\partial x}(\rho Y^o u) + \frac{\partial}{\partial y}(\rho Y^o v) = \frac{\partial}{\partial y} \left(\rho D^o \frac{\partial Y^o}{\partial y} \right) + s^o \Omega \quad (25)$$

148

149

$$\frac{\partial}{\partial x}(\rho Y^F u) + \frac{\partial}{\partial y}(\rho Y^F v) = \frac{\partial}{\partial y} \left(\rho D^F \frac{\partial Y^F}{\partial y} \right) + s^F \Omega \quad (26)$$

150

151

152 Using the mass conservation equation 1, we can write the Species conser-
153 vation as:

154

155

$$\rho u \frac{\partial Y^o}{\partial x} + \rho v \frac{\partial Y^o}{\partial y} = \frac{\partial}{\partial y} \left(\rho D^o \frac{\partial Y^o}{\partial y} \right) + s^o \Omega \quad (27)$$

156

157

$$\rho u \frac{\partial Y^F}{\partial x} + \rho v \frac{\partial Y^F}{\partial y} = \frac{\partial}{\partial y} \left(\rho D^F \frac{\partial Y^F}{\partial y} \right) + s^F \Omega \quad (28)$$

158

159

160

161

Momentum conservation:

162

163

$$\frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j \quad \text{for } i, j = 1, 2, 3 \quad (29)$$

164

165

166

$$\tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (30)$$

167

168

169

$$\frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \quad (31)$$

170

171

172

$$\frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial}{\partial x_j} \left(\lambda \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left(\mu \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i} \left(\mu \frac{\partial u_j}{\partial x_i} \right) \quad (32)$$

173

174

175

$$\frac{\partial \tau_{i1}}{\partial x_i} = \frac{\partial}{\partial x} \left(\lambda \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} \left(\lambda \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (33)$$

176

177

178

$$\frac{\partial \tau_{i1}}{\partial x_i} = \frac{1}{L_c^2}(\lambda_c U_c) + \frac{1}{L_c^3}(\lambda_c \delta U_c) + \frac{1}{L_c^2}(\mu_c U_c) + \frac{1}{L_c^2}(\mu_c U_c) + \frac{1}{L_c^2}(\mu_c U_c) + \frac{1}{\delta^2}(\mu_c U_c) + \quad (34)$$

179

180

181

$$\frac{\partial \tau_{i1}}{\partial x_i} = \frac{1}{L_c^2}(\lambda_c U_c) \xrightarrow{\approx 0} \frac{1}{L_c^3}(\lambda_c \delta U_c) \xrightarrow{\approx 0} \frac{1}{L_c^2}(\mu_c U_c) \xrightarrow{\approx 0} \frac{1}{L_c^2}(\mu_c U_c) \xrightarrow{\approx 0} \frac{1}{L_c^2}(\mu_c U_c) \xrightarrow{\approx 0} \frac{1}{\delta^2}(\mu_c U_c) + \quad (35)$$

182

183

184

$$\frac{\partial \tau_{i1}}{\partial x_i} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (36)$$

185

186

187

$$\frac{\partial \tau_{i2}}{\partial x_i} = \frac{\partial}{\partial y} \left(\lambda \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\lambda \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial x} \left(\mu \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial v}{\partial y} \right) \quad (37)$$

188

189

190 Now using the characteristic length of the problem we found:

191

192

$$\frac{\partial \tau_{i2}}{\partial x_i} = \frac{1}{\delta L_c}(\lambda_c U_c) + \frac{1}{\delta L_c}(\lambda_c U_c) + \frac{1}{\delta L_c}(\mu_c U_c) + \frac{1}{\delta L_c}(\mu_c U_c) + \frac{1}{L_c^3}(\mu_c \delta U_c) + \frac{1}{\delta L_c}(\mu_c U_c) \quad (38)$$

193

194 Multiply 38 by δ^2

195

196

$$\frac{\partial \tau_{i2}}{\partial x_i} = \left(\frac{\delta}{L_c} \right) (\lambda_c U_c) + \left(\frac{\delta}{L_c} \right) (\lambda_c U_c) + \left(\frac{\delta}{L_c} \right) (\mu_c U_c) + \left(\frac{\delta}{L_c} \right) (\mu_c U_c) + \left(\frac{\delta}{L_c} \right)^3 (\mu_c \delta U_c) + \left(\frac{\delta}{L_c} \right) (\mu_c U_c) \quad (39)$$

197

198

199

$$\frac{\partial \tau_{i2}}{\partial x_i} = \left(\frac{\delta}{L_c} \right) (\lambda_c U_c) + \left(\frac{\delta}{L_c} \right) (\lambda_c U_c) + \left(\frac{\delta}{L_c} \right) (\mu_c U_c) + \left(\frac{\delta}{L_c} \right) (\mu_c U_c) + \left(\frac{\delta}{L_c} \right)^3 (\mu_c \delta U_c) + \left(\frac{\delta}{L_c} \right) (\mu_c U_c) \quad (40)$$

200

201

202

$$\frac{\partial \tau_{i2}}{\partial x_i} \approx 0 \quad (41)$$

203

204

205 Now Momentum conservation equation in x direction can be wrote:

206

207

$$\frac{\partial}{\partial x} (\rho u u) + \frac{\partial}{\partial y} (\rho v u) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + \rho f_x \quad (42)$$

208

209

210 Using the characteristic length of the problem, we can see that the oth-
211 ers terms are important:

212

213

$$\frac{1}{L_c} (\rho_c U_c^2) + \frac{1}{L_c} (\rho_c U_c^2) = \frac{p_c}{L_c} + \frac{1}{\delta^2} (\mu_c U_c) + \rho_c f_{xc}. \quad (43)$$

214

In the viscous term multiple by $\frac{L_c}{U_c^2 \rho_c}$:

$$\frac{L_c}{U_c^2 \rho_c} \frac{1}{\delta^2} (\mu_c U_c) = \frac{\mu_c}{\rho_c \delta} \frac{L_c}{\delta U_c} = \frac{\mu_c}{\rho_c V_c \delta} \quad (44)$$

215

In order to keep the approximation $\delta_c \ll L_c$ this term:

$$\frac{\mu_c}{\rho_c V_c \delta} \approx 1 = Re \quad (45)$$

216

217

218 Using the mass conservation equation 16 in 43:

219

220

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + \rho f_x \quad (46)$$

221

222

223 The Momentum equation in y direction can be wrote as:

224

225

$$\frac{\partial}{\partial x} (\rho u v) + \frac{\partial}{\partial y} (\rho v v) = - \frac{\partial p}{\partial y} + \rho f_y \quad (47)$$

226

227

228 Using the characteristic length of the problem, we can see that the oth-
229 ers terms are important:

230

231

$$\frac{1}{L_c} \frac{\delta}{L_c} (\rho_c U_c^2) + \frac{1}{L_c} \frac{\delta}{L_c} (\rho_c U_c^2) = - \frac{p_c}{L_c} + \rho_c f_{yc}. \quad (48)$$

232

233

234

$$\cancel{\frac{1}{L_c} \frac{\delta}{L_c} (\rho_c U_c^2)} \approx 0 + \cancel{\frac{1}{L_c} \frac{\delta}{L_c} (\rho_c U_c^2)} \approx 0 = - \frac{p_c}{L_c} + \rho_c f_{yc}. \quad (49)$$

235

236

237 Then the y momentum equation using the boundary layer assumption can
238 be wrote as:

239

240

$$\frac{\partial p}{\partial y} = \rho f_y. \quad (50)$$

241

242

²⁴³ This equation means that the only variations of the pressure in y is determined by the body forces in that direction.

245

246 Energy conservation:

247

$$\frac{\partial}{\partial x_i}(\rho u_i h) = u_i \frac{\partial p}{\partial x_i} + \frac{\partial u_j}{\partial x_i} \tau_{ij} - \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + Q\Omega \quad (51)$$

248

249

$$\tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (52)$$

250

251

$$\frac{\partial u}{\partial x_i} \tau_{i1} = \frac{\partial u}{\partial x_i} \lambda \delta_{i1} \frac{\partial u_k}{\partial x_k} + \frac{\partial u}{\partial x_i} \mu \left(\frac{\partial u_i}{\partial x} + \frac{\partial u}{\partial x_i} \right) \quad (53)$$

253

354

$$\frac{\partial u}{\partial x_i} \tau_{i1} = \frac{\partial u}{\partial x} \lambda \frac{\partial u_k}{\partial x_k} + \frac{\partial u}{\partial x_i} \mu \left(\frac{\partial u_i}{\partial x} + \frac{\partial u}{\partial x_i} \right) \quad (54)$$

256

257

$$\frac{\partial u}{\partial x_i} \tau_{i1} = \frac{\partial u}{\partial x} \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \frac{\partial u}{\partial x_i} \frac{\partial u_i}{\partial x} + \mu \left(\frac{\partial u}{\partial x_i} \right)^2 \quad (55)$$

259

260

$$\frac{\partial u}{\partial x} \cdot \tau_{i1} = \lambda \left(\frac{\partial u}{\partial x} \right)^2 + \lambda \frac{\partial u}{\partial x} \frac{\partial v}{\partial u} + \mu \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \mu \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \mu \left(\frac{\partial u}{\partial x} \right)^2 + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (56)$$

262

248

$$\frac{\partial u}{\partial x_i} \tau_{i1} = \lambda \left(\frac{\partial u}{\partial x} \right)^2 + \lambda \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \mu \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \mu \left(\frac{\partial u}{\partial x} \right)^2 + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (57)$$

265

266

267

$$\frac{\partial u}{\partial x_i} \tau_{i1} \approx \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (58)$$

268

269

270

$$\frac{\partial v}{\partial x_i} \tau_{i2} = \frac{\partial v}{\partial x_i} \lambda \delta_{i2} \frac{\partial u_k}{\partial x_k} + \frac{\partial v}{\partial x_i} \mu \left(\frac{\partial u_i}{\partial y} + \frac{\partial v}{\partial x_i} \right) \quad (59)$$

271

272

273

$$\frac{\partial v}{\partial x_i} \tau_{i2} = \lambda \frac{\partial v}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \frac{\partial v}{\partial x_i} \frac{\partial u_i}{\partial y} + \left(\mu \frac{\partial v}{\partial x_i} \right)^2 \quad (60)$$

274

275

276

$$\frac{\partial v}{\partial x_i} \tau_{i2} = \lambda \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial v}{\partial y} \right)^2 + \mu \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \mu \left(\frac{\partial v}{\partial y} \right)^2 + \mu \left(\frac{\partial v}{\partial x} \right)^2 + \mu \left(\frac{\partial v}{\partial y} \right)^2 \quad (61)$$

277

278

279

$$\frac{\partial v}{\partial x_i} \tau_{i2} = \lambda \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} + \overbrace{\left(\frac{\partial v}{\partial y} \right)^2}^{\approx 0} + \mu \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \overbrace{\mu \left(\frac{\partial v}{\partial y} \right)^2}^{\approx 0} + \mu \left(\frac{\partial v}{\partial x} \right)^2 + \overbrace{\left(\frac{\partial v}{\partial y} \right)^2}^{\approx 0} \quad (62)$$

280

281

282

283

284

$$\frac{\partial v}{\partial x_i} \tau_{i2} \approx 0 \quad (63)$$

285

Finally for the heat diffusion:

$$\frac{\partial kT}{\partial x_i} = k \frac{\partial T}{\partial x} + k \frac{\partial T}{\partial y} \quad (64)$$

²⁸⁶ Using the characteristic lengths:

$$\frac{\partial kT}{\partial x_i} = k \frac{T_c}{L_c} + k \frac{T_c}{\delta_c} \quad (65)$$

²⁸⁷

²⁸⁸

²⁸⁹ Hence, the energy equation for steady flow become:

²⁹⁰

$$\frac{\partial}{\partial x_i}(\rho u_i h) = u_i \frac{\partial p}{\partial x_i} + \frac{\partial u_j}{\partial x_i} \tau_{ij} - \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + Q\Omega \quad (66)$$

²⁹¹

²⁹²

$$\frac{\rho_c U_c h_c}{L_c} + \frac{\rho_c U_c \delta_c h_c}{L_c \delta_c} = U_c \frac{p_c}{L_c} + U_c \frac{p_c}{L_c} + \frac{U_c \delta_c p_c}{L_c \delta_c} + \mu \frac{U_c^2}{\delta^2} + k \frac{T_c}{L_c^2} + k \frac{T_c}{\delta_c^2} + \dots \quad (67)$$

²⁹³

$$\frac{1}{L_c} + \frac{1}{L_c} = \frac{p_c}{\rho_c h_c} \frac{1}{L_c} + \frac{p_c}{\rho_c h_c} \frac{1}{L_c} + \mu \frac{U_c}{\rho_c h_c} \frac{1}{\delta^2} + k \cancel{\frac{1}{\rho_c U_c h_c} \frac{T_c}{L_c^2}} \overset{\approx 0}{\rightarrow} + k \frac{1}{\rho_c U_c h_c} \frac{T_c}{\delta_c^2} + \dots \quad (68)$$

²⁹⁴

²⁹⁵ Then the Energy conservation gets:

²⁹⁶

$$\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) = u_i \frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial u}{\partial y} \right)^2 - k \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + Q\Omega \quad (69)$$

²⁹⁷

²⁹⁸ Using the mass conservation equation:

²⁹⁹

$$h \left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) \right) \overset{\approx 0}{\rightarrow} + \rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u_i \frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial u}{\partial y} \right)^2 - k \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + Q\Omega \quad (70)$$

³⁰⁰

³⁰¹ The equations that govern the Steady Reacting Mixing Layer are:

³⁰²

³⁰³

$$\rho u \frac{\partial Y^o}{\partial x} + \rho v \frac{\partial Y^o}{\partial y} = \frac{\partial}{\partial y} \left(\rho D^o \frac{\partial Y^o}{\partial y} \right) + s^o \Omega \quad (72)$$

³⁰⁴

³⁰⁵

$$\rho u \frac{\partial Y^F}{\partial x} + \rho v \frac{\partial Y^F}{\partial y} = \frac{\partial}{\partial y} \left(\rho D^F \frac{\partial Y^F}{\partial y} \right) + s^F \Omega \quad (73)$$

306

307

308

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + \rho f_x \quad (74)$$

309

310

311

$$\frac{\partial p}{\partial y} = \rho f_y. \quad (75)$$

312

313

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319

320

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + Q\Omega \quad (76)$$

321

322

323

324

325

326

327

328

329

$$\rho u \frac{\partial Y^o}{\partial x} + \rho v \frac{\partial Y^o}{\partial y} = \frac{\partial}{\partial y} \left(\rho D^o \frac{\partial Y^o}{\partial y} \right) + s^o \Omega \quad (77)$$

$$\rho u \frac{\partial Y^F}{\partial x} + \rho v \frac{\partial Y^F}{\partial y} = \frac{\partial}{\partial y} \left(\rho D^F \frac{\partial Y^F}{\partial y} \right) + s^F \Omega \quad (78)$$

326

327

328

329

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (79)$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + Q\Omega \quad (80)$$

330

331

332

$$p = \rho R T \quad (81)$$

333

334

$$h = c_p T \quad (82)$$

335

336

$$\Omega = \beta Y_F Y_O \exp\left(\frac{E}{RT}\right) \quad (83)$$

337 **Boundary conditions**

338 $\mathbf{x} \rightarrow +\infty$

339 $u = u_{+\infty}$

340 $v = v_{+\infty} = 0$

341 $p = p_{+\infty} = \text{constant} = C$

342 $\rho = \rho_{+\infty} = 0$

343 $Y_F = Y_{F+\infty} = 1$

344 $Y_O = Y_{O+\infty} = 0$

345

346 $\mathbf{x} \rightarrow -\infty$

347 $u = u_{-\infty}$

348 $v = v_{-\infty} = 0$

349 $p = p_{-\infty} = \text{constant} = C$

350 $\rho = \rho_{-\infty} = 0$

351 $Y_F = Y_{F+\infty} = 0$

352 $Y_O = Y_{O+\infty} = 1$

353 2. DIMENSIONLESS EQUATIONS

354 The dimesionless equations for a reacting mixing layer are obtained using the
355 positive freestream variables, L_c characteritic lenghts:

356

357

$$\bar{x} = \frac{x}{L_c}, \quad \bar{y} = \frac{y}{\delta}, \quad (84)$$

358

359

$$U = \frac{u}{u_\infty}, \quad V = \frac{v}{v_\infty}, \quad \text{but} \quad v_\infty = \frac{u_\infty \delta_c}{L_c}, \quad \text{then} \quad V = \frac{L_c}{\delta} \frac{v}{u_\infty}, \quad (85)$$

360

361

$$\bar{\rho} = \frac{\rho}{\rho_\infty}, \quad \bar{T} = \frac{T}{T_\infty}, \quad P = \frac{p}{P_\infty}, \quad (86)$$

362

363

$$\psi_F = \frac{Y^F}{Y^{F\infty}}, \quad \psi_O = \frac{Y^O}{Y^{O\infty}}, \quad (87)$$

364

365

$$\frac{\mu}{\mu_\infty} = \frac{k}{k_\infty} = \frac{c_v}{c_{v\infty}} = \frac{c_p}{c_{p\infty}} = \frac{D^F}{D^{F\infty}} = \frac{D^0}{D^{O\infty}} = \frac{R}{R_\infty} = \bar{T}^n, \quad (88)$$

366

367

368 **Mass conservation:**

369

370

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad (89)$$

371

372

373

$$\frac{\rho_\infty U_\infty}{L_c} \frac{\partial}{\partial \bar{x}}(\varrho U) + \frac{\rho_\infty U_\infty}{L_c} \frac{\partial}{\partial \bar{y}}(\varrho V) = 0 \quad (90)$$

374

375

376

$$\frac{\partial}{\partial \bar{x}}(\varrho U) + \frac{\partial}{\partial \bar{y}}(\varrho V) = 0 \quad (91)$$

377

378

379 **Momemtum equation:**

380

381

$$\frac{\rho_\infty U_\infty^2}{L_c} \varrho U \frac{\partial U}{\partial \bar{x}} + \frac{\rho_\infty U_\infty^2}{L_c} \varrho V \frac{\partial U}{\partial \bar{y}} = \frac{\mu_\infty U_\infty}{\delta^2} \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial U}{\partial \bar{y}} \right) \quad (92)$$

382

383 We do not known the relation between characteristics lenghts of the problem,
 384 δ and L_c . For the above equation have the same same order, we can defined:
 385

386

387

$$\frac{\rho_\infty U_\infty}{L_c} = \frac{\mu_\infty}{\delta^2} \quad (93)$$

388

389

390

$$\frac{\delta^2}{L_c} = \frac{\mu_\infty}{\rho_\infty U_\infty} \quad (94)$$

391

391
392
393

$$\frac{\delta^2}{L_c} = L_c \frac{\mu_\infty}{\rho_\infty L_c U_\infty} \quad (95)$$

394

395

396 Defining:

397

398

$$Re_{L_c} = \frac{\mu_\infty}{\rho_\infty L_c U_\infty} \quad (96)$$

399

400

401 as the Reynolds number respect to L_c , we get:

402

403

$$\frac{\delta^2}{L_c} = L_c \frac{1}{Re_{L_c}} \quad (97)$$

404

405

406 then we obtain:

407

408

$$\delta = L_c \sqrt{\frac{1}{Re_{L_c}}} \quad (98)$$

409

410

411 Inserting this relation in the dimensionless Momemtum equation 92:

412

413

$$\frac{\rho_\infty U_\infty^2}{L_c} \varrho U \frac{\partial U}{\partial \bar{x}} + \frac{\rho_\infty U_\infty^2}{L_c} \varrho V \frac{\partial U}{\partial \bar{y}} = \frac{\mu_\infty U_\infty}{\delta^2} \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial U}{\partial \bar{y}} \right) \quad (99)$$

414

415

416 we obtain:

417

418

$$\varrho U \frac{\partial U}{\partial \bar{x}} + \varrho V \frac{\partial U}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial U}{\partial \bar{y}} \right) \quad (100)$$

419

420 Species conservation:

421

422

$$\frac{\rho_\infty U_\infty Y^{O\infty}}{L_c} \varrho U \frac{\partial \psi_o}{\partial \bar{x}} + \frac{\rho_\infty U_\infty Y^{O\infty}}{L_c} \varrho V \frac{\partial \psi_o}{\partial \bar{y}} + \frac{\rho_\infty D^{O\infty} Y^{O\infty}}{\delta_c^2} \frac{\partial}{\partial \bar{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \bar{y}} \right) + s^o \Omega \quad (101)$$

423

424

$$\frac{\rho_\infty U_\infty Y^{F\infty}}{L_c} \varrho U \frac{\partial \psi_F}{\partial \bar{x}} + \frac{\rho_\infty U_\infty Y^{F\infty}}{L_c} \varrho V \frac{\partial \psi_F}{\partial \bar{y}} + \frac{\rho_\infty D^{F\infty} Y^{F\infty}}{\delta^2} \frac{\partial}{\partial \bar{y}} \left(\rho D^F \frac{\partial \psi_F}{\partial \bar{y}} \right) + s^F \Omega \quad (102)$$

425

426

427

$$\varrho U \frac{\partial \psi_o}{\partial \bar{x}} + \varrho V \frac{\partial \psi_o}{\partial \bar{y}} + \frac{L_c}{\delta_c^2} \frac{D^{O\infty}}{U_\infty} \frac{\partial}{\partial \bar{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \bar{y}} \right) + s^o \Omega \quad (103)$$

428

429

430

$$\varrho U \frac{\partial \psi_F}{\partial \bar{x}} + \varrho V \frac{\partial \psi_F}{\partial \bar{y}} + \frac{L_c}{\delta_c} \frac{1}{\delta_c} \frac{D^{F\infty}}{U_\infty} \frac{\partial}{\partial \bar{y}} \left(\rho D^F \frac{\partial \psi_F}{\partial \bar{y}} \right) + s^F \Omega \quad (104)$$

431 Using the definition of the Prandtl, Lewis numbers and the characteristic
 432 lenghts relation, we can write:

$$Pr = \frac{\nu}{\alpha} = \frac{\mu_\infty c_{p\infty}}{k_\infty} \quad Le_n = \frac{\alpha}{D^{n\infty}} = \frac{k_\infty}{\rho_\infty c_{p\infty}} \frac{1}{D^{n\infty}} \quad (105)$$

433

where n is the specie, fuel (F) or oxidizer (O).

$$\frac{L_c}{\delta_c^2} \frac{D^{O\infty}}{U_\infty} = \frac{Re_{L_c}}{L_c} \frac{\mu_\infty}{U_\infty \rho_\infty} \frac{\rho_\infty D^{O\infty}}{\mu_\infty} = \frac{\rho_\infty D^{O\infty} c_{p\infty}}{k_\infty} \frac{k_\infty}{\mu_\infty c_{p\infty}} = \frac{1}{Le_O} \frac{1}{Pr} \quad (106)$$

434

435

$$\frac{L_c}{\delta_c} \frac{D^{F\infty}}{\delta_c U_\infty} = \frac{1}{Le_F Pr}, \quad (107)$$

436

437

438 Then we get:

439

440

$$\varrho U \frac{\partial \psi_o}{\partial \bar{x}} + \varrho V \frac{\partial \psi_o}{\partial \bar{y}} + \frac{1}{Le_o Pr} \frac{\partial}{\partial \bar{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \bar{y}} \right) + s^o \Omega \quad (108)$$

441

442

443

$$\varrho U \frac{\partial \psi_F}{\partial \bar{x}} + \varrho V \frac{\partial \psi_F}{\partial \bar{y}} + \frac{1}{Le_F Pr} \frac{\partial}{\partial \bar{y}} \left(\rho D^F \frac{\partial \psi_F}{\partial \bar{y}} \right) + s^F \Omega. \quad (109)$$

444

445

446

447

448 Before non-dimensionalization the energy equation, we use the equation that
449 relate the entaphy and the temperature for a perfect gas, we get:

450

451

$$\rho u \frac{\partial}{\partial x} (c_p T) + \rho v \frac{\partial}{\partial y} (c_p T) = \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + Q \Omega \quad (110)$$

452

453

454 Using the non-dimensional variables:

455

456

$$\frac{\rho_\infty U_\infty T_\infty c_{p\infty}}{L_c} \varrho U \frac{\partial}{\partial \bar{x}} (c_p \bar{T}) + \frac{\rho_\infty U_\infty T_\infty c_{p\infty}}{L_c} \varrho V \frac{\partial}{\partial \bar{y}} (c_p \bar{T}) = \frac{\mu_\infty U_\infty^2}{\delta^2} \mu \frac{\partial^2 U}{\partial \bar{y}^2} + \frac{k_\infty T_\infty}{\delta^2} \frac{\partial}{\partial \bar{y}} \left(k \frac{\partial \bar{T}}{\partial \bar{y}} \right) + Q \Omega \quad (111)$$

457

458

459

$$\varrho U \frac{\partial}{\partial \bar{x}} (c_p \bar{T}) + \varrho V \frac{\partial}{\partial \bar{y}} (c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial \bar{y}^2} + \frac{1}{Pr} \frac{\partial}{\partial \bar{y}} \left(k \frac{\partial \bar{T}}{\partial \bar{y}} \right) + Q \Omega \quad (112)$$

460

461

462 where:

463

464

$$M = \frac{U_\infty}{a_{0\infty}} \quad \text{Mach Number with} \quad a_{0\infty} = \sqrt{\gamma_\infty R_\infty T_\infty} \quad (113)$$

465

466

467

$$Pr = \frac{\mu_\infty c_p}{k_\infty} \quad \text{Prandtl Number} \quad (114)$$

468

469

470 Finally the equation for a perfect gas became:

471

472

$$1 = \rho \bar{T} \quad (115)$$

473

474

475

476 **Non-dimensional Reacting Mixing layer equation:**

477

478

479

$$\frac{\partial}{\partial \bar{x}}(\varrho U) + \frac{\partial}{\partial \bar{y}}(\varrho V) = 0 \quad (116)$$

480

481

482

483

$$\varrho U \frac{\partial \psi_o}{\partial \bar{x}} + \varrho V \frac{\partial \psi_o}{\partial \bar{y}} = \frac{1}{Le_o} \frac{\partial}{\partial \bar{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \bar{y}} \right) + s^o \Omega \quad (117)$$

484

485

486

$$\varrho U \frac{\partial \psi_F}{\partial \bar{x}} + \varrho V \frac{\partial \psi_F}{\partial \bar{y}} = \frac{1}{Le_F} \frac{\partial}{\partial \bar{y}} \left(\rho D^F \frac{\partial \psi_F}{\partial \bar{y}} \right) + s^F \Omega. \quad (118)$$

487

488

489

$$\varrho U \frac{\partial U}{\partial \bar{x}} + \varrho V \frac{\partial U}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial U}{\partial \bar{y}} \right) \quad (119)$$

490

491

492

$$\varrho U \frac{\partial}{\partial \bar{x}}(c_p \bar{T}) + \varrho V \frac{\partial}{\partial \bar{y}}(c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial \bar{y}^2} + \frac{1}{Pr} \frac{\partial}{\partial \bar{y}} \left(k \frac{\partial \bar{T}}{\partial \bar{y}} \right) + Q \Omega \quad (120)$$

493

494

495

$$1 = \varrho \bar{T} \quad (121)$$

496

497

498

499

3. SELF-SIMILARITY SOLUTION FOR REACTING MIXING LAYER EQUATIONS

500 To find a similarity solutions for a reacting mixing layer equations, firstly we
501 need to use the Howarth-Dorodnitsyn 225 transformation to find am incom-

502 pressible form of the compressibles equations.

503

504 Definig the stream function for compressible flow as:

505

506

$$\frac{\partial \Psi}{\partial x} = -\varrho V \quad \frac{\partial \Psi}{\partial y} = \varrho U, \quad (122)$$

507

508

509 The mass conservation equation is identically satisfied by the introduction of
510 this stream function. Now defining a vertical stretching as:

511

512

$$x = \bar{x} \quad y = \int_0^{\bar{y}} \varrho d\bar{y} \quad (123)$$

513

514

515 and using the chain rule:

516

517

$$f(\bar{x}, \bar{y}) \quad \bar{x} = \bar{x}(x, y) \quad \bar{y} = \bar{y}(x, y) \quad (124)$$

518

519

520

$$\frac{\partial f}{\partial \bar{x}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{x}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{x}} \quad (125)$$

521

522

523

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{y}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{y}} \quad (126)$$

524

525

526

$$\frac{\partial f}{\partial \bar{x}} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \int_0^{\bar{y}} \frac{\partial \varrho}{\partial \bar{x}} dy \quad (127)$$

527

528

529

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y} \varrho \quad (128)$$

530

531

532 so that:

533

534

$$U = \frac{\partial \Psi}{\partial y} = \hat{U}, \quad (129)$$

535

536

537 that have the same form as incompressible flow, in the same way we can
538 defined:

539

540

$$\hat{V} = -\frac{\partial \Psi}{\partial x}, \quad (130)$$

541

542

543 Using the definition of V :

544

545

$$\varrho V = -\frac{\partial \Psi}{\partial \bar{x}} \quad (131)$$

546

547

548 with 229 we can write:

549

550

$$\varrho V = - \left(\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{\partial y}{\partial \bar{x}} \right) \quad (132)$$

551

552

553

$$\varrho V = \left(\hat{V} - U \frac{\partial y}{\partial \bar{x}} \right) \quad (133)$$

554

555

556

$$\hat{V} = \varrho V + U \frac{\partial y}{\partial \bar{x}} \quad (134)$$

557

558

559 Now we can used the equation 225, 229, 230 and 236 in the Mixing layer
560 equations, 218 to 223, obtain:

561

562 Mass conservation

563

564

$$\frac{\partial}{\partial \bar{x}}(\varrho U) + \frac{\partial}{\partial \bar{y}}(\varrho V) = 0 \quad (135)$$

565

566

567

$$\frac{\partial}{\partial x}(\varrho U) + \frac{\partial}{\partial y}(\varrho U) \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial}{\partial y}(\varrho V) = 0 \quad (136)$$

568

569

570

$$\frac{\partial}{\partial x}(\varrho U) + \frac{\partial}{\partial y}(\varrho U) \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial}{\partial y} \left(\hat{V} - U \frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (137)$$

571

572

573

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial \hat{V}}{\partial y} - \varrho \frac{\partial}{\partial \bar{y}} \left(U \frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (138)$$

574

575

576

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial \hat{V}}{\partial \bar{y}} - \varrho \frac{\partial U}{\partial \bar{y}} \frac{\partial y}{\partial \bar{x}} - \varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (139)$$

577

578

579

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \cancel{\varrho \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}}} + \varrho \frac{\partial \hat{V}}{\partial y} - \cancel{\varrho \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}}} - \varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (140)$$

580

581

582

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial \hat{V}}{\partial y} - \varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (141)$$

583

584

585 the last term of the above equation can be write as:

586

587

$$\varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = U \varrho \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = U \frac{\partial}{\partial \bar{y}} \left(\frac{\partial y}{\partial \bar{x}} \right) = U \frac{\partial \varrho}{\partial \bar{x}} = U \left(\frac{\partial \varrho}{\partial x} + \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} \right) \quad (142)$$

588

589

590 hence:

591

592

$$\varrho \frac{\partial U}{\partial x} + \varrho \frac{\partial \hat{V}}{\partial y} = 0 \quad (143)$$

593

594

595

$$\frac{\partial \hat{U}}{\partial x} + \frac{\partial \hat{V}}{\partial y} = 0 \quad (144)$$

596

597

598 Species Conservation

599

600 Oxidizer conservation

601

602

$$\varrho U \frac{\partial \psi_o}{\partial \bar{x}} + \varrho V \frac{\partial \psi_o}{\partial \bar{y}} = \frac{1}{Le_o} \frac{\partial}{\partial \bar{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \bar{y}} \right) + s^o \Omega \quad (145)$$

603

604

605

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho U \frac{\partial \psi_o}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 V \frac{\partial \psi_o}{\partial y} = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (146)$$

606

607

608

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho U \frac{\partial \psi_o}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 \frac{\partial \psi_o}{\partial y} \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (147)$$

609

610

611

612

613

614

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho U \frac{\partial \psi_o}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 \frac{\partial \psi_o}{\partial y} \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (148)$$

615

616

617

618

619

620

621

622

623

624

625

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho \hat{V} \frac{\partial \psi_o}{\partial y} = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (149)$$

626

627

628

629

630

620 Fuel conservation

621

622 In the same way as in oxidizer conservartion, using the transform coordi-

623 nates x, y, \hat{U} and \hat{V} in the Fuel conservations species, we get:

624

625

$$\hat{U} \frac{\partial \psi_F}{\partial x} + \hat{V} \frac{\partial \psi_F}{\partial y} = \frac{1}{Le_F} \frac{\partial}{\partial y} \left(\varrho^2 D^F \frac{\partial \psi_F}{\partial y} \right) + \frac{s^F \Omega}{\varrho} \quad (150)$$

626

627

628 **Momemtum conservation**

629

630

620

621

622

623

624

625

626

627

628

629

630

$$\varrho U \frac{\partial U}{\partial \bar{x}} + \varrho V \frac{\partial U}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial U}{\partial \bar{y}} \right) \quad (152)$$

631

632

633

$$\varrho U \frac{\partial U}{\partial x} + \varrho U \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 V \frac{\partial U}{\partial y} = \varrho \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial U}{\partial y} \right) \quad (153)$$

634

635

636

$$\varrho U \frac{\partial U}{\partial x} + \varrho U \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) \frac{\partial U}{\partial y} = \varrho \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial U}{\partial y} \right) \quad (154)$$

637

638

639

$$\varrho U \frac{\partial U}{\partial x} + \varrho \hat{V} \frac{\partial U}{\partial y} = \varrho \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial U}{\partial y} \right) \quad (155)$$

640

641

642

$$\hat{U} \frac{\partial \hat{U}}{\partial x} + \hat{V} \frac{\partial \hat{U}}{\partial y} = \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial \hat{U}}{\partial y} \right) \quad (156)$$

643

644

645 **Energy Equation**

646

647

$$\varrho U \frac{\partial}{\partial \bar{x}} (c_p \bar{T}) + \varrho V \frac{\partial}{\partial \bar{y}} (c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial \bar{y}^2} + \frac{1}{Pr} \frac{\partial}{\partial \bar{y}} \left(k \frac{\partial \bar{T}}{\partial \bar{y}} \right) + Q \Omega \quad (157)$$

648

649

650 Using the chain rule to change the second order derivative of \bar{y} in the energy

651 equation.

652

653

$$\frac{\partial^2 f}{\partial \bar{y}^2} = \frac{\partial}{\partial \bar{y}} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{y}} \right) \frac{\partial y}{\partial \bar{y}} \quad (158)$$

654

655

656

$$\frac{\partial^2 f}{\partial \bar{y}^2} = \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial \bar{y}} \right)^2 + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial \bar{y}^2} \quad (159)$$

657

658

659

$$\frac{\partial^2 f}{\partial y^2} = \varrho^2 \frac{\partial^2 f}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} \frac{\partial f}{\partial y} \quad (160)$$

660

661

662

$$\begin{aligned} & \varrho U \frac{\partial}{\partial x} (c_p \bar{T}) + \varrho U \frac{\partial}{\partial y} (c_p \bar{T}) \frac{\partial y}{\partial \bar{x}} + \varrho^2 V \frac{\partial}{\partial y} (c_p \bar{T}) = \\ & \varrho^2 (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial U}{\partial y} + \frac{\varrho}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + Q \Omega \end{aligned} \quad (161)$$

663

664

665

$$\begin{aligned} & \varrho U \frac{\partial}{\partial x} (c_p \bar{T}) + \varrho U \frac{\partial}{\partial y} (c_p \bar{T}) \frac{\partial y}{\partial \bar{x}} + \varrho^2 \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) \frac{\partial}{\partial y} (c_p \bar{T}) = \\ & \varrho^2 (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial U}{\partial y} + \frac{\varrho}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + Q \Omega \end{aligned} \quad (162)$$

666

667

668

$$\begin{aligned} & \hat{U} \frac{\partial}{\partial x} (c_p \bar{T}) + \hat{V} \frac{\partial}{\partial y} (c_p \bar{T}) = \varrho (\gamma - 1) M^2 \mu \frac{\partial^2 \hat{U}}{\partial y^2} + \\ & \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial \hat{U}}{\partial y} + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + \frac{Q \Omega}{\varrho} \end{aligned} \quad (163)$$

669

670

671

$$\hat{U} \frac{\partial}{\partial x} (c_p \bar{T}) + \hat{V} \frac{\partial}{\partial y} (c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial}{\partial y} \left(\varrho \frac{\partial \hat{U}}{\partial y} \right) + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + \frac{Q \Omega}{\varrho} \quad (164)$$

672

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674

675

To transform completely the compressible equations to incompressible form we need to used the Chapman's approximate viscosity law:

676

677

$$\mu = C_w T \quad (165)$$

678

679

680 Multiply by ϱ and using the gas perfect equation, we get:

681

682

$$\mu\varrho = C_w \quad (166)$$

683

684

685 where C_w is a function of the temperature $C_w = C_w(T)$.

686

687

$$\frac{\partial \hat{U}}{\partial x} + \frac{\partial \hat{V}}{\partial y} = 0 \quad (167)$$

688

689

690

$$\hat{U} \frac{\partial \psi_o}{\partial x} + \hat{V} \frac{\partial \psi_o}{\partial y} = \frac{1}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + \frac{s^o \Omega}{\varrho} \quad (168)$$

691

692

693

$$\hat{U} \frac{\partial \psi_F}{\partial x} + \hat{V} \frac{\partial \psi_F}{\partial y} = \frac{1}{Le_F} \frac{\partial}{\partial y} \left(\varrho^2 D^F \frac{\partial \psi_F}{\partial y} \right) + \frac{s^F \Omega}{\varrho} \quad (169)$$

694

695

696

$$\hat{U} \frac{\partial \hat{U}}{\partial x} + \hat{V} \frac{\partial \hat{U}}{\partial y} = \frac{\partial}{\partial y} \left(C_w(T) \frac{\partial \hat{U}}{\partial y} \right) \quad (170)$$

697

698

699

$$\hat{U} \frac{\partial}{\partial x} (c_p \bar{T}) + \hat{V} \frac{\partial}{\partial y} (c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial}{\partial y} \left(\varrho \frac{\partial \hat{U}}{\partial y} \right) + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + \frac{Q \Omega}{\varrho} \quad (171)$$

4. CROCCO BUSEMANN

701 In this section a special solution to boundary layer equation is found using
702 the definition of the total enthalpy.

703 From the boundary layer equation in two dimensional flow

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad (172)$$

$$(\rho u)\frac{\partial u}{\partial x} + (\rho v)\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad (173)$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{\partial h}{\partial y} = u\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \quad (174)$$

704 where the Prantdl number is defined as $Pr = \mu c_p/k$.

705 Since the total enthalpy is defined as

$$H = h + \frac{1}{2}u^2 \quad (175)$$

706 the total enthalpy equation using both the energy and momentum equa-
707 tion is

$$(\rho u)\frac{\partial H}{\partial x} + (\rho v)\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\left(1 - \frac{1}{Pr}\right)\mu u\frac{\partial u}{\partial y}\right] \quad (176)$$

708 This equation allows a simplest solution $H = h + \frac{1}{2}u^2 = \text{constant}$ as long
709 as $Pr = 1$, which leads to

$$\frac{\partial H}{\partial y} = \frac{\partial h_w}{\partial y} = 0 \quad (177)$$

710 Representing the not heat transfer at the wall.

711 $Pr = 1$ implies in a perfect balance between viscous dissipation and
712 heat conduction so as keep the the stagnation enthalpy constant in adiabatic
713 boundary layer and also is a good approximation for gases.

714 Another solution can be obtained if the pressure gradient is neglected in
715 the boundary layer equation. With this, the momentum equation and the
716 energy equation get very similar, it seems as if u and h could be interchanged
717 except for the dissipation term.

$$(\rho u) \frac{\partial u}{\partial x} + (\rho v) \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (178)$$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{\partial h}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\mu}{Pr} \frac{\partial h}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (179)$$

⁷¹⁸ Then a solution of the form

$$\frac{\partial h}{\partial y} = \frac{dh}{du} \frac{\partial u}{\partial y} \quad (180)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial h}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{dh}{du} \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial u} \left(\frac{dh}{du} \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y} \quad (181)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial h}{\partial y} \right) = \frac{\partial}{\partial u} \left(\frac{dh}{du} \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y} = \frac{dh}{du} \frac{\partial u^2}{\partial y^2} + \frac{d^2 h}{du^2} \left(\frac{\partial u}{\partial y} \right)^2 \quad (182)$$

$$(\rho u) \frac{\partial u}{\partial x} + (\rho v) \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (183)$$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{du} \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\mu}{Pr} \frac{dh}{du} \frac{\partial u}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (184)$$

⁷¹⁹ Assuming $Pr = 1$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{dy} = \frac{\partial}{\partial y} \left(\mu \frac{dh}{dy} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (185)$$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{dy} = \mu \frac{\partial}{\partial y} \left(\frac{dh}{dy} \right) + \frac{dh}{dy} \frac{\partial \mu}{\partial y} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (186)$$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{dy} = \mu \frac{dh}{du} \frac{\partial u^2}{\partial y^2} + \mu \frac{d^2 h}{du^2} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{dh}{dy} \frac{\partial \mu}{\partial y} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (187)$$

$$(\rho u) \frac{dh}{du} \frac{\partial u}{\partial x} + (\rho v) \frac{dh}{du} \frac{\partial u}{\partial y} - \mu \frac{dh}{du} \frac{\partial u^2}{\partial y^2} - \frac{dh}{du} \frac{\partial u}{\partial y} \frac{\partial \mu}{\partial y} = \left[\frac{d^2 h}{du^2} + 1 \right] \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (188)$$

$$\frac{dh}{du} \left[(\rho u) \frac{\partial u}{\partial x} + (\rho v) \frac{\partial u}{\partial y} - \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \right] = \left[\frac{d^2 h}{du^2} + 1 \right] \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (189)$$

⁷²⁰ Using the momentum equation

$$\left[\frac{d^2 h}{du^2} + 1 \right] \mu \left(\frac{\partial u}{\partial y} \right)^2 = 0 \quad (190)$$

⁷²¹ Then

$$\frac{d^2 h}{du^2} = -1 \quad (191)$$

⁷²² Integrating

$$h = -\frac{u^2}{2} + c_1 u + c_2 \quad (192)$$

⁷²³ For mixing layer these constant can be found, using the values at the
⁷²⁴ boundaries. For the upper stream:

$$h_1 = -\frac{u_1^2}{2} + c_1 u_1 + c_2 \quad (193)$$

⁷²⁵ Similarly for the lower stream

$$h_2 = -\frac{u_2^2}{2} + c_1 u_2 + c_2 \quad (194)$$

⁷²⁶ This is two unknown and two equations.

⁷²⁷ For the equation 295

$$c_2 = h_1 + \frac{u_1^2}{2} - c_1 u_1 \quad (195)$$

⁷²⁸ Using it in 296

$$h_2 = -\frac{u_2^2}{2} + c_1 u_2 + h_1 + \frac{u_1^2}{2} - c_1 u_1 \quad (196)$$

$$c_1 (u_2 - u_1) = h_2 - h_1 + \frac{u_2^2}{2} - \frac{u_1^2}{2} \quad (197)$$

$$c_1 = \frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \quad (198)$$

⁷²⁹ and

$$c_2 = h_1 + \frac{u_1^2}{2} - \left[\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right] u_1 \quad (199)$$

730 Therefore

$$h = -\frac{u^2}{2} + c_1 u + c_2 \quad (200)$$

$$h = -\frac{u^2}{2} + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) u + h_1 + \frac{u_1^2}{2} - \left[\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right] u_1 \quad (201)$$

$$h = \frac{u_1^2}{2} - \frac{u^2}{2} + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) (u - u_1) + h_1 \quad (202)$$

$$h = \frac{1}{2}(u_1 + u)(u_1 - u) + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) (u - u_1) + h_1 \quad (203)$$

$$h = (u_1 - u) \left[\frac{1}{2}(u_1 + u) - \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) \right] + h_1 \quad (204)$$

$$h = (u_1 - u) \left[\frac{1}{2}(u - u_2) - \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} \right) \right] + h_1 \quad (205)$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2 \frac{(u_1 - u)}{(u_2 - u_1)} + h_1 \left(\frac{(u_1 - u)}{(u_2 - u_1)} \right) + h_1 \quad (206)$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2 \frac{(u_1 - u)}{(u_2 - u_1)} + h_1 \left(1 + \left(\frac{(u_1 - u)}{(u_2 - u_1)} \right) \right) \quad (207)$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2 \frac{(u_1 - u)}{(u_2 - u_1)} + h_1 \frac{(u_2 - u)}{(u_2 - u_1)} \quad (208)$$

731 Assuming $c_p = \text{constant}$, the enthalpy can be related with the temper-
732 ature with the relation $h = c_p T$

$$T = \frac{1}{2} \frac{1}{c_p} (u - u_2)(u_1 - u) - T_2 \frac{(u_1 - u)}{(u_2 - u_1)} + T_1 \frac{(u_2 - u)}{(u_2 - u_1)} \quad (209)$$

$$T = T_1 \frac{(u - u_2)}{(u_1 - u_2)} + T_2 \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{1}{c_p} (u_1 - u)(u - u_2) \quad (210)$$

⁷³³ The last term and the temperature are dimensional and depend on the
⁷³⁴ non dimensional parameters.

⁷³⁵ Using $T = T/T_1$ and $U = U/U_1$.

$$TT_1 = T_1 \frac{(u - u_2)}{(u_1 - u_2)} + T_2 \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{U_1^2}{c_p} (u_1 - u)(u - u_2) \quad (211)$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1} \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{U_1^2}{T_1 c_p} (u_1 - u)(u - u_2) \quad (212)$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1} \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{U_1^2 \gamma_1 R}{\gamma_1 R T_1 c_p} (u_1 - u)(u - u_2) \quad (213)$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1} \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{M^2 \gamma_1 R}{c_p} (u_1 - u)(u - u_2) \quad (214)$$

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} \quad (215)$$

⁷³⁶ $\beta_t = T_2/T_1$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \beta_t \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} M^2 (\gamma_1 - 1) (u_1 - u)(u - u_2) \quad (216)$$

⁷³⁷ Using $T = T/T_1$ and $U = U/a_1$.

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \beta_t \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{\gamma - 1}{2} (u_1 - u)(u - u_2) \quad (217)$$

⁷³⁸ **Non-dimensional Reacting Mixing layer equation:**

$$\frac{\partial}{\partial x} (\varrho U) + \frac{\partial}{\partial y} (\varrho V) = 0 \quad (218)$$

740

741

742

$$\varrho U \frac{\partial \psi_o}{\partial \bar{x}} + \varrho V \frac{\partial \psi_o}{\partial \bar{y}} = \frac{1}{Le_o} \frac{\partial}{\partial \bar{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \bar{y}} \right) + s^o \Omega \quad (219)$$

743

744

745

$$\varrho U \frac{\partial \psi_F}{\partial \bar{x}} + \varrho V \frac{\partial \psi_F}{\partial \bar{y}} = \frac{1}{Le_F} \frac{\partial}{\partial \bar{y}} \left(\rho D^F \frac{\partial \psi_F}{\partial \bar{y}} \right) + s^F \Omega. \quad (220)$$

746

747

748

$$\varrho U \frac{\partial U}{\partial \bar{x}} + \varrho V \frac{\partial U}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial U}{\partial \bar{y}} \right) \quad (221)$$

749

750

751

$$\varrho U \frac{\partial}{\partial \bar{x}} (c_p \bar{T}) + \varrho V \frac{\partial}{\partial \bar{y}} (c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial \bar{y}^2} + \frac{1}{Pr} \frac{\partial}{\partial \bar{y}} \left(k \frac{\partial \bar{T}}{\partial \bar{y}} \right) + Q \Omega \quad (222)$$

752

753

754

$$1 = \varrho \bar{T} \quad (223)$$

755

756

757 5. SELF-SIMILARITY SOLUTION FOR REACTING 758 MIXING LAYER EQUATIONS

759 To find a similarity solutions for a reacting mixing layer equations, firstly we
760 need to use the Howarth-Dorodnitsyn 225 transformation to find am incom-
761 pressible form of the compressibles equations.

762 Definig the stream function for compressible flow as:
763

764 37

765

$$\frac{\partial \Psi}{\partial x} = -\varrho V \quad \frac{\partial \Psi}{\partial y} = \varrho U, \quad (224)$$

766

767

768 The mass conservation equation is identically satisfied by the introduction of
769 this stream function. Now defining a vertical stretching as:

770

771

$$x = \bar{x} \quad y = \int_0^{\bar{y}} \varrho d\bar{y} \quad (225)$$

772

773

774 and using the chain rule:

775

776

$$f(\bar{x}, \bar{y}) \quad \bar{x} = \bar{x}(x, y) \quad \bar{y} = \bar{y}(x, y) \quad (226)$$

777

778

779

$$\frac{\partial f}{\partial \bar{x}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{x}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{x}} \quad (227)$$

780

781

782

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{y}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{y}} \quad (228)$$

783

784

785

$$\frac{\partial f}{\partial \bar{x}} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \int_0^{\bar{y}} \frac{\partial \varrho}{\partial \bar{x}} dy \quad (229)$$

786

787

788

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y} \varrho \quad (230)$$

789

790

791 so that:

792

793

$$U = \frac{\partial \Psi}{\partial y} = \hat{U}, \quad (231)$$

794

795

796 that have the same form as incompressible flow, in the same way we can
797 defined:

798

799

$$\hat{V} = -\frac{\partial \Psi}{\partial x}, \quad (232)$$

800

801

802 Using the definition of V :

803

804

$$\varrho V = -\frac{\partial \Psi}{\partial \bar{x}} \quad (233)$$

805

806

807 with 229 we can write:

808

809

$$\varrho V = - \left(\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{\partial y}{\partial \bar{x}} \right) \quad (234)$$

810

811

812

$$\varrho V = \left(\hat{V} - U \frac{\partial y}{\partial \bar{x}} \right) \quad (235)$$

813

814

815

$$\hat{V} = \varrho V + U \frac{\partial y}{\partial \bar{x}} \quad (236)$$

816

817

818 Now we can used the equation 225, 229, 230 and 236 in the Mixing layer

819 equations, 218 to 223, obtain:

820

821 Mass conservation

822

823

$$\frac{\partial}{\partial \bar{x}}(\varrho U) + \frac{\partial}{\partial \bar{y}}(\varrho V) = 0 \quad (237)$$

824

825

826

$$\frac{\partial}{\partial x}(\varrho U) + \frac{\partial}{\partial y}(\varrho U) \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial}{\partial y}(\varrho V) = 0 \quad (238)$$

827

828

829

$$\frac{\partial}{\partial x}(\varrho U) + \frac{\partial}{\partial y}(\varrho U) \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial}{\partial y} \left(\hat{V} - U \frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (239)$$

830

831

832

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial \hat{V}}{\partial y} - \varrho \frac{\partial}{\partial \bar{y}} \left(U \frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (240)$$

833

834

835

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial \hat{V}}{\partial y} - \varrho \frac{\partial U}{\partial \bar{y}} \frac{\partial y}{\partial \bar{x}} - \varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (241)$$

836

837

838

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \cancel{\varrho \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}}} + \varrho \frac{\partial \hat{V}}{\partial y} - \cancel{\varrho \frac{\partial U}{\partial \bar{y}} \frac{\partial y}{\partial \bar{x}}} - \varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (242)$$

839

840

841

$$U \frac{\partial \varrho}{\partial x} + \varrho \frac{\partial U}{\partial x} + U \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho \frac{\partial \hat{V}}{\partial y} - \varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = 0 \quad (243)$$

842

843

844 the last term of the above equation can be write as:

845

846

$$\varrho U \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = U \varrho \frac{\partial}{\partial y} \left(\frac{\partial y}{\partial \bar{x}} \right) = U \frac{\partial}{\partial \bar{y}} \left(\frac{\partial y}{\partial \bar{x}} \right) = U \frac{\partial \varrho}{\partial \bar{x}} = U \left(\frac{\partial \varrho}{\partial x} + \frac{\partial \varrho}{\partial y} \frac{\partial y}{\partial \bar{x}} \right) \quad (244)$$

847

848

849 hence:

850

851

$$\varrho \frac{\partial U}{\partial x} + \varrho \frac{\partial \hat{V}}{\partial y} = 0 \quad (245)$$

852

853

854

$$\frac{\partial \hat{U}}{\partial x} + \frac{\partial \hat{V}}{\partial y} = 0 \quad (246)$$

855

856

857 Species Conservation

858

859 Oxidizer conservation

860

861

$$\varrho U \frac{\partial \psi_o}{\partial \bar{x}} + \varrho V \frac{\partial \psi_o}{\partial \bar{y}} = \frac{1}{Le_o} \frac{\partial}{\partial \bar{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \bar{y}} \right) + s^o \Omega \quad (247)$$

862

863

864

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho U \frac{\partial \psi_o}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 V \frac{\partial \psi_o}{\partial y} = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (248)$$

865

866

867

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho U \frac{\partial \psi_o}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 \frac{\partial \psi_o}{\partial y} \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (249)$$

868

869

870

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho U \frac{\partial \psi_o}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 \frac{\partial \psi_o}{\partial y} \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (250)$$

871

872

873

$$\varrho U \frac{\partial \psi_o}{\partial x} + \varrho \hat{V} \frac{\partial \psi_o}{\partial y} = \frac{\varrho}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + s^o \Omega \quad (251)$$

874

875

876

$$\hat{U} \frac{\partial \psi_o}{\partial x} + \hat{V} \frac{\partial \psi_o}{\partial y} = \frac{1}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + \frac{s^o \Omega}{\varrho} \quad (252)$$

877

878

879 Fuel conservation

880 In the same way as in oxidizer conservartion, using the transform coordi-
881 nates x, y, \hat{U} and \hat{V} in the Fuel conservations species, we get:

883

884

$$\hat{U} \frac{\partial \psi_F}{\partial x} + \hat{V} \frac{\partial \psi_F}{\partial y} = \frac{1}{Le_F} \frac{\partial}{\partial y} \left(\varrho^2 D^F \frac{\partial \psi_F}{\partial y} \right) + \frac{s^F \Omega}{\varrho} \quad (253)$$

885

886

887 **Momemtum conservation**

888

889

$$\varrho U \frac{\partial U}{\partial \bar{x}} + \varrho V \frac{\partial U}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}} \left(\mu \frac{\partial U}{\partial \bar{y}} \right) \quad (254)$$

890

891

892

$$\varrho U \frac{\partial U}{\partial x} + \varrho U \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 V \frac{\partial U}{\partial y} = \varrho \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial U}{\partial y} \right) \quad (255)$$

893

894

895

$$\varrho U \frac{\partial U}{\partial x} + \varrho U \frac{\partial U}{\partial y} \frac{\partial y}{\partial \bar{x}} + \varrho^2 \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) \frac{\partial U}{\partial y} = \varrho \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial U}{\partial y} \right) \quad (256)$$

896

897

898

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900

901

$$\varrho U \frac{\partial U}{\partial x} + \varrho \hat{V} \frac{\partial U}{\partial y} = \varrho \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial U}{\partial y} \right) \quad (257)$$

902

903

904 Energy Equation

905

906

$$\hat{U} \frac{\partial \hat{U}}{\partial \bar{x}} + \hat{V} \frac{\partial \hat{U}}{\partial y} = \frac{\partial}{\partial y} \left(\mu \varrho \frac{\partial \hat{U}}{\partial y} \right) \quad (258)$$

907

908

909 Using the chain rule to change the second order derivative of \bar{y} in the energy

910 equation.

911

912

$$\frac{\partial^2 f}{\partial \bar{y}^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{y}} \right) \frac{\partial y}{\partial \bar{y}} \quad (260)$$

913

914

915

$$\frac{\partial^2 f}{\partial \bar{y}^2} = \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial \bar{y}} \right)^2 + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial \bar{y}^2} \quad (261)$$

916

917

918

$$\frac{\partial^2 f}{\partial \bar{y}^2} = \varrho^2 \frac{\partial^2 f}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} \frac{\partial f}{\partial y} \quad (262)$$

919

920

921

$$\begin{aligned} \varrho U \frac{\partial}{\partial x} (c_p \bar{T}) + \varrho U \frac{\partial}{\partial y} (c_p \bar{T}) \frac{\partial y}{\partial \bar{x}} + \varrho^2 V \frac{\partial}{\partial y} (c_p \bar{T}) = \\ \varrho^2 (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial U}{\partial y} + \frac{\varrho}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + Q \Omega \end{aligned} \quad (263)$$

922

923

924

$$\begin{aligned} \varrho U \frac{\partial}{\partial x} (c_p \bar{T}) + \varrho U \frac{\partial}{\partial y} (c_p \bar{T}) \frac{\partial y}{\partial \bar{x}} + \varrho^2 \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \bar{x}} \right) \frac{\partial}{\partial y} (c_p \bar{T}) = \\ \varrho^2 (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial U}{\partial y} + \frac{\varrho}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + Q \Omega \end{aligned} \quad (264)$$

925

926

927

$$\begin{aligned} \hat{U} \frac{\partial}{\partial x} (c_p \bar{T}) + \hat{V} \frac{\partial}{\partial y} (c_p \bar{T}) = \varrho (\gamma - 1) M^2 \mu \frac{\partial^2 \hat{U}}{\partial y^2} + \\ \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial \hat{U}}{\partial y} + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + \frac{Q \Omega}{\varrho} \end{aligned} \quad (265)$$

928

929

930

$$\hat{U} \frac{\partial}{\partial x} (c_p \bar{T}) + \hat{V} \frac{\partial}{\partial y} (c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial}{\partial y} \left(\varrho \frac{\partial \hat{U}}{\partial y} \right) + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + \frac{Q \Omega}{\varrho} \quad (266)$$

931

932

To transform completely the compressible equations to incompressible form we need to used the Chapman's approximate viscosity law:

935

936

$$\mu = C_w T \quad (267)$$

937

938

Multiply by ϱ and using the gas perfect equation, we get:

940
941

$$\mu\varrho = C_w \quad (268)$$

942
943

944 where C_w is a function of the temperature $C_w = C_w(T)$.

945
946

$$\frac{\partial \hat{U}}{\partial x} + \frac{\partial \hat{V}}{\partial y} = 0 \quad (269)$$

947
948

949

$$\hat{U} \frac{\partial \psi_o}{\partial x} + \hat{V} \frac{\partial \psi_o}{\partial y} = \frac{1}{Le_o} \frac{\partial}{\partial y} \left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y} \right) + \frac{s^o \Omega}{\varrho} \quad (270)$$

950
951

952

$$\hat{U} \frac{\partial \psi_F}{\partial x} + \hat{V} \frac{\partial \psi_F}{\partial y} = \frac{1}{Le_F} \frac{\partial}{\partial y} \left(\varrho^2 D^F \frac{\partial \psi_F}{\partial y} \right) + \frac{s^F \Omega}{\varrho} \quad (271)$$

953
954

955

$$\hat{U} \frac{\partial \hat{U}}{\partial x} + \hat{V} \frac{\partial \hat{U}}{\partial y} = \frac{\partial}{\partial y} \left(C_w(T) \frac{\partial \hat{U}}{\partial y} \right) \quad (272)$$

956
957

958

$$\hat{U} \frac{\partial}{\partial x} (c_p \bar{T}) + \hat{V} \frac{\partial}{\partial y} (c_p \bar{T}) = (\gamma - 1) M^2 \mu \frac{\partial}{\partial y} \left(\varrho \frac{\partial \hat{U}}{\partial y} \right) + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \bar{T}}{\partial y} \right) + \frac{Q \Omega}{\varrho} \quad (273)$$

959

6. CROCCO BUSEMANN

960 In this section a special solution to boundary layer equation is found using
961 the definition of the total enthalpy.

962 From the boundary layer equation in two dimensional flow

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad (274)$$

$$(\rho u)\frac{\partial u}{\partial x} + (\rho v)\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad (275)$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{\partial h}{\partial y} = u\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \quad (276)$$

963 where the Prantdl number is defined as $Pr = \mu c_p/k$.
 964 Since the total enthalpy is defined as

$$H = h + \frac{1}{2}u^2 \quad (277)$$

965 the total enthalpy equation using both the energy and momentum equa-
 966 tion is

$$(\rho u)\frac{\partial H}{\partial x} + (\rho v)\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\left(1 - \frac{1}{Pr}\right)\mu u\frac{\partial u}{\partial y}\right] \quad (278)$$

967 This equation allows a simplest solution $H = h + \frac{1}{2}u^2 = \text{constant}$ as long
 968 as $Pr = 1$, which leads to

$$\frac{\partial H}{\partial y} = \frac{\partial h_w}{\partial y} = 0 \quad (279)$$

969 Representing the not heat transfer at the wall.

970 $Pr = 1$ implies in a perfect balance between viscous dissipation and
 971 heat conduction so as keep the the stagnation enthalpy constant in adiabatic
 972 boundary layer and also is a good approximation for gases.

973 Another solution can be obtained if the pressure gradient is neglected in
 974 the boundary layer equation. With this, the momentum equation and the
 975 energy equation get very similar, it seems as if u and h could be interchanged
 976 except for the dissipation term.

$$(\rho u)\frac{\partial u}{\partial x} + (\rho v)\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad (280)$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \quad (281)$$

⁹⁷⁷ Then a solution of the form

$$\frac{\partial h}{\partial y} = \frac{dh}{du} \frac{\partial u}{\partial y} \quad (282)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial h}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{dh}{du} \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial u} \left(\frac{dh}{du} \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y} \quad (283)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial h}{\partial y} \right) = \frac{\partial}{\partial u} \left(\frac{dh}{du} \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y} = \frac{dh}{du} \frac{\partial u^2}{\partial y^2} + \frac{d^2 h}{du^2} \left(\frac{\partial u}{\partial y} \right)^2 \quad (284)$$

$$(\rho u) \frac{\partial u}{\partial x} + (\rho v) \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (285)$$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{du} \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{dh}{du} \frac{\partial u}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (286)$$

⁹⁷⁸ Assuming $Pr = 1$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{dy} = \frac{\partial}{\partial y} \left(\mu \frac{dh}{dy} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (287)$$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{dy} = \mu \frac{\partial}{\partial y} \left(\frac{dh}{dy} \right) + \frac{dh}{dy} \frac{\partial \mu}{\partial y} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (288)$$

$$(\rho u) \frac{\partial h}{\partial x} + (\rho v) \frac{dh}{dy} = \mu \frac{dh}{du} \frac{\partial u^2}{\partial y^2} + \mu \frac{d^2 h}{du^2} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{dh}{dy} \frac{\partial \mu}{\partial y} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (289)$$

$$(\rho u) \frac{dh}{du} \frac{\partial u}{\partial x} + (\rho v) \frac{dh}{du} \frac{\partial u}{\partial y} - \mu \frac{dh}{du} \frac{\partial u^2}{\partial y^2} - \frac{dh}{du} \frac{\partial u}{\partial y} \frac{\partial \mu}{\partial y} = \left[\frac{d^2 h}{du^2} + 1 \right] \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (290)$$

$$\frac{dh}{du} \left[(\rho u) \frac{\partial u}{\partial x} + (\rho v) \frac{\partial u}{\partial y} - \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \right] = \left[\frac{d^2 h}{du^2} + 1 \right] \mu \left(\frac{\partial u}{\partial y} \right)^2 \quad (291)$$

⁹⁷⁹ Using the momentum equation

$$\left[\frac{d^2 h}{du^2} + 1 \right] \mu \left(\frac{\partial u}{\partial y} \right)^2 = 0 \quad (292)$$

980 Then

$$\frac{d^2h}{du^2} = -1 \quad (293)$$

981 Integrating

$$h = -\frac{u^2}{2} + c_1 u + c_2 \quad (294)$$

982 For mixing layer these constant can be found, using the values at the
983 boundaries. For the upper stream:

$$h_1 = -\frac{u_1^2}{2} + c_1 u_1 + c_2 \quad (295)$$

984 Similarly for the lower stream

$$h_2 = -\frac{u_2^2}{2} + c_1 u_2 + c_2 \quad (296)$$

985 This is two unknown and two equations.

986 For the equation 295

$$c_2 = h_1 + \frac{u_1^2}{2} - c_1 u_1 \quad (297)$$

987 Using it in 296

$$h_2 = -\frac{u_2^2}{2} + c_1 u_2 + h_1 + \frac{u_1^2}{2} - c_1 u_1 \quad (298)$$

$$c_1 (u_2 - u_1) = h_2 - h_1 + \frac{u_2^2}{2} - \frac{u_1^2}{2} \quad (299)$$

$$c_1 = \frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \quad (300)$$

988 and

$$c_2 = h_1 + \frac{u_1^2}{2} - \left[\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right] u_1 \quad (301)$$

989 Therefore

$$h = -\frac{u^2}{2} + c_1 u + c_2 \quad (302)$$

$$h = -\frac{u^2}{2} + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) u + h_1 + \frac{u_1^2}{2} - \left[\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right] u_1 \quad (303)$$

$$h = \frac{u_1^2}{2} - \frac{u^2}{2} + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) (u - u_1) + h_1 \quad (304)$$

$$h = \frac{1}{2}(u_1 + u)(u_1 - u) + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) (u - u_1) + h_1 \quad (305)$$

$$h = (u_1 - u) \left[\frac{1}{2}(u_1 + u) - \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1) \right) \right] + h_1 \quad (306)$$

$$h = (u_1 - u) \left[\frac{1}{2}(u - u_2) - \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} \right) \right] + h_1 \quad (307)$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2 \frac{(u_1 - u)}{(u_2 - u_1)} + h_1 \left(\frac{(u_1 - u)}{(u_2 - u_1)} \right) + h_1 \quad (308)$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2 \frac{(u_1 - u)}{(u_2 - u_1)} + h_1 \left(1 + \left(\frac{(u_1 - u)}{(u_2 - u_1)} \right) \right) \quad (309)$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2 \frac{(u_1 - u)}{(u_2 - u_1)} + h_1 \frac{(u_2 - u)}{(u_2 - u_1)} \quad (310)$$

⁹⁹⁰ Assuming $c_p = \text{constant}$, the enthalpy can be related with the temper-
⁹⁹¹ ature with the relation $h = c_p T$

$$T = \frac{1}{2} \frac{1}{c_p} (u - u_2)(u_1 - u) - T_2 \frac{(u_1 - u)}{(u_2 - u_1)} + T_1 \frac{(u_2 - u)}{(u_2 - u_1)} \quad (311)$$

$$T = T_1 \frac{(u - u_2)}{(u_1 - u_2)} + T_2 \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{1}{c_p} (u_1 - u)(u - u_2) \quad (312)$$

992 The last term and the temperature are dimensional and depend on the
 993 non dimensional parameters.

994 Using $T = T/T_1$ and $U = U/U_1$.

$$TT_1 = T_1 \frac{(u - u_2)}{(u_1 - u_2)} + T_2 \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{U_1^2}{c_p} (u_1 - u)(u - u_2) \quad (313)$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1} \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{U_1^2}{T_1 c_p} (u_1 - u)(u - u_2) \quad (314)$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1} \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{U_1^2 \gamma_1 R}{\gamma_1 R T_1 c_p} (u_1 - u)(u - u_2) \quad (315)$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1} \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} \frac{M^2 \gamma_1 R}{c_p} (u_1 - u)(u - u_2) \quad (316)$$

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} \quad (317)$$

995 $\beta_t = T_2/T_1$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \beta_t \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2} M^2 (\gamma_1 - 1) (u_1 - u)(u - u_2) \quad (318)$$

996 Using $T = T/T_1$ and $U = U/a_1$.

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \beta_t \frac{(u_1 - u)}{(u_1 - u_2)} + \frac{\gamma - 1}{2} (u_1 - u)(u - u_2) \quad (319)$$

997 [1] Olivier H Planche. *A numerical investigation of the compressible reacting
 998 mixing layer*. PhD thesis, Stanford University, 1992.

- 999 [2] Herbert Brian Squire. On the stability for three-dimensional dis-
1000 turbances of viscous fluid flow between parallel walls. *Proceed-*
1001 *ings of the Royal Society of London. Series A, Containing Pa-*
1002 *pers of a Mathematical and Physical Character*, 142(847):621–628,
1003 11 1933. ISSN 0950-1207. doi: 10.1098/rspa.1933.0193. URL
1004 <https://doi.org/10.1098/rspa.1933.0193>.

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