Reacting Mixing Layer

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Abstract

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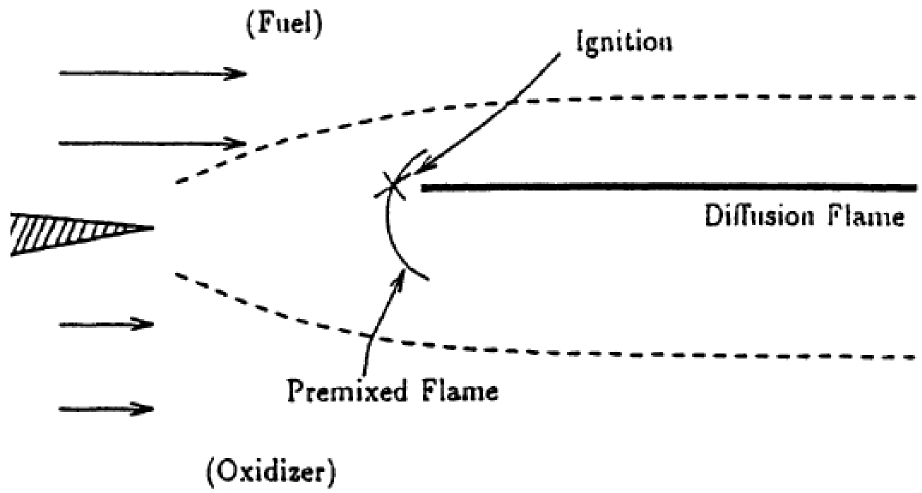

Figure 1: Reacting Mixing Layer

Reacting Mixing Layer

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1. Reacting Mixing Layer Equations

1.1. Conservation's Equations

In a compressible, conservative form and in a perfect gas the conservation's equations are:

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \quad \text{for} \quad i = 1, 2, 3, \tag{1}$$

Species conservation:

$$\frac{\partial}{\partial t}(\rho Y^m) + \frac{\partial}{\partial x_i}(\rho Y^m(u_i + V^m)) = s^m \Omega \quad \text{for} \quad i = 1, 2, 3 \quad \text{for} \quad m = \text{Oxidizer, Fuel} \tag{2}$$

Using a constitutive equation for V^m

$$V^m = -\rho D^m \frac{\partial Y^m}{\partial x_i} \tag{3}$$

where D^m is the diffusion coefficient of specie respect to the most abundant specie.

who is Ω?

Momemtum conservation:

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j \quad \text{for} \quad i, j = 1, 2, 3 \tag{4}$$

where

$$\tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{5}$$

and using the Stoke's relation:

$$\lambda = -\frac{2}{3}\mu \tag{6}$$

$$\tau_{ij} = \mu \left(-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \tag{7}$$

Energy conservation:

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x_i}(\rho u_i h) = \frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\tau_{ij} - \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i} \right) + Q\Omega \tag{8}$$

Perfect gas equation:

$$p = \rho T R \tag{9}$$

$$h = c_p T \tag{10}$$

In the above equations ρ is the density, $u_i = (u, v, w)$ are the different velocities in the three different dimensions x, y and z. p is the pressure of the flow, T is the temperature of the flow, f_j is the body force acting on the flow. Y^m are the different mass fraction for the two different species, oxygen and fuel. μ, D^m and k are the viscosity, diffusion coefficient for the different species and the gas thermal conductivity. Q is the heat release for the chemical reaction and Ω which is the mass reaction rate.

Base Flow equations:

We are interesting in the a two dimensional compressible mixing layer, because the tridimensional effect can be neglected for the study of instabilities in the first stages, it states that for *the onset of linear instability in parallel shear flows, the least stable (i.e., first to become unstable) disturbances are two-dimensiona* [2]. In the incompressible case, the process of transition to turbulence in a mixing layer is dictated by: the growth of two dimensional coherent structures and the development of secondary instability, the merging of the large-structures and finally the breakdown into small-scales three dimensional turbulence [1].

The mean flow for two dimensional mixing layer which separates two fluids, the oxidizer and the fuel, at different speeds and temperature with zero pressure gradient can be assumed that is governed by boundary layer equations. This mean that gradients for the different properties are in the y direction, in Cartesian coordinates.

To applied the boundary layer equations is necessary to defined a characteristic length of the flow. In this problem exist two characteristics lengths, one can be defined as the thickness of the mixing layer δ and other can be defined as the distance that the mixing layer needs to grow L_c.

Assume that δ is sufficient small compared with the x length where the mixing layer is development, it mean that $\delta/L_c << 1$. The mean velocity scale can be approximate, in the x direction is the velocity of order of U_c, and $\partial/\partial x$ is of order of $1/L_c$ and assuming that ρ is of order of ρ_c then:

$$\frac{\partial}{\partial x}(\rho u) \sim \frac{\rho_c U_c}{L_c} \tag{11}$$

Using the conservation of mass equation 1 in two dimensions (x, y), the order of

$$\frac{\partial}{\partial y}(\rho v) \sim \frac{\rho_c U_c}{L_c} \tag{12}$$

We know that in the mixing layer the velocity v is smaller than u, then we can assume that

$$v \sim \frac{\delta}{L_c} U_c \qquad \text{being} \qquad \frac{\partial}{\partial y} \sim \frac{1}{\delta}. \tag{13}$$

With this assumptions the terms in the conservation equations of order:

$$\frac{\partial^2}{\partial x^2} \sim \frac{1}{L_c^2}, \tag{14}$$

are smaller that term of order

$$\frac{\partial^2}{\partial y^2} \sim \frac{1}{\delta^2}. \tag{15}$$

Thereby, using the different length scales in the others conservations equations and assuming steady state to the mean flow we get:

Mass conservation:

$$\frac{\partial}{\partial x}(\rho u) = \frac{\partial}{\partial y}(\rho v) \tag{16}$$

Species conservation:

$$\frac{\partial}{\partial x}(\rho Y^o(u + V^o)) + \frac{\partial}{\partial y}(\rho Y^o(v + V^o)) = s^o\Omega \tag{17}$$

$$\frac{\partial}{\partial x}(\rho Y^F(u + V^F)) + \frac{\partial}{\partial y}(\rho Y^F(v + V^F)) = s^F\Omega \tag{18}$$

Using a constitutive equation for V^m

$$V^m = -\rho D^m \frac{\partial}{\partial x_i}(ln(Y^m)) \quad \text{for} \quad m = 1, 2 \tag{19}$$

Hence

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{\partial}{\partial x_i}\left(\rho D^m \frac{\partial Y^m}{\partial x_i}\right) \quad \text{for} \quad m = 1, 2 \tag{20}$$

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{\partial}{\partial x}\left(\rho D^m \frac{\partial Y^m}{\partial x}\right) - \frac{\partial}{\partial y}\left(\rho D^m \frac{\partial Y^m}{\partial y}\right) \quad \text{for} \quad m = 1, 2 \tag{21}$$

Using the characteristic length scales:

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{1}{L_c}\frac{\rho_c D_c^m Y_c^m}{L_c} - \frac{1}{\delta}\frac{\rho_c D_c^m Y_c^m}{\delta} \quad \text{for} \quad m = 1, 2 \tag{22}$$

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\frac{\rho_c D_c^m Y_c^m}{L_c^2} - \frac{\rho_c D_c^m Y_c^m}{\delta^2} \quad \text{for} \quad m = 1, 2 \tag{23}$$

$$\frac{\partial}{\partial x_i}(\rho Y^m V^m) = -\cancelto{\approx 0}{\frac{\rho_c D_c^m Y_c^m}{L_c^2}} - \frac{\rho_c D_c^m Y_c^m}{\delta^2} \quad \text{for} \quad m = 1, 2 \tag{24}$$

The Species conservation can be wrote as:

$$\frac{\partial}{\partial x}(\rho Y^o u) + \frac{\partial}{\partial y}(\rho Y^o v) = \frac{\partial}{\partial y}\left(\rho D^o \frac{\partial Y^o}{\partial y}\right) + s^o \Omega \tag{25}$$

$$\frac{\partial}{\partial x}(\rho Y^F u) + \frac{\partial}{\partial y}(\rho Y^F v) = \frac{\partial}{\partial y}\left(\rho D^F \frac{\partial Y^F}{\partial y}\right) + s^F\Omega \tag{26}$$

Using the mass conservation equation 1, we can wrote the Species conservation as:

$$\rho u \frac{\partial Y^o}{\partial x} + \rho v \frac{\partial Y^o}{\partial y} + \frac{\partial}{\partial y}\left(\rho D^o \frac{\partial Y^o}{\partial y}\right) + s^o\Omega \tag{27}$$

$$\rho u \frac{\partial Y^F}{\partial x} + \rho v \frac{\partial Y^F}{\partial y} + \frac{\partial}{\partial y}\left(\rho D^F \frac{\partial Y^F}{\partial y}\right) + s^F\Omega \tag{28}$$

Momemtum conservation:

$$\frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j \quad \text{for} \quad i, j = 1, 2, 3 \tag{29}$$

$$\tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{30}$$

$$\frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\lambda \delta_{ij} \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) \tag{31}$$

$$\frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(\mu \frac{\partial u_j}{\partial x_i}\right) \tag{32}$$

$$\frac{\partial \tau_{i1}}{\partial x_i} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\lambda \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{33}$$

$$\frac{\partial \tau_{i1}}{\partial x_i} = \frac{1}{L_c^2}(\lambda_c U_c) + \frac{1}{L_c^3}(\lambda_c \delta U_c) + \frac{1}{L_c^2}(\mu_c U_c) + \frac{1}{L_c^2}(\mu_c U_c) + \frac{1}{L_c^2}(\mu_c U_c) + \frac{1}{\delta^2}(\mu_c U_c) + \tag{34}$$

$$\frac{\partial \tau_{i1}}{\partial x_i} = \cancelto{\approx 0}{\frac{1}{L_c^2}(\lambda_c U_c)} + \cancelto{\approx 0}{\frac{1}{L_c^3}(\lambda_c \delta U_c)} + \cancelto{\approx 0}{\frac{1}{L_c^2}(\mu_c U_c)} + \cancelto{\approx 0}{\frac{1}{L_c^2}(\mu_c U_c)} + \cancelto{\approx 0}{\frac{1}{L_c^2}(\mu_c U_c)} + \frac{1}{\delta^2}(\mu_c U_c) + \tag{35}$$

$$\frac{\partial \tau_{i1}}{\partial x_i} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{36}$$

$$\frac{\partial \tau_{i2}}{\partial x_i} = \frac{\partial}{\partial y}\left(\lambda \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) \tag{37}$$

Now using the characteristic length of the problem we found:

$$\frac{\partial \tau_{i2}}{\partial x_i} = \frac{1}{\delta L_c}(\lambda_c U_c) + \frac{1}{\delta L_c}(\lambda_c U_c) + \frac{1}{\delta L_c}(\mu_c U_c) + \frac{1}{\delta L_c}(\mu_c U_c) + \frac{1}{L_c^3}(\mu_c \delta U_c) + \frac{1}{\delta L_c}(\mu_c U_c) \tag{38}$$

Multiply 38 by δ^2

$$\frac{\partial \tau_{i2}}{\partial x_i} = \left(\frac{\delta}{L_c}\right)(\lambda_c U_c) + \left(\frac{\delta}{L_c}\right)(\lambda_c U_c) + \left(\frac{\delta}{L_c}\right)(\mu_c U_c) + \left(\frac{\delta}{L_c}\right)(\mu_c U_c) + \left(\frac{\delta}{L_c}\right)^3(\mu_c \delta U_c) + \left(\frac{\delta}{L_c}\right)(\mu_c U_c) \tag{39}$$

$$\frac{\partial \tau_{i2}}{\partial x_i} = \cancelto{\approx 0}{\left(\frac{\delta}{L_c}\right)(\lambda_c U_c)} + \cancelto{\approx 0}{\left(\frac{\delta}{L_c}\right)(\lambda_c U_c)} + \cancelto{\approx 0}{\left(\frac{\delta}{L_c}\right)(\mu_c U_c)} + \cancelto{\approx 0}{\left(\frac{\delta}{L_c}\right)(\mu_c U_c)} + \cancelto{\approx 0}{\left(\frac{\delta}{L_c}\right)^3(\mu_c \delta U_c)} + \cancelto{\approx}{\left(\frac{\delta}{L_c}\right)(\mu_c U_c)} \tag{40}$$

$$\frac{\partial \tau_{i2}}{\partial x_i} \approx 0 \tag{41}$$

Now Momentum conservation equation in x direction can be wrote:

$$\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \rho f_x \tag{42}$$

Using the characteristic length of the problem, we can see that the others terms are important:

$$\frac{1}{L_c}(\rho_c U_c^2) + \frac{1}{L_c}(\rho_c U_c^2) = \frac{p_c}{L_c} + \frac{1}{\delta^2}(\mu_c U_c) + \rho_c f_{xc}. \tag{43}$$

In the viscous term multiple by $\dfrac{L_c}{U_c^2 \rho_c}$:

$$\frac{L_c}{U_c^2 \rho_c}\frac{1}{\delta^2}(\mu_c U_c) = \frac{\mu_c}{\rho_c \delta}\frac{L_c}{\delta U_c} = \frac{\mu_c}{\rho_c V_c \delta} \tag{44}$$

In order to keep the approximation $\delta_c << L_c$ this term:

$$\frac{\mu_c}{\rho_c V_c \delta} \approx 1 = Re \tag{45}$$

Using the mass conservation equation 16 in 43:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \rho f_x \tag{46}$$

The Momentum equation in y direction can be wrote as:

$$\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v v) = -\frac{\partial p}{\partial y} + \rho f_y \tag{47}$$

Using the characteristic length of the problem, we can see that the others terms are important:

$$\frac{1}{L_c}\frac{\delta}{L_c}(\rho_c U_c^2) + \frac{1}{L_c}\frac{\delta}{L_c}(\rho_c U_c^2) = -\frac{p_c}{L_c} + \rho_c f_{yc}. \tag{48}$$

$$\cancelto{\approx 0}{\frac{1}{L_c}\frac{\delta}{L_c}(\rho_c U_c^2)} + \cancelto{\approx 0}{\frac{1}{L_c}\frac{\delta}{L_c}(\rho_c U_c^2)} = -\frac{p_c}{L_c} + \rho_c f_{yc}. \tag{49}$$

Then the y momentum equation using the boundary layer assumption can be wrote as:

$$\frac{\partial p}{\partial y} = \rho f_y. \tag{50}$$

This equation means that the only variations of the pressure in y is determined by the body forces in that direction.

Energy conservation:

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x_i}(\rho u_i h) = \frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\tau_{ij} - \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + Q\Omega \tag{51}$$

$$\tau_{ij} = \lambda \delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{52}$$

$$\frac{\partial u}{\partial x_i}\tau_{i1} = \frac{\partial u}{\partial x_i}\lambda\delta_{i1}\frac{\partial u_k}{\partial x_k} + \frac{\partial u}{\partial x_i}\mu\left(\frac{\partial u_i}{\partial x} + \frac{\partial u}{\partial x_i}\right) \tag{53}$$

$$\frac{\partial u}{\partial x_i}\tau_{i1} = \frac{\partial u}{\partial x}\lambda\frac{\partial u_k}{\partial x_k} + \frac{\partial u}{\partial x_i}\mu\left(\frac{\partial u_i}{\partial x} + \frac{\partial u}{\partial x_i}\right) \tag{54}$$

$$\frac{\partial u}{\partial x_i}\tau_{i1} = \frac{\partial u}{\partial x}\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\frac{\partial u}{\partial x_i}\frac{\partial u_i}{\partial x} + \mu\left(\frac{\partial u}{\partial x_i}\right)^2 \tag{55}$$

$$\frac{\partial u}{\partial x_i}\tau_{i1} = \lambda\left(\frac{\partial u}{\partial x}\right)^2 + \lambda\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \mu\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \mu\left(\frac{\partial u}{\partial x}\right)^2 + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{56}$$

$$\frac{\partial u}{\partial x_i}\tau_{i1} = \lambda\cancelto{\approx 0}{\left(\frac{\partial u}{\partial x}\right)^2} + \lambda\cancelto{\approx 0}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} + \mu\cancelto{\approx 0}{\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}} + \mu\cancelto{\approx 0}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} + \mu\cancelto{\approx 0}{\left(\frac{\partial u}{\partial x}\right)^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{57}$$

$$\frac{\partial u}{\partial x_i}\tau_{i1} \approx \mu \left(\frac{\partial u}{\partial y}\right)^2 \tag{58}$$

$$\frac{\partial v}{\partial x_i}\tau_{i2} = \frac{\partial v}{\partial x_i}\lambda\delta_{i2}\frac{\partial u_k}{\partial x_k} + \frac{\partial v}{\partial x_i}\mu\left(\frac{\partial u_i}{\partial y} + \frac{\partial v}{\partial x_i}\right) \tag{59}$$

$$\frac{\partial v}{\partial x_i}\tau_{i2} = \lambda\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\frac{\partial v}{\partial x_i}\frac{\partial u_i}{\partial y} + \left(\mu\frac{\partial v}{\partial x_i}\right)^2 \tag{60}$$

$$\frac{\partial v}{\partial x_i}\tau_{i2} = \lambda\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y}\right)^2 + \mu\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \mu\left(\frac{\partial v}{\partial y}\right)^2 + \mu\left(\frac{\partial v}{\partial x}\right)^2 + \mu\left(\frac{\partial v}{\partial y}\right)^2 \tag{61}$$

$$\frac{\partial v}{\partial x_i}\tau_{i2} = \lambda\cancelto{\approx 0}{\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}} + \cancelto{\approx 0}{\left(\frac{\partial v}{\partial y}\right)^2} + \mu\cancelto{\approx 0}{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} + \mu\cancelto{\approx 0}{\left(\frac{\partial v}{\partial y}\right)^2} + \mu\cancelto{\approx 0}{\left(\frac{\partial v}{\partial x}\right)^2} + \cancelto{\approx 0}{\left(\frac{\partial v}{\partial y}\right)^2} \tag{62}$$

$$\frac{\partial v}{\partial x_i}\tau_{i2} \approx 0 \tag{63}$$

Finally for the heat diffusion:

$$\frac{\partial kT}{\partial x_i} = k\frac{\partial T}{\partial x} + k\frac{\partial T}{\partial y} \tag{64}$$

Using the characteristic lengths:

$$\frac{\partial kT}{\partial x_i} = k\frac{T_c}{L_c} + k\frac{T_c}{\delta_c} \tag{65}$$

Hence, the energy equation for steady flow become:

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x_i}(\rho u_i h) = \frac{\partial p}{\partial t} + u_i\frac{\partial p}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\tau_{ij} - \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + Q\Omega \tag{66}$$

$$\frac{\rho_c U_c h_c}{L_c} + \frac{\rho_c U_c h_c}{L_c} + \frac{\rho_c U_c \delta_c h_c}{L_c \delta_c} = U_c\frac{p_c}{L_c} + U_c\frac{p_c}{L_c} + \frac{U_c \delta_c}{L_c}\frac{p_c}{\delta_c} + \mu\frac{U_c^2}{\delta^2} + k\frac{T_c}{L_c^2} + k\frac{T_c}{\delta_c^2} + ... \tag{67}$$

$$\frac{1}{L_c} + \frac{1}{L_c} + \frac{1}{L_c} = \frac{p_c}{\rho_c h_c}\frac{1}{L_c} + \frac{p_c}{\rho_c h_c}\frac{1}{L_c} + \frac{p_c}{\rho_c h_c}\frac{1}{L_c} + \mu\frac{U_c}{\rho_c h_c}\frac{1}{\delta^2} + k\frac{1}{\rho_c U_c h_c}\frac{T_c}{L_c^2} + k\frac{1}{\rho_c U_c h_c}\frac{T_c}{\delta_c^2} + ... \tag{68}$$

Energy conservation:

$$\frac{\partial}{\partial x_i}(\rho u_i h) = u_i\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + Q\Omega \tag{69}$$

Using the mass conservation equation 16 in 69:

$$\rho u\frac{\partial h}{\partial x} + \rho v\frac{\partial h}{\partial y} = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \mu\frac{\partial^2 u}{\partial y^2} + \cancelto{\approx 0}{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q\Omega \tag{70}$$

$$\rho u\frac{\partial h}{\partial x} + \rho v\frac{\partial h}{\partial y} = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \mu\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q\Omega \tag{71}$$

The equations that govern the Steady Reacting Mixing Layer are:

$$\rho u\frac{\partial Y^o}{\partial x}+\rho v\frac{\partial Y^o}{\partial y}+\frac{\partial}{\partial y}\left(\rho D^o\frac{\partial Y^o}{\partial y}\right)+s^o\Omega \tag{72}$$

$$\rho u\frac{\partial Y^F}{\partial x}+\rho v\frac{\partial Y^F}{\partial y}+\frac{\partial}{\partial y}\left(\rho D^F\frac{\partial Y^F}{\partial y}\right)+s^F\Omega \tag{73}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\rho f_x \tag{74}$$

$$\frac{\partial p}{\partial y}=\rho f_y. \tag{75}$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+\mu\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+Q\Omega \tag{76}$$

The above equations can be more simplified if we omit the body force in all direction and with a zero pressure gradient, became:

$$\rho u\frac{\partial Y^o}{\partial x}+\rho v\frac{\partial Y^o}{\partial y}+\frac{\partial}{\partial y}\left(\rho D^o\frac{\partial Y^o}{\partial y}\right)+s^o\Omega \tag{77}$$

$$\rho u\frac{\partial Y^F}{\partial x}+\rho v\frac{\partial Y^F}{\partial y}+\frac{\partial}{\partial y}\left(\rho D^F\frac{\partial Y^F}{\partial y}\right)+s^F\Omega \tag{78}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{79}$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + Q\Omega \tag{80}$$

$$p = \rho R T \tag{81}$$

$$h = c_p T \tag{82}$$

$$\Omega = \beta Y_F Y_O \exp\left(\frac{E}{RT}\right) \tag{83}$$

Boundary conditions

$\mathbf{x} \longrightarrow +\infty$

$u = u_{+\infty}$
$v = v_{+\infty} = 0$
$p = p_{+\infty} = constant = C$
$\rho = \rho_{+\infty} = 0$
$Y_F = Y_{F+\infty} = 1$
$Y_O = Y_{O+\infty} = 0$

$\mathbf{x} \longrightarrow -\infty$

$u = u_{-\infty}$
$v = v_{-\infty} = 0$
$p = p_{-\infty} = constant = C$
$\rho = \rho_{-\infty} = 0$
$Y_F = Y_{F+\infty} = 0$
$Y_O = Y_{O+\infty} = 1$

2. Dimensionless Equations

The dimesionless equations for a reacting mixing layer are obtined using the positive freestream variables,L_c and δ characteritic lenghts:

$$\overline{x} = \frac{x}{L_c} \qquad \overline{y} = \frac{y}{\delta} \tag{84}$$

$$U = \frac{u}{u_\infty} \qquad V = \frac{L_c}{\delta}\frac{v}{u_\infty} \qquad \overline{\rho} = \frac{\rho}{\rho_\infty} \qquad \overline{T} = \frac{T}{T_\infty} \qquad P = \frac{p}{P_\infty} \tag{85}$$

$$\psi_F = \frac{Y^F}{Y^{F\infty}} \qquad \psi_O = \frac{Y^O}{Y^{O\infty}} \tag{86}$$

$$\frac{\mu}{\mu_\infty} = \frac{k}{k_\infty} = \frac{c_v}{c_{v\infty}} = \frac{c_p}{c_{p\infty}} = \frac{D^F}{D^{F\infty}} = \frac{D^0}{D^{O\infty}} = \frac{R}{R_\infty} = \overline{T}^n \tag{87}$$

Mass conservation:

$$\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) = 0 \tag{88}$$

$$\frac{\rho_\infty U_\infty}{L_c}\frac{\partial}{\partial \overline{x}}(\varrho U) + \frac{\rho_\infty U_\infty}{L_c}\frac{\partial}{\partial \overline{y}}(\varrho V) = 0 \tag{89}$$

$$\frac{\partial}{\partial \overline{x}}(\varrho U) + \frac{\partial}{\partial \overline{y}}(\varrho V) = 0 \tag{90}$$

Species conservation:

$$\frac{\rho_\infty U_\infty Y^{O\infty}}{L_c}\varrho U\frac{\partial \psi_o}{\partial \overline{x}} + \frac{\rho_\infty U_\infty Y^{O\infty}}{L_c}\varrho V\frac{\partial \psi_o}{\partial \overline{y}} + \frac{\rho_\infty D^{O\infty} Y^{O\infty}}{L_c^2}\frac{\partial}{\partial \overline{y}}\left(\rho D^o\frac{\partial \psi_o}{\partial \overline{y}}\right) + s^o\Omega \tag{91}$$

$$\frac{\rho_\infty U_\infty Y^{F\infty}}{L_c}\varrho U\frac{\partial \psi_F}{\partial \overline{x}} + \frac{\rho_\infty U_\infty Y^{F\infty}}{L_c}\varrho V\frac{\partial \psi_F}{\partial \overline{y}} + \frac{\rho_\infty D^{F\infty} Y^{F\infty}}{L_c^2}\frac{\partial}{\partial \overline{y}}\left(\rho D^F\frac{\partial \psi_F}{\partial \overline{y}}\right) + s^F\Omega \tag{92}$$

$$\varrho U\frac{\partial \psi_o}{\partial \overline{x}} + \varrho V\frac{\partial \psi_o}{\partial \overline{y}} + \frac{D^{O\infty}}{L_c U}\frac{\partial}{\partial \overline{y}}\left(\rho D^o\frac{\partial \psi_o}{\partial \overline{y}}\right) + s^o\Omega \tag{93}$$

$$\varrho U\frac{\partial \psi_F}{\partial \overline{x}} + \varrho V\frac{\partial \psi_F}{\partial \overline{y}} + \frac{D^{F\infty}}{L_c U}\frac{\partial}{\partial \overline{y}}\left(\rho D^F\frac{\partial \psi_F}{\partial \overline{y}}\right) + s^F\Omega \tag{94}$$

$$\frac{D^{O\infty}}{L_c U} = \frac{1}{Le_O}, \qquad Le_O \text{ is the Lewis number for the oxidizer.} \tag{95}$$

$$\frac{D^{F\infty}}{L_c U} = \frac{1}{Le_F}, \qquad Le_F \text{ is the Lewis number for the fuel.} \tag{96}$$

Then we get:

$$\varrho U \frac{\partial \psi_o}{\partial \overline{x}} + \varrho V \frac{\partial \psi_o}{\partial \overline{y}} + \frac{1}{Le_o} \frac{\partial}{\partial \overline{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \overline{y}} \right) + s^o \Omega \tag{97}$$

$$\varrho U \frac{\partial \psi_F}{\partial \overline{x}} + \varrho V \frac{\partial \psi_F}{\partial \overline{y}} + \frac{1}{Le_F} \frac{\partial}{\partial \overline{y}} \left(\rho D^F \frac{\partial \psi_F}{\partial \overline{y}} \right) + s^F \Omega. \tag{98}$$

Momemtum equation:

$$\frac{\rho_\infty U_\infty^2}{L_c} \varrho U \frac{\partial U}{\partial \overline{x}} + \frac{\rho_\infty U_\infty^2}{L_c} \varrho V \frac{\partial U}{\partial \overline{y}} = \frac{\mu_\infty U_\infty}{\delta^2} \frac{\partial}{\partial \overline{y}} \left(\mu \frac{\partial U}{\partial \overline{y}} \right) \tag{99}$$

We do not known the relation between characteristics lenghts of the problem, δ and L_c. For the above equation have the same same order, we can defined:

$$\frac{\rho_\infty U_\infty}{L_c} = \frac{\mu_\infty}{\delta^2} \tag{100}$$

$$\frac{\delta^2}{L_c} = \frac{\mu_\infty}{\rho_\infty U_\infty} \tag{101}$$

$$\frac{\delta^2}{L_c} = L_c \frac{\mu_\infty}{\rho_\infty L_c U_\infty} \tag{102}$$

$$\frac{\delta^2}{L_c} = L_c \frac{\mu_\infty}{\rho_\infty L_c U_\infty} \tag{103}$$

Defining:

$$Re = \frac{\mu_\infty}{\rho_\infty L_c U_\infty} \tag{104}$$

as the Reynolds number respect to L_c, we get:

$$\delta^2 = L_c^2 \frac{1}{Re} \tag{105}$$

At the beginning we assume

$$\frac{\delta}{L_c} << 1, \qquad \text{or in others words} \qquad Re_{L_c} >> 1, \tag{106}$$

then we obtain:

$$\delta = L_c \sqrt{\frac{1}{Re}} \tag{107}$$

inserting this relation in the dimensionless Momemtum equation 108:

$$\frac{\rho_\infty U_\infty^2}{L_c}\varrho U\frac{\partial U}{\partial \overline{x}} + \frac{\rho_\infty U_\infty^2}{L_c}\varrho V\frac{\partial U}{\partial \overline{y}} = \frac{\mu_\infty U_\infty}{\delta^2}\frac{\partial}{\partial \overline{y}}\left(\mu\frac{\partial U}{\partial \overline{y}}\right) \tag{108}$$

we obtain:

$$\varrho U\frac{\partial U}{\partial \overline{x}} + \varrho V\frac{\partial U}{\partial \overline{y}} = \frac{\partial}{\partial \overline{y}}\left(\mu\frac{\partial U}{\partial \overline{y}}\right) \tag{109}$$

Before non-dimensionalization the energy equation, we use the equation that relate the entaphy and the temperature for a perfect gas, we get:

$$\rho u\frac{\partial}{\partial x}(c_pT) + \rho v\frac{\partial}{\partial y}(c_pT) = \mu\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q\Omega \tag{110}$$

Using the non-dimensional variables:

$$\frac{\rho_\infty U_\infty T_\infty c_{p\infty}}{L_c}\varrho U\frac{\partial}{\partial \overline{x}}(c_p\overline{T}) + \frac{\rho_\infty U_\infty T_\infty c_{p\infty}}{L_c}\varrho V\frac{\partial}{\partial \overline{y}}(c_p\overline{T}) = \frac{\mu_\infty U_\infty^2}{\delta^2}\mu\frac{\partial^2 U}{\partial \overline{y}^2} + \frac{k_\infty T_\infty}{\delta^2}\frac{\partial}{\partial \overline{y}}\left(k\frac{\partial \overline{T}}{\partial \overline{y}}\right) + Q\Omega \tag{111}$$

$$\varrho U\frac{\partial}{\partial \overline{x}}(c_p\overline{T}) + \varrho V\frac{\partial}{\partial \overline{y}}(c_p\overline{T}) = (\gamma-1)M^2\mu\frac{\partial^2 U}{\partial \overline{y}^2} + \frac{1}{Pr}\frac{\partial}{\partial \overline{y}}\left(k\frac{\partial \overline{T}}{\partial \overline{y}}\right) + Q\Omega \tag{112}$$

where:

$$M = \frac{U_\infty}{a_{0\infty}} \qquad \text{Mach Number with} \qquad a_{0\infty} = \sqrt{\gamma_\infty R_\infty T_\infty} \tag{113}$$

$$Pr = \frac{\mu_\infty c_p}{k_\infty} \qquad \text{Prandtl Number} \tag{114}$$

Finaly the equation for a perfect gas became:

$$1 = \varrho \overline{T} \tag{115}$$

Non-dimensional Reacting Mixing layer equation:

$$\frac{\partial}{\partial \overline{x}}(\varrho U) + \frac{\partial}{\partial \overline{y}}(\varrho V) = 0 \tag{116}$$

$$\varrho U \frac{\partial \psi_o}{\partial \overline{x}} + \varrho V \frac{\partial \psi_o}{\partial \overline{y}} = \frac{1}{Le_o} \frac{\partial}{\partial \overline{y}} \left(\rho D^o \frac{\partial \psi_o}{\partial \overline{y}} \right) + s^o \Omega \tag{117}$$

$$\varrho U \frac{\partial \psi_F}{\partial \overline{x}} + \varrho V \frac{\partial \psi_F}{\partial \overline{y}} = \frac{1}{Le_F} \frac{\partial}{\partial \overline{y}} \left(\rho D^F \frac{\partial \psi_F}{\partial \overline{y}} \right) + s^F \Omega. \tag{118}$$

$$\varrho U \frac{\partial U}{\partial \overline{x}} + \varrho V \frac{\partial U}{\partial \overline{y}} = \frac{\partial}{\partial \overline{y}} \left(\mu \frac{\partial U}{\partial \overline{y}} \right) \tag{119}$$

$$\varrho U \frac{\partial}{\partial \overline{x}}(c_p \overline{T}) + \varrho V \frac{\partial}{\partial \overline{y}}(c_p \overline{T}) = (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial \overline{y}^2} + \frac{1}{Pr} \frac{\partial}{\partial \overline{y}} \left(k \frac{\partial \overline{T}}{\partial \overline{y}} \right) + Q\Omega \tag{120}$$

$$1 = \varrho \overline{T} \tag{121}$$

3. Self-Similarity solution for Reacting Mixing Layer equations

To find a similarity solutions for a reacting mixing layer equations, firstly we need to use the Howarth-Dorodnitsyn 123 transformation to find am incompressible form of the compressibles equations.

Definig the stream function for compressible flow as:

$$\frac{\partial \Psi}{\partial x} = -\varrho V \qquad \frac{\partial \Psi}{\partial y} = \varrho U, \tag{122}$$

The mass conservation equation is identically satisfied by the introduction of this stream function. Now defining a vertical stretching as:

$$x = \overline{x} \qquad y = \int_0^{\overline{y}} \varrho d\overline{y} \tag{123}$$

and using the chain rule:

$$f(\overline{x}, \overline{y}) \qquad \overline{x} = \overline{x}(x, y) \qquad \overline{y} = \overline{y}(x, y) \tag{124}$$

$$\frac{\partial f}{\partial \overline{x}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{x}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{x}} \tag{125}$$

$$\frac{\partial f}{\partial \overline{y}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{y}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{y}} \tag{126}$$

$$\frac{\partial f}{\partial \overline{x}} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\int_0^{\overline{y}} \frac{\partial \varrho}{\partial \overline{x}} dy \tag{127}$$

$$\frac{\partial f}{\partial \overline{y}} = \frac{\partial f}{\partial y}\varrho \tag{128}$$

so that:

$$U = \frac{\partial \Psi}{\partial y} = \hat{U}, \tag{129}$$

that have the same form as incompresible flow, in the same way we can defined:

$$\hat{V} = -\frac{\partial \Psi}{\partial x}, \tag{130}$$

Using the definition of V:

$$\varrho V = -\frac{\partial \Psi}{\partial \overline{x}} \tag{131}$$

with 127 we can write:

$$\varrho V = -\left(\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial \overline{x}}\right) \tag{132}$$

$$\varrho V = \left(\hat{V} - U\frac{\partial y}{\partial \overline{x}}\right) \tag{133}$$

$$\hat{V} = \varrho V + U\frac{\partial y}{\partial \overline{x}} \tag{134}$$

Now we can used the equation 123, 127, 128 and 134 in the Mixing layer equations, 116 to 121, obtain:

Mass conservation

$$\frac{\partial}{\partial \overline{x}}(\varrho U) + \frac{\partial}{\partial \overline{y}}(\varrho V) = 0 \tag{135}$$

$$\frac{\partial}{\partial x}(\varrho U) + \frac{\partial}{\partial y}(\varrho U)\frac{\partial y}{\partial \overline{x}} + \varrho\frac{\partial}{\partial y}(\varrho V) = 0 \tag{136}$$

$$\frac{\partial}{\partial x}(\varrho U) + \frac{\partial}{\partial y}(\varrho U)\frac{\partial y}{\partial \overline{x}} + \varrho\frac{\partial}{\partial y}\left(\hat{V} - U\frac{\partial y}{\partial \overline{x}}\right) = 0 \tag{137}$$

$$U\frac{\partial \varrho}{\partial x} + \varrho\frac{\partial U}{\partial x} + U\frac{\partial \varrho}{\partial y}\frac{\partial y}{\partial \overline{x}} + \varrho\frac{\partial U}{\partial y}\frac{\partial y}{\partial \overline{x}} + \varrho\frac{\partial \hat{V}}{\partial y} - \varrho\frac{\partial}{\partial \overline{y}}\left(U\frac{\partial y}{\partial \overline{x}}\right) = 0 \tag{138}$$

$$U\frac{\partial \varrho}{\partial x} + \varrho\frac{\partial U}{\partial x} + U\frac{\partial \varrho}{\partial y}\frac{\partial y}{\partial \overline{x}} + \varrho\frac{\partial U}{\partial y}\frac{\partial y}{\partial \overline{x}} + \varrho\frac{\partial \hat{V}}{\partial \overline{y}} - \varrho\frac{\partial U}{\partial \overline{y}}\frac{\partial y}{\partial \overline{x}} - \varrho U\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial \overline{x}}\right) = 0 \tag{139}$$

$$U\frac{\partial \varrho}{\partial x} + \varrho\frac{\partial U}{\partial x} + U\frac{\partial \varrho}{\partial y}\frac{\partial y}{\partial \overline{x}} + \cancel{\varrho\frac{\partial U}{\partial y}\frac{\partial y}{\partial \overline{x}}}^{0} + \varrho\frac{\partial \hat{V}}{\partial y} - \cancel{\varrho\frac{\partial U}{\partial \overline{y}}\frac{\partial y}{\partial \overline{x}}} - \varrho U\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial \overline{x}}\right) = 0 \tag{140}$$

$$U\frac{\partial\varrho}{\partial x}+\varrho\frac{\partial U}{\partial x}+U\frac{\partial\varrho}{\partial y}\frac{\partial y}{\partial\overline{x}}+\varrho\frac{\partial\hat{V}}{\partial y}-\varrho U\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial\overline{x}}\right)=0 \tag{141}$$

the last term of the above equation can be write as:

$$\varrho U\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial\overline{x}}\right)=U\varrho\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial\overline{x}}\right)=U\frac{\partial}{\partial\overline{y}}\left(\frac{\partial y}{\partial\overline{x}}\right)=U\frac{\partial\varrho}{\partial\overline{x}}=U\left(\frac{\partial\varrho}{\partial x}+\frac{\partial\varrho}{\partial y}\frac{\partial y}{\partial\overline{x}}\right) \tag{142}$$

hence:

$$\varrho\frac{\partial U}{\partial x}+\varrho\frac{\partial\hat{V}}{\partial y}=0 \tag{143}$$

$$\frac{\partial\hat{U}}{\partial x}+\frac{\partial\hat{V}}{\partial y}=0 \tag{144}$$

Species Conservation

Oxidizer conservation

$$\varrho U\frac{\partial\psi_o}{\partial\overline{x}}+\varrho V\frac{\partial\psi_o}{\partial\overline{y}}=\frac{1}{Le_o}\frac{\partial}{\partial\overline{y}}\left(\rho D^o\frac{\partial\psi_o}{\partial\overline{y}}\right)+s^o\Omega \tag{145}$$

$$\varrho U\frac{\partial\psi_o}{\partial x}+\varrho U\frac{\partial\psi_o}{\partial y}\frac{\partial y}{\partial\overline{x}}+\varrho^2 V\frac{\partial\psi_o}{\partial y}=\frac{\varrho}{Le_o}\frac{\partial}{\partial y}\left(\varrho^2 D^o\frac{\partial\psi_o}{\partial y}\right)+s^o\Omega \tag{146}$$

$$\varrho U\frac{\partial\psi_o}{\partial x}+\varrho U\frac{\partial\psi_o}{\partial y}\frac{\partial y}{\partial\overline{x}}+\varrho^2\frac{\partial\psi_o}{\partial y}\left(\frac{1}{\varrho}\hat{V}-\frac{1}{\varrho}U\frac{\partial y}{\partial\overline{x}}\right)=\frac{\varrho}{Le_o}\frac{\partial}{\partial y}\left(\varrho^2D^o\frac{\partial\psi_o}{\partial y}\right)+s^o\Omega \tag{147}$$

$$\varrho U\frac{\partial\psi_o}{\partial x}+\varrho U\frac{\partial\psi_o}{\partial y}\frac{\partial y}{\partial\overline{x}}+\varrho^2\frac{\partial\psi_o}{\partial y}\left(\frac{1}{\varrho}\hat{V}-\frac{1}{\varrho}U\frac{\partial y}{\partial\overline{x}}\right)=\frac{\varrho}{Le_o}\frac{\partial}{\partial y}\left(\varrho^2D^o\frac{\partial\psi_o}{\partial y}\right)+s^o\Omega \tag{148}$$

$$\varrho U\frac{\partial\psi_o}{\partial x}+\varrho\hat{V}\frac{\partial\psi_o}{\partial y}=\frac{\varrho}{Le_o}\frac{\partial}{\partial y}\left(\varrho^2D^o\frac{\partial\psi_o}{\partial y}\right)+s^o\Omega \tag{149}$$

$$\hat{U}\frac{\partial\psi_o}{\partial x}+\hat{V}\frac{\partial\psi_o}{\partial y}=\frac{1}{Le_o}\frac{\partial}{\partial y}\left(\varrho^2D^o\frac{\partial\psi_o}{\partial y}\right)+\frac{s^o\Omega}{\varrho} \tag{150}$$

Fuel conservation

In the same way as in oxidizer conservartion, using the transform coordinates $x, y, \hat{U}$ and $\hat{V}$ in the Fuel conservations species, we get:

$$\hat{U}\frac{\partial\psi_F}{\partial x}+\hat{V}\frac{\partial\psi_F}{\partial y}=\frac{1}{Le_F}\frac{\partial}{\partial y}\left(\varrho^2D^F\frac{\partial\psi_F}{\partial y}\right)+\frac{s^F\Omega}{\varrho} \tag{151}$$

Momemtum conservation

$$\varrho U\frac{\partial U}{\partial\overline{x}}+\varrho V\frac{\partial U}{\partial\overline{y}}=\frac{\partial}{\partial\overline{y}}\left(\mu\frac{\partial U}{\partial\overline{y}}\right) \tag{152}$$

$$\varrho U\frac{\partial U}{\partial x}+\varrho U\frac{\partial U}{\partial y}\frac{\partial y}{\partial \overline{x}}+\varrho^2 V\frac{\partial U}{\partial y}=\varrho\frac{\partial}{\partial y}\left(\mu\varrho\frac{\partial U}{\partial y}\right) \tag{153}$$

$$\varrho U\frac{\partial U}{\partial x}+\varrho U\frac{\partial U}{\partial y}\frac{\partial y}{\partial \overline{x}}+\varrho^2\left(\frac{1}{\varrho}\hat{V}-\frac{1}{\varrho}U\frac{\partial y}{\partial \overline{x}}\right)\frac{\partial U}{\partial y}=\varrho\frac{\partial}{\partial y}\left(\mu\varrho\frac{\partial U}{\partial y}\right) \tag{154}$$

$$\varrho U\frac{\partial U}{\partial x}+\varrho\hat{V}\frac{\partial U}{\partial y}=\varrho\frac{\partial}{\partial y}\left(\mu\varrho\frac{\partial U}{\partial y}\right) \tag{155}$$

$$\hat{U}\frac{\partial \hat{U}}{\partial x}+\hat{V}\frac{\partial \hat{U}}{\partial y}=\frac{\partial}{\partial y}\left(\mu\varrho\frac{\partial \hat{U}}{\partial y}\right) \tag{156}$$

Energy Equation

$$\varrho U\frac{\partial}{\partial \overline{x}}(c_p\overline{T})+\varrho V\frac{\partial}{\partial \overline{y}}(c_p\overline{T})=(\gamma-1)M^2\mu\frac{\partial^2 U}{\partial \overline{y}^2}+\frac{1}{Pr}\frac{\partial}{\partial \overline{y}}\left(k\frac{\partial \overline{T}}{\partial \overline{y}}\right)+Q\Omega \tag{157}$$

Using the chain rule to change the second order derivative of $\overline{y}$ in the energy equation.

$$\frac{\partial^2 f}{\partial \overline{y}^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{y}}\right)\frac{\partial y}{\partial \overline{y}} \tag{158}$$

$$\frac{\partial^2 f}{\partial \overline{y}^2}=\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial \overline{y}}\right)^2+\frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial \overline{y}^2} \tag{159}$$

$$\frac{\partial^2 f}{\partial \overline{y}^2} = \varrho^2 \frac{\partial^2 f}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} \frac{\partial f}{\partial y} \tag{160}$$

$$\begin{aligned} \varrho U \frac{\partial}{\partial x}(c_p \overline{T}) + \varrho U \frac{\partial}{\partial y}(c_p \overline{T}) \frac{\partial y}{\partial \overline{x}} + \varrho^2 V \frac{\partial}{\partial y}(c_p \overline{T}) = \\ \varrho^2 (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial U}{\partial y} + \frac{\varrho}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \overline{T}}{\partial y} \right) + Q\Omega \end{aligned} \tag{161}$$

$$\begin{aligned} \varrho U \frac{\partial}{\partial x}(c_p \overline{T}) + \varrho U \frac{\partial}{\partial y}(c_p \overline{T}) \frac{\partial y}{\partial \overline{x}} + \varrho^2 \left(\frac{1}{\varrho} \hat{V} - \frac{1}{\varrho} U \frac{\partial y}{\partial \overline{x}} \right) \frac{\partial}{\partial y}(c_p \overline{T}) = \\ \varrho^2 (\gamma - 1) M^2 \mu \frac{\partial^2 U}{\partial y^2} + \varrho \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial U}{\partial y} + \frac{\varrho}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \overline{T}}{\partial y} \right) + Q\Omega \end{aligned} \tag{162}$$

$$\begin{aligned} \hat{U} \frac{\partial}{\partial x}(c_p \overline{T}) + \hat{V} \frac{\partial}{\partial y}(c_p \overline{T}) = \varrho (\gamma - 1) M^2 \mu \frac{\partial^2 \hat{U}}{\partial y^2} + \\ \frac{\partial \varrho}{\partial y} (\gamma - 1) M^2 \mu \frac{\partial \hat{U}}{\partial y} + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \overline{T}}{\partial y} \right) + \frac{Q\Omega}{\varrho} \end{aligned} \tag{163}$$

$$\hat{U} \frac{\partial}{\partial x}(c_p \overline{T}) + \hat{V} \frac{\partial}{\partial y}(c_p \overline{T}) = (\gamma - 1) M^2 \mu \frac{\partial}{\partial y} \left(\varrho \frac{\partial \hat{U}}{\partial y} \right) + \frac{1}{Pr} \frac{\partial}{\partial y} \left(k \varrho \frac{\partial \overline{T}}{\partial y} \right) + \frac{Q\Omega}{\varrho} \tag{164}$$

To transform completely the compressible equations to incompressible form we need to used the Chapman's approximate viscosity law:

$$\mu = C_w T \tag{165}$$

Multiply by ϱ and using the gas perfect equation, we get:

$$\mu\varrho = C_w \tag{166}$$

where C_w is a function of the temperature $C_w = C_w(T)$.

$$\frac{\partial \hat{U}}{\partial x} + \frac{\partial \hat{V}}{\partial y} = 0 \tag{167}$$

$$\hat{U}\frac{\partial \psi_o}{\partial x} + \hat{V}\frac{\partial \psi_o}{\partial y} = \frac{1}{Le_o}\frac{\partial}{\partial y}\left(\varrho^2 D^o \frac{\partial \psi_o}{\partial y}\right) + \frac{s^o\Omega}{\varrho} \tag{168}$$

$$\hat{U}\frac{\partial \psi_F}{\partial x} + \hat{V}\frac{\partial \psi_F}{\partial y} = \frac{1}{Le_F}\frac{\partial}{\partial y}\left(\varrho^2 D^F \frac{\partial \psi_F}{\partial y}\right) + \frac{s^F\Omega}{\varrho} \tag{169}$$

$$\hat{U}\frac{\partial \hat{U}}{\partial x} + \hat{V}\frac{\partial \hat{U}}{\partial y} = \frac{\partial}{\partial y}\left(C_w(T)\frac{\partial \hat{U}}{\partial y}\right) \tag{170}$$

$$\hat{U}\frac{\partial}{\partial x}(c_p\overline{T}) + \hat{V}\frac{\partial}{\partial y}(c_p\overline{T}) = (\gamma - 1)M^2\mu\frac{\partial}{\partial y}\left(\varrho\frac{\partial \hat{U}}{\partial y}\right) + \frac{1}{Pr}\frac{\partial}{\partial y}\left(k\varrho\frac{\partial \overline{T}}{\partial y}\right) + \frac{Q\Omega}{\varrho} \tag{171}$$

4. Crocco Busemann

In this section a special solution to boundary layer equation is found using the definition of the total enthalpy.

From the boundary layer equation in two dimensional flow

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{172}$$

$$(\rho u)\frac{\partial u}{\partial x} + (\rho v)\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{173}$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{\partial h}{\partial y} = u\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{174}$$

where the Prantdl number is defined as $Pr = \mu c_p / k$.

Since the total enthalpy is defined as

$$H = h + \frac{1}{2}u^2 \tag{175}$$

the total enthalpy equation using both the energy and momentum equation is

$$(\rho u)\frac{\partial H}{\partial x} + (\rho v)\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\left(1 - \frac{1}{Pr}\right)\mu u \frac{\partial u}{\partial y}\right] \tag{176}$$

This equation allows a simplest solution $H = h + \frac{1}{2}u^2 =$ constant as long as $Pr = 1$, which leads to

$$\frac{\partial H}{\partial y} = \frac{\partial h_w}{\partial y} = 0 \tag{177}$$

Representing the not heat transfer at the wall.

$Pr = 1$ implies in a perfect balance between viscous dissipation and heat conduction so as keep the the stagnation enthalpy constant in adiabatic boundary layer and also is a good approximation for gases.

Another solution can be obtained if the pressure gradient is neglected in the boundary layer equation. With this, the momentum equation and the energy equation get very similar, it seems as if u and h could be interchanged except for the dissipation term.

$$(\rho u)\frac{\partial u}{\partial x} + (\rho v)\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{178}$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{179}$$

Then a solution of the form

$$\frac{\partial h}{\partial y} = \frac{dh}{du}\frac{\partial u}{\partial y} \tag{180}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{dh}{du}\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{dh}{du}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y} \tag{181}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{dh}{du}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y} = \frac{dh}{du}\frac{\partial u^2}{\partial y^2} + \frac{d^2 h}{du^2}\left(\frac{\partial u}{\partial y}\right)^2 \tag{182}$$

$$(\rho u)\frac{\partial u}{\partial x} + (\rho v)\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{183}$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{dh}{du}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{dh}{du}\frac{\partial u}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{184}$$

Assuming $Pr = 1$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{dh}{dy} = \frac{\partial}{\partial y}\left(\mu\frac{dh}{dy}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{185}$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{dh}{dy} = \mu\frac{\partial}{\partial y}\left(\frac{dh}{dy}\right) + \frac{dh}{dy}\frac{\partial \mu}{\partial y} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{186}$$

$$(\rho u)\frac{\partial h}{\partial x} + (\rho v)\frac{dh}{dy} = \mu\frac{dh}{du}\frac{\partial u^2}{\partial y^2} + \mu\frac{d^2 h}{du^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{dh}{dy}\frac{\partial \mu}{\partial y} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{187}$$

$$(\rho u)\frac{dh}{du}\frac{\partial u}{\partial x} + (\rho v)\frac{dh}{du}\frac{\partial u}{\partial y} - \mu\frac{dh}{du}\frac{\partial u^2}{\partial y^2} - \frac{dh}{du}\frac{\partial u}{\partial y}\frac{\partial \mu}{\partial y} = \left[\frac{d^2 h}{du^2} + 1\right]\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{188}$$

$$\frac{dh}{du}\left[(\rho u)\frac{\partial u}{\partial x} + (\rho v)\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right] = \left[\frac{d^2 h}{du^2} + 1\right]\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{189}$$

Using the momentum equation

$$\left[\frac{d^2h}{du^2}+1\right]\mu\left(\frac{\partial u}{\partial y}\right)^2=0 \tag{190}$$

Then

$$\frac{d^2h}{du^2}=-1 \tag{191}$$

Integrating

$$h=-\frac{u^2}{2}+c_1u+c_2 \tag{192}$$

For mixing layer these constant can be found, using the values at the boundaries. For the upper stream:

$$h_1=-\frac{u_1^2}{2}+c_1u_1+c_2 \tag{193}$$

Similarly for the lower stream

$$h_2=-\frac{u_2^2}{2}+c_1u_2+c_2 \tag{194}$$

This is two unknown and two equations.

For the equation 193

$$c_2=h_1+\frac{u_1^2}{2}-c_1u_1 \tag{195}$$

Using it in 194

$$h_2=-\frac{u_2^2}{2}+c_1u_2+h_1+\frac{u_1^2}{2}-c_1u_1 \tag{196}$$

$$c_1\left(u_2-u_1\right)=h_2-h_1+\frac{u_2^2}{2}-\frac{u_1^2}{2} \tag{197}$$

$$c_1=\frac{(h_2-h_1)}{(u_2-u_1)}+\frac{1}{2}(u_2+u_1) \tag{198}$$

and

$$c_2 = h_1 + \frac{u_1^2}{2} - \left[\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1)\right] u_1 \tag{199}$$

Therefore

$$h = -\frac{u^2}{2} + c_1 u + c_2 \tag{200}$$

$$h = -\frac{u^2}{2} + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1)\right) u + h_1 + \frac{u_1^2}{2} - \left[\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1)\right] u_1 \tag{201}$$

$$h = \frac{u_1^2}{2} - \frac{u^2}{2} + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1)\right)(u - u_1) + h_1 \tag{202}$$

$$h = \frac{1}{2}(u_1 + u)(u_1 - u) + \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1)\right)(u - u_1) + h_1 \tag{203}$$

$$h = (u_1 - u)\left[\frac{1}{2}(u_1 + u) - \left(\frac{(h_2 - h_1)}{(u_2 - u_1)} + \frac{1}{2}(u_2 + u_1)\right)\right] + h_1 \tag{204}$$

$$h = (u_1 - u)\left[\frac{1}{2}(u - u_2) - \left(\frac{(h_2 - h_1)}{(u_2 - u_1)}\right)\right] + h_1 \tag{205}$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2\frac{(u_1 - u)}{(u_2 - u_1)} + h_1\left(\frac{(u_1 - u)}{(u_2 - u_1)}\right) + h_1 \tag{206}$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2\frac{(u_1 - u)}{(u_2 - u_1)} + h_1\left(1 + \left(\frac{(u_1 - u)}{(u_2 - u_1)}\right)\right) \tag{207}$$

$$h = \frac{1}{2}(u - u_2)(u_1 - u) - h_2\frac{(u_1 - u)}{(u_2 - u_1)} + h_1\frac{(u_2 - u)}{(u_2 - u_1)} \tag{208}$$

Assuming c_p = constant, the enthalpy can be related with the temperatute with the relation $h = c_p T$

$$T = \frac{1}{2}\frac{1}{c_p}(u - u_2)(u_1 - u) - T_2\frac{(u_1 - u)}{(u_2 - u_1)} + T_1\frac{(u_2 - u)}{(u_2 - u_1)} \tag{209}$$

$$T = T_1\frac{(u - u_2)}{(u_1 - u_2)} + T_2\frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2}\frac{1}{c_p}(u_1 - u)(u - u_2) \tag{210}$$

The last term and the temperature are dimensional and depend on the non dimensional parameters.

Using $T = T/T_1$ and $U = U/U_1$.

$$TT_1 = T_1\frac{(u - u_2)}{(u_1 - u_2)} + T_2\frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2}\frac{U_1^2}{c_p}(u_1 - u)(u - u_2) \tag{211}$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1}\frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2}\frac{U_1^2}{T_1 c_p}(u_1 - u)(u - u_2) \tag{212}$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1}\frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2}\frac{U_1^2\gamma_1 R}{\gamma_1 R T_1 c_p}(u_1 - u)(u - u_2) \tag{213}$$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \frac{T_2}{T_1}\frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2}\frac{M^2\gamma_1 R}{c_p}(u_1 - u)(u - u_2) \tag{214}$$

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma} \tag{215}$$

$\beta_t = T_2/T_1$

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \beta_t\frac{(u_1 - u)}{(u_1 - u_2)} + \frac{1}{2}M^2(\gamma_1 - 1)(u_1 - u)(u - u_2) \tag{216}$$

Using $T = T/T_1$ and $U = U/a_1$.

$$T = \frac{(u - u_2)}{(u_1 - u_2)} + \beta_t\frac{(u_1 - u)}{(u_1 - u_2)} + \frac{\gamma - 1}{2}(u_1 - u)(u - u_2) \tag{217}$$

References

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