Ejemplo

Suponga que $y_1, \cdots, y_n|\theta$ son variables distribuidas normal e independientemente con media θ y varianza σ^2 conocida. Suponga que $p(\theta) \propto 1$ es la distribución a priori uniforme (impropia) sobre los números reales. Encuentre la distribución posterior de θ.

Verosimilitud

$$p(\underset{\sim}{y}|\theta) \propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2\right]$$

$$p(\bar{y}|\theta) \propto \exp\left[-\frac{n}{2\sigma^2}(\bar{y}-\theta)^2\right]$$

La distribución posterior

$$p(\theta|\bar{y}) \propto p(\bar{y}|\theta)\,p(\theta)$$

$$\propto \exp\left[-\frac{n}{\sigma^2}(\bar{y}-\theta)^2\right] \times (1)$$

$$\theta|\bar{y} \sim N(\bar{y}, \sigma^2/n)$$

Definición

Sea $p(x|\theta)$ la densidad de x dado θ. La información de Fisher es definida como:

$$\mathbf{I}(\theta) = -E\left[\frac{\partial^2 \log(p(x|\theta))}{\partial\theta^2}\right]$$

Si $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)$, entonces:

$$\mathbf{I}(\boldsymbol{\theta}) = -E\left[\frac{[\partial^2(\log(p(x|\boldsymbol{\theta})))}{\partial\theta_i\partial\theta_j}\right]_{pxp},$$

en este caso $\mathbf{I}(\boldsymbol{\theta})$ es una matriz de dimensión pxp.

$$p(\boldsymbol{\theta}) \propto |\mathbf{I}(\boldsymbol{\theta})|^{1/2}$$

Ejemplo

Se tienen v.a. independientes $y_1, \cdots, y_n$ Bernoulli con parámetro θ. Encontraremos la distribución a priori de Jeffreys para θ.

$$p(y|\theta) = \underline{\theta^y} \; \underline{(1-\theta)^{1-y}}$$

$$y = 0, 1 \qquad E(y) = \theta$$

$$V(y) = \theta(1-\theta)$$

$$\log p(y|\theta) = y\log(\theta) + (1-y) \times \log(1-\theta)$$

$$\frac{\partial \log p(y|\theta)}{\partial \theta} = \frac{y}{\theta} + (-1)\frac{(1-y)}{1-\theta}$$

$$= \frac{y}{\theta} - \frac{(1-y)}{1-\theta}$$

$$\frac{\partial \log p(y|\theta)}{\partial \theta^2} = -\frac{y}{\theta^2} - \frac{(1-y)}{(1-\theta)^2}$$

$$I(\theta) = -E\left[-\frac{y}{\theta^2} - \frac{(1-y)}{(1-\theta)^2}\right]$$

$$= \frac{E(y)}{\theta^2} + \frac{E(1-y)}{(1-\theta)^2}$$

$$= \frac{\cancel{\theta}}{\theta^2} + \frac{\cancel{1-\theta}}{(1-\theta)^2} = \frac{1}{\theta} + \frac{1}{(1-\theta)}$$

La a priori de Jeffreys

$$p(\theta) \propto \left(\frac{1}{\theta} + \frac{1}{1-\theta}\right)^{1/2}$$

$$\propto \left(\frac{1 \cancel{-\theta} \cancel{+\theta}}{\theta(1-\theta)}\right)^{1/2} = \theta^{-1/2}(1-\theta)^{-1/2}$$

$$= \theta^{1/2-1}(1-\theta)^{1/2-1}$$

$\therefore \quad \Theta \sim \text{Beta}(1/2, 1/2)$

La a priori de Jeffreys es propia

Ejemplo

Se tiene $X \sim N(\mu, \sigma^2)$. *Calculemos la distribución a priori de Jeffreys para* (μ, σ).

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

$$\log p(x \mid \mu, \sigma^2) = -\frac{1}{2}\log(2\pi) - \log(\sigma) - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{\partial \log p(x \mid \mu, \sigma^2)}{\partial \mu} = \frac{(x-\mu)}{\sigma^2}$$

$$\frac{\partial \log p(x \mid \mu, \sigma^2)}{\partial \sigma} = -\frac{1}{\sigma} + \frac{(x-\mu)^2}{\sigma^3}$$

$$\frac{\partial^2 \log p(x \mid \mu, \sigma^2)}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

$$\frac{\partial^2 \log p(x \mid \mu, \sigma^2)}{\partial \sigma^2} = \frac{1}{\sigma^2} - \frac{3}{\sigma^4}(x-\mu)^2$$

$$\frac{\partial^2 \log p(x \mid \mu, \sigma^2)}{\partial \sigma \, \partial \mu} = -\frac{2}{\sigma^3}(x-\mu)$$

$$I\left(\begin{pmatrix} \mu \\ \sigma \end{pmatrix}\right) = -\mathcal{E}\begin{bmatrix} -\frac{1}{\sigma^2} & -\frac{2(x-\mu)}{\sigma^3} \\ -\frac{2(x-\mu)}{\sigma^3} & \frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 2/\sigma^2 \end{bmatrix}$$

$$-\mathcal{E}\left(-\frac{2(x-\mu)}{\sigma^3}\right) = \frac{2}{\sigma^3}\left[\overbrace{\mathcal{E}(x)}^{\mu} - \mu\right] = 0$$

$$-E\left(\frac{1}{\sigma^2}-\frac{3(X-\mu)^2}{\sigma^4}\right)$$

$$=-\frac{1}{\sigma^2}+\frac{3}{\sigma^4}E[(X-\mu)^2]$$

$$=-\frac{1}{\sigma^2}+\frac{3}{\sigma^{\cancel{4}}}\cancel{\sigma}^2=-\frac{1}{\sigma^2}+\frac{3}{\sigma^2}=\frac{2}{\sigma^2}$$

La a priori de Jeffreys está dada por

$$p(\mu,\sigma)\propto\left|I\left(\binom{\mu}{\sigma}\right)\right|^{1/2}$$

$$\propto\left(\frac{2}{\sigma^4}\right)^{1/2}$$

$$\propto\frac{1}{\sigma^2}$$

Esta distribución a priori de Jeffreys es impropia

Ejemplo

Suponga $X \sim N(\mu, 1)$ *y* $\psi(\mu) = e^{\mu}$. *Encuentre la apriori de Jeffreys para* $\psi(\mu)$.

$$(\mathbf{I}(\theta))^{1/2} = (\mathbf{I}(\psi(\theta)))^{1/2} \left| \frac{\partial \psi \theta}{\partial \theta} \right|$$

$$p(\theta) = p(\psi(\theta)) \left| \frac{\partial \psi(\theta)}{\partial \theta} \right|$$

$$p(x \mid \mu, 1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-\mu)^2\right]$$

$$\log p(x \mid \mu, 1) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}(x-\mu)^2$$

$$\frac{\partial \log p(x \mid \mu, 1)}{\partial \mu} = (x - \mu)$$

$$\frac{\partial^2 \log p(x \mid \mu, 1)}{\partial \mu^2} = -1$$

$$I(\mu) = -E(-1) = 1$$

$$\therefore \quad p(\mu) \propto 1$$

La a priori de Jeffreys para μ

Tenemos $\varphi(\mu)$ es una transformación uno a uno de μ. La a priori de Jeffreys para $\varphi(\mu)$ es:

$$\left(I(\varphi(\mu))\right)^{1/2} = \left(I(\mu)\right)^{1/2} \left|\frac{d\varphi(\mu)}{d\mu}\right|^{-1}$$

$$= 1\left(e^{\mu}\right)^{-1}$$

$$= e^{-\mu}$$

$$\therefore \quad p(\varphi(\mu)) \propto e^{-\mu}$$