② Notate que: $f(x|\theta) = \dfrac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1}(1-x)^{\theta-1} = \theta x^{\theta-1}$

ⓐ Ahora, $\left(\dfrac{-1}{3\ln(x)}, \dfrac{-2}{3\ln(x)}\right)$, Para la Prob de cubrimiento está dado por

$P\left(\dfrac{-1}{3\ln(x)} < \theta < \dfrac{-2}{3\ln(x)}\right)$ para $0<x<1$, $\theta>0$, pero como $0<x<1$, $\ln(x)$ es un # negativo.

$\therefore P\left(\dfrac{-1}{3\theta} > \ln(x) > \dfrac{-2}{3\theta}\right) \Rightarrow$ Recordemos que, si a,b son #s en $\mathbb{R}$ $\therefore$

para $a>b$

$a>b \Leftrightarrow e^{a}>e^{b} \Leftrightarrow e^{-a}<e^{-b}$ por decrecimiento exponencial.

$\therefore P(e^{-\frac{1}{3}\theta} < x < e^{-2/3\theta}) \Leftrightarrow P(\underset{a}{e^{\frac{-2}{3\theta}}} < x < \underset{b}{e^{\frac{-1}{3\theta}}})$, calculando $\rightarrow$ Prob. de cubrimiento

$\displaystyle\int_a^b \theta x^{\theta-1}\,dx = x^{\theta}\Big|_a^b = e^{-\frac{1}{3}} - e^{-2/3} \approx 0{,}2031\ldots$

$\inf_{\theta}\left(\theta \in \left(\dfrac{-1}{3\ln(x)}, \dfrac{-2}{3\ln(x)}\right)\right) = \inf_{\theta}\left(e^{-1/3} - e^{-2/3}\right) \approx 0{,}2031$ Será el coeficiente de confianza

③ ⓐ $H_0 = \beta = \beta_0$ vs $H_1 = \beta \neq \beta_0$

$\therefore L(\beta|x,\lambda) = \displaystyle\prod_{i=1}^{n} \frac{1}{\Gamma(\lambda)\beta^{\lambda}} x^{\lambda-1} e^{-\frac{x_i}{\beta}} = (\Gamma(\lambda)\beta^{\lambda})^{-n}\left(\prod_{i=1}^{n} x_i\right)^{\lambda-1} \cdot e^{-\frac{1}{\beta}\sum x_i}$

$\ell(\beta) = -n\ln(\Gamma(\lambda)) - n\lambda\ln(\beta) + (\lambda-1)\sum\ln(x_i) - \frac{1}{\beta}\sum x_i$

$\ell'(\beta) = \dfrac{-n\lambda}{\beta} + \dfrac{1}{\beta^2}\sum x_i = \dfrac{1}{\beta}\left[\dfrac{\sum x_i}{\beta} - n\lambda\right] = \dfrac{1}{\beta}\left[\dfrac{\sum x_i - n\lambda\beta}{\beta}\right]$

$\therefore \hat{\beta} = \dfrac{\sum x_i}{n\lambda} = \dfrac{\bar{x}}{\lambda}$, luego $S(\beta) = \dfrac{\sum x_i - n\lambda\beta}{\beta^2}$ $\rightarrow$ ☺

$I_n(\beta) = -E\left[\dfrac{\partial}{\partial\beta}\left(\dfrac{-n\lambda}{\beta} + \dfrac{1}{\beta^2}\sum x_i\right)\right]$

$$-E\left[\frac{n\lambda}{\beta^2}-\frac{2}{\beta^3}\sum x_i\right]=\frac{-n\lambda}{\beta^2}+\frac{2}{\beta^3}\sum E(x_i)\longrightarrow x_i\sim\text{Gama}$$

$$=\frac{-n\lambda}{\beta^2}+\frac{2n\lambda\beta}{\beta^3}=\frac{-n\lambda+2n\lambda}{\beta^2}=\frac{n\lambda}{\beta^2}\quad \text{II}$$

$\therefore E(x_i)=\lambda\beta$, Asi entonces

Bajo H_0 y Tomando $n\bar{x}=\sum x_i$

$$Z_s=\frac{S(\beta_0)}{\sqrt{I_n(\beta_0)}}=\frac{\dfrac{\sum x_i-n\lambda\beta_0}{\beta_0^2}}{\sqrt{\dfrac{n\lambda}{\beta_0^2}}}\overset{\Uparrow}{=}\frac{\dfrac{n\bar{x}-n\lambda\beta_0}{\beta_0^2}}{\dfrac{\sqrt{n}\sqrt{\lambda}}{\beta_0}}$$

$$=\frac{n(\bar{x}-\beta_0\lambda)}{\sqrt{n}\sqrt{\lambda}\,\beta_0}=\frac{\sqrt{n}(\bar{x}-\lambda\beta_0)}{\sqrt{\lambda}\,\beta_0}\xrightarrow{d}n(0,1)$$

Asi, La region de rechazo bajo H_0 serà $|Z_s|>Z_{\frac{\alpha}{2}}$ y la region de aceptación para hallar un Ic aprox al $(100)(1-\alpha)\%$ para β serà

$$1-\alpha\cong P\left(Z_s \mid |Z_s|<Z_{\alpha/2}\right)$$

$$1-\alpha\cong P\Bigg(\underbrace{\left|\frac{\sqrt{n}(\bar{x}-\lambda\beta_0)}{\sqrt{\lambda}\,\beta_0}\right|\leq Z_{\frac{\alpha}{2}}}_{\text{III}}\Bigg)$$

Tomando III $\therefore$ $$-Z_{\frac{\alpha}{2}}<\frac{\sqrt{n}(\bar{x}-\lambda\beta_0)}{\sqrt{\lambda}\,\beta_0}<Z_{\frac{\alpha}{2}}$$

$$\frac{-Z_{\frac{\alpha}{2}}\sqrt{\lambda}}{\sqrt{n}}<\frac{\bar{x}}{\beta_0}-\frac{\lambda\beta_0}{\beta_0}<\frac{Z_{\frac{\alpha}{2}}\sqrt{\lambda}}{\sqrt{n}}$$

$$\lambda-\frac{Z_{\frac{\alpha}{2}}\sqrt{\lambda}}{\sqrt{n}}<\frac{\bar{x}}{\beta_0}\leq\lambda+\frac{Z_{\alpha/2}\sqrt{\lambda}}{\sqrt{n}}$$

$$\frac{\sqrt{n}}{\sqrt{n}\lambda+Z_{\alpha/2}\sqrt{\lambda}}<\frac{\beta_0}{\bar{x}}<\frac{\sqrt{n}}{\sqrt{n}\lambda-Z_{\frac{\alpha}{2}}\sqrt{\lambda}}$$

$$\underbrace{\frac{\sqrt{n}\,\bar{x}}{\sqrt{\lambda}(\sqrt{n\lambda}+z_{\frac{\alpha}{2}})}}_{=D} < \beta_0 < \underbrace{\frac{\sqrt{n}\,\bar{x}}{\sqrt{\lambda}(\sqrt{n\lambda}-z_{\frac{\alpha}{2}})}}_{=(}$$

Por tanto queda demostrado que

Un IC aprox al $100(1-\alpha)\%$ para β es $\left(=D \,,\, =(\right)$

(B) Tenemos $\bar{x} = 1{,}07875$, $z_{\frac{\alpha}{2}} = 1{,}96$, $n = 80$, reemplazando en

Tendremos $\left(\dfrac{\sqrt{80}\,(1{,}078)}{\sqrt{2}(\sqrt{80(2)}+1{,}96)} , \dfrac{\sqrt{80}\,(1{,}078)}{\sqrt{2}\sqrt{80(2)}-1{,}96} \right)$ $\left(=D \,,\, =(\right)$

$= (0{,}46668;\ 0{,}6378)$

(I) Calculemos la verosimilitud.

$L(p|x) = \prod_{i=1}^{20} \left(p^{x_i}(1-p)^{1-x_i}\right) = p^{\sum x_i}(1-p)^{n-\sum x_i}$, luego, aplicando $\ln$

$\ell(p|x) = \sum_{i=1}^{20} x_i \ln(p) + (20-\sum x_i)\ln(1-p) =$

$= \sum_{i=1}^{n} x_i \ln(p) + 20\ln(1-p) - \sum x_i \ln(1-p) = \ell(p|x)$

$\ell'(p|x) = \dfrac{\sum^{n} x_i}{p} + \dfrac{(-n+\sum x_i)}{(1-p)} = 0 = \dfrac{\sum x_i - pn}{p(1-p)} = 0$

$= \sum_{i=1}^{n} x_i = pn \rightarrow \hat{p} = \bar{x}$, verificando con 2do derivada

$d^{II}\ell(p|x) = p^{-1}\sum x_i + (-n+\sum x_i)(1-p)^{-1}$

$= \dfrac{\sum_{i=1}^{n} x_i}{p^2} + \dfrac{(\sum x_i - n)}{(1-p)^2} < 0$, por tanto $\hat{p}$ es el MLE.

$\therefore$ LRT será $\circledast$ $\quad n=20$

$\lambda(x) = \dfrac{L(p_0|x)}{L(\bar{x}|x)} \geq c$ con $p_0 = 1/2 \rightarrow \circledast = \dfrac{\frac{1}{2}\sum x_i \left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\sum x_i} = \left(\frac{1}{2}\right)^n$

$\star = L(\bar{x}|x) = \dfrac{\bar{x}^{\sum x_i}(1-\bar{x})^{(1-x_i)}}{(1-\bar{x})^{\sum x_i}}$ para $\bar{x}$, se usa $-2\ln(\lambda p)$ entonces,

$$-2\ln(\lambda(P)) = -2\ln\left(\frac{(1-\bar{x})^{\Sigma x_i}}{(\bar{x})^{\Sigma x}(1-\bar{x})^{2}}\right)$$

$$= -2\ln(1-\bar{x})\sum^{n} x_i - 2\left[\frac{(\Sigma x_i)\ln(\bar{x}) + \ln(1-\bar{x})}{2}\right]$$

Suponiendo que se cumplen las condiciones de regularidad, entonces,

$$= -2\ln(\lambda(\bar{x})) \xrightarrow{d} \chi^2_{\circ}(1)$$, luego bajo $H_0 = \frac{1}{2}$, $n = 20$

se cumple con n suficientemente grande entonces para una confianza al 95%, buscamos en la tabla de la ji-cuadrado $\chi^2_{0{,}05}(1) = 3{,}841$.

Finalmente se rechaza $H_0 = \frac{1}{2}$ cuando $-2\ln(\lambda(\bar{x})) > 3{,}841$

Nota: claramente en el ejercicio, $n=20$.