Ejemplo

Suponga $x_1, \cdots, x_n$ *son variables aleatorias i.i.d. Bernoulli*(θ) *y sea* $y = \sum x_i$. *Considere la transformación* $\tau = \frac{\theta}{(1-\theta)}$, $\tau =$ *odds ratio.*

a) *Determine la distribución de probabilidad de* y. *Escribala en función de* τ.

b) *Encuentre la distribución a priori de Jeffreys para* τ.

c) *Muestre que la distribución a priori de Jeffreys para* θ *es invariante con respecto a la transformación* $\tau = \frac{\theta}{1-\theta}$.

Definición

Sea $p(x|\theta)$ *la densidad de* x *dado* θ. *La información de Fisher es definida como:*

$$\mathbf{I}(\theta) = -E\left[\frac{\partial^2 \log(p(x|\theta))}{\partial\theta^2}\right]$$

Si $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)$, *entonces:*

$$\mathbf{I}(\boldsymbol{\theta}) = -E\left[\frac{[\partial^2(\log(p(x|\boldsymbol{\theta})))}{\partial\theta_i\partial\theta_j}\right]_{pxp},$$

en este caso $\mathbf{I}(\boldsymbol{\theta})$ *es una matriz de dimensión pxp.*

$$p(\boldsymbol{\theta}) \propto |\mathbf{I}(\boldsymbol{\theta})|^{1/2}$$

$$(\mathbf{I}(\theta))^{1/2} = (\mathbf{I}(\psi(\theta)))^{1/2}\left|\frac{\partial\psi\theta}{\partial\theta}\right|$$

$$p(\theta) = p(\psi(\theta))\left|\frac{\partial\psi(\theta)}{\partial\theta}\right|$$

a) $Y \sim \text{Binomial}(n, \theta)$

$$p(y|\theta) = \binom{n}{y} \theta^{y} (1-\theta)^{n-y}$$

$$\tau = \frac{\theta}{1-\theta}$$

$$\tau - \tau\theta = \theta$$

$$\tau = \theta + \tau\theta$$

$$\frac{\tau}{1+\tau} = \theta$$

$$p(y|\tau) = \binom{n}{y} \left(\frac{\tau}{1+\tau}\right)^{y} \left(1 - \frac{\tau}{1+\tau}\right)^{n-y}$$

$$= \binom{n}{y} \left(\frac{\tau}{1+\tau}\right)^{y} \left(\frac{1+\cancel{\tau}-\cancel{\tau}}{1+\tau}\right)^{n-y}$$

$$= \binom{n}{y} \frac{\tau^{y}}{(1+\tau)^{y}} \frac{1}{(1+\tau)^{n-y}}$$

$$= \binom{n}{y} \tau^{y} (1+\tau)^{-n}$$

b) $$\ln p(y|\tau) = \ln \binom{n}{y} + y \ln \tau - n \ln(1+\tau)$$

$$\frac{\partial}{\partial \tau} \ln p(y|\tau) = \frac{y}{\tau} - \frac{n}{1+\tau}$$

$$\frac{\partial^2}{\partial \tau^2} \ln p(y|\tau) = -\frac{y}{\tau^2} + \frac{n}{(1+\tau)^2}$$

$$-E\left[\frac{\partial^2}{\partial \tau^2} \ln p(y|\tau)\right]$$

$$= -E\left[-\frac{y}{\tau^2} + \frac{n}{(1+\tau)^2}\right]$$

$$= \frac{n\theta}{\tau^2} - \frac{n}{(1+t)^2} = \frac{n\cancel{\tau}}{(1+\tau)\tau^{\cancel{2}}} - \frac{n}{(1+\tau)^2}$$

$$= \frac{n}{(1+\tau)\tau} - \frac{n}{(1+\tau)^2} = \frac{1}{(1+\tau)}\left[\frac{n}{\tau} - \frac{n}{1+\tau}\right]$$

$$= \frac{1}{1+\tau}\left[\frac{n + \cancel{n\tau} - \cancel{n\tau}}{\tau(1+\tau)}\right] = \frac{n}{\tau(1+\tau)^2}$$

$$p(\tau) \propto \left(\frac{1}{\tau(\tau+1)^2} \right)^{1/2}$$

$$= \tau^{-1/2} (\tau+1)^{-1}$$

c) Vamos a calcular la a priori de Jeffreys para θ

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$$\log p(y|\theta) = \log \binom{n}{y}$$

$$+ y \log(\theta) + (n-y) \log(1-\theta)$$

$$\frac{\partial}{\partial \theta} \log p(y|\theta) = \frac{y}{\theta} - \frac{n-y}{1-\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\log p(y|\theta) = -\frac{y}{\theta^2} - \frac{(n-y)}{(1-\theta)^2}$$

$$-E\left[\frac{\partial^2}{\partial\theta^2}\log p(y|\theta)\right]$$

$$= -E\left(-\frac{y}{\theta^2}\right) - E\left[-\frac{(n-y)}{(1-\theta)^2}\right]$$

$$= \frac{n\cancel{\theta}}{\theta^{\cancel{2}}} + \frac{n - n\theta}{(1-\theta)^2}$$

$$= \frac{n}{\theta} + \frac{n\cancel{(1-\theta)}}{(1-\theta)^{\cancel{2}}}$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta} = \frac{n - \cancel{n\theta} + \cancel{n\theta}}{\theta(1-\theta)}$$

$$= \frac{n}{\theta(1-\theta)}$$

$$p(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2}$$

$$\frac{d\tau}{d\theta} = \frac{d}{d\theta}\left(\frac{\theta}{1-\theta}\right)$$

$$= \frac{(1-\cancel{\theta}) + \cancel{\theta}}{(1-\theta)^2} = \frac{1}{(1-\theta)^2}$$

Ya teníamos que:

$$p(\tau) \propto \tau^{-1/2} (\tau+1)^{-1}$$

Sabemos que $\tau = \dfrac{\theta}{1-\theta}$

$\therefore$

$$p(\tau) \propto \left(\frac{\theta}{1-\theta}\right)^{-1/2} \left(\frac{\theta}{1-\theta} + 1\right)^{-1}$$

$$= \left(\frac{\theta}{1-\theta}\right)^{-1/2} \left(\frac{\cancel{\theta} + 1 - \cancel{\theta}}{1-\theta}\right)^{-1}$$

$$= \left(\frac{(1-\theta)}{\theta}\right)^{1/2} (1-\theta)^{1}$$

$$= (1-\theta)^{3/2} (1/\theta)^{1/2}$$

Entonces

$$p(\tau)\left|\frac{d\tau}{d\theta}\right|$$

$$= \frac{(1-\theta)^{3/2}}{\theta^{1/2}} \times \frac{1}{(1-\theta)^2} = \theta^{-1/2}(1-\theta)^{-1/2}$$

y recuerden que

$$p(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2}$$