④ Sea $X_1, X_2 \ldots X_n$ una muestra aleatoria de una población

$$f(x|\theta) = \frac{\theta}{(1+x)^{1+\theta}} \; ; \; x>0, \; \theta>0$$

Ⓐ Halle est. LRT para probar $H_0 = \theta = \theta_0$ vs $H_1: \theta = \theta \neq \theta_0$

Ⓑ Halle el Test UMP y la respectiva región de Rechazo para el dado, para probar las hipótesis $H_0 = 3$ vs $H_1 = \theta = 2$

Solución: El espacio paramétrico Θ estará definido por

$\Theta = \mathbb{R}^+$, donde $\Theta_0 = \{\theta_0\}$ y $\Theta_0^c = (0, \theta_0) \cup (\theta_0, \infty)$

Para hallar el LRT es necesario calcular el MLE.

$$L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{\theta}{(1+x_i)^{1+\theta}} = \frac{\theta^n}{\prod_{i=1}^{n}(1+x_i)^{1+\theta}}$$

$$\ell(L(\theta|x)) = n\ln(\theta) - (1+\theta)\sum_{i=1}^{n} \ln(1+x_i)$$

De ser necesario, por si no se ve claro si es max.

Calculando puntos críticos, luego evaluando para ver si es max, min.

$$\frac{\partial}{\partial\theta}(\ell(\theta|x)) = \frac{n}{\theta} - \sum_{i=1}^{n} \ln(x_i+1) \Leftrightarrow \frac{n}{\theta} = \sum_{i=1}^{n} \ln(x_i+1)$$

$$\hat{\theta}^* = \frac{n}{\sum_{i=1}^{n} \ln(x_i+1)}$$

Por criterio de segunda derivada

$$\frac{\partial^2}{\partial\theta^2}(\ell(\theta|x)) = \frac{-n}{\theta^2} < 0,$$ por tanto es $\hat{\theta}^*$ MLE para θ

$$\therefore \sup_{\theta \subset \Theta_0} L(\theta|x) = L(\theta_0|x) \;\wedge\; \sup_{\theta \subset \Theta} L(\theta|x) = L(\hat{\theta}^*|x)$$

$$\therefore \lambda(x) = \frac{L(\theta_0|x)}{L(\hat{\theta}^*|x)} = \frac{\dfrac{\theta_0^n}{\left[\prod_{i=1}^{n}(1+x_i)\right]^{1+\theta_0}}}{\dfrac{\left(\dfrac{n}{\sum_{i=1}^{n}(\ln(x_i+1))}\right)^n}{\left[\prod_{i=1}^{n}(1+x_i)\right]^{1+\hat{\theta}^*}}}$$

se deja indicado por lo larga de la expresión, pero $\hat{\theta}^*$ es el MLE

$$= \frac{\theta^n \left[\prod_{i=1}^{n}(1+x_i)\right]^{1+\hat{\theta}^*}}{\left(\frac{n}{\sum_{i=1}^{n}\ln(1+x_i)}\right)^n \left(\prod_{i=1}^{n}(1+x_i)\right)^{1+\theta_0}} = \frac{\theta^n \left[\prod_{i=1}^{n}(1+x_i)\right]^{\hat{\theta}^*-\theta_0}}{\frac{n^n}{\left(\sum \ln(1+x_i)\right)^n}}$$

$$= \left(\frac{\theta}{n}\right)^n \left[\prod_{i=1}^{n}(1+x_i)\right]^{\hat{\theta}^*-\theta_0} \cdot \left[\sum_{i=1}^{n}\ln(x_i+1)\right]^n \text{ Est LRT}$$

(B) Usando N-P tenemos que: $\frac{F(x_1 \dots x_n \mid 1)}{F(x_1 \dots x_n \mid 3)} > k$

$$\therefore \frac{\frac{1^n}{\left[\prod_{i=1}^{n}(1+x_i)\right]^{1+1}}}{\frac{3^n}{\left[\prod_{i=1}^{n}(1+x_i)\right]^{1+3}}} > k \Leftrightarrow \frac{\left[\prod_{i=1}^{n}(x_i+1)\right]^4}{3^n\left[\prod(1+x_i)\right]^2} > k$$

$$\frac{\left(\prod_{i=1}^{n}(1+x_i)\right)^2}{3^n} > k \Leftrightarrow \left[\prod_{i=1}^{n}(x_i+1)\right]^2 > k3^n \therefore \prod_{i=1}^{n}(1+x_i) > \sqrt{k3^n}$$

$$\sum_{i=1}^{n}\ln(1+x_i) = \frac{1}{2}\ln(3^n k)$$

Sea $Y_i = \ln(1+x_i)$, entonces $\mathbb{R}$ (region de rechazo) estará dado por

$$\mathbb{R} = \left\{(x_1 \dots x_n) \mid \sum_{i=1}^{n} Y_i > \frac{1}{2}\ln(k3^n)\right\}; \text{ si } P_\theta(X \in R) = \star$$

$$\star = P\left(\sum_{i=1}^{n} Y_i > \frac{1}{2}\ln(k3^n)\right) \Rightarrow$$ Para calcular esta Prob necesitamos saber como se distribuye Y

Cambio de variable

$Y_i = \ln(1+x_i) \Leftrightarrow x_i = e^{Y_i}-1 \wedge \frac{dx_i}{dY_i} = e^{Y_i}$, Entonces, por T camb. de v.

$$F_{Y_i}(Y_i) = f_x(e^{Y_i}-1)\left|\frac{dx_i}{dY_i}\right| = \frac{\theta}{(e^{Y_i})^{1+\theta}} e^{Y_i} = \theta e^{-\theta Y_i} \sim e(\theta)$$

$\therefore Y_i \sim \exp(\theta)$, y por lemos visto en clase, $\sum Y_i \sim \text{Gamma}(n, \frac{1}{\theta})$

Si $S = \sum_{i=1}^{n} Y_i \sim S \sim Gamma(n, 1/\theta)$; Finalmente

$\alpha = P(S > \frac{1}{2}\ln(k3^n) \mid \theta = 3) \Rightarrow (H_0 = \theta = 3)$ donde

S_α un cuantil superior α de una Gamma y $S_\alpha = \frac{1}{2}\ln(k3^n)$

Así, $S_\alpha \Rightarrow 2S_\alpha = \ln(k3^n) \Rightarrow e^{2S_\alpha} = 3^n k$; $k = \dfrac{e^{2S_\alpha}}{3^n}$

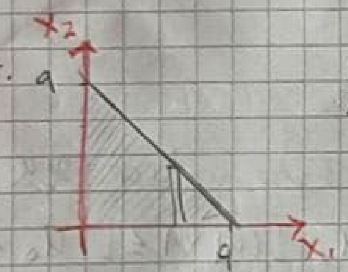
$\therefore R = \{S \mid S > S_\alpha\}$ con $S_\alpha = \frac{1}{2}\ln(k3^n)$

(2) $C = \{(x_1, x_2) \mid x_1 + x_2 > 9\}$ $\quad X_i \sim \exp(1/\theta)$ $\quad \Theta = (0, \infty)$

$= X_i \sim \frac{1}{\theta} e^{-x/\theta}$

Como piden la función de potencia,

$\beta(\theta) = P_\theta(x_1, x_2 \in C) = P(x_1 + x_2 > 9] = 1 - P(x_1 + x_2 \leq 9)$



$\therefore$ Como es una muestra de tamaño 2, tenemos que:

$$= 1 - \int_0^9 \int_0^{9-x_2} \frac{1}{\theta} e^{-x_1/\theta} \cdot \frac{1}{\theta} e^{-\frac{x_2}{\theta}} dx_1\, dx_2$$

$$= 1 - \int_0^9 \int_0^{9-x_2} \frac{1}{\theta^2} e^{-\frac{1}{\theta}(x_1+x_2)} dx_1\, dx_2 \,,$$ sea $u = -\dfrac{x_1}{\theta}$ $\quad du = -\dfrac{1}{\theta} dx_1$

$$= 1 - \int_0^9 -\frac{1}{\theta} e^{-\frac{x_2}{\theta}} \int_0 e^u(-\theta du) = 1 - \int_0^9 -\frac{1}{\theta} e^{-\frac{x_2}{\theta}} \left[e^{-x_1/\theta} \right] \Big|_0^{9-x_2} dx_2$$

$$= 1 + \frac{1}{\theta} \int_0^9 e^{\left(\frac{-x_2 - 9 + x_2}{\theta}\right)} - e^{-x_2/\theta} dx_2 = 1 + \frac{1}{\theta} \int_0^9 e^{-9/\theta} - e^{-\frac{x_2}{\theta}} dx_2$$

$$= 1 + \frac{1}{\theta}\left(9e^{-9/\theta}(x_2) \Big|_0^9 + \theta \left(e^{-x_2/\theta} \right) \Big|_0^9 \right) = \cancel{1} + \frac{1}{\theta}\left(e^{-9/\theta} \cdot 9 + \theta(e^{-9/\theta} - \cancel{1}\right)$$

$$= \frac{9}{\theta} e^{-9/\theta} + e^{-9/\theta} = e^{-9/\theta}\left(\frac{9}{\theta} + 1\right) = \beta(\theta)$$

(2) El LRT está definido por $\lambda(x) = \dfrac{\sup_{H_0} L(\theta|x)}{L(\hat{\theta}|x)}$ con $\hat{\theta}$ - MLE

$$L(\theta|x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\sigma^2 (x_i-u)^2} = \left(\frac{1}{2\pi}\right)^{n/2} \theta^{-n} \cdot e^{-1/2\sigma^2 \sum_{i=1}^{n}(x_i-u)^2}$$

, Necesitamos saber cuando se maximiza esta función, pero $L(\theta|x)$ no converge por tanto ...

Suponga $u \leq u_0$ con $\Theta = (-\infty, u_0]$

$\therefore$ Como $u \leq u_0$ bajo H_0, entonces

$$-u \geq u_0 \iff x_i - u_0 \leq x_i - u$$

$$\sum_{i=1}^{n}(x_i-u_0)^2 \leq \sum_{i=1}^{n}(x_i-u)^2$$

$$-\sum_{i=1}^{n}(x_i-u_0)^2 \geq -\sum_{i=1}^{n}(x_i-u)^2$$

$$\frac{-1}{2\sigma^2}\sum(x_i-u_0)^2 \geq \frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-u)^2$$

$$e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-u_0)^2} \geq e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-u)^2} \Rightarrow$$ Esto implica que $\forall u,\ L(u|x) \leq L(u_0|x)$

Por tanto $\sup_{H_0} L(u|x) = L(u_0|x)$, Así

$$\lambda(x) = \frac{L(u_0|x)}{L(\bar{x}|x)} = \frac{\left(\frac{1}{2\pi}\right)^{n/2} \sigma^{-n} e^{\frac{-1}{2\sigma^2}\sum(x_i-u_0)^2}}{\left(\frac{1}{2\pi}\right)^{n/2} \sigma^{-n} e^{\frac{-1}{2\sigma^2}\sum(x_i-\bar{x})^2}}$$

$$= \frac{\exp\left(\frac{-1}{2\sigma^2}\sum(x_i-u_0)^2\right)}{\exp\left(\frac{-1}{2\sigma^2}\sum(x_i-\bar{x})^2\right)} = \exp\left(\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2 - \sum_{i=1}^{n}(x_i-u_0)^2\right)\right)$$

Ahora, se rechaza H_0 si

$$\lambda(x) \leq c \ ;\ \exp\left(\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2 - \sum(x_i-u_0)^2\right)\right) \leq c$$

$$\left(\sum(x_i-\bar{x})^2-\sum(x_i-u_0)^2\right) \leq 2\sigma^2\ln(c)$$

$$\sum x_i^2-2\bar{x}\sum x_i+n\bar{x}^2-\sum x_i^2+2u_0\sum x_i-nu_0^2 \leq 2\sigma^2\ln(c)$$

$$\sum x_i^2-2n\bar{x}\sum\frac{x_i}{n}+n\bar{x}^2-\sum x_i^2+2u_0\sum x_i-nu_0^2 \leq 2\sigma^2\ln(c)$$

$$\sum x_i^2-n\bar{x}^2-\sum x_i^2+2u_0\sum x_i-nu^2 \leq 2\sigma^2\ln(c)$$

$$-n\bar{x}^2+2u_0\sum x_i \leq 2\sigma^2\ln(c)+nu_0^2$$

$$-n\bar{x}^2+2nu_0\bar{x} \leq 2\sigma^2\ln(c)+nu_0^2$$

$$\bar{x}^2-2u_0\bar{x} \geq \frac{-1}{n}\left(2\sigma^2\ln(c)+nu_0^2\right)$$

$$\bar{x}^2-2u_0\bar{x} \geq \frac{-1}{n}\left(2\sigma^2\ln(c)+nu_0^2\right)$$

$$\bar{x}-2u_0\bar{x}+u_0^2 \geq \frac{-1}{n}\left(2\sigma^2\ln(c)+nu_0^2\right)+u_0^2$$

$$(\bar{x}-u_0)^2 \geq \frac{-2\sigma^2\ln(c)}{n}-u_0^2+u_0^2$$

$$(\bar{x}-u_0)^2 \geq \frac{\sigma^2}{n}(-\ln(c)2) \iff |\bar{x}-u_0| \geq \frac{\sigma}{\sqrt{n}}\sqrt{-\ln(c)2}$$

$$\therefore \frac{|\bar{x}-u_0|}{\frac{\sigma}{\sqrt{n}}} \geq \underbrace{\sqrt{-\ln(c)2}}_{C_0} \Rightarrow \in (0,1)\text{, Raiz bien definida}$$

Sea $z=\dfrac{\bar{x}-u_0}{\frac{\sigma}{\sqrt{n}}} \sim n(0,1)$, se rechaza Ho si $z > C_0$

(Por hipótesis solo la se fuz en parte positiva de la función) [sic] cierto

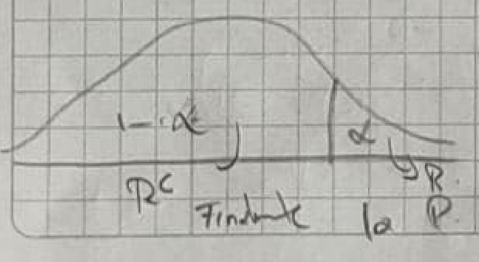
Aho, se fija la P. error tipo 1, $\iff$ se fija α

$$\alpha = P(z > C_0 \mid u_0 < u) = P(N(0,1)) > C_0$$

$$P(z > C_0) = 1-P(z \leq C_0) = P(\text{error tipo } 1)$$

$$= P(\text{Rechazo Ho} \mid \text{Ho es cierto})$$

con $z_\alpha = P(z \leq C_0)$



Rechazo Ho si $z_c > z_\alpha$

Finalmente la P de rechazar Ho dado α es

$$\therefore \ln(\lambda \mid x, y) = -n\ln(\lambda) - \frac{1}{\lambda}\sum_{i=1}^{n} x_i - m\ln(\lambda) - \frac{1}{\lambda}\sum_{j=1}^{m} y_j$$

$$= (n+m)\ln(\lambda) - \frac{1}{\lambda}\left(\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right)$$

$$\frac{\partial}{\partial \lambda} = \frac{-(n+m)}{\lambda} + \frac{1}{\lambda^2}\left(\sum x_i + \sum y_j\right) = 0, \quad \hat{\lambda} = \left(\frac{\sum^{n} x_i + \sum^{m} y_j}{(n+m)}\right)$$

Por tanto...

$$\lambda(x, y) = \frac{\dfrac{(n+m)^n}{\left(\sum^{n} x_i + \sum^{m} y_i\right)^n} \cdot \dfrac{(n+m)^m\, e^{-(n+m)}}{\left(\sum^{n} x_i + \sum^{m} y_j\right)^m}}{\dfrac{1}{\left(\frac{1}{n}\sum x_i\right)^n} \cdot \dfrac{1}{\left(\frac{1}{m}\sum y_j\right)^m} \cdot \cancel{e^{-n} \cdot e^{-m}}}$$

$$= \left(\frac{n+m}{n}\right)^n \left(\frac{n+m}{m}\right)^m \left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j}\right)^n \cdot \left(\frac{\sum_{j=1}^{m} y_j}{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j}\right)^m$$

↳ LRT para H_0 vs H_1

B $T = \dfrac{\sum^{n} x_i}{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j}$; si $\lambda(x, y) = \dfrac{(n+m)^{n+m}}{n^n m^m} T^n (1-T)^m$, se escribe como un estadístico de prueba ya H_0 es acep es $1-\alpha$; pero si rechazo H_0 es Error tipo 1; por tanto <u>NO</u> se rechaza y esta seria Region rechazo

(4) $X_1 \ldots X_n \sim exp(1/\theta)$, $\theta > 0$, $Y_1 \ldots Y_n \sim exp(1/\lambda)$; $\lambda > 0$

Notar que
$$H_0 = \theta = \lambda \quad vs \quad H_1 = \theta \neq \lambda.$$

Calculemos el LRT

$$\lambda(X) = \frac{\sup_{\theta \in \Theta_0} L(\theta \mid X)}{\sup_{\theta \in \Theta} L(\theta \mid X)}$$ Por definición, por tanto Necesitamos el MLE

$$\therefore L(\theta, \lambda \mid x, y) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \cdot \prod_{i=1}^{m} \frac{1}{\lambda} e^{-y_i/\lambda}$$

$$= \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \cdot \frac{1}{\lambda} e^{-\frac{1}{\lambda}\sum_{j=1}^{m} y_j} = \frac{1}{\theta^n \lambda^m} e^{-\frac{1}{\theta}\sum x_i - \frac{1}{\lambda}\sum^{m} y_j}$$

$$\ell(L(\theta, \lambda \mid X, Y)) = -n\ln(\theta) - m\ln(\lambda) - \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{\lambda}\sum_{j=1}^{m} y_j$$

Ahora, $\frac{d}{d\theta} = \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0 \to \hat{\theta}^* = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$

$\frac{d^2}{d^2\theta} = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum x_i < 0$; MLE para $\hat{\theta}$ de θ

Analogamente... $\frac{\partial}{\partial\lambda} = \frac{-m}{\lambda} + \frac{1}{\lambda^2}\sum_{j=1}^{m} y_j = 0 \to \hat{\lambda} = \frac{1}{m}\sum_{j=1}^{m} y_j = \overline{Y}$

$\frac{\partial^2}{\partial^2\lambda} = \frac{1}{\lambda^2} - \frac{2m}{\lambda^3}\sum_{j=1}^{m} y_j < 0$, $\hat{\lambda} = \overline{Y}$, el MLE para $\cancel{X}$

Ahora, el sup bajo H_0, con MLE $\hat{\lambda}$ →

$$L(\lambda \mid x, y) = \lambda^{-n} e^{\left(-1/\lambda \sum_{i=1}^{n} x_i\right)} \lambda^{-m} e^{\left(-\frac{1}{\lambda}\sum_{j=1}^{m} y_i\right)}$$