A Journey in Signal Processing with IPython

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November 4, 2015

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%run ../nbinit.ipy
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   ... scipy.signal as sig
... Importing widgets, display, HTML, Image, Javascript
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Chapter 1

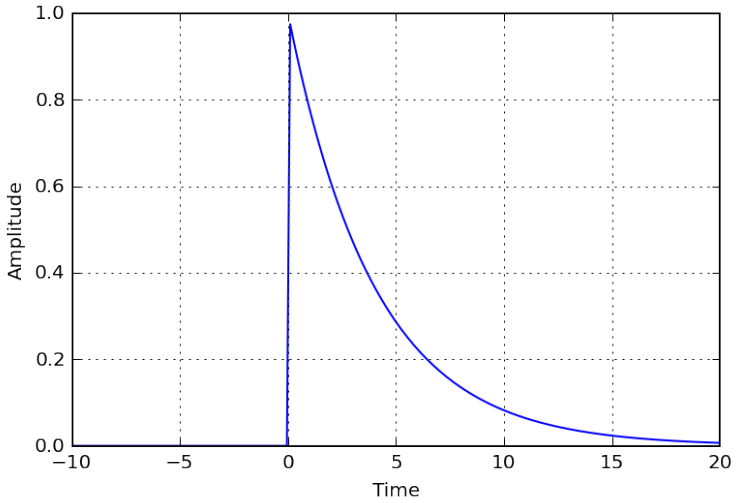
A basic introduction to signals and systems

1.1 Effects of delays and scaling on signals

In this simple exercise, we recall the effect of delays and scaling on signals. It is important for students to experiments with that to ensure that they master these simple transformations.

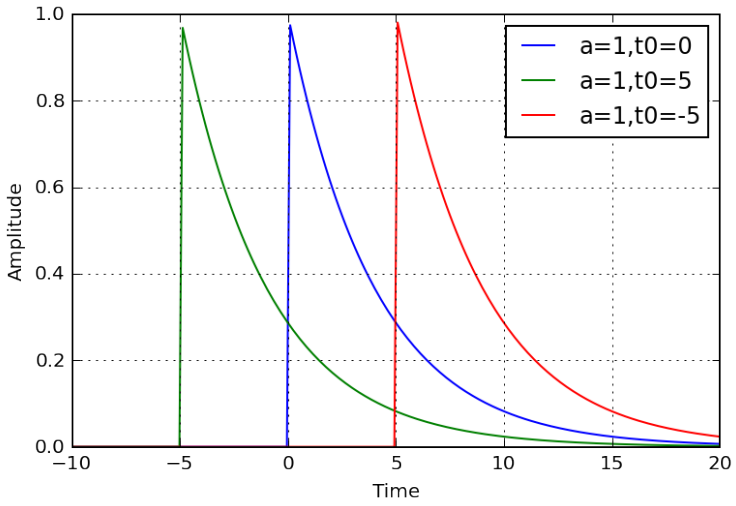
Study the code below and experiment with the parameters

```
# Define a simple function
def f(t):
    return np.exp(-0.25*t) if t>0 else 0


T= np.linspace(-10,20,200)
L=len(T)
x=np.zeros(L)  # reserve some space for x

t0=0; a=1 # initial values

# Compute x as f(a*t+t0)
k=0
for t in T:
    x[k]=f(a*t+t0)
    k=k+1

# Plotting the signal
plt.plot(T,x)
plt.xlabel("Time")
plt.ylabel("Amplitude")
plt.grid(b='on')


# Experiment with several values of a and t0:
# a=1 t0=0
# a=1 t0=+5  (advance)
# a=1 t0=-5  (delay)
# a=-1 t0=0  (time reverse)
# a=-1 t0=5  (time reverse + advance)
# a=-1 t0=-5 (...)
```



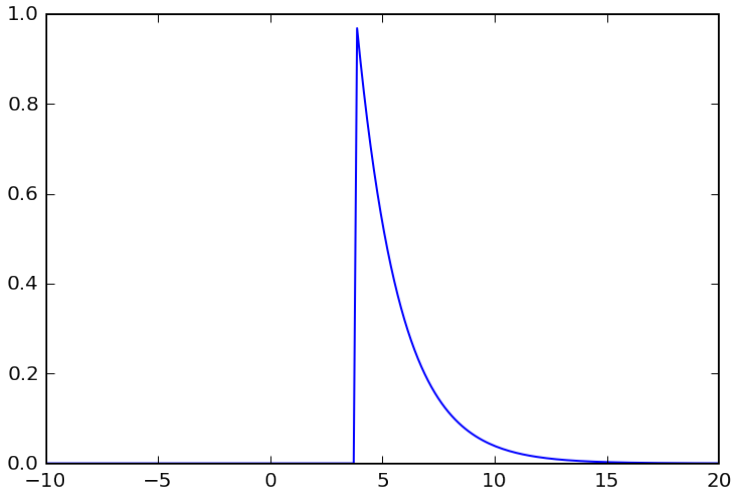
This to show that you do automatically several tests and plot the results all together.

```
def compute_x(a,t0):
    k=0
    for t in T:
        x[k]=f(a*t+t0)
        k=k+1
    return x

list_tests=[(1,0),(1,5), (1,-5)]#,(-1,0),(-1,3), (-1,-3) ]
for (a,t0) in list_tests:
    x=compute_x(a,t0)
    plt.plot(T,x,label="a={},t0={}".format(a,t0))

plt.xlabel("Time")
plt.ylabel("Amplitude")
plt.grid(b='on')
plt.legend()
```



And finally an interactive version

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

def f(t):
    out=np.zeros(len(t))
    tpos=np.where(t>0)
    out[tpos]=np.exp(-0.25*t[tpos])
    return out

t= np.linspace(-10,20,200)
L=len(t)
x=np.zeros(L)

def compute_xx(t0,a):
        x=f(a*t+t0)
        len(t)
        len(x)
        plt.plot(t,x)

s_t0=widgets.FloatSlider(min=-20,max=20,step=1)
s_a=widgets.FloatSlider(min=0.1,max=5,step=0.1,value=2)
_=interact(compute_xx,t0=s_t0,a=s_a)
```




```
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1.2 A basic introduction to filtering

Through examples, we define several operations on signals and show how they transform them. Then we define what is a filter and the notion of impulse response.

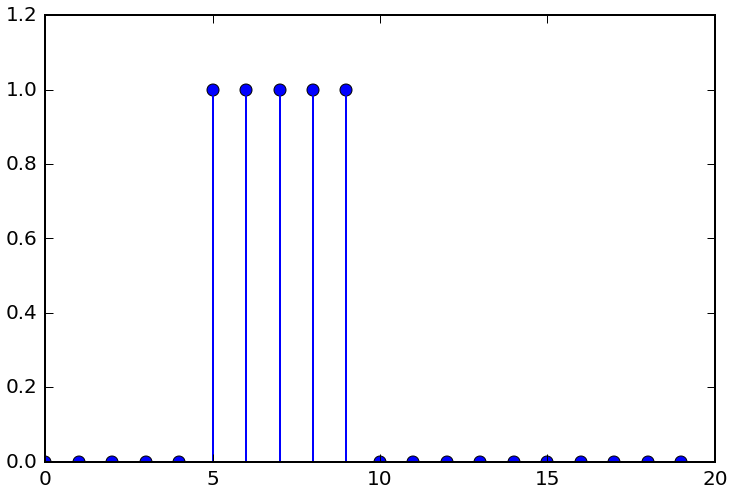
- Transformations of signals - Examples of difference equations
- Filters
- Notion of impulse response

1.2.1 Transformations of signals - Examples of difference equations

We begin by defining a test signal.

```
# rectangular pulse
N=20; L=5; M=10
r=np.zeros(N)

r[L:M]=1
#
plt.stem(r)
_=plt.ylim([0, 1.2])
```

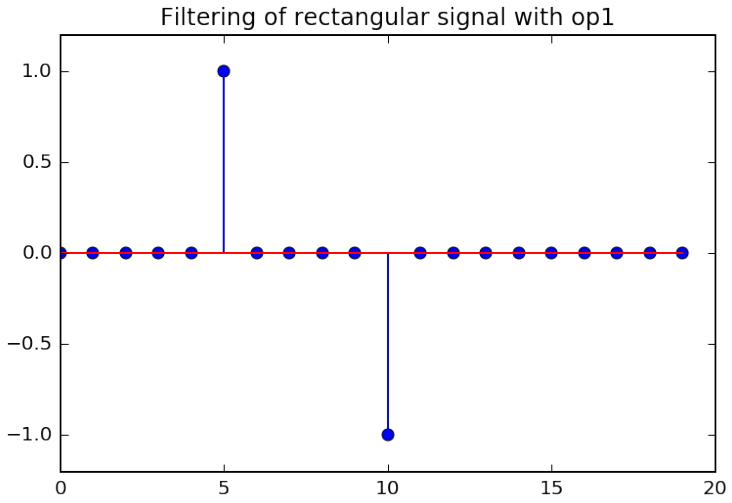


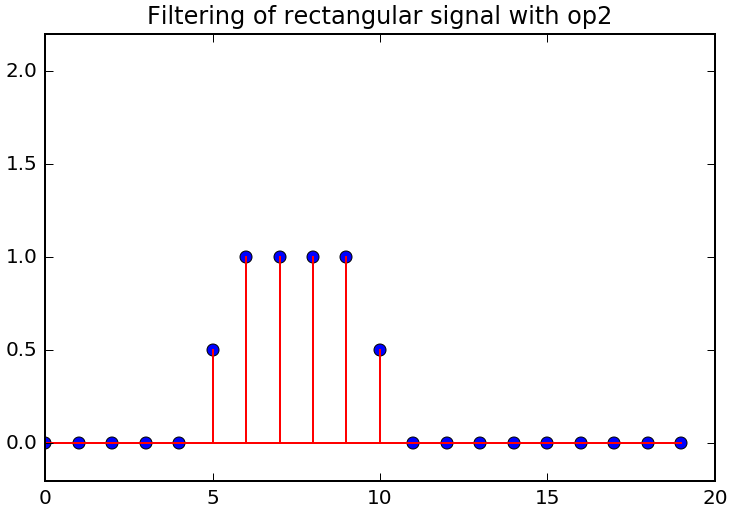

```
def op1(signal):
    transformed_signal=np.zeros(np.size(signal))
    for t in np.arange(np.size(signal)):
        transformed_signal[t]=signal[t]-signal[t-1]
    return transformed_signal

def op2(signal):
    transformed_signal=np.zeros(np.size(signal))
    for t in np.arange(np.size(signal)):
        transformed_signal[t]=0.5*signal[t]+0.5*signal[t-1]
    return transformed_signal
```

```
plt.figure()
plt.stem(op1(r))
_=plt.ylim([-1.2, 1.2])
```

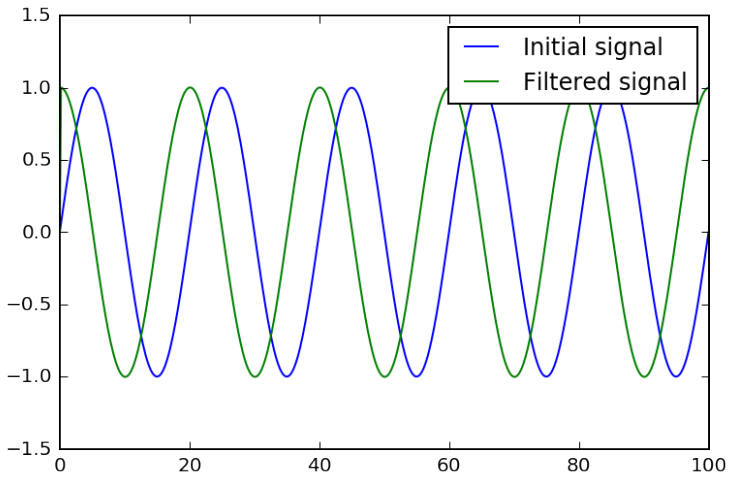
```
plt.title("Filtering of rectangular signal with op1")
plt.figure()
plt.stem(op2(r),'r')
_=plt.ylim([-0.2, 2.2])
plt.title("Filtering of rectangular signal with op2")
```





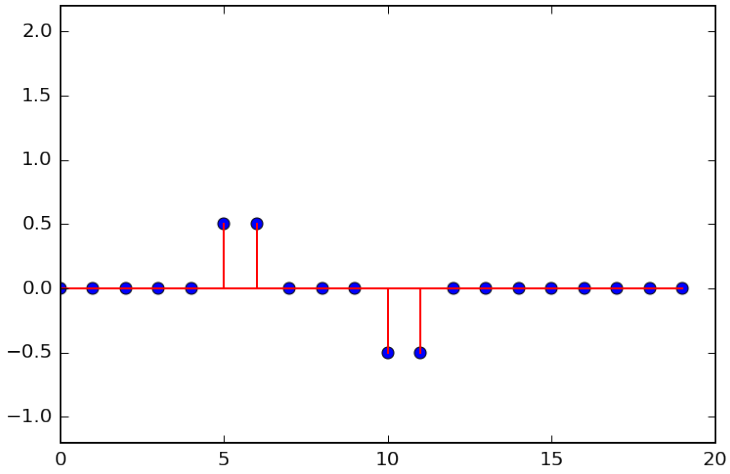
We define a sine wave and check that the operation implemented by "op1" seems to be a derivative...

```
t=np.linspace(0,100,500)
sig=np.sin(2*pi*0.05*t)
plt.plot(t,sig, label="Initial signal")
plt.plot(t,5/(2*pi*0.05)*op1(sig), label="Filtered signal")
plt.legend()
```



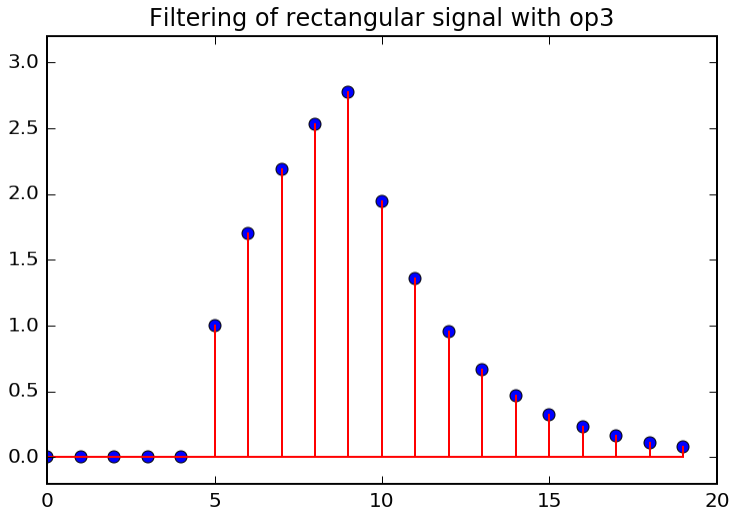
Composition of operations:

```
plt.stem(op1(op2(r)),'r')
_=plt.ylim([-1.2, 2.2])
```




```
def op3(signal):
    transformed_signal=np.zeros(np.size(signal))
    for t in np.arange(np.size(signal)):
        transformed_signal[t]= 0.7*transformed_signal[t-1]+signal[t]
    return transformed_signal

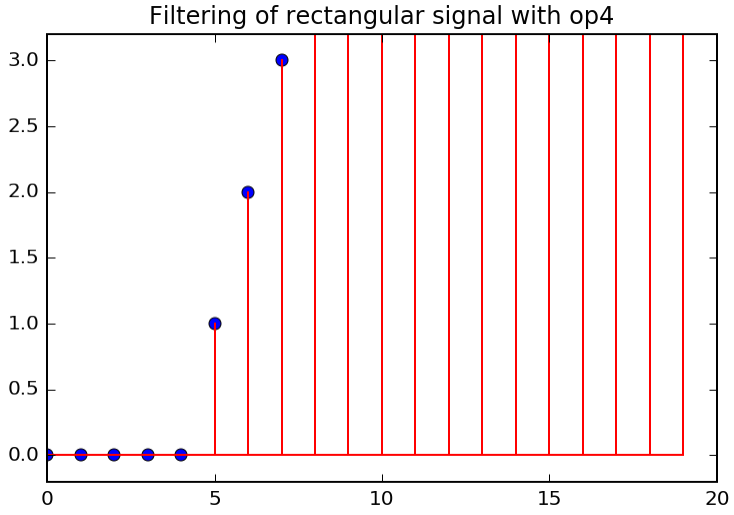
plt.stem(op3(r),'r')
plt.title("Filtering of rectangular signal with op3")
_=plt.ylim([-0.2, 3.2])
```

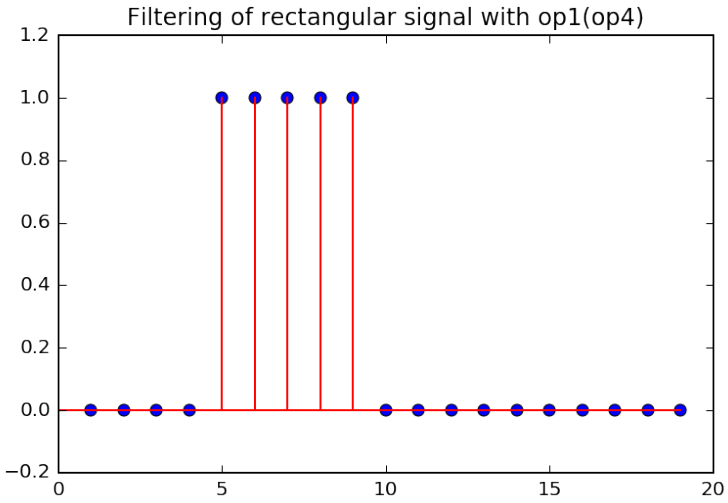


A curiosity

```
def op4(signal):
    transformed_signal=np.zeros(np.size(signal))
    for t in np.arange(np.size(signal)):
        transformed_signal[t]= 1*transformed_signal[t-1]+signal[t]
    return transformed_signal

plt.stem(op4(r),'r')
plt.title("Filtering of rectangular signal with op4")
_=plt.ylim([-0.2, 3.2])
# And then..
plt.figure()
plt.stem(op1(op4(r)),'r')
plt.title("Filtering of rectangular signal with op1(op4)")
_=plt.ylim([-0.2, 1.2])
```





1.2.2 Filters

Definition A filter is a time-invariant linear system.

- Time invariance means that if $y(n)$ is the response associated with an input $x(n)$, then $y(n-n_0)$ is the response associated with the input $x(n-n_0)$.
- Linearity means that if $y_1(n)$ and $y_2(n)$ are the outputs associated with $x_1(n)$ and $x_2(n)$, then the output associated with $a_1x_1(n)+a_2x_2(n)$ is $a_1y_1(n)+a_2y_2(n)$ (superposition principle)

Exercise 1. *Check whether the following systems are filters or not.*

- $x(n) \to 2x(n)$
- $x(n) \to 2x(n)+1$
- $x(n) \to 2x(n)+x(n-1)$
- $x(n) \to x(n)^2$

Notion of impulse response

Definition A Dirac impulse (or impulse for short) is defined by

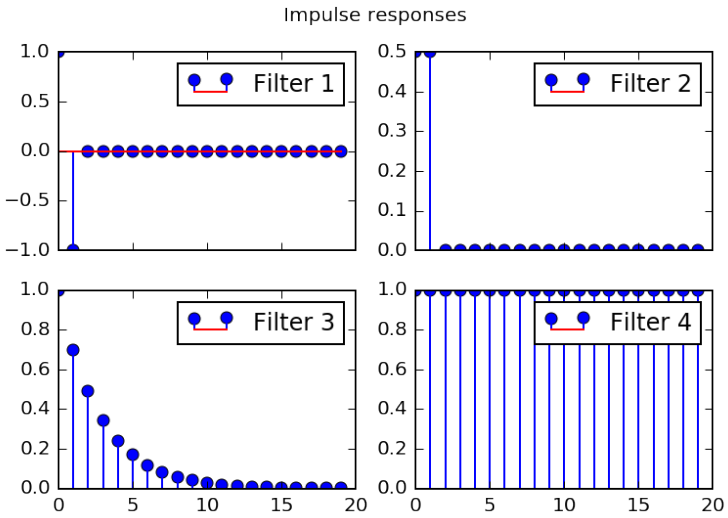
$$\delta(n) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{elsewhere} \end{cases} \tag{1.1}$$

Definition The impulse response of a system is nothing but the output of the system excited by a Dirac impulse. It is often denoted $h(h)$.

$$\delta(n) \to \text{System} \to h(n) \tag{1.2}$$

```
def dirac(n):
# dirac function
    return 1 if n==0 else 0
def dirac_vector(N):
    out = np.zeros(N)
    out[0]=1
    return out
```

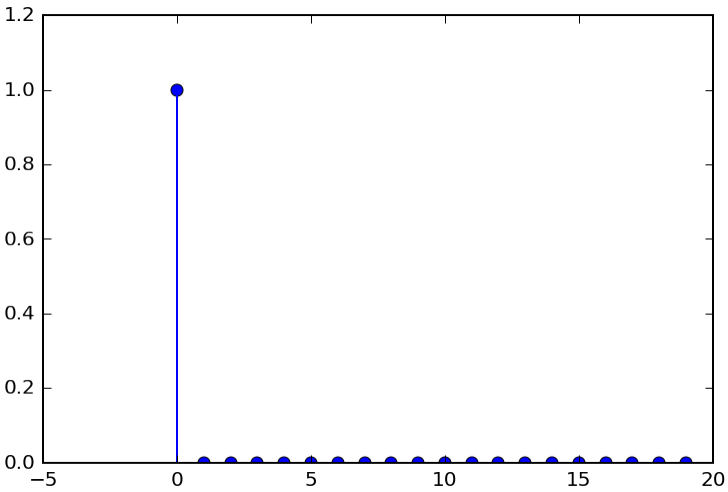
```
d=dirac_vector(20)
fig,ax=plt.subplots(2,2,sharex=True)
plt.subplot
ax[0][0].stem(op1(d),   label="Filter 1")
ax[0][0].legend()
ax[0][1].stem(op2(d),   label="Filter 2")
ax[0][1].legend()
ax[1][0].stem(op3(d), label="Filter 3")
ax[1][0].legend()
ax[1][1].stem(op4(d), label="Filter 4")
ax[1][1].legend()
plt.suptitle("Impulse responses")
```



Curiosity (continued)

The impulse response of op4(op1) is given by

```
h=op4(op1(dirac_vector(20)))
plt.stem(h, label="Filter 4(1)")
_=plt.axis([-5, 20, 0, 1.2])
```



This is nothing but a Dirac impulse! We already observed that op4(op1(signal))=signal; that is the filter is an identity transformation. In other words, op4 acts as the "inverse" of op1. Finally, we note that the impuse response of the indentity filter is a Dirac impulse.

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```

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Chapter 2

Introduction to the Fourier representation

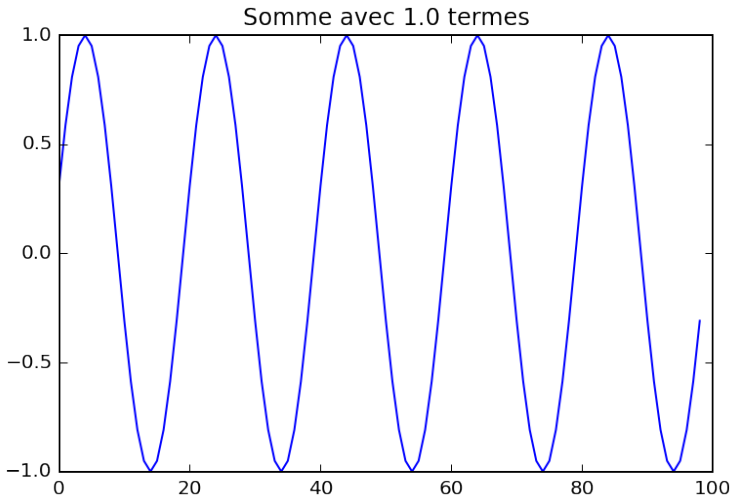
We begin by a simple example which shows that the addition of some sine waves, with special coefficients, converges constructively. We then explain that any periodic signal can be expressed as a sum of sine waves. This is the notion of Fourier series. After an illustration (denoising of a corrupted signal) which introduces a notion of filtering in the frequency domain, we show how the Fourier representation can be extended to aperiodic signals.

- Simple examples
- Decomposition on basis - scalar producs
- Decomposition of periodic functions – Fourier series
- Complex Fourier series
- Computer experiment
- Towards Fourier transform

2.1 Simple examples

Read the script below, execute (CTRL-Enter), experiment with the parameters.

```
N=100
L=20
s=np.zeros(N-1)

for k in np.arange(1,3,2):
    s=s+1/float(k)*sin(2*pi*k/L*np.arange(1,N,1))
plt.plot(s)
plt.title("Somme avec "+str((k-1)/2+1)+" termes")
```

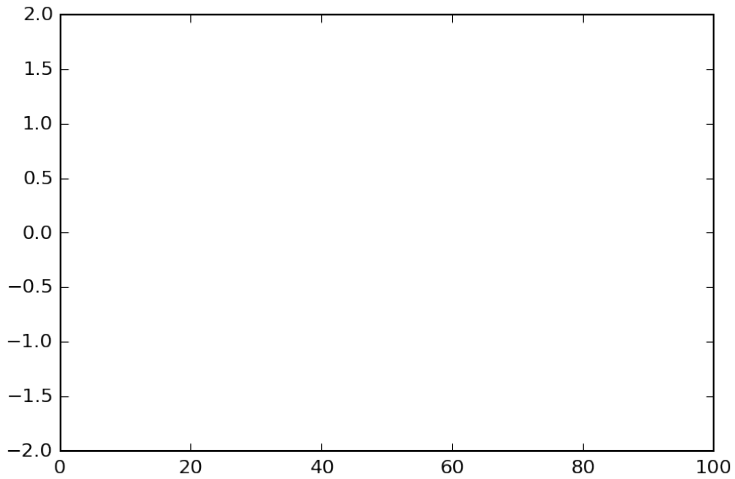


The next example is more involved in that it sums sin a cos of different frequencies and with different amplitudes. We also add widgets (sliders) which enable to interact more easily with the program.

```
def sfou_exp(Km):
    clear_output(wait=True)
    Kmax=int(Km)
    L=400
    N=1000
    k=0
    s=np.zeros(N-1)
    #plt.clf()
    for k in np.arange(1,Kmax):
        ak=0
        bk=1.0/k if (k % 2) == 1 else 0 # k odd

       # ak=0 #if (k % 2) == 1 else -2.0/(pi*k**2)
       # bk=-1.0/k if (k % 2) == 1 else 1.0/k #

        s=s+ak*cos(2*pi*k/L*np.arange(1,N,1))+bk*sin(2*pi*k/L*np.arange(1,
            N,1))
    ax = plt.axes(xlim=(0, N), ylim=(-2, 2))
    ax.plot(s)
    plt.title("Sum with {} terms".format(k+1))

### ----------------------------------------------------------
```

```
fig = plt.figure()
ax = plt.axes(xlim=(0, 100), ylim=(-2, 2))

# ---- Widgets ----------------------
# slider=widgets.FloatSlider(max=100,min=0,step=1,value=1)
slide=widgets.IntSlider(max=100,min=0,step=1,value=1)
val=widgets.IntText(value='1')

#----- Callbacks des widgets ----------
def sfou1_Km(name,Km):
        val.value=str(Km)
```

```
        sfou_exp(Km)

def sfou2_Km(name, Km):
        slide.value=Km
        #sfou_exp(Km.value)


# ---- Display ----------------
display(slide)
display(val)
slide.on_trait_change(sfou1_Km,'value')
val.on_trait_change(sfou2_Km,'value')
```

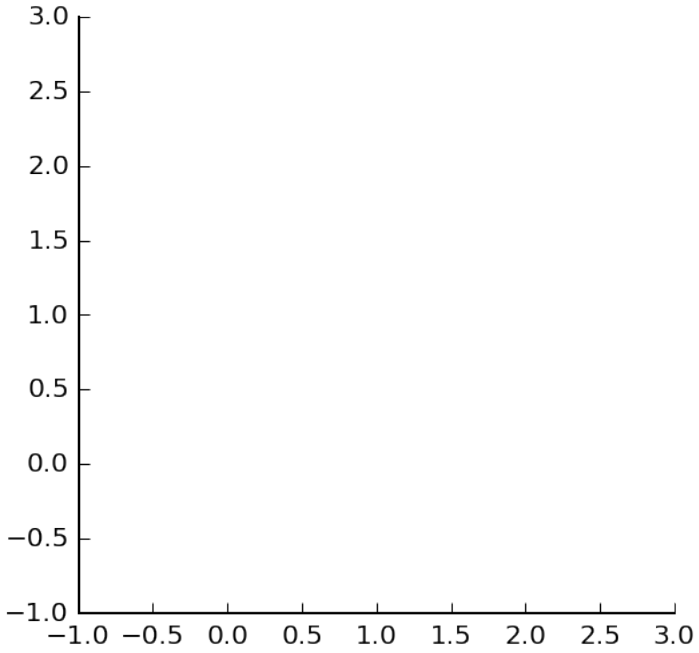


2.1.1 Decomposition on basis - scalar producs

We recall here that any vector can be expressed on a orthonormal basis, and that the coordinates are the scalar product of the vector with the basis vectors.

```
z=array([1,2])
u=array([0,1])
v=array([1,0])
u1=array([1,1])/sqrt(2)
v1=array([-1,1])/sqrt(2)

f,ax=subplots(1,1,figsize=(4,4))
ax.set_xlim([-1,3])
ax.set_ylim([-1,3])
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
#ax.spines['bottom'].set_position('center')
ax.quiver(0,0,z[0],z[1],angles='xy',scale_units='xy',scale=1,color='green'
    )
ax.quiver(0,0,u[0],u[1],angles='xy',scale_units='xy',scale=1,color='black'
    )
ax.quiver(0,0,v[0],v[1],angles='xy',scale_units='xy',scale=1,color='black'
    )
ax.quiver(0,0,u1[0],u1[1],angles='xy',scale_units='xy',scale=1,color='red'
    )
ax.quiver(0,0,v1[0],v1[1],angles='xy',scale_units='xy',scale=1,color='red'
    )
```

```
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')
```



From a coordinate system to another: Take a vector (in green in the illustration). Its coordinates in the system (u, v) are [1,2]. In order to obtain the coordinates in the new system (O, u_1, v_1), we have to project the vector on u_1 and u_2. This is done by the scalar products:

```
x=z.dot(u1)
y=z.dot(v1)
print('New coordinates: ',x,y)
```

```
New coordinates:  2.12132034356 0.707106781187
```

2.2 Decomposition of periodic functions – Fourier series

This idea can be extended to (periodic) functions. Consider the set of all even periodic functions, with a given period, say L. The cosine wave functions of all the multiple or the *fundamental* frequency $1/L$ constitute a basis of even periodic functions with period T. Let us check that these functions are normed and ortogonal with each other.

```
L=200
k=8
l=3
sk=sqrt(2/L)*cos(2*pi/L*k*np.arange(0,L))
sl=sqrt(2/L)*cos(2*pi/L*l*np.arange(0,L))
```

```
sl.dot(sl)
```

```
1.0000000000000004
```

Except in the case $l = 0$ where a factor 2 entails

```
l=0
sl=sqrt(2/L)*cos(2*pi/L*l*np.arange(0,L))
sl.dot(sl)
```

```
2.0000000000000013
```

Therefore, the decomposition of any even periodic function $x(n)$ with period L on the basis of cosines expresses as

$$x(n) = \sqrt{\frac{2}{L}} \left(\frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos(2\pi k/Ln) \right) \tag{2.1}$$

with

$$a_k = \sqrt{\frac{2}{L}} \sum_{n \in [L]} x(n) \cos(2\pi k/Ln). \tag{2.2}$$

Regrouping the factors, the series can also be expressed as

$$x_{\text{even}}(n) = \left(\frac{a_0}{2} + \sum_{k=1}^{L-1} a_k \cos(2\pi k/Ln) \right) \tag{2.3}$$

with

$$a_k = \frac{2}{L} \sum_{n \in [L]} x(n) \cos(2\pi k/Ln), \tag{2.4}$$

where the notation $n \in [L]$ indicates that the sum has to be done on any length-L interval. The very same reasoning can be done for odd functions, which introduces a decomposition into sine waves:

$$x_{\text{odd}}(n) = \sum_{k=0}^{L-1} b_k \sin(2\pi k/Ln) \tag{2.5}$$

with

$$b_k = \frac{2}{L} \sum_{n \in [L]} x(n) \sin(2\pi k/Ln), \tag{2.6}$$

Since any function can be decomposed into an odd + even part

$$x(n) = x_{\text{even}}(n) + x_{\text{odd}}(n) = \frac{x(n) + x(-n)}{2} + \frac{x(n) - x(-n)}{2}, \tag{2.7}$$

we have the sum of the decompositions:

$$\boxed{x(n) = \frac{a_0}{2} + \sum_{k=1}^{L-1} a_k \cos(2\pi k/Ln) + \sum_{k=1}^{+\infty} b_k \sin(2\pi k/Ln)} \tag{2.8}$$

with

$$\boxed{\begin{cases} a_k = \frac{2}{L} \sum_{n \in [L]} x(n) \cos(2\pi k/Ln), \\ b_k = \frac{2}{L} \sum_{n \in [L]} x(n) \sin(2\pi k/Ln), \end{cases}} \tag{2.9}$$

This is the definition of the Fourier series, and this is no more compicated than that... A remaining question is the question of convergence. That is, does the series converge to the true function? The short answer is Yes: the equality in the series expansion is a true equality, not an approximation. This is a bit out of scope for this course, but you may have a look at this article.

There of course exists a continuous version, valid for time-continuous dignals.

2.3 Complex Fourier series

2.3.1 Introduction

Another series expansion can be defined for complex valued signals. In such case, the trigonometric functions will be replaced by complex exponentials $\exp(j2\pi k/Ln)$. Let us check that they indeed form a basis of signals:

```
L=200
k=8
l=3
sk=sqrt(1/L)*exp(1j*2*pi/L*k*np.arange(0,L))
sl=sqrt(1/L)*exp(1j*2*pi/L*l*np.arange(0,L))
print("scalar product between sk and sl: ",np.vdot(sk,sl))
print("scalar product between sk and sl (i.e. norm of sk): ",np.vdot(sk,sk
    ))
```

```
scalar product between sk and sl:  (-1.04083408559e-17+3.25260651746e-18j)
scalar product between sk and sl (i.e. norm of sk):  (1+0j)
```

It is thus possible to decompose a signal as follows:

$$\boxed{\begin{aligned} x(n) &= \sum_{k=0}^{L-1} c_k e^{j2\pi\frac{kn}{L}} \\ \text{with } c_k &= \frac{1}{L}\sum_{n\in[L]} x(n) e^{-j2\pi\frac{kn}{L}} \end{aligned}} \tag{2.10}$$

where c_k is the dot product between $x(n)$ and $\exp(j2\pi k/Ln)$, i.e. the 'coordinate' of x with respect to the 'vector' $\exp(j2\pi k/Ln)$. This is nothing but the definition of the `complex Fourier series`.

Exercise – Show that c_k is periodic with period L; i.e. $c_k = c_{k+L}$.

Since c_k is periodic in k of period L, we see that in term or the *"normalized frequency"* k/L, it is periodic with period 1.

Relation of the complex Fourier Series with the standard Fourier Series

It is easy to find a relation between this complex Fourier series and the classical Fourier series. The series can be rewritten as

$$x(n) = c_0 + \sum_{k=1}^{+\infty} c_k e^{j2\pi k/Ln} + c_{-k} e^{-j2\pi k/Ln}. \tag{2.11}$$

By using the Euler formulas, developping and rearranging, we get

$$\begin{aligned} x(n) = c_0 + \sum_{k=1}^{+\infty} &\mathcal{R}\{c_k + c_{-k}\}\cos(2\pi k/Ln) + \mathcal{I}\{c_{-k} - c_k\}\sin(2\pi k/Ln) \\ &+ j\left(\mathcal{R}\{c_k - c_{-k}\}\sin(2\pi k/Ln) + \mathcal{I}\{c_k + c_{-k}\}\cos(2\pi k/Ln)\right). \end{aligned} \tag{2.12}$$

Suppose that $x(n)$ is real valued. Then by direct identification, we have

$$\begin{cases} a_k = \mathcal{R}\{c_k + c_{-k}\} \\ b_k = \mathcal{I}\{c_{-k} - c_k\} \end{cases} \tag{2.13}$$

and, by the cancellation of the imaginary part, the following symmetry relationships for real signals:

$$\begin{cases} \mathcal{R}\{c_k\} = \mathcal{R}\{c_{-k}\} \\ \mathcal{I}\{c_k\} = -\mathcal{I}\{c_{-k}\}. \end{cases} \tag{2.14}$$

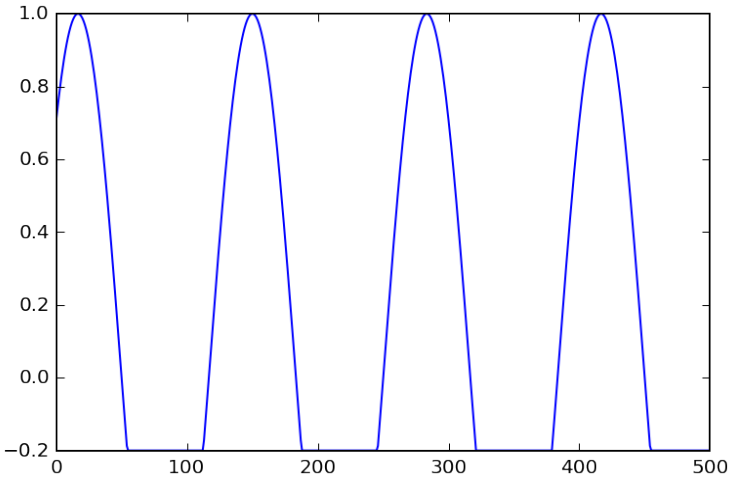
This symmetry is called 'Hermitian symmetry'.

2.3.2 Computer experiment

Experiment. Given a signal, computes its decomposition and then reconstruct the signal from its individual components.

```
%matplotlib inline
L=400
N=500
t=np.arange(N)
s=sin(2*pi*3*t/L+pi/4)
x=[ss if ss>-0.2 else -0.2 for ss in s]
plt.plot(t,x)
```

```
[<matplotlib.lines.Line2D at 0x7ff684dd6908>]
```



A function for computing the Fourier series coefficients

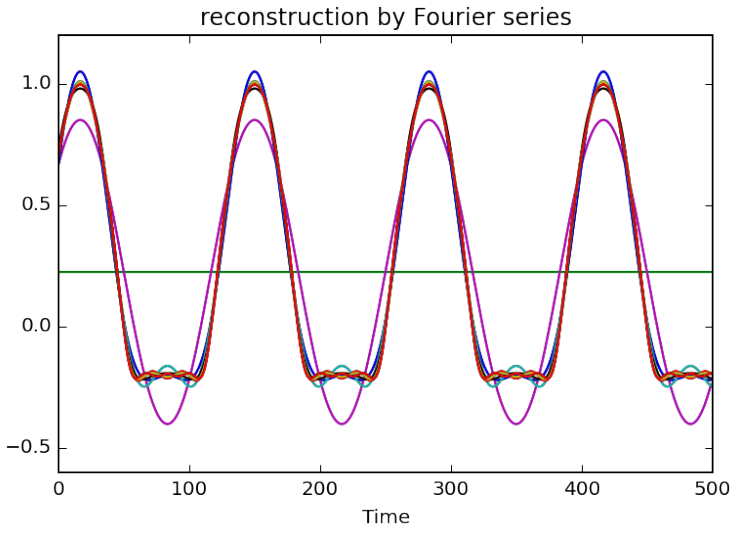
```
# compute the coeffs ck
def coeffck(x,L,k):
    assert np.size(x)==L, "input must be of length L"
    karray=[]
    res=[]
    if isinstance(k,int):
        karray.append(k)
    else:
        karray=np.array(k)

    for k in karray:
        res.append(np.vdot(exp(1j*2*pi/L*k*np.arange(0,L)),x))
    return 1/L*np.array(res)
```

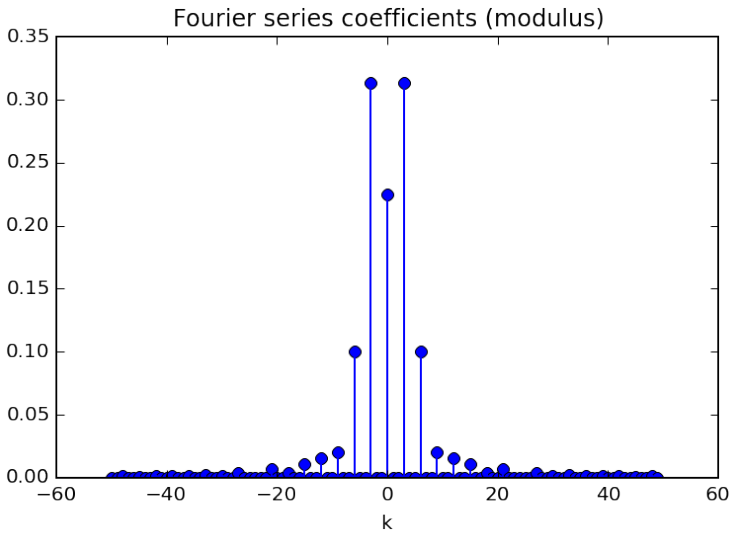
```
#test: coeffck(x[0:L],L,[-12,1,7])
# --> array([  1.51702135e-02 +4.60742555e-17j,
#      -1.31708229e-05 -1.31708229e-05j,   1.37224241e-05 -1.37224241e-05j
    ])
```

Now let us compute the coeffs for actual signal

```
# compute the coeffs for actual signal
c1=coeffck(x[0:L],L,np.arange(0,100))
c2=coeffck(x[0:L],L,np.arange(0,-100,-1))
s=c1[0]*np.ones((N))
for k in np.arange(1,25):
    s=s+c1[k]*exp(1j*2*pi/L*k*np.arange(0,N))+c2[k]*exp(-1j*2*pi/L*k*np.
        arange(0,N))
    plt.plot(t,np.real(s))
plt.title("reconstruction by Fourier series")
plt.xlabel("Time")


plt.figure()
kk=np.arange(-50,50)
c=coeffck(x[0:L],L,kk)
plt.stem(kk,np.abs(c))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("k")
msg="""In the frequency representation, the x axis corresponds to the
    frequencies k/L
of the complex exponentials.
Therefore, if a signal is periodic of period M, the corresponding
    fundamental frequency
is 1/M. This frequency then appears at index ko=L/M (if this ratio is an
    integer).
Harmonics will appear at multiples of ko."""
print(msg)
```

```
In the frequency representation, the x axis corresponds to the frequencies k/L
of the complex exponentials.
Therefore, if a signal is periodic of period M, the corresponding fundamental frequency
is 1/M. This frequency then appears at index ko=L/M (if this ratio is an integer).
Harmonics will appear at multiples of ko.
```



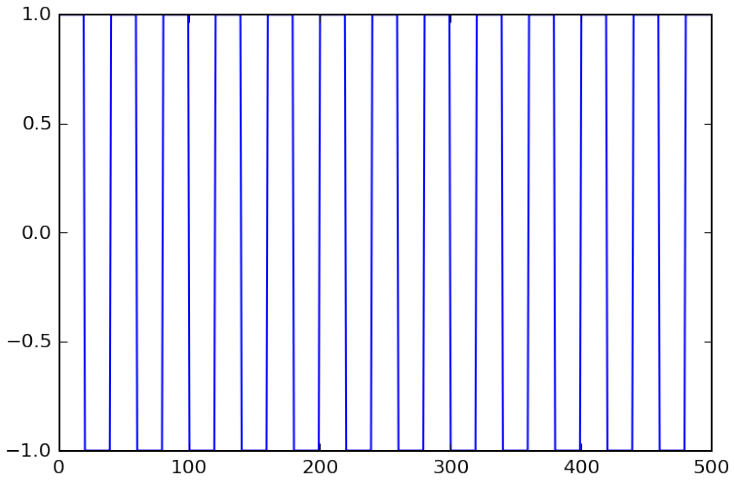


A pulse train corrupts our original signal

```
L=400
# define a pulse train which will corrupt our original signal
def sign(x):
    if isinstance(x,(int,float)):
        return 1 if x>=0 else -1
    else:
        return np.array([1 if u>=0 else -1 for u in x])

#test: sign([2, 1, -0.2, 0])


def repeat(x,n):
    if isinstance(x,(np.ndarray,list,int,float)):
        return np.array([list(x)*n]).flatten()
    else:
        raise('input must be an array,list,or float/int')

#t=np.arange(N)
```

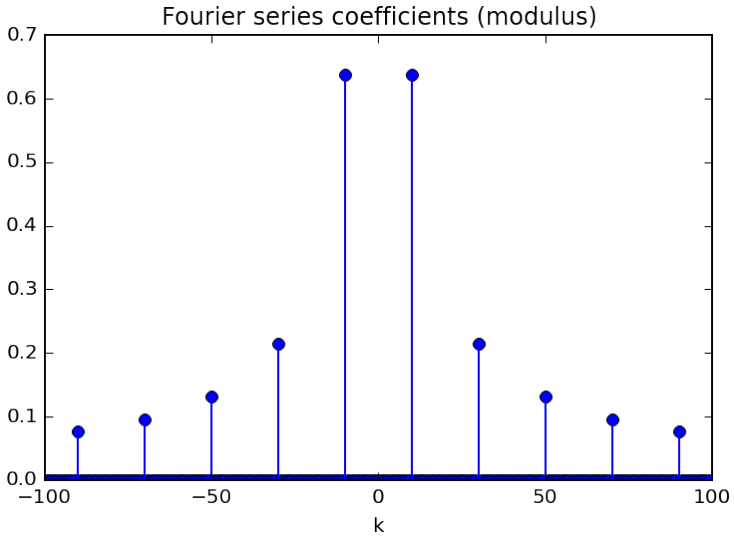
```
#sig=sign(sin(2*pi*10*t/L))

rect=np.concatenate((np.ones(20),-np.ones(20)))
#[1,1,1,1,1,-1,-1,-1,-1,-1]

sig=repeat(rect,15)
sig=sig[0:N]
plt.plot(sig)
```

```
[<matplotlib.lines.Line2D at 0x7ff684189eb8>]
```



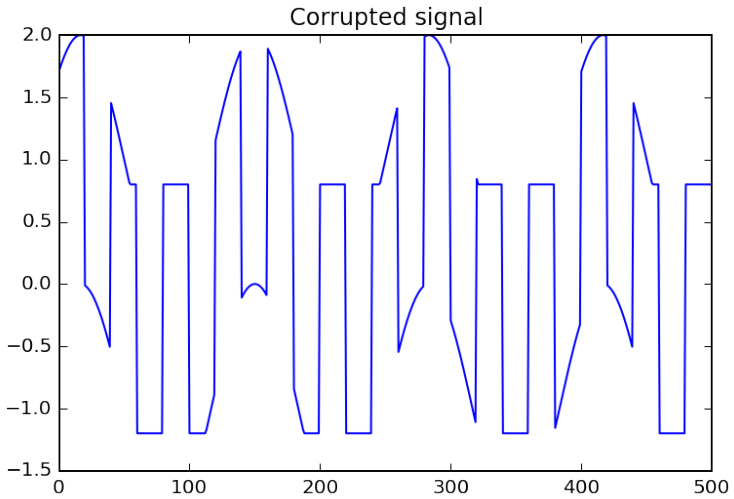
Compute and represent the Fourier coeffs of the pulse train

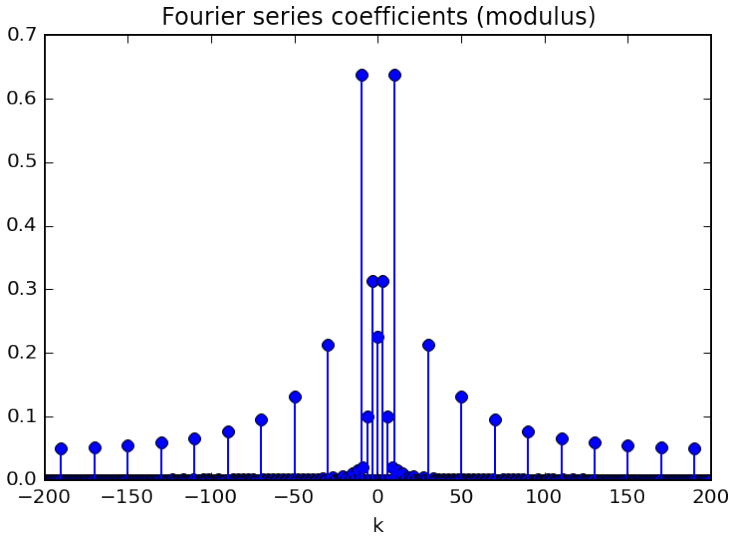
```
kk=np.arange(-100,100)
c=coeffck(sig[0:L],L,kk)
plt.figure()
plt.stem(kk,np.abs(c))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("k")
```



The fundamental frequency of the pulse train is 1 over the length of the pulse, that is 1/40 here. Since The Fourier series is computed on a length L=400, the harmonics appear every 10 samples (ie at indexes k multiples of 10).

```
z=x+1*sig
plt.plot(z)
plt.title("Corrupted signal")

kk=np.arange(-200,200)
cz=coeffck(z[0:L],L,kk)
plt.figure()
plt.stem(kk,np.abs(cz))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("k")
```

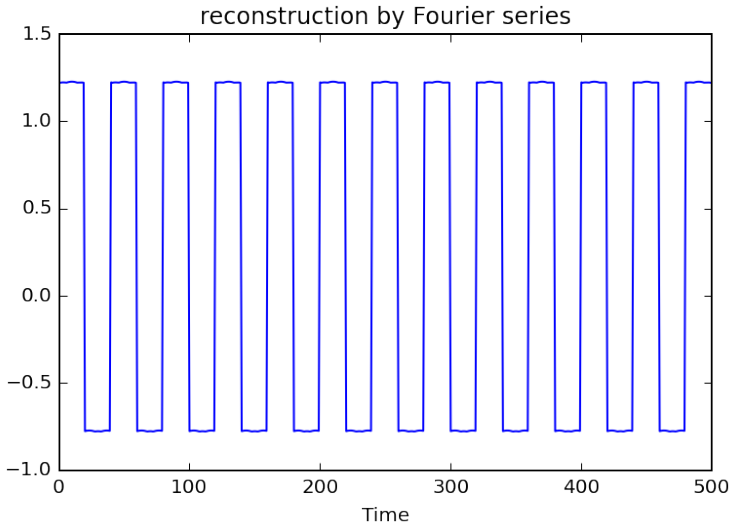


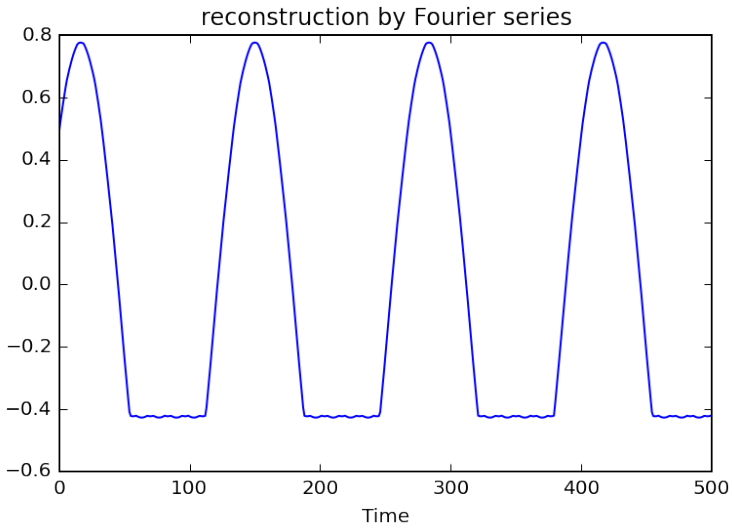


Now, we try to kill all the frequencies harmonics of 10 (the fundamental frequency of the pulse train), and reconstruct the resulting signal...

```
# kill frequencies harmonics of 10 (the fundamental frequency of the pulse
    train)
```

```
# and reconstruct the resulting signal

s=np.zeros(N)
kmin=np.min(kk)
for k in kk:
    if not k%10:  #true if k is multiple of 10
        s=s+cz[k+kmin]*exp(1j*2*pi/L*k*np.arange(0,N))
plt.figure()
plt.plot(t,np.real(s))
plt.title("reconstruction by Fourier series")
plt.xlabel("Time")

plt.figure()
plt.plot(t,z-np.real(s))
plt.title("reconstruction by Fourier series")
plt.xlabel("Time")
```





Chapter 3

From Fourier Series to Fourier transforms

In this section, we go from the Fourier series to the Fourier transform for discrete signal. So doing, we also introduce the notion of Discrete Fourier Transform that we will study in more details later. For now, we focus on the representations in the frequency domain, detail and experiment with some examples.

3.1 Introduction and definitions

Suppose that we only have an observation of length N. So no periodic signal, but a signal of size N. We do not know if there were data before the first sample, and we do not know if there were data after sample N. What to do? Facing to such situation, we can still - imagine that the data are periodic outside of the observation interval, with a period N. Then the formulas for the Fourier series **are** valid, for n in the observation interval. Actually there is no problem with that. The resulting transformation is called the *Discrete Fourier Transform* . The corresponding formulas are

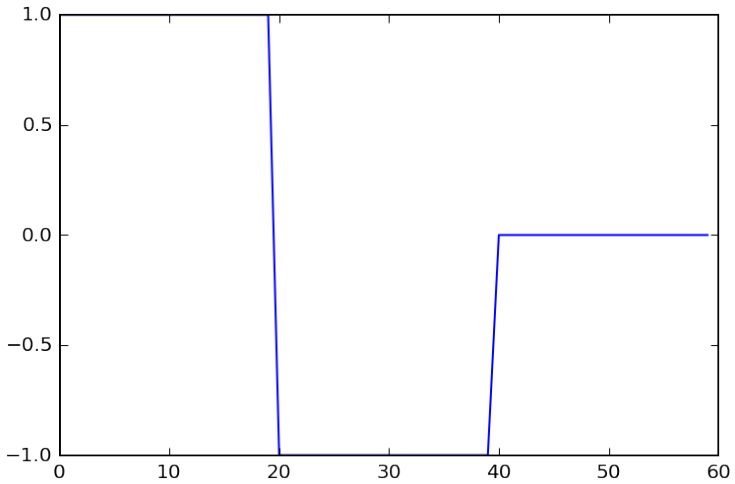
$$\begin{aligned} x(n) &= \sum_{k=0}^{N-1} X(k) e^{j2\pi \frac{kn}{N}} \\ \text{with } X(k) &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}} \end{aligned} \tag{3.1}$$

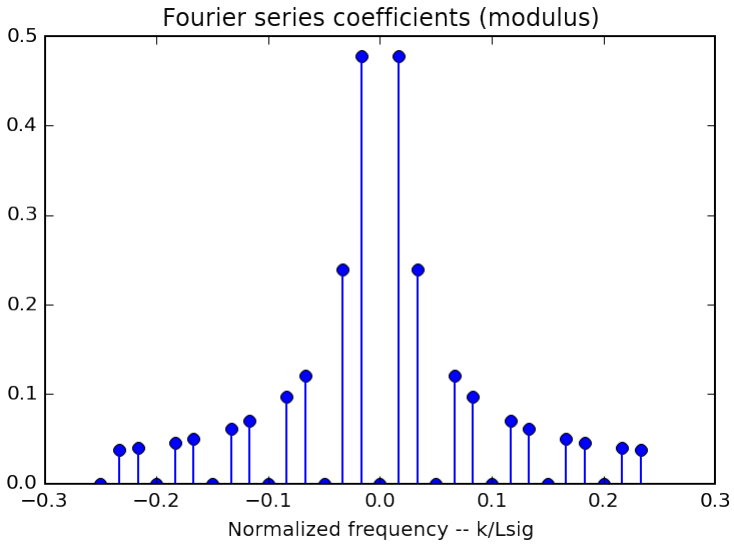
- we may also consider that there is nothing –that is zeros, outside of the observation interval. In such condition, we can still imagine that we have a periodic signal, but with an infinite period. Since the separation of two harmonics in the Fourier series is Δf=1/period, we see that $\Delta f \to 0$. Then the Fourier representation becomes continuous. This is illustrated below.

```
# compute the coeffs ck
def coeffck(x,L,k):
    assert np.size(x)==L, "input must be of length L"
    karray=[]
    res=[]
    if isinstance(k,int):
       karray.append(k)
    else:
```

```
        karray=np.array(k)

    for k in karray:
        res.append(np.vdot(exp(1j*2*pi/L*k*np.arange(0,L)),x))
    return 1/L*np.array(res)

#test: coeffck(x[0:L],L,[-12,1,7])
# --> array([  1.51702135e-02 +4.60742555e-17j,
#       -1.31708229e-05 -1.31708229e-05j,    1.37224241e-05 -1.37224241e-05j
    ])
```

```
Lpad=20 # then 200, then 2000
# define a rectangular pulse
rect=np.concatenate((np.ones(20),-np.ones(20)))
# Add zeros after:
rect_zeropadded=np.concatenate((rect,np.zeros(Lpad)))
sig=rect_zeropadded
plt.plot(sig)
# compute the Fourier series for |k/Lsig|<1/4
Lsig=np.size(sig)
fmax=int(Lsig/4)
kk=np.arange(-fmax,fmax)
c=coeffck(sig[0:Lsig],Lsig,kk)
# plot it
plt.figure()
plt.stem(kk/Lsig,np.abs(c))
plt.title("Fourier series coefficients (modulus)")
plt.xlabel("Normalized frequency -- k/Lsig")
```





Hence we obtain a formula where the discrete sum for reconstructing the time-series $x(n)$ becomes a continuous sum, since f is now continuous:

$$\begin{aligned} x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{kn}{N}} = \sum_{k/N=0}^{1-1/N} NX(k) e^{j2\pi \frac{kn}{N}} \frac{1}{N} \\ \rightarrow x(n) = \int_0^1 X(f) e^{j2\pi fn} \mathrm{d}f \end{aligned} \tag{3.2}$$

Finally, we end with what is called the **Discrete-time Fourier transform** :

$$\boxed{\begin{aligned} x(n) &= \int_0^1 X(f) e^{j2\pi fn} \mathrm{d}f \\ \text{with } X(f) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn} \end{aligned}} \tag{3.3}$$

Even before exploring the numerous properties of the Fourier transform, it is important to stress that

The Fourier transform of a discrete signal is periodic with period one.

Check it as an exercise! Begin with the formula for $X(f)$ an compute $X(f+1)$. use the fact that n is an integer and that $\exp(j2\pi n) = 1$.

3.2 Examples

Exercise 2. .

- *Compute the* Fourier transform *of a rectangular window given on N points. The result is called a (discrete) cardinal sine (or sometimes Dirichlet kernel). Sketch a plot, and study the behaviour of this function with N.*
- *Experiment numerically... {} See below the provided functions.*
- *Compute the Fourier transform of a sine wave* $\sin(2\pi f_0 n)$ *given on N points.*
- *Examine what happens when the N and f_0 vary.*

3.2.1 The Fourier transform of a rectangular window

The derivation of the formula will be done in class. Let us see the experimental part.

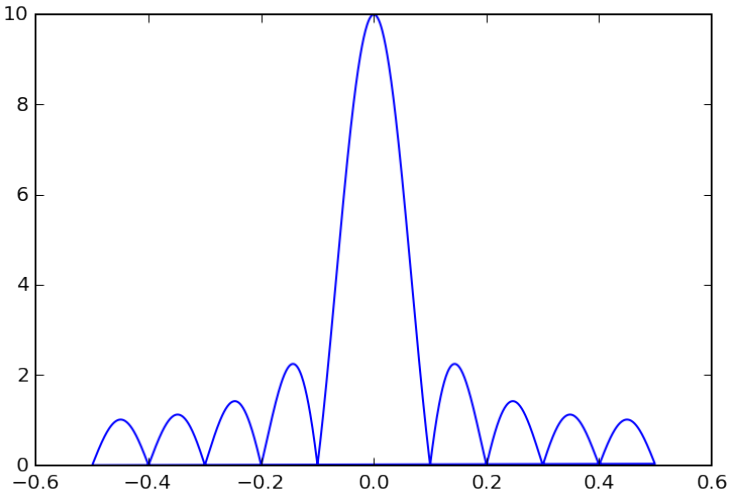
For the numerical experiments, import the fft (Fast Fourier Transform) function,

```
from numpy.fft import fft, ifft
```

define a sine wave, complute and plot its Fourier transform. As the FFT is actually an implementation of a discrete Fourier transform, we will have an approximation of the true Fourier transform by using zero-padding (check that a parameter in the fft enables to do this zero-padding).

```
from numpy.fft import fft, ifft

#Define a rectangular window, of length L
#on N points, zeropad to NN=1000
# take eg L=100, N=500
NN=1000
L=10 # 10, then 6, then 20, then 50, then 100...
r=np.ones(L)
Rf=fft(r,NN)
f=fftfreq(NN)
plt.plot(f,np.abs(Rf))
```

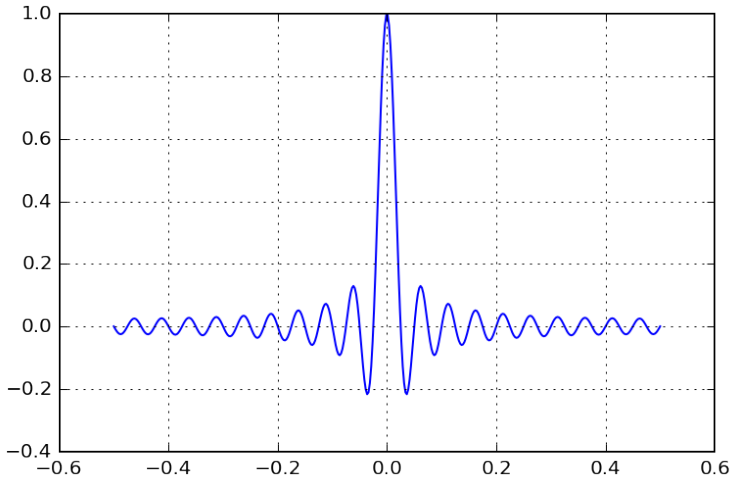
```
[<matplotlib.lines.Line2D at 0x7fa3f24d90f0>]
```



It remain to compare this to a discrete cardinal sinus. First we define a function and then compare the results.

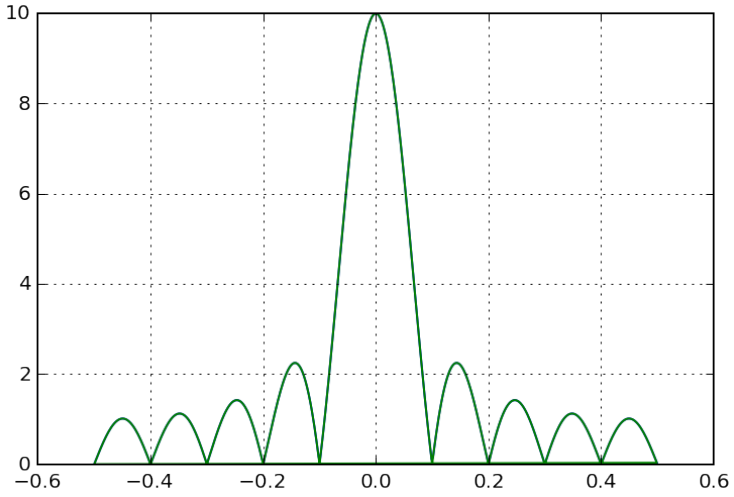
```
def dsinc(x,L):
    if isinstance(x,(int,float)): x=[x]
    x=np.array(x)
    out=np.ones(np.shape(x))
    I=np.where(x!=0)
    out[I]=np.sin(x[I])/(L*np.sin(x[I]/L))
    return out
```

```
N=1000
L=40
f=np.linspace(-0.5,0.5,400)
plt.plot(f,dsinc(pi*L*f,L))
plt.grid(b='on')
```



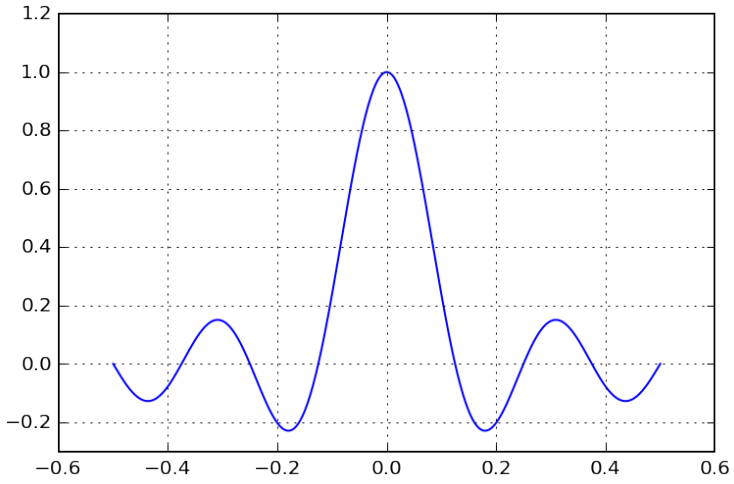
Comparison of the Fourier transform of a rectangle and a cardinal sine:

```
NN=1000
L=10 # 10, then 6, then 20, then 50, then 100...
r=np.ones(L)
Rf=fft(r,NN)

N=1000
f=np.linspace(-0.5,0.5,400)
plt.plot(f,L*np.abs(dsinc(pi*L*f,L)))
f=fftfreq(NN)
plt.plot(f,np.abs(Rf))
plt.grid(b='on')
```



Interactive versions...

```
# using %matplotlib use a backend that allows external figures
# using %matplotlib inline plots the results in the notebook
%matplotlib inline
slider=widgets.FloatSlider(min=0.1,max=100,step=0.1,value=8)
display(slider)


#------ Callbacks des widgets ----------
def pltsinc(name,L):
        plt.clf()
        clear_output(wait=True)
        #val.value=str(f)
        f=np.linspace(-0.5,0.5,400)
        plt.plot(f,dsinc(pi*L*f,L))
        plt.ylim([-0.3, 1.2])
        plt.grid(b='on')

pltsinc('Width',8)
slider.on_trait_change(pltsinc,'value')
```



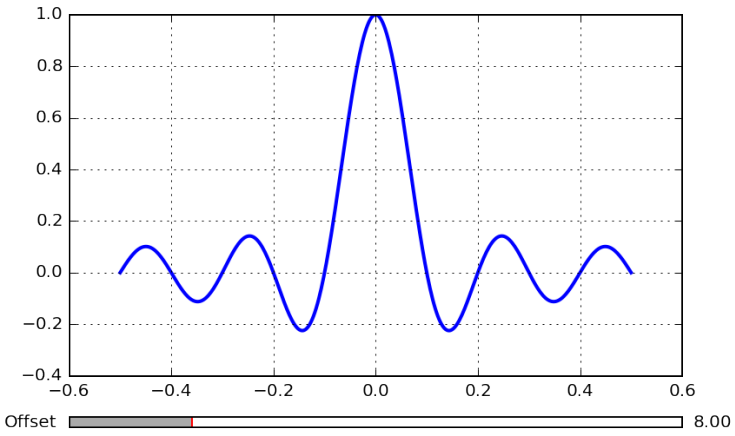
This is an example with matplotlib widgets interactivity, (instead of html widgets). The docs can be found at

```
%matplotlib
from matplotlib.widgets import Slider

fig, ax = plt.subplots()
fig.subplots_adjust(bottom=0.2, left=0.1)

slider_ax = plt.axes([0.1, 0.1, 0.8, 0.02])
slider = Slider(slider_ax, "Offset", 0, 40, valinit=8, color='#AAAAAA')
L=10
f=np.linspace(-0.5,0.5,400)

line, = ax.plot(f,dsinc(pi*L*f,L), lw=2)
#line2, = ax.plot(f,sinc(pi*L*f), lw=2)
#line2 is in order to compare with the "true" sinc
ax.grid(b='on')

def on_change(L):
```

```
    line.set_ydata(dsinc(pi*L*f,L))
#     line2.set_ydata(sinc(pi*L*f))


slider.on_changed(on_change)
```

```
Using matplotlib backend: TkAgg
```

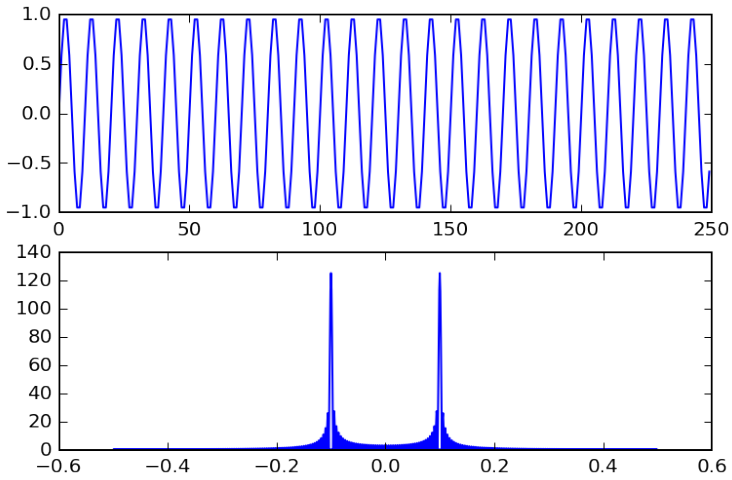
```
0
```



3.2.2 Fourier transform of a sine wave

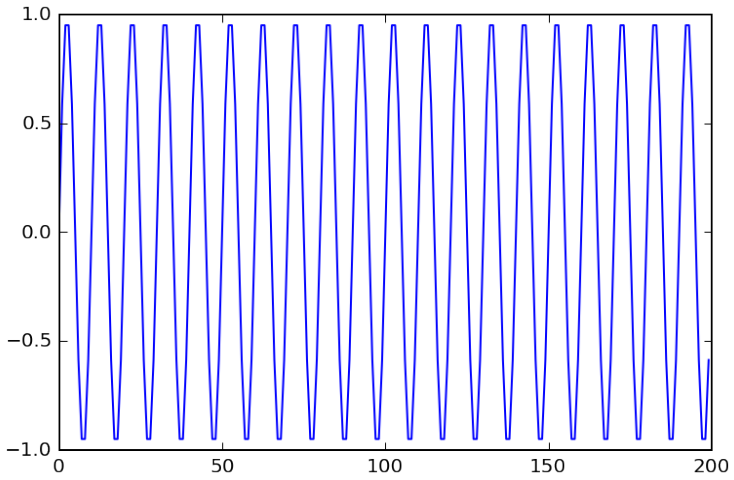
Again, the derivation will be done in class.

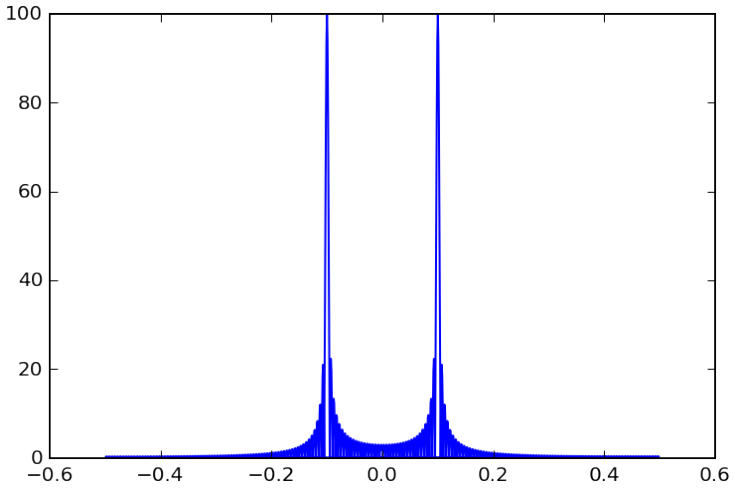
```
%matplotlib inline
from numpy.fft import fft, ifft
N=250; f0=0.1; NN=1000
fig,ax=plt.subplots(2,1)
def plot_sin_and_transform(N,f0,ax):
    t=np.arange(N)
    s=np.sin(2*pi*f0*t)
    Sf=fft(s,NN)
    ax[0].plot(t,s)
    f=np.fft.fftfreq(NN)
    ax[1].plot(f,np.abs(Sf))
plot_sin_and_transform(N,f0,ax)
```



Interactive versions

```
# using %matplotlib use a backend that allows external figures
# using %matplotlib inline plots the results in the notebook
%matplotlib inline
sliderN=widgets.IntSlider(min=1,max=1000,step=1,value=200)
sliderf0=widgets.FloatSlider(min=0,max=0.5,step=0.01,value=0.1)

display(sliderN)
display(sliderf0)
N=500; f0=0.1;
t=np.arange(N)
s=np.sin(2*pi*f0*t)
Sf=fft(s,NN)
f=np.fft.fftfreq(NN)


#------ Callbacks des widgets ----------
def pltsin(dummy):
        clear_output(wait=True)
        N=sliderN.value
        f0=sliderf0.value
        t=np.arange(N)
        s=np.sin(2*pi*f0*t)
        Sf=fft(s,NN)
        f=np.fft.fftfreq(NN)
        plt.figure(1)
        plt.clf()
        plt.plot(t,s)
        plt.figure(2)
        plt.clf()
        plt.plot(f,np.abs(Sf))

pltsin(8)
sliderN.on_trait_change(pltsin)
sliderf0.on_trait_change(pltsin)
```

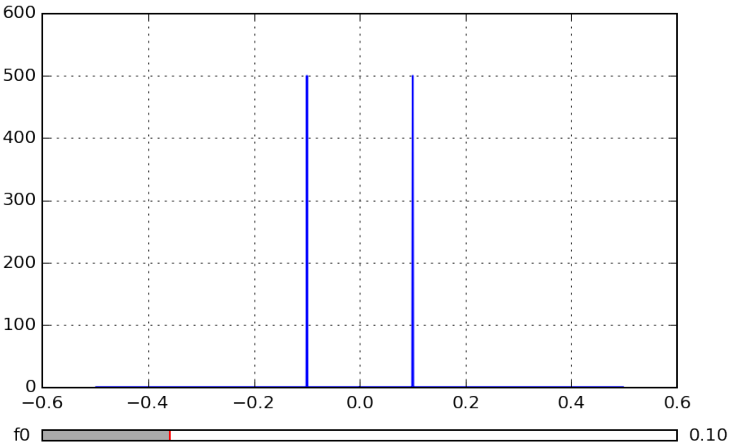





```
%matplotlib tk
from matplotlib.widgets import Slider

fig, ax = plt.subplots()
fig.subplots_adjust(bottom=0.2, left=0.1)

slider_ax = plt.axes([0.1, 0.1, 0.8, 0.02])
slider = Slider(slider_ax, "f0", 0, 0.5, valinit=0.1, color='#AAAAAA')
f=np.linspace(-0.5,0.5,400)
N=1000
t=np.arange(N)
s=np.sin(2*pi*f0*t)
Sf=fft(s,NN)
f=np.fft.fftfreq(NN)
line, = ax.plot(f,np.abs(Sf))

ax.grid(b='on')

def on_change(f0):
    s=np.sin(2*pi*f0*t)
    Sf=fft(s,NN)
    line.set_ydata(np.abs(Sf))
```

```
#     line2.set_ydata(sinc(pi*L*f))


slider.on_changed(on_change)
```

0



Some definitions

(3.4)

```
%run nbinit.ipy
js_addon()
```

```
... Configuring matplotlib formats
... Configuring matplotlib with inline figures
... Importing numpy as np, scipy as sp, pyplot as plt, scipy.stats as stats
   ... scipy.signal as sig
... Importing widgets, display, HTML, Image, Javascript
... Some LaTeX definitions

... Defining figures captions

... Loading customized Javascript for interactive solutions (show/hide)
```

3.3 Symmetries of the Fourier transform.

Consider the Fourier pair

$$x(n) \rightleftharpoons X(f). \tag{3.5}$$

When $x(n)$ is complex valued, we have

$$\boxed{x^*(n) \rightleftharpoons X^*(-f)}. \tag{3.6}$$

This can be easily checked beginning with the definition of the Fourier transform:

$$\begin{aligned} \mathrm{FT}\{x^*(n)\} &= \sum_n x^*(n)e^{-j2\pi fn}, \\ &= \left(\int_{[1]} x(n)e^{j2\pi fn}\mathrm{d}f\right)^*, \\ &= X^*(-f). \end{aligned}$$

In addition, for any signal $x(n)$, we have

$$\boxed{x(-n) \rightleftharpoons X(-f)}. \tag{3.7}$$

This last relation can be derived directly from the Fourier transform of $x(-n)$

$$\mathrm{FT}\{x(-n)\} = \int_{-\infty}^{+\infty} x(-n)\ e^{-j2\pi ft}\ \mathrm{d}t, \tag{3.8}$$

using the change of variable $-t \to t$, we get

$$\begin{aligned} \mathrm{FT}\{x(-n)\} &= \int_{-\infty}^{+\infty} x(n)\ e^{j2\pi tf}\ \mathrm{d}t, \\ &= X(-f). \end{aligned}$$

using the two last emphasized relationships, we obtain

$$\boxed{x^*(-n) \rightleftharpoons X^*(f)}. \tag{3.9}$$

To sum it all up, we have

$$\boxed{\begin{array}{lcl} x(n) & \rightleftharpoons & X(f) \\ x(-n) & \rightleftharpoons & X(-f) \\ x^*(n) & \rightleftharpoons & X^*(-f) \\ x^*(-n) & \rightleftharpoons & X^*(f) \end{array}} \tag{3.10}$$

These relations enable to analyse all the symetries of the Fourier transform. We begin with the *Hermitian symmetry* for **real signals**:

$$\boxed{X(f) = X^*(-f)} \tag{3.11}$$

from that, we observe that if $x(n)$ is real, then

- the real part of $X(f)$ is *even*,
- the imaginary part of $X(f)$ is *odd*,
- the modulus of $X(f)$, $|X(f)|$ is *even*,
- the phase of $X(f)$, $\theta(f)$ is *odd*.

Moreover, if $x(n)$ is odd or even ($x(n)$ is not necessarily real), we have

$$\begin{array}{llcll} \text{[even]} & x(n) = x(-n) & \rightleftharpoons & X(f) = X(-f) & \text{[even]} \\ \text{[odd]} & x(n) = -x(-n) & \rightleftharpoons & X(f) = -X(-f) & \text{[odd]} \end{array} \tag{3.12}$$

The following table summarizes the main symmetry properties of the Fourier transform:\

$\mathbf{x(n)}$	Symmetry	Time	Frequency	consequence on $X(f)$
real	any	$x(n) = x^*(n)$	$X(f) = X^*(-f)$	Re. even, Im. odd
real	even	$x(n) = x^*(n) = x(-n)$	$X(f) = X^*(-f) = X(-f)$	Real and even
real	odd	$x(n) = x^*(n) = -x(-n)$	$X(f) = X^*(-f) = -X(-f)$	Imaginary and odd
imaginary	any	$x(n) = -x^*(n)$	$X(f) = -X^*(-f)$	Re. odd, Im. even
imaginary	even	$x(n) = -x^*(n) = x(-n)$	$X(f) = -X^*(-f) = X(-f)$	Imaginary and even
imaginary	odd	$x(n) = -x^*(n) = -x(-n)$	$X(f) = -X^*(-f) = -X(-f)$	Real and odd

(3.13)

Finally, we have

$$\begin{array}{lcl} \text{Real even} + \text{imaginary odd} & \rightleftharpoons & \text{Real} \\ \text{Real odd} + \text{imaginary even} & \rightleftharpoons & \text{Imaginary} \end{array} \tag{3.14}$$

3.4 Table of Fourier transform properties

The following table lists the main properties of the Discrete time Fourier transform. The table is adapted from the article on discrete time Fourier transform on Wikipedia.

Property	Time domain $x(n)$	Frequency domain $X(f)$
Linearity	$ax(n) + by(n)$	$aX(f) + bY(f)$
Shift in time	$x(n - n_0)$	$X(f)e^{-j2\pi f n_0}$
Shift in frequency (modulation)	$x(n)e^{j2\pi f_0 n}$	$X(f - f_0)$
Time scaling	$x(n/k)$	$X(kf)$
Time reversal	$x(-n)$	$X(-f)$
Time conjugation	$x(n)^*$	$X(-f)^*$
Time reversal & conjugation	$x(-n)^*$	$X(f)^*$
Sum of $x(n)$	$\sum_{n=-\infty}^{\infty} x(n)$	$X(0)$
Derivative in frequency	$\frac{n}{i}x(n)$	$\frac{dX(f)}{df}$
Integral in frequency	$\frac{i}{n}x(n)$	$\int_{[1]} X(f)df$
Convolve in time	$x(n) * y(n)$	$X(f) \cdot Y(f)$
Multiply in time	$x(n) \cdot y(n)$	$\int_{[1]} X(f_1) \cdot Y(f - f_1)df_1$
Area under $X(f)$	$x(0)$	$\int_{[1]} X(f)\, df$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x(n) \cdot y^*(n)$	$\int_{[1]} X(f) \cdot Y^*(f)df$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2$	$\int_{[1]} \lvert X(f)\rvert^2 df$

(3.15)

Some examples of Fourier pairs are collected below:

Time domain	Frequency domain
$x[n]$	$X(f)$
$\delta[n]$	$X(f) = 1$
$\delta[n-M]$	$X(f) = e^{-j2\pi fM}$
$\sum_{k=-\infty}^{\infty} \delta[n-kM]$	$\frac{1}{M}\sum_{k=-\infty}^{\infty} \delta\left(f-\frac{k}{M}\right)$
$u[n]$	$X(f) = \frac{1}{1-e^{-j2\pi f}} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\delta(f-k)$
$a^n u[n]$	$X(f) = \frac{1}{1-ae^{-j2\pi f}}$
$e^{-j2\pi f_a n}$	$X(f) = \delta(f+f_a)$
$\cos(2\pi f_a n)$	$X(f) = \frac{1}{2}[\delta(f+f_a)+\delta(f-f_a)]$
$\sin(2\pi f_a n)$	$X(f) = \frac{1}{2j}[\delta(f+f_a)-\delta(f-f_a)]$
$\mathrm{rect}_M\left[(n-(M-1)/2)\right]$	$X(f) = \frac{\sin[\pi fM]}{\sin(\pi f)} e^{-j\pi f(M-1)}$
$\begin{cases} 0 & n=0 \\ \frac{(-1)^n}{n} & \text{elsewhere} \end{cases}$	$X(f) = j2\pi f$
$\begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n \text{ odd} \end{cases}$	$X(f) = \begin{cases} j & f<0 \\ 0 & f=0 \\ -j & f>0 \end{cases}$

(3.16)

```
%run nbinit.ipy
js_addon()
```

```
... Configuring matplotlib formats
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... Some LaTeX definitions

... Defining figures captions

... Loading customized Javascript for interactive solutions (show/hide)
```

Chapter 4

Filters and convolution

4.1 Representation formula

Any signal $x(n)$ can be written as follows:

$$\boxed{x(n) = \sum_{m=-\infty}^{+\infty} x(m)\delta(n-m).} \tag{4.1}$$

It is very important to understand the meaning of this formula.

- Since $\delta(n-m) = 1$ if and only if $n = m$, then all the terms in the sum cancel, excepted the one with $n = m$ and therefore we arrive at the identity $x(n) = x(n)$.
- The set of delayed Dirac impulses $\delta(n-m)$ form a basis of the space of discrete signals. Then the coordinate of a signal on this basis is the scalar product $\sum_{n=-\infty}^{+\infty} x(n)\delta(n-m) = x(m)$. Hence, the reprentation formula just expresses the decomposition of the signal on the basis, where the $x(m)$ are the coordinates.

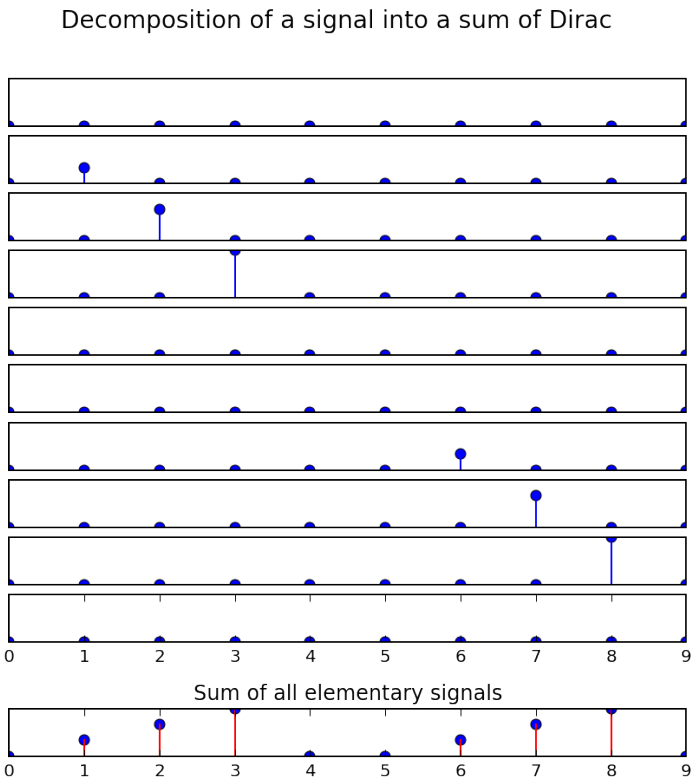
This means that $x(n)$, as a waveform, is actually composed of the sum of many Dirac impulses, placed at each integer, with a weight $x(m)$ which is nothing but the amplitude of the signal at time $m = n$. The formula shows how the signal can be seen as the superposition of Dirac impulses with the correct weights. Lets us illustrate this with a simple Python demonstration:

```
L=10
z=np.zeros(L)
x=np.zeros(L)
x[5:9]=range(4)
x[0:4]=range(4)
print("x=",x)
s=np.zeros((L,L))
for k in range(L):
        s[k][k]=x[k]
# this is equivalent as s=np.diag(x)
f,ax=plt.subplots(L+2,figsize=(7,7))
for k in range(L):
    ax[k].stem(s[k][:])
    ax[k].set_ylim([0,3])
    ax[k].get_yaxis().set_ticks([])
    if k!=L-1: ax[k].get_xaxis().set_ticks([])

ax[L].axis('off')
```

```
ax[L+1].get_yaxis().set_ticks([])
ax[L+1].stem(x,'r')
ax[L+1].set_title("Sum of all elementary signals")
#f.tight_layout()
f.suptitle("Decomposition of a signal into a sum of Dirac",fontsize=14)
```

```
x= [ 0.  1.  2.  3.  0.  0.  1.  2.  3.  0.]
```



4.2 The convolution operation

4.2.1 Definition

Using previous elements, we are now in position of characterizing more precisely the *filters*. As already mentioned, a filter is a linear and time-invariant system, see Intro_Filtering.

The system being time invariant, the output associated with $x(m)\delta(n-\tau)$ is $x(m)h(n-m)$, if h is the impulse response.

$$x(m)\delta(n-m) \rightarrow x(m)h(n-mu). \tag{4.2}$$

Since we know that any signal $x(n)$ can be written as (representation formula)

$$x(n) = \sum_{m=-\infty}^{+\infty} x(m)\delta(n-m), \tag{4.3}$$

we obtain, by linearity –that is superposition of all outputs, that

$$\boxed{y(n) = \sum_{m=-\infty}^{+\infty} x(m)h(n-m) = [x*h](n).} \tag{4.4}$$

This relation is called *convolution* of x and h, and this operation is denoted $[x * h](t)$, so as to indicate that the *result* of the convolution operation is evaluated at time n and that the variable m is simply a dummy variable that disappears by the summation.

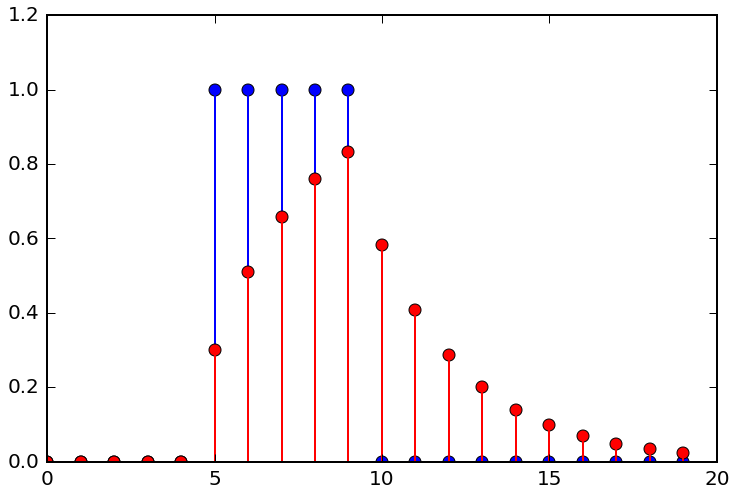
The convolution operation is important since it enables to compute the output of the system using only its impulse response. It is not necessary to know the way the system is build, its internal design and so on. The only thing one must have is its impulse response. Thus we see that the knowledge of the impulse response enable to fully characterize the input-output relationships.

4.2.2 Illustration

We show numerically that the output of a system is effectively the weightened sum of delayed impulse responses. This indicates that the output of the system can be computed either by using its difference equation, or by the convolution of its input with its impulse response.

Direct response

```
def op3(signal):
    transformed_signal=np.zeros(np.size(signal))
    for t in np.arange(np.size(signal)):
        transformed_signal[t]= 0.7*transformed_signal[t-1]+0.3*signal[t]
    return transformed_signal
#
# rectangular pulse
N=20; L=5; M=10
r=np.zeros(N)
r[L:M]=1
#
plt.stem(r)
plt.stem(op3(r),linefmt='r-',markerfmt='ro')
_=plt.ylim([0, 1.2])
```



Response by the sum of delayed impulse responses

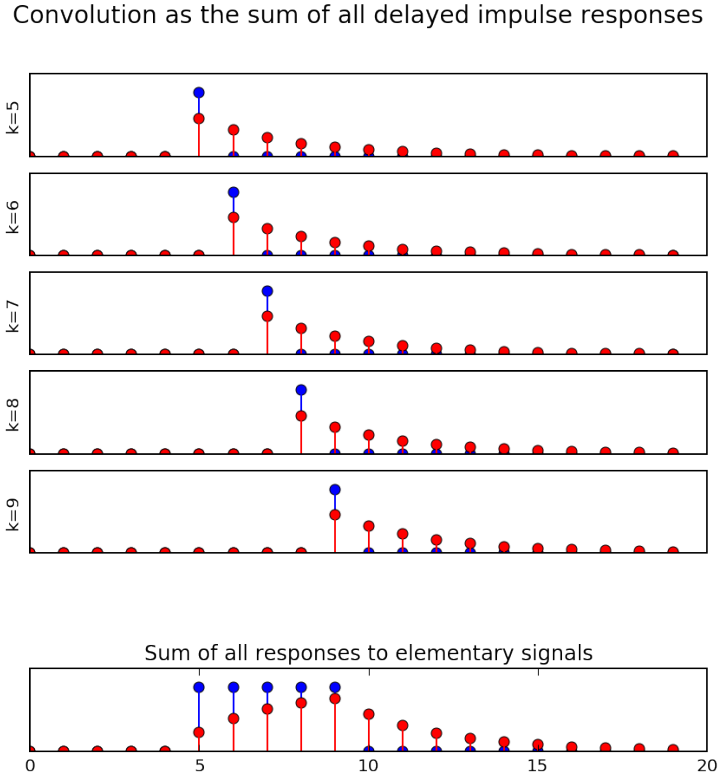
```
s=np.zeros((N,N))
for k in range(N):
```

```
        s[k][k]=r[k]
# this is equivalent to s=np.diag(x)
ll=range(5,10)
llmax=ll[-1]
f,ax=plt.subplots(len(ll)+2,figsize=(7,7))
u=0
sum_of_responses=np.zeros(N)
for k in ll:
    ax[u].stem(s[k][:])
    ax[u].stem(2*op3(s[k][:]),linefmt='r-',markerfmt='ro')
    ax[u].set_ylim([0,1.3])
    ax[u].set_ylabel('k={}'.format(k))
    ax[u].get_yaxis().set_ticks([])
    sum_of_responses+=op3(s[k][:])
    if u!=llmax-1: ax[u].get_xaxis().set_ticks([])
    u+=1

ax[u].axis('off')

ax[u+1].get_yaxis().set_ticks([])
ax[u+1].stem(r,linefmt='b-',markerfmt='bo')
ax[u+1].stem(sum_of_responses,linefmt='r-',markerfmt='ro')
ax[u+1].set_ylim([0,1.3])
ax[u+1].set_title("Sum of all responses to elementary signals")

#
#f.tight_layout()
f.suptitle("Convolution as the sum of all delayed impulse responses",
    fontsize=14)
```



4.2.3 Exercises

Exercise 3. *1. Compute* by hand *the convolution between two rectangular signals,*

2. propose a python program that computes the result, given two arrays. Syntax:

```
def myconv(x,y):
    return z
```

3. Of course, convolution functions have already be implemented, in many languages, by many people and using many algorithms. Implementations also exist in two or more dimensions. So, we do need to reinvent the wheel. Consult the help of `np.convolve` *and of* `sig.convolve` *(respectively from numpy and scipy modules).*

4. use this convolution to compute and display the convolution between two rectangular signals

```
def myconv(x,y):
    L=np.size(x)
    # we do it in the simpla case where both signals have the same length
    assert np.size(x)==np.size(y), "The two signals must have the same
        lengths"
    # as an exercise, you can generalize this

    z=np.zeros(2*L-1)
    #
    ## -> FILL IN
    #
    return z
# test it:
z=myconv(np.ones(L),np.ones(L))
print('z=',z)
```

```
z= [ 0.  0.  0.  0.  0.  0.  0.  0.  0.]
```

```
def myconv(x,y):
    L=np.size(x)
    # we do it in the simpla case where both signals have the same length
    assert np.size(x)==np.size(y), "The two signals must have the same
        lengths"
    # as an exercise, you can generalize this

    z=np.zeros(2*L-1)
    # delay<L
    for delay in np.arange(0,L):
        z[delay]=np.sum(x[0:delay+1]*y[-1:-1-delay-1:-1])
    # delay>=L
    for delay in np.arange(L,2*L-1):
        z[delay]=np.sum(x[delay+1-L:L]*y[-delay-1+L:0:-1])
    return z
# test it:
z=myconv(np.ones(L),np.ones(L))
print('z=',z)
```

```
z= [ 1.  2.  3.  4.  5.  4.  3.  2.  1.]
```

Convolution with legacy convolve:

```
#help(np.convolve)
# convolution between two squares of length L
L=10
z=sig.convolve(np.ones(L),np.ones(L))
plt.stem(z)
plt.title("Convolution between two rectangular pulses")
plt.xlabel("Delay")
```