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Elliptic PDEs -Problem Sheet One

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Exercise 1. *Suppose that $u \in H^1(B_\rho(x_0))$ is a weak solution of*

$$-div(A(x,u)\nabla u) = 0 \text{ in } B_\rho(x_0),$$

where A is a symmetric matrix valued map for which there exists $0 < \lambda < \Lambda < \infty$ such that for a.e. $x \in B_\rho(x_0)$ and a.e. $y \in \mathbb{R}$ and any $\zeta \in \mathbb{R}^d$

$$\lambda \zeta \cdot \zeta \le A_{ij}(x,y)\zeta_i\zeta_j \le \Lambda \zeta \cdot \zeta.$$

Show that for any $0 < \rho' < \rho$

$$\int_{B_{\rho'}(x_0)} |\nabla u|^2 \le \frac{\Lambda}{\lambda}\frac{1}{(\rho' - \rho)^2}\int_{B_\rho(x_0)} u^2.$$

Proof. Testing the equation with $\eta^2 v \in H_0^1(B_\rho)$, where η is the radial cut-off function equal to 1 on $B_{\rho'}(x_0)$, vanishing outside $B_\rho(x_0)$ and with a bounded gradient $\nabla\eta \le \frac{1+\varepsilon}{\rho-\rho'}$, then we have

$$\begin{aligned}
0 =& \int_{B_{\rho(x_0)}} A\nabla u \cdot \nabla(\eta^2 u) \\
=& \int_{B_{\rho(x_0)}} A\nabla u \cdot \nabla u \cdot \eta^2 + 2\int_{B_{\rho(x_0)}} A\nabla u \cdot u \cdot \nabla\eta \cdot \eta \\
=& \int_{B_{\rho(x_0)}} A\nabla(\eta u)\cdot\nabla(\eta u) - \int_{B_{\rho(x_0)}} u^2 A\nabla\eta\cdot\nabla\eta \\
\ge& \lambda\int_{B_{\rho(x_0)}} \nabla(\eta u)\cdot\nabla(\eta u) - \frac{\Lambda(1+\varepsilon)^2}{(\rho-\rho')^2}\int_{B_{\rho(x_0)}} u^2 \\
\ge& \lambda\int_{B_{\rho'(x_0)}} |\nabla u|^2 - \frac{\Lambda(1+\varepsilon)^2}{(\rho-\rho')^2}\int_{B_{\rho(x_0)}} u^2.
\end{aligned}$$

Therefore,

$$\int_{B_{\rho'(x_0)}} |\nabla u|^2 \leq \frac{\Lambda}{\lambda} \frac{(1+\varepsilon)^2}{(\rho-\rho')^2} \int_{B_{\rho(x_0)}} u^2,$$

and take a limit as $\varepsilon \to 0$. $\square$

Exercise 2. *Let $u_n \in H^1(B_\rho(x_0))$ be a sequence of weak solutions of*

$$-div(A(x)\nabla u_n + b(x)u_n) + c(x) \cdot \nabla u_n + d(x)u_n = f_n \text{ in } B_\rho(x_0),$$

where a.e. $x \in B_\rho(x_0)$ and a.e. $y \in \mathbb{R}$ and any $\zeta \in \mathbb{R}^d$

$$\lambda\zeta \cdot \zeta \leq A_{ij}(x,y)\zeta_i, \zeta_j \leq \Lambda\zeta \cdot \zeta, \text{ and } |b(x)| + |c(x)| + |d(x)| \leq M$$

for some positive constants λ, Λ and M. Assume furthermore that $f_n \to f \in L^2(B_\rho(x_0))$, and for all n,

$$\int_{B_\rho} u_n^2 \leq C \text{ for all } n \in \mathbb{N}.$$

Given $0 < \rho' < \rho$, show that there exists a subsequence u_m such that

$$u_m \to u \text{ in } H^1(B_{\rho'}),$$

where u satisfies

$$-div(A(x)\nabla u + b(x)u) + c(x) \cdot \nabla u + d(x)u = f \text{ in } B_\rho(x_0).$$

Hint: start by proving the appropriate Caccioppoli inequality.

Proof.

- Proof of the Caccioppoli inequality for this equation:

  Testing against $u\eta^2$ where η is the radial cut-off function as defined in Exercise 1, we have

  $$\int_{B_\rho(x_0)} (A(x)\nabla u + b(x)u) \cdot \nabla(u\eta^2) + c(x)\nabla u \cdot u\eta^2 + du^2\eta^2 = \int_\Omega fu\eta^2$$

  Define

  $$I_1 := \int_{B_\rho(x_0)} A(x)\nabla u \cdot \nabla(u\eta^2)$$

  and

  $$I_2 := \int_{B_\rho(x_0)} b(x)u \cdot \nabla(u\eta^2) + c(x)\nabla u \cdot u\eta^2 + du^2\eta^2.$$

  Then

$$
\begin{aligned}
I_1 =& \int_{B_\rho(x_0)} A(x)\nabla u \cdot \nabla(u\eta)\cdot\eta + \int_{B_\rho(x_0)} A(x)\nabla u\cdot\nabla\eta u\eta \\
=& \int_{B_\rho(x_0)} A(x)\nabla(u\eta)\nabla(u\eta) - \int_{B_\rho(x_0)} A(x)\nabla\eta\nabla(u\eta)u + \int_{B_\rho(x_0)} A(x)\nabla u\nabla\eta u\eta \\
=& \int_{B_\rho(x_0)} A(x)\nabla(u\eta)\nabla(u\eta) - \int_{B_\rho(x_0)} A(x)\nabla\eta\nabla\eta u^2 \\
-& \int_{B_\rho(x_0)} A(x)\nabla\eta\cdot\nabla u\eta u + \int_{B_\rho(x_0)} A(x)\nabla u\cdot\nabla\eta u\eta \\
=& \int_{B_\rho(x_0)} A(x)\nabla(u\eta)\cdot\nabla(u\eta) - \int_{B_\rho(x_0)} (A(x)\nabla\eta\cdot\nabla\eta)u^2 + \int_{B_\rho(x_0)} ((A-A^T)\nabla u\nabla\eta)\cdot(u\eta)
\end{aligned}
$$

Note that

$$
\int_{B_\rho(x_0)} ((A-A^T)\nabla u\nabla\eta)\cdot(u\eta) = \int_{B_\rho(x_0)} ((A-A^T)\nabla(u\eta)\cdot\nabla\eta\cdot u - \int_{B_\rho(x_0)} (A-A^T)\nabla\eta\cdot\nabla\eta u^2
$$

Thus,

$$
\begin{aligned}
I_1 =& \int_{B_\rho(x_0)} A(x)\nabla(u\eta)\cdot\nabla(u\eta) + \int_{B_\rho(x_0)} ((A^T-2A)\nabla\eta\cdot\nabla\eta)u^2 + \int_{B_\rho(x_0)} (A-A^T)\nabla(u\eta)\cdot\nabla\eta\cdot u \\
\geq& \lambda\int_{B_\rho(x_0)} |\nabla(\eta u)|^2 - 3\|A\|_\infty \int_{B_\rho(x_0)} |\nabla\eta|^2u^2 - 2\|A\|_\infty \int_{B_\rho(x_0)} |\nabla(u\eta)||\nabla\eta||u|
\end{aligned}
$$

By Young's inequality, we have

$$
2\|A\|_\infty|\nabla(u\eta)||\nabla\eta||u| \leq \frac{\lambda}{4}|\nabla(u\eta)|^2 + \frac{4\|A\|_\infty^2}{\lambda}|\nabla\eta|^2|u|^2.
$$

Thus

$$
I_1 \geq \frac{3}{4}\lambda\int_{B_\rho(x_0)} |\nabla(u\eta)|^2 - \|A\|_\infty(3+\frac{4\|A\|_\infty}{\lambda})\int_{B_\rho(x_0)} |\nabla\eta|^2|u|^2.
$$

$$
\begin{aligned}
I_2 = &\int_{B_\rho(x_0)} b(x)u \cdot \nabla(u\eta^2) + c(x)\nabla u \cdot u\eta^2 + du^2\eta^2 \\
= &\int_{B_\rho(x_0)} u\eta b \cdot \nabla(u\eta) + u^2\eta b \cdot \nabla\eta \\
&+ \int_{B_\rho(x_0)} u\eta(c \cdot \nabla(u\eta)) - u^2\eta(c \cdot \nabla\eta) + \int_{B_\rho(x_0)} du^2\eta^2 \\
\leq &M(\int_{B_\rho(x_0)} \nabla(\eta u)|u\eta| + u^2\eta^2)
\end{aligned}
$$

Note that

$$
M|\nabla(\eta u)||u\eta| \leq \frac{\lambda}{4}|\nabla(u\eta)|^2 + \frac{4M^2}{\lambda}|u\eta|^2
$$

Thus

$$
\begin{aligned}
\frac{\lambda}{2}\int_{B_\rho(x_0)} |\nabla(u\eta)|^2 \leq &M(3 + \frac{4M}{\lambda})\int_{B_\rho(x_0)} (|\nabla\eta|^2 + 1)|u\eta|^2 + \int_{B_\rho(x_0)} fu\eta^2 \\
\leq &M(3 + \frac{4M}{\lambda})\int_{B_\rho(x_0)} (|\nabla\eta|^2 + 1)|u\eta|^2 + \frac{1}{2}(\|f\|^2_{L^2(B_\rho(x_0))} + \|u\eta\|^2_{L^2(B_\rho(x_0))})
\end{aligned}
$$

That is,

$$
\int_{B_\rho(x_0)} |\nabla(u\eta)|^2 \leq C(M, \lambda, \rho, \rho')\left(\int_{B_{\rho(x_0)}} |\eta u|^2 + |f|^2\right)
$$

- The compactness argument:

  As the sequence (u_n) is bounded in $L^2(B_\rho(x_0))$, there exists a weakly converging subsequence (we call it u_m as well) such that

  $$u_m \rightharpoonup u \text{ in } L^2(B_\rho(x_0)).$$

  Passing to the limit in the equation (in the weak sense), u is a weak solution to

  $$-div(A(x)\nabla u + b(x)u) + c(x) \cdot \nabla u + d(x)u = f.$$

  Choose $\rho' < \rho'' < \rho$, we have

  $$\int_{B_{\rho''}} |\nabla u_m - \nabla u| \leq C(M, \lambda, \rho, \rho')\left(\int_{B_\rho(x_0)} |u_m - u|^2 + |f_m - f|^2\right).$$

  $u_m - u$ is bounded in $H^1(B_{\rho''}(x_0))$ and converges weakly to 0, thus by Rellich Kondrachov theorem, we know that

  $$u_m \to u \text{ in } L^2(B_{\rho''(x_0)}).$$

Now apply the Caccioppoli inequality again,

$$\int_{B_{\rho'}(x_0)} |\nabla(u_m - u)|^2 \leq \left(\int_{B''_\rho(x_0)} |u_m - u|^2 + |f_m - f|^2 \right) \to 0$$

as $f_m \to f$ in $L^2(B_\rho(x_0))$ and $u_m \to u$ in $L^2(B''_\rho(x_0))$.
Thus, $u_m \to u$ in $H^1(B'_\rho(x_0))$.

□

Exercise 3. *Suppose that $d \geq 3$. Given $u \in H^1(\Omega)$ a weak solution of*

$$div(A\nabla u) + b \cdot \nabla u = f,$$

where $A, b, c \in L^\infty(\Omega)$, $A\xi \cdot \xi \geq \lambda \xi \cdot \xi$ and $|A(x)|_\infty + |b(x)|_\infty \leq M$ a.e. in Ω, and $f \in L^d(\Omega)$, show that for all $B_\rho \subset \Omega$ with $|B_\rho| \leq 1$,

$$\int_{B_\rho} |\nabla(|u|^{\frac{p+1}{2}}\eta)|^2 \leq C(p+1)^2 \left(\int_{B_\rho} (|\nabla\eta|^2 + 1)|u|^{p+1} + \|f\|^{p+1}_{L^d(B_\rho)} \right)$$

with a constant C depending on λ, M and d only.

Proof. We take $\eta \in C_c^\infty(B_\rho)$ to be the usual radial cut-off function with $|\eta| \leq 1$ and bounded gradient $|\nabla\eta| \leq \frac{1}{|\rho-\rho'|}$. Give $K > 0$, let $T_K(x) = \min(K, \max(x, 0))$ and consider $T_K(u)$. Clearly, $T_K(u) \in H^1(\Omega)$ is bounded and $T_K(u) = u$ on $\Omega_K^+ = \{x \in \Omega : 0 \leq u \leq K\}$. Testing against $T_K(u)^{p-1}u_+\eta^2$ (similarly for u_-), we have

$$-\int_\Omega A\nabla u \nabla(T_K(u)^{p-1}u_+\eta^2) + \int_\Omega b \cdot \nabla u T_K(u)^{p-1}u_+\eta^2 = \int_\Omega f T_K(u)^{p-1}u_+\eta^2.$$

That is

$$\begin{aligned}
0 &= \int_\Omega A\nabla u \nabla(T_K(u)^{p-1}u_+\eta^2) - \int_\Omega b \cdot \nabla u T_K(u)^{p-1}u_+\eta^2 + \int_\Omega f T_K(u)^{p-1}u_+\eta^2 \\
&= \int_{\Omega_K^+} A\nabla T_K(u)\nabla(T_K(u)^p\eta^2) + K^{p-1}\int_{\Omega\setminus\Omega_K^+} A\nabla u_+\nabla(u_+\eta^2) \\
&- \int_{\Omega_K^+} b \cdot \nabla T_K(u)(T_K(u)^p\eta^2) - K^{p-1}\int_{\Omega\setminus\Omega_K^+} b\nabla u_+(u_+\eta^2) \\
&+ \int_{\Omega_K^+} f(T_K(u)^p\eta^2) + K^{p-1}\int_{\Omega\setminus\Omega_K^+} f(u_+\eta^2)
\end{aligned}$$

Note that we only consider the integrals on Ω_K^+ since integrals on $\Omega \setminus \Omega_K^+$ would vanish as $K \to \infty$. Write $v := T_K(u)$. Then

$$
\begin{aligned}
I_1 =& \int_{\Omega_K^+} A\nabla T_K(u)\nabla(T_K(u)^p\eta^2) \\
=& \int_{\Omega_K^+} A\nabla v\nabla(v^p\eta^2) \\
=& 2\int_{\Omega_K^+} A\nabla v\cdot\nabla\eta v^p\eta + \frac{4p}{(p+1)^2}\int_{\Omega_K^+} A(\nabla v^{\frac{p+1}{2}})(\nabla v^{\frac{p+1}{2}})\eta^2 \\
=& \frac{4p}{(p+1)^2}\int_{\Omega_K^+} A\nabla(v^{\frac{p+1}{2}}\eta)\nabla(v^{\frac{p+1}{2}}\eta) - \frac{4p}{(p+1)^2}\int_{\Omega_K^+} A\nabla\eta\nabla\eta v^{p+1} \\
&+ \frac{4(1-p)}{(p+1)^2}\int_{\Omega_K^+} A\nabla v^{\frac{p+1}{2}}\nabla\eta v^{\frac{p+1}{2}}\eta \\
=& \frac{4p}{(p+1)^2}\int_{\Omega_K^+} A\nabla(v^{\frac{p+1}{2}}\eta)\nabla(v^{\frac{p+1}{2}}\eta) - \frac{4}{(p+1)^2}\int_{\Omega_K^+} A\nabla\eta\nabla\eta v^{p+1} \\
&+ \frac{4(1-p)}{(p+1)^2}\int_{\Omega_K^+} A\nabla(v^{\frac{p+1}{2}}\eta)\nabla\eta v^{\frac{p+1}{2}} \\
\geq& \frac{4p\lambda}{(p+1)^2}\|\nabla(v^{\frac{p+1}{2}}\eta)\|^2_{L^2(\Omega_K^+)} - \frac{4\|A\|_\infty}{(p+1)^2}\|\nabla\eta v^{\frac{p+1}{2}}\|^2_{L^2(\Omega_K^+)} \\
&- \frac{4p}{(p+1)^2}\|A\|_\infty\|\nabla(v^{\frac{p+1}{2}}\eta)\|_{L^2(\Omega_K^+)}\|\nabla\eta v^{\frac{p+1}{2}}\|_{L^2(\Omega_K^+)}.
\end{aligned}
$$

Applying Young's inequality, we have

$$
I_1 \geq \left(\frac{4p\lambda}{(p+1)^2} - \frac{4p\varepsilon}{(p+1)^2}\right)\|\nabla(v^{\frac{p+1}{2}}\eta)\|^2_{L^2(\Omega_K^+)} - \frac{C(\varepsilon, M)}{(p+1)^2}\|\nabla\eta v^{\frac{p+1}{2}}\|^2_{L^2(\Omega_K^+)}.
$$

Now consider the term involving b,

$$
\begin{aligned}
I_2 =& \int_{\Omega_K^+} b\cdot\nabla v v^p\eta^2 \\
=& \frac{2}{p+1}\int_{\Omega_K^+} b\cdot\nabla(v^{\frac{p+1}{2}}\eta)v^{\frac{p+1}{2}}\eta - \frac{2}{p+1}\int_{\Omega_K^+} b\cdot\nabla\eta v^{p+1}\eta \\
\leq& \frac{2}{p+1}\|b\|_\infty\|\nabla(v^{\frac{p+1}{2}}\eta)\|_{L^2(\Omega_K^+)}\|v^{\frac{p+1}{2}}\eta\|_{L^2(\Omega_K^+)} \\
&+ \frac{2}{p+1}\|b\|_\infty\|\nabla\eta v^{\frac{p+1}{2}}\|_{L^2(\Omega_K^+)}\|v^{\frac{p+1}{2}}\eta\|_{L^2(\Omega_K^+)} \\
\leq& \frac{2}{p+1}\left(\frac{\varepsilon}{4}\|\nabla(v^{\frac{p+1}{2}}\eta)\|_{L^2(\Omega_K^+)} + \left(\frac{\|b\|^2_\infty}{\varepsilon} + \frac{\|b\|_\infty}{|\rho-\rho'|}\right)\|v\eta\|^{p+1}_{L^{p+1}(\Omega_K^+)}\right)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_{\Omega_K^+} f v^p \eta^2 \leq & C(d) \|v^{\frac{p+1}{2}}\eta\|_{L^q(\Omega_K^+)} \|f\eta\|_{L^r(\Omega_K^+)} \|v^{\frac{p-1}{2}}\eta\|_{L^s(\Omega_K^+)} \text{ for } \frac{1}{s}+\frac{1}{q}+\frac{1}{r}=1 \\
\leq & C'(d) \|\nabla(v^{\frac{p+1}{2}}\eta)\|_{L^2(\Omega_K^+)} \|f\eta\|_{L^r(\Omega_K^+)} \|v^{\frac{p-1}{2}}\eta\|_{L^s(\Omega_K^+)} \text{ taking } q=\frac{2d}{d-2} \\
\leq & \varepsilon \|\nabla(v^{\frac{p+1}{2}}\eta)\|^2_{L^2(\Omega_K^+)} + \frac{C'(d)}{\varepsilon} \|f\eta\|^2_{L^r(\Omega_K^+)} \|v^{\frac{p-1}{2}}\eta\|^2_{L^s(\Omega_K^+)} \\
\leq & \varepsilon \|\nabla(v^{\frac{p+1}{2}}\eta)\|^2_{L^2(\Omega_K^+)} + \frac{C'(d)}{\theta\varepsilon} \|f\eta\|^{2\theta}_{L^r(\Omega_K^+)} + \frac{C'(d)}{\theta'\varepsilon} \|v^{\frac{p-1}{2}}\eta\|^{2\theta'}_{L^s(\Omega_K^+)}
\end{aligned}
$$

with $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Note that the second line follows from Sobolev inequality. We observe that $\|v^{\frac{p-1}{2}}\|^{2\theta'}_{L^s} = \|v\|^{\theta'(p-1)}_{L^{\frac{s(p-1)}{2}}}$, so we can choose $2\theta = p+1$ and $\frac{s(p-1)}{2} = p+1$. Then we have $\theta = \frac{p+1}{2}$, $s = \frac{2(p+1)}{(p-1)}$ and $r = \frac{d(p+1)}{p+1+d} < d$. Thus, we can deduce that

$$
\|f\eta\|_{L^r(\Omega_K^+)} \leq |B_\rho|^{\frac{1}{r}-\frac{1}{d}} \|f\|_{L^d}.
$$

Note that $|B_\rho| < 1$, then we have

$$
\int_{\Omega_K^+} f v^p \eta^2 \leq \varepsilon \|\nabla(v^{\frac{p+1}{2}}\eta)\|^2_{L^2(\Omega_K^+))} + C(d,\varepsilon)\|f\|^{p+1}_{L^d(\Omega_K^+))} + \tilde{C}\|v\|^{p+1}_{L^{p+1}}.
$$

Thus,

$$
\begin{aligned}
\left(\frac{4p\lambda}{(p+1)^2} - \frac{4p\varepsilon}{(p+1)^2} - \frac{\varepsilon}{2(p+1)} - \varepsilon\right) \|\nabla(v^{\frac{p+1}{2}}\eta)\|^2_{L^2(\Omega_K^+)} \leq & \frac{C(\varepsilon, M)}{(p+1)^2} \|\nabla\eta v^{\frac{p+1}{2}}\|^2_{L^2(\Omega_K^+)} \\
& + \frac{2}{p+1}\left(\frac{M^2}{\varepsilon} + \frac{M}{|\rho-\rho'|}\right) \|v\|^{p+1}_{L^{p+1}(\Omega_K^+)} \\
& + C(d,\varepsilon)\|f\|^{p+1}_{L^d(\Omega_K^+)} + \|v\|^{p+1}_{L^{p+1}(\Omega_K^+)}.
\end{aligned}
$$

We do the same for u_- and add the results together, then

$$
\int_{B_\rho} |\nabla(|u|^{\frac{p+1}{2}}\eta)|^2 \leq C(p+1)^2 \left(\int_{B_\rho} (|\nabla\eta|^2+1)|u|^{p+1} + \|f\|^{p+1}_{L^d(B_\rho)}\right)
$$

with a constant C depending on λ, M and d only. □

Exercise 4. *Let Φ be a convex and locally Lipschitz continuous function on some interval I. Suppose $u \in H^1(\Omega)$ takes its values in I.*

- *Assume that $\Phi' \geq 0$. Suppose that for all $v \in H_0^1(\Omega)$, with $v \geq 0$*

$$
\int_\Omega A\nabla u \cdot \nabla v \leq 0,
$$

(which we will refer to as a subsolution). Assuming that $\Phi(u) \in H^1(\Omega)$*, show that for all* $v \in H_0^1(\Omega)$*, with* $v \geq 0$

$$\int_\Omega A\nabla\Phi(u) \cdot \nabla v \leq 0.$$

- *Assume that* $\Phi' \leq 0$*. Suppose that for all* $v \in H_0^1(\Omega)$*, with* $v \geq 0$

$$\int_\Omega A\nabla u \cdot \nabla v \geq 0,$$

(which we will refer to as a supersolution). Assuming that $\Phi(u) \in H^1(\Omega)$*, show that for all* $v \in H_0^1(\Omega)$*, with* $v \geq 0$

$$\int_\Omega A\nabla\Phi(u) \cdot \nabla v \leq 0.$$

- *Thus show that if* u *is a subsolution, then* $u^+ = \max(0, u)$ *is also a subsolution.*

Proof. Since Φ is a convex function, by *Alexandrov's Theorem*, we know that Φ has second derivative almost everywhere.

- If Φ satisfies $\Phi' \geq 0$, then

$$\begin{aligned}
\int_\Omega A\nabla\Phi(u) \cdot \nabla v &= \int_\Omega A\Phi'(u)\nabla u \cdot \nabla v \\
&= \int_\Omega A\nabla u \cdot \nabla(\Phi'(u)v) - \int_\Omega A\nabla u \cdot (\nabla\Phi'(u))v \\
&= \int_\Omega A\nabla u \cdot \nabla(\Phi'(u)v) - \int_\Omega A\nabla u \cdot \nabla u\Phi''(u)v \\
&\leq 0 - \int_\Omega A\nabla u \cdot \nabla u\Phi''(u)v \text{ since } \Phi'(u)v \geq 0 \text{ and } \Phi'(u)v \in H_0^1 \\
&\leq 0 - \lambda \int_\Omega |\nabla u|^2\Phi''(u)v \text{ (by ellipticity)} \\
&\leq 0
\end{aligned}$$

since $\Phi'' > 0$ by convexity and $v \geq 0$ by assumption.

- If Φ satisfies $\Phi' \leq 0$, then

$$
\begin{aligned}
\int_\Omega A\nabla\Phi(u)\cdot\nabla v =& -\int_\Omega A(-\Phi'(u))\nabla u\cdot\nabla v \\
=& -\int_\Omega A\nabla u\cdot\nabla((-\Phi'(u))v) - \int_\Omega A\nabla u\cdot(\nabla\Phi'(u))v \\
=& -\int_\Omega A\nabla u\cdot\nabla((-\Phi'(u))v) - \int_\Omega A\nabla u\cdot\nabla u\Phi''(u)v \\
\leq& 0 - \int_\Omega A\nabla u\cdot\nabla u\Phi''(u)v \text{ since } (-\Phi'(u))v \geq 0 \text{ and } (-\Phi'(u))v \in H_0^1 \\
\leq& 0 - \lambda\int_\Omega |\nabla u|^2\Phi''(u)v \text{ (by ellipticity)} \\
\leq& 0
\end{aligned}
$$

since $\Phi'' > 0$ by convexity and $v \geq 0$ by assumption.

- Note that $u^+ = \max(u, 0)$ is convex and $u^{+\prime} \geq 0$, so applying the proof for the first part of this question, we have $\int_\Omega A\nabla u^+\cdot\nabla v \leq 0$ for all $v \in H_0^1$ with $v \geq 0$.

□

Exercise 5. *Check that if $u \in H_0^1(\Omega)$ is a weak sub-solution, that is,*

$$\int_\Omega A\nabla u\cdot\nabla\varphi \leq 0 \textit{ for all } \varphi \geq 0 \textit{ s.t. } \varphi \in H_0^1,$$

then

$$u \leq \left(\frac{\Lambda}{\lambda}\right)^{\frac{d}{4}} C(d)\left(\frac{1}{|B_R|}\int_{B_R(x_0)} u^2 dx\right)^{\frac{1}{2}}.$$

Proof. Since we only need to prove that u is bounded above, it is sufficient to prove $u_+ = \max(u, 0)$ satisfies

$$|u_+| \leq \left(\frac{\Lambda}{\lambda}\right)^{\frac{d}{4}} C(d)\left(\frac{1}{|B_R|}\int_{B_R(x_0)} u^2 dx\right)^{\frac{1}{2}}.$$

Note that since u is a weak sub-solution, u_+ is a weak subsolution by Exercise 1.4. We first show that $u_+ \in L^p_{loc}(\Omega)$ for all $p > 2$. This is trivially true for $d \leq 2$ by Sobolev embeddings. For $d \geq 3$, we need to use the following lemma:

Lemma 1. *$u \in H_0^1(\Omega)$ is a subsolution and A is symmetric. Let $p > 1$ and $B_{\rho(x_0)} \subset \Omega$ be such that $|u_+|^{p+1} \in L^1(B_{\rho(x_0)})$. Then for any $\eta \in C_c^\infty(B_\rho(x_0))$, $|u_+|^{\frac{p+1}{2}}\eta \in H_0^1(B_\rho(x_0))$ and*

$$\int_{B_\rho} |\nabla(|u_+|^{\frac{p+1}{2}}\eta)|^2 \leq \frac{\Lambda}{\lambda}\int_{B_\rho} |\nabla\eta|^2|u_+|^{p+1}.$$

Proof. Give $M > 0$, let $T_M(x) = \min(M, \max(x, 0))$ and consider $T_M(u)$. Clearly, $T_M(u) \in H^1(\Omega)$ is bounded and $T_M(u) = u$ on $\Omega_M^+ = \{x \in \Omega : 0 \le u \le M\}$. Consider $\varphi = T_M(u_+)^{p-1}u_+\eta^2$, then $\varphi \in H_0^1(\Omega)$ since

$$\nabla\varphi = \mathbb{1}_{\Omega_M^+}(2\eta\nabla\eta u_+^p + \eta^2 p u_+^{p-1}\nabla u_+ + \mathbb{1}_{\Omega\setminus\Omega_M^+}M^{p-1}(2\eta\nabla\eta u_+ + \eta^2\nabla u_+) \in L^2(\Omega)$$

where $\mathbb{1}_{\Omega_M^+}$ is the characteristic function of the set Ω_M^+.
Testing the equation against φ, we obtain

$$\begin{aligned}0 \ge &\int_\Omega A\nabla u \cdot \nabla(T_M(u)^{p-1}u_+\eta^2)\\ =&\int_{\Omega_M^+} A\nabla T_M(u)\cdot\nabla(T_M(u)^p\eta^2) + M^{p-1}\int_{\Omega\setminus\Omega_M^+} A\nabla u_+\nabla(u_+\eta^2).\end{aligned}$$

For simplicity, we write v for $T_M(u)$.

$$\begin{aligned}
I =&\int_{\Omega_M^+} A\nabla T_M(u)\cdot\nabla(T_M(u)^p\eta^2)\\
=&\int_{\Omega_M^+} A\nabla v\cdot\nabla(v^p\eta^2)\\
=&2\int_{\Omega_M^+} A\nabla v\cdot\nabla\eta v^p\eta + p\int_{\Omega_M^+}(A\nabla v\cdot\nabla v)v^{p-1}\eta^2\\
=&2\int_{\Omega_M^+} A\nabla v\cdot\nabla\eta v^p\eta + \frac{4p}{(p+1)^2}\int_{\Omega_M^+}(A\nabla v^{\frac{p+1}{2}}\cdot\nabla v^{\frac{p+1}{2}})\eta^2\\
=&2\int_{\Omega_M^+} A\nabla v\cdot\nabla\eta v^p\eta + \frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla v^{\frac{p+1}{2}\eta}\cdot\nabla v^{\frac{p+1}{2}\eta}\\
&-\frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla\eta\cdot\eta v^{p+1} - \frac{4p}{(p+1)}\int_{\Omega_M^+} A\nabla v\cdot\nabla\eta v^p\eta\\
=&\frac{2(1-p)}{(p+1)}\int_{\Omega_M^+} A\nabla v\cdot\nabla\eta v^p\eta + \frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla(v^{\frac{p+1}{2}}\eta) - \frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla\eta\cdot\eta v^{p+1}\\
=&\frac{4(1-p)}{(p+1)^2}\int_{\Omega_M^+}(A\nabla v^{\frac{p+1}{2}}\nabla\eta)\cdot v^{\frac{p+1}{2}}\eta + \frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla(v^{\frac{p+1}{2}}\eta) - \frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla\eta\cdot\nabla\eta v^{p+1}\\
=&\frac{4(1-p)}{(p+1)^2}\int_{\Omega_M^+}(A\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla\eta)v^{\frac{p+1}{2}} - \frac{4(1-p)}{(p+1)^2}\int_{\Omega_M^+} A\nabla\eta\cdot\nabla\eta v^{p+1}\\
&+\frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla(v^{\frac{p+1}{2}}\eta) - \frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla\eta\cdot\nabla\eta v^{p+1}\\
=&\frac{4(1-p)}{(p+1)^2}\int_{\Omega_M^+}(A\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla\eta)v^{\frac{p+1}{2}} - \frac{4}{(p+1)^2}\int_{\Omega_M^+} A\nabla\eta\cdot\nabla\eta v^{p+1}\\
&+\frac{4p}{(p+1)^2}\int_{\Omega_M^+} A\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla(v^{\frac{p+1}{2}}\eta)
\end{aligned}$$

This implies that

$$\begin{aligned}\frac{(p+1)^2}{4}I =&(1-p)\int_{\Omega_M^+}(A\cdot\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla\eta)v^{\frac{p+1}{2}}\\&-\int_{\Omega_M^+}A\nabla\eta\cdot\nabla\eta v^{p+1}+p\int_{\Omega_M^+}A\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla(v^{\frac{p+1}{2}}\eta)\end{aligned}$$

Use Cauchy-Schwarz to get rid of the cross term

$$\begin{aligned}&|(p-1)\int_{\Omega^+}(A\cdot\nabla(v^{\frac{p+1}{2}}\eta)\cdot\nabla\eta)v^{\frac{p+1}{2}}|\\\leq&(p-1)\int_{\Omega_M^+}\sqrt{A\nabla(v^{\frac{p+1}{2}})\cdot\nabla(v^{\frac{p+1}{2}})}\cdot\sqrt{v^{p+1}A\nabla\eta\cdot\nabla\eta}\\\leq&\frac{p-1}{2}\int_{\Omega_M^+}A\nabla(\eta v^{\frac{p+1}{2}})\cdot\nabla(\eta v^{\frac{p+1}{2}})+\frac{p-1}{2}\int_{\Omega_M^+}(A\nabla\eta\cdot\nabla\eta)v^{p+1}\end{aligned}$$

and obtain

$$\begin{aligned}&\frac{(p+1)^2}{4}I+(p-1)\int_{\Omega_M^+}A\cdot\nabla(v^{\frac{p+1}{2}}\eta)\nabla\eta v^{p+1}\\\leq&\frac{(p+1)^2}{4}I+\frac{p-1}{2}\int_{\Omega_M^+}A\nabla(\eta v^{\frac{p+1}{2}})\cdot\nabla(\eta v^{\frac{p+1}{2}})+\frac{p-1}{2}\int_{\Omega_M^+}(A\nabla\eta\cdot\nabla\eta)v^{p+1}.\end{aligned}$$

Thus,

$$\frac{(p+1)^2}{4}I\geq\frac{p+1}{2}\int_{\Omega_M^+}A\nabla(\eta v^{\frac{p+1}{2}})\cdot\nabla(\eta v^{\frac{p+1}{2}})-\frac{p+1}{2}\int_{\Omega_M^+}(A\nabla\eta\cdot\nabla\eta)v^{p+1}$$

which implies that

$$I\geq\frac{p+1}{2}\int_{\Omega_M^+}A\nabla(\eta v^{\frac{p+1}{2}})\cdot\nabla(\eta v^{\frac{2}{p+1}})-\frac{2}{p+1}\int_{\Omega_M^+}(A\nabla\eta\cdot\nabla\eta)v^{p+1}.$$

$$\begin{aligned}II=&M^{p-1}\int_{\Omega\backslash\Omega_M^+}A\nabla u_+\cdot\nabla(u_+\eta^2)\\=&M^{p-1}\int_{\Omega\backslash\Omega_M^+}A\nabla u_+\cdot\nabla u_+\eta^2+2M^{p-1}\int_{\Omega\backslash\Omega_M^+}A\nabla u_+\cdot\nabla\eta u_+\eta\\=&M^{p-1}\int_{\Omega\backslash\Omega_M^+}A\nabla(u_+\eta)\cdot\nabla(u_+\eta)-M^{p-1}\int_{\Omega\backslash\Omega_M^+}A\nabla\eta\cdot\nabla\eta u_+^2\end{aligned}$$

Then we have

$$
\begin{aligned}
0 \geq & I + II \\
\geq & \frac{2}{p+1} \int_{\Omega_M^+} A\nabla(\eta v^{\frac{p+1}{2}} \cdot \nabla(\eta v^{\frac{p+1}{2}}) - \frac{2}{p+1} \int_{\Omega_M^+} A\nabla\eta \cdot \nabla\eta v^{p+1} \\
& + M^{p-1} \int_{\Omega \setminus \Omega_M^+} A\nabla(u_+\eta) \cdot \nabla(u_+\eta) - \int_{\Omega \setminus \Omega_M^+} A\nabla\eta \cdot \nabla\eta u_+^2
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \frac{2}{p+1} \int_{\Omega_M^+} A\nabla(\eta T_M(u_+)^{\frac{p+1}{2}}) \cdot (\eta T_M(u_+)^{\frac{p+1}{2}}) + M^{p-1} \int_{\Omega \setminus \Omega_M^+} A\nabla(u_+\eta) \cdot \nabla(u_+\eta) \\
\leq & \frac{2}{p+1} \int_{\Omega_M^+} A\nabla\eta \cdot \nabla\eta u_+^{p+1} + M^{p-1} \int_{\Omega \setminus \Omega_M^+} A\nabla\eta \cdot \nabla\eta u_+^2 \\
\leq & \frac{2}{p+1} \int_{\Omega_M^+} A\nabla\eta \cdot \nabla\eta u_+^{p+1} + \int_{\Omega \setminus \Omega_M^+} A\nabla\eta \cdot \nabla\eta u_+^{p+1}.
\end{aligned}
$$

This implies that

$$
\int_{\Omega_M^+} A\nabla(\eta T_M(u_+)^{\frac{p+1}{2}}) \cdot (\eta T_M(u_+)^{\frac{p+1}{2}}) \leq \int_{\Omega_M^+} A\nabla\eta \cdot \nabla\eta u_+^{p+1} + \frac{p+1}{2} \int_{\Omega \setminus \Omega_M^+} A\nabla\eta\nabla\eta u_+^{p+1}
$$

Since $\lambda\xi \cdot \xi \leq A\xi \cdot \xi \leq \Lambda\xi \cdot \xi$,

$$
\int_{\Omega_M^+} |\nabla(\eta T_M(u_+)^{\frac{p+1}{2}}|^2 \leq \frac{\Lambda}{\lambda} \left(\int_{\Omega_M^+} |\nabla\eta|^2 u_+^{p+1} + \frac{p+1}{2} \int_{\Omega \setminus \Omega_M^+} |\nabla\eta|^2 u_+^{p+1} \right)
$$

$\eta T_M(u_+)^{\frac{p+1}{2}}$ is bounded in H_0^1, so there exists a subsequence. we call it $\eta T_M(u_+)^{\frac{p+1}{2}}$ as well, converging to ξ weakly in $H_0^1(\Omega)$. By Rellich-Kondrachov Theorem, $\eta T_M(u_+)^{\frac{p+1}{2}} \to \xi$ in L^2. However, $\eta T_M(u_+)^{\frac{p+1}{2}} \to \eta u_+^{\frac{p+1}{2}}$ almost surely, so $\xi = \eta u_+^{\frac{p+1}{2}}$ by uniqueness of weak limit. Therefore,

$$
\int_\Omega |\nabla(\eta u_+^{\frac{p+1}{2}}|^2 \leq \liminf \int_\Omega |\nabla(\eta u_+^{\frac{p+1}{2}})|^2 \leq \frac{\Lambda}{\lambda} \int_\Omega |\nabla\eta|^2 u_+^{p+1}.
$$

□

If $u_+ \in H_0^1(\Omega)$ and Ω is bounded in $\mathbb{R}^d (d \geq 3)$, then by Sobolev embedding, we have

$$
\|u_+\|_{L^{2^\star}((\Omega))}^2 \leq C(d) \|\nabla u_+\|_{L^2(\Omega)}^2 \text{ where } 2^\star = \frac{2d}{d-2}.
$$

Then

$$\begin{aligned}\left(\int_{B_\rho}(u_+^{\frac{p+1}{2}}\eta\right)^{\frac{2d}{d-2}} \leq & C(d)\int_{B_\rho}|\nabla(u_+^{\frac{p+1}{2}})|^2 \\ \leq & C(d)\frac{\Lambda}{\lambda}\int_{B_\rho}|\nabla\eta|^2|u_+|^{p+1} \\ \leq & C(d)\frac{\Lambda}{\lambda}\int_{B_\rho}|\nabla\eta|^2|u|^{p+1}\end{aligned}$$

and using the radial cut-off function introduced in Exercise 1, we have for $\rho' < \rho$

$$\left(\int_{B_{\rho'}(x_0)} u_+^{(p+1)\frac{d}{d-2}}\right)^{\frac{d-2}{d}} \leq \frac{\Lambda}{\lambda}C(d)(\frac{1}{\rho-\rho'})^2\int_{B_\rho}|u|^{p+1}.$$

Take $q = p+1, \tau = \frac{d}{d-2} > 1$, then

$$\|u_+\|^q_{L^{q\tau}(B_{\rho'}(x_0))} \leq \frac{\Lambda}{\lambda}C(d)(\frac{1}{\rho-\rho'})^2\int_{B_\rho}|u|^{p+1}.$$

That is,

$$\|u_+\|_{L^{q\tau}(B_{\rho'}(x_0))} \leq \left(\frac{\Lambda}{\lambda}\right)^{\frac{1}{q}} C(d)(\frac{1}{\rho-\rho'})^{\frac{2}{q}}\|u\|_{L^q(B_\rho(x_0))}.$$

Define $q_n := \tau^n q$, $q_0 := q$, $rho_n = \frac{R}{2} + \frac{R}{2^{n+1}}$ such that $\rho_0 = R$ and $\rho_n \downarrow \frac{R}{2}$ as $n \to \infty$.

$$\begin{aligned}\|u_+\|^{q_n}_L(B_{\rho_n}(x_0)) \leq & C(d)\left(\frac{\Lambda}{\lambda}\right)^{\frac{1}{q_{n-1}}}\left(\frac{1}{\frac{R}{2^n}-\frac{R}{2^{n+1}}}\right)^{\frac{2}{q_{n-1}}}\|u\|_{L^{q_{n-1}}(B_{\rho_{n-1}}(x_0))} \\ \leq & C(d)\left(\frac{\Lambda}{\lambda}\frac{4^{n+1}}{R^2}\right)^{\frac{1}{q_{n-1}}}\|u\|_{L^{q_{n-1}}(B_{\rho_{n-1}}(x_0))} \\ \leq & C(d)\left(\frac{\Lambda}{\lambda}\frac{4^{n+1}}{R^2}\right)^{\frac{1}{\tau^{n-1}q}}\|u\|_{L^{q_{n-1}}(B_{\rho_{n-1}}(x_0))} \\ \leq & \left(\prod_1^n C(d)\left(\frac{\Lambda}{\lambda}\frac{4^{n+1}}{R^2}\right)^{\frac{1}{\tau^{k-1}}}\right)^{\frac{1}{q}}\|u\|_{L^q(B_R(x_0))}\end{aligned}$$

We may check that the constant is bounded when $n \to \infty$. Therefore

$$
\begin{aligned}
\|u_+\|_{L^\infty(B_{\frac{R}{2}}(x_0))} &\leq \left[\left(\frac{\Lambda}{\lambda} \right)^{\frac{1}{1-\tau^{-1}}} C(d) \frac{1}{R^{2(1-\tau^{-1})}} \int_{B_R(x_0)} u^2 dx \right]^{\frac{1}{2}} \\
&\leq \left[\left(\frac{\Lambda}{\lambda} \right)^{\frac{d}{2}} C(d) \frac{1}{|B_R|} \int_{B_R(x_0)} u^2 dx \right]^{\frac{1}{2}}
\end{aligned}
$$

□

Exercise 6. *Suppose that there exists a constant C depending on m, d, ε and ρ such that, for any $u \in H^m(B_\rho)$, there holds*

$$\|u\|_{H^{m-1}}(B_\rho) \leq \varepsilon \|u\|_{H^m(B_\rho)} + C\|u\|_{L^2(B_\rho)}.$$

Show that the norm $N_{m,\rho}(u) := \|D^m u\|_{L^2(B_\rho)} + \|u\|_{L^2(B_\rho)}$ is equivalent to the canonical norm of $H^m(B_\rho)$. In the inequality

$$N_{m-1,\rho}(u) \leq \varepsilon N_{m,\rho}(u) + C\|u\|_{L^2(B_\rho)},$$

how does C depend on ρ?

Claim 1. *There exists a constant $C = C(\varepsilon, \rho, m, d) > 0$ such that for all $u \in H^m(B_\rho)$,*

$$\|u\|_{H^{m-1}}(B_\rho) \leq \varepsilon \|u\|_{H^m(B_\rho)} + C\|u\|_{L^2(B_\rho)}.$$

Proof. Assume that there exists $(u_n) \in H^m(B_\rho)$ such that $\|u_n\|_{H^m(B_\rho)} = 1$ and

$$\|u_n\|_{H^{m-1}(B_\rho)} > \varepsilon + n\|u_n\|_{L^2(B_\rho)} \geq n\|u_n\|_{L^2(B_\rho)}.$$

Since (u_n) is uniformly bounded in H^m, then $u_n \to u$ weakly in H^m for some u,and thus $u_n \to u$ in H^{m-1} by Rellich-Kondrachov Theorem. But $u_n \to 0$ in L^2, thus $u \equiv 0$, which contradicts to $\|u\|_{H^{m-1}} \geq \varepsilon$. □

Claim 2. *The norm $N_{m,\rho}(u) := \|D^m u\|_{L^2(B_\rho)} + \|u\|_{L^2(B_\rho)}$ is equivalent to $\| \cdot \|_{H^m(B_\rho)}$.*

Proof.

$$N_{m,\rho}^2 \leq 2(\|D^m u\|_{L^2}^2 + \|u\|_{L^2}^2) \leq 2\|u\|_{H^m}^2.$$

$$
\begin{aligned}
\|u\|_{H^m}^2 =&\|u\|_{H^{m-1}}^2 + \|D^m u\|_{L^2}^2 \\
\leq&(\varepsilon\|u\|_{H^m} + C\|u\|_{L^2})^2 + \|D^m u\|_{L^2}^2 \\
\leq&2\varepsilon\|u\|_{H^m}^2 + 2C\|u\|_{L^2}^2 + \|D^m u\|_{L^2}^2
\end{aligned}
$$

Thus

$$\|u\|_{H^m}^2 \leq \tilde{C} N_{m,\rho}(u)^2.$$

□

Claim 3. *The constant $C \sim O(\rho^{1-m})$.*

Proof. Note that the inequality implies that

$$\|D^{m-1}u\|_{L^2(B_\rho)} \leq \varepsilon\|D^m u\|_{L^2(B_\rho)} + (C(\varepsilon, \rho, m, d) + \varepsilon - 1)\|u\|_{L^2(B_\rho)}.$$

For $u \in H^m(B_1)$, look at $v := u(\frac{\cdot}{\rho}) \in H^m(B_\rho)$, then

$$\|D^{m-1}v\|_{L^2(B_\rho)} \leq \varepsilon\|D^m v\|_{L^2(B_\rho)} + (C(\varepsilon, \rho, m, d) + \varepsilon - 1)\|v\|_{L^2(B_\rho)}.$$

Rescaling gives

$$\rho^{1-m}\|D^{m-1}u\|_{L^2(B_1)} \leq \varepsilon\rho^{-m}\|D^m u\|_{L^2(B_1)} + (C(\varepsilon, \rho, m, d) + \varepsilon - 1)\|u\|_{L^2(B_1)}.$$

That is,

$$\|D^{m-1}u\|_{L^2(B_1)} \leq \varepsilon\rho^{-1}\|D^m u\|_{L^2(B_1)} + (C(\varepsilon, \rho, m, d) + \varepsilon - 1)\rho^{m-1}\|u\|_{L^2(B_1)}.$$

Thus $C(\varepsilon, m, d) \approx (C(\frac{\varepsilon}{\rho}, \rho, m, d) + \frac{\varepsilon}{\rho} - 1)\rho^{1-m} + 1 - \varepsilon$. □