University of Oxford

Elliptic PDEs -Problem Sheet Two

Aili Shao

Theorem 1. *(Stampacchia) Suppose $d \geq 3$, $A\xi \cdot \xi \geq \alpha\xi \cdot \xi$ and $|A| \leq M$. Suppose that $u \in H_0^1(\Omega)$ is the weak solution of*

$$div(A\nabla u) = f \ in \ \mathcal{D}'(\Omega)$$

for some $f \in L^p(\Omega)$, $p > d/2$, Then

$$\|u\|_{L^\infty(\Omega)} \leq C(\Omega, \alpha, d, p)\|f\|_{L^p(\Omega)}.$$

Remark 1. *The proof is unchanged if A depends on $u, \nabla u$ as well.*

Exercise 1. *Let $\phi\colon [0, \infty) \to [0, \infty)$ be non-increasing function such that, for some M,$\gamma > 0$ and $\delta > 1$ there holds*

$$\phi(y) \leq \frac{M\phi(x)^\delta}{|y - x|^\gamma} \ for \ all \ y > x > 0.$$

Show that

$$\phi(d) = 0,$$

where

$$d^\gamma = M\phi(0)^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}}.$$

Hint: consider $d_n = d(1 - 2^{-n})$, and show that $\phi(d_n) \leq \phi(0)2^{-\frac{n\gamma}{\delta-1}}$.

Proof.

- *Direct Method*

  Consider $d_n = d(1 - 2^{-n})$, then

  $$d_n - d_{n-1} = 2^{-n}d > 0,$$

  and

  $$|d_n - d_{n-1}|^\gamma = 2^{-n\gamma}d^\gamma = 2^{-n\gamma}M\phi(0)^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}}.$$

Then

$$
\begin{aligned}
\phi(d_n) \leq & \frac{M\phi(d_{n-1})^\delta}{2^{-n\gamma}M\phi(0)^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}}} \\
=& 2^{n\gamma}\cdot 2^{-\frac{\delta\gamma}{\delta-1}}\phi(0)^{1-\delta}\phi(d_{n-1})^\delta \\
\leq & 2^{n\gamma}\cdot 2^{-\frac{\delta\gamma}{\delta-1}}\phi(0)^{1-\delta}\left[2^{(n-1)\gamma}\cdot 2^{-\frac{\delta\gamma}{\delta-1}}\phi(0)^{1-\delta}\phi(d_{n-2})^\delta\right]^\delta \\
\leq & \cdots \text{ iteratively} \\
=& \phi(0)^\alpha 2^\beta
\end{aligned}
$$

where

$$\alpha = \sum_{k=0}^{n-1}\delta^k(1-\delta)+\delta^n = 1$$

and

$$\beta = \sum_{k=0}^{n-1}(n-k)\delta^k\gamma - \frac{\delta\gamma}{\delta-1}\cdot\delta^k.$$

Note that

$$
\begin{aligned}
\beta =& \frac{\delta\gamma}{\delta-1}\sum_{k=0}^{n-1}(\delta-1)(n-k)\delta^{k-1}-\delta^k \\
=& \frac{\delta\gamma}{\delta-1}\sum_{k=0}^{n-1}(n-k-1)\delta^k-(n-k)\delta^{k-1} \\
=& \frac{\delta\gamma}{\delta-1}\left[\sum_{k=1}^{n}(n-k)\delta^{k-1}-\sum_{k=0}^{n-1}(n-k)\delta^{k-1}\right] \\
=& \frac{\delta\gamma}{\delta-1}\left(-\frac{n}{\delta}\right) \\
=& -\frac{n\gamma}{\delta-1}.
\end{aligned}
$$

Thus

$$\phi(d_n) \leq \phi(0)2^{-\frac{\gamma n}{\gamma-1}} \text{ for all } n\in\mathbb{N}.$$

Since $d_n = d(1-2^{-n}) \to d$ almost surely as $n\to\infty$, we have

$$\phi(d) \leq \liminf_{n\to\infty}\phi(d_n) \leq \phi(0)\liminf_{n\to\infty}2^{-\frac{\gamma n}{\delta-1}} = 0$$

provided that $\phi(0)$ is bounded. Also $\phi \geq 0$ by definition, so $\phi(d) = 0$.

- *Proof by induction*

  Alternatively, we can achieve the same result by an induction argument:

For $n = 1$,

$$\phi(d_1) = \phi(\frac{d}{2}) \le \frac{M\phi(x)^\delta}{|\frac{d}{2} - x|^\gamma} \le \frac{M\phi(0)^\delta}{|\frac{d}{2} - x|^\gamma} \text{ for all } \frac{d}{2} > x > 0.$$

Take $x \downarrow 0$,

$$\phi(d_1) \le \frac{M\phi(0)^\delta}{d^\gamma 2^{-\gamma}} = \phi(0)2^{-\frac{\gamma}{\delta-1}}.$$

Now assume that the statement holds for some fixed $n \in \mathbb{N}$, we prove for $n+1$,

$$\phi(d_{n+1}) \le \frac{M\phi(d_n)^\delta}{|d_{n+1} - d_n|^\gamma} \le \frac{M\phi(0)^\delta 2^{-\frac{n\delta\gamma}{\delta-1}}}{d^\gamma 2^{-\gamma(n+1)}} = \phi(0)2^{-\frac{(n+1)\gamma}{\delta-1}}.$$

□

Exercise 2. *Let $G \in C^1(\mathbb{R})$ be such that $G(0) = 0$ and $|G'(s)| \le M$ for all $s \in \mathbb{R}$. Given $u \in W_0^{1,p}(\Omega)$, then check that*

$$G(u) \in W_0^{1,p}(\Omega) \text{ and } \partial_i(G(u)) = G'(u)\partial_i u \text{ a.e.}$$

Show that this is also true for the piecewise C^1 functions G_k given by with $G_k(x) = -k$ when $x \le -k$, $G_k(x) = k$ when $x \ge k$, and $G_k(x) = x$ otherwise.

Proof. Let $u \in W_0^{1,p}(\Omega)$, then we approximate u by $\{u_m\} \in C_c^\infty$. That is

$$u_m \to u \text{ in } W^{1,p}(\Omega).$$

We know that $G(u_m) \in C_c^1(\Omega)$ and

$$\nabla G(u_m) = G'(u_m)\nabla u_m.$$

Note that LHS $\nabla G(u_m) \to \nabla G(u)$ in $\mathcal{D}'(\Omega)$ as $G(u_m) \to G(u)$ strongly in $L^p(\Omega)$ as a result of the following inequality

$$\|G(u_m) - G(u)\|_{L^p(\Omega)} \le M\|u_m - u\|_{L^p(\Omega)}.$$

RHS $G'(u_m)\nabla u_m \to G'(u)\nabla u$ in $L^p(\Omega)$ since

$$\begin{aligned}\|G'(u_m)\nabla u_m - G'(u)\nabla u\|_{L^p(\Omega)} \le& \|G'(u_m)(\nabla u_m - \nabla u)\|_{L^p(\Omega)} + \|(G'(u_m) - G'(u))\nabla u\|_{L^p(\Omega)} \\ \le& M\|\nabla u_m - \nabla u\|_{L^p(\Omega)} \text{ (2nd term vanishes by DCT)} \\ \to& 0.\end{aligned}$$

Note that we can apply DCT to 2nd term since $u_m \to u$ in $L^p(\Omega)$ implies that there exists a subsequence $u_{m_k} \to u$ almost everywhere, and thus $G(u_{m_k}) \to G(u)$ almost everywhere.
Thus

$$\nabla G(u) = G'(u)\nabla u \text{ in } \mathcal{D}'(\Omega)$$

due to uniqueness of limit.
Now we need to check that $G(u) \in W_0^{1,p}$.

- $G(u) \in L^p(\Omega)$ since $\int_\Omega |G(u)|^p \leq M^p \int_\Omega |u|^p < \infty$.
- $G'(u)\nabla u \in L^p(\Omega)$.
- $tr(G(u)) = 0$ since $0 = tr(G(u_m)) \to tr(G(u))$ in $L^p(\partial\Omega)$.

Therefore, $G(u) \in W_0^{1,p}$ with $\nabla G(u) = G'(u)\nabla u$ a.e.
Now we proceed to the second part of the proof.

Lemma 1. *If $u \in W_0^{1,p}(\Omega)$, then $u_+ \in W_0^{1,p}(\Omega)$ and $\nabla u_+ = (\nabla u)\mathbb{1}_{\{u>0\}} = (\nabla u)\mathbb{1}_{\{u\geq 0\}}$.*

Proof. We approximate $u_+ := \max(u, 0)$ by $u_\varepsilon = \begin{cases} \sqrt{u^2+\varepsilon^2} - \varepsilon & \text{if } u > 0, \\ 0 & \text{if } u \leq 0. \end{cases}$

Then $u_\varepsilon(0) = 0$ and $u_\varepsilon \in C^1$ with $\|u'_\varepsilon\|_\infty < \infty$ since $u'_\varepsilon(u) = \begin{cases} \frac{u}{\sqrt{u^2+\varepsilon^2}} & \text{if } u > 0, \\ 0 & \text{if } u \leq 0. \end{cases}$

By first part of this exercise, we know that $u_\varepsilon(u) \in W_0^{1,p}(\Omega)$, with $\nabla u_\varepsilon(u) = u'_\varepsilon(u)\nabla u$. Then for all $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega u_\varepsilon(u)\nabla\varphi = -\int_\Omega u'_\varepsilon(u)\nabla u\varphi = -\int_\Omega \mathbb{1}_{\{u>0\}}\frac{u}{\sqrt{u^2+\varepsilon^2}}\nabla u\varphi.$$

$LHS = \int_\Omega u_\varepsilon(u)\nabla\varphi = \int_\Omega \mathbb{1}_{\{u>0\}}(\sqrt{u^2+\varepsilon^2} - \varepsilon)\nabla\varphi \to \int_\Omega \mathbb{1}_{\{u>0\}}u\nabla\varphi$ by DCT as the integrand is bounded by $\sqrt{u^2+1}|\nabla\varphi| \in L^1(\Omega)$.
Similarly, $RHS = -\int_\Omega \mathbb{1}_{\{u>0\}}\frac{u}{\sqrt{u^2+\varepsilon^2}}\nabla u\varphi \to -\int_\Omega \mathbb{1}_{\{u>0\}}\nabla u\varphi$ by DCT since the integrand $\mathbb{1}_{\{u>0\}}\frac{u}{\sqrt{u^2+\varepsilon^2}}\nabla u\varphi \leq |\nabla u||\varphi| \in L^1(\Omega)$.
Since $0 = tr(u_\varepsilon(u)) \to tr(u_+)$ in $L^p(\partial\Omega)$, $tr(u_+) = 0$. We can thus conclude that $u_+ \in W_0^{1,p}$ and $\nabla u_+ = \mathbb{1}_{\{u>0\}}(\nabla u)$. □

Corollary 1. *$G_k(u) \in W_0^{1,p}(\Omega)$, and $\nabla G_k(u) = (\nabla u)\mathbb{1}_{\{|u|<k\}}$.*

Proof. $G_k(u) = (u_+ - k)_- + (u_- + k)_+$. By the lemma above, we know that $u_+, u_- \in W_0^{1,p}(\Omega)$ with

$$\nabla u_+ = \mathbb{1}_{\{u>0\}}(\nabla u)$$

and

$$\nabla u_- = \mathbb{1}_{\{u<0\}}(\nabla u).$$

Note that $u_+ - k \in W^{1,p}(\Omega)$ with $tr(u_+ - k) = -k$ and

$$\nabla(u_+ - k) = \mathbb{1}_{\{u>0\}}(\nabla u)$$

Thus $(u_+ - k)_- \in W_0^{1,p}(\Omega)$ with

$$\nabla[(u_+ - k)_-] = \mathbb{1}_{\{u_+<k\}}(\nabla u_+).$$

Similarly, Thus $(u_- + k)_+ \in W_0^{1,p}(\Omega)$ with

$$\nabla[(u_- + k)_+] = \mathbb{1}_{\{u_->-k\}}(\nabla u_-).$$

Therefore, we have

$$\begin{aligned}\nabla G_k(u) =&\nabla u_+ \mathbb{1}_{\{u_+<k\}} + \nabla u_- \mathbb{1}_{\{u_->-k\}}\\ =&\nabla u \mathbb{1}_{\{0<u<k\}} + \nabla u \mathbb{1}_{\{0>u>-k\}}\\ =&\nabla u \mathbb{1}_{\{|u|\leq k\}}.\end{aligned}$$

$\square$

$\square$

Exercise 3. *Testing the equation against $G_1(u)$ with $G_1(x) = x - G_k(x)$, writing $2^\star = \frac{2d}{d-2}$ and $2_\star = \frac{2d}{d+2}$, show that if $A_k := \{x : |u(x)| \geq k\}$,*

$$\left(\int_{A_k} (G_1(u))^{2^\star}\right)^{\frac{1}{2^\star}} \leq \frac{C(p,d)}{\alpha} \left(\int_{A_k} |f|^{2_\star}\right)^{\frac{1}{2_\star}}.$$

Deduce that

$$|A_h| \leq \left(\frac{C(p,d)}{\alpha}\|f\|_{L^p(\Omega)}\right)^{2^\star} \frac{|A_k|^{\frac{2^\star}{2_\star}-\frac{2^\star}{p}}}{|h-k|^{2^\star}} \text{ for } h > k > 0$$

and conclude the proof of the Theorem.

Proof. Testing the equation with $G_1(u)$, we have

$$\int_\Omega A\nabla u \cdot \nabla(G_1(u)) = \int_\Omega f G_1(u).$$

$\nabla(G_1(u)) = \nabla(u - G_k(u)) = \nabla u - G_k'(u)\nabla u = \nabla u \mathbb{1}_{\{|u|\geq k\}}$ a.e. by Exercise 2. Consider $A_k := \{x : |u(x)| \geq k\}$,

$$\int_{A_k} A\nabla(G_1(u)) \cdot \nabla(G_1(u)) = \int_{A_k} f G_1(u).$$

By ellipticity of A and Hölder's inequality, we have

$$\alpha\|\nabla(G_1(u))\|^2_{L^2(A_k)} \leq \int_{A_k} f G_1(u) \leq \|f\|_{L^{2_\star}(A_k)}\|G_1(u)\|_{L^{2^\star}(A_k)}.$$

Gagliardo-Nirenberg inequality tells us that

$$\alpha\|G_1(u)\|^2_{L^{2^\star}(A_k)} \leq C(d,p)\alpha\|\nabla(G_1(u))\|^2_{L^2(A_k)} \leq C(d,p)\|f\|_{L^{2_\star}(A_k)}\|G_1(u)\|_{L^{2^\star}(A_k)}.$$

Thus,

$$\|G_1(u)\|_{L^{2^\star}(A_k)} \leq \frac{C(d,p)}{\alpha}\|f\|_{L^{2_\star}(A_k)}.$$

Note that

$$RHS = \frac{C(d,p)}{\alpha}\|f\|_{L^{2_\star}(A_k)} \leq \frac{C(d,p)}{\alpha}\|f\|_{L^p(A_k)}|A_k|^{\frac{1}{2_\star}-\frac{1}{p}}$$

by Hölder inequality.

$$LHS = \|G_1(u)\|_{L^{2^\star}(A_k)} \geq \|G_1(u)\|_{L^{2^\star}(A_h)} \geq \left(\int_{A_h} |h-k|^{2^\star}\right)^{\frac{1}{2^\star}} = |A_h|^{\frac{1}{2^\star}}|h-k|$$

for $h > k > 0$. Therefore

$$|A_h| \leq \left(\frac{C(p,d)}{\alpha}\|f\|_{L^p(\Omega)}\right)^{2^\star} \frac{|A_k|^{\frac{2^\star}{2_\star}-\frac{2^\star}{p}}}{|h-k|^{2^\star}} \text{ for } h > k > 0.$$

To conclude the Theorem using Exercise 1, we define a non-increasing function ϕ by $\phi(h) = |A_h|$. Take $\gamma = 2^\star$, $\delta = \frac{2^\star}{2_\star} - \frac{2^\star}{p} > 1$ as $p > \frac{d}{2}$. Take $M := \left(\frac{C(p,d)}{\alpha}\|f\|_{L^p(\Omega)}\right)^{2^\star}$, then $\phi(h) \leq \frac{M\phi(k)^\delta}{|h-k|^\gamma}$ for $h > k > 0$.
By Exercise 1, we know that $|A_l| = 0$ if $l^\gamma = M\phi(0)^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}} = M|\Omega|^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}}$. This implies that $\|u\|_{L^\infty(\Omega)} \leq l = (M|\Omega|^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}})^{\frac{1}{\gamma}} \leq C(\Omega,\alpha,d,p)\|f\|_{L^p(\Omega)}$. $\square$

Exercise 4. *Let $\theta \in (0,1)$, $A \geq 0$ be given. Show that there exists $\epsilon_0 > 0$ such that if*

$$\rho^m\|u\|_{H^m(B_{\theta\rho}(z))} \leq \epsilon_0\rho^m\|u\|_{H^m(B_\rho(z))} + A$$

for all $\rho \leq R$ and $B_\rho(z) \subset B_R(x_0)$, then

$$\|u\|_{H^m(B_{\theta R}(x_0))} \leq C\frac{A}{R^m},$$

for some constant C depending on θ, m and d.

Proof. Define $s := \sup\{\rho^m\|u\|_{H^m(B_{\theta\rho}(z))} \colon B_\rho(z) \subset B_R(x_0)\}$. Applying the inequality in the assumption, we have

$$(\theta\rho)^m\|u\|_{H^m(B_{\theta^2\rho}(z))} \leq \epsilon_0(\theta\rho)^m\|u\|_{H^m(B_{\theta\rho}(z))} + A \leq \epsilon_0\theta^m s + A \text{ for all } B_\rho(z) \subset B_R(x_0).$$

Fix $B_\rho(z) \subset B_R(x_0)$ and cover $B_{\theta\rho}(z)$ by n balls that are contained in $B_\rho(z)$,

$$\{B_{\theta^2(1-\theta)\rho}(y_1), B_{\theta^2(1-\theta)\rho}(y_2), \cdots, B_{\theta^2(1-\theta)\rho}(y_n)\}$$

such that $y_1, \cdots, y_n \in B_{\theta\rho}(z)$. Then

$$\begin{aligned}
\rho^m\|u\|_{H^m(B_{\rho\theta}(z))} \leq& \rho^m\sum_{i=1}^n \|u\|_{H^m(B_{\theta^2(1-\theta)\rho}(y_i))} \\
=& (\theta(1-\theta))^{-m}\sum_{i=1}^n(\theta(1-\theta)\rho)^m\|u\|_{H^m(B_{\theta^2(1-\theta)\rho}(y_i))} \\
\leq& (\theta(1-\theta))^{-m}n(\epsilon_0\theta^m s + A)
\end{aligned}$$

for all $B_\rho(z) \subset B_R(x_0)$. Take sup over $B_\rho(z) \subset B_R(x_0)$, we have

$$s \leq \tilde{C}(\epsilon_0 s + A)$$

where $\tilde{C} = \tilde{C}(\theta, n, m) = \tilde{C}(\theta, d, m)$. For $\epsilon_0 \leq \frac{1}{2}\tilde{C}^{-1}$, the s term on RHS can be absorbed to LHS, and thus

$$s \leq 2\tilde{C}A.$$

That is,

$$\rho^m \|u\|_{H^m(B_{\rho\theta}(z))} \leq 2\tilde{C}A$$

for all $B_\rho(z) \subset B_R(x_0)$. Take $B_\rho(z) = B_R(x_0)$ to see that

$$\|u\|_{H^m(B_{\theta R}(x_0))} \leq C\frac{A}{R^m},$$

where $C = C(\theta, m, d)$. □

Exercise 5. *Show that if $u \in H^m_{loc}(\Omega)$ satisfies*

$$\sum_{|\alpha|,|\beta| \leq m} D^\beta((-1)^{|\beta|} a_{\alpha,\beta} D^\alpha u) = \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta f_\beta \text{ in } \mathcal{D}'(\Omega),$$

with $f_\beta \in L^2_{loc}(\Omega)$, $a_{\alpha,\beta} \in L^\infty(\Omega)$, with $\sup_{\alpha,\beta,x} a_{\alpha,\beta}(x) \leq M$ a.e. in Ω, and there exists a constant $\lambda > 0$ such that

$$\sum_{\alpha,\beta |\alpha|=|\beta|=m} a_{\alpha,\beta}\zeta_\alpha\zeta_\beta \geq \lambda|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^d \text{ and a.e. } x \in \Omega,$$

for all balls $B_{\rho'} \subset B_\rho \subset \Omega$ the bound

$$\|u\|_{H^m(B_{\rho'})} \leq \epsilon\|u\|_{H^m(B_\rho)} + C(\frac{M}{\lambda}, \epsilon, \rho, \rho')\left(\|u\|_{L^2(B_\rho)} + \|\sum_{\beta \leq m} |f_\beta|\|_{L^2(B_\rho)}\right)$$

holds, then there is $\tilde{C}(\frac{M}{\lambda}, \epsilon)$ such that for each $\theta \in (0,1)$ and $\rho < 1$,

$$\|u\|_{H^m(B_{\theta\rho})} \leq \epsilon\|u\|_{H^m(B_\rho)} + \frac{1}{\rho^m}C(\frac{M}{\lambda}, \epsilon, \theta)\left(\|u\|_{L^2(B_\rho)} + \|\sum_{\beta \leq m} |f_\beta|\|_{L^2(B_\rho)}\right).$$

Proof. For $u \in H^m(\Omega)$ and $k \leq m$, $D^k u = (D^\alpha u)_{|\alpha|=k}$,and

$$\|D^k u\|_{L^2(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Define $|||u|||_{H^m(\Omega)} := \sum_{k=1}^m \|D^k u\|_{L^2(\Omega)}$.

Take $y = \rho x$, $u_\rho = u(y) = u(\rho x)$, then our new PDE is

$$\sum_{|\alpha|,|\beta|\le m} D^\beta((-1)^{|\beta|}\rho^{-|\beta|-|\alpha|}a_{\alpha,\beta}D^\alpha u_\rho) = \sum_{|\beta|\le m}(-1)^{|\beta|}\rho^{-|\beta|}D^\beta f_\beta$$

Multiplying ρ^{2m} on both sides, we get the same PDE with

$$\tilde{a}_{\alpha,\beta} = \rho^{2m-|\alpha|-|\beta|}a_{\alpha,\beta}, \tilde{f}_\beta = \rho^{2m-\beta}f_\beta \text{ and } \frac{\tilde{M}}{\tilde{\lambda}} = \frac{M}{\lambda}.$$

By assumption, we know that

$$\|u_\rho\|_{H^m(B_\theta)} \le \epsilon\|u_\rho\|_{H^m(B_1)} + C(\frac{M}{\lambda},\epsilon,\theta)\left(\|u_\rho\|_{L^2(B_1)} + \|\sum_{\beta\le m}|f_\beta|\|_{L^2(B_1)}\right).$$

Note that $\|u\|_{H^m(B_\rho)} \le |||u|||_{H^m(B_\rho)} := \sum_{k=1}^m \|D^k u\|_{L^2(\Omega)} \le \sqrt{m}\|u\|_{H^m(B_\rho)}$. $|||u|||_{H^m(B_\rho)}$ is an equivalent norm on $H^m(B_\rho)$ with bounds independent of ρ, so it suffices to prove the estimate with $|||\cdot|||_{H^m(B_\rho)}$ instead.
Now rescale it back to B_ρ,

$$\sum_{k=0}^m \rho^{k-\frac{d}{2}}\|D^k u\|_{L^2(B_{\rho\theta})} \le \epsilon\sum_{k=0}^m \rho^{k-\frac{d}{2}}\|D^k u\|_{L^2(B_\rho)} + \rho^{-\frac{d}{2}}C(\frac{M}{\lambda},\epsilon,\theta)\left(\|u\|_{L^2(B_\rho)} + \|\sum_{\beta\le m}|f_\beta|\|_{L^2(B_\rho)}\right).$$

Now multiply by $\rho^{-m+\frac{d}{2}}$ on both sides

$$\sum_{k=0}^m \rho^{k-m}\|D^k u\|_{L^2(B_{\rho\theta})} \le \epsilon\sum_{k=0}^m \rho^{k-m}\|D^k u\|_{L^2(B_\rho)} + \rho^{-m}C(\frac{M}{\lambda},\epsilon,\theta)\left(\|u\|_{L^2(B_\rho)} + \|\sum_{\beta\le m}|f_\beta|\|_{L^2(B_\rho)}\right).$$

Since $\rho \ge 1$, $|||u|||_{H^m(B_{\rho\theta})} \le \sum_{k=0}^m \rho^{k-m}\|D^k u\|_{L^2(B_{\rho\theta})}$ and

$$\sum_{k=0}^m \rho^{k-m}\|D^k u\|_{L^2(B_\rho)} \le \|D^m u\|_{L^2(B_\rho)} + C\rho^{-m}\|u\|_{L^2(B_\rho)}.$$

Thus

$$\begin{aligned}|||u|||_{H^m(B_{\rho\theta})} \le& \epsilon\|D^m u\|_{L^2(B_\rho)} + \rho^{-m}C(\frac{M}{\lambda},\epsilon,\theta)\left(\|u\|_{L^2(B_\rho)} + \|\sum_{\beta\le m}|f_\beta|\|_{L^2(B_\rho)}\right)\\ \le& \epsilon|||u|||_{H^m(B_\rho)} + \rho^{-m}C(\frac{M}{\lambda},\epsilon,\theta)\left(\|u\|_{L^2(B_\rho)} + \|\sum_{\beta\le m}|f_\beta|\|_{L^2(B_\rho)}\right).\end{aligned}$$

□