Paper Review: Regularity of Minimizers of Semilinear Elliptic Problems Up to Dimension 4 by Xavier Cabré

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1 Introduction

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function and F a primitive of f, i.e. $F' = f$. Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain.
Consider the semilinear PDE

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Its energy functional is

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)dx. \tag{1.2}$$

- A function $u \in C_0^1(\bar{\Omega})$ is a local minimizer of (1.2) if there exists $\varepsilon > 0$ such that

  $$E(u) \leq E(u + \xi)$$

  for every $\xi \in C_0^1(\bar{\Omega})$ such that $\|\xi\|_{C_1(\bar{\Omega})} \leq \varepsilon$.

- A classical solution $u \in C^2(\bar{\Omega})$ of (1.1) is *semi-stable* if

  $$Q_u[\xi] := \int_\Omega |\nabla \xi|^2 - f'(u)\xi^2 dx \geq 0 \;\; \forall\, \xi \in C_0^1(\bar{\Omega}) \tag{1.3}$$

Remark 1. • *By elliptic regularity, every local minimizer u is a C^∞ classical solution to (1.1).*

- *The semistability of a solution u is equivalent to the condition $\lambda_1 \geq 0$ where λ_1 is the first Dirichlet eigenvalue of the linearized operator $-\Delta - f'(u)$ as*

$$\lambda_1 = \frac{\int_\Omega |\nabla u|^2 - f'(u)u^2 dx}{\int_\Omega u^2}.$$

- *A local minimizer is always semi-stable. Note that is u is a minimizer, then*

$$\frac{d^2}{dt^2}\Big|_{t=0} E[u+tv] \geq 0.$$

$$\frac{d}{dt}E[u+tv] = \int_\Omega (\nabla u + t\nabla v)\nabla v) - f(u+tv)v$$
$$\frac{d^2}{dt^2}E[u+tv] = \int_\Omega |\nabla v|^2 - f'(u+tv)v^2$$

$$\frac{d^2}{dt^2}\Big|_{t=0} E[u+tv] = \int_\Omega |\nabla v|^2 - f'(u)v^2 \geq 0.$$

2 Main Estimate and Proof

Theorem 1. *Let f be any C^∞ function and $\Omega \subset \mathbb{R}^n$ be any smooth and bounded domain. Assume $2 \leq n \leq 4$. Let $u \in C_0^1(\bar{\Omega})$ with $u > 0$ in Ω be a local minimizer of (1.2) or more generally a positive classical semi-stable solution of (1.1). Then for every $t > 0$,*

$$\|u\|_{L^\infty(\Omega)} \leq t + \frac{C}{t}|\Omega|^{\frac{4-n}{2n}} \left(\int_{\{u<t\}} |\nabla u|^4 dx \right)^{\frac{1}{2}}, \tag{2.1}$$

where C is a universal constant (in particular, independent of f, Ω and u).

Key Ingredients Needed to Prove Theorem 1.

- *Sard Lemma* (proved in Aili's 3rd Year Extended Essay)

  Let Ω be an open subset of $\mathbb{R}^n$ and $u\colon \Omega \to \mathbb{R}^n$ be $C^1(\Omega)$, then the measure of the set of critical of u values is zero. In particular, the set of regular values of u is dense in $\mathbb{R}^n$.

- *Coarea Formula* (Proof involves Fubini Theorem)

  Let Ω be an open set in $\mathbb{R}^n$. and u is a real-valued Lipschitz function on Ω. Then for $g \in L^1(\Omega)$,

$$\int_\Omega g(x)|\nabla u(x)|dx = \int_0^T \int_{\Gamma_s} g(x) dV_s ds$$

  where $T := \max_\Omega u = \|u\|_{L^\infty(\Omega)}$, $\Gamma_s := \{x \in \Omega\colon u(x) = s\}$.

- *Simon Michael Sobolev Inequality*

  Let $M \subset \mathbb{R}^{m+1}$ be C^∞ immersed, m-dimensional compact hypersurface without boundary. Then for every $p \in [1,m)$, there exists a constant $C = C(m,p)$ such that for every C^∞ function $v\colon M \to \mathbb{R}$,

$$\left(\int_M |v|^{p^\star} dV\right)^{\frac{1}{p^\star}} \leq C(m,p)\left(\int_M |\nabla v|^p + |Hv|^p dV\right)^{\frac{1}{p}}, \tag{2.2}$$

  where H is the mean curvature of M and $p^\star = \frac{mp}{m-p}$.

- *Curvature Inequality*

  $|H| \leq |A|$ where H is the mean curvature of a surface M defined as $H := \frac{1}{n-1}\sum_1^{n-1} k_l$ while $A := (\sum_1^{n-1} k_l^2)^{1/2}$

- *Sternberg and Zumbrun Inequality* (proved in class)

  Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and u is a smooth, positive , semi-stable solution of (1.1). Then for every Lipschitz continuous function η in $\bar{\Omega}$ with $\eta \mid_{\partial\Omega} = 0$,

$$\int_{\Omega\cap\{|\nabla u|>0\}} \left(|\nabla_T |\nabla u||^2 + |A|^2|\nabla u|^2\right)\eta^2 dx \leq \int_\Omega |\nabla u|^2 |\nabla \eta|^2 dx, \tag{2.3}$$

  where ∇_T denotes the tangential or Riemannian gradient along a level set of u and A is defined as above.

Proof. Note that

$$Q_u = \int_\Omega |\nabla \xi|^2 - f'(u)\xi^2 dx \geq 0$$

holds for every Lipschitz function ξ in $\bar{\Omega}$ with $\xi \mid_{\partial\Omega} = 0$ as $C_0^1(\bar{\Omega})$ is dense in this space. Take $\xi = c\eta$ in the above inequality where c is a smooth function while η is Lipschitz continuous in $\bar{\Omega}$ and $\eta \mid_{\partial\Omega} = 0$.

$$\begin{aligned}
Q_u[c\eta] &= \int_\Omega |\nabla(c\cdot\eta)|^2 - f'(u)c^2\eta^2 dx \\
&= \int_\Omega |\nabla c\cdot\eta + c\cdot\nabla\eta|^2 - f'(u)c^2\eta^2 dx \\
&= \int_\Omega c^2|\nabla\eta|^2 + 2\int_\Omega \nabla c\cdot c\nabla\eta\cdot\eta + \int_\Omega \eta^2|\nabla c|^2 - \int_\Omega f'(u)c^2\eta^2 dx \\
&= \int_\Omega c^2|\nabla\eta|^2 + \cancel{\int_\Omega \nabla(\eta^2\nabla c\cdot c)} - \int_\Omega \eta^2\Delta c\cdot c - \int_\Omega f'(u)c^2\eta^2 dx \\
&= \int_\Omega c^2|\nabla\eta|^2 - (\Delta c + f'(u)c)c\eta^2 dx.
\end{aligned}$$

Thus the semi-stability condition gives

$$Q_u[c\eta] = \int_\Omega c^2|\nabla\eta|^2 - (\Delta c + f'(u)c)c\eta^2 dx \geq 0. \tag{2.4}$$

Take $c = \sqrt{|\nabla u|^2 + \varepsilon^2}$ for a give $\varepsilon > 0$. c is smooth.

$$\Delta u + f(u) = 0 \text{ in } \Omega.$$
$$\Delta u_j + f'(u)u_j = 0 \text{ in } \Omega.$$
$$c_j = \frac{1}{2}\frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}} \cdot 2\sum_{i=1}^n u_i u_{ij} = \frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}} \cdot \sum_{i=1}^n u_i u_{ij}.$$

$$\begin{aligned} c_j j =& \frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}} \cdot \sum_{i=1}^n u_{ij}^2. + \frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}} \cdot \sum_{i=1}^n u_i u_{ijj} \\ &+ \frac{1}{(\sqrt{|\nabla u|^2+\varepsilon^2})^3} \cdot (-\frac{1}{2}) \sum_{i=1}^n 2u_i u_{ij} \cdot \sum_{i=1}^n u_i u_{ij} \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^n c_{jj} =& \frac{1}{|\nabla u|^2+\varepsilon^2}\left[\sum_{i,j}^n u_{ij}^2\sqrt{|\nabla u|^2+\varepsilon^2} + \sum_{i=1}^n u_i \Delta u_i \sqrt{|\nabla u|^2+\varepsilon^2}\right] \\ &- \frac{1}{|\nabla u|^2+\varepsilon^2}\left[\sum_{j=1}^n (\sum_{i=1}^n u_{ij}u_i)^2)\frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right] \end{aligned}$$

That is,

$$\begin{aligned} \Delta c =& \frac{1}{|\nabla u|^2+\varepsilon^2}\left[-f'(u)|\nabla u|^2\sqrt{|\nabla u|^2+\varepsilon^2} + \sum_{i,j}^n u_{ij}^2\sqrt{|\nabla u|^2+\varepsilon^2}\right] \\ &- \frac{1}{|\nabla u|^2+\varepsilon^2}\left[(\sum_{i=1}^n u_{ij}u_i)^2)\frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right]. \end{aligned}$$

Thus,

$$\begin{aligned} \Delta c + f'(u)c =& f'(u)\sqrt{|\nabla u|^2+\varepsilon^2} - \frac{f'(u)|\nabla u|^2}{sqrt|\nabla u|^2+\varepsilon^2} \\ &+ \frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}}\left[\sum_{i,j}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right] \\ =& f'(u)\frac{\varepsilon^2}{\sqrt{|\nabla u|^2+\varepsilon^2}} + \frac{1}{\sqrt{|\nabla u|^2+\varepsilon^2}}\left[\sum_{ij}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right] \end{aligned}$$

Using the semi-stable inequality (2.4), we deduce

$$\begin{aligned}
\int_\Omega (|\nabla u|^2+\varepsilon^2)|\nabla\eta|^2 dx &= \int_\Omega c^2|\nabla\eta|^2 dx \\
&\geq \int_\Omega (\Delta c + f'(u)c)c\eta^2) dx \\
&= \int_\Omega f'(u)\varepsilon^2\eta^2 dx + \int_\Omega \left[\sum_{ij}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right]\eta^2 dx
\end{aligned}$$

The integrand in the last integral is non-negative, so we have

$$\begin{aligned}
&\int_\Omega \left[\sum_{ij}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right]\eta^2 dx \\
\geq &\int_{\Omega\cap\{|\nabla u|>0\}} \left[\sum_{ij}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right]\eta^2 dx \\
\geq &\int_{\Omega\cap\{|\nabla u|>0\}} \left[\sum_{ij}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{|\nabla u|}\right)^2\right]\eta^2 dx
\end{aligned}$$

Thus,

$$\int_\Omega (|\nabla u|^2+\varepsilon^2)|\nabla\eta|^2 dx \geq \int_\Omega f'(u)\varepsilon^2\eta^2 dx + \int_{\Omega\cap\{|\nabla u|>0\}} \left[\sum_{ij}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{|\nabla u|}\right)^2\right]\eta^2 dx$$

Letting $\varepsilon\to 0$, we have

$$\int_\Omega |\nabla u|^2|\nabla\eta|^2 dx \geq \int_{\Omega\cap\{|\nabla u|>0\}} \left[\sum_{ij}^n u_{ij}^2 - \sum_j^n\left(\sum_i^n u_{ij}\frac{u_i}{|\nabla u|}\right)^2\right]\eta^2 dx$$

It remains to show that

$$\sum_{ij}^n u_{ij}^2 - |\nabla|\nabla u||^2 = |\nabla_T|\nabla u||^2 + |A|^2|\nabla u|^2. \tag{2.5}$$

Proof of (2.5): Fix x_0 such that $|\nabla u(x_0)|\neq 0$. Define $\tau_n := \frac{\nabla u(x_0)}{|\nabla u(x_0)|}$ be the normal direction and $\tau_1,\tau_2,\cdots,\tau_{n-1}$ be the tangential directions.

$$
\begin{aligned}
\nabla u_j =&(\nabla u_j\cdot\tau_n)\tau_n+\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)\tau_i\\
=&(\nabla u_j\cdot\frac{\nabla u}{|\nabla u|})\tau_n+\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)\tau_i\\
=&\frac{1}{2}(|\nabla u|)_j\frac{1}{|\nabla u|}\tau_n+\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)\tau_i\\
=&(|\nabla u|)_j\tau_n+\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)\tau_i
\end{aligned}
$$

Thus $|\nabla u_j|^2=(|\nabla u|)_j^2+\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)^2$. This implies that

$$
\sum_{i,j=1}^{n}u_{ij}^2=\sum_{j=1}^{n}|\nabla u_j|^2=|\nabla|\nabla u||^2+\sum_{j=1}^{n}\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)^2.
$$

Now it remains to show that

$$
\sum_{j=1}^{n}\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)^2=|A|^2|\nabla u|^2+|\nabla_T|\nabla u||^2.
$$

Note that since $\nabla u=|\nabla u|\tau_n$, we have

$$
\nabla u_j\tau_i=\frac{\partial}{\partial x_j}(|\nabla u|\tau_n)\tau_i=|\nabla u|\tau_{n.j}\tau_i
$$

where $\tau_{n,j}=\frac{\partial}{\partial x_j}(\tau_n)$. Therefore,

$$
\begin{aligned}
\sum_{j=1}^{n}\sum_{i=1}^{n-1}(\nabla u_j\cdot\tau_i)^2=&|\nabla u|^2\sum_{j=1}^{n}\sum_{i=1}^{n-1}(\tau_{n,j}\tau_i)^2\\
=&|\nabla u|^2\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(\tau_{n,j}\tau_i)^2+|\nabla u|^2\sum_{i=1}^{n-1}(\tau_{n,n}\tau_i)^2\\
=&|\nabla u|^2\sum_{i,j=1}^{(}n-1)h_{ij}^2+|\nabla_T|\nabla u||^2\\
=&|A|^2|\nabla u|^2+|\nabla_T|\nabla u||^2.
\end{aligned}
$$

$\square$

- *Geometric Inequality for* Γ_s

$$|\Gamma_s|^{\frac{n-2}{n-1}} \le C(n) \int_{\Gamma_s} |H| dV_s$$

It follows from the *Simon Michael Sobolev Inequality* by taking $v \equiv 1$, $m = n-1 > 1 = p$, $M = \Gamma_s$. This inequality also holds if Γ_s is not connected.

- *Isoperimetric Inequality*

$$V(s) := |\{u > s\}| \le C(n) |\Gamma_s|^{\frac{n}{n-1}}$$

Now we proceed to the proof of Theorem 1.

Proof.

- *Step 1: Set Up.*

By elliptic regularity theory, $u \in C^\infty(\bar{\Omega})$. Recall that $u > 0$ in Ω. We define

$$T := \max_{\Omega} u = \|u\|_{L^\infty(\Omega)}.$$

For $s \in (0,T), \Gamma_s := \{x \in \Omega \colon u(x) = s\}$.

By Sard's Lemma, almost every $s \in (0,T)$ is a regular value of u. By definition, $|\nabla u(x)| > 0$ for all $x \in \Gamma_s$. In particular, if s is a regular value, Γ_s is a C^∞-immersed compact hypersurface of $\mathbb{R}^n$ without boundary. (Later we will apply the Simon Michael Sobolev inequality with $M = \Gamma_s$. Note that Γ_s could have a finite number of connected components, the Simon Michael Sobolev inequality still holds.)

- *Step 2: Apply Semi-stability Condition and Sternberg Zumbrun Inequality.*

Since u is a semi-stable solution, we can apply the Sternberg Zumbrun inequality. Take

$$\eta(x) = \varphi(u(x)) \text{ for } x \in \Omega$$

where φ is a Lipschitz function on $[0,T]$ with $\varphi(0) = 0$.

Now the RHS of the Sternberg Zumbrun Inequality becomes

$$\begin{aligned}\int_\Omega |\nabla u|^2 |\nabla \eta|^2 dx &= \int_\Omega |\nabla u|^4 \varphi'(u)^2 dx \\ &= \int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 dV_s \right) \varphi'(s)^2 ds \text{ By Coarea Formula}\end{aligned}$$

The integral in ds is over the regular values of u whose complement is of zero measure in $(0,T)$.

For the LHS of the inequality, we integrate ove $\Omega \cap \{|\nabla u| > \delta\}$ for a given $\delta > 0$, then the inequality still holds. That is,

$$\begin{aligned}\int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 dV_s\right) \varphi'(s)^2 ds &\geq \int_{\Omega\cap\{|\nabla u|>\delta\}} \left(|\nabla_T|\nabla u||^2 + |A|^2|\nabla u|^2\right) \varphi(u)^2 dx \\ &= \int_0^T \left(\int_{\Gamma_s\cap\{|\nabla u|>\delta\}} \frac{1}{|\nabla u|} \left(|\nabla_T|\nabla u||^2 + |A|^2|\nabla u|^2\right) dV_s\right) \varphi(s)^2 ds \\ &= \int_0^T \left(\int_{\Gamma_s\cap\{|\nabla u|>\delta\}} 4\left(|\nabla_T|\nabla u|^{1/2}|^2 + (|A||\nabla u|^{1/2})^2\right) dV_s\right) \varphi(s)^2 ds\end{aligned}$$

Letting $\delta \to 0$, by Monotone Convergence Theorem, we have

$$\int_0^T h_1(s)\varphi(s)^2 ds \leq \int_0^T h_2(s)\varphi'(s)^2 ds \tag{2.6}$$

for all Lipschitz functions $\varphi\colon [0,T] \to \mathbb{R}$ with $\varphi(0) = 0$ where

$$h_1(s) := \int_{\Gamma_s} 4\left(|\nabla_T|\nabla u|^{1/2}|^2 + (|A||\nabla u|^{1/2})^2\right) dV_s$$

and

$$h_2(s) := \int_{\Gamma_s} |\nabla u|^3 dV_s$$

for every regular value s of u.

- *Step 3: Apply the Sobolev Inequality*

In this step, we apply the Simon Michael Sobolev inequality to argue the reason for restricting $n \leq 4$. Take $M = \Gamma_s, p = 2 < m = n-1, v = |\nabla u|^{1/2}$, then

$$\left(\int_{\Gamma_s} |\nabla u|^{\frac{n-1}{n-3}} dV_s\right)^{\frac{n-1}{n-3}} \leq C(n) \int_{\Gamma_s} |\nabla_T|\nabla u|^{1/2}|^2 + (|H||\nabla u|^{1/2})^2 dV_s.$$

Then

$$\left(\int_{\Gamma_s} |\nabla u|^{\frac{n-1}{n-3}} dV_s\right)^{\frac{n-1}{n-3}} \leq c(n)h_1(s) \tag{2.7}$$

as $|H| \leq |A|$. Combining this with (2.6) , we have

$$\int_0^T \left(\int_{\Gamma_s} |\nabla u|^{\frac{n-1}{n-3}} dV_s\right)^{\frac{n-1}{n-3}} \varphi(s)^2 ds \leq C(n) \int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 dV_s\right) \varphi'(s)^2 ds \tag{2.8}$$

for all Lipschitz functions φ in $[0,T]$ with $\varphi(0) = 0$. So we need $\frac{n-1}{n-3} \geq 3$. That is $n \leq 4$.

Now we define

$$B_t := \frac{1}{t^2} \int_{\{u<t\}} |\nabla u|^4 dx = \frac{1}{t^2} \int_0^t h_2(s) ds$$

where the last equality follows from the Coarea Formula.

- *Step 4: Proof for $n = 4$*

  When $n = 4$, $\frac{n-1}{n-3} = 3$. Then (2.7) gives

$$h_2^{1/3} \leq Ch_1 \text{ a.e. in } (0, T) \tag{2.9}$$

where C is a universal constant. For every regular value s of u, we have $0 < h_2(s)$, $h_1(s) < \infty$, so $\frac{h_1}{h_2} \in (0, \infty)$ a.e. in $(0, T)$.

$$g_k(s) := \min\{k, \frac{h_1(s)}{h_2(s)}\}$$

for regular values s and for a postive integer k, we have that $g_k \in L^\infty(0, T)$ and $g_k(s) \to \frac{h_1(s)}{h_2(s)} \in (0, \infty)$ as $k \to \infty$. for a.e. $s \in (0, T)$. Since $g_k \in L^\infty(0, T)$,

$$\varphi_k(s) := \begin{cases} s/t & \text{if } s \leq t; \\ \exp\left(\frac{1}{\sqrt{2}} \int_t^s \sqrt{g_k(\tau)} d\tau\right) & \text{if } t \leq s \leq T \end{cases}$$

is well defined and Lipschitz continuous in $[0, T]$ with $\varphi_k(0) = 0$.

Since $h_2(\varphi_k')^2 = h_2 \frac{1}{2} g_k \varphi_k^2 \leq \frac{1}{2} h_1 \varphi_k^2$ in (t, T), by inequality (2.6) ($\varphi = \varphi_k$), we have

$$\begin{aligned} \int_t^T h_1 \varphi_k^2 ds \leq & \int_t^T h_2(\varphi_k')^2 ds + \int_0^t h_2(\varphi_k')^2 ds \\ \leq & \frac{1}{2} \int_t^T h_1 \varphi_k^2 ds + \frac{1}{t^2} \int_0^t h_2 ds \end{aligned}$$

Thus,

$$\begin{aligned} \int_t^T h_1 \varphi_k^2 ds \leq & \frac{2}{t^2} \int_0^t h_2 ds \\ = & \frac{2}{t^2} \int_{\{u<t\}} |\nabla u|^4 dx \\ = & 2B_t \end{aligned}$$

Note that we need to establish

$$T - t \leq CB_t^{1/2}.$$

$$
\begin{aligned}
T-t=&\int_t^T ds\\
=&\sup_{k\geq 1}\int_t^T \sqrt[4]{\frac{h_2}{h_1}g_k}ds\\
=&\int_t^T (\sqrt{h_1}\varphi_k)(\sqrt[4]{\frac{h_2 g_k}{h_1^3}\frac{1}{\varphi_k}})ds\\
\leq&(2B_t)^{1/2}\left[\int_t^T (\sqrt{\frac{h_2 g_k}{h_1^3}\frac{1}{\varphi_k^2}})ds\right]^{1/2}.\\
\leq&(2B_t)^{1/2}\left[C\int_t^T \sqrt{g_k}\frac{1}{\varphi_k^2}ds\right]^{1/2}
\end{aligned}
$$

since $h_2 \leq Ch_1^3$. Finally, we need to bound the integral on the RHS,

$$
\begin{aligned}
\int_t^T \sqrt{g_k}\frac{1}{\varphi_k^2}ds=&\int_t^T \sqrt{g_k}\frac{1}{\varphi_k^2}\frac{\varphi_k^{'}}{\frac{1}{\sqrt{2}}\sqrt{g_k}\varphi_k}ds\\
=&\sqrt{2}\int_t^T \frac{\varphi_k^{'}}{\varphi_k^3}ds\\
=&\frac{\sqrt{2}}{2}[\varphi_k^{-2}(s)]_{s=T}^{s=t}\\
\leq&\frac{\sqrt{2}}{2}\varphi_k^{-2}(t)\\
=&\frac{\sqrt{2}}{2}
\end{aligned}
$$

This implies that $\int_t^T(\sqrt{h_1}\varphi_k)(\sqrt[4]{\frac{h_2 g_k}{h_1^3}\frac{1}{\varphi_k}})ds \leq \sqrt{2}B_t^{1/2}\frac{\sqrt{2}}{2} = B_t^{1/2}$. That is

$$
T-t\leq B_t^{1/2}.
$$

Thus,

$$
\|u\|_{L^\infty(\Omega)} \leq t+\frac{1}{t}\left(\int_{\{u<t\}}|\nabla u|^4 dx\right)^{1/2}.
$$

This completes the proof for $n=4$.

- *Step 5: Proof for* $n=2,3$

  Now we consider a simple test function

$$
\varphi(s)=\begin{cases} s/t & \text{if } s\leq t;\\ 1 & \text{if } s>t.\end{cases}
$$

By definition of $h_1(s)$, we have $h_1(s) \geq \int_{\Gamma_s} |A|^2 |\nabla u| dV_s$. Inequality (2.6) leads to

$$\begin{aligned}
\int_t^T \int_{\Gamma_s} |A|^2 |\nabla u| dV_s ds \leq & \int_0^T h_1(s)\varphi(s)^2 ds \\
\leq & \int_0^T h_2(s)(\varphi'(s))^2 ds \\
= & \frac{1}{t^2} \int_0^t h_2(s) ds \\
= & \frac{1}{t^2} \int_{\{u<t\}} |\nabla u|^4 dx \\
= & B_t
\end{aligned}$$

This equality holds for ecery dimension n. It is at the end of the proof that we will need to assume $n \leq 3$. Now we use the geometric inequality for Γ_s,

$$|\Gamma_s|^{\frac{n-2}{n-1}} \leq C(n) \int_{\Gamma_s} |H| dV_s$$

and the *isoperimetric inequality*

$$V(s) := |\{u > s\}| \leq C(n) |\Gamma_s|^{\frac{n}{n-1}}$$

to deduce an inequality about $V(s)$.

$$\begin{aligned}
V(s)^{\frac{n-2}{n}} \leq & C(n) |\Gamma_s|^{\frac{n-2}{n-1}} \\
\leq & C(n) \int_{\Gamma_s} |H| dV_s \\
\leq & C(n) \left[\int_{\Gamma_s} |A|^2 |\nabla u| dV_s \right]^{1/2} \left[\int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \right]^{1/2}
\end{aligned}$$

for all regular values s by Cauchy Schwarz and since $|H| \leq |A|$. Then

$$\begin{aligned}
T - t = & \int_t^T ds \\
\leq & \int_t^T C(n) \left[\int_{\Gamma_s} |A|^2 |\nabla u| dV_s \right]^{1/2} \left[\int_{\Gamma_s} V(s)^{\frac{2(2-n)}{n}} \frac{dV_s}{|\nabla u|} \right]^{1/2} \\
\leq & C(n) \left[\int_t^T \int_{\Gamma_s} |A|^2 |\nabla u| dV_s \right]^{1/2} \left[\int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \right]^{1/2} \\
\leq & C(n) B_t^{1/2} \left[\int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \right]^{1/2}
\end{aligned}$$

Finally since $V(s) = |\{u > s\}|$ is non-increasing, it is differentiable a.e. by *Coarea Formula*, we have

$$-V(s) = \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \text{ for a.e. } s \in (0, T).$$

In addition, for $n \leq 3$. $V(s)$ is non-increasing in s and thus it is total variation satisfies

$$\begin{aligned}
|\Omega|^{\frac{4-n}{n}} \geq & V(t)^{\frac{4-n}{n}} \\
= & \left[V(s)^{\frac{4-n}{n}}\right]_{s=T}^{s=t} \\
\geq & \int_t^T \frac{4-n}{n} V(s)^{\frac{2(2-n)}{n}} (-V'(s)) ds \\
= & \frac{4-n}{n} \int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV}{|\nabla u|} ds
\end{aligned}$$

Thus,

$$T - t \leq C(n) B_t^{1/2} |\Omega|^{\frac{4-n}{2n}} \text{ for } n \leq 3.$$

Note that this argument gives nothing for $n \geq 4$ since the integral

$$\int_t^T V(s)^{\frac{2(2-n)}{n}} (-V'(s)) ds = \int_0^{V(t)} \frac{dr}{r^{\frac{2(n-2)}{n}}},$$

is not convergent at $s = T (r = 0)$ because $\frac{2(n-2)}{n} \geq 1$.

$\square$

3 Relevant Results and Applications

Theorem 2. *Let f be any C^∞ function and $\Omega \subset \mathbb{R}^n$ any C^∞ bounded domain. Assume that $2 \leq n \leq 4$ and that Ω is convex in the case $n \in \{3, 4\}$. Let $u \in L^1(\Omega)$ be a positive weak solution of (1.1) and suppose that u is the $L^1(\Omega)$ limit of a sequence of classical positive semistable solutions of (1.1). We then have the following:*

1. *If $f \geq 0$ in $[0, \infty)$, then $u \in L^\infty(\Omega)$.*
2. *Assume that $f(s) \geq c_1 > 0$ and $f(s) \geq \mu s - c_2$ for all $s \in [0, \infty)$ for some positive constants c_1 and c_2 and for $\mu > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$ in Ω. Then,*

   $$\|u\|_{L^\infty(\Omega)} \leq C(\Omega, \mu, c_1, c_2, \|f\|_{L^\infty([0, \bar{C}(\Omega, \mu, c_2)])}),$$

   where $C(\cdot)$ and $\bar{C}(\cdot)$ are constants depending only on the quantities within the parentheses.

Before we proceed to the proof of Theorem 2, we prove the following propositions and lemma first:

Proposition 1. *Let f be any C^∞ function. Let $\Omega \subset \mathbb{R}^n$ be any smooth bounded domain. Assume that $2 \leq n \leq 4$. Let u be a classical semi-stable solution of (1.1). Assume that*

$$u \geq c_3 \mathrm{dist}(\cdot, \partial\Omega) \text{ in } \Omega \tag{3.1}$$

and

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \leq c_4 \text{ where } \Omega_\varepsilon = \{x \in \Omega\colon : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}, \tag{3.2}$$

for some positve constants ε, c_3 and c_4. Then,

$$\|u\|_{L^\infty(\Omega)} \leq C(\Omega, \varepsilon, c_3, c_4, \|f\|_{L^\infty([0,c_4])}), \tag{3.3}$$

where $C(\dot{})$ is a constant depending only on the quantities within the parentheses.

Proof. By taking ε smaller if necessary, we may assume that

$$\Omega_\delta = \{x \in \Omega\colon \mathrm{dist}(x, \partial\Omega) < \delta\}$$

is smooth for every $0 < \delta \leq \varepsilon$. We use Theorem 1 with the choice

$$t = c_3 \frac{\varepsilon}{2}.$$

Note that if $x \in \{u < t\}$, by (3.1), we have

$$c_3 \mathrm{dist}(x, \partial\Omega) < t = c_3 \frac{\varepsilon}{2}.$$

This implies

$$\mathrm{dist}(x, \partial\Omega) < \frac{\varepsilon}{2}.$$

Thus $\{u < t\} \subset \Omega_{\varepsilon/2}$. Now it suffices to bound $\|u\|_{W^{1,4}\left(\Omega_{\varepsilon/2}\right)}$.
u is a solution of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega. \end{cases}.$$

By (3.2), $\|u\|_{L^\infty(\Omega_\varepsilon)} \leq c_4$ and thus the RHS of the PDE satisfies

$$\|f(u)\|_{L^\infty(\Omega_\varepsilon)} \leq \|f\|_{L^\infty([0,c_4])}.$$

Note that $f \in L^\infty(\Omega_\varepsilon) \subset L^4(\Omega_{\varepsilon/2})$, so by elliptic regularity, $\|u\|_{W^{2,4}(\Omega_{\varepsilon/2})} < \infty$. We can thus conclude that $\|u\|_{W^{1,4}(\Omega_{\varepsilon/2})}$ is bounded. $\square$

Note that the L^∞ bound in (3.2) holds every Lipschitz nonlinear function f when Ω is a convex domain (for $n \geq 2$). The precise statement is the following:

Proposition 2. *Let f be any locally Lipschitz function and let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let u be any positive classical solution of (1.1).*
If Ω is convex, then there exist positive constants ρ and γ depending only on the domain Ω such that for every Ω with $\operatorname{dist}(x, \partial\Omega) < \rho$, there exists a set $I_x \subset \Omega$ with the following properties:

$$|I_x| \geq \gamma \text{ and } u(x) \leq u(y) \text{ for all } y \in I_x. \tag{3.4}$$

As a consequence,

$$\|u\|_{L^\infty(\Omega_\rho)} \leq \frac{1}{\gamma}\|u\|_{L^1(\Omega)} \text{ where } \Omega_\rho = \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) < \rho\}. \tag{3.5}$$

If Ω is not convex but we assume that $n = 2$ and $f \geq 0$, then (3.5) also holds for some constants ρ and γ depending only on Ω.

Proof. Use the Method of Moving Plane □

Lemma 1. *If u is solution to*

$$\begin{cases} -\Delta u = f(u) \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases},$$

then

$$\frac{u}{\delta} \geq c\|f(u)\|_{L^1_\delta(\Omega)} \text{ in } \Omega \tag{3.6}$$

for some positive constants c and ρ depending only on Ω. Note that

$$\|f(u)\|_{L^1_\delta(\Omega)} = \int_\Omega f(u)\delta dx.$$

Proof.

- *Step 1*: For any compact set $K \subset \Omega$, we show that

  $$u(x) \geq c\int_\Omega f(u)\delta \text{ for all } x \in K \tag{3.7}$$

  where c is a positive constant depending only on K and Ω.
  To prove (3.7), we first define $\rho := \frac{\operatorname{dist}(K, \partial\Omega)}{2}$ and then take n balls of radius ρ such that

  $$K \subset B_\rho(x_1) \cup \cdots \cup B_\rho(x_m) \subset \Omega.$$

  This is possible by compactness of K. Let $\xi_1, \xi_2, \cdots, \xi_n$ be the solution of

  $$\begin{cases} -\Delta \xi_i = \chi_{B_\rho(x_i)} & \text{in } \Omega, \\ \xi_i = 0 & \text{on } \partial\Omega \end{cases}$$

  where χ_A denotes the characteristic function of A. The Hopf boundary lemma implies that there exist $c > 0$ such that

  $$\xi_i(x) \geq c\delta(x) \text{ for all } x \in \Omega, 1 \leq i \leq m.$$

Here and in the rest of the proof, c denotes various constants depending only on K and Ω.

Let $x \in K$, take a ball $B_\rho(x_i)$ containing x , then

$$B_\rho(x_i) \subset B_{2\rho}(x) \subset \Omega.$$

$$\begin{aligned}
u(x) \geq & \frac{1}{|B_{2\rho}(x)|} \int_{|B_{2\rho}(x)|} u(x) \text{ (By Mean Value Formula)} \\
= & c \int_{B_{2\rho}(x)} u(x) \\
\geq & c \int_{B_\rho(x_i)} u(x) \\
= & c \int_\Omega u(-\Delta \xi_i) \\
= & c \int_\Omega f(u)\xi_i \\
\geq & c \int_\Omega f(u)\delta
\end{aligned}$$

- *Step 2:* Fix a smooth compact set $K \subset \Omega$, by (3.7),

$$u \geq c \int_\Omega f(u)\delta \text{ in } K$$

so that it suffices to prove (3.7) for $x \in \Omega \setminus K$.

Let w be solution of

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \setminus K \\ w = 0 & \text{on } \partial\Omega \\ w = 1 & \text{on } \partial K \end{cases}$$

Then Hopf lemma implies that $w(x) \geq c\delta(x)$ for all $x \in \Omega \setminus K$.

u is a superharmonic and $u((x) \geq c \left(\int_\Omega f\delta\right) w(x) \geq c \left(\int_\Omega f\delta\right) \delta(x)$ for $x \in \Omega \setminus K$. This completes the proof.

$\square$

Now we are ready to prove Theorem 2:

Proof. Assume $f \geq 0$ and Ω is convex in the case $n \in \{3, 4\}$. Let u_k be a sequence of classical positive semistable solutions of (1.1) converging to u in $L^1(\Omega)$.
For $x \in \Omega$ and $v \colon \Omega \to \mathbb{R}$, define

$$\delta(x) := \operatorname{dist}(x, \partial\Omega) \text{ and } \|v\|_{L^1_\delta(\Omega} = \|v\delta\|_{L^1(\Omega)}.$$

By Proposition 2,

$$\|u_k\|_{L^\infty(\Omega_\rho)} \leq \frac{1}{\gamma}\|u_k\|_{L^1(\Omega)} \to \frac{1}{\gamma}\|u\|_{L^1(\Omega}, \tag{3.8}$$

as $k \to \infty$ where ρ and γ are positive constants depending only on Ω.
By Lemma 1, we know that

$$\frac{u_k}{\delta} \geq c\|f(u_k)\|_{L^1_\delta(\Omega)} \text{ in } \Omega \tag{3.9}$$

for some positive c depending only on Ω.
Multiply (1.1) (with u replaced by u_k) by the first Dirichlet eigenfucntion of $-\Delta$ in Ω and integrate twice by parts:

$$\int -\Delta u_k\phi = \int f(u_k)\phi$$

$$\int -u_k\Delta\phi = \int f(u_k)\phi$$

$$\int -\lambda_1 u_k\phi = \int f(u_k)\phi$$

We deduce that $\|u_k\|_{L^1_\delta(\Omega)}$ and $\|f(u_k)\|_{L^1_\delta(\Omega)}$ are comparable up to multiplicative constants depending only on Ω.
Similarly, by multiplying (1.1) by the solution w of

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

$$\int u_k = \int f(u_k)w \tag{3.10}$$

we also deduce that $\|u_k\|_{L^1(\Omega)}$ is comparable to $\|u_k\|_{L^1_\delta(\Omega)}$ and $\|f(u_k)\|_{L^1_\delta(\Omega)}$.
Recall that $\|u_k\|_{L^1(\Omega)} \to \|u\|_{L^1(\Omega)} > 0$. The RHS of (3.9) is bounded below by a positive constant indepedent of k. As a consequence of this lower bound and of (3.8), Proposition 1 gives a uniform $L^\infty(\Omega)$ estimated for all u_k. Letting $k \to \infty, u \in L^\infty(\Omega)$. Part (i) of Theorem 2 is thus proved.
To prove part(ii), we simply take more precise the constants in (3.8) and (3.9). Since we now assume $f \geq c_1 > 0$, from (3.10) $u_k \geq c_1 w \geq c_1 c\delta = c_1 c\mathrm{dist}(\cdot, \partial\Omega)$
Finally, multiply (1.1) (u_k for u) by the first Dirichlet eigenfunction $-\Delta$ in Ω and integrate twice by parts. Using the fact taht $f(s) \geq \mu s - c_2$ for all $s, \mu > \lambda_1$,

$$\int -\Delta u_k\phi = \int f(u_k)\phi$$

$$\int -u_k\Delta\phi = \int f(u_k)\phi$$

$$\int -\lambda_1 u_k\phi = \int f(u_k)\phi$$

$$\int -\lambda_1 u_k \phi \geq \int (\mu u_k - c_2)\phi$$

$$\int (\mu - \lambda_1) u_k \phi \leq \int c_2 \phi$$

This shows that $\|u_k\|_{L^1_\delta(\Omega)} \leq \bar{C}(\Omega, \mu, c_2)$ and also for $\|u_k\|_{L^1(\Omega)}$. By (3.8),

$$\|u\|_{L^\infty(\Omega_\rho} \leq \frac{1}{\gamma}\bar{C}(\Omega, \mu, c_2)$$

. Then the result of (ii) of Proposition 2 gives the desired result for Theorem 2. □

The main application of Theorem 1 is the following PDE:

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is smooth bounded domain, $n \geq 2, \lambda \geq 0$ and the nonlinearity $g\colon [-, \infty) \to \mathbb{R}$ satisfies

$$g \in C^1, \text{ nondecreasing } g(0).0, \text{and } \lim_{u \to \infty} \frac{g(u)}{u} = \infty. \tag{3.11}$$

Theorem 3. *Let g satisfy (3.11) and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Assume that $2 \leq n \leq 4$ and Ω is convex in the case $n \in \{3, 4\}$. Let $u^\star$ be the extremal solution of the above problem, then $u^\star \in L^\infty(\Omega)$.*

Proof.

1. *Step 1:*

   We extend g in C^1 manner to all of $\mathbb{R}$ with g non-decreasing and $g \geq g(0)/2$ in $\mathbb{R}$. Recall that the extremal solution $u^\star$ in the increasing L^1 limit as $\lambda \to \lambda^\star$, of the minimal solutions u_λ of the eigenvalue problem. In addition, for $\lambda < \lambda^\star$, u_λ is C^2-semistable solution of the eigenvalue problem.

2. *Step 2:*

   If g is C^∞, we simply apply part (ii) of Theorem 2 with $f = \lambda g$ for $\lambda^\star/2 < \lambda^\star$. Using that g satisfies (3.11), and $f = \lambda g$, we know that

   $$f(s) \geq \frac{\lambda g(0)}{2} = c_1 > 0 \text{ and } f(s) = \lambda g(s) \geq \mu s - c_2.$$

   By Theorem 2, $\|u_\lambda\|_{L^\infty(\Omega)}$ are uniformly bounded in λ. Letting, $\lambda \to \lambda^\star$, $u^\star \in L^\infty(\Omega)$.

3. *Step 3:* If $g \in C^1$ but not C^∞, we use mollifier. Let ρ_k be a C^∞ mollifier with support in $(0, 1/k)$ of the form

   $$\rho_k(\beta) = k\rho(k\beta).$$

We replace g by

$$g_k(s) = \int_{s-1/k}^{s} g(\tau)\rho_k(s-\tau)d\tau = \int_0^1 g(s-\beta/k)\rho(\beta)d\beta.$$

For all k, we have $g_k \leq g_{k+1} \leq g$ in $\mathbb{R}$. g_k is C^∞, nondecreasing, and satisfies (3.11).

Since $g(u^\star) \geq g_k(u^\star)$, $u^\star$ is a super -solution to

$$\begin{cases} -\Delta u = \lambda g_k(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the monotone iteration procedure, the extremal parameter for g_k, $\lambda_k^\star$ satisfies $\lambda^\star \leq \lambda_k^\star$. Hence $u^k_{\lambda^\star-1/k}$ is a classical solution to

$$-\Delta u = (\lambda^\star - 1/k)g_k.$$

Thus, we can apply Theorem 1.2 with $f = \lambda g_k$ and $\lambda = \lambda^\star - 1/k$ to obtain an $L^\infty(\Omega)$ bound for $u^k_{\lambda^\star-1/k}$ independent of k. Note that $u^k_{\lambda^\star-1/k} \leq u^k_{\lambda^\star-1/(k+1)}$ and that , since $g_l \leq g_{k+1} \leq g$, $u^k_{\lambda^\star-1/(k+1)} \leq u^{k+1}_{\lambda^\star-1/(k+1)} \leq u_{\lambda^\star} = u^\star$. Thus, $u^k_{\lambda^\star-1/k}$ increases in $L^1(\Omega)$ towards a solution of $-\Delta u = \lambda^\star g(u)$ smaller or equal to $u^\star$, and hence identically $u^\star$. From the $L^\infty(\Omega)$ bound for $u^k_{\lambda^\star-1/k}$ independent of k, we conclude that $u^\star \in L^\infty(\Omega)$.

□

4 Open Problems and Future Work

1. Theorem 3 for nonconvex domains
2. The boundedness of $u^\star$ in the dimensions $5 \leq n \leq 9$