EXERCISES

1. Problem Sheet One

Exercise 1.1. Suppose that $u \in H^1(B_\rho(x_0))$ is a weak solution of

$$-\operatorname{div}(A(x,u)\nabla u) = 0 \text{ in } B_\rho(x_0),$$

where A is a symmetric matrix valued map for which there exists $0 < \lambda < \Lambda < \infty$ such that for a.e. $x \in B_\rho(x_0)$ and a.e. $y \in \mathbb{R}$ and any $\zeta \in \mathbb{R}^d$

$$\lambda\zeta \cdot \zeta \le A_{ij}(x,y)\zeta_i\zeta_j \le \Lambda\zeta \cdot \zeta. \tag{1.1}$$

Show that for any $0 < \rho' < \rho$

$$\int_{B_{\rho'}(x_0)} |\nabla u|^2 \le \frac{\Lambda}{\lambda}\frac{1}{(\rho' - \rho)^2}\int_{B_\rho(x_0)} u^2.$$

Exercise 1.2. Let $w_n \in H^1(B_\rho(x_0))$ be a sequence a weak solutions of

$$-\operatorname{div}(A(x)\nabla u_n + b(x)u_n) + c(x) \cdot \nabla u_n + d(x)u_n = f_n \text{ in } B_\rho(x_0),$$

where a.e. $x \in B_\rho(x_0)$ and a.e. $y \in \mathbb{R}$ and any $\zeta \in \mathbb{R}^d$

$$\lambda\zeta \cdot \zeta \le A_{ij}(x,y)\zeta_i\zeta_j \le \Lambda\zeta \cdot \zeta, \text{ and } |b(x)| + |c(x)| + |d(x)| \le M \tag{1.2}$$

for some positive constants λ, Λ and M . Assume furthermore that $f_n \to f \in L^2(B_\rho(x_0))$, and for all n ,

$$\int_{B_\rho} u_n^2 \le C \text{ for all } n \in \mathbb{N}.$$

Given $0 < \rho' < \rho$, show that there exists a subsequence u_m such that

$$u_m \to u \text{ in } H^1(B_{\rho'}),$$

where u satisfies

$$-\operatorname{div}(A(x)\nabla u) + b(x) \cdot \nabla u + c(x)u = f \text{ in } B_\rho.$$

Hint: Start by proving the appropriate Caccioppoli inequality.

Exercise 1.3. Suppose $d \ge 3$. Given $u \in H^1(\Omega)$ a weak solution of

$$\operatorname{div}(A\nabla u) + b \cdot \nabla u = f,$$

with $A, b, c \in L^\infty(\Omega)$, $A\xi \cdot \xi \ge \lambda\xi \cdot \xi$ and $|A(x)|_\infty + |b(x)|_\infty \le M$ a.e. in Ω , and $f \in L^d(\Omega)$, show that for all $B_\rho \subset \Omega$ with $|B_\rho| \le 1$,

$$\int_{B_\rho} \left|\nabla\left(|u|^{\frac{p+1}{2}}\eta\right)\right|^2 \le C(p+1)^2\left(\int_{B_\rho}\left(|\nabla\eta|^2 + 1\right)|u|^{p+1} + \|f_-\|_{L^d(B_\rho)}^{p+1}\right),$$

with a constant C depending on λ, M and d only.

Exercise 1.4. Let Φ be a convex and locally Lipschitz continuous function on some interval I . Suppose $u \in H^1(\Omega)$ takes its values in I .

- Assume that $\Phi' \geq 0$. Suppose that for all $v \in H_0^1(\Omega),$ with $v \geq 0$
$$\int_\Omega A\nabla u \cdot \nabla v \leq 0,$$
(which we will refer to as a *subsolution*). Assuming that $\Phi(u) \in H^1(\Omega)$, show that for all $v \in H_0^1(\Omega),$ with $v \geq 0$
$$\int_\Omega A\nabla \Phi(u) \cdot \nabla v \leq 0.$$
- Assume that $\Phi' \leq 0$. Suppose that for all $v \in H_0^1(\Omega),$ with $v \geq 0$
$$\int_\Omega A\nabla \cdot \nabla v \geq 0,$$
(which we will refer to as a *supersolution*). Assuming that $\Phi(u) \in H^1(\Omega)$, show that for all $v \in H_0^1(\Omega),$ with $v \geq 0$
$$\int_\Omega A\nabla \Phi(u) \cdot \nabla v \leq 0.$$
- Thus show that if u is a subsolution, then $u^+ = \max(0, u)$ is also a subsolution.

Exercise 1.5. Check that if $u \in H^1(\Omega)$ is a weak sub-solution, that is,
$$\int_\Omega A\nabla u \cdot \nabla \psi \leq 0 \text{ for all } \psi \geq 0 \text{ s.t. } \psi \in H_0^1(\Omega),$$
then
$$u \leq \left(\frac{\Lambda}{\lambda}\right)^{d/2} C(d) \left(\frac{1}{|B_R|} \int_{B_R(x_0)} u^2 dx\right)^{\frac{1}{2}}.$$

Exercise 1.6. Suppose that there exists a constant C depending on m , d , ϵ and ρ such that, for any $u \in H^m(B_\rho)$, there holds
$$\|u\|_{H^{m-1}(B_\rho)} \leq \epsilon \|u\|_{H^m(B_\rho)} + C\|u\|_{L^2(B_\rho)}.$$

Show that the norm $N_{m,\rho}(u) := \|D^m u\|_{L^2(B_\rho)} + \|u\|_{L^2(B_\rho)}$ is equivalent to the canonical norm of $H^m(B_\rho)$.

In the inequality
$$N_{m-1,\rho}(u) \leq \epsilon N_{m,\rho}(u) + C\|u\|_{L^2(B_\rho)},$$
How does C depend on ρ ?

2. Problem Sheet Two

We are going to give an alternative proof of the L^∞ regularity result, using Stampacchia's method. The main advantage of this method is that its estimates do not involve upper bounds on A .

Theorem (Stampacchia)**.** *Suppose* $d \geq 3$, $A\xi \cdot \xi \geq \alpha\xi \cdot \xi$ *and* $|A| \leq M$. *Suppose that* $u \in H_0^1(\Omega)$ *is the weak solution of*

$$div\,(A\nabla u) = f \text{ in } \mathcal{D}'(\Omega)$$

for some $f \in L^p(\Omega)$, $p > d/2$. *Then,*

$$\|u\|_{L^\infty(\Omega)} \leq C(\Omega, \alpha, d, p)\|f\|_{L^p(\Omega)}.$$

Remark. The proof is unchanged if A depends on $u, \nabla u$ as well.

Exercise 2.1. Let $\phi : [0, \infty) \to [0, \infty)$ be a non-increasing function such that, for some $M, \gamma > 0$ and $\delta > 1$ there holds

$$\phi(y) \leq \frac{M\phi(x)^\delta}{|y - x|^\gamma} \text{ for all } y > x > 0.$$

Show that

$$\phi(d) = 0,$$

where

$$d^\gamma = M\phi(0)^{\delta-1}2^{\frac{\delta\gamma}{\delta-1}}.$$

Hint: consider $d_n = d(1 - 2^{-n})$, and show that $\phi(d_n) \leq \phi(0)2^{-\frac{n\gamma}{\delta-1}}$.

Exercise 2.2. Let $G \in C^1(\mathbb{R})$ be such that $G(0) = 0$ and $|G'(s)| \leq M$ for all $s \in \mathbb{R}$.

Given $u \in W_0^{1,p}(\Omega)$, then check that

$$G \circ u \in W_0^{1,p}(\Omega) \text{ and } \partial_i\,(G \circ u) = (G' \circ u)\,\partial_i u \text{ a.e.}$$

Show that this is also true for the piecewise C^1 functions G_k given by with $G_k(x) = -k$ when $x \leq -k$, $G_k(x) = k$ when $x \geq k$, and $G_k(x) = x$ otherwise.

Exercise 2.3. Testing the equation against $G_1(u)$ with $G_1(x) = x - G_k(x)$, writing $2^* = \frac{2d}{d-2}$ and $2_* = \frac{2d}{d+2}$, show that, if $A_k := \{x : |u(x)| \geq k\}$,

$$\left(\int_{A_k} (G_k(u))^{2^*}\right)^{\frac{1}{2^*}} \leq \frac{C(p,d)}{\alpha}\left(\int_{A_k} |f|^{2_*}\right)^{\frac{1}{2_*}}.$$

Deduce that

$$|A_h| \leq \left(\frac{C(p,d)}{\alpha}\|f\|_{L^p(\Omega)}\right)^{2*} \frac{|A_k|^{\frac{2^*}{2_*} - \frac{2^*}{p}}}{|h-k|^{2^*}} \text{ for } h > k > 0$$

and conclude the proof of the Theorem.

Exercise 2.4. Let $\theta \in (0,1)$, $A \geq 0$ be given. Show that there exists $\epsilon_0 > 0$ such that if

$$\rho^m \|u\|_{H^m(B_{\theta\rho}(z))} \leq \epsilon_0 \rho^m \|u\|_{H^m(B_\rho(z))} + A$$

for all $\rho \leq R$ and $B_\rho(z) \subset B_R(x_0)$, then

$$\|u\|_{H^m(B_{\theta R}(x_0))} \leq C \frac{A}{R^m},$$

for some constant C depending on θ, R , m and d .

Exercise 2.5. Show that if $u \in H^m_{\text{loc}}(\Omega)$ satisfies

$$\sum_{|\alpha|,|\beta| \leq m} D^\beta \left((-1)^{|\beta|} a_{\alpha\beta} D^\alpha u \right) = \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta f_\beta \text{ in } D'(\Omega),$$

with $f_\beta \in L^2_{\text{loc}}(\Omega)$, $a_{\alpha\beta} \in L^\infty(\Omega)$, with $\sup_{\alpha\beta,x} a_{\alpha,\beta}(x) \leq M$ a.e. in Ω, and there exists a constant $\lambda > 0$ such that

$$\sum_{\substack{\alpha,\beta \\ |\alpha|=|\beta|=m}} a_{\alpha\beta} \zeta_\alpha \zeta_\beta \geq \lambda |\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^d \text{ and a.e.} x \in \Omega,$$

for all balls $B_{\rho'} \subset B_\rho \subset \Omega$ the bound

$$\|u\|_{H^m\left(B_{\rho'}\right)} \leq \epsilon \|u\|_{H^m(B_\rho)} + C\left(\frac{M}{\lambda}, \epsilon, \rho, \rho'\right) \left(\|u\|_{L^2(B_\rho)} + \| \sum_{\beta \leq m} \left|f^\beta\right| \|_{L^2(B_\rho)} \right)$$

holds, then there is $\tilde{C}\left(\frac{M}{\lambda}, \epsilon\right)$ such that for each $\theta \in (0,1)$ and $\rho < 1$,

$$\|u\|_{H^m(B_{\theta\rho})} \leq \epsilon \|u\|_{H^m(B_\rho)} + \frac{1}{\rho^m} C\left(\frac{M}{\lambda}, \epsilon, \theta\right) \left(\|u\|_{L^2(B_\rho)} + \| \sum_{\beta \leq m} \left|f^\beta\right| \|_{L^2(B_\rho)} \right),$$

3. Problem Sheet 3

Exercise 3.1. Given Ω a bounded open set in $\mathbb{R}^d$, and $p \in L^1(\Omega)$ such that $p > \alpha > 0$ a.e. in Ω , we define

$$H^1(p,\Omega) = \{u \in L^2(\Omega) \,:\, \nabla u \in L^1_{\text{loc}}(\Omega) \text{ and } p\nabla u \in L^2(\Omega)\}.$$

We endow $H^1(p,\Omega)$ with the norm

$$\|u\|^2_{H^1(p,\Omega)} = \|u\|^2_{L^2(\Omega;\mathbb{R})} + \|p\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Show that $H^1(p,\Omega) \subset H^1(\Omega)$
Show that $H^1(p,\Omega)$ with the above norm is a Hilbert Space.
We set

$$H^1_0(p,\Omega) = H^1(p,\Omega) \cap H^1_0(\Omega)$$

Check that $H^1_0(p,\Omega)$ is a closed linear subspace of $H^1(p,\Omega)$.
Given $h \in L^2(\Omega)$, show that there exists a unique $u \in H^1_0(p,\Omega)$ such that

$$\int_\Omega p^2 \nabla u \cdot \nabla v = \int_\Omega hv \text{ for all } v \in H^1_0(p,\Omega).$$

Suppose $d \geq 3$, $A \in W^{1,p_1}\left(B_R;\mathbb{R}^{d\times d}\right) \cap L^\infty\left(B_R;\mathbb{R}^{d\times d}\right)$, $B_1, B_2 \in W^{1,p_2}\left(B_R;\mathbb{R}^d\right) \cap L^\infty\left(B_R;\mathbb{R}^d\right)$, $c \in W^{1,p_3}(B_R;\mathbb{R}) \cap L^\infty(B_R;\mathbb{R})$ and $f \in W^{1,2}(B_R;\mathbb{R})$ with the usual coercivity hypothesis. Consider a solution $u \in H^1(B_R)$ of

$$-\text{div}\,(A\nabla u + B_1 u) + B_2 \cdot \nabla u + cu = f \text{ in } \mathcal{D}'(B_R).$$

We wish to find p_1, p_2, p_3 such that $u \in W^{2,2}(B_R)$ for all $R' < R$.

Exercise 3.2. Show that when $p_1 = \infty$, $p_2 = d$ and $p_3 = \frac{d}{2}$ then indeed $u \in W^{2,2}(B_{R'})$ for all $R' < R$.

Exercise 3.3. Suppose $p_1 > d$, $B_1, B_2, c = 0$ and $A(x) = a(x)I_d$ where $a \in W^{1,p_1}(B_R;\mathbb{R}) \cap L^\infty(B_R;\mathbb{R})$. Using the Schauder Method, show that in a sufficiently small ball within B_R , we have $Du \in L^{2^*}$, and then remove the small ball assumption. (Hint : you may want to use that a is Hölder continuous and therefore close to a constant locally).

Establish the same result when $A \in W^{1,p_1}\left(B_R;\mathbb{R}^{d\times d}\right) \cap L^\infty\left(B_R;\mathbb{R}^{d\times d}\right)$ is a symmetric matrix.

Exercise 3.4. Show that $p_1 > d$, $p_2 = d$ and $p_3 = \frac{d}{2}$ (for large d) then $u \in W^{2,2}(B_{R'})$. Find also lower p_2 and p_3 (when possible).

Exercise 3.5. Supposing $B_1 = B_2 = c = 0$ and $d = 3$, show that $A \in H^2\left(B_R;\mathbb{R}^{d\times d}\right) \cap L^\infty\left(B_R;\mathbb{R}^{d\times d}\right)$ implies $u \in H^3(B_{R'})$.

Exercise 3.6. Suppose that $B_1 = B_2 = C = 0$, and $A \in W^{1,p}\left(B_R;\mathbb{R}^{d\times d}\right) \cap L^\infty\left(B_R;\mathbb{R}^{d\times d}\right)$ with $p > d$. Show that if $u \in H^1_0(B_R)$ then $u \in C^{0,\alpha}(B_R)$.

Exercise 3.7. Prove the weak maximum principle for L_{ND} on a bounded domain (the case $c = 0$, $Lu < 0$, was done in the lectures and hints are given in the lecture notes).

Exercise 3.8. Suppose that Ω is an arbitrary open set in $\mathbb{R}^d$. Show that if $u \in H^1(\Omega) \cap C(\bar{\Omega})$ is a weak solution of $\operatorname{div}(A\nabla u) + u = f$ in $\mathcal{D}'(\Omega)$, with A elliptic and $f \in L^2(\Omega)$, then

$$\min\left(\inf_{\partial\Omega} u, \inf_{\Omega} f\right) \leq u \leq \max\left(\sup_{\partial\Omega} u, \sup_{\Omega} f\right).$$

Hint: use Stampacchia's truncations, $G \in C^1(\mathbb{R})$, $G'(x) > 0$ for $x > K$, $\lim_{x\to\infty} G(x) \to \infty$ and $G(x) = 0$ for $x \leq K$, with $K = \max\left(\sup_{\partial\Omega} u, \sup_{\Omega} f\right)$ ($< \infty$).

Exercise 3.9. Find a counter example for the Maximum principle for a fourth order operator, in one dimension.

4. Problem Sheet 4

Exercise 4.1. Suppose $d \geq 3$. Using the non-linear approach introduced in Section 3.3, show that there is at most one solution in $H_0^1(\Omega)$ to

$$-\mathrm{div}(A\nabla u) + H(x, \nabla u) + u = f \text{ in } \mathcal{D}'(\Omega)$$

with $f \in H^{-1}(\Omega)$ and $c \geq 0$.

Adapt this proof to show that there is at most one solution in $H_0^1(\Omega)$ to

$$-\mathrm{div}\,(A\nabla u) + b \cdot \nabla u + cu = f \text{ in } \mathcal{D}'(\Omega)$$

with $b \in L^d(\Omega)$ and $c \in L^{d/2}(\Omega)$ with $c \geq 0$.

Exercise 4.2. Let $\Omega = (0, \pi)^d$. Show the functions

$$\left(\frac{2}{\pi}\right)^{\frac{d}{2}} \sin(k_1 x_1) \cdot \ldots \cdot \sin(k_d x_d)$$

with $k_i \in \mathbb{N} \backslash \{0\}$ form a complete orthonormal set for $L^2(\Omega)$.

(You may want to check that if you have two orthonormal basis on $L^2(X)$ and $L^2(Y)$ then the product of the elements gives a basis on $L^2(X \times Y)$).

Exercise 4.3. Deduce from Exercise 7.1 (and the results shown in the lecture) that all eigensolutions of $-\Delta u = \lambda u$ in $H_0^1(\Omega)$ are of the form

$$\sin(k_1 x_1) \cdot \ldots \cdot \sin(k_d x_d).$$

and give a characterization of the eigenvalues (as sum of squares).

Exercise 4.4. Show that for any $\lambda \in \mathbb{R}$, $\lambda \geq d$ the number $N(\lambda)$ of positive integers such that

$$\sum_{j=1}^{d} n_j^2 \leq \lambda$$

is bounded by

$$\frac{1}{c(d)} \lambda^{d/2} \leq N(\lambda) \leq c(d) \lambda^{d/2}.$$

(for example, note that this is the number of lattice points included in the closed ball of radius $\sqrt{\lambda}$ and compare the ball to a cube).

Exercise 4.5. Prove Lemma 5.6.