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Elliptic PDEs -Problem Sheet Four

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Exercise 1. *Suppose $d \geq 3$. Using the non-linear approach introduced in Section 3.3, show that there is at most one solution in $H_0^1(\Omega)$ to*

$$-div(A\nabla u) + H(x, \nabla u) + u = f \text{ in } \mathcal{D}'(\Omega)$$

with $f \in H^{-1}(\Omega)$ and $c \geq 0$.

Adapt this proof to show that there is at most one solution in $H_0^1(\Omega)$ to

$$-div(A\nabla u) + b \cdot \nabla u + cu = f \text{ in } \mathcal{D}'(\Omega)$$

with $b \in L^d(\Omega)$ and $c \in L^{d/2}(\Omega)$ with $c \geq 0$.

Proof.

Claim 1. *Suppose that $v \in H_0^1(\Omega)$ satisfies*

$$-div(A\nabla v) + cv \leq b(x)|\nabla v| \text{ in } \mathcal{D}'(\Omega)$$

for $c \geq 0$, then $v \leq 0$.

proof of claim 1: For $k > 0$, test the problem against $(v-k)^+$ to obtain

$$\int_\Omega A\nabla v \nabla (v-k)^+ + \int_\Omega cv(v-k)^+ \leq \int_\Omega b(x)|\nabla v||(v-k)^+|.$$

Define $B_k := \{x \in \Omega \colon v \geq k \text{ and } |\nabla v| > 0\}$. Then by ellipticity of A, we have

$$\lambda \int_{B_k} |\nabla (v-k)^+|^2 + \int_{B_k} ck(v-k)^+ \leq \int_{B_k} b(x)|\nabla v||(v-k)^+|.$$

Note that by applying Hölder's inequality and Sobolev embedding, we have

$$\begin{aligned}\int_{B_k} b(x)|\nabla v||(v-k)^+| \le& \left(\int_{B_k} b^d\right)^{\frac{1}{d}} \left(\int_{B_k} |\nabla v|^2\right)^{\frac{1}{2}} \left(\int_{B_k} |(v-k)^+|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{2d}} \\ \le& C(d)\left(\int_{B_k} b^d\right)^{\frac{1}{d}} \left(\int_{B_k} |\nabla v|^2\right)^{\frac{1}{2}} \left(\int_{B_k} |\nabla (v-k)^+|^2\right)^{\frac{1}{2}}\end{aligned}$$

This implies that

$$\lambda \int_{B_k} |\nabla (v-k)^+|^2 \le C(d)\left(\int_{B_k} b^d\right)^{\frac{1}{d}} \left(\int_{B_k} |\nabla v|^2\right)^{\frac{1}{2}} \left(\int_{B_k} |\nabla (v-k)^+|^2\right)^{\frac{1}{2}} \tag{1}$$

Let $M = \sup v$. We first assume that $0 < M < \infty$ and take $k < M$. Then we have

$$\lambda \int_{B_k} |\nabla v|^2 \le C(d)\left(\int_{B_k} b^d\right)^{\frac{1}{d}} \left(\int_{B_k} |\nabla v|^2\right)^{\frac{1}{2}} |M-k||B_k|^{\frac{d-2}{2d}}$$

Thus

$$\left(\int_{B_k} |\nabla v|^2\right)^{\frac{1}{2}} \le \frac{C(d)}{\lambda}\left(\int_{B_k} b^d\right)^{\frac{1}{d}} |M-k||\Omega|^{\frac{d-2}{2d}} \to 0 \text{ as } k \to M.$$

Note that v is fixed and does not depend on k, thus either $\lim_{k\uparrow M} |B_k| = 0$ or $\nabla v \equiv 0$ on the limit set, which is a contradiction.

If $\sup v = \infty$, then as $k \to \infty$, $|B_k| = 0$ as v is integrable.

Altogether this mean that for some $k_0 < M$ we have

$$C(d)\left(\int_{B_{k_0}} b^d\right)^{\frac{1}{d}} < \frac{\lambda}{2}.$$

Therefore equality (1) shows that $\int_{B_{k_0}} |\nabla(v-k)|^2 = 0$. This implies that $|\{x \in \Omega \colon v \ge k_0\}| = 0$ which yields a contradiction as $k_0 < M$. Thus $M \le 0$, which implies that $v \le 0$.

Now we assume for contradiction that both $u_1, u_2 \in H_0^1(\Omega)$ are solutions to

$$-div(A\nabla u) + H(x, \nabla u) + u = f \text{ in } \mathcal{D}'(\Omega).$$

That is, for all $\varphi \in C_c^\infty(\Omega)$(we can assume that $\varphi \in H_0^1(\Omega)$ by a density argument),

$$\int_\Omega A\nabla u_1 \nabla\varphi + \int_\Omega H(x, \nabla u_1)\cdot\varphi + \int_\Omega u_1\varphi = \int_\Omega f\varphi$$

and

$$\int_\Omega A\nabla u_2 \nabla\varphi + \int_\Omega H(x, \nabla u_2)\cdot\varphi + \int_\Omega u_2\varphi = \int_\Omega f\varphi.$$

Note that we assume $\varphi \geq 0$ here since we test against $(v-k)^+$ in the previous proof. Consider $v = u_1 - u_2$, then v satisfies

$$\int_\Omega A\nabla v \nabla \varphi + \int_\Omega v\varphi \leq \int_\Omega b(x)|\nabla v|\varphi$$

That is,

$$-div(A\nabla v) + v = H(x, \nabla u_2) - H(x, \nabla u_1) \leq b(x)|\nabla v| \text{ in } \mathcal{D}'(\Omega).$$

We need $b \in L^d(\Omega; \mathbb{R}^+)$ so that

$$\int_\Omega b(x)|\nabla v|\varphi \leq \|b\|_{L^d(\Omega)} \|\nabla v\|_{L^2(\Omega)} \|\varphi\|_{L^{\frac{2d}{d-2}}(\Omega)} < \infty.$$

By the claim proved, we have $v = u_1 - u_2 \leq 0$. If we swap the roles of u_1 and u_2, we can also get $u_2 - u_1 \leq 0$. Therefore $u_1 \equiv u_2$.

Similarly, assume for contradiction that both $u_1, u_2 \in H_0^1(\Omega)$ are solutions to

$$-div(A\nabla u) + b \cdot \nabla u + cu = f \text{ in } \mathcal{D}'(\Omega).$$

That is,

$$\int_\Omega A\nabla u_1 \nabla \varphi + \int_\Omega b \cdot \nabla u_1 \varphi + \int_\Omega c u_1 \varphi = \int_\Omega f\varphi$$

and

$$\int_\Omega A\nabla u_2 \nabla \varphi + \int_\Omega b \cdot \nabla u_2 \varphi + \int_\Omega c u_2 \varphi = \int_\Omega f\varphi.$$

Then consider the difference $v = u_1 - u_2$, we have

$$-div(A\nabla v) + cv \leq |b(x)||\nabla v| \text{ in } \mathcal{D}'(\Omega).$$

Note that we need $b \in L^d(\Omega)$ and $c \in L^{d/2}(\Omega)$ so that

$$\int_\Omega b \cdot \nabla v\varphi \leq \|b\|_{L^d(\Omega)} \|\nabla v\|_{L^2(\Omega)} \|\varphi\|_{L^{\frac{2d}{d-2}}(\Omega)} < \infty$$

and

$$\int_\Omega cv\varphi \leq \|c\|_{L^{d/2}(\Omega)} \|v\|_{L^{\frac{2d}{d-2}}(\Omega)} \|\varphi\|_{L^{\frac{2d}{d-2}}(\Omega)} < \infty.$$

The rest of the proof is exactly the same as before. □

Exercise 2. *Let $\Omega = (0, \pi)^d$. Show the functions*

$$\left(\frac{2}{\pi}\right)^{\frac{d}{2}} \sin(k_1 x_1) \cdot \ldots \cdot \sin(k_d x_d)$$

with $k_i \in \mathbb{N} \setminus \{0\}$ form a complete orthonormal set of $L^2(\Omega)$.

(You may want to check that if you have two orthonormal bases on $L^2(X)$ and $L^2(Y)$, then the product of the elements gives a basis on $L^2(X \times Y)$).

Proof.

Claim 2. *If $\{e_m : m \in \mathbb{N}\}$ and $\{\hat{e}_n : n \in \mathbb{N}\}$ are two bases on $L^2(X)$ and $L^2(Y)$, then $\{e_m \cdot \hat{e}_n : (m,n) \in \mathbb{N}^2\}$ is an orthonormal basis on $L^2(X \times Y)$.*

Proof of Claim 2: We first show that $\{e_m \cdot \hat{e}_n : (m,n) \in \mathbb{N}^2\}$ is an orthonormal set on $L^2(X \times Y)$.

$$\begin{aligned}
\langle e_m \hat{e}_n, e_p \hat{e}_q \rangle =& \int_Y \int_X e_m \hat{e}_n e_p \hat{e}_q d\mu(x) d\nu(y) \\
=& \int_Y \left(\int_X e_m e_p d\mu(x) \right) \hat{e}_n \hat{e}_q d\nu(y) \\
=& \int_Y \delta_{m,p} \hat{e}_n \hat{e}_q d\nu(y) \\
=& \delta_{m,p} \delta_{n,q} \\
=& \delta_{(m,p),(n,q)}
\end{aligned}$$

Now we show the completeness. If $h \in \{e_m \cdot \hat{e}_n : (m,n) \in \mathbb{N}^2\}^\perp$, then we have

$$0 = \int_X \left(\int_Y h(x,y) \hat{e}_n(y) d\nu(y) \right) e_m(x) d\mu(x).$$

This implies that

$$x \longmapsto \int_Y h(x,y) \hat{e}_n(y) d\nu(y) \text{ is zero almost everywhere.}$$

Define

$$E_n := \{x \in X : \int_Y h(x,y) \hat{e}_n(y) d\nu(y) \neq 0\}.$$

E_n is a null set for each n, and so its countable union $E = \cup_n E_n$. This implies that

$$\int_Y h(x,y) \hat{e}_n(y) d\nu(y) = 0 \text{ almost everywhere for all } n.$$

Then $h(x,y) \equiv 0$ almost everywhere, and

$$\int_{X \times Y} |h(x,y)|^2 d(\mu) = \int_{X \setminus E} \int_Y |h(x,y)|^2 d\nu(y) d\mu(x) = 0.$$

With this result, it is sufficient to prove that $\frac{\sqrt{2}}{\sqrt{\pi}} \sin(kx)$ is a complete orthonormal set on $L^2(0,\pi)$ and conclude the desired result using induction.

Note that

$$\begin{aligned}
\frac{2}{\pi} \int_0^\pi \sin(kx) \sin(lx) dx =& \frac{2}{\pi} \int_0^\pi \frac{1}{2} [\cos((k-l)x) - \cos((k+l)x)] dx \\
=& \frac{1}{\pi} \left[\frac{\sin((k-l)x)}{k-l} - \frac{\sin((k+l)x)}{k+l} \right]_0^\pi \\
=& 0 \text{ if } k \neq l
\end{aligned}$$

and

$$\begin{aligned}\frac{2}{\pi}\int_0^\pi \sin^2(kx) =& \frac{2}{\pi}\int_0^\pi \frac{1}{2}-\frac{\cos(2x)}{2}dx \\ =& \frac{2}{\pi}\left[\frac{x}{2}-\frac{\sin(2x)}{4}\right]_0^\pi \\ =& 1.\end{aligned}$$

Thus $\{\frac{\sqrt{2}}{\sqrt{\pi}}\sin(kx)\colon k \in \mathbb{N}\setminus\{0\}\}$ form an orthonormal set in $L^2(0,\pi)$.

To show completeness, we assume that $f \in L^2(0,\pi)$ such that f is orthogonal to $\{\frac{\sqrt{2}}{\sqrt{\pi}}\sin(kx)\colon k \in \mathbb{N}\setminus\{0\}\}$.

Define $\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 \le x < \pi, \\ -f(-x) & \text{if } -\pi \le x < 0. \end{cases}$ Then

$$\int_{-\pi}^{\pi}\tilde{f}(x)\frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\int_0^\pi f(x)dx - \frac{1}{\sqrt{\pi}}\int_{-\pi}^0 f(-x)dx = 0,$$

and

$$\int_{-\pi}^{\pi}\tilde{f}(x)\cos(kx)dx = 0$$

as $\tilde{f}$ is an odd function while $\cos(kx)$ is even for all $k \in \mathbb{N}\setminus\{0\}$.

$$\int_{-\pi}^{\pi}\tilde{f}(x)\sin(kx)dx = 2\int_0^\pi f(x)\sin(kx)dx = 0$$

by assumption.

Since $\overline{span\{e^{inx}\colon n \in \mathbb{Z}\}} = \overline{span\{1, \sin(nx), \cos(nx)\colon n \in \mathbb{N}\setminus\{0\}\}} = L^2(-\pi,\pi)$, $\tilde{f}(x) \equiv 0$. This implies that $f(x) \equiv 0$.

Then we can conclude that

$$\left(\frac{2}{\pi}\right)^{\frac{d}{2}}\sin(k_1x_1)\cdot\ldots\cdot\sin(k_dx_d)$$

with $k_i \in \mathbb{N}\setminus\{0\}$ form a complete orthonormal set for $L^2(\Omega)$. □

Exercise 3. *Deduce that all eigenfunctions of $-\Delta u = \lambda u$ in $H_0^1(\Omega)$ are of the form*

$$\sin(k_1x_1)\cdot\ldots\cdot\sin(k_dx_d)$$

and give a characterization of the eigenvalues (as sum of squares).

Proof. First note that for $\phi_k = sin(k_1x_1)\cdot\ldots\cdot\sin(k_dx_d)$,

$$-\Delta\phi_k = -(\sum_{i=1}^{d}\frac{\partial^2\phi_k}{\partial x_k^2}) = (\sum_{i=1}^{d}k_i^2)\phi_k.$$

Thus $\phi_k = \sin(k_1 x_1) \cdot \ldots \cdot \sin(k_d x_d)$ are eigenfunctions of $-\Delta u = \lambda_k u$ with $\lambda_k = \sum_{i=1}^d k_i^2$. By Exercise 4.2, we know that

$$\left(\frac{2}{\pi}\right)^{\frac{d}{2}} \sin(k_1 x_1) \cdot \ldots \cdot \sin(k_d x_d)$$

with $k_i \in \mathbb{N} \setminus \{0\}$ form a complete orthonormal set of $L^2(\Omega) \supset H_0^1(\Omega)$. Note that $\phi_k \in H_0^1(\Omega)$ as $\sin(kx) = 0$ for $x = k$ or $x = \pi$. Thus $e_k := \sin(k_1 x_1) \cdot \ldots \cdot \sin(k_d x_d)$ for $k_i \in \mathbb{N} \setminus \{0\}$ form an orthogonal basis of $H_0^1(\Omega)$.

We claim that all the eigenvalues are of the form $\lambda_k = \sum_{i=1}^d k_i^2$ for $k = 1, 2 \cdots$. Assume for contradiction that there exists another eigenvalue μ with corresponding eigenvector ϕ, then by definition ϕ is orthogonal to the eigenspace E_k (generated by e_k) corresponding to λ_k for each $k = 1, 2 \cdots$. We know that $\{e_k \colon k \in \mathbb{N}\}$ form an orthogonal basis of $H_0^1(\Omega)$. This implies $\phi \in \overline{span\{e_k \colon k \in \mathbb{N}\}}^{\perp} = 0$.

Now we show that all the eigensolutions of $-\Delta u = \lambda u$ must be this form with eigenvalues $\lambda_k = \sum_{i=1}^d k_i^2$. For each eigenvalue λ_k, assume for contradiction that ϕ is a corresponding eigenfunction of different form. But we can write ϕ as

$$\phi = \sum_{n=1}^{\infty} (e_n, \phi)_{L^2} \, e_n.$$

By orthogonality, we know that the inner product $(e_n, \phi)_{L^2}$ is zero except for the term e_k, which shares the same eigenvalue with ϕ. This shows that $\phi = \alpha e_k$ for some constant α.

Thus, all eigensolutions of $-\Delta u = \lambda u$ in $H_0^1(\Omega)$ are of the form $\sin(k_1 x_1) \cdot \ldots \cdot \sin(k_d x_d)$ with corresponding eigenvalues $\lambda_k = \sum_{i=1}^d k_i^2$ for $k \in \mathbb{N} \setminus \{0\}$.

□

Exercise 4. *Show that for any $\lambda \in \mathbb{R}, \lambda \geq d$ the number $N(\lambda)$ of positive integers such that*

$$\sum_{j=1}^{d} n_j^2 \leq \lambda$$

is bounded by

$$\frac{1}{c(d)} \lambda^{d/2} \leq N(\lambda) \leq c(d) \lambda^{d/2}.$$

(For example, note that this is the number of lattice points included in the closed ball of radius $\sqrt{\lambda}$ and compare the ball to a cube.)

Proof. Note that $N(\lambda)$ is the number of lattice points included in the closed ball of radius $\sqrt{\lambda}$, so we need to consider the maximal cube inscribed in this ball to get the upper bound. The length of this cube is $\sqrt{d}\sqrt{\lambda}$. That is, for each n_j, it has no more than $\sqrt{d}\sqrt{\lambda}$ integer solutions. Then

$$N(\lambda) \leq (\sqrt{d}\sqrt{\lambda})^d = c(d) \lambda^{d/2}$$

where $c(d) = d^{d/2}$. If each $n_j \le \frac{\sqrt{\lambda}}{\sqrt{d}}$, then $\sum_{j=1}^{d} n_j^2 \le \sum_{j=1}^{d} \frac{\lambda}{d} = \lambda$. This means that the lattice points in the cubic with length $\frac{\sqrt{\lambda}}{\sqrt{d}}$ are included in the set of lattice points represented by $N(\lambda)$. Thus, we have

$$N(\lambda) \ge \left(\frac{\sqrt{\lambda}}{\sqrt{d}}\right)^{d/2} = \frac{1}{c(d)}\lambda^{d/2}.$$

Combining these two inequalities, the result holds. □

Remark 1. *Note that the question asks for positive solutions only, so we only need to consider the lattice points in the positive quadrant.*

Exercise 5. *Prove that there exists $C(d)$ such that the kth eigenvalue of $-\Delta u = \lambda u$ satisfies*

$$C^{-1}k^{2/d} \le \lambda_k \le Ck^{2/d}.$$

Proof. By Exercise 4.3, we know that $\lambda_k = \sum_{j=1}^{d} k_j^2$. Then applying the result proved in Exercise 4.4, we have

$$\frac{1}{c(d)}\lambda_k^{d/2} \le N(\lambda_k) \le c(d)\lambda_k^{d/2}$$

where $c(d) = d^{d/2}$.

Note that the number of eigenvalues of $-\Delta$ that are less than or equal to the k-th eigenvalue λ_k is $N(\lambda_k)$. This implies that $N(\lambda_k) \ge k$. It follows that

$$k \le N(\lambda_k) \le c(d)\lambda_k^{d/2},$$

that is

$$\lambda_k \ge c(d)^{d/2}k^{2/d}.$$

Note that $N(\lambda_k)$ is the number of lattice points included in the closed ball of radius $\sqrt{\lambda_k}$. Now we decrease the radius so that the number of lattice points is smaller than k. That is, $N(\lambda_k - 1) \le k$. Then we have

$$\frac{1}{c(d)}(\lambda_k - 1)^{d/2} \le N(\lambda_k - 1) \le k.$$

It follows that

$$\lambda_k \le 1 + k^{2/d}c(d)^{2/d} \le 2c(d)^{2/d}k^{2/d}.$$

Therefore,

$$C^{-1}k^{2/d} \le \lambda_k \le Ck^{2/d}$$

where $C = 2c(d)^{2/d}$. □

Alternatively, we can get the upper bound for λ_k in the following way. By Exercise 4.4, we have

$$N(c(d)^{2/d}k^{2/d}) \geq \frac{1}{c(d)}(c(d)k) = k.$$

Then

$$c^{2/d}k^{2/d} \geq \lambda_{N(c^{2/d}k^{2/d})} \geq \lambda_k.$$