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Elliptic PDEs -Problem Sheet Three

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Exercise 1. *Given Ω a bounded open set in $\mathbb{R}^d$, and $p \in L^1(\Omega)$ such that $p > \alpha > 0$ a.e. in Ω, we define*

$$H^1(p,\Omega) = \{u \in L^2(\Omega) \colon \nabla u \in L^1_{loc}(\Omega) \text{ and } p\nabla u \in L^2(\Omega)\}.$$

We endow $H^1(p,\Omega)$ with the norm

$$\|u\|^2_{H^1(p,\Omega)} = \|u\|^2_{L^2(\Omega;\mathbb{R})} + \|p\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Show that $H^1(p,\Omega) \subset H^1(\Omega)$. Show that $H^1(p,\Omega)$ with the above norm is a Hilbert Space. We set

$$H^1_0(p,\Omega) = H^1(p,\Omega) \cap H^1_0(\Omega).$$

Check that $H^1_0(p,\Omega)$ is a closed linear subspace of $H^1(p,\Omega)$. Given $h \in L^2(\Omega)$, show that there exists a unique $u \in H^1_0(p,\Omega)$ such that

$$\int_\Omega p^2 \nabla u \cdot \nabla v = \int_\Omega hv \text{ for all } v \in H^1_0(p,\Omega).$$

Proof.

- *Claim 1: $H^1(p,\Omega) \subset H^1(\Omega)$.*

  If $u \in H^1(p,\Omega)$, $\|u\|_{L^2(\Omega)} < \infty$ and $\|p\nabla u\|_{L^2(\Omega)} < \infty$. Then

  $$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega \frac{1}{p}\nabla u \cdot p\nabla u \le \int_\Omega |\frac{1}{p}\nabla u||p\nabla u| \le \frac{1}{4\varepsilon}\|p\nabla u\|^2_{L^2(\Omega)} + \frac{\varepsilon}{\alpha^2}\|\nabla u\|^2_{L^2(\Omega)}.$$

  Choosing ε small enough, then we have

  $$\|\nabla u\|_{L^2(\Omega)} \le C(\alpha,\varepsilon)\|p\nabla u\|_{L^2(\Omega)} < \infty.$$

  That is, $u \in H^1(\Omega)$. Thus, $H^1(p,\Omega) \subset H^1(\Omega)$.

- *Claim 2:* $H^1(p,\Omega)$ with the above norm is a Hilbert space.

  Define the corresponding inner product to be

$$\langle u, v\rangle_{H^1(p,\Omega)} := \int_\Omega u \cdot v + \int_\Omega p^2 \nabla u \cdot \nabla v.$$

  It is easy to check that $\langle \cdot, \cdot\rangle_{H^1(p,\Omega)}$ is a well-defined inner product on $H^1(p,\Omega)$, so it suffices to show completeness. Assume $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $H^1(p,\Omega)$, then $\{u_m\}_{m=1}^\infty$ and $\{p\nabla u_m\}_{m=1}^\infty$ are Cauchy sequences in $L^2(\Omega)$. By completeness of $L^2(\Omega)$, we know that there exist $u, v \in L^2(\Omega)$ such that

$$u_m \to u \text{ in } L^2(\Omega) \text{ and } p\nabla u_m \to v \text{ in } L^2(\Omega).$$

  Now we aim to show that $p\nabla u = v$ a.e. By claim 1, we know that $\{\nabla u_m\}_{m=1}^\infty$ is also a Cauchy sequence in $L^2(\Omega)$ and thus converges to $w \in L^2(\Omega)$. Note that for $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega u \nabla \varphi = \lim_{m\to\infty} \int_\Omega u_m \nabla \varphi = -\lim_{m\to\infty} \int_\Omega \nabla u_m \varphi = -\int_\Omega w\varphi.$$

  Thus $w = \nabla u$, that is $\nabla u_m \to \nabla u$ in $L^2(\Omega)$. Passing to a subsequence, we have $\nabla u_{m_k} \to \nabla u$ almost surely, thus $p\nabla u_{m_k} \to p\nabla u$ almost surely. By uniqueness of limit, $v = p\nabla u$. Therefore, we have $u_m \to u$ in $H^1(p,\Omega)$ as required.

- *Claim 3:* $H_0^1(p,\Omega) = H^1(p,\Omega) \cap H_0^1(\Omega)$ is a closed linear subspace of $H^1(p,\Omega)$.

  Assume that $\{u_m\}_{m=1}^\infty \in H_0^1(p,\Omega)$ is a sequence such that $u_m \to u$ in $H^1(p,\Omega)$. We need to show that $u \in H_0^1(p,\Omega)$, that is $tr(u) = 0$. Note that

$$0 = tr(u_m) \to tr(u) \text{ in } L^2(\partial\Omega)$$

  by trace theorem. Thus, $H_0^1(p,\Omega)$ is a closed linear subspace of $H^1(p,\Omega)$. As a consequence, $H_0^1(p,\Omega)$ is a Hilbert space as well.

- *Claim 4:* Given $h \in L^2(\Omega)$, there exists unique $u \in H_0^1(p,\Omega)$ such that

$$\int_\Omega p^2 \nabla u \cdot \nabla v = \int_\Omega hv \text{ for all } v \in H_0^1(p,\Omega).$$

  We define the bilinear form as

$$a(u, v) = \int_\Omega p^2 \nabla u \cdot \nabla v.$$

  $a(\cdot,\cdot)$ is continuous as

$$a(u, v) \le \|p\nabla u\|_{L^2(\Omega)} \|p\nabla v\|_{L^2(\Omega)} \le \|u\|_{H^1(p,\Omega)} \|v\|_{H^1(p,\Omega)}.$$

$a(\cdot,\cdot)$ is coercive as

$$a(u,u) = \|p\nabla u\|^2_{L^2(\Omega)} \geq \frac{1}{2}\|p\nabla u\|^2_{L^2(\Omega)} + C\|\nabla u\|_{L^2(\Omega)} \geq \tilde{C}(\|p\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})$$

where the second inequality follows from claim 1 and the last inequality follows from Poincaré inequality.

We can take the right hand side of the integral in the claim as $F(v)$ where $F \in H^1_0(p,\Omega)^\star$. Then by Lax-Milgram theorem, there exists unique u such that $a(u,v) = F(v)$.

□

Suppose $d \geq 3$, $A \in W^{1,p_1}(B_R;\mathbb{R}^{d\times d}) \cap L^\infty(B_R;\mathbb{R}^{d\times d})$, $B_1, B_2 \in W^{1,p_2}(B_R;\mathbb{R}^d) \cap L^\infty(B_R;\mathbb{R}^d)$, $c \in W^{1,p_3}(B_R;\mathbb{R}) \cap L^\infty(B_R;\mathbb{R})$ and $f \in W^{1,2}(B_R;\mathbb{R})$ with the usual coercivity hypothesis. Consider a solution $u \in H^1(B_R)$ of

$$-div(A\nabla u + B_1 u) + B_2 \cdot \nabla u + cu = f \text{ in } \mathcal{D}'(B_R).$$

We wish to find p_1, p_2, p_3 such that $u \in W^{2,2}(B_R)$ for $R' < R$.

Exercise 2. *Show that when $p_1 = \infty$, $p_2 = d$ and $p_3 = \frac{d}{2}$ then indeed $u \in W^{2,2}(B_{R'})$ for all $R' < R$.*

Proof. We adapt the notation from *Gilbarg and Trudinger's* book and write $a^{ij}(x) = A(x)$, and

$$Lu := -D_i(a^{ij}(x)D_j u + B^i_1(x)u) + B^i_2(x)D_i u + c(x)u = f.$$

Testing against $v \in H^1_0(B_R)$ (or simply $v \in C^\infty_c(B_R)$ and then conclude using a density argument), we have

$$\int_{B_R} a^{ij} D_j u D_i v dx = \int_{B_R} g v dx$$

where $g := (B^i_1 - B^i_2)D_i u + (D_i B^i_1 - c)u + f$. For $|2h| < dist(\text{supp}v, \partial B_R)$, let us replace v by its difference quotient $\Delta^{-h} v = \Delta^{-h}_k v = \frac{v(x-he_k)-v(x)}{h}$ for some k, $1 \leq k \leq d$, we then obtain

$$\int_{B_R} \Delta^h(a^{ij}D_j u)D_i v = -\int_{B_R} a^{ij} D_j u D_i \Delta^{-h} v dx = -\int_{B_R} g\Delta^{-h} v dx.$$

Note that

$$\begin{aligned}
\Delta^h(a^{ij}D_j u) =& \frac{1}{h}\left(a^{ij}(x+e_k h)D_j u(x+e_k h) - a^{ij}(x)D_j u(x)\right) \\
=& \frac{1}{h}\left[a^{ij}(x+e_k h)D_j u(x+e_k h) - a^{ij}(x+e_k h)D_j u(x)\right] \\
+& \frac{1}{h}\left[a^{ij}(x+e_k h)D_j u(x) - a^{ij}(x)D_j u(x)\right] \\
=& a^{ij}(x+e_k h)\Delta^h D_j u(x) + \Delta^h a^{ij}(x) D_j u(x).
\end{aligned}$$

Then we have

$$
\begin{aligned}
\int_{B_R} a^{ij}(x+e_kh)D_j\Delta^h u D_i v =& \int_{B_R} -\Delta^h a^{ij} D_j u D_i v - \int_{B_R} g\Delta^{-h} v dx \\
=& -\int_{B_R} \bar{\mathbf{g}} \cdot Dv + g\Delta^{-h} v dx
\end{aligned}
$$

where $\bar{\mathbf{g}} = (\bar{g_1}, \cdots, \bar{g_n})$ with $\bar{g_i} = \Delta^h a^{ij} D_j u$. Since $A \in W^{1,\infty}(B_R;\mathbb{R}^{d\times d}) \cap L^\infty(B_R;\mathbb{R}^{d\times d})$, $\|\Delta^h a^{ij}\|_{L^\infty(B_R)}$ is bounded.

Let $K := \max\left(\|a^{ij}\|_{L^\infty(B_R)}, \|\nabla a^{ij}\|_{L^\infty(B_R)}, \|B_1^i\|_{L^\infty(B_R)}, \|B_2^i\|_{L^\infty(B_R)}, \|c\|_{L^\infty(B_R)}\right)$. Then

$$
\begin{aligned}
\int_{B_R} \bar{\mathbf{g}} \cdot Dv \le& \|\Delta^h a^{ij}\|_{L^\infty(B_R)} \|Du\|_{L^2(B_R)} \|Dv\|_{L^2(B_R)} \\
\le& K\|u\|_{H^1(B_R)} \|Dv\|_{L^2(B_R)}
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_\Omega g\Delta^{-h} v =& \int_{B_R} \left[(B_2^i - B_1^i)D_i u + cu - D_i B_1^i u - f\right] \Delta^{-h} v \\
\le& 2K\|Du\|_{L^2(B_R)}\|Dv\|_{L^2(B_R)} + K\|u\|_{L^2(B_R)}\|Dv\|_{L^2(B_R)} \\
&+ \|DB_1^i\|_{L^d(B_R)} \|u\|_{L^{\frac{2d}{d-2}}(B_R)} \|Dv\|_{L^2(B_R)} + \|f\|_{L^2(B_R)}\|Dv\|_{L^2(B_R)} \\
\le& \left[C(d,K)\|u\|_{H^1(B_R)} + \|f\|_{L^2(B_R)}\right] \|Dv\|_{L^2(B_R)}
\end{aligned}
$$

Where $C(d,K)$ is a constant depending on d, K. Note that the middle line follows from Sobolev embedding and the condition that $B_1 \in W^{1,d}(B_R)$.

To proceed further let us take a cut-off function $\eta \in C_c^\infty(B_R)$ such that $0 \le \eta \le 1$, $\eta = 1$ on $B_{R'}$ but vanishes outside B_R with a bounded gradient $D\eta < \frac{2}{R-R'}$. Set $v = \eta^2 \Delta^h u$, then by ellipticity of $A(x)(i.e. A\xi\cdot\xi \ge \lambda\xi\cdot\xi)$ and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\lambda \int_{B_R} |\eta D\Delta^h u|^2 dx \le& \int_{B_R} \eta^2 a^{ij}(x+e_kh)\Delta^h D_i u \Delta^h D_j u \\
=& \int_{B_R} a^{ij}(x+e_kh)\Delta^h D_i u \Delta^h D_j v - 2\int_{B_R} a^{ij}(x+he_k) D_j\Delta^h u \Delta^h u \eta D_i \eta \\
\le& \left[C(d,K)\|u\|_{H^1(B_R)} + \|f\|_{L^2(B_R)}\right] \|Dv\|_{L^2(B_R)} \\
&+ \tilde{C}(d,K)\|\eta D\Delta^h u\|_{L^2(B_R)} \|\Delta^h u D\eta\|_{L^2(B_R)} \\
\le& \left[C(d,K)\|u\|_{H^1(B_R)} + \|f\|_{L^2(B_R)}\right] \left[\|\eta D\Delta^h u\|_{L^2(B_R)} + 2\|\Delta^h u D\eta\|_{L^2(B_R)}\right] \\
&+ \tilde{C}(d,K)\|\eta D\Delta^h u\|_{L^2(B_R)} \|\Delta^h u D\eta\|_{L^2(B_R)}
\end{aligned}
$$

By Young's inequality, we have

$$\begin{aligned}
\int_{B_R} |\eta D\Delta^h u|^2 dx \leq & \frac{C(d,K)}{4\lambda\varepsilon}\|u\|^2_{H^1(B_R)} + \frac{\varepsilon}{\lambda}\|\eta D\Delta^h u\|^2_{L^2(B_R)} \\
& + \frac{1}{4\lambda\varepsilon}\|f\|^2_{L^2(B_R)} + \frac{\varepsilon}{\lambda}\|\eta D\Delta^h u\|^2_{L^2(B_R)} \\
& + \frac{C(d,K)}{\lambda}\|u\|^2_{H^1(B_R)} + \frac{1}{\lambda}\|\Delta^h u D\eta\|^2_{L^2(B_R)} \\
& + \frac{1}{\lambda}\|f\|^2_{L^2(B_R)} + \frac{1}{\lambda}\|\Delta^h u D\eta\|^2_{L^2(B_R)} \\
& + \frac{C(\tilde{d},K)}{4\lambda\varepsilon}\|\Delta^h u D\eta\|^2_{L^2(B_R)} + \frac{\varepsilon}{\lambda}\|\eta D\Delta^h u\|^2_{L^2(B_R)}
\end{aligned}$$

By rearranging and taking ε small, we have

$$\begin{aligned}
\|\eta\Delta^h Du\|_{L^2(B_R)} \leq & C(d,K,\lambda)\left(\|u\|_{H^1(B_R)} + \|f\|_{L^2(B_R)} + \|\Delta^h u D\eta\|_{L^2(B_R)}\right) \\
& \leq C(d,K,\lambda)\left(1 + \sup_{B_R}|D\eta|\right)\left(\|u\|_{H^1(B_R)} + \|f\|_{L^2(B_R)}\right).
\end{aligned}$$

Thus

$$\|\Delta^h Du\|_{L^2(B_{R'})} \leq \tilde{C}(d,K,\lambda,R,R')\left(\|u\|_{H^1(B_R)} + \|f\|_{L^2(B_R)}\right) < \infty.$$

This implies that $Du \in H^1(B_{R'})$, and thus $u \in W^{2,2}(B_{R'})$. □

Exercise 3. *Suppose $p_1 > d$, $B_1, B_2, c = 0$ and $A(x) = a(x)Id$ where $a \in W^{1,p_1}(B_R;\mathbb{R}) \cap L^\infty(B_R;\mathbb{R})$. Using the Schauder Method, show that in a sufficiently small ball within B_R, we have $Du \in L^{2^\star}$, and then remove the small ball assumption. (Hint: you may want to use that a is Hölder continuous and therefore close to a constant locally).*

Establish the same result when $A \in W^{1,p_1}(B_R;\mathbb{R}^{d\times d}) \cap L^\infty(B_R;\mathbb{R}^{d\times d})$ is a symmetric matrix.

Proof. Note that $A = a(x)Id$ with $a \in W^{1,p_1}(B_R;\mathbb{R}) \cap L^\infty(B_R;\mathbb{R})$, then $a \in C^{0,\gamma}(B_R)$ by Morrey's inequality. This implies that a is close to a constant locally. Consider a cut-off function $\eta \in C_c^\infty(B_R)$, $0 \leq \eta \leq 1$ such that $\eta \equiv 1$ on $B_{r'}$ while vanishes outside B_r for $0 < r' < r < R$ with bounded gradient. Then $u\eta \in H_0^1(B_R)$ and

$$\begin{aligned}
-div(A\nabla(u\eta)) = & - div(A\nabla u \cdot \eta) - div(A\nabla\eta \cdot u) \\
= & - \eta div(A\nabla u) - 2A\nabla u \cdot \nabla\eta - div(A\nabla\eta)u \\
= & \eta f - 2A\nabla u \cdot \nabla\eta - div(A\nabla\eta)u.
\end{aligned}$$

We decompose A as $a_0 Id + (a(x) - a_0)Id$ for some constant a_0 for which $a(x)$ is very close to a_0 on $B_{r'}$. Then we obtain

$$-a_0\Delta(\eta u) = div((a(x) - a_0)Id\nabla(u\eta)) + \eta f - 2A\nabla u \cdot \nabla\eta - div(A\nabla\eta)u.$$

That is,

$$-\Delta(u\eta) = div(B\nabla(u\eta)) + \tilde{f} \text{ in } \mathcal{D}'(B_R)$$

where $\tilde{f} := \frac{1}{a_0}\eta f - \frac{2}{a_0}A\nabla u \cdot \nabla\eta - \frac{1}{a_0}div(A\nabla\eta)u$ and $B := \frac{1}{a_0}(a(x) - a_0)Id$. Theorem 2.3 from lecture notes shows that $-\Delta$ is an isomorphism between $W_0^{1,q}(B_R)$ and $W^{-1,q}(B_R)$. If we write by Δ_0^{-1} the solution map, that is

$$\Delta_0^{-1} \colon W^{-1,q}(B_R) \to W_0^{1,q}(B_R)$$

$$f \to w$$

where w is the unique solution in $W_0^{1,q}(B_R)$, that is $-\Delta w = f$ in B_R. Note that in this problem $q = 2^\star$. We can write

$$u\eta = \Delta_0^{-1}(div(B\nabla(u\eta))) + \Delta_0^{-1}(\tilde{f}).$$

If we write $v = u\eta \in H_0^1(B_R)$, then

$$v = \Delta_0^{-1}(div(B\nabla v) + \Delta_0^{-1}(\tilde{f}).$$

$$(Id - T)v = h$$

where $T \colon W_0^{1,q}(B_R) \to W_0^{1,q}(B_R)$ and $h = \Delta_0^{-1}(\tilde{f})$. Provided that we show that

- $\tilde{f} \in L^2(B_R)$,
- $\|h\|_{W_0^{1,q}(B_R)} \leq C\|\tilde{f}\|_{L^2(B_R)}$,
- and $\|T\|_{W_0^{1,q}(B_r)} \leq K < 1$

the conclusion follows as $Id - T$ is classically invertible with

$$(Id - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

First, it is easy to show that $\tilde{f} \in L^2(B_R)$. $\frac{1}{a_0}\eta f \in L^2(B_R)$ since $f \in L^2(B_R)$ and $\eta \in C_c^\infty(B_R)$ is bounded by 1. $\frac{2}{a_0}A\nabla u\nabla\eta \in L^2(B_R)$ as both $A, \nabla\eta \in L^\infty(B_R)$. $\frac{1}{a_0}div(A\nabla\eta)u \in L^2(B_R)$ since u is in $L^2(B_R)$ while $div(A\nabla\eta)$ is bounded on B_R.

Now we show $\|h\|_{W_0^{1,q}(B_R)} \leq C\|\tilde{f}\|_{L^2(B_R)}$. Note that $-\Delta$ maps $W^{2,2}(B_R) \cap W_0^{1,2}(B_R)$ to $L^2(B_R)$. Thus $h \in W^{2,2}(B_R) \cap W_0^{1,2}(B_R)$. Thanks to Sobolev embedding, we can conclude that

$$\|h\|_{W^{1,q}(B_R)} \leq C(2, d, B_R)\|\tilde{f}\|_{L^2(B_R)}$$

for $q = 2^\star$.

Next, we show that $\|T\|_{W_0^{1,q}} \leq K < 1$. Given $g \in W_0^{1,q}(B_r)$, we have

$$\|B\nabla g\|_{L^q(B_r)} \leq \|B\|_{L^\infty(B_r)}\|\nabla g\|_{L^q(B_r)} \leq \tilde{C}(B_r)\|\nabla g\|_{L^q(B_r)}$$

where $\tilde{C}(B_r)$ is the constant depending on B_r. Thus

$$\|div(B\nabla g)\|_{W^{-1,q}(B_r)} = \|B\nabla g\|_{L^q(B_r)} \leq \tilde{C}(B_r)\|g\|_{W_0^{1,q}(B_r)}.$$

This implies that

$$\|Tg\|_{W_0^{1,q}(B_r)} \leq C(2,d,B_R)\tilde{C}(B_r)\|g\|_{W_0^{1,q}(B_r)}$$

and we can choose $r > r' > 0$ small enough such that $C(2,d,B_R)\tilde{C}(B_r) = K < 1$.

Then we have $\|v\|_{W_0^{1,q}(B_r)} \leq C\|\tilde{f}\|_{L^2(B_R)}$. Thus

$$\|u\|_{W_0^{1,q}(B_{r'})} \leq C\|\tilde{f}\|_{L^2(B_R)}$$

and $Du \in L^{2^\star}$ on a small ball.

Note that we need to show that $\tilde{C}(B_r)$ does not blow up as $r \to 0$.

An alternate formulation of the PDE:

Testing the original PDE against $\eta\phi$, we have

$$\int_{B_R} a(x)\nabla u \cdot \nabla(\phi\eta) = \int_{B_R} f\eta\phi.$$

$$LHS = \int_{B_R} a(x)\nabla u \cdot \nabla\phi\eta + \int_{B_R} a(x)\phi\nabla u \cdot \nabla\eta.$$

Also,

$$\begin{aligned} -div(a\nabla(u\eta)) &= -\, div(a\eta\nabla u) - div(au\nabla\eta) \text{ in } \mathcal{D}'(B_R) \\ &= f\eta - a\nabla u \cdot \nabla\eta - div(au\nabla\eta) \end{aligned}$$

$$-div(B\nabla(u\eta)) = \frac{1}{a(x_0)}\left[f\eta - a\nabla u \cdot \nabla\eta - div(au\nabla\eta)\right].$$

Then $u\eta \in W^{1,p^\star}(B_\rho)$ if $\|B - Id\|_\infty < \varepsilon$.

For each $B_{R'} \subset B_R$, we can cover it with finitely many small balls $\{B_{r_i}\}_{i=1}^\infty$ such that $B_{r_i} \subset B_R$ and $Du \in L^{2^\star}$ in each B_{r_i}. Then $Du \in L^{2^\star}(B_{R'})$.

When $A \in W^{1,p_1}(B_R, \mathbb{R}^{d\times d}) \cap L^\infty(B_R, \mathbb{R}^{d\times d})$, A is Hölder continuous, and thus close to a constant matrix $A(x_0)$ on a small ball. As $A(x)$ is symmetric, $A(x_0)$ is symmetric, and thus we can write $A(x_0) = PDP^T$ via eigenvalue decomposition with $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_d)$. We apply change of coordinates here, that is, take $\tilde{u}(y) = u(QPy) = u(x)$ where $Q = diag(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \cdots, \frac{1}{\sqrt{\lambda_d}})$. Then $A(y) = Q^TP^TA(x)PQ$ and the corresponding matrix $A(y_0) = Q^TP^TA(x_0)PQ = Id$. $\tilde{f}(y) = f(QPy) = f(x)$. we decompose $A(y)$ to be $(A(y) - Id) + Id = B(y) + Id$. Then our new PDE becomes

$$-\Delta u = div(B\nabla u) + \tilde{f}$$

It follows that $\tilde{f} \in L^2(B_R)$. Then we only need to check that $\|B\|_{L^\infty(QPB_r)}$ is small on an ellipsoid QPB_r obtained from change of coordinates and then apply the proof for the case $A = a(x)Id$. Note that

$$\|B\|_{L^\infty(QPB_r)} \leq C \sup_{x \in B_r} |A(x) - A(x_0)| < \tilde{C}(B_r)$$

where $\tilde{C}(B_r)$ is the constant we can control by adjusting the small ball B_r. $\square$

Remark 1. *The same result holds for a general matrix* $A \in W^{1,p}(B_R; \mathbb{R}^{d\times d}) \cap L^\infty(B_R, \mathbb{R}^{d\times d})$.

Proof. For any square matrix A, we can write it as $A = S + \tilde{S}$ where S is symmetric but $\tilde{S}$ is skew-symmetric. Same as Exercise 3.3, we know that A is close to a corresponding constant matrix $S(x_0) + \tilde{S}(x_0)$. Here we need to use the fact that a constant skew-symmetric matrix is divergence-free, that is $div(\tilde{S}\nabla u) = \tilde{S}_{ij} D_{ij} u$ has zero trace if $\tilde{S}$ is a constant skew symmetric matrix. In this case, our PDE is

$$-\Delta u = div(B \nabla u) + \tilde{f} \text{ in } \mathcal{D}'$$

where

$$B := S(y) - Q^T P^T S(x_0) PQ + \tilde{S}(y) - Q^T P^T \tilde{S}(x_0) PQ = Q^T P^T (A(x) - A(x_0)) PQ.$$

Then the rest of the proof follows from Exercise 3.3.

$\square$

Exercise 4. *Show that* $p_1 > d, p_2 = d$ *and* $p_3 = \frac{d}{2}$ *(for large d) then* $u \in W^{2,2}(B_{R'})$. *Find also lower p_2 and p_3 (when possible).*

Proof. The proof is the same as Exercise 3.2 except the approximation for

$$-\int_{B_R} \Delta^h a^{ij} D_j u D_i v.$$

If we can prove the result in Exercise 3.3 for a general (not necessarily symmetric) matrix $A \in W^{1,p_1}(B_R, \mathbb{R}^{d\times d}) \cap L^\infty(B_R; \mathbb{R}^{d\times d})$, non-zero $B_1, B_2 \in W^{1,d}(B_R; \mathbb{R}^d) \cap L^\infty(B_R; \mathbb{R}^d)$ and non-zero $c \in W^{1,d/2}(B_R; \mathbb{R}) \cap L^\infty(B_R; \mathbb{R})$, we can obtain

$$\begin{aligned}\int_{B_{R'}} \Delta^h a^{ij} D_j u D_i v \leq & \|\Delta^h a^{ij}\|_{L^d(B_{R'})} \|Du\|_{L^{2^\star}(B_{R'})} \|Dv\|_{L^2(B_{R'})} \\ \leq & C(B_R, d, p_1) \|\Delta^h a^{ij}\|_{L^{p_1}(B_{R'})} \|Du\|_{L^{2^\star}(B_{R'})} \|Dv\|_{L^2(B_{R'})} \\ < & \infty\end{aligned}$$

where $2^\star = \frac{2d}{d-2}$ and $p_1 > d$.

Now we prove that $Du \in L^{2^\star}(B_{R'})$ for $0 < R' < R$.

We reformulate the PDE as

$$-div(A\nabla u) = g$$

where $g := f - cu - B_2 \cdot \nabla u + div(B_1 u)$.

In order to apply the Schauder method to prove the result in the remark after Exercise 3.3, we need to check that

$$g \in W^{-1,q}(B_R) = (W^{1,q'}(B_R))^\star \text{ where } q' = \frac{2d}{d+2}.$$

We know that $f \in L^2(B_R)$, so it is sufficient to check that $div(B_1 u) - B_2 \cdot \nabla u - cu \in (W^{1,\frac{2d}{d+2}}(B_R))^\star$. It is trivially true that $B_2 \cdot \nabla u - cu \in L^2(B_R)$ since B_1 and c are in $L^\infty(B_R)$ while $u \in H^1(B_R)$. For a function $\phi \in W^{1,\frac{2d}{d+2}}(B_R)$, we have

$$\int_{B_R} B_1^i u D_i \phi \leq \|B_1\|_{L^\infty(B_R)} \|u\|_{L^{\frac{2d}{d-2}}(B_R)} \|\phi\|_{W^{1,\frac{2d}{d+2}}(B_R)} < \infty,$$

thus $div(B_1 v) \in (W^{1,\frac{2d}{d+2}}(B_R))^\star$. Combining these estimates together, we know that $g \in W^{-1,q}(B_R)$, so we can apply the result in the remark after Exercise 3.3 to conclude that $u \in W^{1,q}(B_{R'})$ where $q = 2^\star = \frac{2d}{d-2}$.

In order to find lower p_2 and p_3, we need to look at the terms related to weak derivatives of B_1, B_2 and c. Note that we do not use weak derivative of c in our proof, so $c \in L^\infty(B_R)$ is good enough. This implies that p_3 can be any positive number.

Look at the term that involves $D_i B_1$, then

$$\int_{B_{R'}} D_i B_1^i u \Delta^{-h} v \leq \|DB_1\|_{L^{p_2}(B_{R'})} \|u\|_{L^{(2^\star)^\star}(B_{R'})} \|Dv\|_{L^2(B_{R'})}$$

since we have proved that $u \in W^{1,2^\star}(B_{R'})$ and Sobolev embedding implies that $u \in L^{(2^\star)^\star}(B_{R'})$ where $(2^\star)^\star = (\frac{2d}{d-2})^\star = \frac{2d}{d-4}$. We need $p_2 \geq \frac{1}{\frac{1}{2} - \frac{d-4}{2d}} = \frac{d}{2}$ for the above inequality to be bounded. Note that we need p large enough to make sure that $\frac{2d}{d-4}$ makes sense. Thus, we can take p_2 to be $\frac{d}{2}$. □

Exercise 5. *Supposing $B_1 = B_2 = c = 0$ and $d = 3$, show that $A \in H^2(B_R; \mathbb{R}^{d\times d}) \cap L^\infty(B_R; \mathbb{R}^{d\times d})$ implies $u \in H^3(B_{R'})$.*

Proof. First note that $H^2(B_R)$ is embedded into $W^{1,6}(B_R)$ by Sobolev embedding. $6 > d$ here, so we can apply the result from Exercise 3.4 to get

$$u \in W^{2,2}(B_{R^{(1)}}) \text{ for } 0 < R^{(1)} < R.$$

Now we aim to show that $Du \in W^{2,2}(B_{R'})$ for $0 < R' < R^{(1)} < R$.

Consider $\tilde{u} := Du \in H^1(B_{R^{(1)}})$, $v = -D\tilde{v}$, then

$$\begin{aligned}\int_{B_{R^{(1)}}} a^{ij} D_j u D_i v &= \int_{B_{R^{(1)}}} a^{ij} D_j u D_i(-D\tilde{v}) \\ &= \int_{B_{R^{(1)}}} D_i(a^{ij} D_j u) D_i \tilde{v} \\ &= \int_{B_{R^{(1)}}} a^{ij} D_i(D_j u) D_i \tilde{v} + \int_{B_{R^{(1)}}} D_i a^{ij} D_j u D_i \tilde{v} \\ &= \int_{B_{R^1}} f v\end{aligned}$$

Thus we have

$$\int_{B_{R^{(1)}}} a^{ij} D_i(D_j u) D_i \tilde{v} = \int_{B_{R^{(1)}}} \tilde{f} \tilde{v}$$

where

$$\tilde{f} := D_i f + D_{ij} a^{ij} D_j u + D_i a^{ij} D_{ij} u.$$

In order to use elliptic regularity theory to conclude that $\tilde{u} \in H^2(B_{R'})$ for $0 < R' < R^{(1)}$, we need to prove that $\tilde{f} \in L^2(B_{R^{(1)}})$.

- It is clearly that $D_i f \in L^2(B_{R^{(1)}})$ since $f \in H^1(B_R)$.
- $D_{ij} a^{ij} D_j u$ is in $L^2(B_{R^{(1)}})$ if $D_j u \in L^\infty(B_{R^{(1)}})$.
- $D_i a^{ij} D_{ij} u$ is in $L^2(B_{R^{(1)}})$ if $D_{ij} u \in L^3(B_{R^{(1)}})$ since Hölder's inequality implies that $\|D_i a^{ij} D_{ij} u\|_{L^2(B_{R^{(1)}})} \leq \|D_i a^{ij}\|_{L^6(B_{R^{(1)}})} \|D_{ij} u\|_{L^3(B_{R^{(1)}})}$.

Thus we need to check that

- $D_{ij} u \in L^3(B_{R^{(1)}})$
- $D_j u \in L^\infty(B_{R^{(1)}})$

We first show that $D_{ij} u \in L^3(B_{R^{(1)}})$ using *Schauder theory*. Note that it is sufficient to prove that $\tilde{u} = Du \in W^{1,3}(B_{R^{(1)}})$. Since A is in $H^2(B_R)$, we may use the result from Exercise 3.3. Note that since our q is 3, so we can achieve the desired result if $\tilde{f} \in L^{3/2}(B_{R^{(2)}})$ for $0 < R' < R^{(1)} < R^{(2)} < R$ as $(\frac{3}{2})^\star = 3$ when $d = 3$. It is obvious that $D_i f \in L^{3/2}(B_{R^{(2)}})$ as $D_i f \in L^2(B_{R^{(2)}})$. $D_{ij} a^{ij} D_j u + D_i a^{ij} D_{ij} u$ are also in $L^{3/2}(B_{R^{(2)}})$ as

$$\int_{B_{R^{(2)}}} (D_i a^{ij} D_{ij} u)^{3/2} \leq \left(\int_{B_{R^{(2)}}} (D_i a^{ij})^{\frac{3}{2} \cdot 4} \right)^{1/4} \left(\int_{B_{R^{(2)}}} (D_{ij} u)^{\frac{3}{2} \cdot \frac{4}{3}} \right)^{3/4} < \infty$$

and

$$\int_{B_{R^{(2)}}} (D_{ij} a^{ij} D_j u)^{3/2} \leq \left(\int_{B_{R^{(2)}}} (D_{ij} a^{ij})^{\frac{3}{2} \cdot \frac{4}{3}} \right)^{3/4} \left(\int_{B_{R^{(2)}}} (D_j u)^{\frac{3}{2} \cdot 4} \right)^{1/4} < \infty.$$

By the result from Exercise 3.3, we can conclude that $D_i u \in W^{1,3}(B_{R^{(1)}})$.

Now we use *Stampacchia theorem* to show that $\tilde{u} = D_i u \in L^\infty(B_{R^{(1)}})$. We may choose a cut-off function η such that $\eta\tilde{u} \in H_0^1(B_{R^{(1)}})$. It is sufficient to check that $\tilde{f} \in L^p(B_{B^{(1)}})$ with $2 > p > 3/2$. Note that $D_i u \in W^{1,3}(B_{R^{(1)}})$ and $d = 3$ implies that $D_i u \in L^q$ for any $1 \le q < \infty$ and $D_{ij} u \in L^3(B_{R^{(1)}})$.

Then

$$\int_{B_{R^{(1)}}} (D_i a^{ij} D_{ij} u)^p \le \left(\int_{B_{R^{(1)}}} (D_i a^{ij})^{\frac{6}{p} \cdot p} \right)^{p/6} \left(\int_{B_{R^{(1)}}} (D_{ij} u)^{p \cdot \frac{6}{6-p}} \right)^{(6-p)/6} < \infty$$

since $\frac{6p}{6-p} < \frac{6 \cdot 2}{6-2} = 3$. Similarly,

$$\int_{B_{R^{(1)}}} (D_{ij} a^{ij} D_i u)^p \le \left(\int_{B_{R^{(1)}}} (D_{ij} a^{ij})^{p \cdot \frac{2}{p}} \right)^{p/2} \left(\int_{B_{R^{(1)}}} (D_i u)^{\frac{2}{2-p} \cdot p} \right)^{(2-p)/2} < \infty$$

since $\frac{2p}{2-p} < \infty$.

Thus, $\tilde{u} = D_i u \in L^\infty(B_{R^{(1)}})$. Together with $D_i u \in W^{1,3}(B_{R^{(1)}})$, we obtain $\tilde{f} \in L^2(B_{R^{(1)}})$. Therefore, by the regularity result from Exercise 3.4, we have $\tilde{u} \in H^2(B_{R'})$, that is, $u \in H^3(B_{R'})$.

□

Exercise 6. *Suppose that $B_1 = B_2 = c = 0$ and $A \in W^{1,p}(B_R; \mathbb{R}^{d\times d}) \cap L^\infty(B_R; \mathbb{R}^{d\times d})$ with $p > d$. Show that if $u \in H_0^1(B_R)$, then $u \in C^{0,\alpha}(B_R)$.*

Proof. Recalling the remark after Exercise 3.3, we deduce that $u \in H^2(B_{R'})$ for a general matrix $A \in W^{1,p}(B_R; \mathbb{R}^{d\times d}) \cap L^\infty(B_R; \mathbb{R}^{d\times d})$. Now we aim to show that $u \in H^2(B_R)$. For each $x_0 \in \partial B_R$, we can find a small ball $B(x_0, r)$ such that the Schauder method can be applied to $B(x_0, r) \cap B_R$. Take a smaller ball $B(x_0, r')$ with $r' < r$ and consider $V_0 := B(x_0, r') \cap B_R$. Note that $u \in H_0^1(B_R)$, $u = 0$ on $B(x_0, r) \cap \partial B_R$, so we can choose a cut-off function $\eta \in C_c^\infty(B_R)$ such that $\eta = 1$ on V_0 but vanishes outside $B(x_0, r) \cap B_R$. Then we can show that $u \in H^2(V_0)$ by using Schauder method as in Exercise 3.3. Since ∂B_R is compact, we can cover it with finitely many sets $V_0, V_1, \cdots V_N$ as defined above. We sum the resulting estimates, along with the interior estimate, to find $u \in H^2(B_R)$. Note that $d = 3$ and $2 > \frac{d}{2}$, so we have $u \in C^{0,\gamma}(B_R)$ by Sobolev inequality. □

Proof. An alternative way to show that $u \in W^{1,p^\star}(B_R)$ globally. Define

$$\tilde{f} = \begin{cases} f & \text{in } B_R \\ 0 & \text{in } \mathbb{R}^d \setminus B_R \end{cases}$$

$$\tilde{A}(x) = \begin{cases} A(x) & \text{if } |x| < R \\ A(\frac{Rx}{|x|}) & \text{if } |x| \ge R \end{cases}$$

and

$$\tilde{u} = \begin{cases} u & \text{in } B_R \\ 0 & \text{in } \mathbb{R}^d \setminus B_R \end{cases}.$$

Then $\tilde{f} \in L^2(\mathbb{R}^d), u \in H^1(\mathbb{R}^d), \tilde{A} \in W^{1,p}_{loc}(\mathbb{R}^d)$ and

$$-div(\tilde{A}(x)\nabla\tilde{u}) = \tilde{f} \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

Consider $B_{R'} \supset\supset B_R$, we have $u \in W^{1,2^\star}_{loc}(B_{R'})$. This implies that $u \in W^{1,2^\star}(B_R) \hookrightarrow C^{0,\frac{1}{2}}(B_R)$. □

Exercise 7. *Prove the weak maximum principle for L_{ND} on a bounded domain (the case $c = 0, Lu < 0$, was done in the lectures and hints are given in the lecture notes).*

Claim : *Suppose Ω is open and bounded and $c \geq 0$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$L_{ND} \leq 0$$

where the coefficients of L_{ND} are continuous on Ω. Then

$$\max_{\bar{\Omega}} \leq \max_{\partial\Omega} u^+.$$

Proof. We first consider $c \equiv 0$. Supposing first that we have a *strict* subsolution, $L_{ND}(u) < 0$, the statement follows by elementary calculus: if u has a local maximum at $x_0 \in \Omega$, then $Du(x_0) = 0$ and $D^2u(x_0) \preceq 0 (i.e. D^2u(x_0)$ is negative semi-definite). Using the ellipticity $A(x) = \{a^{ij}(x)\}_{i,j=1}^n \geq \theta > 0$, some linear algebra (details can be found in Evans) shows that $a(x)^{ij}(x_0)u_{ij}(x_0) \leq 0$, so $L_{ND}u(x_0) \geq 0$, a contradiction.

Now suppose only $L_{ND}u \leq 0$ in Ω. Given $\varepsilon > 0$, set $u_\varepsilon := u(x) + \varepsilon e^{\lambda x_1}$ for some $\lambda > 0$ to be chosen. Compute

$$L_{ND}(u_\varepsilon) = L_{ND}(u) + \varepsilon e^{\lambda x_1}(-\lambda^2 a^{11}(x) + \lambda b^1(x)) \leq \varepsilon e^{\lambda x_1}\left(-\lambda^2\theta + \|b\|_{L^\infty(\Omega)}\lambda\right) < 0$$

in Ω for λ large enough. So by the above,

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon$$

and then letting $\varepsilon \searrow 0$, $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$.

Now suppose $c(x) \geq 0$. If $u(x) \leq 0$, we are done. Otherwise, consider the shifted operator $K := L_{ND} - c$ on the non-empty open set $V := \{x \in \Omega \mid u(x) > 0\}$, on which $Ku = L_{ND}u - cu \leq 0$. So by the above, $\max_{\bar{V}} u = \max_{\partial V} u$. If $x \in \partial V$, then either $x \in \partial\Omega$ or else $u(x) = 0$, and so

$$\max_{\bar{\Omega}} u = \max_{\bar{V}} u = \max_{\partial\Omega} u.$$

□

Exercise 8. *Suppose that Ω is an arbitrary open set in $\mathbb{R}^d$. Show that if $u \in H^1(\Omega) \cap C(\bar{\Omega})$ is a weak solution of $-div(A\nabla u)+u = f$ in $\mathcal{D}'(\Omega)$, with A elliptic and $f \in L^2(\Omega)$, then*

$$\min\left(\inf_{\partial\Omega} u, \inf_{\Omega} f\right) \leq u \leq \max\left(\sup_{\partial\Omega}, \sup_{\Omega} f\right).$$

Hint: use Stampacchia's truncations, $G \in C^1(\mathbb{R}), G'(x) > 0$ for $x > K$, $\lim_\infty G(x) \to \infty$ and $G(x) = 0$ for $x \leq K$, with $K = \max\left(\sup_{\partial\Omega}, \sup_\Omega f\right) < \infty$.

Proof. We first prove the upper bound. If $\max\left(\sup_{\partial\Omega}, \sup_\Omega f\right) = \infty$, we are done. Otherwise, define $K := \max\left(\sup_{\partial\Omega}, \sup_\Omega f\right) < \infty$ and consider $v = u - K$. Then

$$L(v) = -div(A\nabla u) + u - K = f - K \leq 0.$$

Testing against $v^+ = \max(v, 0)$, we have

$$\int_\Omega A\nabla v \cdot \nabla v^+ + \int_\Omega v v^+ \leq 0.$$

But

$$\int_\Omega A\nabla v \nabla v^+ + \int_\Omega v v^+ = \int_\Omega A\nabla v^+ \nabla v^+ + \int_\Omega v^+ v^+ \geq \theta \int_\Omega |\nabla v^+|^2 + \int_\Omega |v^+|^2 \geq 0.$$

Thus $v^+ = (u-K)^+ = 0$ in Ω, that is $u \leq K$ in Ω. Now we apply the same technique to prove the lower bound. Similarly, define $k := \min\left(\inf_{\partial\Omega)} u, \inf_\Omega f\right)$. Consider $v = u - k$, then $L(-v) \leq 0$. By the above result, we can conclude that $-v \leq 0$ in Ω, that is, $u \geq k$. □

Proof. We need to consider both $|\Omega| < \infty$ and $|\Omega| = \infty$. For $|\Omega| < \infty$, the proof is the same as above. $\sup_\Omega f < \infty$, $f \in L^2$, we have $\sup_\Omega f > 0$. Now test against $(u - k')^+$ where $k' > K$ is arbitrary and then take $k' \to K$. □

Exercise 9. *Find a counter example for the Maximum principle for a fourth order operator, in one dimension.*

Proof. Consider $u = e^{-x^2}$ on $\Omega := (-\frac{1}{2}, \frac{1}{2})$. Define $Lu = -u''''$. Then $Lu = -4e^{-x^2}(4x^4 + 2x^2 + 3) < 0$ in Ω. But the maximum of u achieves at $x = 0$ instead of the boundary points $x = \{\frac{1}{2}, -\frac{1}{2}\}$ as $1 = e^0 > e^{-\frac{1}{4}}$. □