Elementary Calculus

Chapter 1. Function

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First Semester

Remark

Throughout this lecture, we let

$$\begin{aligned}\mathbb{N} &= \text{set of natural number},\\ \mathbb{Q} &= \text{set of rational number},\\ \mathbb{Q}^c &= \text{set of irrational number},\\ \mathbb{R} &= \text{set of real number}.\end{aligned}$$

Remark

Throughout this lecture, we set for $a, b \in \mathbb{R}$,

$$\begin{aligned}(a,b) &= \{x \in \mathbb{R} : a < x < b\}\\ (a,b] &= \{x \in \mathbb{R} : a < x \leq b\}\\ [a,b) &= \{x \in \mathbb{R} : a \leq x < b\}\\ [a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\}\end{aligned}$$

Definition (Function)

A **function** f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.

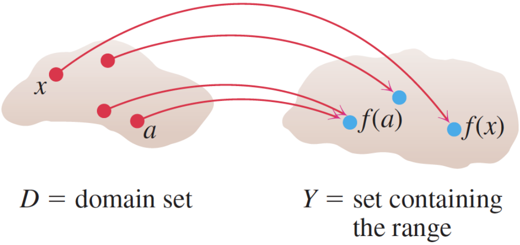


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

Remark

We say that "y is a function of x" and write this symbolically as

$$y = f(x).$$

In this notation, the symbol f represents the function, the letter x is the **independent variable** representing the input value of f, and y is the **dependent variable** or output value of f at x.

Remark

The set D of all possible input values is called the **domain** of the function. The set of all output values of $f(x)$ as x varies throughout D is called the **range** of the function.

Remark

When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real $x-$values for which the formula gives real $y-$values, which is called the **natural domain**.

Remark

When the range of a function is a set of real numbers, the function is said to be **real-valued**.

Example (Natural domain)

Let's verify the natural domains and associated ranges of some simple functions.

Function	Domain (D)	Range (Y)
$y = x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Definition (Graphs of Functions)

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

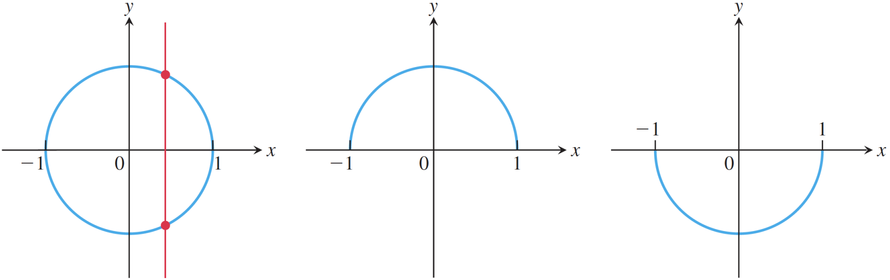
$$\{(x, f(x)) : x \in D\}.$$

Example

1. Graph the function $y = x^2$ over the interval $[-2, 2]$.
2. Graph the function $y = \sqrt{1 - x^2}$ over the interval $[-1, 1]$.

Remark (The Vertical Line Test for a Function)

- Not every curve in the coordinate plane can be the graph of a function.
- A function f can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once.
- If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.



(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

Definition (Piecewise-Defined Functions)

Piecewise-defined function is described in pieces by using different formulas on different parts of its domain.

Example (Absolute value function)

The absolute value function is a piecewise-defined function.

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

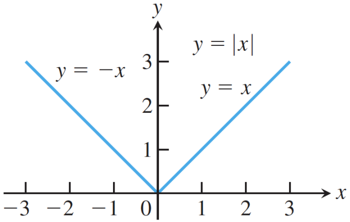


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Example

Find the definition of **Rectified Linear Unit (ReLU)** and graph the function.

Definition (Increasing and Decreasing Functions)

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) \geq f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) \leq f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Definition (Even Functions and Odd Functions)

A function $y = f(x)$ is an

1. **even function** of x if $f(-x) = f(x)$,
2. **odd function** of x if $f(-x) = -f(x)$,

for every x in the function's domain.

Remark (Symmetry)

1. The graph of an even function is symmetric about the $y-$axis.
2. The graph of an odd function is symmetric about the origin.

Definition

1. A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**.
2. The function $f(x) = x$ is called the **identity function**.
3. Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k.

Definition

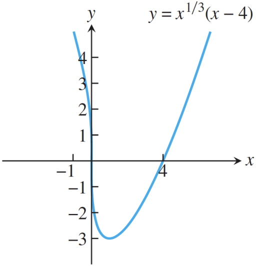
1. A function $f(x) = x^a$, where a is a constant, is called a **power function**.
2. A function $p(x)$ is a **polynomial** if

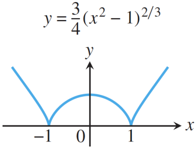
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

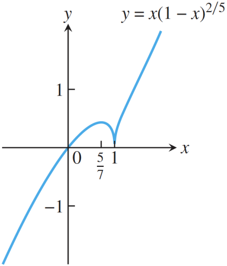
   where n is a natural number and $a_0, a_1, a_2, \cdots, a_n$ are real constants (called the **coefficients** of the polynomial).
3. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**.
4. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3.

Definition

1. A **rational function** is a quotient or ratio
$$f(x) = \frac{p(x)}{q(x)},$$
where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$.
2. Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**.







Definition

1. Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**.
2. Functions of the form $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant, are called **logarithmic functions**. They are the inverse functions of the exponential functions.
3. These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well.
4. The six basic trigonometric functions will be considered later.

Definition (Composite Function)

If f and g are functions, the composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

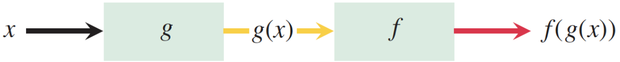


FIGURE 1.27 A composite function $f \circ g$ uses the output $g(x)$ of the first function g as the input for the second function f.

Definition (The Six Basic Trigonometric Functions)

Define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle of radius r

$$\begin{aligned} \text{sine} : \sin\theta &= \frac{y}{r} & \text{cosecant} : \csc\theta &= \frac{r}{y} \\ \text{cosine} : \cos\theta &= \frac{x}{r} & \text{secant} : \sec\theta &= \frac{r}{x} \\ \text{tangent} : \tan\theta &= \frac{y}{x} & \text{cotangent} : \cot\theta &= \frac{x}{y} \end{aligned}$$

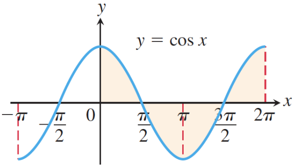
Remark

Notice also that whenever the quotients are defined,

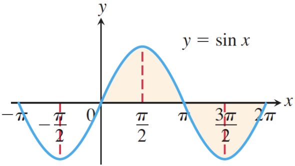
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}, \quad \text{etc.}$$

Definition (Periodic Function)

1. A function $f(x)$ is **periodic** if there is a positive number p such that $f(x+p) = f(x)$ for every value of x.
2. The smallest such value of p is the **period** of f.



Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



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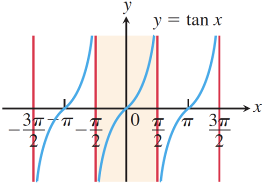
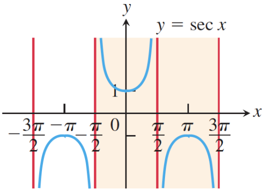
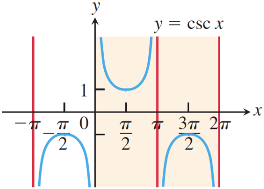


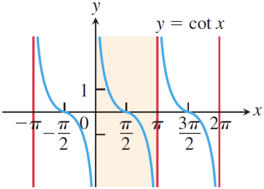
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

Theorem (Trigonometric Identities)

For any θ,

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta.$$

Proof.

Since $\cos\theta = x/r$ and $\sin\theta = y/r$,

$$\cos^2\theta + \sin^2\theta = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1.$$

Moreover, since $\tan\theta = y/x$ and $\sec\theta = r/x$,

$$1 + \tan^2\theta = 1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2} = \sec^2\theta.$$

$\square$

Theorem (Addition Formulas)

For any $x, y \in \mathbb{R}$,

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

Proof.

The proof is homework. $\square$

Corollary (Double-Angle Formulas)

For any θ,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \textit{and} \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Proof.

Applying $x = y = \theta$ to the addition formulas, we can evaluate that

$$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.$$

□

Corollary (Half-Angle Formulas)

For any θ,

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \textit{and} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}.$$

Proof.

Since

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (1-\sin^2\theta) - \sin^2 = 1 - 2\sin^2\theta,$$

we can obtain

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \text{and} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}.$$

□

Theorem (The Law of Cosines)

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

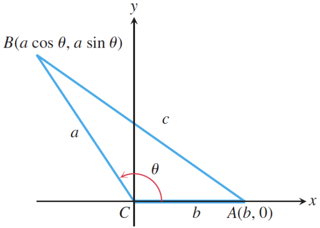
This equation is called the ***law of cosines****.*

Proof.

Based on the Figure and the definition of the distance between two points, we can examine that

$$\begin{aligned} c^2 &= (b - a\cos\theta)^2 + (a\sin\theta)^2 \\ &= b^2 - 2ab\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

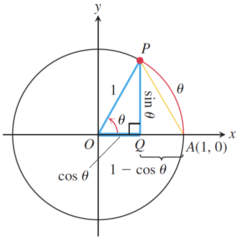
□



Theorem (Two Special Inequalities)

For any angle θ measured in radians, the sine and cosine functions satisfy

$$-|\theta| \leq \sin\theta \leq |\theta|, \quad -|\theta| \leq 1-\cos\theta \leq |\theta|.$$



Proof.

Since $\overline{PQ} = \sin\theta$ and $\overline{AQ} = 1-\cos\theta$,

$$\overline{PQ}^2 + \overline{AQ}^2 = \sin^2\theta + (1-\cos\theta)^2 = \overline{AP}^2 \leq \theta^2.$$

The terms on the left-hand side of above are both positive so that

$$\sin^2\theta \leq \theta^2 \quad \text{and} \quad (1-\cos\theta)^2 \leq \theta^2.$$

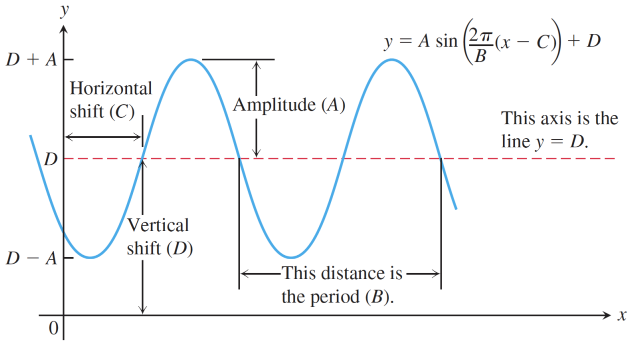
By taking square roots, this is equivalent to saying that $-|\theta| \leq \sin\theta \leq |\theta|$ and $-|\theta| \leq 1-\cos\theta \leq |\theta|$. $\square$

Observation

The transformation rules applied to the sine function give the **general sine function** or **sinusoid formula**

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the amplitude, $|B|$ is the period, C is the horizontal shift, and D is the vertical shift.



Definition (Exponential Function: Revisited)

In general, if $a \neq 1$ is a positive constant, the function

$$f(x) = a^x$$

is the **exponential function** with base a.

Remark (Evaluating the value of an exponential function)

1. If $x = n$ is a positive integer, the number $a^x = a^n$ is given by multiplying a by itself n times:
$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a.}_{n \text{ factors}}$$
2. If $x = 0$, then $a^0 = 1$, and if $x = -n$ for some positive integer n, then
$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n .$$
3. If $x = p/q$ is any rational number, then
$$a^x = a^{p/q} = a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p .$$
4. If x is irrational, we cannot do anything at this moment (we will study this later).

Theorem (Rules for Exponents)

If $a, b > 0$, the following rules hold true for all real numbers x and y

$$a^x \cdot a^y = a^{x+y}, \quad \frac{a^x}{a^y} = a^{x-y}, \quad (a^x)^y = a^{xy},$$
$$a^x \cdot b^x = (ab)^x, \quad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

Definition (Natural Exponential Function)

The exponential function with base e

$$f(x) = e^x$$

is called the **natural exponential function**.

Definition (One-to-One Function)

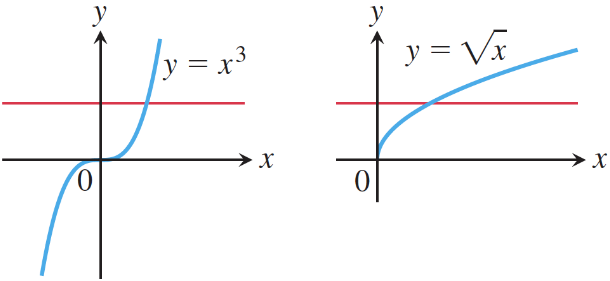
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Example

1. $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers.
2. $f(x) = \sin x$ is not one-to-one on the interval $[0, \pi]$ but one-to-one on the interval $[0, \pi/2]$.

Remark (The Horizontal Line Test for One-to-One Functions)

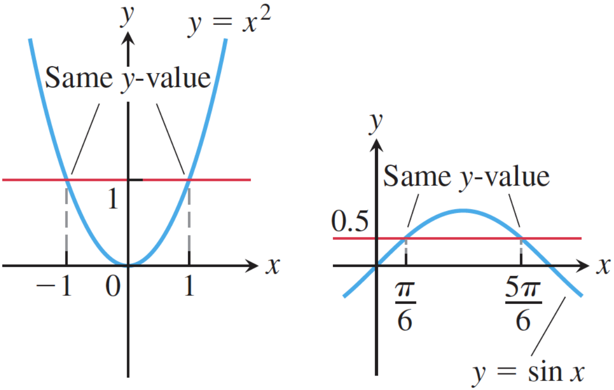
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



(a) One-to-one: Graph meets each horizontal line at most once.

Remark (The Horizontal Line Test for One-to-One Functions)

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

Definition (Inverse Function)

Suppose that f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by

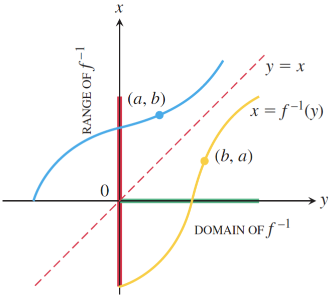
$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

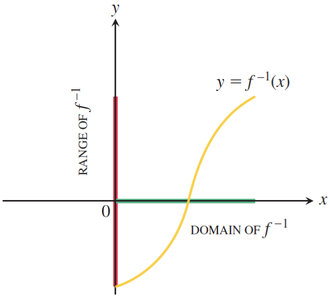
Remark (Process for finding inverse function)

The process of passing from f to f^{-1} can be summarized as

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

Example

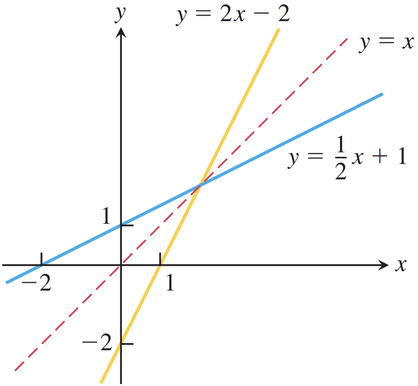
Find the inverse of

$$y = \frac{1}{2}x + 1,$$

expressed as a function of x.

Answer.

$y = 2x - 2.$ □



Example

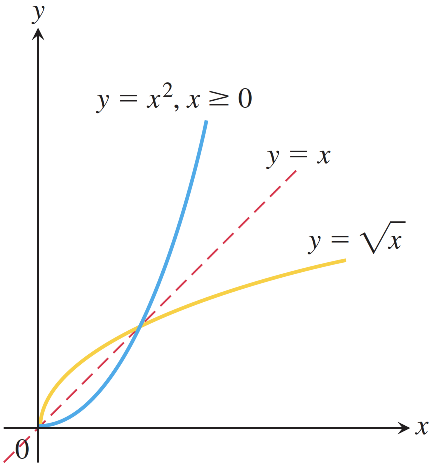
Find the inverse of

$$y = x^2, \quad x \geq 0,$$

expressed as a function of x.

Answer.

$y = \sqrt{x}$, $x \geq 0$. □



Definition (Logarithmic Functions)

1. The **logarithm function** (or common logarithm function) with base a,

$$y = \log_a x,$$

is the inverse of the base a exponential function $y = a^x$ $(a > 0,\ a \neq 1)$.

2. The function

$$y = \log_e x = \ln x$$

is called the **natural logarithm function**.

Theorem (Algebraic Properties of the Natural Logarithm)

For any numbers $a > 0$ *and* $x > 0$*, the natural logarithm satisfies the following rules:*

1. *Product Rule:* $\ln ax = \ln a + \ln x$
2. *Quotient Rule:* $\ln(a/x) = \ln a - \ln x$
3. *Reciprocal Rule:* $\ln(1/x) = -\ln x$
4. *Power Rule:* $\ln x^a = a \ln x$.

Remark (Caution!)

$$\ln(x + y) \neq \ln x + \ln y.$$

Theorem (Inverse Properties)

1. $a^{\log_a x} = x$, $\log_a a^x = x$, $a > 0, a \neq 1, x > 0$
2. $e^{\ln x} = x$, $\ln e^x = x$, $x > 0$.

Remark

Every exponential function is a power of the natural exponential function.

$$a^x = e^{x \ln a}.$$

That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

Theorem (Change of Base Formula)

Every logarithmic function is a constant multiple of the natural logarithm.

$$\log_a x = \frac{\ln x}{\ln a}, \quad a > 0, a \neq 1.$$

Proof.

Based on the properties of a^x and $\log_a x$,

$$\ln x = \ln(a^{\log_a x}) = (\log_a x) \ln a \quad \text{implies} \quad \log_a x = \frac{\ln x}{\ln a}.$$

$\square$

Observation (Half-Life)

- The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay.
- It is a notable fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.
- Let let y_0 be the number of radioactive nuclei initially present in the sample. Then the number $y = y(t)$ present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$y_0 e^{-kt} = \frac{1}{2} y_0 \quad \text{implies} \quad t = \frac{\ln 2}{k}.$$

- This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not have any effect.

Example (Half-Life)

- The effective radioactive lifetime of polonium -210 is so short that we measure it in days rather than years.
- The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5\times 10^{-3}t}.$$

- To evaluate the element's half-life, we seek the value of t

$$y_0 e^{-5\times 10^{-3}t} = \frac{1}{2}y_0 \quad \text{implies} \quad 5\times 10^{-3}t = \ln 2.$$

- Thus, the element's half-life is

$$t = \frac{\ln 2}{5\times 10^{-3}} \approx 139 \text{ days}.$$

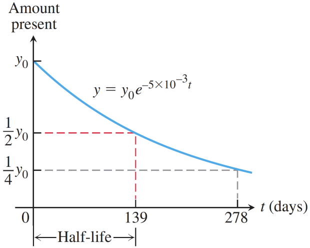
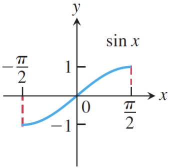
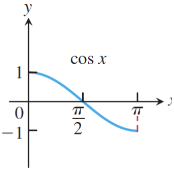


FIGURE 1.63 Amount of polonium-210 present at time t, where y_0 represents the number of radioactive atoms initially present (Example 7).

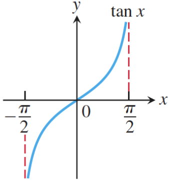
Domain restrictions that make the trigonometric functions one-to-one



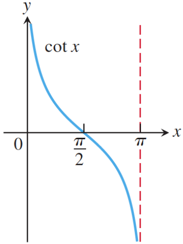
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



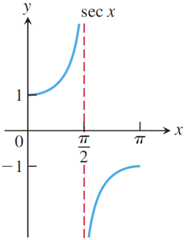
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



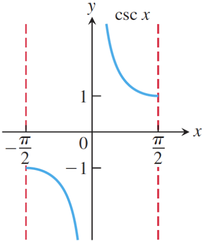
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

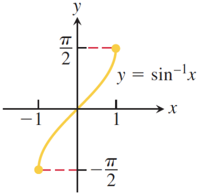
Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{aligned}
y &= \sin^{-1} x \qquad \text{or} \qquad y = \arcsin x \\
y &= \cos^{-1} x \qquad \text{or} \qquad y = \arccos x \\
y &= \tan^{-1} x \qquad \text{or} \qquad y = \arctan x \\
y &= \cot^{-1} x \qquad \text{or} \qquad y = \operatorname{arccot} x \\
y &= \sec^{-1} x \qquad \text{or} \qquad y = \operatorname{arcsec} x \\
y &= \csc^{-1} x \qquad \text{or} \qquad y = \operatorname{arccsc} x
\end{aligned}
$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

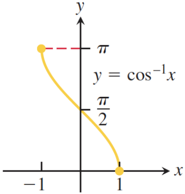
Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



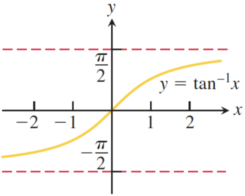
(a)

Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$



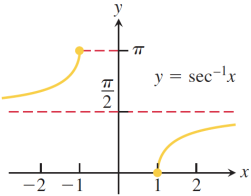
(b)

Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



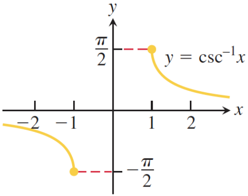
(c)

Domain: $x \leq -1$ or $x \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$



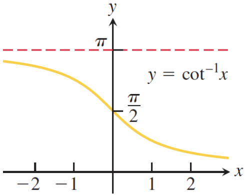
(d)

Domain: $x \leq -1$ or $x \geq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



(e)

Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



(f)

Definition (Arccosine and Arcsine Functions)

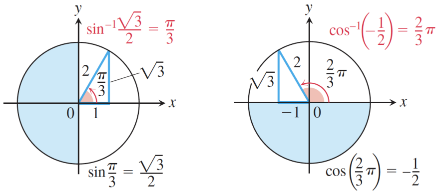
1. $y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.
2. $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.
3. Definition of $\tan^{-1} x$ and $\sec^{-1} x$ will be considered later.

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

Example

Examine that

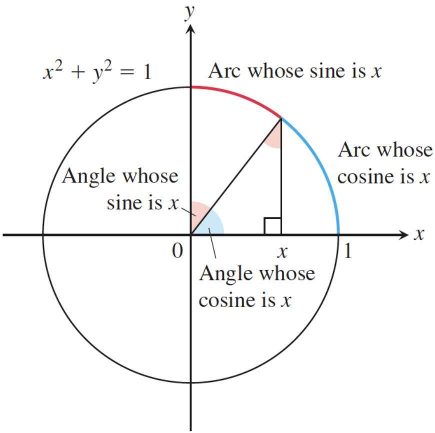
$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$
$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$



(a) (b)

Remark (The "Arc" in Arcsine and Arccosine)

- For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure.
- If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



Theorem

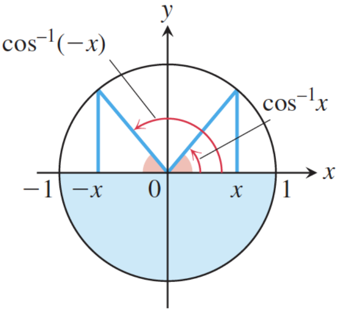
$$\cos^{-1} x + \cos^{-1}(-x) = \pi.$$

Proof.

Let $\cos^{-1} x = \theta$ and $\cos^{-1}(-x) = \phi$. Then $\cos\theta = x$ and $\cos\phi = -x = \cos(\pi - \theta)$. Hence, $\phi = \pi - \theta$ implies

$$\cos^{-1} x + \cos^{-1}(-x) = \theta + (\pi - \theta) = \pi.$$

$\square$



Theorem

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

Proof.

Let $\sin^{-1} x = \theta$ and $\cos^{-1} x = \phi$. Then since

$$\sin\theta = \cos\phi = x \quad \text{and} \quad \frac{\pi}{2} + \theta + \phi = \pi,$$

we can examine that

$$\sin^{-1} x + \cos^{-1} x = \theta + \phi = \frac{\pi}{2}.$$

□