What Does Multi-agent Path-finding Tell Us About Intelligent Intersections

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Keywords: Multi-agent Systems, Centralized Planning, Cooperative Navigation, Intelligent Intersection.

Abstract: In this paper, we study the problem of an intelligent intersection. There are many studies that present an algorithm that tries to efficiently coordinate many agents in a given intersection, however, in this paper, we study the layout of the intersection and its implications to the quality of the plan. We start with two of today commonly used road intersections (4-way intersection and roundabout) and compare them with an intersection that is less restrictive on the movements of the agents. This means that the agents do not have to use predefined lanes or follow a prescribed driving direction. We also study the effect of granularity of the intersection. The navigation of the agents in a given intersection is solved as an instance of (online) multi-agent path-finding problem.

1 INTRODUCTION

Intersections are nowadays one of the most problematic parts of our roads. According to the statistics (Huang et al., 2008), intersections are the place where most accidents happen. Navigating through an intersection is problematic because the paths of the vehicles cross each other and so not all vehicles can move at the same time. This means that some vehicles must stand still and yield the others. If the traffic around the intersection is dense enough, then a situation can happen when every newly arriving vehicle must spend more and more time waiting to get through the intersection.

In order to make the intersection effective, and above all safe, various intersection management mechanisms have been utilized. In low-traffic intersections, following the traffic regulations and respect the traffic signs is enough. In busier intersections, we encounter traffic lights that can control traffic in all directions. However, these mechanisms are not efficient. For example, if we arrive at an empty intersection but have a red light, we must not continue our journey, although there is no one to whom we are giving way. In this case, the light signal control does not behave optimally at all, because otherwise, it would recognize that we are alone at the intersection and let us through.

If we take into account the rise of autonomous vehicles, we can further develop the idea of an intelligent intersection in such a way that it can communicate with the vehicles that want to go through. Instead of the intersection recognizing a vehicle and changing lights, the vehicle asks for permission to pass through the intersection at a given time and the intersection navigates it safely with other present vehicles.

Note that while we are using the terminology from car navigation, the same problems can be found in other areas of agent navigation. For example warehouse robots – navigation between shelves, where movement in only one direction is allowed, is easy using only sensors. However, navigation through a crossing of these corridors is hard due to the presence of other robots. Some other examples include ship navigation within harbors in comparison with navigation on the open sea, and airplane navigation in the proximity to an airport with comparison to flights in predefined flight corridors. For this reason, we will use the more general term *agents*.

Several studies present an algorithm that navigates agents through a fixed intersection. In contrast, we will study the effect of the layout of the intersection on the quality of plans. The intersections can be abstracted as a shared graph and the navigation part is then solved as an instance of multi-agent path-finding (MAPF) problem.

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2 MAPF PROBLEM

Offline MAPF. An *instance of offline MAPF* problem can be formally defined as a pair (G,A), where G is a graph and A is a set of agents. The graph G can be directed or undirected. Each agent $a \in A$ is defined as a pair (a_s, a_g), where a_s and a_g correspond to a starting location and goal location respectively in graph G.

The time is assumed discretized and in each timestep, all of the agents can perform either a move action to one of the neighboring vertices or a wait action to remain in the current vertex. All of the agents are moving simultaneously.

A *plan* for a single agent i is a sequence of locations where the agent is located at a given time. Let π_i denote a plan for agent i, where $\pi_i(j) = v$ represents that agent i is located in vertex v at timestep j. We denote $|\pi_i|$ as the length of the plan.

A plan π_i is a *valid plan* if $\pi_i(0)$ and $\pi_i(|\pi_i|)$ are the initial and goal locations of a_i, respectively, and for every $j \in \{0,\dots,|\pi_i|\}$ either $\pi_i(j) = \pi_i(j+1)$ or $(\pi_i(j), \pi_i(j+1)) \in E$. This means that at each timestep, the agent is either waiting in the current vertex or moves to a neighboring vertex.

A *joint plan* π is a set of valid plans, one for each agent. The joint plan must further satisfy the following constraints:

- $\forall i,j \in A, \forall t : \pi_i(t) \neq \pi_j(t)$ Meaning that no two agents can occupy the same vertex at the same time.
- $\forall i,j \in A, \forall t : \pi_i(t) \neq \pi_j(t+1) \vee \pi_i(t+1) \neq \pi_j(t)$ Meaning that no two agents can use the same edge in the opposite direction at the same time.

In the context of MAPF, we call these constraints a vertex conflict and a swapping conflict (Stern et al., 2019). On the other hand, we allow two agents to closely follow each other. This means that agent a can move into a vertex that is currently occupied by some other agent, provided that the vertex will be empty before the arrival of the agent a. This situation is for example forbidden in the setting of MAPF that is called Pebble Motion (Kornhauser et al., 1984).

When creating a plan, we often prefer to find an optimal plan with respect to some cost function. The two functions often used in the literature are *makespan* and *sum of costs*.

The makespan function is defined as:

$$\max_{i \in A} |\pi_i|$$

The sum of costs function is defined as:

$$\sum_{i \in A} |\pi_i|$$

While there are many polynomial-time algorithms that can find a feasible solution (de Wilde et al., 2014; Surynek, 2009), the task to find either a makespan optimal solution or a sum of costs optimal solution is an NP-Hard problem (Ratner and Warmuth, 1990; Yu and LaValle, 2013).

Online MAPF. As opposed to the offline setting, where all of the agents are known in advance, we can define an *online MAPF* (Švancara et al., 2019), where new agents can appear over time, while agents that reached their goal location disappear. This setting very closely represents the intersection model we are interested in.

Formally an *online MAPF instance* includes in addition to the offline instance a set of new agents A_{new}. Each new agent a is a triple (a_s, a_g, t), where t is the timestep at which the agent wants to enter the graph. If it is not possible for the agent to enter at the desired time (because it would cause an unavoidable collision), it is possible to delay the agent.

The solution to the online MAPF is then a sequence of valid joint plans $\Pi = (\pi^0, \dots, \pi^m)$, where $m = |A_{new}|$ and π^0 is the joint plan for the offline part of the instance. Each time a new agent enters the graph, we produce a new plan including that agent. Depending on the solver strategy, we may forbid or allow to change plans for the agents that were already present in the graph.

Note that the online nature of the problem means that there is no solver that can compute the overall optimal solution (Švancara et al., 2019).

3 INTERSECTION MODEL

An intersection is a part of a shared environment where several roads with simple navigation rules cross each other. Intersections are typically hard for agents to navigate through due to the presence of many other agents, whose desired paths cross each other.

A way to deal with this challenge is for the agents to be coordinated by some centralized *autonomous intersection management* that organizes the agents in such a way that they do not collide and, furthermore, the time spend crossing the intersection is minimized (Dresner and Stone, 2008b). There are some basic logical requirements for this intersection management such as that all of the independent agents (*driver agent*) respect the commands of the central agent (*intersection manager*). For simplicity, we assume that all of the driver agents are homogeneous and are travelling at the same speed.

The intersection is characterized by some locations, where an agent can enter and some locations where the agent may leave the intersection. Once the agent leaves the intersection, it is again an independent agent.

The driver agents send a request for traversing the intersection with the desired entrance time and trajectory. The intersection manager then either approves or rejects this request. A simple strategy to coordinate the intersection can be described as first-come, first-served (Dresner and Stone, 2008b). In this strategy, the intersection manager simply checks if the request can be executed without causing a collision with previously approved trajectory and if so it approves the request.

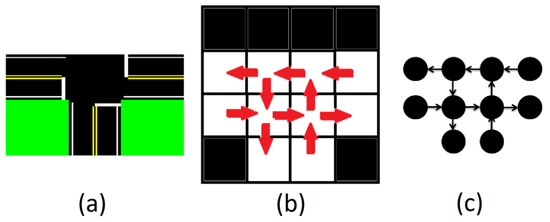

Figure 1: The process of creating an input graph for MAPF from real intersection: (a) real intersection (b) tiles representation (c) oriented graph.

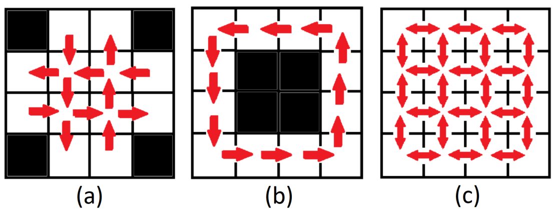

Figure 2: Intersection types: (a) corridors intersection (b) roundabout intersection (c) free movement intersection.

In this paper, we try to improve the intersection manager by letting it decide the trajectory of the agent. We will also consider a model, where the intersection manager can change the already approved trajectories of the agents that are already traversing the intersection. This approach can be used also for emergencies when an accident happens in the intersection and we need to change the approved trajectories to avoid the collision (Dresner and Stone, 2008a).

4 COORDINATING INTERSECTION USING MAPF

Intersection Representation. We divide the space of the intersection into a grid of $n \times n$ tiles, we call n the granularity of the intersection. In every timestep, an agent can occupy only one tile and a tile can be occupied by at most one agent. We suppose that the agents move synchronously and in every discrete timestep they move to one of their neighboring tiles, they are not allowed to stay on the same tile in two consequent timesteps (wait actions are not allowed).

Every tile has at most 4 neighboring tiles —- they can be situated to the north, to the south, to the east and to the west of that tile. The diagonal tiles are not neighbors. The space of the intersection is described as a graph in this way: The tiles correspond to the vertices of the graph and the edges represent all couples of the neighboring edges. An example of translating a real intersection into tiles representation and then into oriented graph representation can be seen in Figure 1. Then we can apply *MAPF* on this graph in order to find paths for agents in the intersection. We reduced the intersection manager to an online MAPF problem.

A tile might have less than 4 neighbors because of two reasons. Firstly, edge tiles do not have naturally some neighbors because they simply do not exist. Secondly, it may be forbidden to perform some moves in the intersection. This can be achieved by forbidding a tile to be a neighbor of any other tile by deleting an appropriate edge from the graph. If the edge is completely missing, two tiles are not neighboring although they lay physically next to each other. In case there is a directed edge between two tiles, it is possible to move between these tiles in the direction of that edge, but movement in the opposite direction is forbidden.

Intersection Models. We created three different intersection types to compare (see Figure 2). Two of them are a sample of intersections used in the real world, while the last one gives the agents more freedom in movement.

The Corridors Intersection is represented by a directed graph. Its edges limit the movement of the agents through the intersection in such a way that the trajectories approximately correspond to those which a human driver would choose. This is a representation of a typical 4-way intersection that is usually coordinated by traffic lights or by signs.

The Roundabout Intersection is also represented by a directed graph. It simulates a roundabout build on the space reserved for the intersection —- it is of the same size but it allows only the movement through the margin tiles of the intersection in the counterclockwise direction. The tiles in the middle of the intersection are unreachable.

The Free Movement Intersection is represented by an undirected graph. It contains an undirected edge between each couple of tiles laying next to each other.

The two previous intersections are used today in traffic because a few simple rules can achieve coordinated movement of agents inside the intersection. The free movement intersection, on the other hand, is not viable to use without some centralized entity that coordinates the movement of agents. The benefit of this intersection is that it allows the agents to move freely through the whole space of the intersection, thus theoretically increasing throughput.

Incoming Requests. Every request consists of input and output direction and of the required time (in abstract discrete units) for entering the intersection. Before every experiment, we always set the *space limit* – maximum delay allowed on every entrance to the intersection. Requests for which there is no free path starting between required time and (required time + maximum delay) are rejected.

Solvers. We always process one new request in one run of the solver although it is possible to process more requests at once. In most cases, we search path for a new request in such a way that we respect the found paths for all previous requests and we do not change them. We shall call this approach *adding-solver*. In some experiments we use also *rescheduling-solver* which, in contrast to the previous one, also might change some paths found before. Such solver is more powerful — it might succeed in some cases in which the adding-solver fails but its computations may take a longer time.

We should mention that it is reasonable to use rescheduling-solver only for the model of free movement intersection. In the rest of the models there is always only one possible path in every direction, no alternative paths exist. Then the rescheduling of previous agents does not bring us any new information.

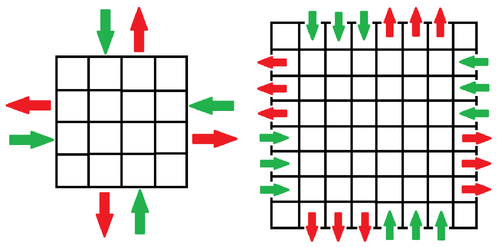
Figure 3: Input tiles for intersections with granularities $n = 4$ and $n = 8$.

Experimental Settings. In our experiments we use two granularities: $n = 4$ and $n = 8$. For $n = 4$ we have one input tile and one output tile for every direction, for $n = 8$ we chose to have 3 input tiles and 3 output tiles in every direction (see Figure 3). In this case, we allow the agents to use arbitrary input and output tile in a requested direction. The requests do not correspond to a particular tile, only to directions. Specific tiles for every agent are chosen by the solver.

Since there are more options for the agents to enter the graph in the 8 by 8 grid, there are more possibilities of how to define the corridors in the intersection (see Figure 4). We can either forbid to change lines once the agent enters the intersection or we can allow it. If the line switching is forbidden then one input tile corresponds exactly to one output tile, thus the agent is forced to enter the graph based on the desired output direction (Figure 4 (a)). On the other hand, if we allow the agent to change lines, the solver has more choices where the agent can enter (Figure 4 (b)). We can increase the number of choices even more by allowing the agents to move even over the input tiles (Figure 4 (c)). Note that the version (c) has still only three input and output tiles in each direction. Also, there is a difference between corridor versions (b) and (c), and the free movement intersection. The free movement intersection allows movement in every direction, while the corridor intersection only allows movement in up to two directions in any given tile.

Increasing the granularity does not change the internal structure of the roundabout intersection and free movement intersection.

5 EXPERIMENTS

In this section, we describe experiments in which we tested the performance of our models at different levels of traffic density. For every experiment, we generated 5 random sequences of input requests corresponding to parameters stated before and we processed them by tested intersections. Every sequence was characterized by minimum and maximum number of requests which could appear in every discrete timestep, by the timestep in which the last request came and by space limit used.

To generate the requests we used a generator with uniform distribution. At first we always randomly chose the number of requests for a timestep (within a given range) and then we generated input and output direction for particular requests independently. The only condition was that the directions can not be equal (we do not support the so-called U-turns).

Since it is not important how a particular instance of a MAPF problem is solved, we do not describe in much detail the MAPF solver that was used in the experiments. The solver is based on reducing the instance into a satisfiability problem using the declar-

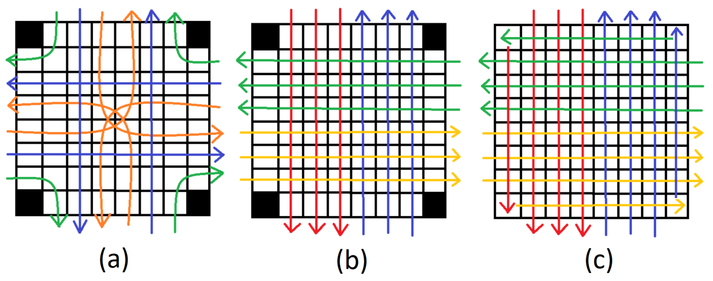
Figure 4: Three different line settings in the 8×8 corridor intersection. (a) strict lines (b) free lines (c) oriented graph.

ative language Picat (Barták et al., 2017). In the adding-solvers, we always compute a path for only one agent (while avoiding the planned agents), thus it is the problem of finding the shortest path. In the rescheduling-solver, we find the makespan optimal solution.

In our tables we always present averages of results of 5 inputs generated with equal parameters. There are three metrics we are interested in:

Plan Length corresponds to the number of discrete timesteps needed to transport all agents to their goals according to the plan. We measure the experiment from time 1.

Delay is the cumulative number of timesteps each agent has to spend in the intersection (or waiting before entering the intersection) over the minimum number of timesteps it would take the agent to reach the goal position if there were no other agents present.

Refused is the number of requests which were not possible to plan within a given space-limit. The requests that were refused did not move through the intersection at all.

Light Traffic. In the first experiment, we worked with intersections with granularity *n=4*. Tested input sequences consisted of exactly one request in every timestep which corresponded to light traffic situation. The space limit was set to 1.

We tested all three models in connection with adding-solver, the free movement intersection model was tested also together with rescheduling-solver.

The results of this experiment are noted in table 1. Almost every request in every model was accepted, the only model which refused some requests was the corridors model. The plan of the roundabout model was a bit longer than the other plans but the difference was not considerable.

The free movement intersection model had a smaller delay than the corridors model, the best result was achieved by free movement intersection model together with rescheduling solver. It means in some situations there existed a better solution than the one

Table 1: Results for light traffic with space limit of 1.

Intersection type	plan length	delay	refused
Roundabout	15.6	1	**0**
Corridor	**13**	2.4	0.2
Free – adding	**13**	1.4	**0**
Free – rescheduling	**13**	**0.8**	**0**

adding-solver found but it had required a change of the plan of some agent(s) planned before.

Medium Traffic. In this experiment, we use the same parameters except that there can be from 0 to 2 requests in every timestep. Generated sequences represent medium traffic density.

The results of this experiment are summarized in table 2. The most successful model was the model of free movement intersection, this time in connection with adding-solver. It happened because rescheduling-solver time-outed on some requests, otherwise it should always find at least the same results as adding-solver.

Table 2: Results for medium traffic with space limit of 1.

Intersection type	plan length	delay	refused
Roundabout	20.2	**2.4**	0.8
Corridor	**17.2**	4.4	1.6
Free – adding	17.4	5.8	**0.4**
Free – rescheduling	18	4.75	1

As in the previous example, the plan length of the roundabout model is worse in comparison with the rest of the models. But we should notice that the delay of this model was the smallest of all models.

Since the rescheduling-solver tends to time-out, we will no longer consider it in further experiments. The experiments will increase in difficulty and the trend of time-outing will be more prominent. The intersection management should be a fast responding entity, the agent can not afford to wait several minutes before receiving approval to enter the intersection.

Heavy Traffic. In this example, there will be heavy traffic consisting of 1 to 3 requests in every timestep. We can expect a bigger number of refused requests.

We run the testing input sequences twice. First, there will be a space limit of 1, in the second run space limit will be 5.

The results of both variants of this experiment are compared in tables 3 and 4. In both cases, we can see that free movement intersection refused the smallest number of agents. With the space limit of 1, there was a shorter plan length and smaller delay when using the corridor model. But we should mention that this number is influenced by the fact that some agents were refused so the resulting plan of corridor model is in reality simpler.

Table 3: Results for heavy traffic with space limit of 1.

Intersection type	plan length	delay	refused
Roundabout	18.4	**5.2**	4.2
Corridor	**14**	8.4	3.4
Free	15	10.8	**2**

Table 4: Results for heavy traffic with space limit of 5.

Intersection type	plan length	delay	refused
Roundabout	18.8	20	0.6
Corridor	16.4	31.4	0.4
Free	**15.4**	**19.2**	**0**

When comparing both tables we can notice that with increasing space limit, the number of refused agents decreases and the delay increases. The length of the plan stays almost the same. Higher space limit adds flexibility to solver so it is able to schedule some extra agents but there is no other possibility than to delay their starts so the delay increases rapidly (almost twice).

Extremely Heavy Traffic. In the following experiment, we have extremely heavy traffic consisting of 2 to 5 requests in every timestep. At first, we allow only a space limit of 2, later we increase it to 5. We expect there will be a huge number of refused requests and the majority of the others will have big delays.

Results of this experiment are available in tables 5, 6. In both cases we can see similar results as in previous experiments: free movement intersection refused the fewest requests and the roundabout model had the largest plan length.

When comparing both tables we can notice that with a space limit of 5 there is a significant decrease in the number of refused requests. Since the length of the plan stayed almost the same, it means the space of the intersection is better utilized.

Table 5: Results for extremely heavy traffic with space limit of 2.

Intersection type	plan length	delay	refused
Roundabout	17.2	**15**	8.2
Corridor	**14**	21.4	9.6
Free	14.8	36.2	**5.8**

Table 6: Results for extremely heavy traffic with space limit of 5.

Intersection type	plan length	delay	refused
Roundabout	19	**47**	3.6
Corridor	17	59.2	3.8
Free	**16.2**	63.6	**1.6**

Unfortunately with a space limit of 5, the delay increased considerably which was caused mainly because of the additional space limit. We can observe it in the results of the roundabout model. With the space limit of 2, there was only 15 timesteps delay in this model. But with the space limit of 5, it increased to 47. Since there are no alternative (and longer) paths in the roundabout model the value of the delay in this model consists only of waiting in front of the intersection. Thus on average, in the intersection with space limit of 5 an agent waits in front of the intersection three times longer than in the intersection with space limit of 2.

It means that if we want the solver to accept as many requests as possible and to plan their paths with minimum delay all at once, we have to reach a compromise.

Long-term Extremely Heavy Traffic. In this experiment, we will simulate the situation of an arising traffic jam. Every timestep there will be between 2 and 5 requests and the experiment will run 3 times longer than all previous experiments. Only free movement intersection will be tested and there will be two runs of the experiment - with a space limit of 2 and 5.

The results of this experiment can be seen in table 7. When using the space limit of 5, there were accepted 2 more agents but otherwise, the result is not very successful. The delay increased more than twice which is quite unsatisfying.

Table 7: Results for long-term extremely heavy traffic over the free-movement intersection solved by the adding-solver.

Space limit	plan length	delay	refused
2	**36**	**169**	32
5	40	350	**30**

Figure 5 shows the number of refused requests in ev-

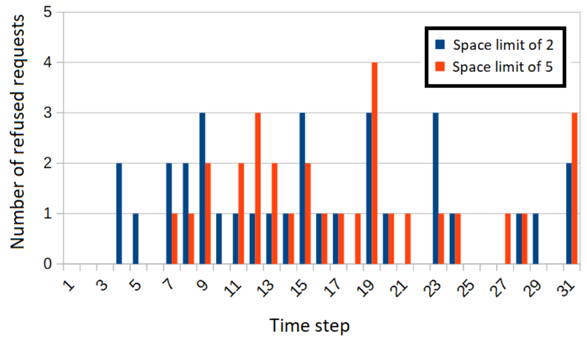

Figure 5: Number of refused requests in individual timesteps in long-term extremely heavy traffic.

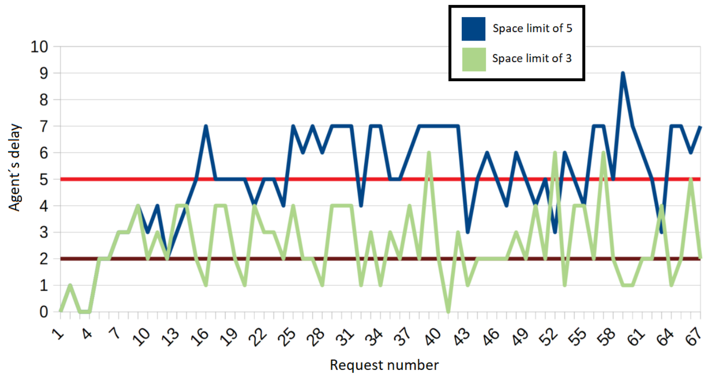

Figure 6: Delay of particular agents in long-term extremely heavy traffic.

ery timestep. We can see that there were 2 requests refused in space limit of 2 before refusal of the first request in space limit of 5. The reason for that was probably the size of the space limit — a smaller time window was filled up sooner so the solver had to start refusing the requests sooner. After a 5-step window was filled up the number of refused requests was almost the same.

In the figure 6 we can observe the origin of the traffic jam. A few agents at the beginning have just small delays but quickly the majority of agents start to have delay approximately the size corresponding to the relevant space limit. This means that the majority of agents are planned at the end of the allowed time window, almost none of them were planned at their required time.

We can conclude that in the longer horizon, a bigger space limit does not increase the throughput of the intersection, in contrast, it only delays particular agents more. On average, an agent has a 2.5 step delay in plan with a space limit of 2 while it has a 5.1 step delay in plan with a space limit of 5.

By increasing the space limit we slightly postponed the formation of the traffic jam but we had not avoided it. So it is reasonable to use a small space limit to increase the throughput of the intersection.

Intersection 8 $\times$ 8. In the last experiment, we use intersection with granularity $n = 8$ and we test several corridor models which we compare with free movement intersection.

After extending the corridor model to an intersection with higher granularity, we can observe two interesting issues. At first such an intersection should have more than one input and output tiles for every direction, otherwise, input tiles would be a bottleneck of the intersection. Furthermore, if we model corridors in a bigger intersection we can have separate lanes for particular directions.

In our experiment, we use 3 input and output tiles for every direction and 3 different types of corridor model described before.

The results of this experiment are described in table 8. The models are ordered from the most restrictive to the most liberal one in the table. The number of refused agents corresponds to this ordering, more liberal models refused fewer agents. The length of the plan is almost equal, the most interesting metric in this experiment is the delay.

The smallest delay had the model of strict lines corridor. It was because its graph has almost no alternative paths and thus the planned paths are optimal. An interesting result is that the free lines corridor has a bigger delay although it has an equal number of refused requests. It is probably the result of freedom which was given to the model, it used some unsuitable paths in the beginning and then the other agents had to wait in front of the intersection.

Table 8: Results for 8×8 intersections based on different line management.

Intersection type	plan length	delay	refused
Corridor – strict lines	17.8	**12.4**	1.2
Corridor – free lines	18.5	15.8	1.2
Corridor – oriented graph	18.4	17	1
Free	**17.6**	13	**0**

6 CONCLUSION

In this paper, we studied an intelligent intersection design. The intersection manager receives requests for traversing the shared environment and its job is to navigate all of the agents through the intersection safely and as efficiently as possible.

Rather than an algorithm that plans and schedules the paths itself, we studied the spatial design of the intersection and its effect on the efficiency of the found plan. The planning itself can be seen as an instance of multi-agent path-finding. We assumed two types of intersections that are commonly used on roads today - 4-way intersection with turning lanes and roundabout. We also added an intersection with less restriction on the movements, where agents can travel in any direction.

The extensive simulation experiments show that while roundabout type intersections do not cause much extra delay to the agents, the traversed path is quite long in comparison with other types. The free movement type intersection has the highest throughput of the agents at the expense of higher delay. This is caused by the higher flexibility of the paths the agents can traverse. If the optimal path in the restricted intersection is occupied, the agent has to wait, however, in the free movement intersection, the agent can still find some less optimal path to go through.

ACKNOWLEDGEMENTS

This research is supported by the Czech Science Foundation under the project P103-19-02183S, by SVV project number 260 453, and by the Charles University Grant Agency under the project 90119.

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