Assignment 5 FTS

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October 13, 2022

1. $y_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$

   (a)
$$\mu = E(y_t) = 0$$
$$Var(E_t) = \gamma_0 = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$

   (b)
$$\gamma_1 = E[(\varepsilon_t - \mu)(\varepsilon_{t-1} - \mu)] = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$
$$\rho_1 = \frac{\gamma_1}{\gamma_0} = (1 + \theta_1^2 + \theta_2^2)^{-1}(1 + \theta_1^2 + \theta_2^2) \tag{1}$$
$$\gamma_2 = E[(\varepsilon_t - \mu)(\varepsilon_{t-2} - \mu)] = -\theta_2\sigma^2$$
$$\rho_2 = \frac{\gamma_2}{\gamma_0} = (1 + \theta_1^2 + \theta_2^2)^{-1}(-\theta_2) \tag{2}$$

   For $i > 2$
$$\gamma_i = E[(\varepsilon_t - \mu)(\varepsilon_{t-i} - \mu)] = 0$$
$$\rho_i = \frac{\gamma_i}{\gamma_0} = 0 \tag{3}$$

   (c) Since it is a MA process, the stationarity always holds. The necessary and sufficient condition for invertibility is that all roots of the eigenequation are in the unit circle.

2. $y_t = c + \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + e_t$, $e_t = \phi_1 e_{t-1} + ... + \phi_m e_{t-m} + \varepsilon_t$. Prove that $y_t \sim AR(m + p)$.
   Pf. Denote $1 - \alpha_1 L - ... - \alpha_p L^p$ as $\alpha(L)$, and $1 - \phi_1 L - ... - \phi_m L^m$ as $\phi(L)$.
   $\varepsilon_t = \phi(L)e_t$, and $e_t = \varepsilon_t\phi(L)^{-1}$, then $\alpha(L)y_t = c + \varepsilon_t\phi(L)^{-1}$, $\alpha(L)\phi(L)y_t = \phi(L)c + \varepsilon_t$, conforming to the expression of AR model.
   $\alpha(L)\phi(L) = \alpha_p L^p \phi_m L^m + ... = \alpha_p\phi_m L^{p+m} + ...$, and thus $y_t \sim AR(m + p)$.

3. $y_t = c + \alpha_1 y_{t-1} + \varepsilon_t - \theta_1\varepsilon t - 1$

   (a) Converting to MA form.
   We rewrite the equation.

$$\begin{aligned}(1 - \alpha_1 L)y_t &= c + (1 - \theta_1 L)\varepsilon_t \\ y_t &= (1 - \alpha_1 L)^{-1}c + (1 - \alpha_1 L)^{-1}(1 - \theta_1 L)\varepsilon_t \\ &= (1 - \alpha_1 L)^{-1}c + (1 + \alpha_1 L + \alpha_1^2 L^2 + ...)(1 - \theta_1 L - \theta_1^2 L^2 - ...)\varepsilon_t \\ &= (1 - \alpha_1 L)^{-1}c + \varepsilon_t - [(\theta_1 - \alpha_1)\varepsilon_{t-1} + \alpha_1(\theta_1 - \alpha_1)\varepsilon_{t-2} + ...]\end{aligned} \tag{4}$$

(b) Converting to AR form.
rewrite the equation.

$$\begin{aligned}(1-\alpha_1 L)y_t &= c + (1-\theta_1 L)\varepsilon_t \\ (1-\alpha_1 L)(1+\theta_1 L+\theta_1^2 L^2+...)y_t &= (1-\theta_1 L)^{-1}c + \varepsilon_t \\ (1-\alpha_1 L)(1-\theta_1 L)y_t &= (1-\theta_1 L)^{-1}c + \varepsilon_t \\ y_t + (\theta_1-\alpha_1)y_{t-1} + \theta_1(\theta_1-\alpha_1)y_{t-2} + ... &= (1-\theta_1 L)^{-1}c + \varepsilon_t\end{aligned}$$

So

$$y_t = (\alpha_1-\theta_1)y_{t-1} + \theta_1(\alpha_1-\theta_1)y_{t-2} + ... + (1-\theta_1 L)^{-1}c + \varepsilon_t \tag{5}$$

(c) If $\alpha_1 = \theta_1$.
Then (4) turns into $y_t = (1-\alpha_1 L)^{-1}c + \varepsilon_t$, and (5) turns into $y_t = (1-\theta_1 L)^{-1}c + \varepsilon_t$.