HW2

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Homework problems are listed below.

- Chapter 10: 2, 6, 10, 12.
- Chapter 11: 5, 9, 14.

1 Chapter 10: Elements of Hilbert space theory

Problem $2, 6, 10, 12$.

1.1 Problem 2

Problem Statement. Show that in any inner product space, the function $y \mapsto \langle x, y\rangle$ is continuous. (x is an arbitrary vector.)
My Solution. Let f be the functional satisfying $f(y) = \langle y, x\rangle$ for all y. It is clear that f is a linear functional since for arbitrary constant α,

$$f(\alpha y) = \langle \alpha y, x\rangle = \alpha\langle y, x\rangle = \alpha f(y).$$

Using the Cauchy-Schwarz inequality, we can say

$$|f(y)| \leq |x||y| = M|y|.$$

Since x is a fixed vector, we can say that $|f(y)|$ is bounded by $M|y|$ with some constant M. Since bounded linear functionals are continuous, we can say f is continuous.

1.2 Problem 6

Problem Statement. Suppose $\{e_j, j \geq 1\}$ is a complete orthonormal sequence in a Hilbert space. Show that if $\{f_j, j \geq 1\}$ is an orthonormal sequence satisfying

$$\sum_{j=1}^{\infty} \|e_j - f_j\|^2 < 1,$$

then $\{f_j, j \geq 1\}$ is also complete.
My Solution. Since e_j is a complete orthonormal sequence in a Hilbert space, for all ϵ there exist N s.t.

$$\|e_n - e_m\| < \frac{\epsilon}{3} \quad \forall n, m > N.$$

Also, since $\sum_{j=1}^{n} \|e_j - f_j\|^2$ is increasing but bounded by 1, it converges and $\|e_j - f_j\|^2$ goes to 0 when n goes to ∞, which means for all ϵ there exists M s.t.

$$\|e_j - f_j\| < \frac{\epsilon}{3} \quad \forall j > M.$$

Then, for all ϵ there exists $K = max(N, M)$ s.t.

$$\|f_n - f_m\| \leq \|f_n - e_n\| + \|e_n - e_m\| + \|e_m - f_m\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall n, m > K,$$

which means $\{f_j\}$ is complete.

1.3 Problem 10

Problem Statement. Suppose $\{e_j, j \geq 1\}$ and $\{f_j, j \geq 1\}$ are orthonormal bases in $\mathcal{H}$. Show that for any Hilbert-Schmidt operators Ψ, Φ

$$\sum_{i=1}^{\infty} \langle \Psi(f_i), \Phi(f_i) \rangle = \sum_{i=1}^{\infty} \langle \Psi(e_i), \Phi(e_i) \rangle.$$

My Solution. Let $x = \sum_{i=1}^{\infty} f_i = \sum_{i=1}^{\infty} e_i$. Then,

$$\langle \Psi(x), \Phi(x) \rangle = \langle \Psi(\sum_{i=1}^{\infty} f_i), \Phi(\sum_{i=1}^{\infty} f_i) \rangle = \sum_{i=1}^{\infty} \langle \Psi(f_i), \Phi(f_i) \rangle,$$

and

$$\langle \Psi(x), \Phi(x) \rangle = \langle \Psi(\sum_{i=1}^{\infty} e_i), \Phi(\sum_{i=1}^{\infty} e_i) \rangle = \sum_{i=1}^{\infty} \langle \Psi(e_i), \Phi(e_i) \rangle,$$

since

$$\langle \Psi(f_i), \Phi(f_j) \rangle = \langle \Psi(e_i), \Phi(e_j) \rangle = 0 \quad for \quad i \neq j.$$

Therefore,

$$\sum_{i=1}^{\infty} \langle \Psi(f_i), \Phi(f_i) \rangle = \langle \Psi(x), \Phi(x) \rangle = \sum_{i=1}^{\infty} \langle \Psi(e_i), \Phi(e_i) \rangle.$$

1.4 Problem 12

Problem Statement. Show that if L is bounded then L^* is also bounded, and

$$\|L^*\|_{\mathcal{L}} = \|L\|_{\mathcal{L}}, \quad \|L^* L\|_{\mathcal{L}} = \|L\|_{\mathcal{L}}^2.$$

My Solution. For an arbitrary vector h in $\mathcal{H}$, with $\|h\| \leq 1$,

$$\|Lh\|^2 = \langle Lh, Lh \rangle = \langle L^* Lh, h \rangle \leq \|L^* Lh\| \|h\| \leq \|L^* L\| \leq \|L^*\| \|L\|.$$

Letting $\|h\| = 1$ gives us

$$\|L\|^2 \leq \|L^* L\| \leq \|L^*\| \|L\|, \quad \|L\| \leq \|L^*\|.$$

We can do the same thing for $\|L\|$, and since $L^{**} = L$,

$$\|L^* h\|^2 = \langle L^* h, L^* h \rangle = \langle LL^* h, h \rangle \leq \|LL^* h\| \|h\| \leq \|LL^*\| \leq \|L\| \|L^*\|,$$

and

$$\|L^*\|^2 \le \|LL^*\| \le \|L\|\|L^*\|, \quad \|L^*\| \le \|L\|.$$

Then,

$$\|L\| = \|L^*\|, \|L^*L\| = \|LL^*\| = \|L\|^2$$

2 Chapter 11: Random Functions

Problem $5, 9, 14$.

2.1 Problem 5

Problem Statement. Suppose $Y_{k,n}, Y_k$ are random varaibles such that for every $M \ge 1$,

$$[Y_{1,n}, Y_{2,n}, \cdots, Y_{M,n}]^T \xrightarrow{P} [Y_1, Y_2, \cdots, Y_M]^T$$

in the Euclidean space $\mathbb{R}^M$. Suppose $\{w_k, k \ge 1\}$ is a sequence of numbers such that

$$\sum_{k=1}^{\infty} |w_k| E|Y_k| < \infty \quad and \quad \sum_{k=1}^{\infty} |w_k| \sup_{n\ge1} E|Y_{k,n} - Y_k| < \infty.$$

Using Theorem 11.1.3, show that $\sum_{k=1}^{\infty} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$.

Theorem 2.1. (Theorem 11.1.3) : Suppose that for each u, $X_n(u) \xrightarrow{d} X(u)(n \to \infty)$, and $X(u) \xrightarrow{d} X(u \to \infty)$. If

$$\lim_{u\to\infty} \limsup_{n\to\infty} P(d(X_n(u), X_n) > \epsilon) = 0,$$

then $X_n \xrightarrow{d} X$.

My Solution. Since $\sum_{k=1}^{m} |w_k| E|Y_k|$ is a increasing sequence and bounded, we can say

$$|w_k| E|Y_k| \to 0 \quad as \quad k \to \infty.$$

Also, since $\sum_{k=1}^{m} |w_k| \sup_{n\ge1} E|Y_{k,n} - Y_k|$ is a increasing sequence and bounded, we can say

$$|w_k| \sup_{n\ge1} E|Y_{k,n} - Y_k| \to 0 \quad as \quad k \to \infty.$$

Using the property of supremum, we can say

$$|w_k| E|Y_{k,n} - Y_k| \to 0 \quad as \quad n \to \infty, .$$

Then by using the fact that $\sum_{k=1}^{m} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{m} w_k Y_k$ as $n \to \infty$ and $\sum_{k=1}^{m} w_k Y_k \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$ as $m \to \infty$, we can say

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(d(\sum_{k=1}^{m} w_k Y_{k,n}, \sum_{k=1}^{m} w_k Y_k) > \epsilon) = 0.$$

Then by using Theorem 11.1.3,

$$\sum_{k=1}^{\infty} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k.$$

2.2 Problem 9

Problem Statement. Suppose $\mathcal{H}$ is an infinite dimensional separable Hilbert space and $\{e_j, j \geq 1\}$ is an orthonormal system. Define the operator Ψ by

$$\Psi(x) = \sum_{j=1}^{\infty} j^{-1} \langle x, e_j \rangle e_j.$$

Show that Ψ is bounded, symmetric and nonnegative definite, but it is not a covariance operator.
My Solution. Since j^{-1} is a decreasing sequence,

$$\|\Psi\| = \sup_{\|x\| \geq 1} \|\Psi(x)\| = 1 < \infty,$$

which means Ψ is bounded. Also, since

$$\langle \Psi(x), y \rangle = \langle \sum_{j=1}^{\infty} j^{-1} \langle x, e_j \rangle e_j, y \rangle = \sum_{j=1}^{\infty} j^{-1} \langle x, e_j \rangle \langle e_j, y \rangle = \langle x, \sum_{j=1}^{\infty} j^{-1} \langle y, e_j \rangle e_j \rangle = \langle x, \Psi(y) \rangle,$$

Ψ is symmetric. Letting $y = x$ for the previous equation gives us the result of

$$\langle \Psi(x), x \rangle = \sum_{j=1}^{\infty} j^{-1} \langle x, e_j \rangle^2 \geq 0,$$

which means Ψ is nonnegative definite. However, Ψ is not a covariance operator since it is not a trace class operator,

$$\sum_{j=1}^{\infty} j^{-1} \not< \infty.$$

2.3 Problem 14

Problem Statement. Suppose X satisfies Definition 11.3.2 and L be a bounded operator. Show that $L(X)$ is Gaussian; find its expected value and covariance operator.

Definition 2.1. (Definition 11.3.2) : A random function X is said to be Gaussian if its characteristic functional has the form

$$\varphi_X(y) = \exp\left\{ i\langle \mu, y \rangle - \frac{1}{2} \langle C(y), y \rangle \right\},$$

where $\mu \in \mathcal{H}$ and C is a covariance operator.

My Solution. Since characteristic functional of a random function $L(X)$ can be expressed as

$$\varphi_{L(X)}(y) = E \exp\{i \langle y, L(X) \rangle\} = E \exp\{i \langle L^*(y), X \rangle\},$$

by using Definition 11.3.2, we can say

$$\varphi_{L(X)}(y) = \exp\left\{ i\langle \mu, L^* y \rangle - \frac{1}{2} \langle C(L^* y), L^* y \rangle \right\} = \exp\left\{ i\langle L(\mu), y \rangle - \frac{1}{2} \langle L(C(L^* y)), y \rangle \right\}.$$

Then, $L(X)$ is indeed a Gaussain. The expected value is $L(\mu)$ and the covariance operator is LCL^*.