FDA
Homework 2 key
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1. **10.6.2**
   Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{H}$. Let $L : \mathcal{H} \to \mathcal{H}$ such that $y \mapsto \langle x, y\rangle$. By CS-inequality,
   $$\|L(y_1 - y_2)\| = \|\langle x, y_1 - y_2\rangle\| \leq \|x\|\|y_1 - y_2\|.$$
   For any $\epsilon > 0$, if $\delta = \epsilon/\|x\|$, $\|y_1 - y_2\| < \delta$ implies $\|L(y_1 - y_2)\| < \epsilon$

2. **10.6.6**
   Let $x \in \mathcal{H}$. Let's rewrite this.
   $$x = \sum_{j=1}^{\infty} \langle x, f_j\rangle f_j + \tilde{x},$$
   where $\langle \tilde{x}, f_j\rangle = 0$ for all $j = 1, 2, \ldots$. Then
   $$\|\tilde{x}\|^2 = \sum_{j=1}^{\infty} \langle \tilde{x}, e_j\rangle^2 = \sum_{j=1}^{\infty} (\langle \tilde{x}, e_j - f_j\rangle + \langle \tilde{x}, f_j\rangle)^2 = \sum_{j=1}^{\infty} \langle \tilde{x}, e_j - f_j\rangle^2 \leq \|\tilde{x}\|^2 \sum_{j=1}^{\infty} \|e_j - f_j\|^2.$$
   The first equation is from Parseval's equality. The second and third equation is from $\langle \tilde{x}, f_j\rangle = 0$. The last inequality is from the CS-inequality. Then by the condition in the question $\sum_{j=1}^{\infty} \|e_j - f_j\|^2 < 1$, we have
   $$\|\tilde{x}\|^2 \leq \|\tilde{x}\|^2 \sum_{j=1}^{\infty} \|e_j - f_j\|^2 < \|\tilde{x}\|^2.$$
   This tells $\|\tilde{x}\|^2 = 0$ and $\tilde{x} = 0$. In other words,
   $$x = \sum_{j=1}^{\infty} \langle x, f_j\rangle f_j.$$

3. **10.6.10**
   Note that using Parseval's equality,
   $$\langle x, y\rangle = \langle \sum_i \langle x, e_i\rangle e_i, \sum_j \langle y, e_j\rangle e_j\rangle = \sum_i \langle x, e_i\rangle\langle x, e_j\rangle,$$
   if $\{e_i\}$ is an orthonormal basis.

   Let Ψ and Φ be two Hilbert-Schmidt operators. The inner product between these two is

$$\begin{aligned}\sum_{i=1}^{\infty}\langle\Psi(f_i),\Phi(f_i)\rangle &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle\Psi(f_i),e_j\rangle\langle\Phi(f_i),e_j\rangle = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\langle f_i,\Psi^*(e_j)\rangle\langle f_i,\Phi^*(e_j)\rangle \\ &= \sum_{j=1}^{\infty}\langle\Psi^*(e_j),\Phi^*(e_j)\rangle = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle e_i,\Psi^*(e_j)\rangle\langle e_i,\Phi^*(e_j)\rangle \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle\Psi(e_i),e_j\rangle\langle\Phi(e_i),e_j\rangle = \sum_{i=1}^{\infty}\langle\Psi(e_i),\Phi(e_i)\rangle\end{aligned}$$

4. **10.6.12**

(a) $\|L^*\|_{\mathcal{L}} = \|L\|_{\mathcal{L}}$

i $\|L^*\|_{\mathcal{L}} \leq \|L\|_{\mathcal{L}}$
For any $x \in \mathcal{H}$,

$$\|L^*(x)\|^2 = \langle L^*(x), L^*(x)\rangle = \langle x, L(L^*(x))\rangle \leq \|x\|\|L(L^*(x))\|$$

By dividing $\|L^*(x)\|$,

$$\|L^*(x)\| \leq \|x\|\left\|L\left(\frac{L^*(x)}{\|L^*(x)\|}\right)\right\| \leq \|x\|\|L\|_{\mathcal{L}}.$$

Thus $\|L^*\|_{\mathcal{L}} \leq \|L\|_{\mathcal{L}}$. Since L is bounded, we can also verify that L^* is bounded.

ii $\|L^*\|_{\mathcal{L}} \geq \|L\|_{\mathcal{L}}$
We can use a similar argument by replacing L with L^* since we verified that L^* is bounded.

$$\|L(x)\|^2 = \langle L(x), L(x)\rangle = \langle x, L^*(L(x)\rangle \leq \|x\|\|L^*(L(x))\|$$

Thus

$$\|L(x)\| \leq \|x\|\|L^*\|_{\mathcal{L}}.$$

(b) To see $\|L^*L\|_{\mathcal{L}} = \|L\|_{\mathcal{L}}^2$, using the above results $\|L^*\|_{\mathcal{L}} = \|L\|_{\mathcal{L}}$,

$$\|L^*L(x)\|^2 \leq \|L^*\|_{\mathcal{L}}\|L(x)\| \leq \|L^*\|_{\mathcal{L}}\|L\|_{\mathcal{L}}\|x\| \leq \|L\|_{\mathcal{L}}^2\|x\|.$$

Thus $\|L^*L\|_{\mathcal{L}} \leq \|L\|^2$, and L^*L is bounded since L is bounded.
On the other hand,

$$\|L(x)\|^2 = \langle L(x), L(x)\rangle = \langle L^*L(x), x\rangle \leq \|L^*L\|_{\mathcal{L}}\|x\|.$$

Thus $\|L\|_{\mathcal{L}}^2 \leq \|L^*L\|_{\mathcal{L}}$. Therefore, $\|L^*L\|_{\mathcal{L}} = \|L\|_{\mathcal{L}}^2$.

5. **11.5.5**

- **Theorem 11.1.3** Suppose that for each u, $X_n(u) \xrightarrow{D} X(u)$ $(n \to \infty)$, and $X(u) \xrightarrow{D} X(u \to \infty)$. If

$$\lim_{u\to\infty} \limsup_{n\to\infty} P(d(X_n(u), X_n) > \epsilon) = 0,$$

then $X_n \xrightarrow{D} X$.

- Let $X_n(M) = \sum_{k=1}^{M} w_k Y_{k,n}$, $X_n = \sum_{k=1}^{\infty} w_k Y_{k,n}$, $X = \sum_{k=1}^{\infty} w_k Y_k$. Then the conditions in the question tells us that $X_n(M) \xrightarrow{D} X(M)$ and $X(M) \xrightarrow{D} X$.

- By Markov inequality,

$$\begin{aligned} P(|X_n(M) - X_n| > \epsilon) &\le \frac{E(|X_n(M) - X_n|)}{\epsilon} \\ &= \frac{E|\sum_{k=M+1}^{\infty} w_k(Y_{k,n} - Y_k + Y_k)}{\epsilon} \\ &\le \underbrace{\frac{\sum_{k=M+1}^{\infty} |w_k| E|Y_{k,n} - Y_k|}{\epsilon}}_{(a)} + \underbrace{\frac{\sum_{k=M+1}^{\infty} |w_k| E|Y_k|}{\epsilon}}_{(b)} \end{aligned}$$

We need to show that

$$\lim_{M\to\infty} \limsup_{n\to\infty}((a) + (b)) = 0.$$

(a)

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{\sum_{k=M+1}^{\infty} |w_k| E|Y_{k,n} - Y_k|}{\epsilon} \le \frac{1}{\epsilon} \lim_{M\to\infty} \sum_{k=M+1}^{\infty} |w_k| \sup_{n\ge 1} E|Y_{k,n} - Y_k| = 0,$$

because $\sum_{k=1}^{\infty} |w_k| \sup_{n\ge1} E|Y_{k,n} - Y_k| < \infty$.

(b)

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{\sum_{k=M+1}^{\infty} |w_k| E|Y_k|}{\epsilon} = \frac{1}{\epsilon} \lim_{M\to\infty} \sum_{k=M+1}^{\infty} |w_k| E|Y_k| = 0$$

because $\sum_{k=1}^{\infty} |w_k| E|Y_k| < \infty$.

6. **11.5.9**

- **Bounded**

$$\|\Psi(x)\|^2 = \langle \Psi(x), \Psi(x) \rangle = \sum_{k=1}^{\infty} j^{-2} \langle x, e_j \rangle^2 \le \sum_{k=1}^{\infty} \langle x, e_j \rangle^2 = \|x\|^2$$

Thus $\|\Psi\|_{\mathcal{L}} \le 1$.

- **Symmetric**

$$\langle \Psi(x), y\rangle = \left\langle \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j, y \right\rangle = \sum_{j=1}^{\infty} \langle x, e_j\rangle y, e_j = \langle x, \Psi(y)\rangle$$

- **Nonnegative**

$$\langle \Psi(x), x\rangle = \sum_{j=1}^{\infty} j^{-1} \langle x, e_j\rangle^2 \geq 0$$

- **Not a covariance operator**

$$\Psi = \sum_{j=1}^{\infty} j^{-1} e_j \otimes e_j.$$

  The eigenvalues are $\{j^{-1}\}_{j=1}^{\infty}$ and $\sum_j j^{-1} = \infty$.

7. **11.5.14**

   The characteristic function of $L(X)$ is

$$\begin{aligned} \phi_{L(X)}(y) &= E \exp(i\langle y, L(X)\rangle) \\ &= E \exp(i\langle L^*(y), X\rangle\rangle) \\ &= \exp\{i\langle \mu, L^*(y)\rangle - \frac{1}{2}\langle C(L^*(y)), L^*(y)\rangle\} \\ &= \exp\{i\langle L(\mu), y\rangle - \frac{1}{2}\langle LCL^*(y), y\rangle \end{aligned}$$

   $L(X)$ is Gaussian with $L^*(\mu)$ and LCL^*.