FDA
Homework 3 key
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1. **12.8.2**
   Suppose that $Lv = \lambda v$ with $\|v\| = 1$. $\|L\|_{\mathcal{L}} = \sup_{\|x\|=1} \|L(x)\| \geq \|L(v)\| = |\lambda|\|v\| = |\lambda|$

2. **12.8.5**
   Hint: $|\langle \hat{v}_j - v_j, v_j \rangle| = \|\hat{v}_j - v_j\|^2/2$.
   The proof is obvious by the following two equations.

$$\|\hat{v}_j - v_j\|^2 = 2 - 2\langle \hat{v}_j, v_j \rangle \tag{1}$$

$$\langle \hat{v}_j - v_j, v_j \rangle = \langle \hat{v}_j, v_j \rangle - 1 \tag{2}$$

   (1) $-\ 2 \times$ (2) gives $|\langle \hat{v}_j - v_j, v_j \rangle| = \|\hat{v}_j - v_j\|^2/2$. $\square$

   Next, using the hint,

$$|N\langle \hat{v}_j - v_j, v_j \rangle| = \|N^{1/2}(\hat{v}_j - v_j)\|^2/2 \xrightarrow{D} \|Z\|^2/2$$

$$C_j = \sum_{k \neq j} \sum_{l \neq j} (\lambda_j - \lambda_k)^{-1}(\lambda_j - \lambda_l)^{-1} \langle \Gamma, v_k \otimes v_j \otimes v_l \otimes v_j \rangle (v_k \otimes v_l)$$

   where $Z \sim N(0, C_j)$ by continuous mapping theorem and the result of Corollary 12.3.1.
   Therefore, $|N\langle \hat{v}_j - v_j, v_j \rangle| = O_p(1)$ $\square$

   $|\langle \hat{v}_j - v_j, v_j \rangle|$ only considers the difference in v_j's direction in $\mathcal{H}$.

3. **12.8.6**
   Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal system in $\mathcal{H}$.

$$\begin{aligned}
\|x \otimes y\|^2_{\mathcal{H}\otimes\mathcal{H}} &= \langle x \otimes y, x \otimes y \rangle_{\mathcal{H}\otimes\mathcal{H}} \\
&= \langle x, x \rangle \langle y, y \rangle \\
&= \|x\|^2 \|y\|^2 \\
&= \sum_{j=1}^{\infty} \langle y, e_j \rangle^2 \|x\|^2 \\
&= \sum_{j=1}^{\infty} \|\langle y, e_j \rangle x\|^2 \\
&= \|\langle y, \cdot \rangle x\|^2_S
\end{aligned}$$

4. **12.8.7** See the lecture 7.

5. **12.8.12**

$$\begin{aligned}
\langle \hat{C} - C, \hat{v}_j \otimes v_k \rangle &= \sum_{i=1}^{\infty} \langle (\hat{C} - C)v_i, (\hat{v}_j \otimes v_k)v_i \rangle \\
&= \langle \hat{C}v_k - Cv_k, \hat{v}_j \rangle \quad \text{because } \langle v_k, v_i \rangle = \delta_{ik} \\
&= \langle \hat{C}v_k - \lambda_k v_k, \hat{v}_j \rangle \\
&= \langle v_k, \hat{C}\hat{v}_j \rangle - \lambda_k \langle v_k, \hat{v}_j \rangle \\
&= \langle v_k, \hat{\lambda}_j \hat{v}_j \rangle - \langle v_k, \lambda_k \hat{v}_j \rangle \\
&= (\hat{\lambda}_j - \lambda_k) \langle v_k, \hat{v}_j \rangle \\
&= (\hat{\lambda}_j - \lambda_k) \langle v_k, \hat{v}_j - v_j \rangle \quad \text{since } \langle v_k, v_j \rangle = 0
\end{aligned}$$

□

For the next part, by Corollary 12.3.1,

$$N^{1/2}(\hat{v}_j - v_j) \xrightarrow{D} N(0, C_j),$$

where

$$C_j = \sum_{i \neq j} \frac{\lambda_i \lambda_j}{(\lambda_j - \lambda_i)^2} (v_i \otimes v_i)$$

Let $L_k : \mathcal{H} \to \mathbb{R}$ such that $L_k(v) = \langle v, v_k \rangle$ which is a continuous linear function. If X is Gaussian with μ, C, $L_k(X)$ is Gaussian with $(L_k(\mu), L_k C L_k^*)$ by Problem 11.5.14. By the continuous mapping theorem,

$$N^{1/2}\langle \hat{v}_j - v_j, v_k \rangle \xrightarrow{D} N(0, L_k C_j L_k^*),$$

Note that $L_k^*(c) = cv_k$ ($L_k : \mathbb{R} \to \mathcal{H}$). To see this, for any $v \in \mathcal{H}, c \in \mathbb{R}$,

$$\langle L_k v, c \rangle = \langle \langle v, v_k \rangle, c \rangle = c \langle v, v_k \rangle = \langle v, cv_k \rangle = \langle v, L_k^*(c) \rangle$$

And $L_k C_j L_k^* = \langle v_k, C_j v_k \rangle$ because $L_k C_j L_k^*(a) = L_k(C_j(av_k)) = a\langle v_k, C_j v_k \rangle$

$$\begin{aligned}
\langle v_k, C_j v_k \rangle &= \langle v_k, \sum_{i \neq j} \frac{\lambda_i \lambda_j}{(\lambda_j - \lambda_i)^2} (v_i \otimes v_i) v_k \rangle \\
&= \langle v_k, \sum_{i \neq j} \delta_{ik} \frac{\lambda_i \lambda_j}{(\lambda_j - \lambda_i)^2} v_i \rangle \\
&= \frac{\lambda_k \lambda_j}{(\lambda_j - \lambda_k)^2}
\end{aligned}$$

Thus,

$$N^{1/2}\langle \hat{v}_j - v_j, v_k \rangle \xrightarrow{D} N\left(0, \frac{\lambda_k \lambda_j}{(\lambda_j - \lambda_k)^2}\right),$$