HW3

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2023. 10. 17.

Homework problems are listed below.

- Chapter 12: 2, 5, 6, 7, 12.

1 Chapter 12: Inference from a random sample

Problem $2, 5, 6, 7, 12$.

1.1 Problem 2

Problem Statement. Show that for every eigenvalue λ of a bounded operator L, we have $|\lambda| \leq \|L\|_{\mathcal{L}}$.

My Solution. For

$$L(x) = \sum_{i=1}^{\infty} \lambda_i \langle x, v_i \rangle v_i,$$

we can say that

$$\|L(x)\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 \langle x, v_i \rangle^2 \leq \lambda_{max}^2 \|x\|^2, \quad \|L(x)\| \leq \lambda_{max} \|x\|.$$

Since

$$\|L\|_{\mathcal{L}} = \sup_{\|x\| \leq 1} \|L(x)\|,$$

by using the definition of supremum we can say

$$\|L\|_{\mathcal{L}} = \lambda_{max} \geq |\lambda|,$$

for all λ.

1.2 Problem 5

Problem Statement.

Assume that $X_1, \dots, X_N$ are iid elements of $L^2[0,1]$ with $E\|X_n\|^4 < \infty$ and whose first p eigenvalues are distinct. Prove that

$$|N \langle \hat{v}_j - v_j, v_j \rangle| = O_P(1), \quad \text{for } j = 1, \dots p.$$

Why is this a seemingly unusual convergence rate? (Hint: $|\langle \hat{v}_j - v_j, v_j\rangle| = \frac{1}{2}\|\hat{v}_j - v_j\|^2$)
My Solution. Since $|\langle \hat{v}_j - v_j, v_j\rangle| = \frac{1}{2}\|\hat{v}_j - v_j\|^2$,

$$|N\langle \hat{v}_j - v_j, v_j\rangle| = \frac{N}{2}\|\hat{v}_j - v_j\|^2.$$

Also, since $X_1, \ldots, X_N$ are iid elements of $L^2[0,1]$ with $E\|X_n\|^4 < \infty$ with distinct p eigenvalues, we can say that

$$N^{1/2}(\hat{v}_j - v_j) = T_{jN} + o_P(1) \quad \text{for} \quad T_{jN} = \sum_{k \neq j} (\lambda_j - \lambda_k)^{-1} \langle Z_N, v_k \otimes v_j\rangle v_k.$$

Then,

$$|N\langle \hat{v}_j - v_j, v_j\rangle| = \frac{1}{2}\|N^{1/2}(\hat{v}_j - v_j)\|^2 = \frac{1}{2}\|T_{jN} + o_P(1)\|^2 = o_P(1) = O_P(1).$$

It seems unusual because the expected values

$$E\|\hat{\mu} - \mu\|^2, E\|\hat{C} - C\|^2, E\|\hat{v}_j - v_j\|^2, E\|\hat{\lambda}_j - \lambda_j\|^2,$$

are all $O(N^{-1})$.

1.3 Problem 6

Problem Statement. Prove Theorem 12.1.3.

Theorem 1.1. (Theorem 12.1.3) : Let $x, y \in \mathcal{H}$, then

$$\|x \otimes y\|_{\mathcal{H}\otimes\mathcal{H}} = \|\langle y, \cdot\rangle x\|_{\mathcal{S}}.$$

My Solution. Since

$$\|x \otimes y\|^2_{\mathcal{H}\otimes\mathcal{H}} = \langle x \otimes y, x \otimes y\rangle_{\mathcal{H}\otimes\mathcal{H}} = \langle x, x\rangle_{\mathcal{H}}\langle y, y\rangle_{\mathcal{H}} = \langle\langle x, y\rangle, \langle x, y\rangle\rangle_{\mathcal{S}} = \langle\langle y, \cdot\rangle x, \langle y, \cdot\rangle x\rangle_{\mathcal{S}} = \|\langle y, \cdot\rangle x\|^2_{\mathcal{S}},$$

we can say that

$$\|x \otimes y\|_{\mathcal{H}\otimes\mathcal{H}} = \|\langle y, \cdot\rangle x\|_{\mathcal{S}}.$$

1.4 Problem 7

Problem Statement. Suppose that the data $\{X_n(t) : t \in [0,1], 1 \leq n \leq N\}$ are expressed using an orthonormal basis $e_1, \ldots, e_j$:

$$X_n(t) = \sum_{j=1}^{J} x_{nj} e_j(t).$$

In this case, the EFPC's, $\hat{v}_i(t)$, can also be expressed as

$$\hat{v}_i(t) = \sum_{j=1}^{J} \hat{v}_{ij} e_j(t).$$

Explain how to obtain the coefficients $\hat{v}_{ij}$ from the x_{nj}. Justify your answer.

My Solution. For

$$X_n(t) = \hat{\mu}(t) + \sum_{i=1}^{\infty} \hat{\xi}_{ni} \hat{v}_i(t),$$

we can say

$$X_n(t) = \hat{\mu}(t) + \sum_{i=1}^{\infty} \hat{\xi}_{ni} \sum_{j=1}^{J} \hat{v}_{ij} e_j(t) = \hat{\mu}(t) + \sum_{j=1}^{J} e_j(t) \times \sum_{i=1}^{\infty} \hat{\xi}_{ni} \hat{v}_{ij} = \sum_{j=1}^{J} x_{nj} e_j(t).$$

Then, we can say that

$$x_{nj} = \sum_{i=1}^{\infty} \hat{\xi}_{ni} \hat{v}_{ij} + \hat{\mu}(t) e_j(t).$$

1.5 Problem 12

Problem Statement. Under the same assumptions as in Problem 12.8.5 show that, for $j \neq k$, and $1 \leq j \leq p$,

$$\langle \hat{v}_j - v_j, v_k \rangle = \frac{\langle \hat{C} - C, \hat{v}_j \otimes v_k \rangle}{\hat{\lambda}_j - \lambda_k}.$$

What can you conclude about the asymptotic distribution of $N^{1/2} \langle \hat{v}_j - v_j, v_k \rangle$?

My Solution.