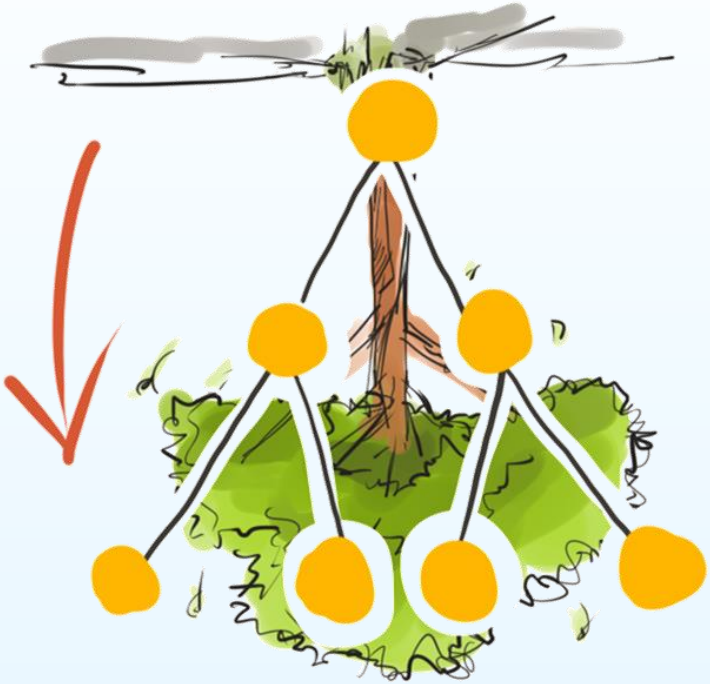

Lecture 4:
Basic Complexity Analysis

01204212 Abstract Data Types and Problem Solving

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Outline

- Mathematic Background
  - Logarithms and Exponents
  - Series
  - Recurrence Relations

- Complexity Analysis
  - Asymptotic Notations
  - Recurrence Relations

Efficiency of Algorithms

- In many situations, you will often have a selection of among possible algorithms or data structures, or even compare them

- It is not possible to simply say that

  "algorithm A is **faster** than algorithm B" => quality

  Why?
  - System dependence: execution time, memory space, compiler, …
  - Application dependence: input data

- An alternative comparison is based on the quantity metric by looking at relatively rates of growth in time requirements as the size of problem increases
  - Use mathematics and elementary calculus

L'Hôpital

If you are attempting to determine

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

but both $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$, it follows

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^{(1)}(n)}{g^{(1)}(n)}$$

Repeat as necessary …

Where $f^{(k)}(n)$ is the k^{th} derivative

Logarithms and Exponents

If $n = e^m$, we define $m = \ln(n)$

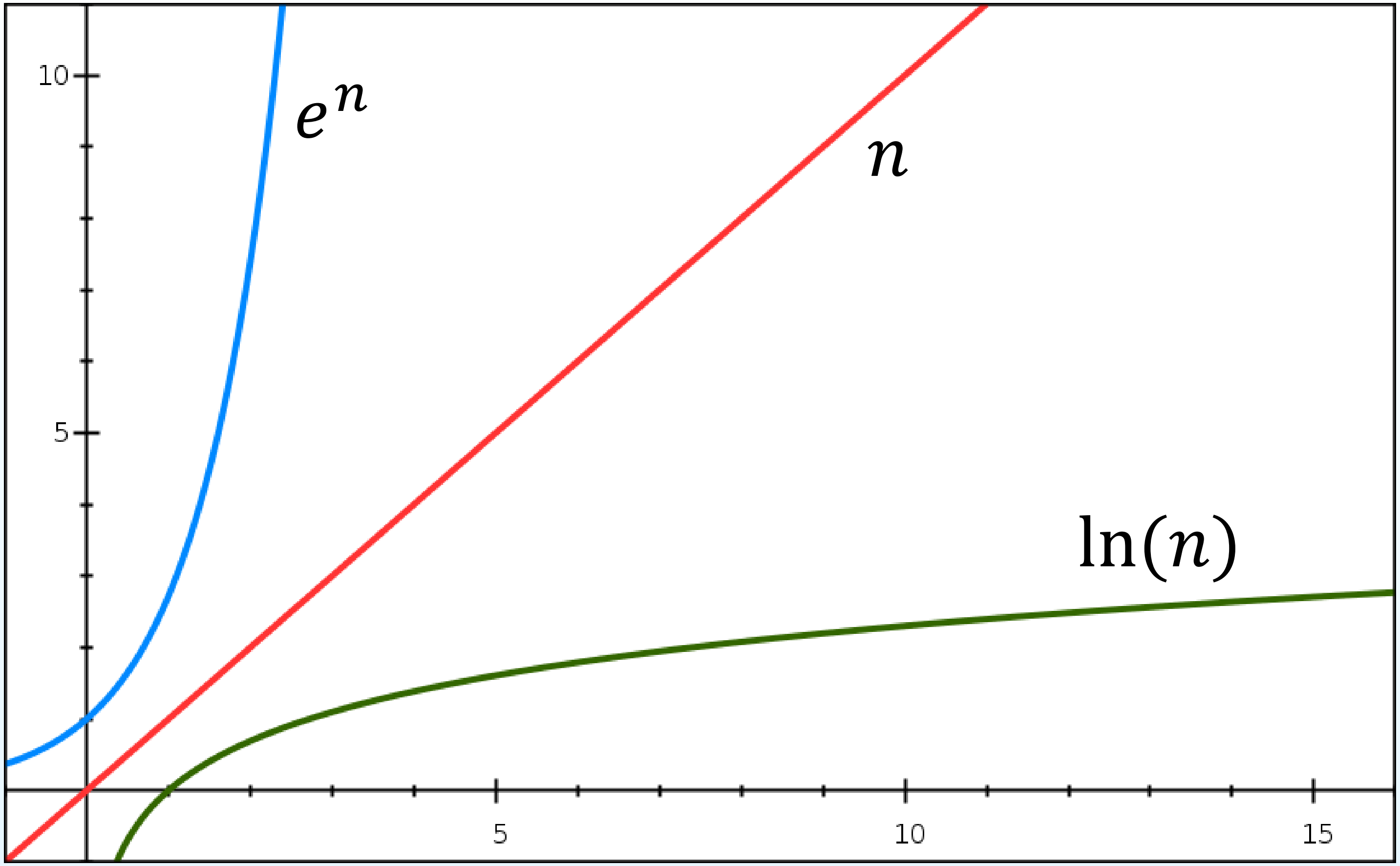
Exponents grow faster than any non-constant polynomial

$$\lim_{n \to \infty} \frac{e^n}{n^d} = \infty \text{ for any } d > 0$$

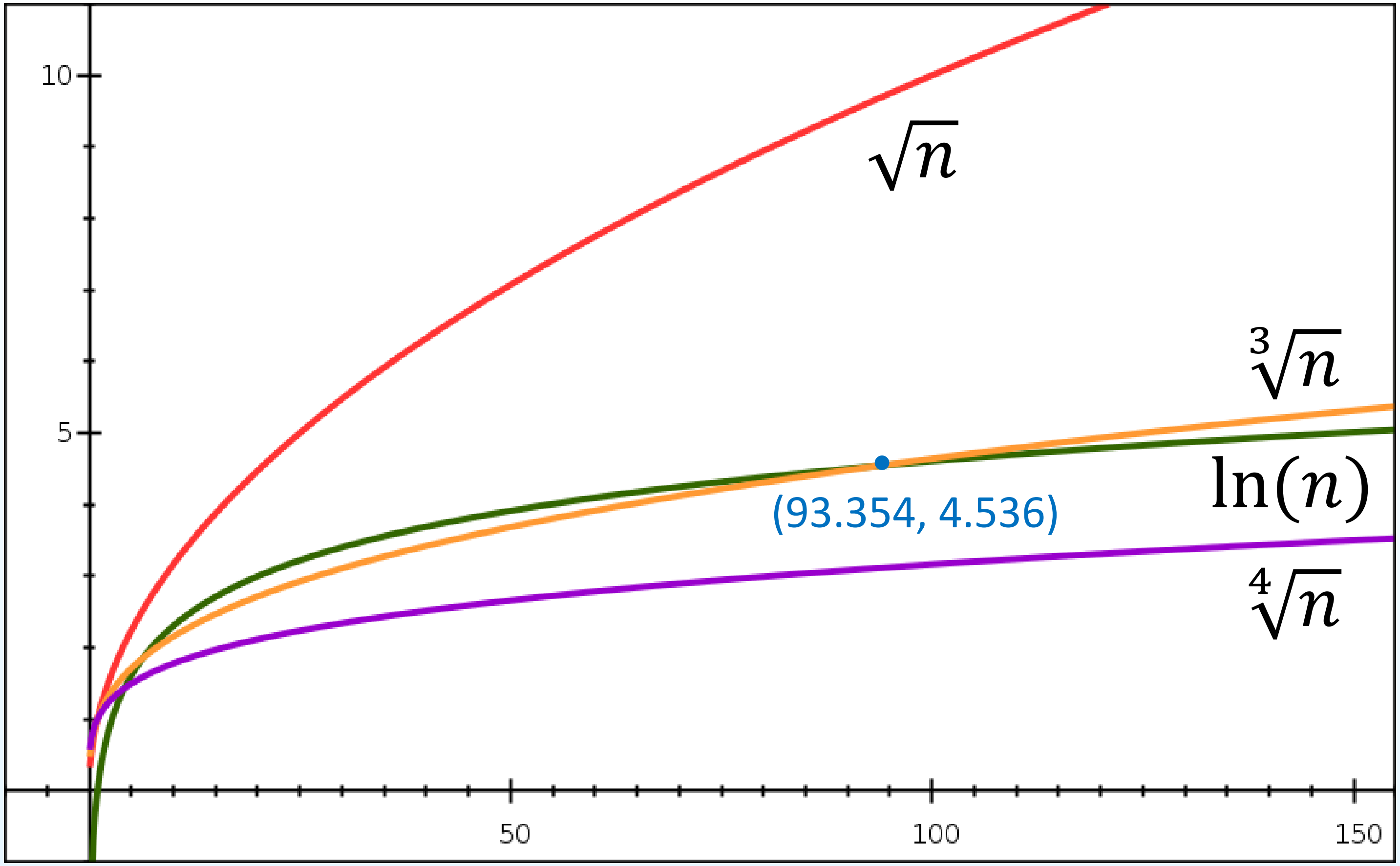
Thus, their inverses (i.e., logarithms) grow slower than any polynomial

$$\lim_{n \to \infty} \frac{\ln(n)}{n^d} = 0$$

Example: Logarithms and Exponents



Example: Logarithms and Exponents



after the point (5503.66, 8.61), $\ln(n)$ will grow slower than $\sqrt[4]{n}$

Logarithm with Different Bases

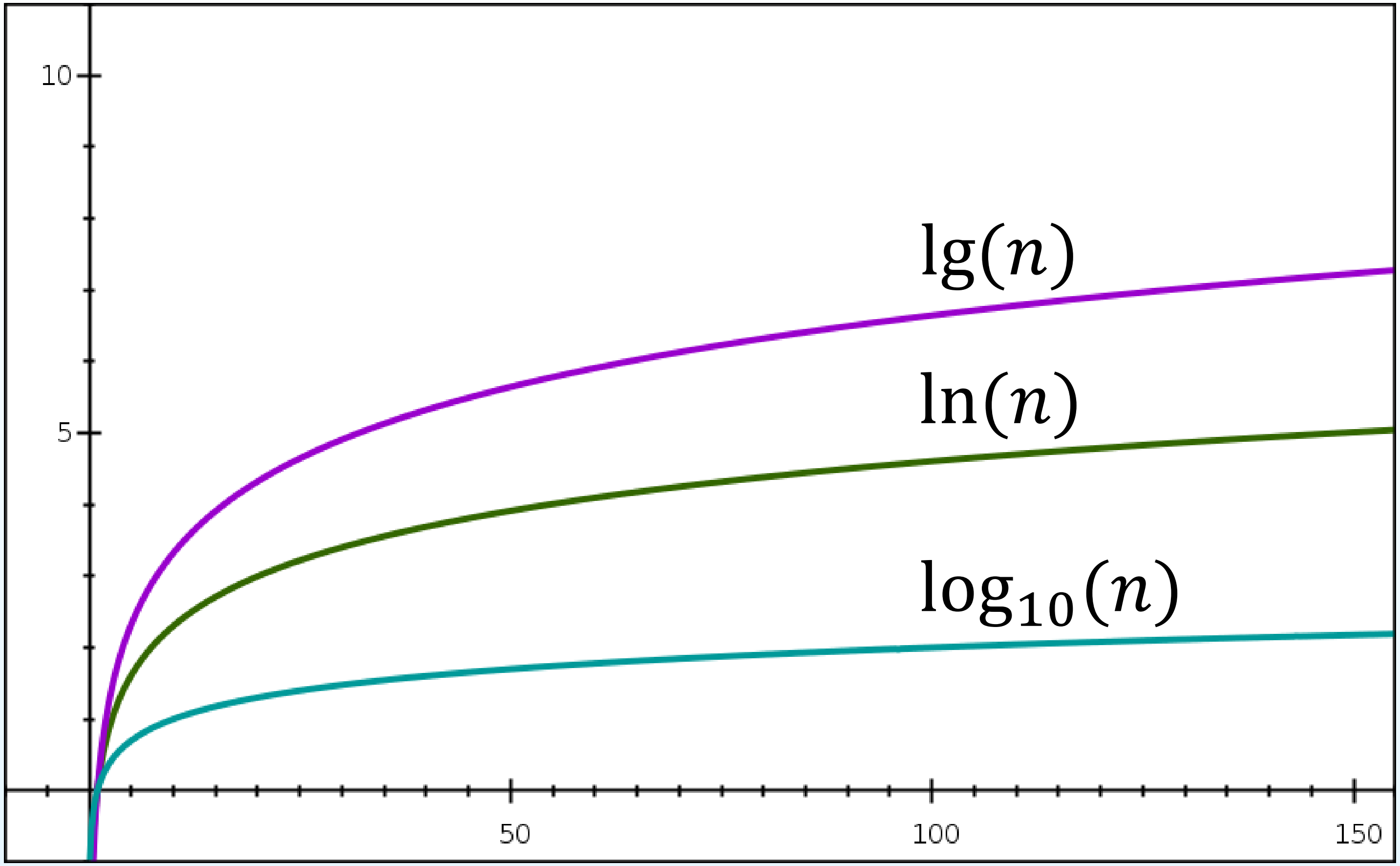
You have seen the formula

$$\log_b(n) = \frac{\log_x(n)}{\log_x(b)} = \frac{\ln(n)}{\ln(b)}$$

constant

So that all logarithms are scalar multiples of each others

Logarithm with Different Bases



Note: the base-2 logarithm $\log_2(n)$ is written as $\lg(n)$

Some Properties of Logarithms

- $\log_b(nm) = \log_b(n) + \log_b(m)$
- $\log_b\left(\frac{n}{m}\right) = \log_b(n) - \log_b(m)$
- $\log_b(n^m) = m\log_b(n)$
- $b^{\log_b(n)} = n$
- $n^{\log_b(m)} = m^{\log_b(n)}$
- $\log_b\log_b(n) < \log_b(n) < n$ for all $n > 0$

Arithmetic Series

Each term in an arithmetic series is increased by a constant value (usually 1):

$$1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof1: Adding the series twice

$$\begin{aligned} S_n &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ S_n &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ 2S_n &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \end{aligned}$$

$$S_n = \tfrac{1}{2}n(n+1)$$

Proof2: By induction

- Basic step: The statement is true for $n = 1$
- Inductive hypothesis: Assume the statement is true for $1 < i \leq n$
- Inductive step: Based on the hypothesis, the statement is also true for $n + 1$

Quickly Algorithm Analysis

Consider the following code fragment:

```
for (i=1; i<=n; i++)
  for (j=1; j<=i; j++)
    printf("Hello\n");
```

How many times is `printf()` executed?

i	j	times
1	1	1
2	1,2	2
3	1,2,3	3
…		
n	1,2,3,…,n	n

arithmetic series

$$= \frac{n(n+1)}{2}$$

Other Polynomial Series

We could repeat the proven process, after all:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Geometric Series

A series for which the ratio of each two consecutive terms ${}^{a_{i+1}}/_{a_i}$ is a constant $|r| < 1$

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

Proof:

$$\begin{aligned}
S_n &= 1 + \ r \ + r^2 + \cdots + r^n \\
rS_n &= r + r^2 + r^3 + \cdots + r^{n+1} \\
S_n - rS_n &= (1 + \ r \ + r^2 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1}) \\
(1 - r)S_n &= 1 - r^{n+1} \\
S_n &= \frac{1 - r^{n+1}}{1 - r}
\end{aligned}$$

Geometric Series

A series for which the ratio of each two consecutive terms $^{a_{i+1}}/_{a_i}$ is a constant $|r| < 1$

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

The sum converges as $n \to \infty$

$$1 + r + r^2 + \cdots = \sum_{i=0}^{\infty} r^i = \frac{1}{1 - r}$$

Recurrence Relations

- Sequences may be defined explicitly
  - For example, the harmonic sequence $x_n = \frac{1}{n}$, we have $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

- A **recurrence relationship** is a means of defining a sequence based on previous values in the sequence
  - Such definitions of sequences are said to be recursive
  - For example,

    the odd number sequence: $x_n = x_{n-1} + 2$ where $x_1 = 1$

    the Fibonacci sequence: $x_n = x_{n-1} + x_{n-2}$ where $x_0 = 0, x_1 = 1$

Recurrence Relations

- In some cases, given the recurrence relation, we can find the explicit formula (closed form)
  - For example,

    the odd number sequence: $x_n = x_{n-1} + 2$ where $x_1 = 1$

    its closed form is given by $x_n = 2n - 1$

    the Fibonacci sequence: $x_n = x_{n-1} + x_{n-2}$ where $x_0 = 0, x_1 = 1$

    its closed form is given by $x_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$

Recurrence Relations

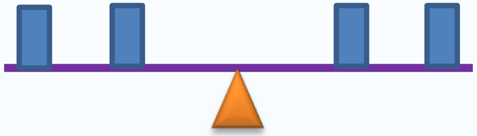
- We may use a functional form for a recurrence relation:

Mathematic	**Function**
$x_1 = 1$	$f(1) = 1$
$x_n = x_{n-1} + 2$	$f(n) = f(n-1) + 2$
$x_n = x_{n-1} + x_{n-2}$	$f(n) = f(n-1) + f(n-2)$

Weighted Averages

Given n objects $x_1, x_2, x_3, \dots, x_n$, the average is

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

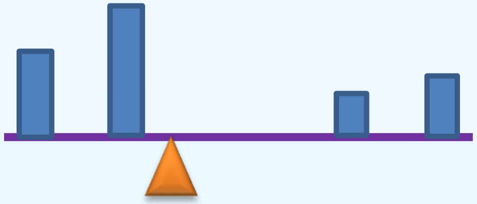


If we are given a sequence of coefficients $c_1, c_2, c_3, \dots, c_n$ where

$$c_1 + c_2 + c_3 + \cdots + c_n = 1$$

then we refer to

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots + c_n x_n$$



as a weighted average

For an average, $c_1 = c_2 = c_3 = \cdots = c_n = \frac{1}{n}$

Combinations

Given n distinct items, in how many ways
can you choose k of these?

The number of ways such items can be chosen is written

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

where $\binom{n}{k}$ is read as "n choose k"

Combinations

You have also seen this in expanding polynomials:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

For example,

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k}$$

The coefficients of Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$= \binom{4}{0} y^4 + \binom{4}{1} xy^3 + \binom{4}{2} x^2y^2 + \binom{4}{3} x^3y + \binom{4}{4} x^4$$

$$= y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + x^4$$

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- Complexity Analysis
  - Asymptotic Notations
  - Recurrence Relations

Algorithm Analysis

In an algorithm analysis, you always have to know as the size of an algorithm's input grows

- Time: How much longer does it run?
- Space: How much memory does it use?

How do you answer these questions?

For now, we will focus on time only.

Problems with Timing

- Why not just code the algorithm and time it?
  - Hardware: processor, memory, etc.
  - OS, programming language, libraries, compiler/interpreter
  - Programs running in the background
  - Choice of input, number of inputs
- Timing does not really evaluate the algorithm but merely evaluates a specific implementation
- At the core of CS, a backbone of theory & mathematics
  - Examine the algorithm itself, not the implementation
  - Reason about performance as a function of n
  - Mathematically proven things about performance
- Yet, timing has its place
  - In real world, we do want to know whether implementation A runs faster than implementation B on data set C, e.g., Benchmarking

Evaluations

Evaluate an algorithm

➡ Use asymptotic analysis

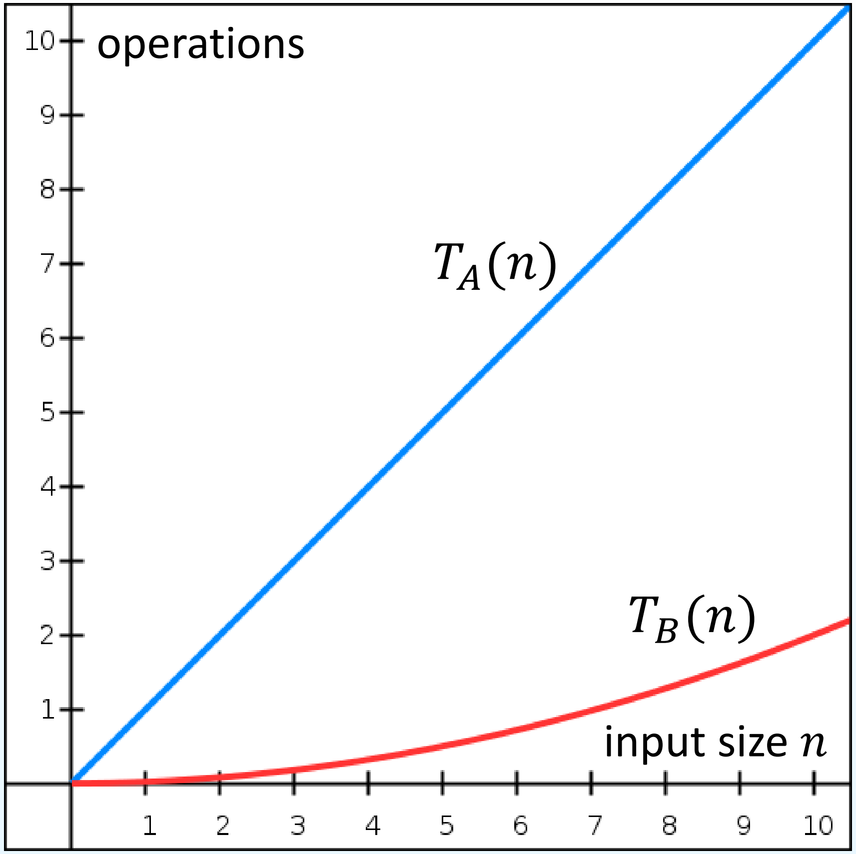
Evaluate an implementation

➡ Use timing

Motivation for Algorithm Analysis

- Suppose you are given two algorithms A and B for solving the problem

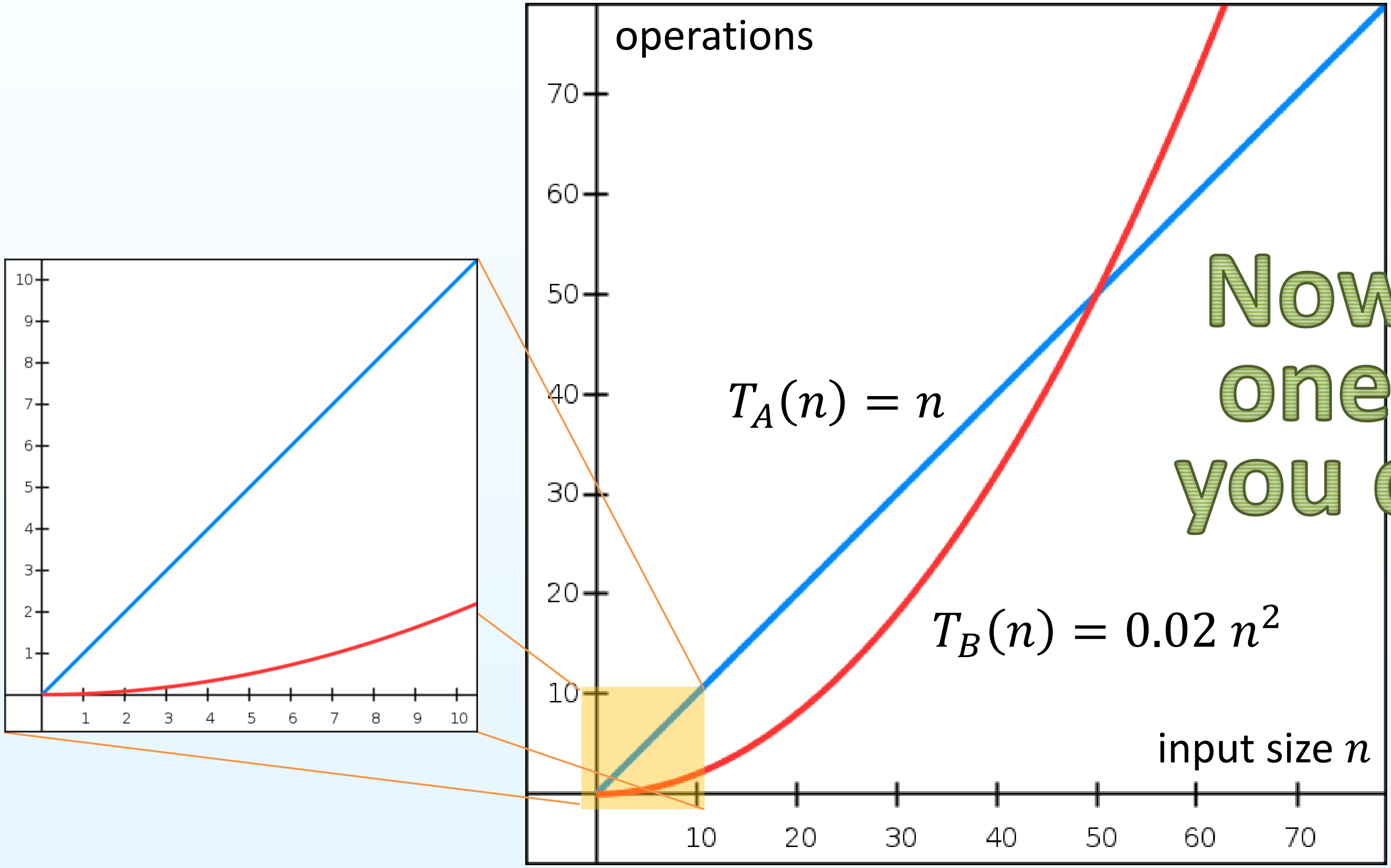
- The running times (operations) $T_A(n)$ and $T_B(n)$ of A and B as a function of input size n are given



Which is better?

Motivation for Algorithm Analysis

- For large n, the running times of A and B are:



Now which one would you choose?

Goals of Algorithm Analysis

- Concentrate on large inputs
  - Some algorithms only work fine for small inputs

- Be independent of hardware, OS, language, etc.

- Be general, not specific in some test cases

Assumptions in Analyzing Code

- Basic operations take a unit (constant) running time, e.g.,
  - Arithmetic
  - Assignment
  - Comparing two simple values
  - Accessing array with an index
- Other operations are summations or products
  - Consecutive statements are summed
  - Loops are (cost of loop body)×(number of loops)

Examples: Analyzing Code

What are the running times for the following codes?

1 $\quad n \quad n$

```
for (i=0; i<n; i++)
  x = x+1;
```

$n \quad n$

$$\approx 1 + 4n$$

$1 + 2n$

```
for (i=0; i<n; i++)
  for (j=0; j<n; j++)
    x = x+1;
```

$n(1 + 4n)$

$$\approx 1 + 2n + n(1 + 4n)$$
$$\approx 1 + 3n + 4n^2$$

$1 + 2n$

```
for (i=0; i<n; i++)
  for (j=0; j<=i; j++)
    x = x+1;
```

$n + 4(1 + 2 + \cdots + n)$

$$\approx 1 + 2n + n + \frac{4n(n+1)}{2}$$
$$\approx 1 + 5n + 2n^2$$

No Need to be so Exact

- Constant coefficients do not matter

  For example: Given $T_A(n) = n^2$ and $T_B(n) = 10n^2$,
  which has the faster growth rate?

$$\lim_{n\to\infty} \frac{n^2}{10n^2} = \lim_{n\to\infty} \frac{1}{10} = 0.1$$

a constant

- Lower-order terms are less important

  For example: Given $T_A(n) = n^2$ and $T_B(n) = 10n^2 + 5n + 2$,
  which has the faster growth rate?

$$\lim_{n\to\infty} \frac{n^2}{10n^2 + 5n + 2} = \lim_{n\to\infty} \frac{2n}{20n + 5} = \lim_{n\to\infty} \frac{1}{10} = 0.1$$

"We will focus on the dominant term only"

Worst-Case Analysis

- In general, we are interested in three types of performance
  - Best-case
  - Average-case
  - Worst-case

- When determining worst-case, we tend to be pessimistic
  - If there is a conditional, count the branch that runs the slowest
  - This will give a loose bound on how slow the algorithm may run

Algorithmic Complexity

How the running time of an algorithm increases with the size of the input *in the limit* (**growth rate**), as the size of the input increase without bound

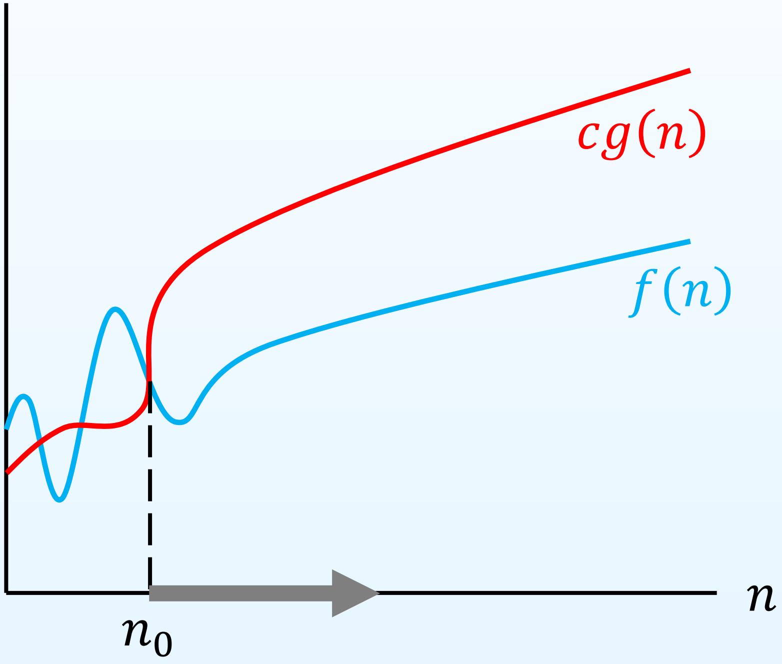
Asymptotic Notation

Notations are used to describe the asymptotic running time (**complexity**) of an algorithm, defined as **functions** whose domains are the set of natural numbers $N = \{0, 1, 2, \ldots\}$

Asymptotic: Big-Oh Notation

Given two functions $f(n)$ and $g(n)$ for inputs n, we say

"$f(n)$ is in $O(g(n))$ iff there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$"



Proof:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0, c \;\; ; c \neq \infty$$

$g(n)$ is an asymptotic upper bound for $f(n)$

Examples: Big-Oh

Are the following statements TRUE or FALSE?

- $4 + 3n$ is in $O(n)$

$$\lim_{n\to\infty} \frac{4+3n}{n} = 3$$

$f(n) = 4 + 3n,\ g(n) = n$
We select $c = 4$ and $n_0 = 4$,
so that $0 \le f(n) \le 4g(n)$
for all $n \ge 4$.

TRUE

- $n + 2\ln(n)$ is in $O(\ln(n))$

$$\lim_{n\to\infty} \frac{n+2\ln(n)}{\ln(n)} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\frac{1}{n}} = \lim_{n\to\infty}(n+2) = \infty$$

FALSE

- n^{50} is in $O(2^n)$

$$\lim_{n\to\infty} \frac{n^{50}}{2^n} = \lim_{n\to\infty} \frac{50n^{49}}{\ln(2)\,2^n} = \lim_{n\to\infty} \frac{50 \cdot 49n^{48}}{\ln^2(2)\,2^n}$$

$$= \cdots = \lim_{n\to\infty} \frac{50!}{\ln^{50}(2)\,2^n} = 0$$

TRUE

Big-Oh Common Comparisons

Increasing running time

Big-Oh	Description
$O(1)$	Constant (or $O(k)$ for constant k)
$O(\log\log n)$	Log log
$O(\log n)$	Logarithmic
$O(\log^2 n)$	Log squared
$O(n)$	Linear
$O(n\log n)$	n log n
$O(n^2)$	Quadratic
$O(n^3)$	Cubic
$O(n^k)$	Polynomial (where k is constant)
$O(k^n)$	Exponential (where constant $k > 1$)

Comment on Notation

- $4 + 3n$ is in $O(n)$?
- $4 + 3n$ is in $O(n \log n)$?
- $4 + 3n$ is in $O(n^2)$?
- $4 + 3n$ is in $O(n^3)$?
- $4 + 3n$ is in $O(n^k)$, for all $k \geq 1$?
- $4 + 3n$ is in $O(k^n)$, for all $k > 1$?

Choose $O(\cdot)$ with the least running time as possible!

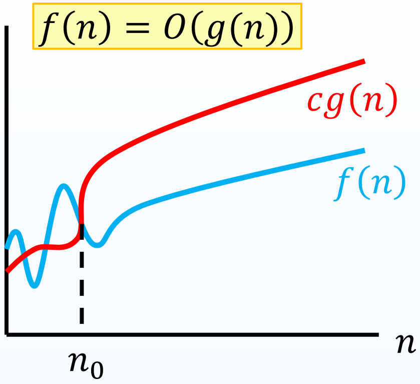
Comment on Notation

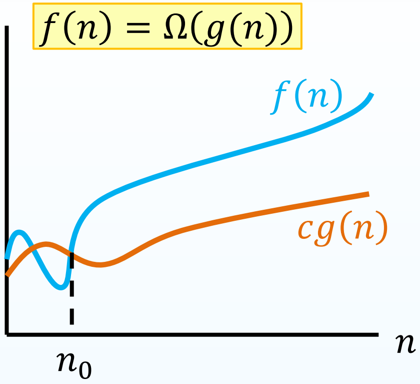
- We say “$3n^2 + 17$ is in $O(n^2)$”
- We may also say/write it as

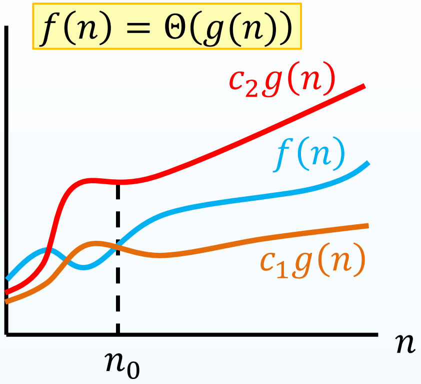
$$3n^2 + 17 \text{ is } O(n^2)$$
$$3n^2 + 17 = O(n^2)$$
$$3n^2 + 17 \in O(n^2)$$

- But ‘=’ does not mean an equality, so that we would never say $O(n^2) = 3n^2 + 17$

Asymptotic Notations







- Big-Oh: upper bound

  $f(n)$ is in $O(g(n))$ iff there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$

- Big-Omega: lower bound

  $f(n)$ is in $\Omega(g(n))$ iff there exist positive constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$

- Big-Theta: tight bound

  $f(n)$ is in $\Theta(g(n))$ iff there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$

Asymptotic Notations

Less common notations

- Little-oh: like Big-Oh but strictly less than

  $f(n)$ is in $o(g(n))$ iff for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$

  For example, $2n = o(n^2)$, but $2n^2 \neq o(n^2)$

- Little-omega: like Big-Omega but strictly greater than

  $f(n)$ is in $\omega(g(n))$ iff for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0$

  For example, $2n^2 = \omega(n)$, but $2n^2 \neq \omega(n^2)$

Asymptotic Notations

$$f(n) = o(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) = O(g(n)) \qquad 0 \leq \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) = \Theta(g(n)) \qquad 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) = \Omega(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$$f(n) = \omega(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

Example: Big-Oh Analysis

Compute the sum of `n` integers stored in the array `a`:

Code Fragment:

```
1: int sum_array(int a[], int n) {
2:   int sum = 0, i;
3:
4:   for (i=0; i<n; i++)
5:     sum += a[i];
6:   return sum;
7: }
```

- Lines 2 and 6 take constant time, i.e., $O(1)$
- Lines 4 and 5 perform `n` iterations, i.e., $O(n)$
- So that the running time is $O(1 + n)$ ➡ $O(n)$
- Actually, $\Theta(n)$ since all `n` integers are exactly accessed

Example: Big-Oh Analysis

Find the value `v` in the array `a` of `n` integers

Code Fragment:

```
1: int find(int a[], int n, int v) {
2:   int i;
3:
4:   for (i=0; i<n; i++)
5:     if (a[i] == v)
6:       return 1;
7:   return -1;
8: }
```

Lines 4-6 are the dominant costs of the running time

- Worst-case: `v` is the last element ➡ $O(n)$
- Best-case: if you are lucky, `v` is the first element ➡ $\Omega(1)$
- Average-case: the probability of `v` stored in each position ➡ …

Example: Big-Oh Analysis

Again, compute the sum of `n` integers stored in the array `a`:

Code Fragment:

```
1: int sum_array(int a[], int n) {
2:   if (n == 1)
3:     return a[n-1];
4:   else
5:     return a[n-1] + sum_array(a, n-1);
6: }
```

- Lines 2-3 take constant $O(1)$
- Let $T(n)$ be the running time of summing all `n` integers
- Then, in lines 4-5, the cost is $O(1) + T(n-1)$
- So that we got the recurrence relation:

$$T(n) = \begin{cases} O(1) & \text{if } n = 1, \\ T(n-1) + O(1) & \text{otherwise} \end{cases}$$

- How can we find $O(T(n))$?

Recurrence Relations

- Substitution method
  - Expand the recurrence and express it as a summation of terms dependent only on n and the initial conditions
- Recursion tree
  - Visualize what happens when a recurrence is iterated
- Master theorem
  - Provide a cookbook method for solving recurrences of the specific form

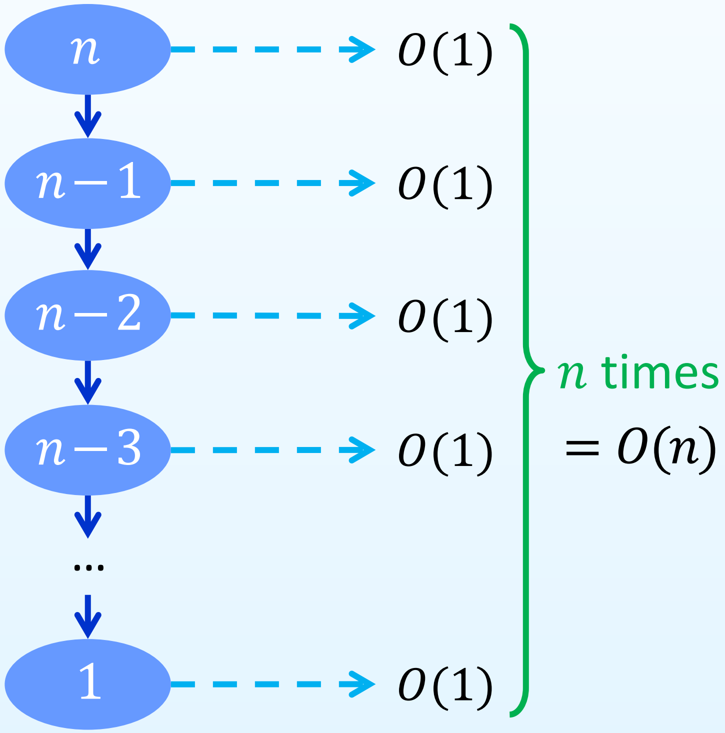
Example: Sum of n Elements in Array

$$T(n) = \begin{cases} O(1) & \text{if } n = 1, \\ T(\underset{\text{branching}}{\underset{\Uparrow}{n-1}}) + \underset{\text{workload}}{\underset{\Uparrow}{O(1)}} & \text{otherwise} \end{cases}$$

- Substitution method

$$\begin{aligned} T(n) &= T(n-1) + O(1) \\ &= T(n-2) + O(1) + O(1) \\ &= T(n-3) + O(1) + O(1) + O(1) \\ &\quad \dots \\ &= \boxed{T(1)} + O(1) + \cdots + O(1) \\ &= \underbrace{O(1) + O(1) + \cdots + O(1)}_{n \text{ times}} \\ &= O(n) \end{aligned}$$

- Recursion tree



Example: Towers of Hanoi

```
#include <stdio.h>

void toh(int n, char from, char to, char aug) {
  if (n == 1)
    printf("Move %d from %c to %c\n", n, from, to); --> base case
  else {
    toh(n-1, from, aug, to); ------------------> branching
    printf("Move %d from %c to %c\n", n, from, to); --> workload
    toh(n-1, aug, to, from); ------------------> branching
  }
}

int main(void) {
  int n = 0; ---------> O(1)

  printf("Enter n: "); ---> O(1)
  scanf("%d", &n);------> O(1)
  toh(n, 'A', 'B', 'C'); --> O(T(n))
  return 0; -----------> O(1)
}
```

$$T(n) = \begin{cases} O(1) & \text{if } n = 1, \\ 2T(n-1) + O(1) & \text{otherwise} \end{cases}$$

Recurrence Relation: Tower of Hanoi

$$T(n) = \begin{cases} O(1) & \text{if } n = 1, \\ 2T(n-1) + O(1) & \text{otherwise} \end{cases}$$

- Substitution method

(n-1) levels

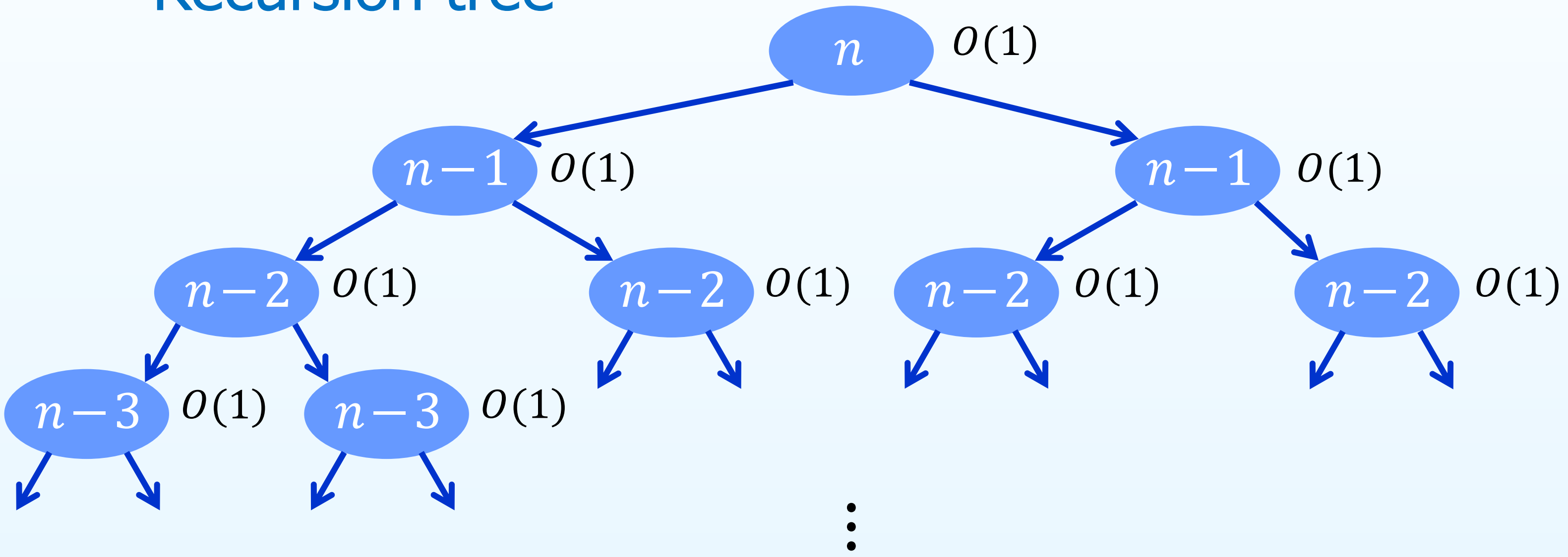
$$\begin{aligned} T(n) &= 2T(n-1) + O(1) \\ &= 2\big(2T(n-2) + O(1)\big) + O(1) = 2^2T(n-2) + 2O(1) + O(1) \\ &= 2^2\big(2T(n-3) + O(1)\big) + 2O(1) + O(1) = 2^3T(n-3) + 2^2O(1) + 2O(1) + O(1) \\ &= 2^4T(n-4) + 2^3O(1) + 2^2O(1) + 2O(1) + O(1) \\ &\quad \dots \\ &= 2^{n-1}T(1) + 2^{n-2}O(1) + \dots + 2^2O(1) + 2O(1) + O(1) \\ &= 2^{n-1}O(1) + 2 + 2^{n-2}O(1) + \dots + 2^2O(1) + 2^1O(1) + 2^0O(1) \\ &= O(1)\sum_{i=0}^{n-1} 2^i \\ &= (2^n - 1)O(1) = O(2^n - 1) = O(2^n) \end{aligned}$$

	n	n-1	…	3	2	1	0	
	0	1	1	1	1	1	1	
=	1	0	0	0	0	0	0	- 1

Recurrence Relation: Tower of Hanoi

$$T(n) = \begin{cases} O(1) & \text{if } n = 1, \\ 2T(n-1) + O(1) & \text{otherwise} \end{cases}$$

- Recursion tree



$==> O(1)$

$==> 2O(1)$

$==> 2^2O(1)$

$==> 2^3O(1)$

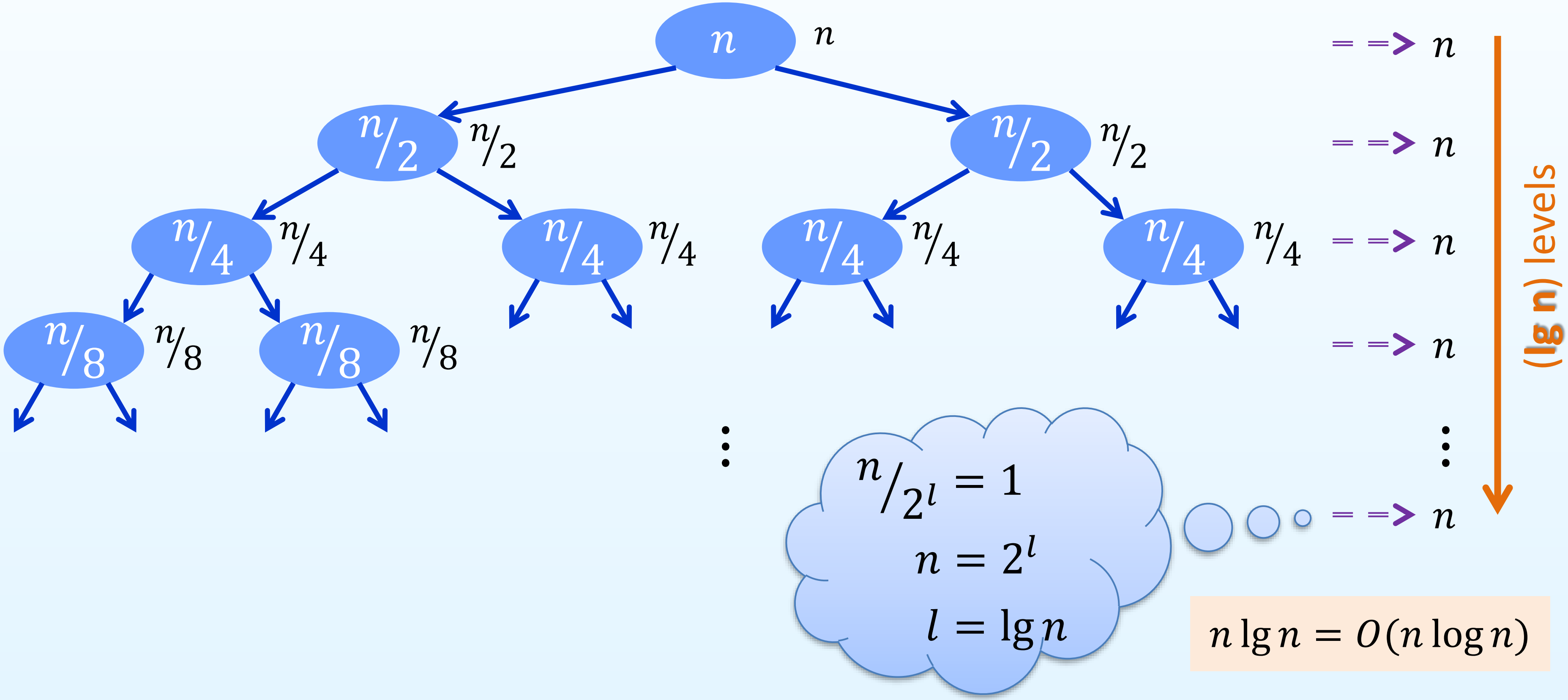
$\vdots$

$=> 2^{n-1}O(1)$

$$O(1)\sum_{i=0}^{n-1} 2^i = O(2^n)$$

Example: Recursion Tree

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



n n

$n/2$ $n/2$ $n/2$ $n/2$

$n/4$ $n/4$ $n/4$ $n/4$ $n/4$ $n/4$ $n/4$ $n/4$

$n/8$ $n/8$ $n/8$ $n/8$

$\vdots$

$= = > n$

$= = > n$

$= = > n$

$= = > n$

$\vdots$

$= = > n$

(lg n) levels

$$n/2^l = 1$$

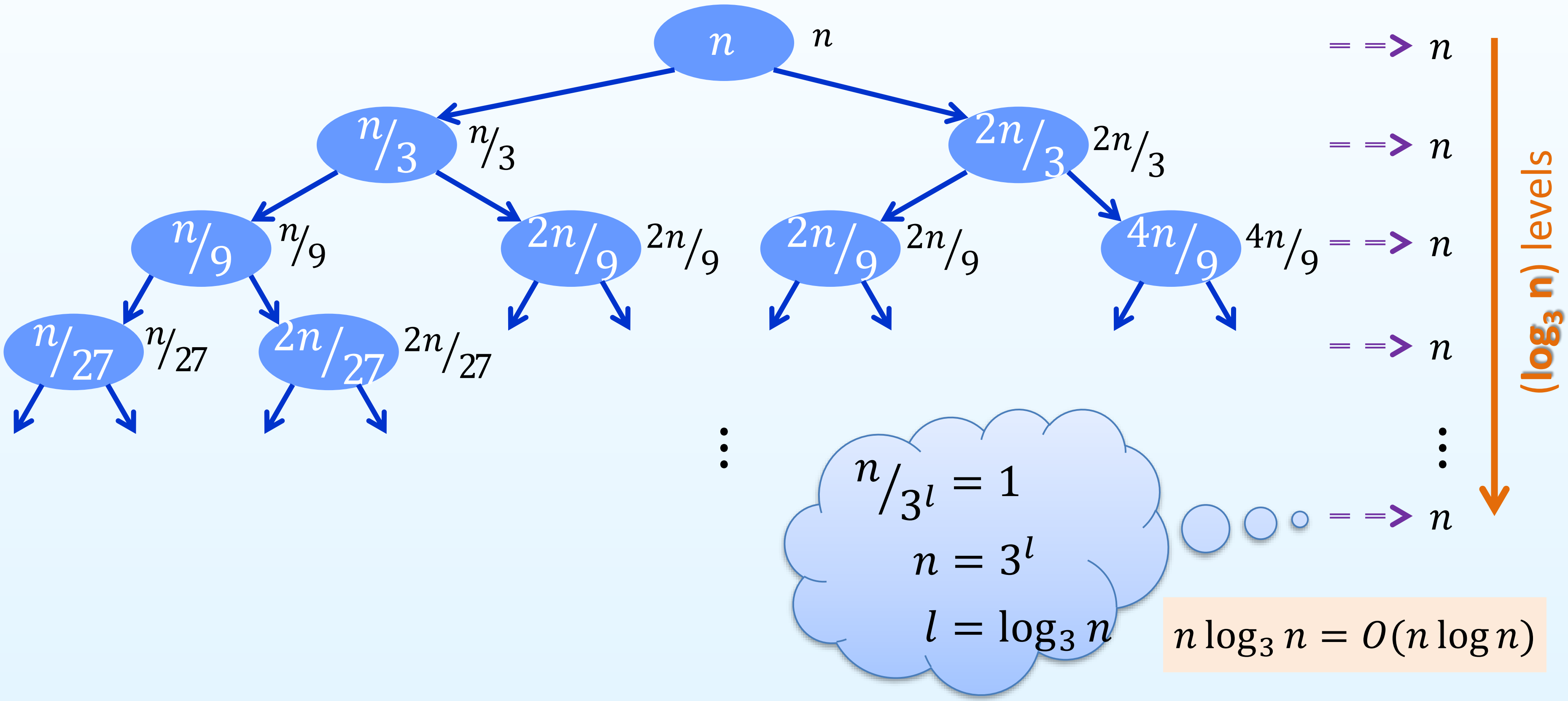
$$n = 2^l$$

$$l = \lg n$$

$$n \lg n = O(n \log n)$$

Example: Recursion Tree

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$



$^{n}/_{3^l} = 1$

$n = 3^l$

$l = \log_3 n$

$n \log_3 n = O(n \log n)$

Recurrence Relations

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \qquad ; a \geq 1, b > 1$$

- Master theorem
  - ▶ If $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ for some constant $\varepsilon > 0$,
    then $T(n) = \Theta\left(n^{\log_b a}\right)$
  - ▶ If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \log n\right)$
  - ▶ If $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some constant $\varepsilon > 0$,
    and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$,
    then $T(n) = \Theta(f(n))$

Example: Master Theorem

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

- We have $a = 9,\ b = 3,\ f(n) = n$
- Thus $n^{\log_b a} = n^{\log_3 9} = n^2$
- Since $f(n) = n = O\left(n^{\log_3 9-\varepsilon}\right)$, where $0 < \varepsilon \leq 1$, we can apply the case 1
- So that $T(n) = \Theta\left(n^{\log_3 9}\right) = \Theta(n^2)$

Example: Master Theorem

$$T(n) = T\left(\frac{n}{3}\right) + 1$$

- We have $a = 1,\ b = 3,\ f(n) = 1$
- Thus $n^{\log_b a} = n^{\log_3 1} = 1$
- Since $f(n) = 1 = \Theta\left(n^{\log_3 1}\right)$, we can apply the case 2
- So that $T(n) = \Theta(n^{\log_3 1} \log n) = \Theta(\log n)$

Example: Master Theorem

$$T(n) = 3T\left(\frac{n}{4}\right) + n\log n$$

- We have $a = 3,\ b = 4,\ f(n) = n\log n$
- Thus $n^{\log_b a} = n^{\log_4 3} \approx n^{0.793}$
- Since $f(n) = n\log n = \Omega\left(n^{\log_3 4+\varepsilon}\right)$, where $0 < \varepsilon \leq 0.207$, the case 3 will apply if we can show $af\left(\frac{n}{b}\right) \leq cf(n), c < 1$
- $3\left(\frac{n}{4}\right)\log\left(\frac{n}{4}\right) = \left(\frac{3}{4}\right)n(\log n - \log 4) \leq \left(\frac{3}{4}\right)n\log n$ for $c = \frac{3}{4}$
- So that $T(n) = \Theta(f(n)) = \Theta(n\log n)$

Any Question?