EE3014/IM3001 Digital Signal Processing

Solutions to Tutorial 11

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Q11.1

Determine a continuous-time frequency $\Omega = \sqrt{3}$ radians maps to which discrete-time frequency ω using the following transformations:

(i) Impulse invariance,

(ii) Bilinear transformation.

Solution to Q11.1

(i) Impulse invariance: $\omega = \Omega \quad (\text{for } -\pi \leq \Omega \leq \pi)$

$$\text{so } \omega = \sqrt{3} \text{ radians}$$

(ii) Bilinear transformation: $\omega = 2\arctan\Omega$

$$= 2\arctan\sqrt{3}$$

$$= 2 \times \frac{\pi}{3} \quad \text{radians}$$

Q11.2

A discrete-time IIR filter $H(z)$ is to be designed using the bilinear transformation method satisfying the specifications:

$$0.9 \le |H(e^{j\omega})| \le 1 \quad \text{for} \quad 0 \le \omega \le 1.55, \quad |H(e^{j\omega})| \le 0.1 \quad \text{for} \quad 2.7 \le \omega \le \pi$$

(i) Determine the required specifications for the continuous-time filter $H_c(s)$.

(ii) Assume $H_c(s) = \dfrac{2}{s^2 + 2s + 2}$ satisfies these specifications.

Determine $H(z)$.

Solution to Q11.2

(i) specifications for $H(z)$: $0.9 \le |H(e^{j\omega})| \le 1$ for $0 \le \omega \le 1.55$

$|H(e^{j\omega})| \le 0.1$ for $2.7 \le \omega \le \pi$

bilinear transformation frequency mapping $\Omega = \tan\left(\dfrac{\omega}{2}\right)$

$\omega = 0 \Rightarrow \Omega = 0$

$\omega = 1.55 \Rightarrow \Omega = \tan(1.55/2) = 0.98 \text{ rad}$

$\omega = 2.7 \Rightarrow \Omega = \tan(2.7/2) = 4.46 \text{ rad}$

$\omega = \pi \Rightarrow \Omega = \infty$

→ specifications for $H_c(s)$: $0.9 \le |H_c(j\Omega)| \le 1$ for $0 \le \Omega \le 0.98$

$|H_c(j\Omega)| \le 0.1$ for $4.46 \le \Omega \le \infty$

Solution to Q11.2 (cont'd)

(ii) given $H_c(s) = \dfrac{2}{s^2 + 2s + 2}$

bilinear transformation mapping $s = \dfrac{1 - z^{-1}}{1 + z^{-1}}$

$$H(z) = \frac{2}{\left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + 2\left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right) + 2}$$

$$= \frac{2(1 + z^{-1})^2}{(1 - z^{-1})^2 + 2(1 - z^{-1})(1 + z^{-1}) + 2(1 + z^{-1})^2}$$

$$= \frac{2 + 4z^{-1} + 2z^{-2}}{5 + 2z^{-1} + z^{-2}}$$

Q11.3

The impulse response for an ideal low-pass filter with cut-off frequency π/2 is $h_d[n] = \dfrac{\sin(n\pi/2)}{n\pi}$ for all n.

(i) Using a rectangular window of length 7, design an FIR filter $h_1[n]$ from this desired impulse response.

(ii) Obtain a causal length 7 FIR filter $h_2[n]$ from $h_1[n]$.

(iii) Find the magnitude response of the second filter $h_2[n]$.

Solution to Q11.3

(i) rectangular window of length 7: $w[n]=\begin{cases}1 & -3\le n\le 3\\ 0 & otherwise\end{cases}$

from $h_d[n]=\dfrac{\sin(n\pi/2)}{n\pi}$

$$h_1[n]=h_d[n].w[n]=\begin{cases}\sin(n\pi/2)/n\pi & -3\le n\le 3\\ 0 & otherwise\end{cases}$$

for $n=\pm3,\quad h_1[\pm3]=\sin(\pm3\pi/2)/(\pm3\pi)=\mp1/(\pm3\pi)=-1/3\pi$

for $n=\pm2,\quad h_1[\pm2]=\sin(\pm2\pi/2)/(\pm2\pi)=0$

for $n=\pm1,\quad h_1[\pm1]=\sin(\pm\pi/2)/(\pm\pi)=\pm1/(\pm\pi)=1/\pi$

for $n=0,\quad h_1[0]=\frac{1}{2}\sin(0)/0$ which in the limit is 1/2

$$h_1[n]=\left[\cdots 0 \quad -\frac{1}{3\pi} \quad 0 \quad \frac{1}{\pi} \quad \underset{\uparrow}{\frac{1}{2}} \quad \frac{1}{\pi} \quad 0 \quad -\frac{1}{3\pi} \quad 0\cdots\right]$$

Solution to Q11.3 (cont'd)

(ii) to obtain a causal filter, delay by 3 samples

$$H_2(z) = z^{-3} H_1(z) \qquad \text{or} \qquad h_2[n] = h_1[n-3]$$

therefore,

$$h_2[n] = \left[\cdots 0 \quad \underset{\uparrow}{-\frac{1}{3\pi}} \quad 0 \quad \frac{1}{\pi} \quad \frac{1}{2} \quad \frac{1}{\pi} \quad 0 \quad -\frac{1}{3\pi} \quad 0 \cdots \right]$$

Solution to Q11.3 (cont'd)

(iii) frequency response

$$H_2(e^{j\omega}) = -\tfrac{1}{3\pi} + \tfrac{1}{\pi}e^{-j2\omega} + \tfrac{1}{2}e^{-j3\omega} + \tfrac{1}{\pi}e^{-j4\omega} - \tfrac{1}{3\pi}e^{-j6\omega}$$

$$= e^{-j3\omega}\left(-\tfrac{1}{3\pi}e^{j3\omega} - \tfrac{1}{3\pi}e^{-j3\omega} + \tfrac{1}{\pi}e^{j\omega} + \tfrac{1}{\pi}e^{-j\omega} + \tfrac{1}{2}\right)$$

$$= e^{-j3\omega}\left(-\tfrac{2}{3\pi}\cos 3\omega + \tfrac{2}{\pi}\cos\omega + \tfrac{1}{2}\right)$$

magnitude response $|H_2(e^{j\omega})| = |-\tfrac{2}{3\pi}\cos 3\omega + \tfrac{2}{\pi}\cos\omega + \tfrac{1}{2}|$

Q11.4

A low-pass FIR filter was designed using the Parks-McClellan algorithm.

The maximum passband error is $\delta_1 = 0.0531$ and the maximum stopband error is $\delta_2 = 0.085$.

The passband and stopband cutoff frequencies are $\omega_p = 0.4\pi$ and $\omega_s = 0.58\pi$.

What was the weighting function $W(\omega)$ that was used in the optimization?

Solution to Q11.4

low pass filter

→ passband 0 to $\omega_p = 0.4\pi$

assume passband weight is 1,

$$W(\omega) = 1 \quad \text{for} \quad 0 \le \omega \le 0.4\pi$$

transition band $\omega_p = 0.4\pi$ to $\omega_s = 0.58\pi$

transition band (don't care) has weight 0,

$$W(\omega) = 0 \quad \text{for} \quad 0.4\pi < \omega < 0.58\pi$$

Solution to Q11.4 (cont'd)

stopband $\omega_s = 0.58\pi$ to π

stopband error × weight = passband error × weight

$$\delta_2 \times W(\omega) = \delta_1 \times 1$$

$$\rightarrow \quad 0.085 \times W(\omega) = 0.0531 \times 1$$

$$\rightarrow \quad W(\omega) = 0.625 \quad \text{for} \quad 0.58\pi \le \omega \le \pi$$

therefore, $$W(\omega) = \begin{cases} 1 & 0 \le \omega \le 0.4\pi \\ 0 & 0.4\pi < \omega < 0.58\pi \\ 0.625 & 0.58\pi \le \omega \le \pi \end{cases}$$

(other solutions are possible)

Q11.5

A discrete-time IIR filter may be obtained from a continuous-time filter using the following transformation:

$$s = \frac{1+z^{-1}}{1-z^{-1}}$$

(i) Show that the above transformation maps the $j\Omega$ axis of the s-plane onto the unit circle of the z-plane.

(ii) Show that the above transformation maps the left-half s-plane to the interior of the unit circle of the z-plane.

(iii) Find the frequency mapping relating Ω to ω. From your result, find what kind of discrete-time filter will be obtained from a continuous-time low-pass filter using this transformation.

Solution to Q11.5

$$s=\frac{1+z^{-1}}{1-z^{-1}} \qquad \Rightarrow s-z^{-1}s=1+z^{-1}$$

$$\Rightarrow -z^{-1}s-z^{-1}=1-s$$

$$\Rightarrow z=\frac{s+1}{s-1}=-\frac{1+s}{1-s}$$

(i) $j\Omega$ axis of s-plane: $\qquad s=j\Omega$

$$z=-\frac{1+j\Omega}{1-j\Omega}=-\frac{\sqrt{1+\Omega^2}\,e^{j\arctan(\Omega)}}{\sqrt{1+\Omega^2}\,e^{j\arctan(-\Omega)}}=-e^{j\times\text{phase}}$$

$\Rightarrow |z|=1$ or, unit circle of the z-plane

Solution to Q11.5 (cont'd)

(ii) left half of s-plane: $s = \sigma + j\Omega, \sigma < 0$

$$z = -\frac{1+\sigma+j\Omega}{1-\sigma-j\Omega} = -\frac{\sqrt{(1+\sigma)^2+\Omega^2}}{\sqrt{(1-\sigma)^2+\Omega^2}} e^{j\times\text{phase}}$$

$$\Rightarrow |z| = \frac{\sqrt{(1+\sigma)^2+\Omega^2}}{\sqrt{(1-\sigma)^2+\Omega^2}}$$

since $\sigma < 0 \Rightarrow (1-\sigma)^2 > (1+\sigma)^2$

it follows that $\sqrt{(1-\sigma)^2+\Omega^2} > \sqrt{(1+\sigma)^2+\Omega^2} \Rightarrow |z| < 1$

or, interior of the unit circle of the z-plane

Solution to Q11.5 (cont'd)

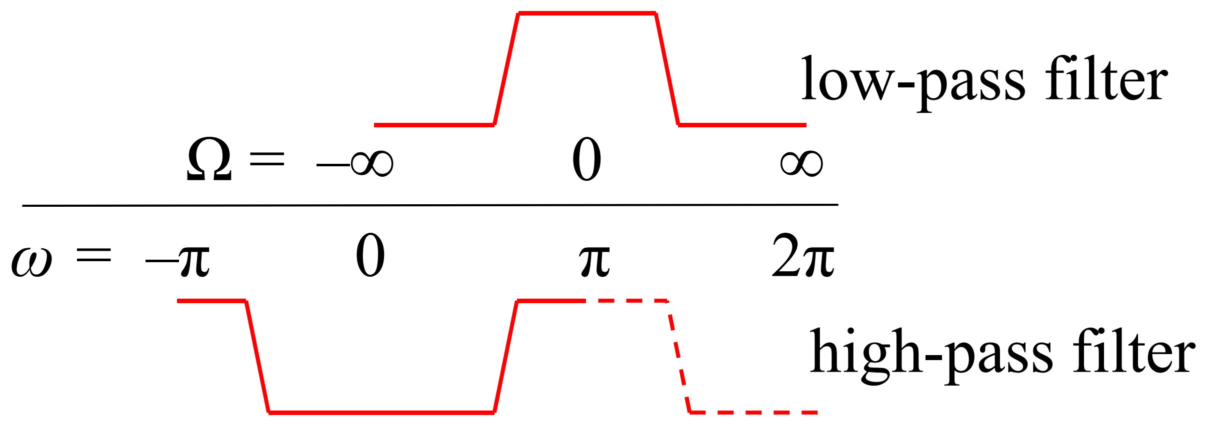
(iii) frequency mapping: $s = j\Omega, z = e^{j\omega}$

$$e^{j\omega} = -\frac{1 + j\Omega}{1 - j\Omega} = -e^{j2\arctan\Omega}$$

from bilinear transformation result

$$= e^{j(\pi + 2\arctan\Omega)} \qquad \Rightarrow \omega = \pi + 2\arctan\Omega$$

mapping: (monotonic) +ve frequency maps to –ve frequency, etc.



Therefore, $H(z)$ is a high-pass filter.