Sparse Dictionary Learning for Image Inpainting

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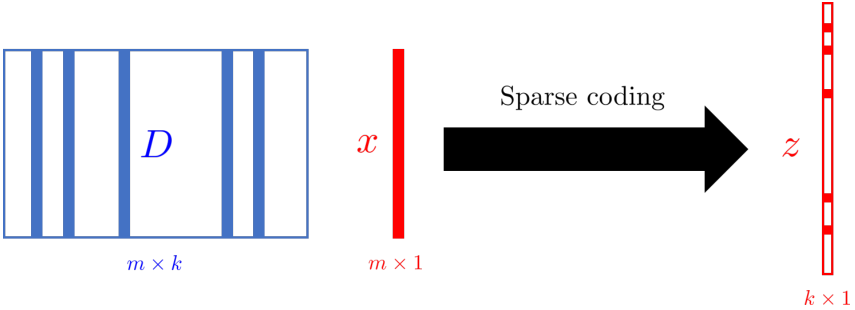
Outline

1. Sparse coding (sparse representation)
2. Sparse dictionary learning
3. Application to image inpainting

Sparse coding problem (sparse representation)

Problem

$\min \|z\|_0$ s.t. $Dz = x$



Sparse coding problem (sparse representation)

SR problem

$$z^* = \arg\min_{z \in \mathbb{R}^k} \left(\frac{1}{2} \|x - Dz\|_2^2 + \lambda \|z\|_0 \right)$$

where

- x is signal vector in $\mathbb{R}^m$
- D is dictionary matrix in $\mathbb{R}^{m \times k}$
- z is sparse coefficient vector in $\mathbb{R}^k$
- $\lambda > 0$ is a penalty parameter

Sparse coding problem (sparse representation)

It is inefficient to compute $\|z\|_0$ directly when n is large. In practice, we will use the ℓ^1 norm instead of the ℓ^0 norm.

SR problem

$$z^* = \arg\min_{z \in \mathbb{R}^k} \left(\frac{1}{2} \|x - Dz\|_2^2 + \lambda \|z\|_1 \right), \lambda > 0.$$

Sparse dictionary learning

SDL problem

$$\min_{D,z_i}\left(\frac{1}{2}\sum_{i=1}^{n}\|x_i - Dz_i\|_2^2 + \lambda\sum_{i=1}^{n}\|z_i\|_1\right)$$
$$\text{subject to } \left\|d_j\right\|_2 \leq 1, \forall\, 1 \leq j \leq k.$$

where

- x_i is a given signals in $\mathbb{R}^m$
- $D = [d_1, d_2, \cdots, d_k]$ is dictionary matrix in $\mathbb{R}^{m\times k}$
- z is sparse coefficient vector in $\mathbb{R}^k$
- $\lambda > 0$ is a penalty parameter

Generate training set

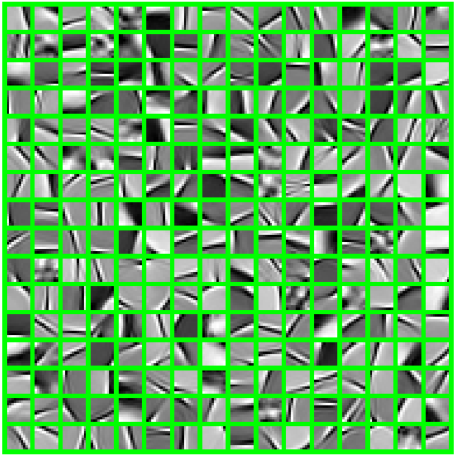
10

10

x_1 $\cdots$ x_n

100×1 100×1

Visualize dictionary



Sparse dictionary learning

SDL method

Given an initial dictionary $D^{(0)}$, for $t = 1, 2, \cdots, T$, we solve the following two sub-problems alternatingly:

$$x^{(t)} = \text{Random drawing from } \{x_1, \cdots, x_n\}$$

$$z^{(t)} = \arg\min_{z \in \mathbb{R}^k} \left(\frac{1}{2} \| x^{(t)} - D^{(t-1)} z \|_2^2 + \lambda \| z \|_1 \right)$$

$$D^{(t)} = \arg\min_{D \in \mathcal{C}} \frac{1}{t} \sum_{i=1}^{t} \left(\frac{1}{2} \| x^{(i)} - D z^{(i)} \|_2^2 + \lambda \| z^{(i)} \|_1 \right)$$

$$\text{where } \mathcal{C} = \{ D \in \mathbb{R}^{m \times k} \mid d_j^\top d_j \le 1, \forall\, 1 \le j \le k \}$$

We iterate until convergence is achieved.

Mini-batch extension

To simplify the formulation of the SDL problem, we define

$$X^{(t)} = \begin{bmatrix} x_1^{(t)}, & x_2^{(t)}, & \cdots, & x_b^{(t)} \end{bmatrix} \in \mathbb{R}^{m \times b}$$
$$Z^{(t)} = \begin{bmatrix} z_1^{(t)}, & z_2^{(t)}, & \cdots, & z_b^{(t)} \end{bmatrix} \in \mathbb{R}^{k \times b}$$

where b is the batch size.

Hence the problem in (SDL) can be rewritten as follows:

$$Z^{(t)} = \operatorname*{arg\,min}_{Z \in \mathbb{R}^{k \times b}} \left(\frac{1}{2} \|X^{(t)} - D^{(t-1)} Z\|_F^2 + \lambda \|Z\|_{1,1} \right) \tag{1}$$

$$D^{(t)} = \operatorname*{arg\,min}_{D \in \mathcal{C}} \frac{1}{t} \sum_{i=1}^{t} \left(\frac{1}{2} \|X^{(i)} - D Z^{(i)}\|_F^2 + \lambda \|Z^{(i)}\|_{1,1} \right) \tag{2}$$

where $\|Z\|_{1,1} := \sum_{j=1}^{b} \|z_j\|_1$.

Solving sparse coding by ADMM

To solve

$$Z^{(t)} = \underset{Z \in \mathbb{R}^{k \times b}}{\arg\min} \left(\frac{1}{2} \|X - DZ\|_F^2 + \lambda \|Z\|_{1,1} \right)$$

We add an auxiliary variable Y and a dual variable U, we define

$$f(Z) = \frac{1}{2} \|X - DZ\|_F^2, \quad g(Y) = \lambda \|Y\|_{1,1}, \quad Z = Y.$$

Then the ADMM for solving (1) is given by

1. $Z^{(i)} = \underset{Z}{\arg\min} \left(\frac{1}{2} \|X - DZ\|_F^2 + \frac{\rho}{2} \|Z - Y^{(i-1)} + U^{(i-1)}\|_F^2 \right)$
2. $Y^{(i)} = \underset{Y}{\arg\min} \left(\lambda \|Y\|_{1,1} + \frac{\rho}{2} \|Z^{(i)} - Y + U^{(i-1)}\|_F^2 \right)$
3. $U^{(i)} = U^{(i-1)} + Z^{(i)} - Y^{(i)}$

Solving Z-subproblem

Define

$$F(Z) = \frac{1}{2}\|X - DZ\|_F^2 + \frac{\rho}{2}\|Z - Y^{(i-1)} + U^{(i-1)}\|_F^2$$

Then

$$\begin{aligned}\nabla F(Z) &= -D^\top (X - DZ) + \rho I \left(Z - Y^{(i-1)} + U^{(i-1)}\right) \\ &= \left(D^\top D + \rho I\right) Z - \left(D^\top X + \rho(Y^{(i-1)} - U^{(i-1)})\right)\end{aligned}$$

Letting $\nabla F(Z) = 0$, we have

$$\left(D^\top D + \rho I\right) Z = \left(D^\top X + \rho(Y^{(i-1)} - U^{(i-1)})\right)$$

Therefore, we obtain the solution

$$Z^{(i)} = \left(D^\top D \ + \rho I\right)^{-1} \left(D^\top X + \rho(Y^{(i-1)} - U^{(i-1)})\right)$$

Solving Z-subproblem

Therefore, we obtain the solution

$$Z^{(i)} = \left(D^\top D \ + \rho I\right)^{-1} \left(D^\top X + \rho(Y^{(i-1)} - U^{(i-1)})\right)$$

Note that if D is flat matrix, we can use Sherman–Morrison formula,

$$\begin{aligned}(D^\top D + \rho I)^{-1} &= \left(D^\top D + \frac{1}{\rho} I\right)^{-1} \\ &= \frac{1}{\rho} I - \frac{1}{\rho^2} D^\top \left(I + \frac{1}{\rho} D D^\top\right)^{-1} D\end{aligned}$$

Since $I + \frac{1}{\rho} DD^\top$ is symmetry positive definite, we can use Cholesky factorization to reduce the time to solve the linear system.

Solving Y-subproblem

To solve

$$Y^{(i)} = \arg\min_{Y} \left(\lambda \|Y\|_{1,1} + \frac{\rho}{2} \|Z^{(i)} - Y + U^{(i-1)}\|_F^2 \right)$$

Using the component-wise soft-thresholding function, the solution of Y-subproblem has the closed form:

$$Y^{(i)} = \mathcal{S}_{\lambda/\rho} \left(Z^{(i)} + U^{(i-1)} \right)$$

where

$$\mathcal{S}_{\lambda/\rho}(V) = \mathrm{sign}(V) \odot \max \left(0, |V| - \frac{\lambda}{\rho} \right)$$

with $\mathrm{sign}(V)$ and $|V|$ are element-wisely applied to the matrix V and $\odot$ is the Hadamard product.

Solving sparse coding by ADMM

Therefore, the iterative scheme can be posed as follows:

1. $Z^{(i)} = \left(D^\top D + \rho I\right)^{-1} \left(D^\top X + \rho(Y^{(i-1)} - U^{(i-1)})\right)$

2. $Y^{(i)} = \mathcal{S}_{\lambda/\rho}\left(Z^{(i)} + U^{(i-1)}\right)$

3. $U^{(i)} = U^{(i-1)} + Z^{(i)} - Y^{(i)}$

Update dictionary

The problem (2) in (SDL) is

$$D^{(t)} = \arg\min_{D \in \mathcal{C}} \frac{1}{t} \sum_{i=1}^{t} \left(\frac{1}{2} \|X^{(i)} - DZ^{(i)}\|_F^2 + \lambda \|Z^{(i)}\|_{1,1} \right)$$

Note that

$$\begin{aligned} & \|X^{(i)} - DZ^{(i)}\|_F^2 \\ = & tr\left[(X^{(i)} - DZ^{(i)})^\top (X^{(i)} - DZ^{(i)}) \right] \\ = & tr({X^{(i)}}^\top X^{(i)}) + tr({Z^{(i)}}^\top D^\top D Z^{(i)}) - 2tr({Z^{(i)}}^\top D^\top X^{(i)}) \end{aligned}$$

Then we have

$$D^{(t)} = \arg\min_{D \in \mathcal{C}} \frac{1}{t} \sum_{i=1}^{t} \left(\frac{1}{2} tr({Z^{(i)}}^\top D^\top D Z^{(i)}) - tr({Z^{(i)}}^\top D^\top X^{(i)}) \right)$$

Update dictionary

Let $A^{(t)} = \sum_{i=1}^{t} Z^{(i)} {Z^{(i)}}^\top$ and $B^{(t)} = \sum_{i=1}^{t} X^{(i)} {Z^{(i)}}^\top$.

We can show that

- $\sum_{i=1}^{t} tr({Z^{(i)}}^\top D^\top D Z^{(i)}) = tr(D^\top D A^{(t)})$
- $\sum_{i=1}^{t} tr({Z^{(i)}}^\top D^\top X^{(i)}) = tr(D^\top B^{(t)})$

Therefore, the problem can be rewritten to

$$D^{(t)} = \arg\min_{D \in \mathcal{C}} \frac{1}{t} \left(\frac{1}{2} tr(D^\top D A^{(t)}) - tr(D^\top B^{(t)}) \right)$$

Update dictionary: projected coordinate descent

To solve

$$D^{(t)} = \arg\min_{D \in \mathcal{C}} \frac{1}{t} \left(\frac{1}{2} tr(D^\top D A^{(t)}) - tr(D^\top B^{(t)}) \right)$$

First, let

$$F(D) = \frac{1}{2} tr(D^\top D A^{(t)}) - tr(D^\top B^{(t)})$$

Then we can get

$$\nabla_{d_i} F(D) = D a_i - b_i$$

where a_i, b_i are the column vector of $A^{(t)}$ and $B^{(t)}$.

Update dictionary: projected coordinate descent

Applying coordinate decent method, d_i can update by

$$d_i^{(t)} = d_i^{(t-1)} - \frac{1}{A_{i,i}}(Da_i - b_i).$$

Since we hope $d_i \leq 1$, $\forall\, 1 \leq i \leq k$,

$$d_i^{(t)} \leftarrow \frac{d_i^{(t)}}{\max(\|d_i\|_2, 1)}$$

Algorithm: dictionary learning

Algorithm Dictionary Learning

Require: $D^{(0)}$, $x_1, \cdots x_n$, λ.
$A^{(0)} \leftarrow 0, B^{(0)} \leftarrow 0$
for $t = 1$ to maxstep **do**
    Random drawing $X^{(t)}$ from $\{x_1, \cdots x_n\}$
    Do the Sparse Coding
    $A^{(t)} \leftarrow A^{(t-1)} + Z^{(t)} {Z^{(t)}}^\top$
    $B^{(t)} \leftarrow B^{(t-1)} + X^{(t)} {Z^{(t)}}^\top$
    Update Dictionary
end for
return D

Algorithm: spare coding

Algorithm Spare Coding

Require: $D, X, Y^{(0)}, U^{(0)}, \lambda, \rho$

for $i = 1$ to maxstep **do**

$\quad Z^{(i)} = \left(D^\top D + \rho I\right)^{-1} \left(D^\top X + \rho(Y^{(i-1)} - U^{(i-1)})\right)$

$\quad Y^{(i)} = \text{sign}\left(Z^{(i)} + U^{(i-1)}\right) \odot \max(0, |Z^{(i)} + U^{(i-1)}| - \frac{\lambda}{\rho})$

$\quad U^{(i)} = U^{(i-1)} + Z^{(i)} - Y^{(i)}$

end for

return Z

Algorithm: update dictionary

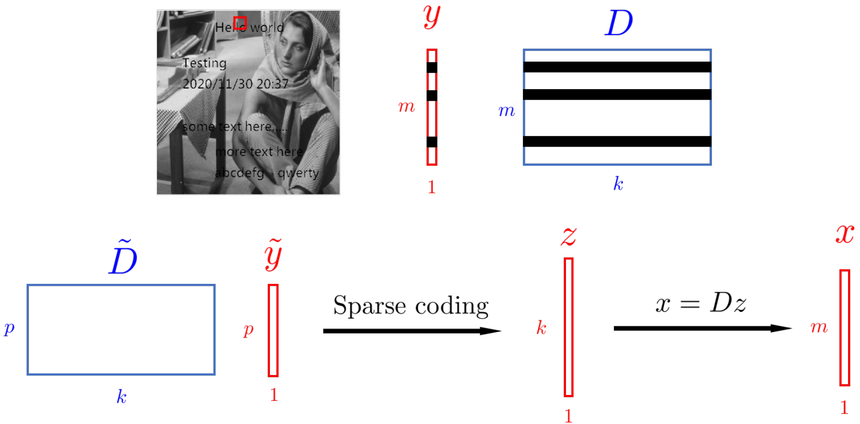
Algorithm Update Dictionary

1: **Require:** $D = [d_1, \cdots d_k]$, $A = [a_1, \cdots a_k]$, $B = [b_1, \cdots b_k]$
2: **repeat**
3: **for** $i = 1$ to k **do**
4: Update the i-th column:

$$u_i \leftarrow d_i - \frac{1}{A_{ii}}(Da_i - b_i)$$
$$d_i \leftarrow \frac{u_i}{\max(\|u_i\|_2, 1)}$$

5: **end for**
6: **until convergence**
7: **return** D

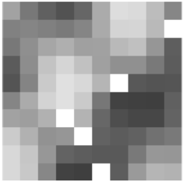
Application to image inpainting



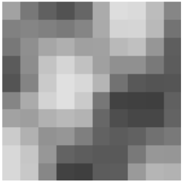
Example of image inpainting

original block

contaminated block



recovered block



- Recovery rate: $80\% \sim 90\%$
- Sparsity: 6%

Numerical result (I) : grayscale image

Before inpainting

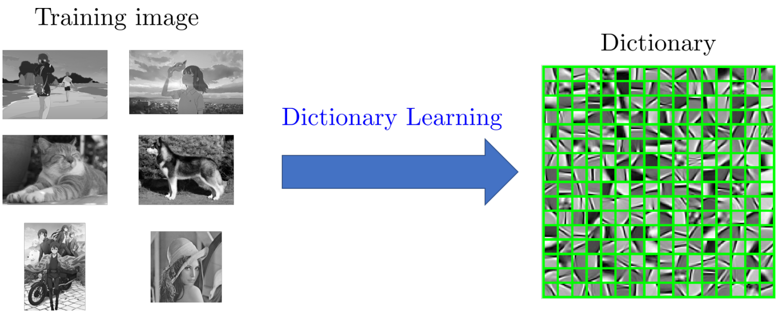
After inpainting

Numerical result (I) : color image

Before inpainting

After inpainting

Dictionary learning from multiple images



Numerical result (II) : grayscale image

Before inpainting

After inpainting

Numerical result (II) : color image

Before inpainting

After inpainting

Reference

- Julien Mairal, Francis Bach, Jean Ponce, Guillermo Sapiro (2009), Online Dictionary Learning for Sparse Coding. https://dl.acm.org/doi/10.1145/1553374.1553463
- Xiaomin Zhang, Jinyu Xia, Jinman Zhao (2015), Image Inpainting using Online Dictionary Learning. http://pages.cs.wisc.edu/~xiaominz/projs/CS532ImageInpainting.pdf
- Po-I Tseng, Suh-Yuh Yang (2020), Sparse Representation and Dictionary Learning with Applications to Image Processing. http://www.math.ncu.edu.tw/~syyang/

THE END

Thanks for listening!