Numerical Methods for Partial Differential Equations H.W.1

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- **Thomas Algorithm function**

```
function x = Thomas(A, d)
n = size(A,1);
a = [0; diag(A,-1)];
b = diag(A);
c = diag(A,1);
x = zeros(n,1);

for i = 2:n
    b(i) = b(i)-a(i)*c(i-1)/b(i-1);
    d(i) = d(i)-a(i)*d(i-1)/b(i-1);
end

x(n) = d(n)/b(n);

for i = n-1:-1:1
    x(i) = (d(i)-c(i)*x(i+1))/b(i);
end
```

- **Finite Difference function**

```
function U = FiniteDifference(nu, x, f, alpha, beta)
h = 1/nu;
U = ones(nu+1,1);
U(1) = alpha; U(end) = beta;
A = diag(-2*ones(nu-1,1)) + diag(ones(nu-2,1), 1) + diag(ones(nu-2,1), -1);

F = f(x(2:end-1))';
F(1) = F(1)-alpha/h^2;
F(end) = F(end)-beta/h^2;

U(2:end-1) = Thomas(A, F)*h^2;

end
```

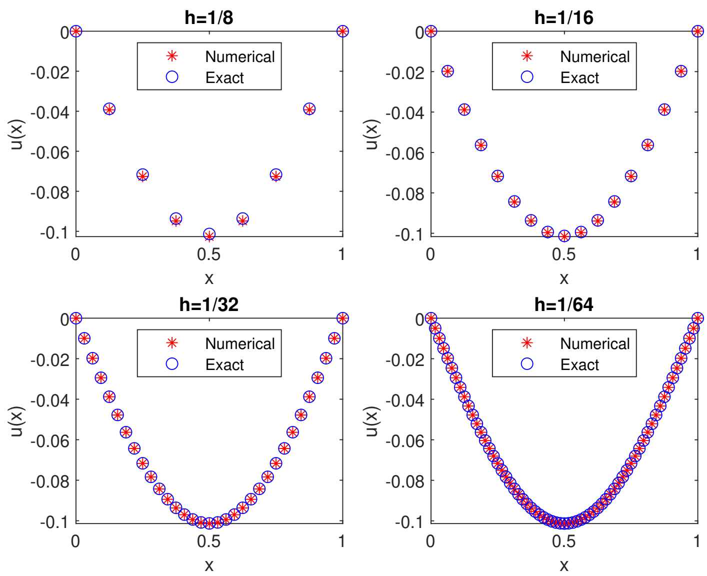
- **Main function**

```matlab
clc; clear; close all;
format long e

f1 =@(x) sin(pi*x);
u1 =@(x) -sin(pi*x)/pi^2;
alpha1 = 0; beta1 = 0;

f2 =@(x) 0*x+1;
u2 =@(x) x.^2/2;
alpha2 = 0; beta2 = 1/2;

f3 =@(x) (heaviside(x-1/2)-heaviside(x-1)).*x.^2 ...
       + (heaviside(x)-heaviside(x-1/2)).*x/2;
u3 =@(x) (heaviside(x-1/2)-heaviside(x-1)).*(x.^4/12-5*x/64-1/192) ...
       +(heaviside(x)-heaviside(x-1/2)).*(x.^3/12 -19*x/192);
alpha3 = 0; beta3 = 0;

%% Exercise 1
close all;

NList = [8 ,16, 32, 64];
ErrorList = zeros(4,1);
RatioList = zeros(3,1);

figure(1);
for i = 1:length(NList)
    nu = NList(i);
    x0 = 0; xn = 1; dx = (xn-x0)/nu; x = x0:dx:xn;

    U = FiniteDifference(nu, x, f1, alpha1, beta1); % numerical solution
    U_hat = u1(x'); % exact solution
    ErrorList(i) = max(abs(U-U_hat)); % infinity norm

    subplot(2,2,i);
    plot(x, U, 'r*', x, U_hat, 'bo');
    title(['h=1/', int2str(nu)]);
    xlabel('x'); ylabel('u(x)');
    legend('Numerical', 'Exact', 'Location', 'N');
    hold on;

    if i > 1
        RatioList(i) = log2(ErrorList(i-1)/ErrorList(i));
    end

end
hold off;

figure(2);
loglog(NList, ErrorList, '-ro');
axis([min(NList), max(NList), ...
    min(ErrorList), max(ErrorList)]);
xlabel('log(h)');
```

```
ylabel('log(error)');

%% Exercise 2
close all;

NList = [8 ,16, 32, 64];
ErrorList = zeros(1,4);
RatioList = zeros(1,3);

figure(1);
for i = 1:length(NList)
    nu = NList(i);
    x0 = 0; xn = 1; dx = (xn-x0)/nu; x = x0:dx:xn;

    U = FiniteDifference(nu, x, f2, alpha2, beta2); % numerical solution
    U_hat = u2(x'); % exact solution
    ErrorList(i) = max(abs(U-U_hat)); % infinity norm

    subplot(2,2,i)
    plot(x, U, 'r*', x, U_hat, 'bo');
    title(['h=1/', int2str(nu)]);
    xlabel('x'); ylabel('u(x)');
    legend('Numerical', 'Exact', 'Location', 'NW');
    hold on;

    if i > 1
        RatioList(i) = log2(ErrorList(i-1)/ErrorList(i));
    end

end

figure(2);
loglog(NList, ErrorList, '-ro');
axis([min(NList), max(NList), ...
    min(ErrorList), max(ErrorList)]);
xlabel('log(h)');
ylabel('log(error)');

%% Exercise 3
close all;

NList = [8 ,16, 32, 64];
ErrorList = zeros(1,4);
RatioList = zeros(1,3);

figure(1);
for i = 1:length(NList)
    nu = NList(i);
    x0 = 0; xn = 1; dx = (xn-x0)/nu; x = x0:dx:xn;

    U = FiniteDifference(nu, x, f3, alpha3, beta3); % numerical solution
    U_hat = u3(x'); % exact solution
    ErrorList(i) = max(abs(U-U_hat)); % infinity norm
```

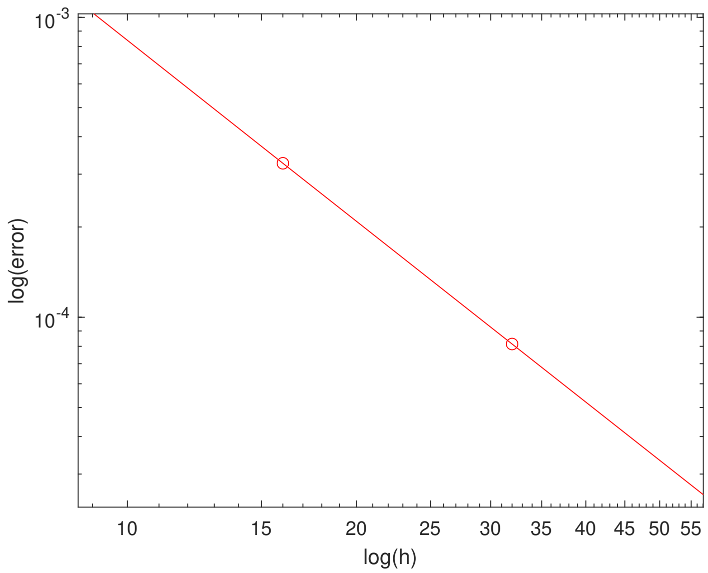
```

    subplot(2,2,i)
    plot(x, U, 'r*', x, U_hat, 'bo');
    title(['h=1/', int2str(nu)]);
    xlabel('x'); ylabel('u(x)');
    legend('Numerical', 'Exact', 'Location', 'N');
    hold on;

    if i > 1
        RatioList(i) = log2(ErrorList(i-1)/ErrorList(i));
    end

end

figure(2);
loglog(NList, ErrorList, '-ro');
axis([min(NList), max(NList), ...
    min(ErrorList), max(ErrorList)]);
xlabel('log(h)');
ylabel('log(error)');
```

(1)
- PDE: $u'' = \sin \pi x$ on $[0, 1]$.
- Boundary conditions: $u(0) = 0, u(1) = 0$.
- Exact solution: $u(x) = -\dfrac{1}{\pi^2} \sin \pi x$.
- Solution:

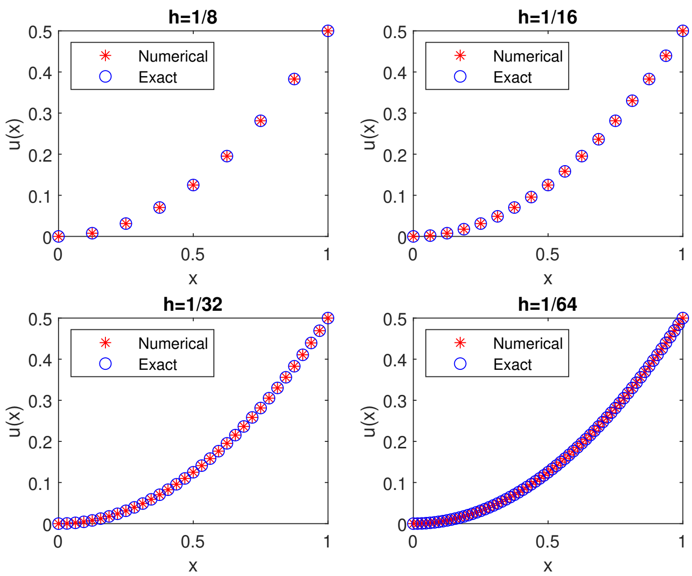


- Error analysis:

h	Error: $\|U - \hat{U}\|_\infty$	$\log_2$ Ratio
$\frac{1}{8}$	0.001312184986913	
$\frac{1}{16}$	3.261492871714650e-04	2.008366739526168
$\frac{1}{32}$	8.141944162229353e-05	2.002087242819560
$\frac{1}{64}$	2.034750345958347e-05	2.000521533817863

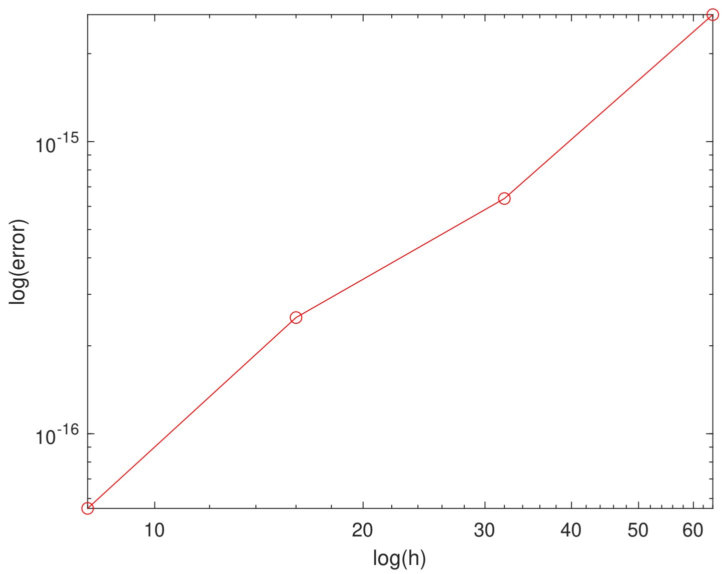


(2)
- PDE: $u'' = 1$ on $[0, 1]$.
- Boundary conditions: $u(0) = 0, u(1) = \frac{1}{2}$.
- Exact solution: $u(x) = \frac{1}{2}x^2$.
- Solution:



- Error analysis:

h	Error: $\Vert U - \hat{U} \Vert_\infty$	$\log_2$ Ratio
$\frac{1}{8}$	5.551115123125783e-17	
$\frac{1}{16}$	2.498001805406602e-16	-2.169925001442313
$\frac{1}{32}$	6.383782391594650e-16	-1.353636954614701
$\frac{1}{64}$	2.720046410331634e-15	-2.091147888058195

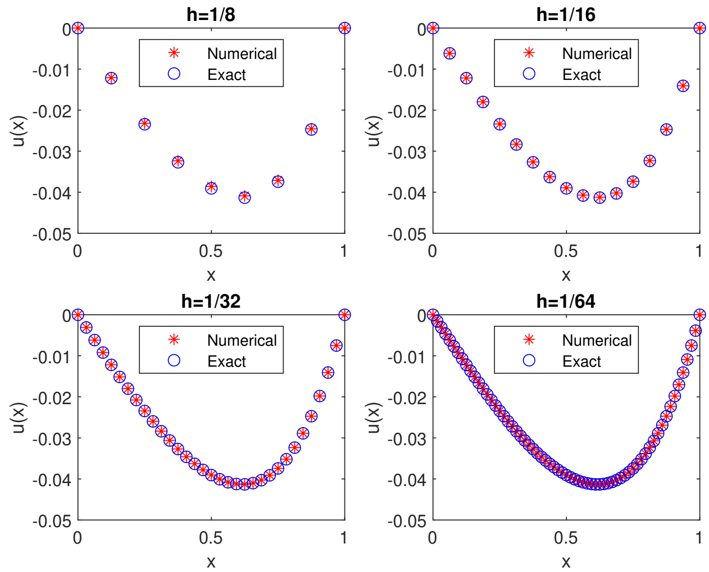


(3) - PDE: $u'' = \begin{cases} \frac{x}{2}, & x < \frac{1}{2} \\ x^2, & x \geq \frac{1}{2} \end{cases}$ on $[0, 1]$.

- Boundary conditions: $u(0) = 0, u(1) = 0$.

- Exact solution: $u(x) = \begin{cases} \frac{x^4}{12} - \frac{5x}{64} - \frac{1}{192}, & \frac{1}{2} \leq x \leq 1 \\ \frac{x^3}{12} - \frac{19x}{192}, & 0 \leq x < \frac{1}{2} \end{cases}$.

- Solution:



- Error analysis:

h	Error: $\|U - \hat{U}\|_\infty$	$\log_2$ Ratio
$\frac{1}{8}$	4.882812499999861e-04	
$\frac{1}{16}$	1.220703125000069e-04	1.999999999999877
$\frac{1}{32}$	3.051757812511796e-05	1.999999999994506
$\frac{1}{64}$	7.629394531590006e-06	1.999999999941283