Numerical Methods for Partial Differential Equations H.W.6

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Let the hat function with support $(-h, h)$,

$$d_h^{(1)} = \begin{cases} \dfrac{h - |x|}{h^2}, & |x| \le h, \\ 0, & \text{otherwise}, \end{cases}$$

a wider hat function with support $(-2h, 2h)$,

$$d_h^{(2)} = \begin{cases} \dfrac{2h - |x|}{4h^2}, & |x| \le 2h, \\ 0, & \text{otherwise}, \end{cases}$$

a perhaps a smoother version such as,

$$d_h^{(3)} = \begin{cases} \dfrac{\cos(\frac{\pi x}{2h}) + 1}{4h}, & |x| \le 2h, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$d_h^{(4)} = \frac{1}{h} \begin{cases} 1 - \left(\dfrac{x}{h}\right)^2, & |x| \le h, \\ 2 - \left|\dfrac{3x}{h}\right| + \left(\dfrac{x}{h}\right)^2, & h \le |x| \le 2h, \\ 0, & \text{otherwise}, \end{cases}$$

- **Exercise 6.1**

  Solve

  $$\begin{cases} u'' = \delta(x - \frac{1}{3}) \text{ on } (0, 1), \\ u(0) = u(1) = 0 \end{cases}.$$

  By using $d_h^{(1)}$, $d_h^{(2)}$, $d_h^{(3)}$, $d_h^{(4)}$, where $h = \frac{1}{16}$.

- **Exercise 6.2**

  Solve

$$\begin{cases} u'' = 1 + \delta(x - \frac{1}{3}) \text{ on } (0,1), \\ u(0) = u(1) = 0 \end{cases}.$$

  By using $d_h^{(1)}$, $d_h^{(2)}$, $d_h^{(3)}$, $d_h^{(4)}$, where $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$.

- **Code**

  1. **Thomas Algorithm function**

```
function x = Thomas(A, d, xn)
n = size(A,1);
a = [0; diag(A,-1)];
b = diag(A);
c = diag(A,1);
x = zeros(n,1);

for i = 2:n
    b(i) = b(i)-a(i)*c(i-1)/b(i-1);
    d(i) = d(i)-a(i)*d(i-1)/b(i-1);
end

if nargin ==3
    x(n) = xn;

else
    x(n) = d(n)/b(n);
end

for i = n-1:-1:1
    x(i) = (d(i)-c(i)*x(i+1))/b(i);
end
```

  2. **Finite Difference function**

```
function U = FDM(m, x, f, alpha, beta, method)
h = (x(end)-x(1))/(m+1);

U = ones(m+2,1);
U(1) = alpha; U(end) = beta;

A = diag(-2*ones(m,1)) + diag(ones(m-1,1), 1) + diag(ones(m-1,1), -1);
Ah = A/h^2;

F = [0; f(x(2:end-1))'; 0];

% 4th order convergence
if nargin == 6
    F(2:end-1) = (F(1:end-2) + 10*F(2:end-1) + F(3:end))/12;
```

```
    F(2) = F(2)+alpha/h^2;
    F(end-1) = F(end-1)-beta/h^2;
end

F = F(2:end-1);
U(2:end-1) = Thomas(Ah, F);

end
```

3. Delta function

```
function f = delta(h, idx, c)

if idx==1
    f =@(x) (heaviside(x+h)-heaviside(x-h)).*(h-abs(x))/h^2;
end

if idx==2
    f =@(x) (heaviside(x+2*h)-heaviside(x-2*h)).*(2*h-abs(x))/(4*h^2);
end

if idx==3
    f =@(x)
        (heaviside(x+2*h)-heaviside(x-2*h)).*(cos(pi*x/(2*h))+1)/(4*h);
end

if idx==4
    f =@(x) ((heaviside(abs(x))-heaviside(abs(x)-h)).*(1-(abs(x)/h).^2)...
    +(heaviside(abs(x)-h)-heaviside(abs(x)-2*h)).*(2 - 3*abs(x)/h +
        (abs(x)/h).^2))/h;
end

if nargin == 3
    f = @(x) f(x-c);
end

end
```

4. Green function

```
function G = GreenFunc(x,c)
G = (heaviside(x)-heaviside(x-c)).*(c-1).*x + ...
    + (heaviside(x-c)-heaviside(x-1)).*c.*(x-1);
end
```

5. PlotLogError

```
function PlotLogError(mList, ErrorList)
loglog(mList, ErrorList, '-ro', 'LineWidth', 1.2);
axis([min(mList), max(mList), ...
    min(ErrorList), max(ErrorList)]);
xlabel('$\log h $', 'interpreter', 'latex');
```

```
ylabel('$\log e$', 'interpreter', 'latex');

end
```

6. main_hw_6.1

```
clc; clear; close all;

c = 1/3; % center
m = 15; % mesh
x0 = 0;    xm_1 = 1;
alpha = 0; beta = 0;
h = (xm_1-x0)/(m+1); x = x0:h:xm_1;

ErrorList = zeros(m+2,4);

% Exact solution
Uhat = GreenFunc(x', c);

%% FDM
for ii =1:4
    % Numerical solution
    f = delta(h, ii, c);
    U = FDM(m, x, f, alpha, beta);
    ErrorList(:,ii) = abs(U-Uhat);

    subplot(2,2,ii)
    plot(x, U, 'ro', x, Uhat, 'b*');
    title(['$d_h$', int2str(ii)], 'interpreter', 'latex');
    xlabel('$x$', 'interpreter', 'latex');
    ylabel('$u(x)$', 'interpreter', 'latex');
    legend('Numerical', 'Exact', 'Location', 'best');
end
```

7. main_hw_6.2

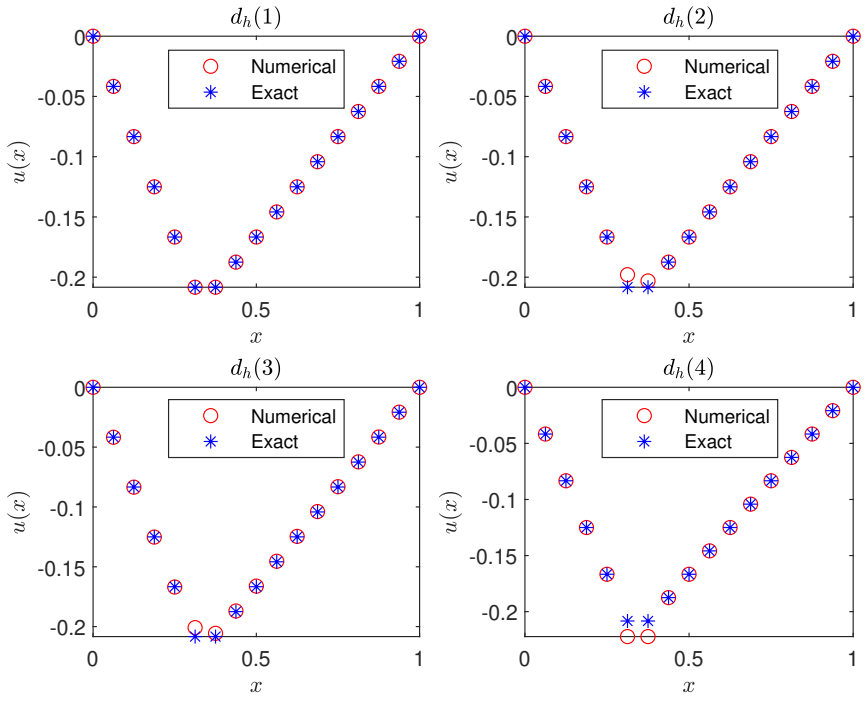
```
clc; clear; close all;

c = 1/3; % center

mList = [15 ,31, 63];
mListLength = length(mList);
ErrorList = zeros(mListLength,4);
RatioList = zeros(mListLength-1,4);

%% FDM
for i = 1:3
    m = mList(i);
    x0 = 0;    xm_1 = 1;
    alpha = 0; beta = 0;
    h = (xm_1-x0)/(m+1); x = x0:h:xm_1;

    % Exact solution
```

```
    Uhat = ((x.^2)/2)' + GreenFunc(x', c) - x'/2;

    for j =1:4
        % Numerical solution
        d = delta(h, j, c);
        f =@(x) d(x)+1;

        U = FDM(m, x, f, alpha, beta);

        ErrorList(i,j) = max(abs(U-Uhat));
        if i>1
            RatioList(i-1,j) = log2(ErrorList(i-1,j)/ErrorList(i,j));
        end

        subplot(3,4,(i-1)*4+j)
        plot(x, U, 'b*', x, Uhat, 'ro');
        title(['$d_h$', int2str(j), ', $h=1/$', int2str(m)],
            'interpreter', 'latex');
        xlabel('$x$', 'interpreter', 'latex');
        ylabel('$u(x)$', 'interpreter', 'latex');
    end
end
```

- **Numerical result**

  Exercise 6.1

  (a) PDE's solution



(b) Error list

j	$d_h^{(1)}$	$d_h^{(2)}$	$d_h^{(3)}$	$d_h^{(4)}$
1	$6.9e-18$	$6.9e-18$	$6.4e-05$	$6.9e-18$
2	$1.4e-17$	$1.4e-17$	0.00013	$1.4e-17$
3	$2.8e-17$	$2.8e-17$	0.00019	0
4	$2.8e-17$	$5.6e-17$	0.00026	0
5	$2.8e-17$	0.01	0.0075	0.014
6	$5.6e-17$	0.0052	0.0027	0.014
7	0	0	0.00057	$2.8e-17$
8	0	0	0.00051	$2.8e-17$
9	$2.8e-17$	$2.8e-17$	0.00045	0
10	$2.8e-17$	$2.8e-17$	0.00038	0
11	$4.2e-17$	$4.2e-17$	0.00032	$1.4e-17$
12	$2.8e-17$	$2.8e-17$	0.00026	$1.4e-17$
13	$1.4e-17$	$1.4e-17$	0.00019	0
14	$6.9e-18$	$6.9e-18$	0.00013	$6.9e-18$
15	$3.5e-18$	$3.5e-18$	$6.4e-05$	$3.5e-18$

Exercise 6.2

(a) Error list

h	$d_h^{(1)}$	$d_h^{(2)}$	$d_h^{(3)}$	$d_h^{(4)}$
$\frac{1}{16}$	$5.6e-17$	0.0104	0.0075	0.0139
$\frac{1}{32}$	$7.2e-16$	0.0052	0.0036	0.0069
$\frac{1}{64}$	$3.1e-15$	0.0026	0.0019	0.0035

(b) Ratio list

$\log_2$ ratio	$d_h^{(1)}$	$d_h^{(2)}$	$d_h^{(3)}$	$d_h^{(4)}$
$\log_2 \frac{e(16)}{e(32)}$	-3.70	1.00	1.07	1.00
$\log_2 \frac{e(32)}{e(64)}$	-2.08	1.00	0.93	1.00