AMATH 251: Assignment #6

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1) Problem 20, section 5.6, page 358.

We want to solve the following IVP,

$$y'' + 0.1y' + y = 1 + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t), \quad y(0) = 0, \quad y'(0) = 0$$

where $u_{k\pi}(t)$ is a translated unit function defined by,

$$u_{k\pi}(t) = u(t - k\pi)$$

By the linearity of the Laplace transform, we know that the Laplace transform of a sum of functions is the sum of the Laplace transforms.

$$(s^2 + 0.1s + 1)Y(s) = \mathcal{L}\{1\} + 2\sum_{k=1}^{n}(-1)^k \mathcal{L}\{u_{k\pi}(t)\}$$

$$(s^2 + 0.1s + 1)Y(s) = \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k \cdot \frac{e^{-k\pi s}}{s}$$

$$Y(s) = \frac{1}{s(s^2 + 0.1s + 1)} \cdot \left(1 + 2\sum_{k=1}^{n}(-1)^k \cdot e^{-k\pi s}\right)$$

By partial fractions we get,

$$Y(s) = [1 + 2\sum_{k=1}^{n}(-1)^k e^{-k\pi s}][\frac{1}{s} - \frac{s + 0.05}{(s + 0.05)^2 + 0.9975} - \frac{0.05}{(s + 0.05)^2 + 0.9975}]$$

And lastly, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, but we will first define a separate function to make the solution cleaner.

$$H(s) = \frac{1}{s} - \frac{s + 0.05}{(s + 0.05)^2 + 0.9975} - \frac{0.05}{(s + 0.05)^2 + 0.9975}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = 1 - e^{-0.05t}\cos(\sqrt{0.9975}t) - \frac{0.05e^{-0.05t}}{\sqrt{0.9975}}\sin(\sqrt{0.9975}t)$$

And we know that the inverse Laplace transform of $e^{-k\pi s}H(s)$ will be the inverse Laplace transform of $H(s)$ shifted by $k\pi$. Thus we have,

$$2\sum_{k=1}^{n}(-1)^k \mathcal{L}^{-1}\{e^{-k\pi s}H(s)\} = 2\sum_{k=1}^{n}(-1)^k h(t - k\pi)$$

And putting it all together, we have

$$y(t) = h(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)h(t - k\pi)$$