Final Report
Arfken's Mathematical Methods for Physicists,
Seventh Edition, Chapter 19, Fourier Series

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Abstract

會先簡介傅立葉的用途，再說他的廣義數學定義和須滿足的條件，接著討論傅立葉的收斂現象以及一些相關定理。

1 Introduction

週期性現象描述了波浪，旋轉機械（諧波運動）或其他重複驅動力。 傅里葉級數是求解帶週期邊界條件的常微分方程（ODE）和偏微分方程（PDE）的基本工具。

General Properties

傅里葉級數定義為一系列正弦和余弦中的一個函數的擴展或表示

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \cos ns \, ds, \qquad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \sin ns \, ds, \qquad n = 1, 2, \ldots,$$

一週期函數可展開成傳立葉，須滿足:

1. 此函數必須是有界的 (bounded)，即對於任意 x，$|f(x)| < M$，M 是一正實數；
2. 在任意區間內，除了有限個不連續點，$f(x)$ 必須是連續函數；
3. 在任意區間內，$f(x)$ 必須僅包含有限個極值；
4. 在一週期內，$|f(x)|$ 的積分必須收斂。

1.1 Sturm-Liouville Theory

Definition 1. A second ordered differential equation of the form

$$-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y+q(x)y=\lambda\omega(x)y, \qquad x\in[a,b] \tag{1}$$

with p, q and ω specified such that $p(x)>0$ and $\omega(x)>0$ for $x\in(a,b)$, is called a *Sturm-Liouville (SL) differential equation.*

Note that SL differential equation is essentially an eigenvalue problem since λ is not specified. While solving SL equation, both λ and y must be determined.

2.2. Discontinuous Functions

傅立葉級數使用來自整個擴展區間的信息，因此可以描述僅通過正交擴展來保證函數表示收斂於均值的函數。 此功能對具有不連續性的功能的擴展起作用，其中不存在擴展必須收斂的唯一值。 但是，對於傅立葉級數，可以證明，如果滿足 Dirichlet 條件的函數在點 x0 處不連續，則在該點求值的傅立葉級數將是左右極限的算術平均值。

Dirichlet:

若函數 $f(x)$ 以及它的導函數 $f'(x)$ 在一個週期內分段連續，則 $f(x)$ 的傅里葉級數(1)在每一點 x 都收斂，並且

(1) 當 x 是 $f(x)$ 的連續點時，級數收斂於 $f(x)$

(2) 當 x 是 $f(x)$ 的間斷點時，級數收斂於 $f(x)$ 在點 x 的右極限與左極限的算數平均值

$$\frac{f(x+0)+f(x-0)}{2}$$

2.3. Periodic Functions

若函數 f(x)對所有實數均有定義，且存在某個正數 P，使得 f(x+p)= f(x)，對所有的 x 均成立。則稱 f(x)為週期函數。此正數 P 為函數 f(x)的週期（period）。

一般 P 指最小值的週期。例如：sinx 的週期函數有 2π、4π、6π、8π・・・・等，其週期為 2π。

週期函數可用三角函數來表示:

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

a_0：$a_0=\dfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$

求 a_0： $\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi}^{\pi}a_0dx+\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}(a_n\cos nx+b_n\sin nx)dx$

$$=a_0\Big|_{-\pi}^{\pi}+\sum_{n=1}^{\infty}\left\{\frac{a_n}{n}\sin nx\Big|_{-\pi}^{\pi}+\frac{-b_n}{n}\cos nx\Big|_{-\pi}^{\pi}\right\}$$

$$=2\pi a_0$$

$\int_{-\pi}^{\pi}f(x)dx=2\pi a_0$ ，所以 $a_0=\dfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$ （π 為正半週期，-π 為負半週期）

a_n： $a_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$ 而 n=1.2.3……

原來 $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

同乘上 cosmx 得 $f(x)\cos mx=a_0\cos mx+\sum_{n=1}^{\infty}(a_n\cos nx\cos mx+b_n\sin nx\cos mx)$

$$\int_{-\pi}^{\pi}f(x)\cos mxdx=\int_{-\pi}^{\pi}a_0\cos mxdx+\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}(a_n\cos nx\cos mx+b_n\sin nx\cos mx)dx$$

$$=\frac{a_0}{m}\sin mx\Big|_{-\pi}^{\pi}+\sum_{n=1}^{\infty}\left\langle\!\left\langle\frac{a_n}{2}\int_{-\pi}^{\pi}[\cos(nx+mx)+\cos(nx-mx)]+\frac{b_n}{2}\int_{-\pi}^{\pi}[\sin(nx+mx)+\sin(nx-mx)]\right\rangle\!\right\rangle dx$$

$=0+a_n\pi+0=a_n\pi$ ，所以 $a_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$

b_n：$b_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$ 而 n=1.2.3……

b_n 求法同 a_n 求法，同乘上 sinmx ，所以 $b_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$

2.4. Operations on Fourier Series

傳立葉級數可用來解熱傳導方程式和訊號分析。也可解微分方程。

2 Gibbs Phenomenon

Gibbs 是一個超射現象(overshooting)，是傳立葉級數和其他特徵函數級數處在一個不連續處的奇特現象。常見於方波。

2.1 Partial Summation of Fourier Series

The complete Fourier series representation of a signal requires an infinite number of terms in general. When the complete series is used, the series converges to the exact value of the signal at every point where the signal is continuous and converges to the midpoint of the discontinuity wherever the signal is discontinuous.

The signal $x(t)$ can be approximated by using a truncated form of the Fourier series, that is, stopping the summation after a finite number N of terms. The approximation may be 'good' or 'bad' in a subjective sense, but it will be the best approximation for a given N in terms of minimizing the mean squared error between the approximation and the actual signal. The approximation is given by

$$\hat{x}(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + \sum_{n=1}^{N} b_n \sin(n\omega_0 t), \quad N < \infty$$

2.2 Square Wave

Let f be a 2π -periodic square wave function so that

$$f = -1 \qquad -\pi \leq x < 0$$

$$f = 1 \qquad 0 \leq x < \pi$$

$S_{2n-1}(x)$ is the $(2n-1)st$ Fourier polynomial of f. Prove that it can be written as:

$$S_{2n-1}(x) = \frac{1}{n\pi} \int_0^{2nx} \frac{\sin t}{\sin \frac{t}{2n}} dt$$

It's obvious that the Fourier-Series can be written as:

$$F_N(x) = \frac{4}{\pi} \sum_{n=1}^{N} \frac{\sin((2n-1)x)}{2n-1}$$

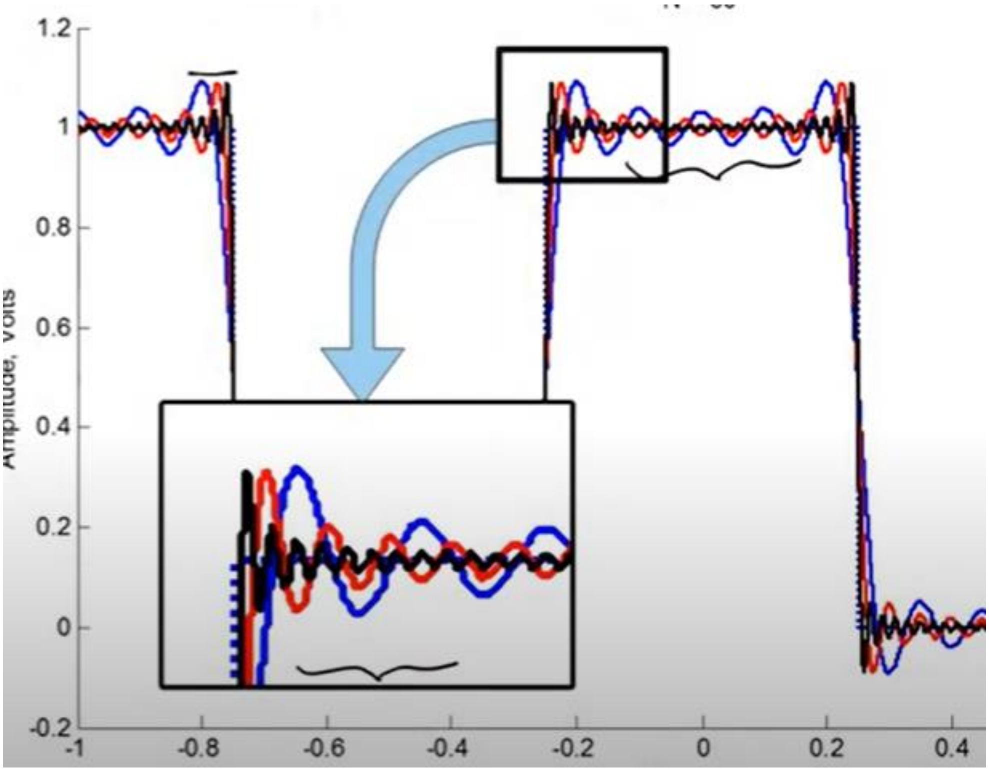
You just have to notice that:

$$\frac{\sin((2n-1)x)}{(2n-1)} = \int_0^x \cos((2n-1)u)\, du$$

hence:

$$F_N(x) = \frac{4}{\pi} \int_0^x \sum_{n=1}^{N} \cos((2n-1)u)\, du = \frac{2}{\pi} \int_0^x \frac{\sin(2Nu)}{\sin u}\, du = \frac{1}{\pi N} \int_0^{2Nx} \frac{\sin t}{\sin \frac{t}{2N}}\, dt.$$

2.3 Calculation of overshoot



Basically, the steps for the canonical example $f(x) = \text{sign}(x)$ on $-\pi < x < \pi$ are:

1. Compute the indicated Fourier series and consider $F_N(x)$, the Nth partial Fourier sum of f.
2. We want to find the value of the first positive local max of $F_N(x)$. Do this by standard calculus followed by some trigonometry to sum the resulting cosine series in terms of a single sine function.
3. From there, it's easy to find the critical number of interest; call this x^*.
4. Evaluating $F_N(x^*)$, the resulting sum can be massaged a bit and then recognized as a Riemann sum.
5. The Riemann sum is one corresponding to $\frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx$.
6. Since N is finite, we have $F_N(x^*) \approx \frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx \approx 0.589\ldots$.
7. This last value is the value of that first positive local max and thus the overshoot above the true value at x^* (which is of course 1/2) is about $0.0589 - 0.5 = 0.089$, i.e., about 8.9% of the unit jump here.

3 Parserval Theorem

$$\int_{-\pi}^{\pi} [f(x)]^2\,dx = \frac{1}{4}a_0^2 \int_{-\pi}^{\pi} dx + a_0 \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [a_n \cos(n\,x) + b_n \sin(n\,x)]\,dx +$$

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [a_n\,a_m \cos(n\,x)\cos(m\,x) + a_n\,b_m \cos(n\,x)\sin(m\,x) +$$

$$a_m\,b_n \sin(n\,x)\cos(m\,x) + b_n\,b_m \sin(n\,x)\sin(m\,x)]\,dx$$

$$= \frac{1}{4}a_0^2\,(2\,\pi) + 0 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [a_n\,a_m\,\pi\delta_{n\,m} + 0 + 0 + b_n\,b_m\,\pi\delta_{n\,m}],$$

4 SUMMARY

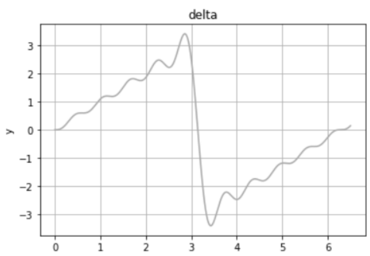
傅立葉是一個很大的學問，以前學習時都只專注於解決微分方程，希望可以更了解相關定理的推導過程，如果有幾何圖形的輔助，或許會更好。

```
import matplotlib.pyplot as plt
import numpy as np
def sigma(first, last, x):
        sum = 2*(np.sin(x)-np.sin(2*x)/2+np.sin(3*x)/3-np.sin(4*x)/4+np.sin(5*x)/5-np.sin(6*x)/6+np.sin(7*x)/7-np.sin(8*x)/8+np.sin(9*x)/9-np.sin(10*x)/10)
        return sum

x = np.arange(0.0, 6.5, 0.001)#x軸上的點，0到1之間以0.001為間隔

plt.plot(x, sigma(0, 6.5, x), color = '.66')

plt.xlabel('x')#x軸標籤
plt.ylabel('y')#y軸標籤
plt.title('delta')#圖的標籤
plt.grid(True)#產生網格
plt.show()#顯示影象
```



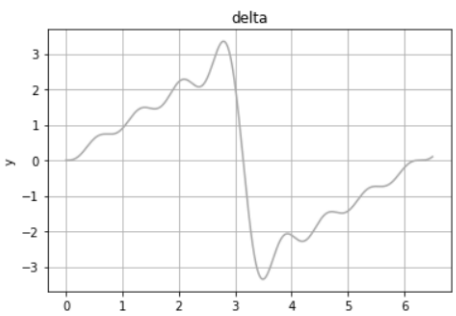
(N=10)

```
import matplotlib.pyplot as plt
import numpy as np
def sigma(first, last, x):
        sum = 2*(np.sin(x)-np.sin(2*x)/2+np.sin(3*x)/3-np.sin(4*x)/4+np.sin(5*x)/5-np.sin(6*x)/6+np.sin(7*x)/7-np.sin(8*x)/8)
        return sum

x = np.arange(0.0, 6.5, 0.001)#x軸上的點，0到1之間以0.001為間隔

plt.plot(x, sigma(0, 6.5, x), color = '.66')

plt.xlabel('x')#x軸標籤
plt.ylabel('y')#y軸標籤
plt.title('delta')#圖的標籤
plt.grid(True)#產生網格
plt.show()#顯示影象
```



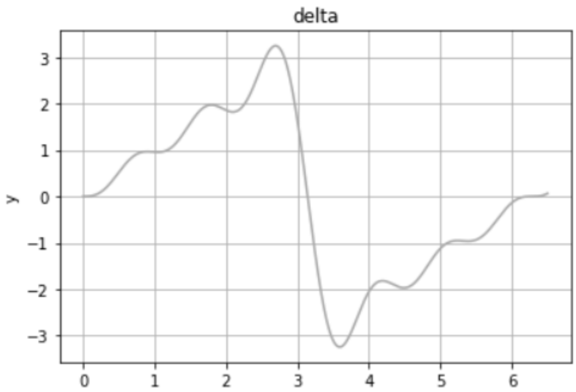
(n=8)

```
import matplotlib.pyplot as plt
import numpy as np
def sigma(first, last, x):
        sum = 2*(np.sin(x)-np.sin(2*x)/2+np.sin(3*x)/3-np.sin(4*x)/4+np.sin(5*x)/5-np.sin(6*x)/6)
        return sum

x = np.arange(0.0, 6.5, 0.001)#x軸上的點，0到1之間以0.001為間隔

plt.plot(x, sigma(0, 6.5, x), color = '.66')

plt.xlabel('x')#x軸標籤
plt.ylabel('y')#y軸標籤
plt.title('delta')#圖的標籤
plt.grid(True)#產生網格
plt.show()#顯示影象
```



N=6

Problem 1. 使用程式畫出 Fig 19.2