G02_HW08

Group 02
HW 08
2019/11/5

ID	Name	Your works	Times you spend	Self score	TA
108202529	葉揚昀	The feature of coupled oscillation Equation of motion of coupled oscillation	10hr	9	
108202009	田家瑋	meaning phase space Arduino	4hr	3	
108202016	張家菖	Draw the sketch of coupled oscillator model Arduino	6hr	7	

Coupled oscillation

1. Definition:

Coupled oscillators are oscillators connected in such a way that energy can be transferred between them. There're two typical example for coupled oscillation, one is two pendulums connected a spring, another one is two masses connected three springs.

2. Feature:

(1) Normal mode:

A normal mode of an oscillating system is the motion in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation.

In following case, we will discuss two pendulums connected a spring (coupled pendulum) Fig.1 in the normal mode.

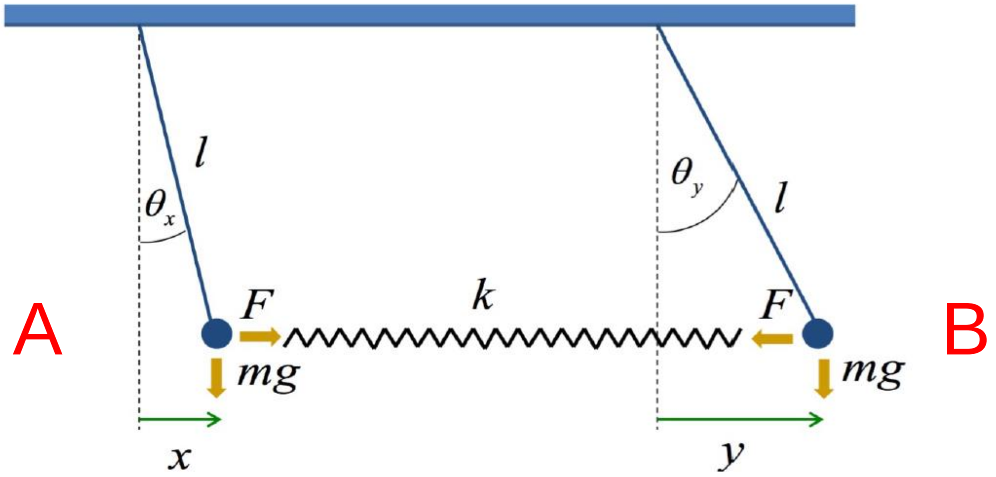


Fig.1 The coupled pendulum
source:
https://www2.physics.ox.ac.uk/sites/default/files/2012-09-04/normalmodes_iandii_pdf_96820.pdf

Coupled oscillation

(2) Express x & y in Fig.1

Assume that the displacements form the equilibrium positions are small enough that the restoring force due to gravity is approximately given by $mg\tan\theta$ and acts along the line of masses.

1. If the coupled pendulum hasn't spring connected:

$$\begin{cases} \text{A: } m\frac{d^2x}{dt^2} = m\ddot{x} = -mg\sin\theta_x = -mg\frac{x}{l} \\ \text{B: } m\frac{d^2y}{dt^2} = m\ddot{y} = -mg\sin\theta_y = -mg\frac{y}{l} \end{cases} \qquad (1)$$

2. Then, we consider that add a spring into the coupled pendulum (modify equation (1)):

$$\begin{cases} \text{A: } m\ddot{x} = -mg\frac{x}{l} - kx + ky = -mg\frac{x}{l} + \mathrm{k(y-x)} \\ \text{B: } m\ddot{y} = -mg\frac{y}{l} - ky + kx = -mg\frac{y}{l} - \mathrm{k(y-x)} \end{cases} \qquad (2)$$

Now, we have two unknown variable x & y.

Coupled oscillation

3. Equation of motion of coupled oscillation:

(1) Normal mode – the first mode (normal coordinate) of vibration:

1. We try to combine $\ddot{x}$ & $\ddot{y}$ by add them (from equation (2)):

$$m(\ddot{x}+\ddot{y}) = -mg\frac{x}{l} + \cancel{\mathrm{k}(\mathrm{y}-\mathrm{x})} - mg\frac{y}{l} - \cancel{\mathrm{k}(\mathrm{y}-\mathrm{x})} = -\frac{mg}{l}(\mathrm{x}+\mathrm{y}) \tag{3}$$

2. Define q_1:

$$q_1 = x + y \rightarrow \ddot{q}_1 = \ddot{x} + \ddot{y} \tag{4}$$

3. Use equation(4) in equation(3):

$$m\ddot{q}_1 = -\frac{mg}{l}q_1 \rightarrow \ddot{q}_1 = -\frac{g}{l}q_1 \tag{5}$$

4. Then we get a form of SHM by equation (5):

$$\ddot{q}_1 + \frac{g}{l}q_1 = 0 \rightarrow \omega_1 = \sqrt{\frac{g}{l}} \rightarrow q_1 = A_1\cos(\omega_1 t + \varphi_1) \tag{6}$$

Coupled oscillation

5. In equation (6) A_1 & φ_1 are arbitrary constants set by the initial or boundary conditions. $q_1 = x + y$ tells us about the coupled motion of the two pendulum in terms of how they oscillate together around a center of mass (shown in Fig.2).

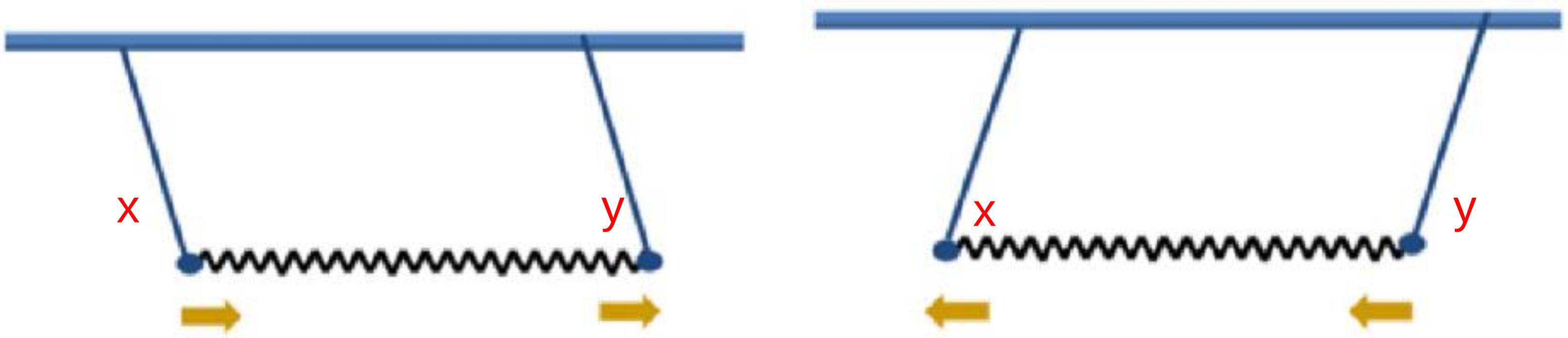


Fig.2 The center of mass motion of the coupled pendulum as described by $q_1 = x + y$
source:
https://www2.physics.ox.ac.uk/sites/default/files/2012-09-04/normalmodes_iandii_pdf_96820.pdf

Coupled oscillation

(2) Normal mode – the second mode (normal coordinate) of vibration:

1. We try to combine $\ddot{x}$ & $\ddot{y}$ by subtract them (from equation (2)):

$$m(\ddot{x}-\ddot{y}) = -mg\frac{x}{l} + \mathrm{k(y-x)} + mg\frac{y}{l} + \mathrm{k(y-x)} = -m(\frac{g}{l}+\frac{2k}{m})(\mathrm{x-y}) \tag{7}$$

2. Define q_2:

$$q_2 = x - y \rightarrow \ddot{q}_2 = \ddot{x} - \ddot{y} \tag{8}$$

3. Use equation(8) in equation(7):

$$m\ddot{q}_2 = -m(\frac{g}{l}+\frac{2k}{m})q_2 \rightarrow \ddot{q}_2 = -(\frac{g}{l}+\frac{2k}{m})q_2 \tag{9}$$

4. Then we get a form of SHM by equation (9):

$$\ddot{q}_1 + (\frac{g}{l}+\frac{2k}{m})q_1 = 0 \rightarrow \omega_2 = \sqrt{\frac{g}{l}+\frac{2k}{m}} \rightarrow q_2 = A_2\cos(\omega_2 t + \varphi_2) \tag{10}$$

Coupled oscillation

5. In equation (10) A_2 & φ_2 are arbitrary constants set by the initial or boundary conditions. $q_2 = x - y$ tells us about the relative motion of the coupled pendulum (shown in Fig.3).

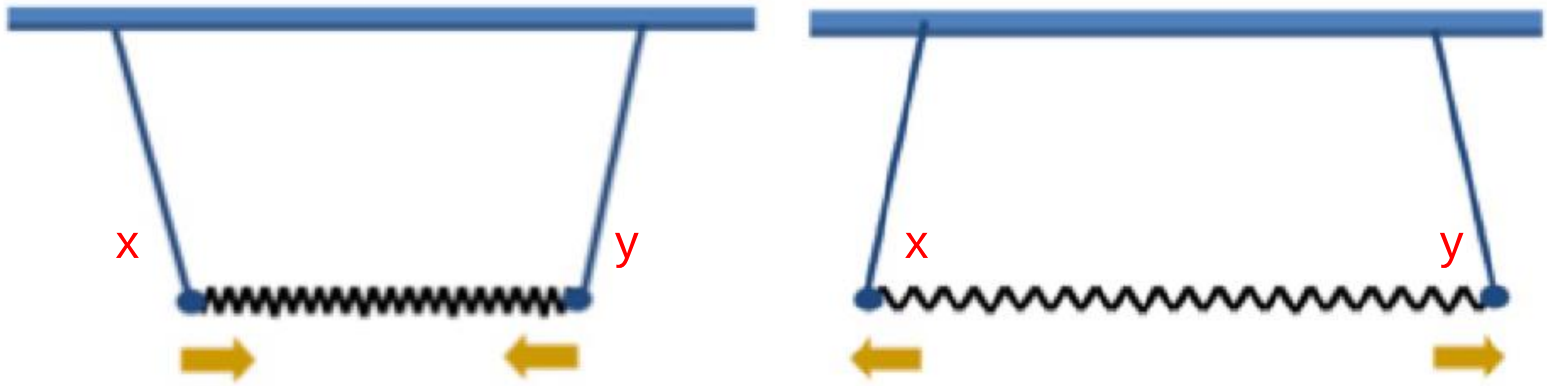


Fig.3 The relative motion of the coupled pendulum as described by $q_2 = x - y$
source:
https://www2.physics.ox.ac.uk/sites/default/files/2012-09-04/normalmodes_iandii_pdf_96820.pdf

Coupled oscillation

(3) Solve functions of x(t) & y(t)

1. Use the results of equation (6) and equation (10). we get:

$$q_1 + q_2 = (x + y) + (x - y) = 2x \rightarrow x = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) \qquad (11)$$

$$q_1 - q_2 = (x + y) - (x - y) = 2y \rightarrow y = A_1 \cos(\omega_1 t + \varphi_1) - A_2 \cos(\omega_2 t + \varphi_2) \qquad (12)$$

2. The variables q_1 and q_2 are the modes or normal coordinates of the system. In any normal mode, only one of these coordinates is active at any one time.

Coupled oscillation

(4) Normal mode_1 x(t) & y(t) (A_1 = 0.1, A_2 = 0, 5 periods), shown in Fig.4

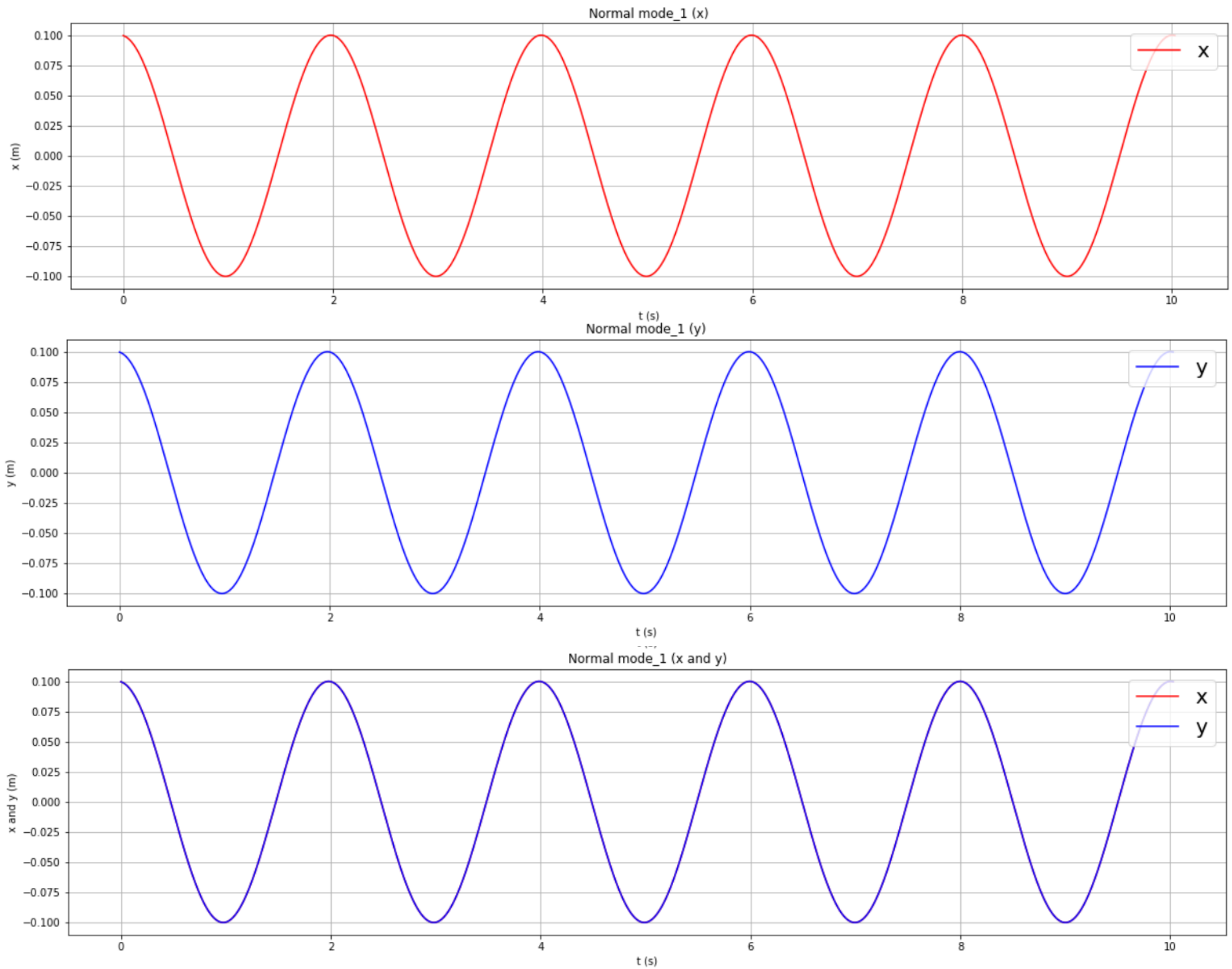


Fig.4
1^{st} mode

Coupled oscillation

(5) Normal mode_2 x(t) & y(t) (A_1 = 0, A_2 = 0.1, 5 periods), shown in Fig.5

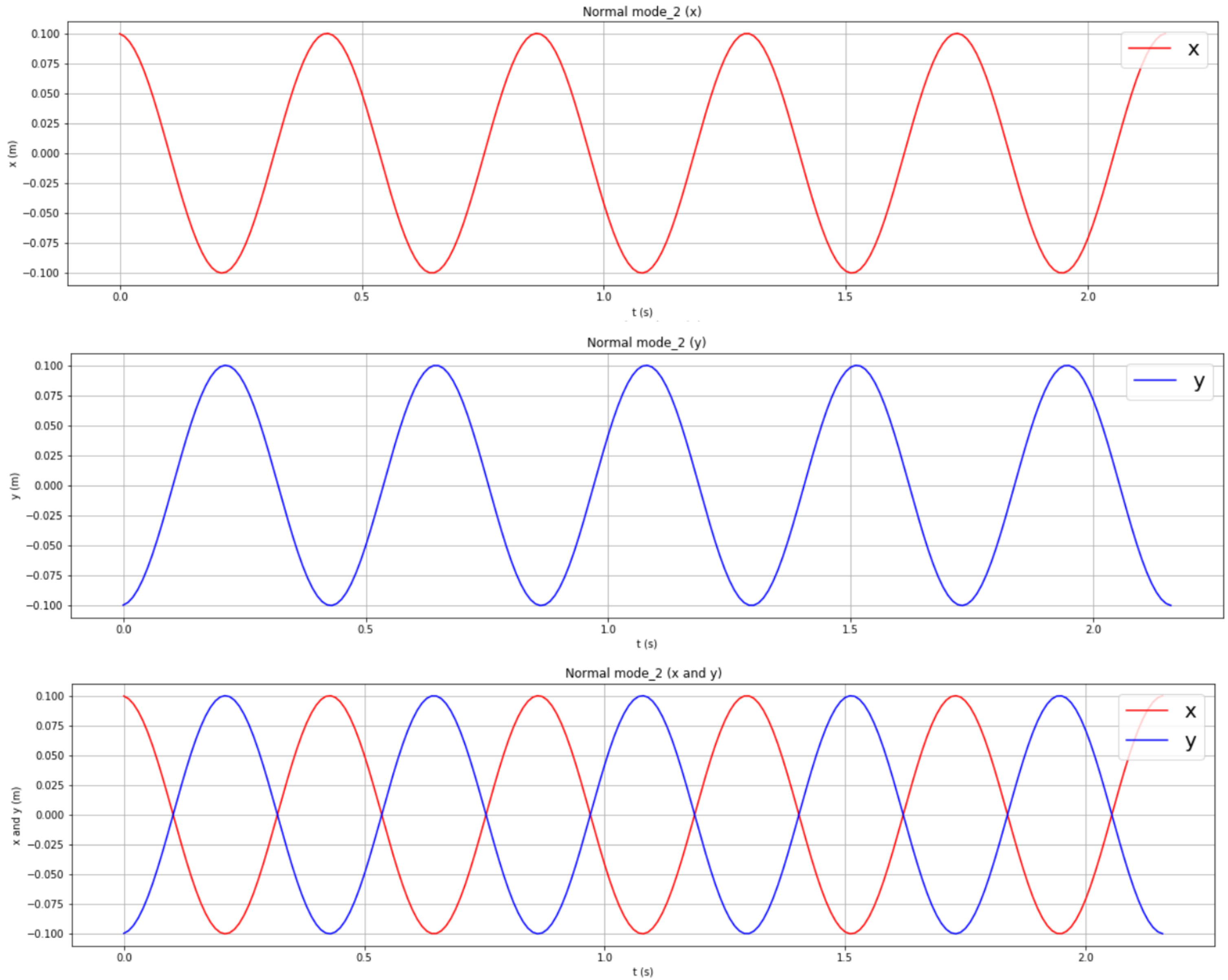


Fig.5
2^{cd} mode

Phase space

Phase space is describing all possible motions of particle . Each possible motion usually use 6 variable $(x , y , z , p_x , p_y , p_z)$ to point on a 6 dimensional space . Phase diagram is a coordinate that involve $\mathrm{r} - p_r$ plot coordinate. And of course we can draw $x - p_x$, $y - p_y$, $z - p_z$ plots to visualize phase space .

Once we can draw a curve on the phase diagram , it means the status when the particle travel around the position

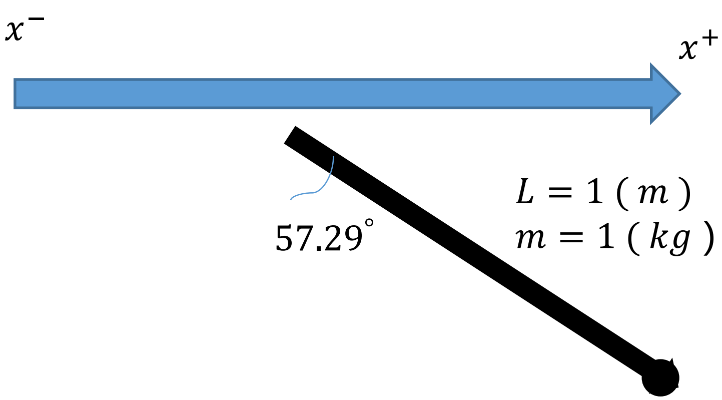


Fig 6: pendulum

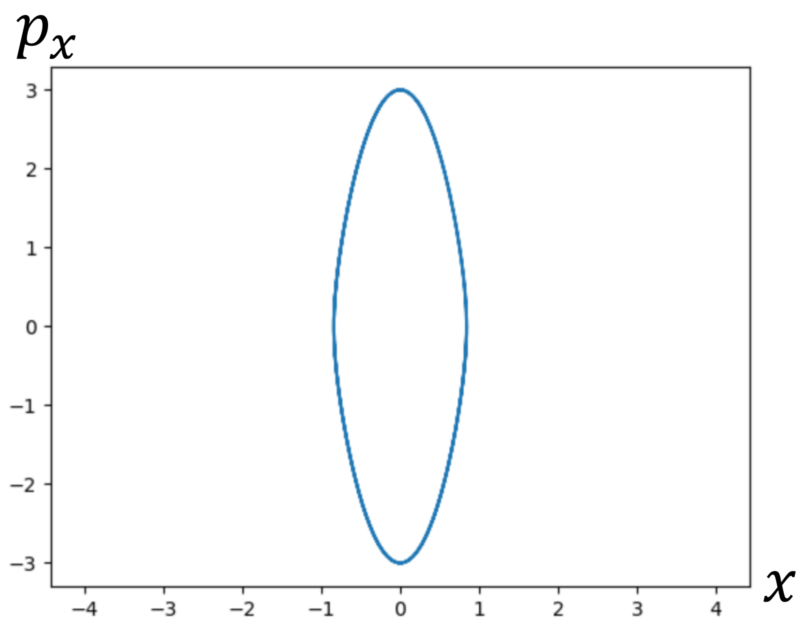


Fig 7: the phase diagram $x - p_x$ of the pendulum (Fig1) of x direction projection

Phase space

Phase space or phase diagram can be used to predict the trajectory of the object in some special case. I take one example , damped pendulum . We can easily draw a phase diagram (Fig 3) , furthermore , In the damped pendulum (we use $\dot{\theta} - \theta$ plot) : $\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \gamma\dot{\theta}$, we can draw vector field of $\left(\frac{d\theta}{dt}, \frac{d\dot{\theta}}{dx}\right)$ which is $(\dot{\theta}, \ddot{\theta})$ to se the movement of the pendulum in another view.

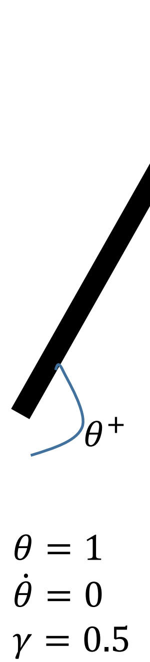


Fig 8: Initial condition of the pendulum

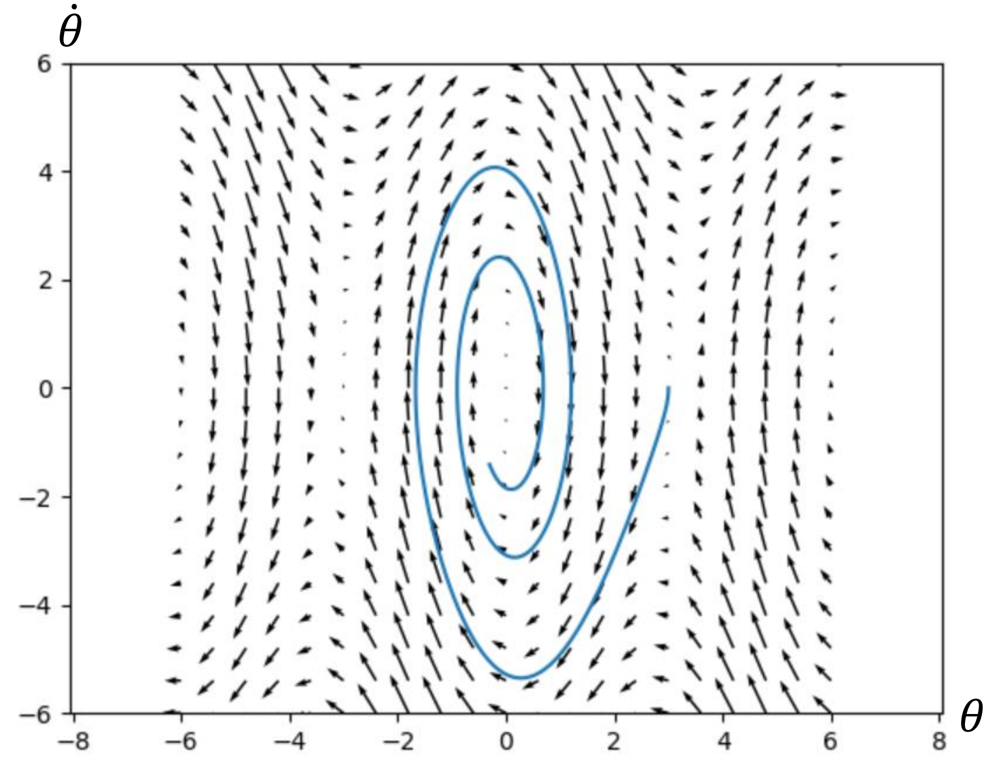


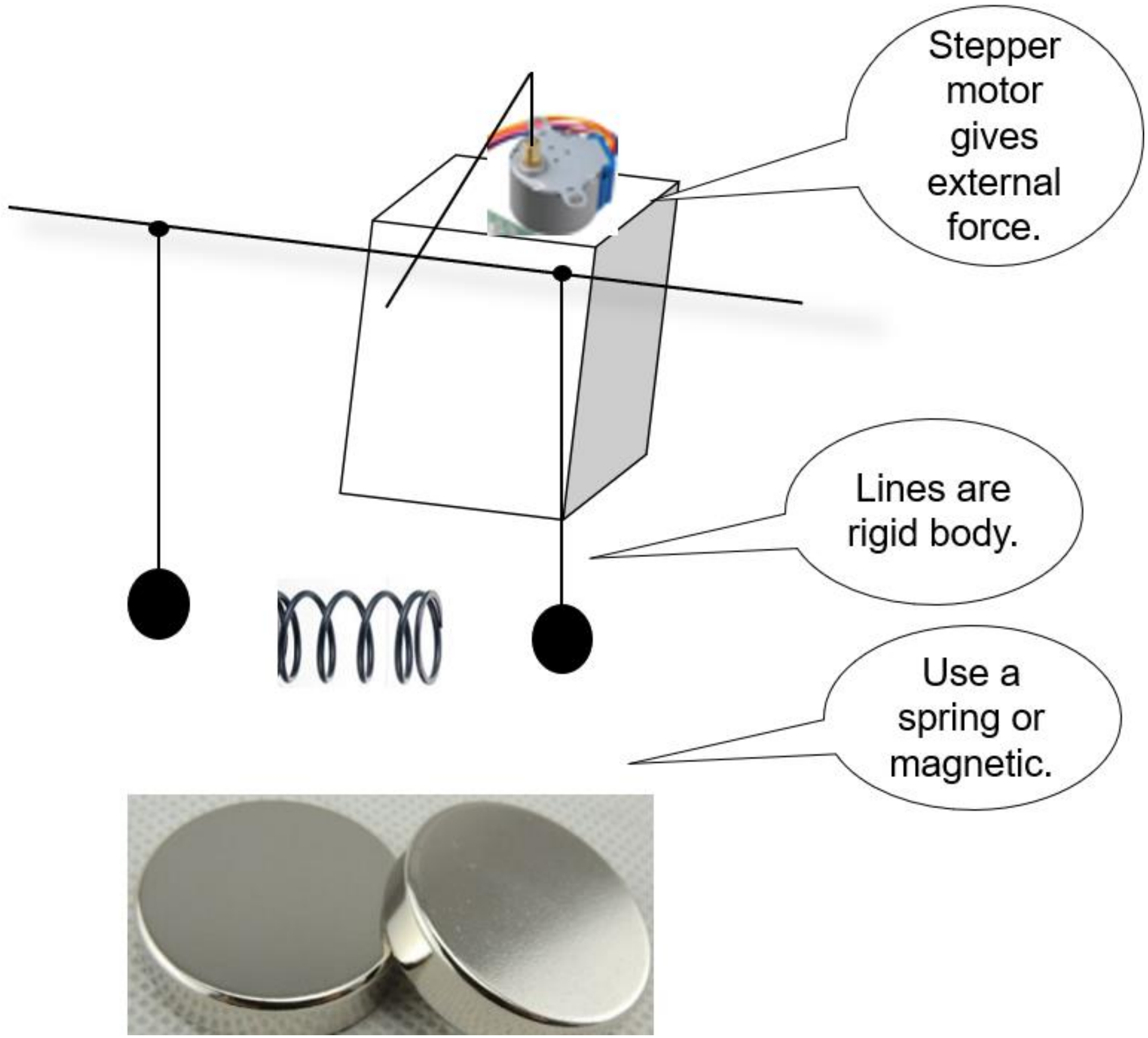
Fig 9: phase space with vector field of $(\dot{\theta}, \ddot{\theta})$

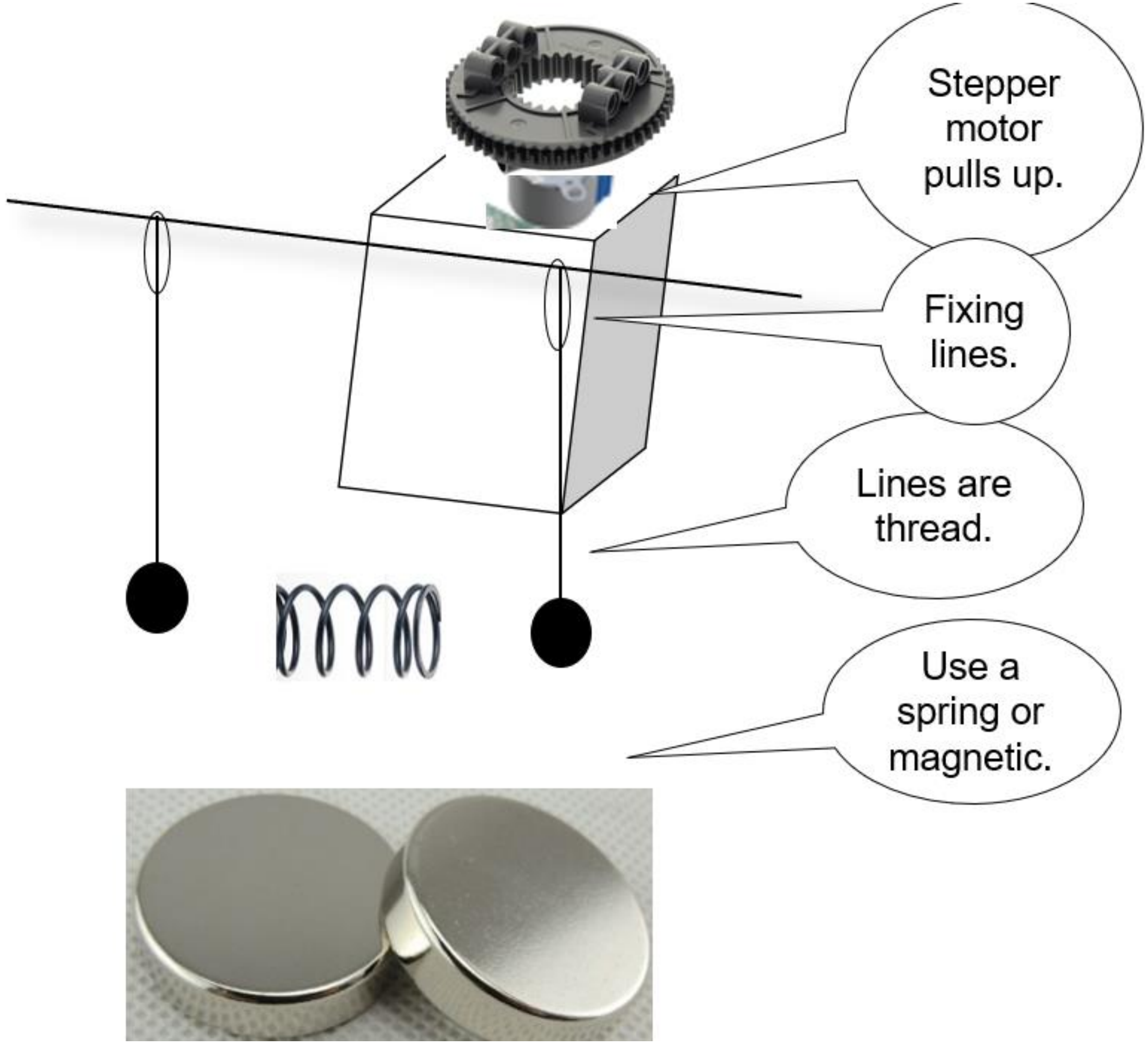
Phase space

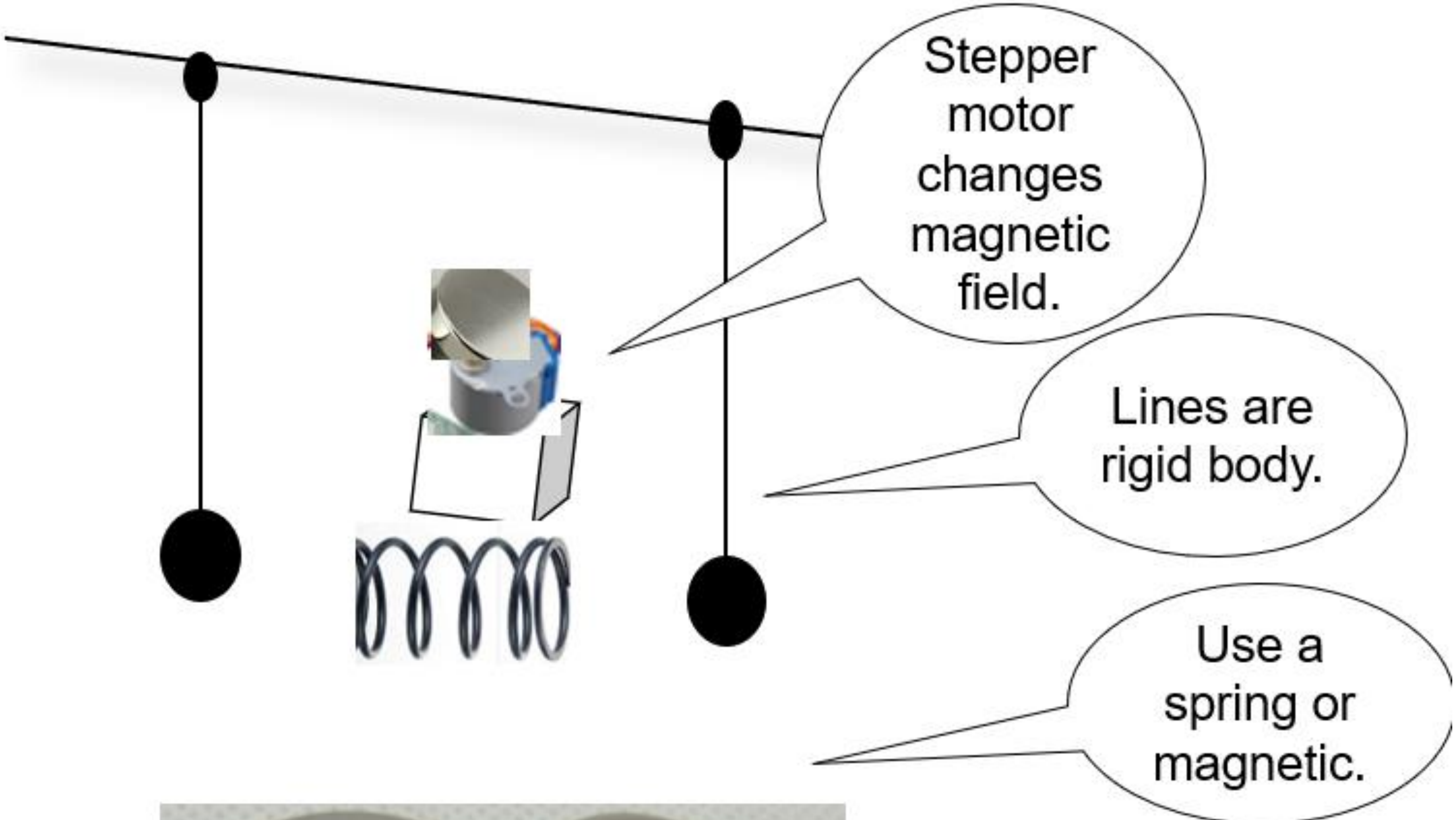
The example shows us another thing --- the slope of the trajectory tells us the acceleration of that moment .

$$slope = \frac{d\dot{\theta}}{d\theta} = \frac{\frac{d\dot{\theta}}{dt}}{\frac{d\theta}{dt}} = \frac{\ddot{\theta}}{\dot{\theta}}$$

And we can get the acceleration by $slope * \dot{\theta}$.


Stepper motor gives external force.
Lines are rigid body.
Use a spring or magnetic.


Stepper motor pulls up.
Fixing lines.
Lines are thread.
Use a spring or magnetic.


Stepper motor changes magnetic field.
Lines are rigid body.
Use a spring or magnetic.

Arduino coding

```
void call(int x){
  if( x == 9){   //創9
  digitalWrite(2, 1);
  digitalWrite(3, 1);
  digitalWrite(4, 1);
  digitalWrite(5, 0);
  digitalWrite(6, 0);
  digitalWrite(7, 1);
  digitalWrite(8, 1);
   }
void setup() {
  pinMode(2, OUTPUT);
  pinMode(3, OUTPUT);
  pinMode(4, OUTPUT);
  pinMode(5, OUTPUT);
  pinMode(6, OUTPUT);
  pinMode(7, OUTPUT);
  pinMode(8, OUTPUT);
  pinMode(9, OUTPUT);
  digitalWrite(9, 0);  // 關閉小數點
```

Arduino coding

```
void loop() {
  int a[9]={1,0,8,2,0,2,0,0,9};   //第一個學號
  int b[9]={1,0,8,2,0,2,0,1,6};   //第二個學號
  int c[9]={1,0,8,2,0,2,5,2,9};   //第三個學號
for(int t=0;t<9;t++){
  call( a[t] );
  delay (500);
  }
  delay (2000);       //中間間隔
for(int t=0;t<9;t++){
  call( b[t] );
  delay (500);
  }
  delay (2000);      //中間間隔
for(int t=0;t<9;t++){
  call( c[t] );
  delay (500);
  }
  delay (2000);      //中間間隔
}
```

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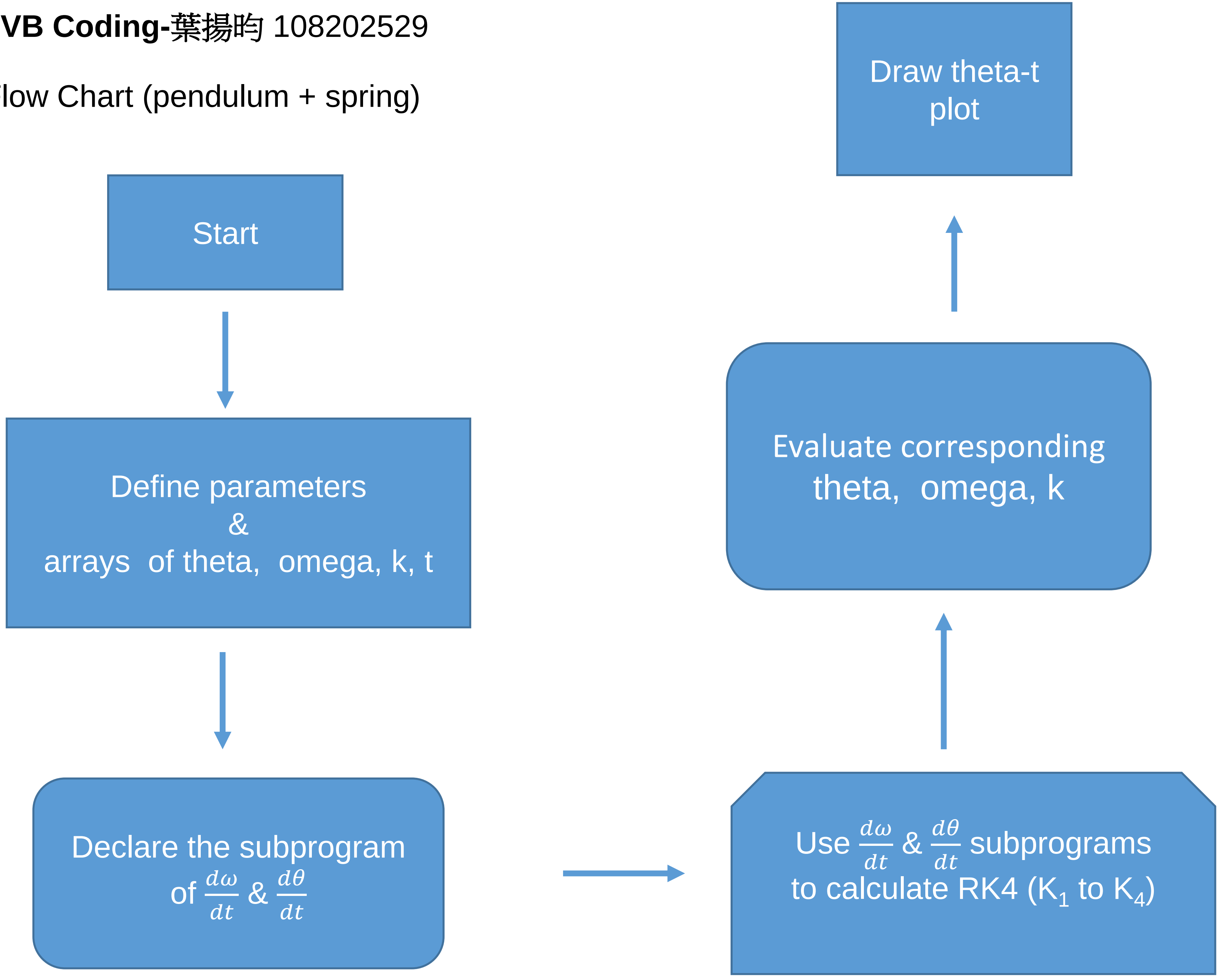
Flow Chart (pendulum + spring)



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Define parameters & arrays of theta, omega, k, t

```
 Dim L As Single = CSng(L_text.Text)
        Dim dt As Single = CSng(time_step_text.Text)
        Dim t(100000.0), theta(100000.0), omega(100000.0) As Single
        Dim k1_theta(100000.0), k2_theta(100000.0), k3_theta(100000.0), k4_theta(100000.0)
As Single
        Dim k1_omega(100000.0), k2_omega(100000.0), k3_omega(100000.0), k4_omega(100000.0)
As Single
        t(0) = 0
        theta(0) = CSng(initial_theta_text.Text)
        omega(0) = CSng(initial_omega_text.Text)
```

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Declare the subprogram of $\frac{d\omega}{dt}$ & $\frac{d\theta}{dt}$

```
 Private Sub Cal_omega_dot(ByVal Cal_omega As Single, ByVal Cal_theta As Single, ByRef Cal_omega_dot As Single)

        Dim g As Single = 9.8

        Dim L As Single = CSng(L_TX.Text)

        Dim c As Single = CSng(c_TX.Text)

        Dim m As Single = CSng(m_TX.Text)

        Dim k As Single = CSng(k_TX.Text)

        Dim Fk As Single = k * L * (1 - Sqrt(((1 - Cos(Cal_theta)) ^ 2) + ((1 - Sin(Cal_theta)) ^ 2)))

        Dim theta_plus_phi As Single = Cal_theta + Atan2((1 - Cos(Cal_theta)), (1 - Sin(Cal_theta)))

        Cal_omega_dot = -g / L * Sin(Cal_theta) - c * Cal_omega - Cos(theta_plus_phi) * Fk / (m * L)

    End Sub


    Private Sub Cal_theta_dot(ByVal Cal_omega As Single, ByRef Cal_theta_dot As Single)

        Cal_theta_dot = Cal_omega

    End Sub
```

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Use $\frac{d\omega}{dt}$ & $\frac{d\theta}{dt}$ subprograms to calculate RK4 (K_1 to K_4)

```
For i As Integer = 1 To 50000 Step 1
    Cal_omega_dot(omega(i - 1), theta(i - 1), k1_omega(i - 1))
    Cal_theta_dot(omega(i - 1), k1_theta(i - 1))
    Cal_omega_dot(omega(i - 1) + dt / 2 * k1_omega(i - 1), theta(i - 1) + dt / 2 * k1_theta(i - 1), k2_omega(i - 1))
    Cal_theta_dot(omega(i - 1) + dt / 2 * k1_omega(i - 1), k2_theta(i - 1))
    Cal_omega_dot(omega(i - 1) + dt / 2 * k2_omega(i - 1), theta(i - 1) + dt / 2 * k2_theta(i - 1), k3_omega(i - 1))
    Cal_theta_dot(omega(i - 1) + dt / 2 * k2_omega(i - 1), k3_theta(i - 1))
    Cal_omega_dot(omega(i - 1) + dt * k3_omega(i - 1), theta(i - 1) + dt * k3_theta(i - 1), k4_omega(i - 1))
    Cal_theta_dot(omega(i - 1) + dt * k3_omega(i - 1), k4_theta(i - 1))
```

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Evaluate corresponding theta, omega, k & draw theta-t plot

```
            omega(i) = omega(i - 1) + 1 / 6 * dt * (k1_omega(i - 1) + 2 * k2_omega(i - 1) + 2 * k3_omega(i - 1) + k4_omega(i - 1))
            theta(i) = theta(i - 1) + 1 / 6 * dt * (k1_theta(i - 1) + 2 * k2_theta(i - 1) + 2 * k3_theta(i - 1) + k4_theta(i - 1))
            t(i) = t(i - 1) + dt


            a.DrawLine(p, 10 * t(i - 1) + 30, -50 * theta(i - 1) + ph_half, 10 * t(i) + 30, -50 * theta(i) + ph_half)
            If 10 * t(i) > (pw - 50) Then Exit For
        Next
    End Sub
```

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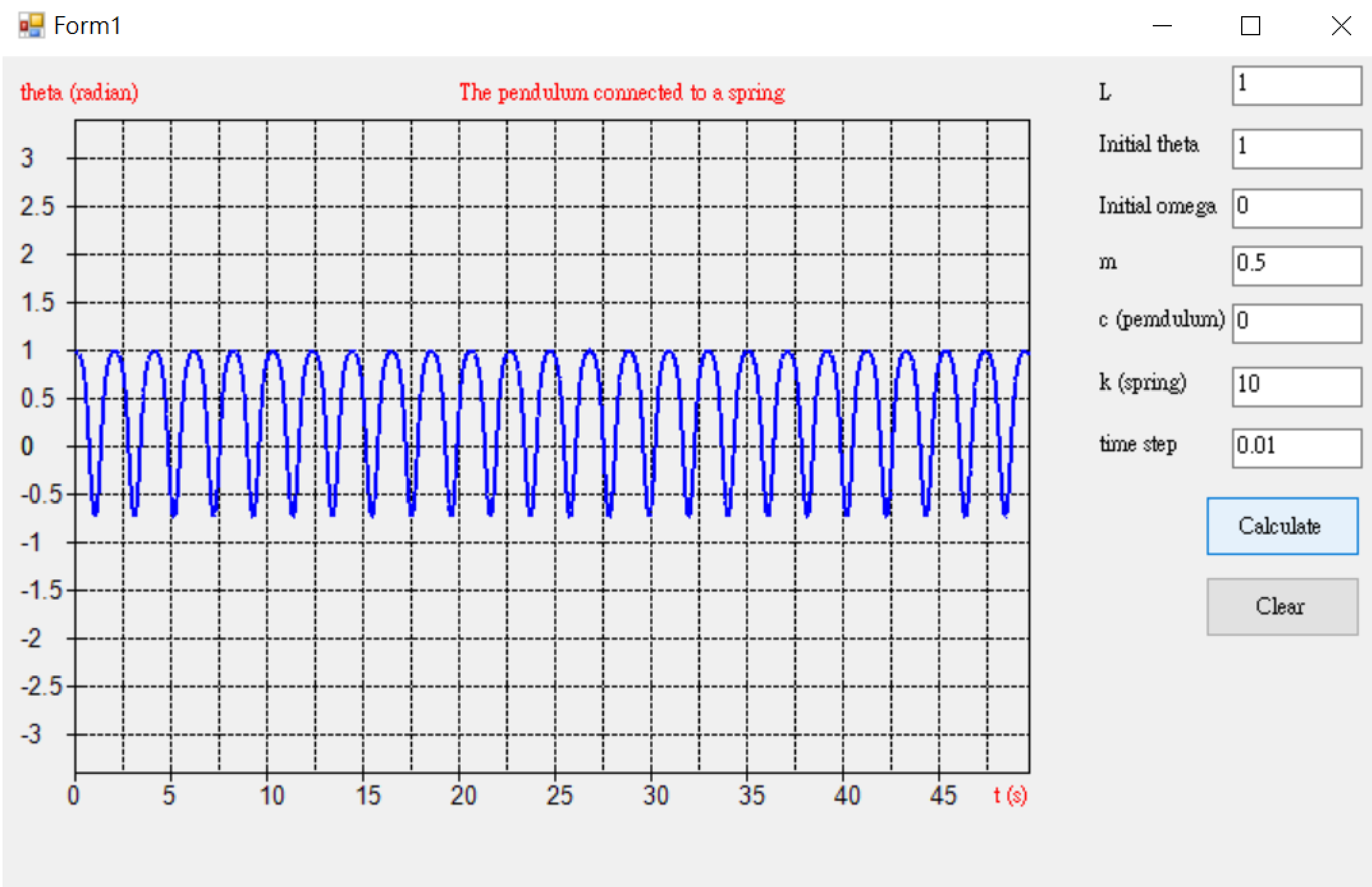
Result (connected a spring k=10)

(L=1(m) , $\theta_0 = 1(radian)$, $\omega_0 = 0(1/s)$, dt = 0.01(s), c=0, k=10, m=0.5(kg))



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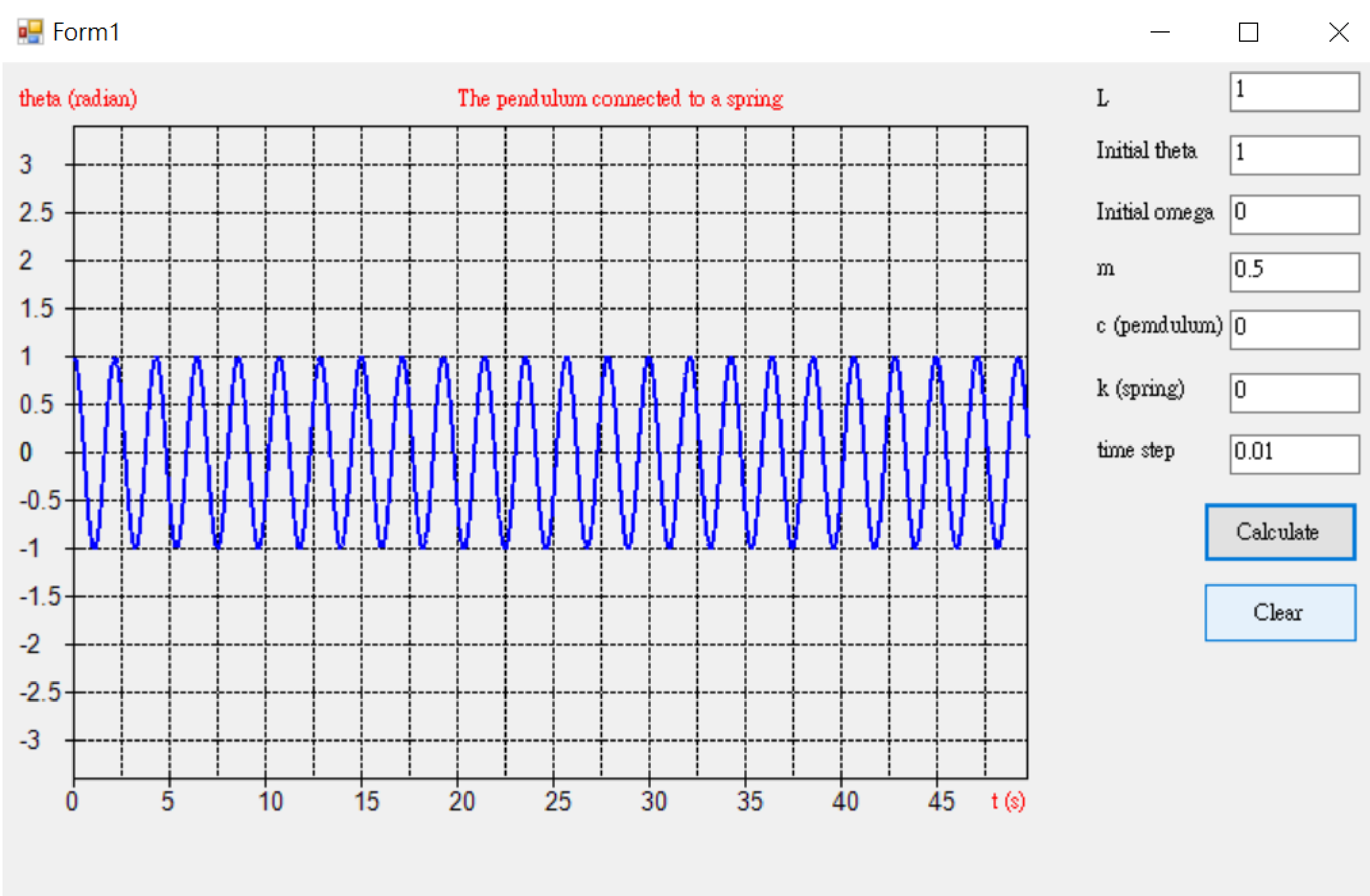
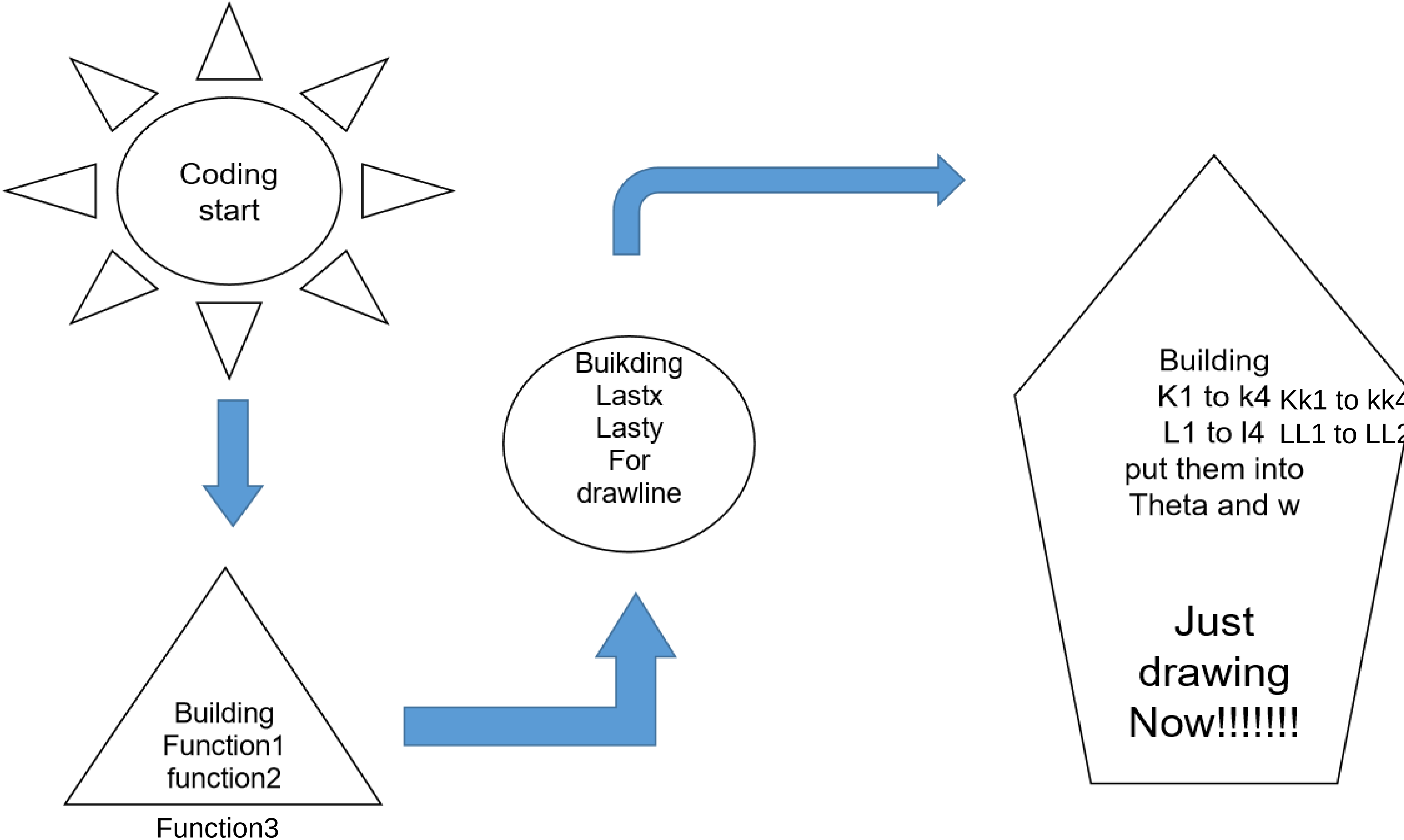
Result (without spring)

(L=1(m) , $\theta_0 = 1(radian)$, $\omega_0 = 0(1/s)$, dt = 0.01(s), c=0, k=0, m=0.5(kg))



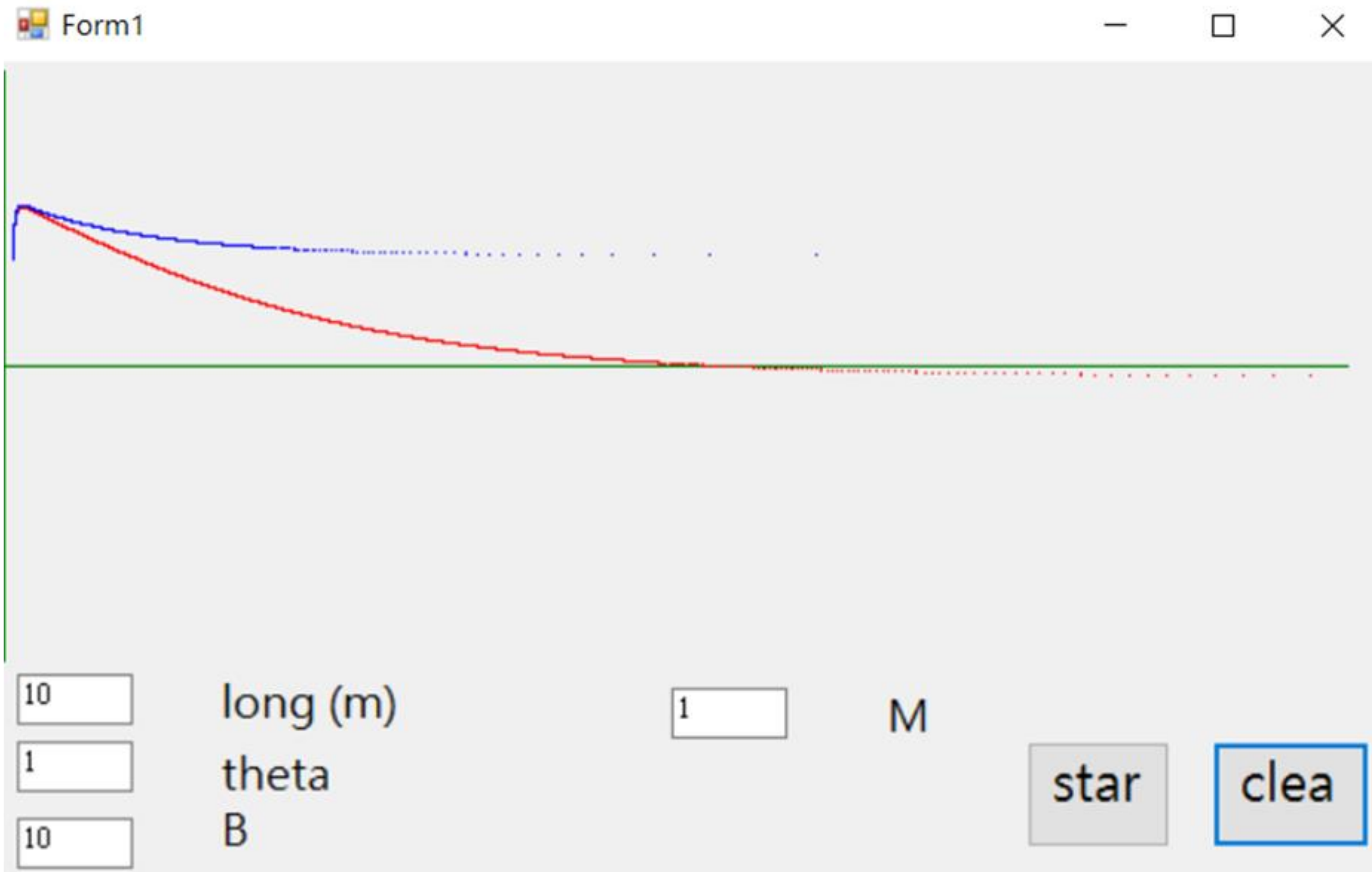

Coding start
Buikding
Lastx
Lasty
For
drawline
Building
K1 to k4 Kk1 to kk4
L1 to l4 LL1 to LL2
put them into
Theta and w
Just
drawing
Now!!!!!!!
Building
Function1
function2
Function3

```
Public Function f1(ByRef theta1 As Decimal, ByRef w1 As Decimal, ByRef time1 As Decimal)
        Dim l As Single = Val(TextBox1.Text) '長度
        Dim b As Single = Val(TextBox3.Text) '阻尼
        Dim theta As Single = Val(TextBox2.Text) '角度(弧度)
        Return w1
    End Function
```

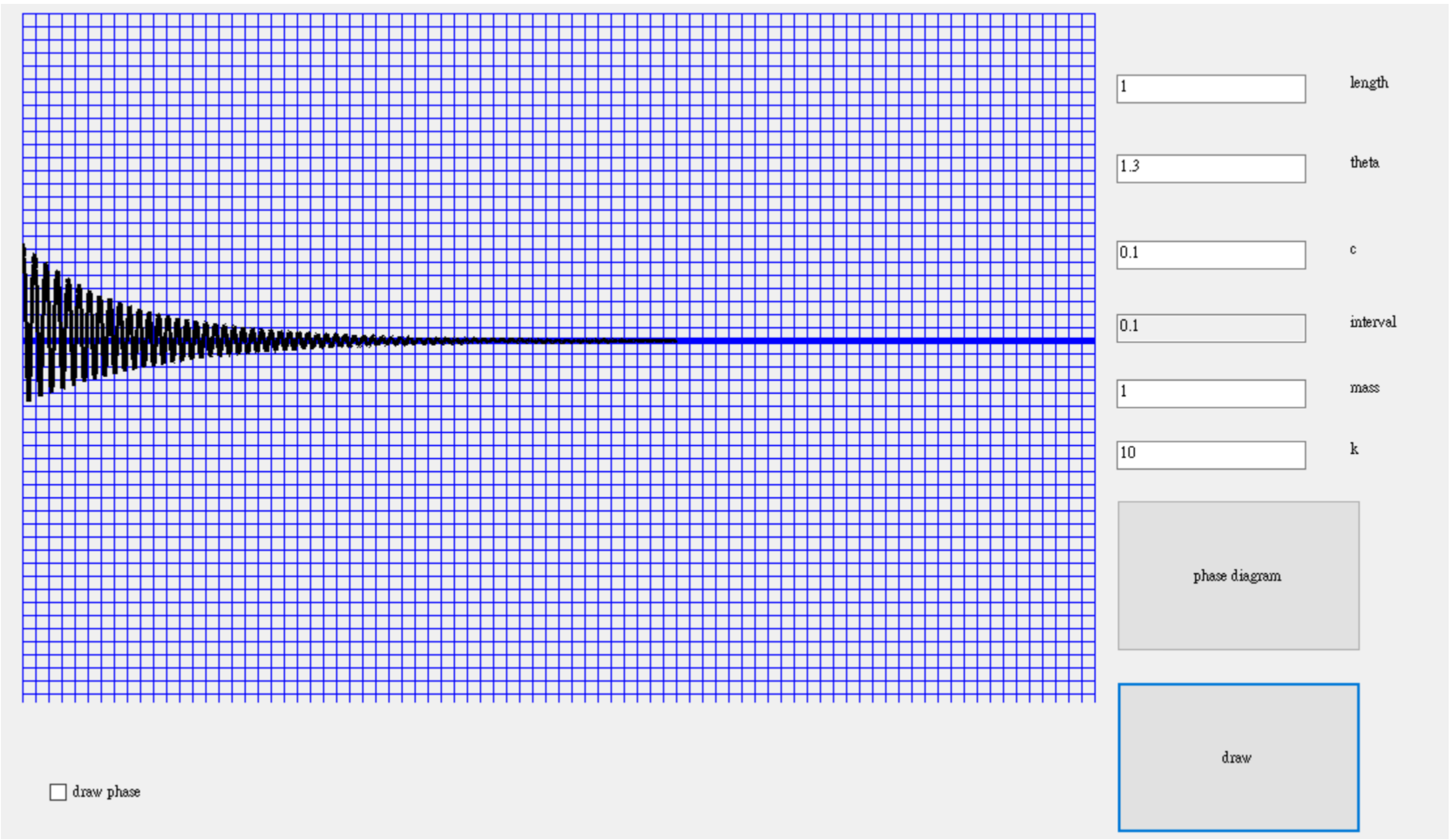
```
Public Function f2(ByRef theta2 As Decimal, ByRef w2 As Decimal, ByRef time2 As Decimal)
        Dim l As Single = Val(TextBox1.Text) '長度
        Dim b As Single = Val(TextBox3.Text) '阻尼
        Dim theta As Single = Val(TextBox2.Text) '角度(弧度)
        Return -9.8 / l * Sin(theta2) - b * w2 - 1 * (l * (1 - Sqrt(3 - 2 * Cos(theta2) * 
Sin(theta2))) / Cos(theta2) * (1 - Sin(theta2)) / Sqrt(3 - 2 * Cos(theta2) * Sin(theta2)))
    End Function
```

```
Public Function f3(ByRef theta2 As Decimal, ByRef w2 As Decimal, ByRef time2 As Decimal)
        Dim l As Single = Val(TextBox1.Text) '長度
        Dim b As Single = Val(TextBox3.Text) '阻尼
        Dim theta As Single = Val(TextBox2.Text) '角度(弧度)
        Return -9.8 / l * Sin(theta2) - b * w2
    End Function
```

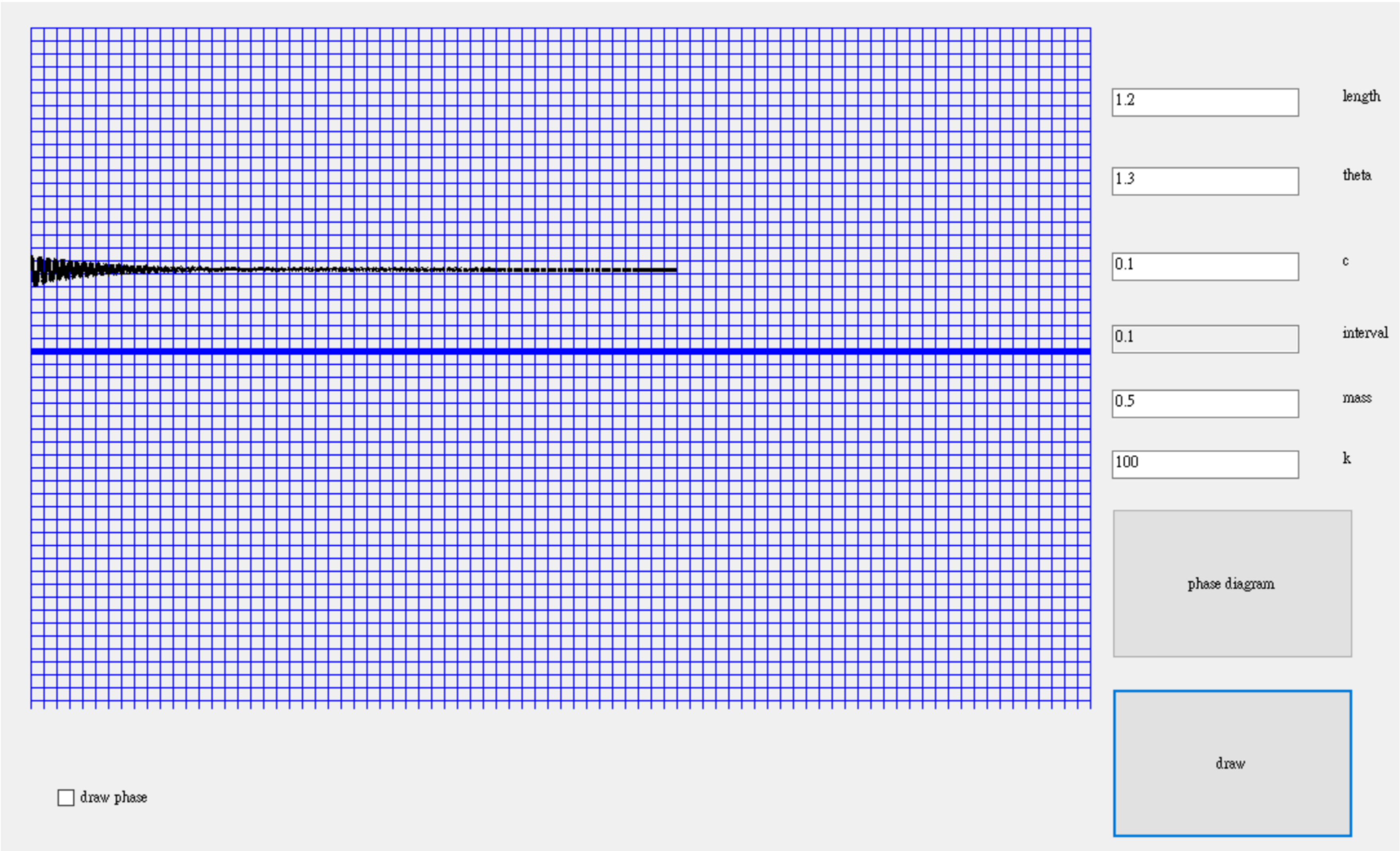
```
For t = 1 To 10000 Step 1
            lasty = thetaa
            lastx = t
            kk1 = f1(thetaa, ww, t) * h
            ll1 = f3(thetaa, ww, t) * h
            kk2 = f1(thetaa + 0.5 * kk1, ww + 0.5 * ll1, t + 0.5) * h
            ll2 = f3(thetaa + 0.5 * kk1, ww + 0.5 * ll1, t + 0.5) * h
            kk3 = f1(thetaa + 0.5 * kk2, ww + 0.5 * ll2, t + 0.5) * h
            ll3 = f3(thetaa + 0.5 * kk2, ww + 0.5 * ll2, t + 0.5) * h
            kk4 = f1(thetaa + kk3, ww + ll3, t + 1) * h
            ll4 = f3(thetaa + kk3, ww + ll3, t + 1) * h
            thetaa = thetaa + (kk1 + 2 * (kk2 + kk3) + kk4) / 6
            ww = ww + (ll1 + 2 * (ll2 + ll3) + ll4) / 6
            g2.DrawLine(pennn, lastx, -50 * lasty, t, -50 * thetaa)
        Next t
        For t = 1 To 10000 Step 1
            lasty = theta
            lastx = t
            k1 = f1(theta, w, t) * h
            l1 = f2(theta, w, t) * h
            k2 = f1(theta + 0.5 * k1, w + 0.5 * l1, t + 0.5) * h
            l2 = f2(theta + 0.5 * k1, w + 0.5 * l1, t + 0.5) * h
            k3 = f1(theta + 0.5 * k2, w + 0.5 * l2, t + 0.5) * h
            l3 = f2(theta + 0.5 * k2, w + 0.5 * l2, t + 0.5) * h
            k4 = f1(theta + k3, w + l3, t + 1) * h
            l4 = f2(theta + k3, w + l3, t + 1) * h
            theta = theta + (k1 + 2 * (k2 + k3) + k4) / 6
            w = w + (l1 + 2 * (l2 + l3) + l4) / 6
            g2.DrawLine(penn, lastx, -50 * lasty, t, -50 * theta)
        Next t
```


Form1
10
long (m)
1
M
1
theta
star
clea
10
B

VB_practice_田家瑋



VB_practice_田家瑋



VB_practice_田家瑋

- For t As Single = dt To times Step dt
- 'theta_0o is point about to puting into the K1'
- theta_0o = theta_0(CInt(t / dt) - 1)
- theta_1o = theta_1(CInt(t / dt) - 1)
- Doftheta = l * Sqrt(3 - 2 * Sin(theta_0o) - 2 * Cos(theta_0o))
- F0 = (Cos(theta_0o) * l * (1 - Sin(theta_0o)) / Doftheta - Sin(theta_0o) * l * (1 - Sin(theta_0o)) / Doftheta) * k * (Doftheta - l)
- 'theta_0I(0) is K1 , theta1I(0) is L1'
- theta_0I(0) = theta_1o
- theta_1I(0) = -g / l * Sin(theta_0o) - c / (M * l) * theta_1o + F0 / M / l
- 'theta_0o is point about to puting into the K2'
- theta_0o = theta_0(CInt(t / dt) - 1) + dt / 2 * theta_0I(0)
- theta_1o = theta_1(CInt(t / dt) - 1) + dt / 2 * theta_1I(0)
- Doftheta = l * Sqrt(3 - 2 * Sin(theta_0o) - 2 * Cos(theta_0o))

VB_practice_田家瑋

- F0 = (Cos(theta_0o) * l * (1 - Sin(theta_0o)) / Doftheta - Sin(theta_0o) * l * (1 - Sin(theta_0o)) / Doftheta) * k * (Doftheta - l)
- 'theta_0I(1) is K2 , theta1I(1) is L2'
- theta_0I(1) = theta_1o
- theta_1I(1) = -g / l * Sin(theta_0o) + -c / M / l * theta_1o + F0 / M / l
- 'theta_0o is point about to puting into the K3'
- theta_0o = theta_0(CInt(t / dt) - 1) + dt / 2 * theta_0I(1)
- theta_1o = theta_1(CInt(t / dt) - 1) + dt / 2 * theta_1I(1)
- Doftheta = l * Sqrt(3 - 2 * Sin(theta_0o) - 2 * Cos(theta_0o))
- F0 = (Cos(theta_0o) * l * (1 - Sin(theta_0o)) / Doftheta - Sin(theta_0o) * l * (1 - Sin(theta_0o)) / Doftheta) * k * (Doftheta - l)
- 'theta_0I(2) is K3 , theta1I(2) is L3'
- theta_0I(2) = theta_1o
- theta_1I(2) = -g / l * Sin(theta_0o) + -c / M / l * theta_1o + F0 / M / l
-

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'theta_0o is point about to puting into the K4'
•    theta_0o = theta_0(CInt(t / dt) - 1) + dt * theta_0I(2)
•    theta_1o = theta_1(CInt(t / dt) - 1) + dt * theta_1I(2)
•    Doftheta = l * Sqrt(3 - 2 * Sin(theta_0o) - 2 * Cos(theta_0o))
•    F0 = (Cos(theta_0o) * l * (1 - Sin(theta_0o)) / Doftheta - Sin(theta_0o) * l * (1 -
Sin(theta_0o)) / Doftheta) * k * (Doftheta - l)
•    'theta_0I(3) is K4 , theta1I(3) is L4'
•    theta_0I(3) = theta_1o
•    theta_1I(3) = -g / l * Sin(theta_0o) + -c / M / l * theta_1o + F0 / M / l
•    'Calculate the average K and L , and then output the nex point'
•    theta_0(CInt(t / dt)) = theta_0(CInt(t / dt) - 1) + dt / 6 * (theta_0I(0) + 2 * theta_0I(1) +
2 * theta_0I(2) + theta_0I(3))
•    theta_1(CInt(t / dt)) = theta_1(CInt(t / dt) - 1) + dt / 6 * (theta_1I(0) + 2 * theta_1I(1) +
2 * theta_1I(2) + theta_1I(3))
•    gra.DrawLine(penn, 5 * (t - dt), CSng(360 / (2 * PI) * theta_0(CInt(t / dt) - 1)), 5 * t,
CSng(360 / (2 * PI) * theta_0(CInt(t / dt))))
•  Next
```

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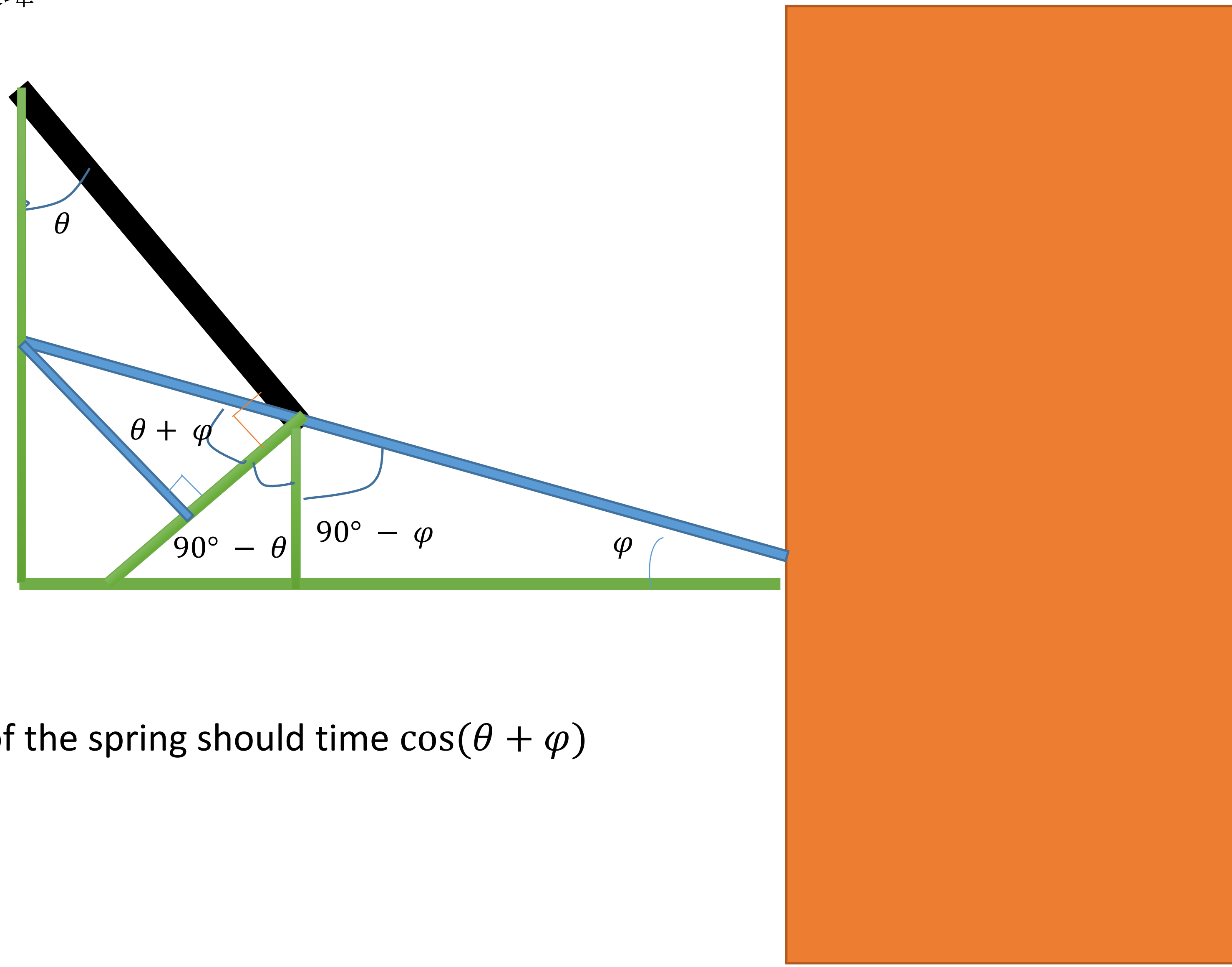

So the force of the spring should time $\cos(\theta + \varphi)$