# Inequalities

Chuah Jia Herng

November 24, 2024

Must-do thing in general:

- Affine transformation on the variables, and smoothing.
- Check equality case to decide how to bound, i.e. which inequality to use, or how to complete the square then $X^2 \geq 0$
- Handle odd and even cases separately, especially when pairing is possible.
- When there is $x_i x_{i+1}$, consider AM-GM/Cauchy, or $(\sum x_i)^2$, or take partial sum, see: IMO SL 2015 A3.
- Change the expression to as symmetric as possible (easier to handle).

## Contents

**Problem 0.1** (IMO SL 2016 A2: https://artofproblemsolving.com/community/c6h1480705p8639281) Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers $a_1, a_2, a_3, a_4, a_5$ (not necessarily distinct), one can always choose distinct subscripts $i, j, k, l$ such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \le C.$$

**Problem 0.2** (IMO SL 2016 A3: https://artofproblemsolving.com/community/c6h1480716p8639312) Find all positive integers $n$ such that the following statement holds: Suppose real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, each of which is either $-1$ or $1$, such that

$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \le 1.$$

**Problem 0.3** (IMO SL 2016 A8: https://artofproblemsolving.com/community/c6h1480693p8639259) Find the largest real constant $a$ such that for all $n \ge 1$ and for all real numbers $x_0, x_1, ..., x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$ we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \ge a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right)$$

**Problem 0.4** (IMO SL 2020 A1: https://artofproblemsolving.com/community/c6h2625913p22698446) Version 1. Let $n$ be a positive integer, and set $N = 2^n$. Determine the smallest real number $a_n$ such that, for all real $x$,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant a_n(x-1)^2 + x.$$

Version 2. For every positive integer $N$, determine the smallest real number $b_N$ such that, for all real $x$,

$$\sqrt[N]{\frac{x^{2N}+1}{2}} \leqslant b_N(x-1)^2 + x.$$

**Problem 0.5** (IMO SL 2022 A4: https://artofproblemsolving.com/community/c6h3107365p28104481) Let $n \geqslant 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $[0, 1]$. Let $s = x_1 + x_2 + \ldots + x_n$, and assume that $s \geqslant 3$. Prove that there exist integers $i$ and $j$ with $1 \leqslant i < j \leqslant n$ such that

$$2^{j-i}x_i x_j > 2^{s-3}.$$

**Problem 0.6** (IMO SL 2020 A3: https://artofproblemsolving.com/community/c6h2625839p22697887) Suppose that $a, b, c, d$ are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

**Problem 0.7** (IMO SL 2019 A4: https://artofproblemsolving.com/community/c6h2279021p17829013)
Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set $A$ by

$$A = \{(i, j) \mid 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}$$

Prove that, if $A$ is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

**Problem 0.8** (IMO SL 2017 A4: https://artofproblemsolving.com/community/c6h1671279p10632313)
A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n} (a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant $M$ such that $|a_n| \leq M$ for all positive integers $n$.

**Problem 0.9** (IMO SL 2017 A5: https://artofproblemsolving.com/community/c6h1671280p10632315)
An integer $n \geq 3$ is given. We call an $n$-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ Shiny if for each permutation $y_1, y_2, \ldots, y_n$ of these numbers, we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geq -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq K$$

holds for every Shiny $n$-tuple $(x_1, x_2, \ldots, x_n)$.

**Problem 0.10** (IMO SL 2018 A7: https://artofproblemsolving.com/community/c6h1876747p12752777)
Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where $a$, $b$, $c$, $d$ are nonnegative real numbers which satisfy $a + b + c + d = 100$.

**Problem 0.11** (IMO SL 2019 A3: https://artofproblemsolving.com/community/c6h2279061p17829510)
Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \dots, a_n)$ be a strictly increasing sequence of $n$ positive real numbers with sum equal to 2. Let $X$ be a subset of $\{1, 2, \dots, n\}$ such that the value of

$$\left|1 - \sum_{i \in X} a_i\right|$$

is minimised. Prove that there exists a strictly increasing sequence of $n$ positive real numbers $(b_1, b_2, \dots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

**Problem 0.12** (IMO SL 2019 A2: https://artofproblemsolving.com/community/c6h2279014p17828913)
Let $u_1, u_2, \dots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \dots, u_{2019})$ and $b = \max(u_1, u_2, \dots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

**Problem 0.13** (IMO SL 2022 C1: https://artofproblemsolving.com/community/c6h3107334p28104276)
A $\pm 1$-sequence is a sequence of 2022 numbers $a_1, \dots, a_{2022}$, each equal to either $+1$ or $-1$. Determine the largest $C$ so that, for any $\pm 1$-sequence, there exists an integer $k$ and indices $1 \leq t_1 < \ldots < t_k \leq 2022$ so that $t_{i+1} - t_i \leq 2$ for all $i$, and

$$\left|\sum_{i=1}^{k} a_{t_i}\right| \geq C.$$

**Problem 0.14** (IMO SL 2021 A1: https://artofproblemsolving.com/community/c6h2882570p25627665)
Let $n$ be a positive integer. Given $A$ is a subset of $\{0, 1, ..., 5^n\}$ with $4n + 2$ elements. Prove that there exist three elements $a < b < c$ from $A$ such that $c + 2a > 3b$.

**Problem 0.15** (IMO SL 2021 A5: https://artofproblemsolving.com/community/c6h2882545p25627517)
Let $n \geq 2$ be an integer and let $a_1, a_2, \dots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1 - a_k} (a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

**Problem 0.16** (IMO SL 2021 A7: https://artofproblemsolving.com/community/c6h2882578p25627708)
Let $n \geqslant 1$ be an integer, and let $x_0, x_1, \dots, x_{n+1}$ be $n + 2$ non-negative real numbers that satisfy $x_i x_{i+1} - x_{i-1}^2 \geqslant 1$ for all $i = 1, 2, \dots, n$. Show that

$$x_0 + x_1 + \cdots + x_n + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

**Problem 0.17** (IMO SL 2023 A1: https://artofproblemsolving.com/community/c6h3359728p31218381)
Professor Oak is feeding his 100 Pokémon. Each Pokémon has a bowl whose capacity is a positive real number of kilograms. These capacities are known to Professor Oak. The total capacity of all the bowls is 100 kilograms. Professor Oak distributes 100 kilograms of food in such a way that each Pokémon receives a non-negative integer number of kilograms of food (which may be larger than the capacity of the bowl). The dissatisfaction level of a Pokémon who received $N$ kilograms of food and whose bowl has a capacity of $C$ kilograms is equal to $|N - C|$.

Find the smallest real number $D$ such that, regardless of the capacities of the bowls, Professor Oak can distribute food in a way that the sum of the dissatisfaction levels over all the 100 Pokémon is at most $D$.

**Problem 0.18** (IMO SL 2023 A5)
Let $a_1, a_2, \ldots, a_{2023}$ be positive integers such that $a_1, a_2, \ldots, a_{2023}$ is a permutation of $1, 2, \ldots, 2023$, and $|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{2022} - a_{2023}|$ is a permutation of $1, 2, \ldots, 2022$. Prove that $\max(a_1, a_{2023}) \geq 507$.

**Problem 0.19** (IMO SL 2011 A7: https://artofproblemsolving.com/community/c6h488538p2737646)
Let $a, b$ and $c$ be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

**Problem 0.20** (IMO SL 2011 A1: https://artofproblemsolving.com/community/c6h418793p2363530)
Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by $s_A$. Let $n_A$ denote the number of pairs $(i, j)$ with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides $s_A$. Find all sets $A$ of four distinct positive integers which achieve the largest possible value of $n_A$.

**Problem 0.21** (IMO SL 2010 A7: https://artofproblemsolving.com/community/c6h356197p1936918)
Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and $s$ be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \quad \text{for all} \quad n > s.$$

Prove there exist positive integers $\ell \leq s$ and $N$, such that

$$a_n = a_\ell + a_{n-\ell} \quad \text{for all} \quad n \geq N.$$

**Problem 0.22** (IMO SL 2010 A8: https://artofproblemsolving.com/community/c6h418684p2362291)
Given six positive numbers $a, b, c, d, e, f$ such that $a < b < c < d < e < f$. Let $a + c + e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\left(S(bd+bf+df) + T(ac+ae+ce)\right)}.$$

# 1 Just algebra

- Application of classical inequalities
- Tangent line trick

**Problem 1.1** (IMO SL 2017 A1: https://artofproblemsolving.com/community/c6h1671272p10632289)
Let $a_1, a_2, \dots a_n, k$, and $M$ be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

**Problem 1.2** (IMO SL 2020 A4: https://artofproblemsolving.com/community/c6h2278647p17821569)
The real numbers $a, b, c, d$ are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

**Lessons —** Expand and compare terms

## 1.1 Convexity

**Problem 1.3** (IMO SL 2021 A4: https://artofproblemsolving.com/community/c6h2625850p22697952)
Show that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \dots x_n$.

# 2 Local bounding, then take union bound

**Problem 2.1** (IMO SL 2010 A3: https://artofproblemsolving.com/community/c6h418680p2362280)
Let $x_1, \dots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \dots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

# 3 Match coefficients

**Problem 3.1** (IMO SL 2010 A2: https://artofproblemsolving.com/community/c6h418679p2362276)
Let the real numbers $a, b, c, d$ satisfy the relations $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that

$$36 \leq 4\left(a^3 + b^3 + c^3 + d^3\right) - \left(a^4 + b^4 + c^4 + d^4\right) \leq 48.$$

**Lessons —** Since $a, b, c, d \in \mathbb{R}$ (we will need to do SOS etc), I want to create $\sum_{\text{cyc}}(a-\lambda)^2 = 0$, but after careful thought, 0 is impossible. I created $\sum_{\text{cyc}}(a-1)^2$. And noting that the expressions desired appear in $(a-1)^4$, I can assure that this will work.

$$4 = \sum_{\text{cyc}} a^2 - 2\sum_{\text{cyc}} a + \sum_{\text{cyc}} 1 = \sum_{\text{cyc}} (a-1)^2$$

Note that $4 = \binom{4}{1}$, so consider

$$\sum_{\text{cyc}}(a-1)^4 = \sum_{\text{cyc}} a^4 - 4\sum_{\text{cyc}} a^3 + 6\sum_{\text{cyc}} a^2 - 4\sum_{\text{cyc}} a + \sum_{\text{cyc}} 1$$

So

$$4\sum_{\text{cyc}} a^3 - \sum_{\text{cyc}} a^4 = 6(12) - 4(6) + 4 - \sum_{\text{cyc}}(a-1)^4 = 52 - \sum_{\text{cyc}}(a-1)^4$$

Then the original inequality is equivalent to

$$4 \le \sum_{\text{cyc}}(a-1)^4 \le 16$$

Note that by expansion, since $(a-1)^2 \ge 0$,

$$16 = 4^2 = \left(\sum_{\text{cyc}}(a-1)^2\right)^2 \ge \sum_{\text{cyc}}(a-1)^4$$

and for the other inequality, by Cauchy-Schwarz,

$$\left(\sum_{\text{cyc}} 1\right)\left(\sum_{\text{cyc}}(a-1)^4\right) \ge \left(\sum_{\text{cyc}}(a-1)^2\right)^2 = 16 \iff \sum_{\text{cyc}}(a-1)^4 \ge 4$$

This finishes the proof.

## 4 Sequences - Algebra

- Rewrite the problem condition into nicer look
- Force telescoping
- Induction

**Problem 4.1** (IMO SL 2015 A1: https://artofproblemsolving.com/community/c6h1268809p6621766)
Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \ge \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer $k$. Prove that $a_1 + a_2 + \ldots + a_n \ge n$ for every $n \ge 2$.

**Problem 4.2** (IMO SL 2022 A1: https://artofproblemsolving.com/community/c6h3107323p28104243)
Let $(a_n)_{n\ge 1}$ be a sequence of positive real numbers with the property that

$$(a_{n+1})^2 + a_n a_{n+2} \le a_n + a_{n+2}$$

for all positive integers $n$. Show that $a_{2022} \le 1$.

### 4.1 Induction-type, Local, Do cases

**Problem 4.3** (IMO SL 2010 A4: https://artofproblemsolving.com/community/c6h418681p2362283)
A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $\forall n \geq 1$, $x_1 + x_2 + \ldots + x_n \geq 0$.

**Lessons** — When there's +1 and -1, consider modulo 2.

**Problem 4.4** (IMO SL 2023 A3: https://artofproblemsolving.com/community/c6h3107339p28104298)
Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geqslant 3034$.

## 5 Sequences - Combinatorial Idea (Extremal etc.)

**Problem 5.1** (IMO 2014 A1: https://artofproblemsolving.com/community/c6h596930p3542095)
Let $a_0 < a_1 < a_2 < \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

**Motivations** — Note that the right inequality is loose, because $a_{n+1}$ is unrelated, except that $a_{n+1} > a_n$. Hence the focus can be put on $a_n$ "solely". If that's the case, we shall pick the first $n$ such that the left inequality holds.

Let $W(n) = (a_0 + ... + a_n) - na_n$. Then

$$W(n+1) - W(n) = a_{n+1} - (n+1)a_{n+1} + na_n = n(a_n - a_{n+1}) < 0$$

So $W(n)$ is strictly decreasing, i.e. the left inequality must fail for all sufficiently large $n$.

Pick $n$ as the largest $n$ such that $W(n) > 0$. By definition, $W(n+1) \leq 0$, i.e.

$$(n+1)a_{n+1} \geq a_0 + ... + a_{n+1}$$

$$\iff a_0 + ... + a_n \leq na_{n+1}$$

which is exactly the right inequality.

**Problem 5.2** (IMO SL 2021 A3: https://artofproblemsolving.com/community/c6h2882539p25627503)
For each integer $n \geq 1$, compute the smallest possible value of

$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$

over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

# 6 Sequences - Consider partial sum, block of sum, cumulative sum

**Problem 6.1** (IMO SL 2014 A3: https://artofproblemsolving.com/community/c6h1113180p5083537)
For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1\le i\le n} |x_1 + \cdots + x_i|.$$

Given $n$ real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price $D$. Greedy George, on the other hand, chooses $x_1$ such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses $x_2$ such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the $i$-th step he chooses $x_i$ among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price $G$.

Find the least possible constant $c$ such that for every positive integer $n$, for every collection of $n$ real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

**Problem 6.2** (IMO SL 2015 A3: https://artofproblemsolving.com/community/c6h1268841p6622146)
Let $n$ be a fixed positive integer. Find the maximum possible value of

$$\sum_{1\le r<s\le 2n} (s-r-n)x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

Solution: https://artofproblemsolving.com/community/c6h1268841p31528092

**Problem 6.3** (IMO SL 2013 A4: https://artofproblemsolving.com/community/c6h597122p3543362)
Let $n$ be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \ge 1$. If

$$a_1 \le a_2 \le \cdots \le a_n \le a_1 + n$$

and

$$a_{a_i} \le n + i - 1 \quad \text{for} \quad i = 1, 2, \ldots, n,$$

prove that

$$a_1 + \cdots + a_n \le n^2.$$

**Motivations —** Since I want to tighten the bound so I consider splitting into block of equal value. Say we have

$$a_1 = a_{b_1} < a_{b_1+b_2} < a_{b_1+b_2+b_3} < \ldots < a_{b_1+\ldots+b_k} = a_n$$

so there're $k$ values, and I can use the smallest $i$. However, in order to bound $a_1 + \ldots + a_n$, I will eventually define/split $a_{a_1}, a_{a_2}, \ldots, a_{a_n}$ in the sequence. And there are a lot of things that I can't tell, so may end up to be a very blind bound. So maybe a crude bound should be enough.

Following the idea, we need to look at $a_{a_1}, a_{a_2}, ..., a_{a_n}$.

$$\begin{gathered}
a_1 + ... + a_{a_1} \le n \cdot a_1 \\
a_{a_1+1} + ... + a_{a_2} \le (n+1)(a_2 - a_1) \\
a_{a_2+1} + ... + a_{a_3} \le (n+2)(a_3 - a_2) \\
\vdots \\
a_{a_{n-1}+1} + ... + a_{a_n} \le (2n-1)(a_n - a_{n-1}) \\
a_{a_n+1} + ... + a_n \le (2n-1)(n - a_n)
\end{gathered}$$

Letting $S = a_1 + ... + a_n$, summing all the inequalities,

$$S \le -(a_1 + ... + a_{n-1}) + n(2n-1)$$

$$\iff 2S \le n(2n-1) + a_n$$

- If $a_n \ge n$, then $2S \le n(2n-1) + n = 2n^2 \implies S \le n^2$.
- If $a_n < n$, then $S \le n \cdots a_n < n^2$.

## 7 Smoothing

**Problem 7.1** (IMO SL 2016 A1: https://artofproblemsolving.com/community/c6h1480690p8639254) Let $a$, $b$, $c$ be positive real numbers such that $\min(ab, bc, ca) \ge 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \le \left(\frac{a+b+c}{3}\right)^2 + 1.$$