Integer Polynomials

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1 Fundamentals

Definition 1.1

A **polynomial** of **degree** n is an algebraic expression of the form

$$P(x) = a_0 + a_1 x + ... a_n x^n$$

where x is a variable, and $a_0, a_1 ..., a_n$ are coefficients of $x^0, x^1, ..., x^n$, and a polynomials is called **monic** if $a_n = 1$.

Theorem 1.2 (Fundamental Theorem of Algebra)

Let $P \in \mathbb{C}[X]$ be a nonzero polynomial of degree n. Then $P(x)$ has exactly n complex roots, up to multiplicity.

Remark. The degree of zero polynomial is $-\infty$.

Theorem 1.3

Every polynomials $P \in \mathbb{C}[X]$ can be factorized into real quadratic and linear factors.

2 Identical Polynomials

Theorem 2.1

If two polynomials f, g coincide for more than $\max\{\deg f, \deg g\}$ times, then they are identical.

Theorem 2.2

Let f be a polynomial such that $f(x) = 0$ for infinitely many x. Then $f \equiv 0$.

Problem 2.3

Find all polynomials P with real coefficients such that $P(x^2 + x) = (x + 1)P(x)$ and $P(1) = 1$.

3 Lagrange Interpolation

Theorem 3.1 (Lagrange Interpolation)

A polynomial of degree n is uniquely determined by $n+1$ values. In particular, if we knew $P(x_0), ..., P(x_n)$, then

$$P(x)\sum_{i=0}^{n} P(x_i)\cdot\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}$$

Problem 3.2 (IMO SL 1997)

Let p be a prime number and f an integer polynomial such that $f(0)=0$, $f(1)=1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $\deg f \geq p-1$.

Problem 3.3 (ELMO 2014 SL N3)

Let t and n be fixed integers each at least 2. Find the largest positive integer m for which there exists a polynomial $P \in \mathbb{Q}[X]$ of degree n such that exactly one of

$$\frac{P(k)}{t^k} \text{ and } \frac{P(k)}{t^{k+1}}$$

is an integer for each $k = 0, 1, ..., m$.

Problem 3.4

Prove that if for a polynomial P, $P(\mathbb{Q}) \subset \mathbb{Q}$. Then P has rational coefficients.

Problem 3.5

Find all polynomials $f \in \mathbb{R}[X]$ such that $x \in \mathbb{Q} \iff f(x) \in \mathbb{Q}$.

Lemma 3.6

ε is fixed. For any polynomial $f : \mathbb{R} \to \mathbb{R}$ with $\deg f > 1$ and positive leading coefficient. Then the difference $f(x+\varepsilon) - f(x)$ grows arbitrarily large.