Functional Equations

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November 28, 2024

General Techniques:

- Try values, get fixed point or at least known point
- Force cancellation
- Injectivity, Surjectivity, Bijectivity
- Exploit as many properties as possible then rewrite original equation.
- Injectivity at a point, injectivity of composition of function
- Cauchy FE

1 IMO SL 2011

Problem 1.1 (https://artofproblemsolving.com/community/c6h488535p2737643)
Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Problem 1.2 (https://artofproblemsolving.com/community/c6h488536p2737644)
Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f)}_{k}(n)\ldots))$.

Problem 1.3 (https://artofproblemsolving.com/community/c6h418798p2363539)
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Problem 1.4 (https://artofproblemsolving.com/community/c6h488539p2737648)
For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly d positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

Problem 1.5 (https://artofproblemsolving.com/community/c6h418981p2365041)
Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

2 IMO SL 2010

Problem 2.1 (https://artofproblemsolving.com/community/c6h356075p1935849)
Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Problem 2.2 (https://artofproblemsolving.com/community/c6h418682p2362286)
Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

Problem 2.3 (https://artofproblemsolving.com/community/c6h418683p2362289)
Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

Problem 2.4 (https://artofproblemsolving.com/community/c6h356076p1935854)
Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

3 IMO SL 2009

Problem 3.1 (https://artofproblemsolving.com/community/c6h289055p1562848)
Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b+f(a)-1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

Problem 3.2 (https://artofproblemsolving.com/community/c6h355779p1932913)
Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x - f(y)) > yf(x) + x$$

Problem 3.3 (https://artofproblemsolving.com/community/c6h355780p1932915)
Find all functions f from the set of real numbers into the set of real numbers which satisfy for all x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2$$

Problem 3.4 (https://artofproblemsolving.com/community/c6h355797p1932942)
Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

4 IMO SL 2008

Problem 4.1 (https://artofproblemsolving.com/community/c6h215430p1191683)
Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 4.2 (https://artofproblemsolving.com/community/c6h287852p1555894)
Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a Spanish Couple on S, if they satisfy the following conditions: (i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all x, $y \in S$ with $x < y$;

(ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple on the set $S = \mathbb{N}$ of positive integers; on the set $S = \{a - \frac{1}{b} : a, b \in \mathbb{N}\}$

Problem 4.3 (https://artofproblemsolving.com/community/c6h287853p1555896)

For an integer m, denote by $t(m)$ the unique number in $\{1, 2, 3\}$ such that $m + t(m)$ is a multiple of 3. A function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(-1) = 0$, $f(0) = 1$, $f(1) = -1$ and $f(2^n + m) = f(2^n - t(m)) - f(m)$ for all integers m, $n \geq 0$ with $2^n > m$. Prove that $f(3p) \geq 0$ holds for all integers $p \geq 0$.

Problem 4.4 (https://artofproblemsolving.com/community/c6h287857p1555902)

Let $f : \mathbb{R} \to \mathbb{N}$ be a function which satisfies $f\left(x + \dfrac{1}{f(y)}\right) = f\left(y + \dfrac{1}{f(x)}\right)$ for all x, $y \in \mathbb{R}$. Prove that there is a positive integer which is not a value of f.

Problem 4.5 (https://artofproblemsolving.com/community/c6h287873p1555934)

For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties: $d(f(x)) = x$ for all $x \in \mathbb{N}$. $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all x, $y \in \mathbb{N}$.