Graphs

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1 Graph Terminologies

- **Graph** is an ordered pair $G = (V, E)$, where V is the set of vertices and E is the set of edges, i.e. the connections between vertices.
- **Order** of a graph is the number of vertices, denoted as $|V|$.
- We say $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
- A graph is **directed** if every edge has a direction. The edge may be represented as (u, v), where u, v are vertices.
- A graph is **undirected** if there is no direction for any edge. The edge may be represented as $\{u, v\}$ where u, v are vertices.
- A graph is **simple** if there is no multiple edges (between any two vertices, there can be at most 1 edge) and self-loop (every edge must have different endpoints).
- **Degree** of a vertex v, denoted as $\deg(v)$, is the number of edges incident to v.

2 Well-known Results

Theorem 2.1 (Hall's Marriage Theorem)

Let $G = (V, E)$ be a bipartite graph with sets of vertices A and B. There exists a perfect matching between A and B if and only if for every $X \subseteq A$, we have

$$|N(X)| \geq |X|$$

Theorem 2.2 (Euler's Theorem)

In a connected planar graph with V vertices, F faces and E edges, then

$$V - E + F = 2$$

Theorem 2.3 (Ramsey's Theorem)

Let $k \geq 2$ be a positive integer and let $n_1, ..., n_k$ be any positive integers. Then there exists a number N such that if the edges of a complete graph of with N vertices are colored in k different colors, then for some $i = 1, ..., k$, the graph contains a complete subgraph with n_i vertices whose vertices are all of color i. Ramsey number $R(n_1, ..., n_k)$ is the smallest possible number N given $n_1, ..., n_k \in \mathbb{N}$.

Theorem 2.4 (Turan's Theorem)
If $G(V, E)$ is a graph of n vertices without k-clique, then

$$|E| \leq \frac{(k-2)n^2}{2(k-1)}$$

3 Problems

Problem 3.1 (St. Petersburg)
Is it possible to choose several points in space and connect some of them by line segments, such that each vertex is the endpoints of exactly 3 segments, and any cycle has size at least 30?

- We generalize 30 to n and construct such a graph.
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Lemma 3.2
Prove that if a graph G with n vertices is a tree (connected graph with no cycle), then it has exactly $n-1$ edges.