P001 — SMMC 2024 B2

Chuah Jia Herng

February 2026

Problem

Determine all continuous functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ such that

$$f(x) = (x+1)f(x^2)$$

for all $x \in \mathbb{R} \setminus \{-1, 1\}$.

Idea

Introduce $g(x) = (x-1)f(x)$ to remove the factor $(x+1)$. Then the equation becomes the self-similarity $g(x) = g(x^2)$. Iterating squaring sends $|x| < 1$ to 0, while iterating square-roots sends $x > 1$ to 1^+. Continuity at -1 forces a right-limit at 1^+, which pins down g for all $x > 1$.

Solution

Define $g(x) = (x-1)f(x)$ for $x \neq 1$. Since f is continuous on $\mathbb{R} \setminus \{1\}$, so is g. For $x \neq \pm 1$,

$$g(x) = (x-1)f(x) = (x-1)(x+1)f(x^2) = (x^2-1)f(x^2) = g(x^2). \tag{1}$$

Step 1: g is constant on $(-1, 1)$. Fix $x \in (-1, 1)$. Then $x^{2^n} \to 0$. By iterating (1),

$$g(x) = g(x^{2^n}) \quad \text{for all } n.$$

Letting $n \to \infty$ and using continuity at 0 gives $g(x) = g(0)$ for all $x \in (-1, 1)$. Write $A := g(0)$. By continuity at -1 (note $-1 \in \mathbb{R} \setminus \{1\}$), we have $g(-1) = A$.

Step 2: the right-limit at 1 equals A. For $t > 1$, set $x = -\sqrt{t} < -1$. Then $x^2 = t$ and (1) gives $g(t) = g(x)$. Let $t \to 1^+$, so $x \to -1^-$. Continuity at -1 yields

$$\lim_{t \to 1^+} g(t) = \lim_{x \to -1^-} g(x) = g(-1) = A. \tag{2}$$

Step 3: g is constant on $(1, \infty)$. Fix $y > 1$ and define $y_n := y^{1/2^n}$, so $y_n \to 1^+$. From (1),

$$g(y) = g(y_n) \quad \text{for all } n.$$

Letting $n \to \infty$ and using (2) gives $g(y) = A$. Hence $g(x) \equiv A$ for all $x > 1$.

Step 4: g is constant on (∞, ∞). From $g(x) = g(x^2) = g(-x)$, since $g(x) = A$ for $x \geq 0$, $g(x) = A$ for $x \leq 0$, i.e. $g(x) \equiv A$ for all $x \neq 1$.

Step 5: Describing f.

$$(x-1)f(x) = A \quad \Rightarrow \quad f(x) = \frac{A}{x-1} \qquad (x \neq 1).$$

This function is continuous on $\mathbb{R} \setminus \{1\}$ and satisfies the given equation.

Answer

$$\boxed{f(x) = \frac{c}{x-1} \text{ for some constant } c \in \mathbb{R}.}$$