P002 — IMO SL 1990 Problem 8

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Problem

For a given positive integer k denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

Idea

Let $s(n)$ be the sum of decimal digits of n, so $f_1(n) = s(n)^2$. Since $s(n) \equiv n \pmod 9$, we get $f_{m+1}(k) \equiv f_m(k)^2 \pmod 9$. For $k = 2^{1990}$ this forces the residues modulo 9 to alternate between 4 and 7. Separately, repeated digit-sum squaring rapidly drives the values below 100, forcing entry into the 2-cycle $169 \leftrightarrow 256$; the parity of m then determines the answer.

Solution

Step 1: A modulo 9 recurrence

Let $s(n)$ denote the sum of decimal digits of n. Then $f_1(n) = s(n)^2$. Since $s(n) \equiv n \pmod 9$ for all integers n, we have

$$f_1(x) = s(x)^2 \equiv x^2 \pmod 9.$$

Hence for all $m \geq 1$,

$$f_{m+1}(k) = f_1(f_m(k)) \equiv f_m(k)^2 \pmod 9. \tag{1}$$

Now $2^6 \equiv 1 \pmod 9$, and $1990 \equiv 4 \pmod 6$, so

$$2^{1990} \equiv 2^4 = 16 \equiv 7 \pmod 9.$$

Therefore

$$f_1(2^{1990}) \equiv (2^{1990})^2 \equiv 7^2 \equiv 4 \pmod 9.$$

Using (1), the residues evolve by squaring:

$$4 \mapsto 4^2 \equiv 7, \qquad 7 \mapsto 7^2 \equiv 4.$$

Thus

$$f_m(2^{1990}) \equiv \begin{cases} 4 \pmod 9, & m \text{ odd}, \\ 7 \pmod 9, & m \text{ even}. \end{cases} \tag{2}$$

Step 2: Rapid descent to small values

The number 2^{1990} has 599 digits, hence

$$s(2^{1990}) \leq 9 \cdot 599 = 5391,$$

and therefore

$$f_1(2^{1990}) = s(2^{1990})^2 \le 5391^2 < 3 \times 10^7.$$

Thus $f_1(2^{1990})$ has at most 8 digits, giving

$$s(f_1(2^{1990})) \le 72 \quad \Rightarrow \quad f_2(2^{1990}) \le 72^2 = 5184.$$

Continuing similarly,

$$f_3(2^{1990}) \le 36^2 = 1296, \qquad f_4(2^{1990}) \le 18^2 = 324, \qquad f_5(2^{1990}) \le 9^2 = 81.$$

Hence $f_5(2^{1990})$ is a perfect square not exceeding 81. By (2), $f_5(2^{1990}) \equiv 4 \pmod 9$, so

$$f_5(2^{1990}) \in \{4, 49\}.$$

Step 3: Entry into a 2-cycle

The subsequent values are forced:

$$f_1(4) = 16, \quad f_1(16) = 49, \quad f_1(49) = 169, \quad f_1(169) = 256, \quad f_1(256) = 169.$$

Thus the sequence eventually enters the 2-cycle

$$169 \leftrightarrow 256.$$

Moreover,

$$169 \equiv 7 \pmod 9, \qquad 256 \equiv 4 \pmod 9,$$

so by (2) we must have $f_m(2^{1990}) = 256$ for odd m and 169 for even m once the cycle is reached.

Since 1991 is odd,

$$f_{1991}(2^{1990}) = 256.$$

Answer

$$\boxed{256}.$$