P005 — Multiples using a prescribed digit set

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February 2026

Problem

Let $b \geq 2$ be an integer base, and let

$$0 \leq u_1 < u_2 < \cdots < u_k \leq b-1$$

for some $k \geq 2$. Consider the digit set $D = \{u_1, u_2, \ldots, u_k\}$.

Determine all $(u_1, \ldots, u_k)$ with the following property:

For every positive integer n, there exists a positive multiple N of n whose base-b representation uses only digits from D, and moreover each digit in D appears at least once in the base-b representation of N.

Idea

If $0 \notin D$, then no number whose digits lie in D can be divisible by b (since a multiple of b must end in digit 0 in base b). Hence $0 \in D$ is necessary.

Assuming $0 \in D$, we first solve the two-digit case $\{0, d\}$ using a classical repunit pigeonhole argument (and appending zeros to handle factors of b). Then for a general digit set $D = \{0, u_2, \ldots, u_k\}$, we construct for each u_i a multiple of n using only digits $\{0, u_i\}$, and concatenate these blocks in base b. Concatenation preserves divisibility and guarantees every digit appears.

Solution

Step 1: Necessity of $u_1 = 0$

Assume the stated property holds. Apply it to $n = b$. Any multiple of b written in base b must end with digit 0, hence its base-b representation must contain the digit 0. Therefore $0 \in D$, i.e. $u_1 = 0$.

Thus $u_1 = 0$ is necessary.

Step 2: A two-digit lemma

Lemma. Let $b \geq 2$, let $d \in \{1, 2, \ldots, b-1\}$, and let $n \geq 1$. Then there exists a positive multiple M of n whose base-b representation consists only of digits 0 and d. Moreover, one can choose M so that both digits 0 and d appear at least once.

Proof. Define the base-b repunits

$$R_k = 1 + b + \cdots + b^{k-1} \qquad (k \geq 1),$$

whose base b representation is the string $11 \cdots 1$ of length k.

First suppose $\gcd(n, b) = 1$. Consider the $n + 1$ residues modulo n:

$$R_0 := 0,\ R_1,\ R_2, \ldots,\ R_n.$$

By the pigeonhole principle, there exist $0 \le i < j \le n$ such that $R_i \equiv R_j \pmod{n}$. Then

$$0 \equiv R_j - R_i = b^i R_{j-i} \pmod{n}.$$

Since $\gcd(n, b) = 1$, we may cancel b^i, obtaining $R_{j-i} \equiv 0 \pmod{n}$. Thus $n \mid R_k$ for some k, hence $n \mid dR_k$, and dR_k uses only digit d in base b.

For a general n, write $n = n_0 n_1$ where n_0 is the largest divisor of n whose prime factors all divide b, so that $\gcd(n_1, b) = 1$. Choose $t \ge 0$ with $n_0 \mid b^t$, and choose k with $n_1 \mid R_k$ by the coprime case. Then

$$M_0 := b^t \cdot dR_k$$

satisfies $n \mid M_0$, and its base-b digits are only 0 and d.

Finally, if M_0 has no digit 0 (this happens only when $t = 0$), replace it by $M := bM_0$. Then $n \mid M$ and the base-b representation appends a trailing zero, so both digits 0 and d appear. □

Step 3: Sufficiency when $u_1 = 0$ and $k \ge 2$

Assume $u_1 = 0$ and $k \ge 2$. Fix any $n \ge 1$.

For each $i = 2, 3, \ldots, k$, apply the lemma with $d = u_i$ to obtain a positive multiple M_i of n whose base-b digits are only 0 and u_i, and such that both digits 0 and u_i occur. Let ℓ_i be the number of base-b digits of M_i.

Define

$$N := M_2\, b^{\ell_3+\cdots+\ell_k} + M_3\, b^{\ell_4+\cdots+\ell_k} + \cdots + M_{k-1} b^{\ell_k} + M_k.$$

Each term is divisible by n, hence $n \mid N$. In base b, multiplying by b^m shifts the representation left by m digits (appending m zeros), so the base-b representation of N is the concatenation of the representations of $M_2, M_3, \ldots, M_k$. Therefore N uses only digits from D, and for each $i \ge 2$, the digit u_i appears in M_i. Also, digit 0 appears in every M_i, hence appears in N.

Thus N is a positive multiple of n, uses only digits from D, and contains every digit of D at least once.

Conclusion

The stated property holds if and only if $u_1 = 0$ and $k \ge 2$.

$$\boxed{\text{All solutions are } (u_1, \ldots, u_k) \text{ with } u_1 = 0 \text{ and } k \ge 2.}$$