P003 — Digit-deletion invariant multiples

Chuah Jia Herng

February 2026

Problem

Find all positive integers n such that the following holds: There exists a multiple N of n, whose decimal representation contain all digits but 0, and for each $i \in \{1, 2, \dots, 9\}$, one can delete a digit i from the decimal representation of N, so that the resulting number is again a multiple of n.

Idea

Deleting a digit from a decimal number changes its value by a highly structured amount. The condition that *every* digit $1, \dots, 9$ can be deleted while preserving divisibility by n forces strong arithmetic constraints on n.

First, simple divisibility arguments exclude factors 3 and 10. Conversely, when n has neither of these factors, we explicitly construct a suitable N by combining modular control modulo $\gcd(n, 10) = 1$ with careful handling of powers of 2 or 5.

Solution

Step 1: Necessary conditions

Claim 1. $10 \nmid n$.

Proof. If $10 \mid n$, then every multiple of n ends in digit 0. This contradicts the requirement that N contains no digit 0. $\square$

Claim 2. $3 \nmid n$.

Proof. Assume $3 \mid n$. Then $3 \mid N$. Since digit 1 appears in N, delete one occurrence of it to obtain N'. Divisibility by 3 depends only on the sum of digits, and deleting a 1 decreases the digit sum by 1. Thus $3 \nmid N'$, contradicting the assumption. $\square$

Hence any admissible n must satisfy

$$10 \nmid n \quad \text{and} \quad 3 \nmid n.$$

Step 2: Two lemmas

Lemma 1. Let d be a positive integer with $\gcd(d, 10) = 1$. Then there exists an integer k such that every residue class modulo d is represented by some k-digit decimal number containing no digit 0.

Proof. For each $k \geq 1$, let S_k be the set of residues modulo d represented by k-digit numbers with no digit 0.

Fix k and choose, for each $r \in S_k$, a representative x_r. Consider the two sets

$$A = \{10^k + x_r : r \in S_k\}, \qquad B = \{2 \cdot 10^k + x_r : r \in S_k\}.$$

All numbers in $A \cup B$ have $k+1$ digits and contain no digit 0, so their residues lie in S_{k+1}.

The maps $r \mapsto 10^k + x_r$ and $r \mapsto 2 \cdot 10^k + x_r$ are injective modulo d. If these two images had complete overlap modulo d, we would obtain $10^k \equiv 0 \pmod{d}$, contradicting $\gcd(d, 10) = 1$. Therefore,

$$|S_{k+1}| \geq \min(d,\, 2|S_k|).$$

Since $|S_1| \geq 1$, repeated doubling implies that for sufficiently large k, $|S_k| = d$. Thus every residue class modulo d is represented by a k-digit number with no digit 0. $\square$

Lemma 2. Let $p \in \{2, 5\}$ and $\alpha \geq 1$. There exists an α-digit number containing no digit 0 that is divisible by p^α.

Proof. We proceed by induction on α.

For $\alpha = 1$, take 2 or 5.

Assume the claim holds for $\alpha = t$ and let M be a t-digit no-zero number with $p^t \mid M$. Consider numbers of the form $B = 10^t a + M$, obtained by prepending a digit a. Then

$$B = p^t \left(\left(\frac{10}{p} \right)^t a + \frac{M}{p^t} \right).$$

Since $\gcd(10/p, p) = 1$, the coefficient $(10/p)^t$ is invertible modulo p. Thus there exists $a \in \{0, 1, \ldots, p-1\}$ such that p divides the bracketed term. If $a = 0$, replace it by p, which is a valid nonzero digit. Hence B is divisible by p^{t+1} and contains no digit 0. $\square$

Step 3: Sufficiency and construction

Assume now that

$$10 \nmid n \quad \text{and} \quad 3 \nmid n.$$

Write

$$n = p^\alpha d,$$

where $p \in \{2, 5\}$ (possibly $\alpha = 0$) and $\gcd(d, 10) = 1$.

Choose $k > \alpha$ as in Lemma 1. We construct

$$N = A_1\, 1\, A_2\, 2 \cdots A_9\, 9\, B,$$

where each A_i is a k-digit no-zero block and B is a $(k + \alpha)$-digit no-zero block. This ensures that N contains digits $1, \ldots, 9$ and no zeros.

Let N_i denote the number obtained by deleting the displayed digit i. A direct computation shows

$$N - N_i = (9X_i + i)\, 10^{\ell_i},$$

where X_i is the prefix before that digit and $\ell_i \geq \alpha$.

Using Lemma 1, choose the blocks $A_1, \ldots, A_9$ inductively so that

$$9X_i + i \equiv 0 \pmod{d} \quad (i = 1, \ldots, 9).$$

Then $d \mid (N - N_i)$ for all i.

Next, by Lemma 2, choose an α-digit no-zero number B_2 divisible by p^α, and write $B = B_1B_2$ with $|B_1| = k$. Since $\gcd(d, 10) = 1$, we may choose B_1 so that $d \mid N$. Thus $n \mid N$.

Finally, since $\ell_i \geq \alpha$, we have $p^\alpha \mid 10^{\ell_i}$, so $n \mid (N - N_i)$ for all i. Therefore $n \mid N_i$ for every digit deletion, as required.

Remarks

The conditions $3 \nmid n$ and $10 \nmid n$ are both necessary and sufficient. The proof shows that the obstruction is purely arithmetic; once these two divisibility constraints are removed, an explicit construction is always possible.