Linear Algebra

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3.2 Evaluation of a determinant using elementary operations

Key Learning in Section 3.2

- **Use elementary row operations to evaluate a determinant.**
- **Use elementary column operations to evaluate a determinant.**
- **Recognize conditions that yield zero determinants.**

- Thm 3.3: (Elementary row operations and determinants)

Let A and B be square matrices.

$$(a) \quad B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e.} \left|r_{ij}(A)\right| = -\left|A\right|)$$

$$(b) \quad B = r_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e.} \left|r_{i}^{(k)}(A)\right| = k\left|A\right|)$$

$$(c) \quad B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e.} \left|r_{ij}^{(k)}(A)\right| = \left|A\right|)$$

- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \qquad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \quad \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \quad \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \quad \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k}\det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

Note:

A row-echelon form of a square matrix is always <u>upper triangular</u>.

- Ex 2: (Evaluation a determinant using elementary row operations)

$$A=\begin{bmatrix} 0 & -7 & 14 \\ 1 & 2 & -2 \\ 0 & 3 & -8 \end{bmatrix} \Rightarrow \det(A)=?$$

Sol:

$$\det(A)=\begin{vmatrix} 0 & -7 & 14 \\ 1 & 2 & -2 \\ 0 & 3 & -8 \end{vmatrix} \overset{r_{12}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 3 & -8 \end{vmatrix}$$

$$\overset{r_2^{(-\frac{1}{7})}}{=} (-1)\left(\frac{1}{\frac{-1}{7}}\right)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{r_{23}^{(-3)}}{=} (7)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{vmatrix} = (7)(1)(1)(-2) = -14$$

- Notes:

$$|EA| = |E||A|$$

$$(1) \quad E = R_{ij} \quad \Rightarrow |E| = \left|R_{ij}\right| = -1$$

$$\Rightarrow |EA| = \left|r_{ij}(A)\right| = -|A| = \left|R_{ij}\right||A| = |E||A|$$

$$(2) \quad E = R_i^{(k)} \quad \Rightarrow |E| = \left|R_i^{(k)}\right| = k$$

$$\Rightarrow |EA| = \left|r_i^{(k)}(A)\right| = k|A| = \left|R_i^{(k)}\right||A| = |E||A|$$

$$(3) \quad E = R_{ij}^{(k)} \quad \Rightarrow |E| = \left|R_{ij}^{(k)}\right| = 1$$

$$\Rightarrow |EA| = \left|r_{ij}^{(k)}(A)\right| = 1|A| = \left|R_{ij}^{(k)}\right||A| = |E||A|$$

- Determinants and elementary column operations

▪ Thm: (Elementary column operations and determinants)

Let A and B be square matrices.

$$(a)\quad B = c_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b)\quad B = c_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e. } |c_{i}^{(k)}(A)| = k|A|)$$

$$(c)\quad B = c_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

- Ex:

$$A=\begin{bmatrix}2 & 1 & -3\\ 4 & 0 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad \det(A)=-8$$

$$A_1=\begin{bmatrix}1 & 1 & -3\\ 2 & 0 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad A_2=\begin{bmatrix}1 & 2 & -3\\ 0 & 4 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad A_3=\begin{bmatrix}2 & 1 & 0\\ 4 & 0 & 1\\ 0 & 0 & 2\end{bmatrix}$$

$$A_1=c_1^{(\frac{1}{2})}(A) \quad \Rightarrow \det(A_1)=\det(c_1^{(4)}(A))=\frac{1}{2}\det(A)=(\frac{1}{2})(-8)=-4$$

$$A_2=c_{12}(A) \quad \Rightarrow \det(A_2)=\det(c_{12}(A))=-\det(A)=-(-8)=8$$

$$A_3=c_{23}^{(3)}(A) \quad \Rightarrow \det(A_3)=\det(c_{23}^{(3)}(A))=\det(A)=-8$$

- Thm 3.4: (Conditions that yield a zero determinant)

If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$.

(a) An entire row (or an entire column) consists of zeros.

(b) Two rows (or two columns) are equal.

(c) One row (or column) is a multiple of another row (or column).

- Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ \boxed{0 \quad 0 \quad 0} \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Note:

Order n	Cofactor Expansion		Row Reduction	
	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

- Ex 5: (Evaluating a determinant)

$$A=\begin{bmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{C_{13}^{(2)}}{=}\begin{vmatrix}-3 & 5 & -4\\ 2 & -4 & 3\\ \boxed{-3 \quad 0 \quad 0}\end{vmatrix}$$

$$=(-3)(-1)^{3+1}\begin{vmatrix}5 & -4\\ -4 & 3\end{vmatrix}=(-3)(-1)=3$$

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{r_{12}^{(\frac{4}{5})}}{=}\begin{vmatrix}-3 & 5 & 2\\ \frac{-2}{5} & 0 & \frac{3}{5}\\ -3 & 0 & 6\end{vmatrix}$$

$$=(5)(-1)^{1+2}\begin{vmatrix}\frac{-2}{5} & \frac{3}{5}\\ -3 & 6\end{vmatrix}=(-5)(-\frac{3}{5})=3$$

- Ex 6: (Evaluating a determinant)

$$A=\begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \underset{\substack{r_{24}^{(1)} \\ r_{25}^{(-1)}}}{=} \begin{vmatrix} 2 & \boxed{0} & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=(1)(-1)^{2+2}\begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{C_{41}^{(-3)}}{=} \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ \boxed{0 \quad 0 \quad 0 \quad 1} \end{vmatrix} = (1)(-1)^{4+4} \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \underset{r_{21}^{(1)}}{=} \begin{vmatrix} \boxed{0 \quad 0 \quad 5} \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3} \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

- **Use elementary row operations to evaluate a determinant.**
- **Use elementary column operations to evaluate a determinant.**
- **Recognize conditions that yield zero determinants.**

- **determinant :** 行列式
- **elementary row operation:** 基本列運算
- **row equivalent:** 列等價
- **elementary matrix:** 基本矩陣
- **elementary column operation:** 基本行運算
- **column equivalent:** 行等價

3. 3 Properties of Determinants

- **Find the determinant of a matrix product and a scalar multiple of a matrix.**
- **Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix.**
- **Find the determinant of the transpose of a matrix.**

- Thm 3.5: (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

- Notes:

(1) $\det(EA) = \det(E)\det(A)$

(2) $\det(A+B) \neq \det(A) + \det(B)$

(3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{22}+b_{22} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Ex 1: (The determinant of a matrix product)

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7 \qquad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}=\begin{bmatrix}8 & 4 & 1\\ 6 & -1 & -10\\ 5 & 1 & -1\end{bmatrix}$$

$$\Rightarrow |AB|=\begin{vmatrix}8 & 4 & 1\\ 6 & -1 & -10\\ 5 & 1 & -1\end{vmatrix}=-77$$

- Check:

$$|AB| = |A|\ |B|$$

- Thm 3.6: (Determinant of a scalar multiple of a matrix)

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

- Ex 2:

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \qquad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$$

Find $|A|$.

Sol:

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- **Thm 3.7: (Determinant of an invertible matrix)**

A square matrix A is invertible (nonsingular) if and only if

$\det(A) \neq 0$

- Ex 3: (Classifying square matrices as singular or nonsingular)

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \quad \Rightarrow$ A has no inverse (it is singular).

$|B| = -12 \neq 0 \quad \Rightarrow$ B has an inverse (it is nonsingular).

- Thm 3.8: (Determinant of an inverse matrix)

$$\text{If } A \text{ is invertible, then } \det(A^{-1}) = \frac{1}{\det(A)}.$$

- Thm 3.9: (Determinant of a transpose)

$$\text{If } A \text{ is a square matrix, then } \det(A^{\mathrm{T}}) = \det(A).$$

- Ex 4:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

(a) $\left|A^{-1}\right| = ?$ (b) $\left|A^{T}\right| = ?$

Sol:

$$\because \quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4$$

$$\therefore \quad \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

$$\left|A^{T}\right| = |A| = 4$$

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Ex 5: Which of the following system has a unique solution?

$$(a)\quad \begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

$$(b)\quad \begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & + & x_3 & = & -4 \end{array}$$

Sol:

(a) $A\mathbf{x}=\mathbf{b}$

$\because$ $|A|=0$

$\therefore$ This system does not have a unique solution.

(b) $B\mathbf{x}=\mathbf{b}$

$\because$ $|B|=-12\neq 0$

$\therefore$ This system has a unique solution.

- **Find the determinant of a matrix product and a scalar multiple of a matrix.**
- **Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix.**
- **Find the determinant of the transpose of a matrix.**

- **determinant:** 行列式
- **matrix multiplication:** 矩陣相乘
- **scalar multiplication:** 純量積
- **invertible matrix:** 可逆矩陣
- **inverse matrix:** 反矩陣
- **nonsingular matrix:** 非奇異矩陣
- **transpose matrix:** 轉置矩陣