Linear Algebra

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1.2 Gaussian Elimination and Gauss-Jordan Elimination

- **Determine the size of a matrix .**
- **Write an augmented or coefficient matrix from a system of linear equations.**
- **Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.**
- **Use matrices and Gauss-Jordan elimination to solve a system of linear equations.**
- **Solve a homogeneous system of linear equations.**

- $m \times n$ matrix:

$$\begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} & m \text{ rows} \\ n \text{ columns} & \end{array}$$

- Notes:

(1) Every **entry** a_{ij} in a matrix is a number.

(2) A matrix with m rows and n columns is said to be of **size** $m \times n$.

(3) If $m = n$, then the matrix is called **square of order** $\boldsymbol{n}$.

(4) For a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called **the main diagonal entries**.

- Ex 1: Matrix / Size

Matrix	Size
$\begin{bmatrix} 2 \end{bmatrix}$	1×1
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	2×2
$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$	1×4
$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$	3×2

- Note:

  One very common use of **matrices** is to represent a system of linear equations.

- a system of *m* equations in *n* variables:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \ldots & + & a_{3n}x_n & = & b_3 \\ & & & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

Matrix form: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Augmented matrix:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ & \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid b]$$

- Coefficient matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = A$$

- Elementary row operation:

(1) Interchange two rows. $r_{ij} : R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero constant. $r_i^{(k)} : (k)R_i \rightarrow R_i,\ k \neq 0$

(3) Add a multiple of a row to another row. $r_{ij}^{(k)} : (k)R_i + R_j \rightarrow R_j$

- Row equivalent:

Two matrices are said to be **row equivalent** if one can be obtained from the other by a finite sequence of **elementary row operation**.

- Ex 2: (Elementary row operation)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

- Ex 3: Using elementary row operations to solve a system

Linear System	Associated Augmented Matrix	Elementary Row Operation
$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	$r_{12}^{(1)} : (1)R_1 + R_2 \to R_2$
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$	$r_{13}^{(-2)} : (-2)R_1 + R_3 \to R_3$

Linear System	Associated Augmented Matrix	Elementary Row Operation
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$	$r_{23}^{(1)} : (1)R_2 + R_3 \to R_3$
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$	$r_{3}^{(\frac{1}{2})} : (\frac{1}{2})R_3 \to R_3$

$$\longrightarrow \quad \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

- Row-echelon form: (1, 2, 3)

- Reduced row-echelon form: (1, 2, 3, 4)

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called **a leading 1**).

(3) For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1.

- Ex 4: (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ (row - echelon form)

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row - echelon form)

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ (row - echelon form)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row - echelon form)

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

- Gaussian elimination:

  The procedure for reducing a matrix to a row-echelon form.

- Gauss-Jordan elimination:

  The procedure for reducing a matrix to a reduced row-echelon form.

- Notes:

  (1) Every matrix has a **unique** reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.

  (Different sequences of row operations can produce different row-echelon forms.)

- Ex: (Procedure of Gaussian elimination and Gauss-Jordan elimination)

Produce leading 1

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -6 & 4 & 12 & 28 \\ 2 & 8 & -1 & 4 & -5 & 4 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} 2 & 8 & -6 & 4 & 12 & 28 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -1 & 4 & -5 & 4 \end{bmatrix}$$

The first nonzero column

leading 1

$$\xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -1 & 4 & -5 & 4 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix}$$

Produce leading 1

Zeros elements below leading 1

The first nonzero column

Submatrix

leading 1

$$\xrightarrow{r_2^{(-\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix} \xrightarrow{r_{23}^{(-5)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & \boxed{3} & \boxed{6} \end{bmatrix}$$

Zeros elements below leading 1

Submatrix

Produce leading 1

Zeros elsewhere

$$\xrightarrow{r_3^{(\frac{1}{3})}} \begin{bmatrix} 1 & 4 & \boxed{-3} & 2 & \boxed{6} & 14 \\ 0 & 0 & 1 & 0 & \boxed{-4} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{31}^{(-6)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

leading 1

(row - echelon form) (row - echelon form)

$$\xrightarrow{r_{32}^{(4)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}^{(3)}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(row - echelon form) (reduced row - echelon form)

- Ex 7: Solve a system by Gauss-Jordan elimination method (only one solution)

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\, r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_3^{(\frac{1}{2})}} \underset{\text{(row-echelon form)}}{\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \xrightarrow{r_{21}^{(2)},\, r_{32}^{(-3)},\, r_{31}^{(-9)}} \underset{\text{(reduced row-echelon form)}}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \longrightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

- Ex 8 : Solve a system by Gauss-Jordan elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + \quad 4x_2 - \quad 2x_3 &= \quad 0 \\ 3x_1 + \quad 5x_2 \qquad\qquad &= \quad 1 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\, r_{12}^{(-3)},\, r_2^{(-1)},\, r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$ (reduced row - echelon form)

the corresponding system of equations is

$$\begin{aligned} x_1 \qquad + \quad 5x_3 &= \quad 2 \\ x_2 \quad - 3x_3 &= -1 \end{aligned}$$

leading variable : x_1, x_2

free variable : x_3

$$\begin{aligned} x_1 &= 2 - 5x_3 \\ x_2 &= -1 + 3x_3 \end{aligned}$$

Let $x_3 = t$

$$\begin{aligned} x_1 &= 2 - 5t, \\ x_2 &= -1 + 3t, \qquad t \in R \\ x_3 &= t, \end{aligned}$$

So this system has <u>infinitely many solutions</u>.

- Homogeneous systems of linear equations:

A system of linear equations is said to be **homogeneous** if **all the constant terms are zero**.

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & 0 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & 0 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & 0
\end{array}
$$

- Trivial solution:

$$x_1 = x_2 = x_3 = \cdots = x_n = 0$$

- Nontrivial solution:

  other solutions

- Notes:

(1) Every homogeneous system of linear equations is consistent.

(2) If the homogenous system has fewer equations than variables, then it must have an infinite number of solutions.

(3) For a homogeneous system, exactly one of the following is true.

(a) The system has only the trivial solution.

(b) The system has infinitely many nontrivial solutions in addition to the trivial solution.

- Ex 9: Solve the following homogeneous system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{r_{12}^{(-2)},\, r_2^{(\frac{1}{3})},\, r_{21}^{(1)}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$ (reduced row - echelon form)

leading variable : x_1, x_2

free variable : x_3

Let $x_3 = t$

$$x_1 = -2t,\ x_2 = t,\ x_3 = t,\ t \in R$$

When $t = 0, x_1 = x_2 = x_3 = 0$ (trivial solution)

- **Determine the size of a matrix .**
- **Write an augmented or coefficient matrix from a system of linear equations.**
- **Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.**
- **Use matrices and Gauss-Jordan elimination to solve a system of linear equations.**
- **Solve a homogeneous system of linear equations.**

- matrix: 矩陣
- row: 列
- column: 行
- entry: 元素
- size: 大小
- square matrix: 方陣
- order: 階
- main diagonal: 主對角線
- augmented matrix: 增廣矩陣
- coefficient matrix: 係數矩陣

2.1 Operations with Matrices

Key Learning in Section 2.1

- **Determine whether two matrices are equal.**
- **Add and subtract matrices and multiply a matrix by a scalar.**
- **Multiply two matrices.**
- **Use matrices to solve a system of linear equations.**
- **Partition a matrix and write a linear combination of column vectors.**

- Matrix:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \in M_{m\times n}$$

(i, j)-th entry: a_{ij}

row: m

column: n

size: $m\times n$

- *i*-th row vector:

$$r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

row matrix

- *j*-th column vector:

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

- Square matrix: $m = n$

- Diagonal matrix:

$$A = diag(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n\times n}$$

- Trace:

$$\text{If } A = [a_{ij}]_{n\times n}$$

$$\text{Then } Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\Rightarrow \quad r_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

$$\Rightarrow \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

- Equal matrix:

$$\text{If} \quad A = [a_{ij}]_{m \times n}, \quad B = [b_{ij}]_{m \times n}$$

$$\text{Then } A = B \quad \text{if and only if} \quad a_{ij} = b_{ij} \quad \forall 1 \leq i \leq m,\ 1 \leq j \leq n$$

- Ex 1: (Equal matrix)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{If} \quad A = B$$

$$\text{Then} \quad a = 1, \quad b = 2, \quad c = 3, \quad d = 4$$

- Matrix addition:

$$\text{If } A=[a_{ij}]_{m\times n},\ B=[b_{ij}]_{m\times n}$$

$$\text{Then } A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$$

- Ex 2: (Matrix addition)

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix}=\begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}+\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}=\begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Scalar multiplication:

$$\text{If } A = [a_{ij}]_{m\times n}, \quad c : \text{scalar}$$

$$\text{Then } cA = [ca_{ij}]_{m\times n}$$

- Matrix subtraction:

$$A - B = A + (-1)B$$

- Ex 3: (Scalar multiplication and matrix subtraction)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find (a) $3A$, (b) $-B$, (c) $3A - B$

Sol:

(a)

$$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b)

$$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c)

$$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

- Matrix multiplication:

$$\text{If } A=[a_{ij}]_{m\times n},\ \ B=[b_{ij}]_{n\times p}$$

$$\text{Then } AB=[a_{ij}]_{m\times n}[b_{ij}]_{n\times p}=[c_{ij}]_{m\times p}$$

Size of AB

where $$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \vdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix}=\begin{bmatrix} & & & & & \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ & & & & & \end{bmatrix}$$

- Notes: (1) $A+B = B+A$, (2) $$AB\neq BA$$

- Ex 4: (Find AB)

$$A=\begin{bmatrix}-1 & 3\\ 4 & -2\\ 5 & 0\end{bmatrix} \qquad B=\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}$$

Sol:

$$AB=\begin{bmatrix}(-1)(-3)+(3)(-4) & (-1)(2)+(3)(1)\\ (4)(-3)+(-2)(-4) & (4)(2)+(-2)(1)\\ (5)(-3)+(0)(-4) & (5)(2)+(0)(1)\end{bmatrix}$$

$$=\begin{bmatrix}-9 & 1\\ -4 & 6\\ -15 & 10\end{bmatrix}$$

- Matrix form of a system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

m linear equations

$$\Downarrow$$

$$\underset{\substack{\shortparallel \\ A}}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{\substack{\shortparallel \\ x}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{\substack{\shortparallel \\ b}}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

Single matrix equation

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

- Partitioned matrices:

submatrix

$$A=\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hdashline a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$A=\left[\begin{array}{c:c:c:c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

- Linear combination of column vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m \times 1} = x_1 \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{c_1} + x_2 \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{c_2} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{c_n}$$

- Ex 7: (Solve a system of linear equations)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 7x_2 & + & 8x_3 & = & 6 \end{array}$$

(infinitely many solutions)

$$\Rightarrow A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}, x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, b=\begin{bmatrix}0\\3\\6\end{bmatrix}, c_1=\begin{bmatrix}1\\4\\7\end{bmatrix}, c_2=\begin{bmatrix}2\\5\\8\end{bmatrix}, c_3=\begin{bmatrix}3\\6\\9\end{bmatrix}$$

$$\Rightarrow Ax=\begin{bmatrix}x_1+2x_2+3x_3\\4x_1+5x_2+6x_3\\7x_1+8x_2+9x_3\end{bmatrix}=x_1\begin{bmatrix}1\\4\\7\end{bmatrix}+x_2\begin{bmatrix}2\\5\\8\end{bmatrix}+x_3\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}0\\3\\6\end{bmatrix}=b$$

$$\Rightarrow 1\begin{bmatrix}1\\4\\7\end{bmatrix}+1\begin{bmatrix}2\\5\\8\end{bmatrix}+(-1)\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}0\\3\\6\end{bmatrix} \quad (\text{one solution}: \ x=\begin{bmatrix}1\\1\\-1\end{bmatrix}, \ \text{i.e. } x_1=1, x_2=1, x_3=-1)$$

- **Determine whether two matrices are equal.**
- **Add and subtract matrices and multiply a matrix by a scalar.**
- **Multiply two matrices.**
- **Use matrices to solve a system of linear equations.**
- **Partition a matrix and write a linear combination of column vectors.**

- **row vector:** 列向量
- **column vector:** 行向量
- **diagonal matrix:** 對角矩陣
- **trace:** 跡數
- **equality of matrices:** 相等矩陣
- **matrix addition:** 矩陣相加
- **scalar multiplication:** 純量乘法**(純量積)**
- **matrix subtraction:** 矩陣相減
- **matrix multiplication:** 矩陣乘法
- **partitioned matrix:** 分割矩陣
- **linear combination:** 線性組合