Linear Algebra

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2.4 Elementary Matrices

- **Factor a matrix into a product of elementary matrices.**
- **Find and use an LU-factorization of a matrix to solve a system of linear equations.**

- Row elementary matrix:

  An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

  (1) $R_{ij} = r_{ij}(I)$ Interchange two rows.

  (2) $R_i^{(k)} = r_i^{(k)}(I)$ $(k \neq 0)$ Multiply a row by a nonzero constant.

  (3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ Add a multiple of a row to another row.

- Note:

  Only do a single elementary row operation.

- Ex 1: (Elementary matrices and nonelementary matrices)

$$(a)\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (b)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \qquad (c)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Yes $(r_2^{(3)}(I_3))$

No (not square)

No (Row multiplication must be by a nonzero constant)

$$(d)\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \qquad (e)\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} \qquad (f)\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}$$

Yes $(r_{23}(I_3))$

Yes $(r_{12}^{(2)}(I_2))$

No (Use two elementary row operations)

- **Thm 2.12: (Representing elementary row operations)**

  Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$

$$r(A) = EA$$

- Notes:

$$(1) \quad r_{ij}(A) = R_{ij}A$$

$$(2) \quad r_i^{(k)}(A) = R_i^{(k)}A$$

$$(3) \quad r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$$

- Ex 2: (Elementary matrices and elementary row operation)

$$(a)\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix}(r_{12}(A)=R_{12}A)$$

$$(b)\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}(r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}(r_{12}^{(2)}(A)=R_{12}^{(2)}A)$$

- Ex 3: (Using elementary matrices)

  Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A=\begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Sol:

$$E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} = B$$

row-echelon form

$$\therefore\ B = E_3 E_2 E_1 A \quad \text{or} \quad B = r_3^{(\frac{1}{2})}(r_{13}^{(-2)}(r_{12}(A)))$$

- Row-equivalent:

Matrix B is **row-equivalent** to A if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- Thm 2.13: (Elementary matrices are invertible)

  If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- Notes:

$$(1)\ \ (R_{ij})^{-1} = R_{ij}$$

$$(2)\ \ (R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$$

$$(3)\ \ (R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$$

- Ex:

Elementary Matrix — Inverse Matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \qquad (R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$ (Elementary Matrix)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)} \qquad (R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)}$$ (Elementary Matrix)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})} \qquad (R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)}$$ (Elementary Matrix)

- **Thm 2.14: (A property of invertible matrices)**

  A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Pf:

(1) Assume that A is the product of elementary matrices.

(a) Every elementary matrix is invertible.

(b) The product of invertible matrices is invertible.

Thus A is invertible.

(2) If A is invertible, $A\mathbf{x} = 0$ has only the trivial solution. (Thm. 2.11)

$$\Rightarrow [A \vdots 0] \rightarrow [I \vdots 0]$$

$$\Rightarrow E_k \cdots E_3 E_2 E_1 A = I$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

Thus A can be written as the product of elementary matrices.

- Ex 4:

Find a sequence of elementary matrices whose product is

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}$$

Sol:

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}\xrightarrow{r_1^{(-1)}}\begin{bmatrix}1 & 2\\ 3 & 8\end{bmatrix}\xrightarrow{r_{12}^{(-3)}}\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix}$$

$$\xrightarrow{r_2^{(\frac{1}{2})}}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\xrightarrow{r_{21}^{(-2)}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Therefore $\quad R_{21}^{(-2)}R_2^{(\frac{1}{2})}R_{12}^{(-3)}R_1^{(-1)}A=I$

$$\text{Thus} \quad A = (R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$$

$$= R_1^{(-1)} R_{12}^{(3)} R_2^{(2)} R_{21}^{(2)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Note:

If A is invertible

$$\text{Then} \quad E_k \cdots E_3 E_2 E_1 A = I$$

$$A^{-1} = E_k \cdots E_3 E_2 E_1$$

$$E_k \cdots E_3 E_2 E_1 [A \vdots I] = [I \vdots A^{-1}]$$

- Thm 2.15: (Equivalent conditions)

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n .

(5) A can be written as the product of elementary matrices.

- *LU*-factorization:

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then A=LU is an LU-factorization of A

$$A = LU$$

L is a lower triangular matrix

U is an upper triangular matrix

- Note:

If a square matrix A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is easy to find an LU-factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU$$

- Ex 5: (LU-factorization)

$$(a)\ \ A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \qquad\qquad (b)\ \ A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}\xrightarrow{r_{12}^{(-1)}}\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix}=U$$

$$\Rightarrow R_{12}^{(-1)}A=U$$

$$\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix} \xrightarrow{r_{23}^{(4)}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = U$$

$$\Rightarrow R_{23}^{(4)} R_{13}^{(-2)} A = U$$

$$\Rightarrow A = (R_{13}^{(-2)})^{-1} (R_{23}^{(4)})^{-1} U = LU$$

$$\Rightarrow L = (R_{13}^{(-2)})^{-1} (R_{23}^{(4)})^{-1} = R_{13}^{(2)} R_{23}^{(-4)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

- Solving $Ax=b$ with an LU-factorization of A:

$$Ax = b \qquad \text{If } \; A = LU\text{, then } \; LUx = b$$

$$\text{Let } \; y = Ux\text{, then } \; Ly = b$$

- Two steps:

(1) Write $y = Ux$ and solve $Ly = b$ for y

(2) Solve $Ux = y$ for x

- Ex 7: (Solving a linear system using *LU*-factorization)

$$\begin{aligned} x_1 - 3x_2 &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $y = Ux$, and solve $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20 - 2y_1 + 4y_2 \\ &= -20 - 2(-5) + 4(-1) = -14 \end{aligned}$$

(2) Solve the following system $Ux = y$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So

$$x_3 = -1$$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

- **Factor a matrix into a product of elementary matrices.**
- **Find and use an LU-factorization of a matrix to solve a system of linear equations.**

- **row elementary matrix:** 列基本矩陣
- **row equivalent:** 列等價
- **lower triangular matrix:** 下三角矩陣
- **upper triangular matrix:** 上三角矩陣
- **LU-factorization:** LU-分解