Linear Algebra

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2.2 Properties of Matrix Operations

- **Use the properties of matrix addition, scalar multiplication, and zero matrices.**
- **Use the properties of matrix multiplication and the identity matrix.**
- **Find the transpose of a matrix.**

- Three basic matrix operators:
  (1) matrix addition
  (2) scalar multiplication
  (3) matrix multiplication

- Zero matrix: $0_{m\times n}$

- Identity matrix of order n: I_n

- Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m\times n}$, $\quad c, d$: scalar

Then (1) $A+B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd) A = c (dA)$

(4) $1A = A$

(5) $c(A+B) = cA + cB$

(6) $(c+d) A = cA + dA$

- Properties of zero matrices:

If $A \in M_{m\times n}$, $c : scalar$

Then (1) $A + 0_{m\times n} = A$

(2) $A + (-A) = 0_{m\times n}$

(3) $cA = 0_{m\times n} \Rightarrow c = 0 \text{ or } A = 0_{m\times n}$

- Notes:

(1) $0_{m\times n}$: **the additive identity** for the set of all $m\times n$ matrices

(2) $-A$: **the additive inverse** of A

- Properties of matrix multiplication:

(1) $A(BC) = (AB)C$

(2) $A(B+C) = AB + AC$

(3) $(A+B)C = AC + BC$

(4) $c(AB) = (cA)B = A(cB)$

- Properties of identity matrix:

If $A \in M_{m\times n}$

Then (1) $AI_n = A$

(2) $I_m A = A$

- Transpose of a matrix:

$$\text{If} \quad A = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2n}} \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ \boxed{a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}} \end{bmatrix} \in M_{m\times n}$$

$$\text{Then} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}$$

- Ex 8: (Find the transpose of the following matrix)

$$(a)\quad A=\begin{bmatrix}2\\8\end{bmatrix}\qquad (b)\quad A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\qquad (c)\quad A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}2\\8\end{bmatrix}\quad\Rightarrow A^T=\begin{bmatrix}2&8\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$$

(c)

$$A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}\quad\Rightarrow A^T=\begin{bmatrix}0&2&1\\1&4&-1\end{bmatrix}$$

- Properties of transposes:

$$(1)\ \ (A^T)^T = A$$

$$(2)\ \ (A+B)^T = A^T + B^T$$

$$(3)\ \ (cA)^T = c(A^T)$$

$$(4)\ \ (AB)^T = B^T A^T$$

- Symmetric matrix:

  A square matrix A is **symmetric** if $A = A^T$

- Skew-symmetric matrix:

  A square matrix A is **skew-symmetric** if $A^T = -A$

- Ex:

  If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c?

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix} \quad \begin{array}{l} A = A^T \\ \Rightarrow \; a = 2, b = 3, c = 5 \end{array}$$

- Ex:

If $A=\begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric, find a, b, c?

Sol:

$$A=\begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix} \qquad -A^T=\begin{bmatrix} 0 & -a & -b \\ -1 & 0 & -c \\ -2 & -3 & 0 \end{bmatrix}$$

$$A=-A^T \quad \Rightarrow \quad a=-1, b=-2, c=-3$$

- Note: AA^T is symmetric

Pf: $$(AA^T)^T=(A^T)^T A^T=AA^T$$

$$\therefore AA^T \text{ is symmetric}$$

- Real number:

$ab = ba$ (Commutative law for multiplication)

- Matrix:

$$\underset{m\times n}{A}\underset{n\times p}{B} \neq BA$$

Three situations:

(1) If $m \neq p$, then AB is defined, BA is undefined.

(2) If $m = p, m \neq n$, then $AB \in M_{m\times m}$, $BA \in M_{n\times n}$ (Sizes are not the same)

(3) If $m = p = n$, then $AB \in M_{m\times m}$, $BA \in M_{m\times m}$

(Sizes are the same, but matrices are not equal)

- Ex 4:

Sow that AB and BA are not equal for the matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

- Note: $AB \neq BA$

- Real number:

$$ac = bc, \;\; c \neq 0$$

$$\Rightarrow \;\; a = b$$

(Cancellation law)

- Matrix:

$$AC = BC \qquad C \neq 0$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Ex 5: (An example in which cancellation is not valid)

Show that $AC=BC$

$$A=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix},\quad C=\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}$$

Sol:

$$AC=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

So $\quad AC=BC$

But $\quad A\neq B$

- **Use the properties of matrix addition, scalar multiplication, and zero matrices.**
- **Use the properties of matrix multiplication and the identity matrix.**
- **Find the transpose of a matrix.**

- **zero matrix:** 零矩陣
- **identity matrix:** 單位矩陣
- **transpose matrix:** 轉置矩陣
- **symmetric matrix:** 對稱矩陣
- **skew-symmetric matrix:** 反對稱矩陣

2.3 The Inverse of a Matrix

- **Find the inverse of a matrix (if it exists).**
- **Use properties of inverse matrices.**
- **Use an inverse matrix to solve a system of linear equations.**

- Inverse matrix:

Consider $A \in M_{n \times n}$

If there exists a matrix $B \in M_{n \times n}$ such that $AB = BA = I_n$,

Then (1) A is **invertible** (or **nonsingular**)

(2) B is **the inverse** of A

- Note:

A matrix that does not have an inverse is called **noninvertible** (or **singular**).

- Thm 2.7: (The inverse of a matrix is unique)

If B and C are both inverses of the matrix A, then $B = C$.

Pf:

$$AB = I$$

$$C(AB) = CI$$

$$(CA)B = C$$

$$IB = C$$

$$B = C$$

Consequently, the inverse of a matrix is unique.

- Notes:

(1) The inverse of A is denoted by A^{-1}

(2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan Elimination}} [I \mid A^{-1}]$$

- Ex 2: (Find the inverse of the matrix)

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

Sol:

$$AX = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11}+4x_{21} & x_{12}+4x_{22} \\ -x_{11}-3x_{21} & -x_{12}-3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rcrcl} x_{11} & + & 4x_{21} & = & 1 \\ -x_{11} & - & 3x_{21} & = & 0 \end{array} \quad (1)$$

$$\begin{array}{rcrcl} x_{12} & + & 4x_{22} & = & 0 \\ -x_{12} & - & 3x_{22} & = & 1 \end{array} \quad (2)$$

$$(1) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3,\ x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4,\ x_{22} = 1$$

Thus

$$X = A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \ (AX = I = AA^{-1})$$

- Note:

$$\underset{A \qquad\qquad I}{\left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]} \xrightarrow[r_{12}^{(1)},\, r_{21}^{(-4)}]{\text{Gauss–Jordan Elimination}} \underset{I \qquad\qquad A^{-1}}{\left[\begin{array}{cc:cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]}$$

If A can't be row reduced to I, then A is singular.

- Ex 3: (Find the inverse of the following matrix)

$$A=\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Sol:

$$[A \;\vdots\; I]=\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{12}^{(-1)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]\xrightarrow{r_{13}^{(6)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]\xrightarrow{r_{3}^{(-1)}}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -1 & -4 & -1 \end{array}\right]$$

$$= [\, I \;\vdots\; A^{-1} \,]$$

So the matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

- Check:

$$AA^{-1} = A^{-1}A = I$$

- Power of a square matrix:

$$(1)A^0 = I$$

$$(2)A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$$

$$(3)A^r \cdot A^s = A^{r+s} \qquad r, s : \text{integers}$$

$$(A^r)^s = A^{rs}$$

$$(4)D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- Thm 2.8: (Properties of inverse matrices)

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

$$(1)\ A^{-1} \text{ is invertible and } (A^{-1})^{-1} = A$$

$$(2)\ A^{k} \text{ is invertible and } (A^{k})^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^{k} = A^{-k}$$

$$(3)\ \mathrm{c}A \text{ is invertible and } (cA)^{-1} = \frac{1}{c}A^{-1},\ c \neq 0$$

$$(4)\ A^{T} \text{ is invertible and } (A^{T})^{-1} = (A^{-1})^{T}$$

- Thm 2.9: (The inverse of a product)

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I$$

If AB is invertible, then its inverse is unique.

So $(AB)^{-1} = B^{-1}A^{-1}$

- Note:

$$(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- Thm 2.10: (Cancellation properties)

  If C is an invertible matrix, then the following properties hold:

  (1) If $AC=BC$, then $A=B$ (Right cancellation property)

  (2) If $CA=CB$, then $A=B$ (Left cancellation property)

Pf:

$$AC = BC$$

$$(AC)C^{-1} = (BC)C^{-1} \qquad \text{(C is invertible, so } C^{-1} \text{ exists)}$$

$$A(CC^{-1}) = B(CC^{-1})$$

$$AI = BI$$

$$A = B$$

- Note:

  If C is not invertible, then cancellation is not valid.

- Thm 2.11: (Systems of equations with unique solutions)

  If A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution given by

$$x = A^{-1}b$$

Pf:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \qquad (A \text{ is nonsingular})$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

If x_1 and x_2 were two solutions of equation $Ax = b$.

then $Ax_1 = b = Ax_2 \Rightarrow x_1 = x_2$ (Left cancellation property)

This solution is unique.

- Note:

For square systems (those having the same number of equations as variables), Theorem 2.11 can be used to determine whether the system has a unique solution.

- Note:

$Ax = b$ (A is an invertible matrix)

$$[A \mid b] \xrightarrow{A^{-1}} [A^{-1}A \mid A^{-1}b] = [I \mid A^{-1}b]$$

$$[A \mid b_1 \mid b_2 \mid \cdots \mid b_n] \xrightarrow{A^{-1}} [I \mid A^{-1}b_1 \mid \cdots \mid A^{-1}b_n]$$

- **Find the inverse of a matrix (if it exists).**
- **Use properties of inverse matrices.**
- **Use an inverse matrix to solve a system of linear equations.**

- **inverse matrix:** 反矩陣
- **invertible:** 可逆
- **nonsingular:** 非奇異
- **noninvertible:** 不可逆
- **singular:** 奇異
- **power:** 冪次