112-1 Introduction to AI Linear Regression

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1. 假設有個資料點 (x_1, y_1), (x_2, y_2), ..., (x_n, y_n)，將課堂中利用最小平方法(Least Squared Method)以求得簡單線性迴歸(Simple Linear Regression)公式 $f(x) = ax + b$ 中的係數(斜率)與(截距)，從已知的資料點來計算與的值。

$$let\ \hat{y} = f(x) = ax + b$$

$Distance:$

$$\begin{aligned} d_n &= \hat{y}_n - y_n \\ &= f(x_n) - y_n \\ &= ax_n + b - y_n \end{aligned}$$

$Loss\ function:$

$$E = \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

$Least\ Squared\ Method:$

$$find\ \frac{\partial E}{\partial a} = 0\ and\ \frac{\partial E}{\partial b} = 0$$

$$\begin{aligned} \frac{\partial E}{\partial a} &= \frac{\partial}{\partial a}\left(\textstyle\sum_{i=1}^{n} (ax_i + b - y_i)^2\right) &= 0 \\ &= \textstyle\sum_{i=1}^{n} \frac{\partial}{\partial a}(ax_i + b - y_i)^2 &= 0 \\ &= \textstyle\sum_{i=1}^{n} 2(ax_i + b - y_i) \cdot x_i &= 0 \\ &= \textstyle\sum_{i=1}^{n} (ax_i + b - y_i) \cdot x_i &= 0 \\ &= a\textstyle\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i y_i &= 0 \\ &= a\textstyle\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i &= \textstyle\sum_{i=1}^{n} x_i y_i \end{aligned}$$

$$\begin{aligned} \frac{\partial E}{\partial b} &= \frac{\partial}{\partial b}\left(\textstyle\sum_{i=1}^{n} (ax_i + b - y_i)^2\right) &= 0 \\ &= \textstyle\sum_{i=1}^{n} \frac{\partial}{\partial b}(ax_i + b - y_i)^2 &= 0 \\ &= \textstyle\sum_{i=1}^{n} 2(ax_i + b - y_i) \cdot 1 &= 0 \\ &= \textstyle\sum_{i=1}^{n} (ax_i + b - y_i) &= 0 \\ &= a\textstyle\sum_{i=1}^{n} x_i + b \cdot n - \sum_{i=1}^{n} y_i &= 0 \\ &= a\textstyle\sum_{i=1}^{n} x_i + b \cdot n &= \textstyle\sum_{i=1}^{n} y_i \end{aligned}$$

$$Let\ Sx = \sum_{i=1}^{n} x_i\ ,\ Sxx = \sum_{i=1}^{n} x_i^2\ ,\ Sy = \sum_{i=1}^{n} y_i\ ,\ Sxy = \sum_{i=1}^{n} x_i y_i$$

$$\begin{cases} a \cdot Sxx + b \cdot Sx &= Sxy \\ a \cdot Sx + b \cdot n &= Sy \end{cases}$$

$$\begin{array}{rr} & a \cdot Sxx \cdot n + b \cdot Sx \cdot n = Sxy \cdot n \\ -) & a \cdot Sx \cdot Sx + b \cdot n \cdot Sx = Sy \cdot Sx \\ \hline \end{array}$$

$$a(Sxx \cdot n - Sx \cdot Sx) = Sxy \cdot n - Sy \cdot Sx$$

$$\begin{array}{rr} & a \cdot Sxx \cdot Sx + b \cdot Sx \cdot Sx = Sxy \cdot Sx \\ -) & a \cdot Sx \cdot Sxx + b \cdot n \cdot Sxx = Sy \cdot Sxx \\ \hline \end{array}$$

$$b(Sx \cdot Sx - n \cdot Sxx) = Sxy \cdot Sx - Sy \cdot Sxx$$

$$\begin{aligned} a &= \frac{Sxy \cdot n - Sy \cdot Sx}{Sxx \cdot n - Sx \cdot Sx} \\ &= \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} y_i \cdot \sum_{i=1}^{n} x_i}{n\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} x_i} \end{aligned}$$

$$\begin{aligned} b &= \frac{Sxy \cdot Sx - Sy \cdot Sxx}{(Sx \cdot Sx - n \cdot Sxx)} \\ &= \frac{\sum_{i=1}^{n} x_i y_i \cdot \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i \cdot \sum_{i=1}^{n} x_i^2}{\left(\sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} x_i - n \cdot \sum_{i=1}^{n} x_i^2\right)} \end{aligned}$$

2.已知個資料點如上題，改採用非線性迴歸的拋物線，其公式 $f(x) = ax^2 + bx + c$。試嘗試利用相同的最小平方法來求得 a, b, c 的係數值。

$let\ \hat{y} = f(x) = ax^2 + bx + c$

$Distance:$

$$\begin{aligned} d_n &= \hat{y}_n - y_n \\ &= f(x_n) - y_n \\ &= ax_n^2 + bx_n + c - y_n \end{aligned}$$

$Loss\ function:$

$$E = \sum_{i=1}^{n} \left(ax_i^2 + bx_i + c - y_i\right)^2$$

$Least\ Squared\ Method:$

$$find\ \frac{\partial E}{\partial a} = 0,\ \frac{\partial E}{\partial b} = 0,\ \frac{\partial E}{\partial c} = 0$$

$$
\begin{aligned}
\frac{\partial E}{\partial a} &= \frac{\partial}{\partial a}\left(\sum\nolimits_{i=1}^{n}\left(ax_i^2+bx_i+c-y_i\right)^2\right) &&= 0\\
&= \sum\nolimits_{i=1}^{n}\frac{\partial}{\partial a}\left(ax_i^2+bx_i+c-y_i\right)^2 &&= 0\\
&= \sum\nolimits_{i=1}^{n}2\left(ax_i^2+bx_i+c-y_i\right)\cdot x_i^2 &&= 0\\
&= \sum\nolimits_{i=1}^{n}\left(ax_i^2+bx_i+c-y_i\right)\cdot x_i^2 &&= 0\\
&= a\sum\nolimits_{i=1}^{n}x_i^4+b\sum\nolimits_{i=1}^{n}x_i^3+c\sum\nolimits_{i=1}^{n}x_i^2-\sum\nolimits_{i=1}^{n}x_i^2y_i &&= 0\\
&= a\sum\nolimits_{i=1}^{n}x_i^4+b\sum\nolimits_{i=1}^{n}x_i^3+c\sum\nolimits_{i=1}^{n}x_i^2 &&= \sum\nolimits_{i=1}^{n}x_i^2y_i
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial b} &= \frac{\partial}{\partial b}\left(\sum\nolimits_{i=1}^{n}\left(ax_i^2+bx_i+c-y_i\right)^2\right) &&= 0\\
&= \sum\nolimits_{i=1}^{n}\frac{\partial}{\partial b}\left(ax_i^2+bx_i+c-y_i\right)^2 &&= 0\\
&= \sum\nolimits_{i=1}^{n}2\left(ax_i^2+bx_i+c-y_i\right)\cdot x_i &&= 0\\
&= \sum\nolimits_{i=1}^{n}\left(ax_i^2+bx_i+c-y_i\right)\cdot x_i &&= 0\\
&= a\sum\nolimits_{i=1}^{n}x_i^3+b\sum\nolimits_{i=1}^{n}x_i^2+c\sum\nolimits_{i=1}^{n}x_i-\sum\nolimits_{i=1}^{n}x_iy_i &&= 0\\
&= a\sum\nolimits_{i=1}^{n}x_i^3+b\sum\nolimits_{i=1}^{n}x_i^2+c\sum\nolimits_{i=1}^{n}x_i &&= \sum\nolimits_{i=1}^{n}x_iy_i
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial c} &= \frac{\partial E}{\partial c}\left(\sum\nolimits_{i=1}^{n}\left(ax_i^2+bx_i+c-y_i\right)^2\right) &&= 0\\
&= \sum\nolimits_{i=1}^{n}\frac{\partial E}{\partial c}\left(ax_i^2+bx_i+c-y_i\right)^2 &&= 0\\
&= \sum\nolimits_{i=1}^{n}2\left(ax_i^2+bx_i+c-y_i\right)\cdot 1 &&= 0\\
&= \sum\nolimits_{i=1}^{n}\left(ax_i^2+bx_i+c-y_i\right) &&= 0\\
&= a\sum\nolimits_{i=1}^{n}x_i^2+b\sum\nolimits_{i=1}^{n}x_i+c\cdot n-\sum\nolimits_{i=1}^{n}x_iy_i &&= 0\\
&= a\sum\nolimits_{i=1}^{n}x_i^2+b\sum\nolimits_{i=1}^{n}x_i+c\cdot n &&= \sum\nolimits_{i=1}^{n}y_i
\end{aligned}
$$

$Let\ Sx=\sum_{i=1}^{n}x_i\ ,\ Sx^2=\sum_{i=1}^{n}x_i^2\ ,\ Sy=\sum_{i=1}^{n}y_i\ ,\ Sxy=\sum_{i=1}^{n}x_iy_i,$

$Sx^4=\sum_{i=1}^{n}x_i^4,\ \ Sx^3=\sum_{i=1}^{n}x_i^3,\ Sx^2y=\sum_{i=1}^{n}x_i^2y_i$

$$
\begin{cases}
a\cdot Sx^4+b\cdot Sx^3+c\cdot Sx^2 = Sx^2y & (1)\\
a\cdot Sx^3+b\cdot Sx^2+c\cdot Sx = Sxy & (2)\\
a\cdot Sx^2+b\cdot Sx+c\cdot n = Sy & (3)
\end{cases}
$$

$$
\begin{array}{c}
\begin{array}{ccc} a & b & c \end{array}\\
\left[\begin{array}{ccc|c}
Sx^4 & Sx^3 & Sx^2 & Sx^2y\\
Sx^3 & Sx^2 & Sx & Sxy\\
Sx^2 & Sx & n & Sy
\end{array}\right]
\end{array}
=
\left[\begin{array}{ccc|c}
Sx^4\cdot Sx^3\cdot Sx^2 & Sx^3\cdot Sx^3\cdot Sx^2 & Sx^2\cdot Sx^3\cdot Sx^2 & Sx^2y\cdot Sx^3\cdot Sx^2\\
Sx^3\cdot Sx^4\cdot Sx^2 & Sx^2\cdot Sx^4\cdot Sx^2 & Sx\cdot Sx^4\cdot Sx^2 & Sxy\cdot Sx^4\cdot Sx^2\\
Sx^2\cdot Sx^3\cdot Sx^4 & Sx\cdot Sx^3\cdot Sx^4 & n\cdot Sx^3\cdot Sx^4 & Sy\cdot Sx^3\cdot Sx^4
\end{array}\right]
$$

$$
=\left[\begin{array}{ccc|c}
Sx^4\cdot Sx^3\cdot Sx^2 & Sx^3\cdot Sx^3\cdot Sx^2 & Sx^2\cdot Sx^3\cdot Sx^2 & Sx^2y\cdot Sx^3\cdot Sx^2\\
0 & Sx^2\cdot Sx^4\cdot Sx^2-Sx^3\cdot Sx^3\cdot Sx^2 & Sx\cdot Sx^4\cdot Sx^2-Sx^2\cdot Sx^3\cdot Sx^2 & Sxy\cdot Sx^4\cdot Sx^2-Sx^2y\cdot Sx^3\cdot Sx^2\\
0 & Sx\cdot Sx^3\cdot Sx^4-Sx^3\cdot Sx^3\cdot Sx^2 & n\cdot Sx^3\cdot Sx^4-Sx^2\cdot Sx^3\cdot Sx^2 & Sy\cdot Sx^3\cdot Sx^4-Sx^2y\cdot Sx^3\cdot Sx^2
\end{array}\right]
$$

$$
=\left[\begin{array}{ccc|c}
Sx^4\cdot Sx^3\cdot Sx^2 & Sx^3\cdot Sx^3\cdot Sx^2 & Sx^2\cdot Sx^3\cdot Sx^2 & Sx^2y\cdot Sx^3\cdot Sx^2\\
0 & Sx^2\cdot Sx^4\cdot Sx^2-Sx^3\cdot Sx^3\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4 & Sx\cdot Sx^4\cdot Sx^2-Sx^2\cdot Sx^3\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4 & Sxy\cdot Sx^4\cdot Sx^2-Sx^2y\cdot Sx^3\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4\\
0 & Sx\cdot Sx^3\cdot Sx^4-Sx^3\cdot Sx^3\cdot Sx^2+Sx^2\cdot Sx^4\cdot Sx^2 & n\cdot Sx^3\cdot Sx^4-Sx^2\cdot Sx^3\cdot Sx^2+Sx^2\cdot Sx^4\cdot Sx^2 & Sy\cdot Sx^3\cdot Sx^4-Sx^2y\cdot Sx^3\cdot Sx^2+Sx^2\cdot Sx^4\cdot Sx^2
\end{array}\right]
$$

$$
=\left[\begin{array}{ccc|c}
Sx^4\cdot Sx^3\cdot Sx^2 & Sx^3\cdot Sx^3\cdot Sx^2 & Sx^2\cdot Sx^3\cdot Sx^2 & Sx^2y\cdot Sx^3\cdot Sx^2\\
0 & Sx^2\cdot Sx^4\cdot Sx^2-Sx^3\cdot Sx^3\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4 & Sx\cdot Sx^4\cdot Sx^2-Sx^2\cdot Sx^3\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4 & Sxy\cdot Sx^4\cdot Sx^2-Sx^2y\cdot Sx^3\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4\\
0 & 0 & n\cdot Sx^3\cdot Sx^4+Sx^2\cdot Sx^4\cdot Sx^2-\left(Sx\cdot Sx^4\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4\right) & Sy\cdot Sx^3\cdot Sx^4+Sx^2\cdot Sx^4\cdot Sx^2-\left(Sxy\cdot Sx^4\cdot Sx^2+Sx\cdot Sx^3\cdot Sx^4\right)
\end{array}\right]
\begin{array}{l} \\ (4)\\ (5)\end{array}
$$

by (5)

$$c = \frac{Sy \cdot Sx^3 \cdot Sx^4 + Sx^2 \cdot Sx^4 \cdot Sx^2 - \left(Sxy \cdot Sx^4 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right)}{n \cdot Sx^3 \cdot Sx^4 + Sx^2 \cdot Sx^4 \cdot Sx^2 - \left(Sx \cdot Sx^4 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right)}$$

$$= \frac{Sy \cdot Sx^3 \cdot Sx^4 - \left(Sxy \cdot Sx^4 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right)}{n \cdot Sx^3 \cdot Sx^4 - \left(Sx \cdot Sx^4 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right)}$$

$$= \frac{Sy \cdot Sx^3 \cdot Sx^4 - Sxy \cdot Sx^4 \cdot Sx^2 - Sx \cdot Sx^3 \cdot Sx^4}{n \cdot Sx^3 \cdot Sx^4 - Sx \cdot Sx^4 \cdot Sx^2 - Sx \cdot Sx^3 \cdot Sx^4}$$

$$= \frac{Sy \cdot Sx^3 \cdot Sx^4 - Sxy \cdot Sx^4 \cdot Sx^2}{n \cdot Sx^3 \cdot Sx^4 - Sx \cdot Sx^4 \cdot Sx^2}$$

$$= \frac{Sy \cdot Sx^3 \cdot Sx^4 - Sxy \cdot Sx^4 \cdot Sx^2}{n \cdot Sx^3 \cdot Sx^4 - Sx \cdot Sx^4 \cdot Sx^2}$$

by (4)

$$b\left(Sx^2 \cdot Sx^4 \cdot Sx^2 - Sx^3 \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right) + c\left(Sx \cdot Sx^4 \cdot Sx^2 - Sx^2 \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right) = Sxy \cdot Sx^4 \cdot Sx^2 - Sx^2y \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4$$

$$b\left(Sx^2 \cdot Sx^4 \cdot Sx^2 - Sx^3 \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right) = Sxy \cdot Sx^4 \cdot Sx^2 - Sx^2y \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4 - c\left(Sx \cdot Sx^4 \cdot Sx^2 - Sx^2 \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right)$$

$$b = \frac{Sxy \cdot Sx^4 \cdot Sx^2 - Sx^2y \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4 - c\left(Sx \cdot Sx^4 \cdot Sx^2 - Sx^2 \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right)}{\left(Sx^2 \cdot Sx^4 \cdot Sx^2 - Sx^3 \cdot Sx^3 \cdot Sx^2 + Sx \cdot Sx^3 \cdot Sx^4\right)}$$

by (3)

$$a \cdot Sx^2 + b \cdot Sx + c \cdot n = Sy$$

$$a \cdot Sx^2 = Sy - c \cdot n - b \cdot Sx$$

$$a = \frac{Sy - c \cdot n - b \cdot Sx}{Sx^2}$$