Linear Algebra - AI1B

Orientation

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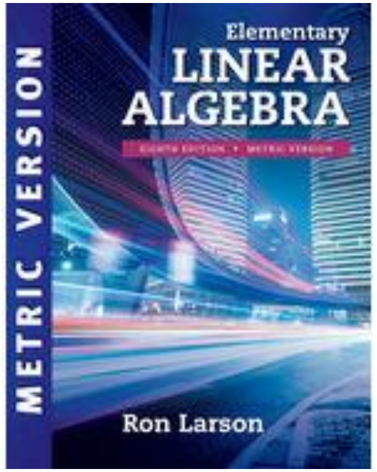
- **Lectures**
  - Tuesday: 11:00 - 12:00
  - Thursday: 08:10 - 10:00
- **Book**
  - Elementary Linear Algebra
- **Grading Policy**
  - Attendance: 20%
  - Assignments: 20%
  - Midterm Exam: 30%
  - Final Exam: 30%
- **Assignments: 7**
  - Some questions of Review Exercises in each chapter
- **Bonus score**
  - Question & Answer
- **Exam(Thursday)**
  - Midterm Exam: **18th of April at 8:10 - 10:00 am**
  - Final Exam: **13th of June at 8:10 - 10:00 am**



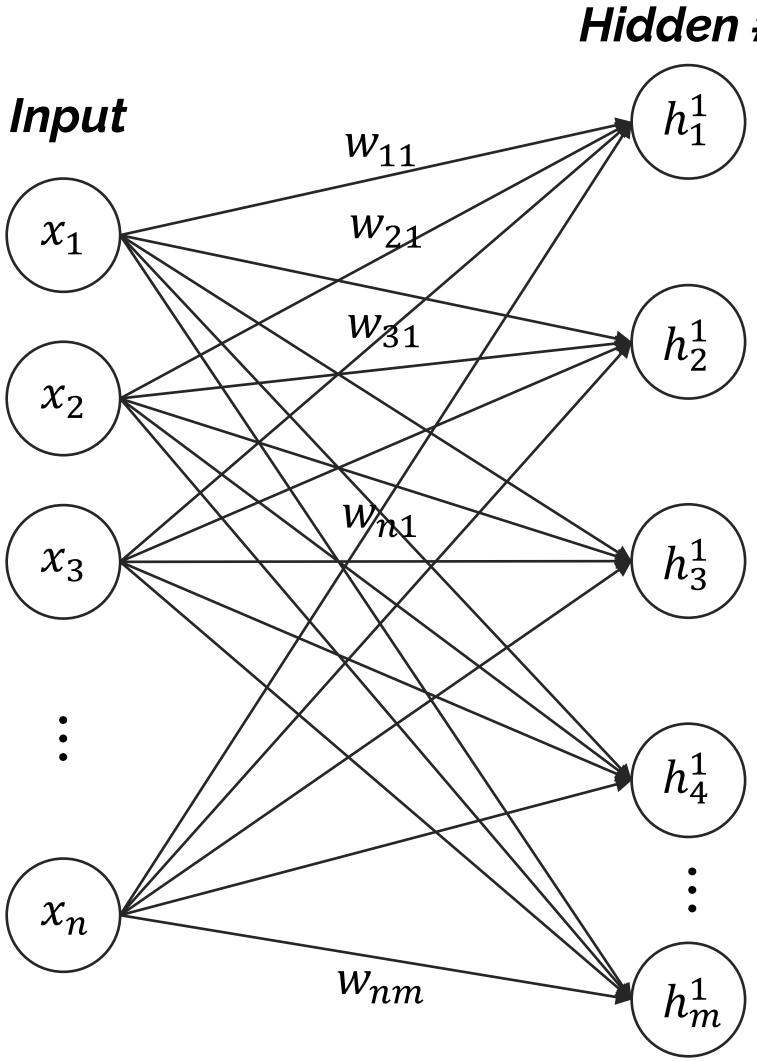
Why Learn Linear Algebra

Why Linear Algebra for AI……?

- **We are learning AI(ML, DL, RL)! Linear Algebra??**
  - Out of the blue!!
  - Suddenly what?!
- **It is enough just to be good Python, right?**
  - Nope!
- **Okay…. I can use AI framework, it is enough, right??**
  - Nope!!
- **Okay…….. I know probability theory, it is enough, right???**
  - Nope!!!

- **Okay………… I think we only need to know a few important formulas for Linear Algebra, right????**
  - Nope!!!!

$bias = \; b_1$



$$h_1^1 = w_{11}x_1 + w_{21}x_2 + w_{31}x_3 + \;\cdots\; + w_{n1}x_n + b_1$$

$$h_2^1 = w_{12}x_1 + w_{22}x_2 + w_{32}x_3 + \;\cdots\; + w_{n2}x_n + b_1$$

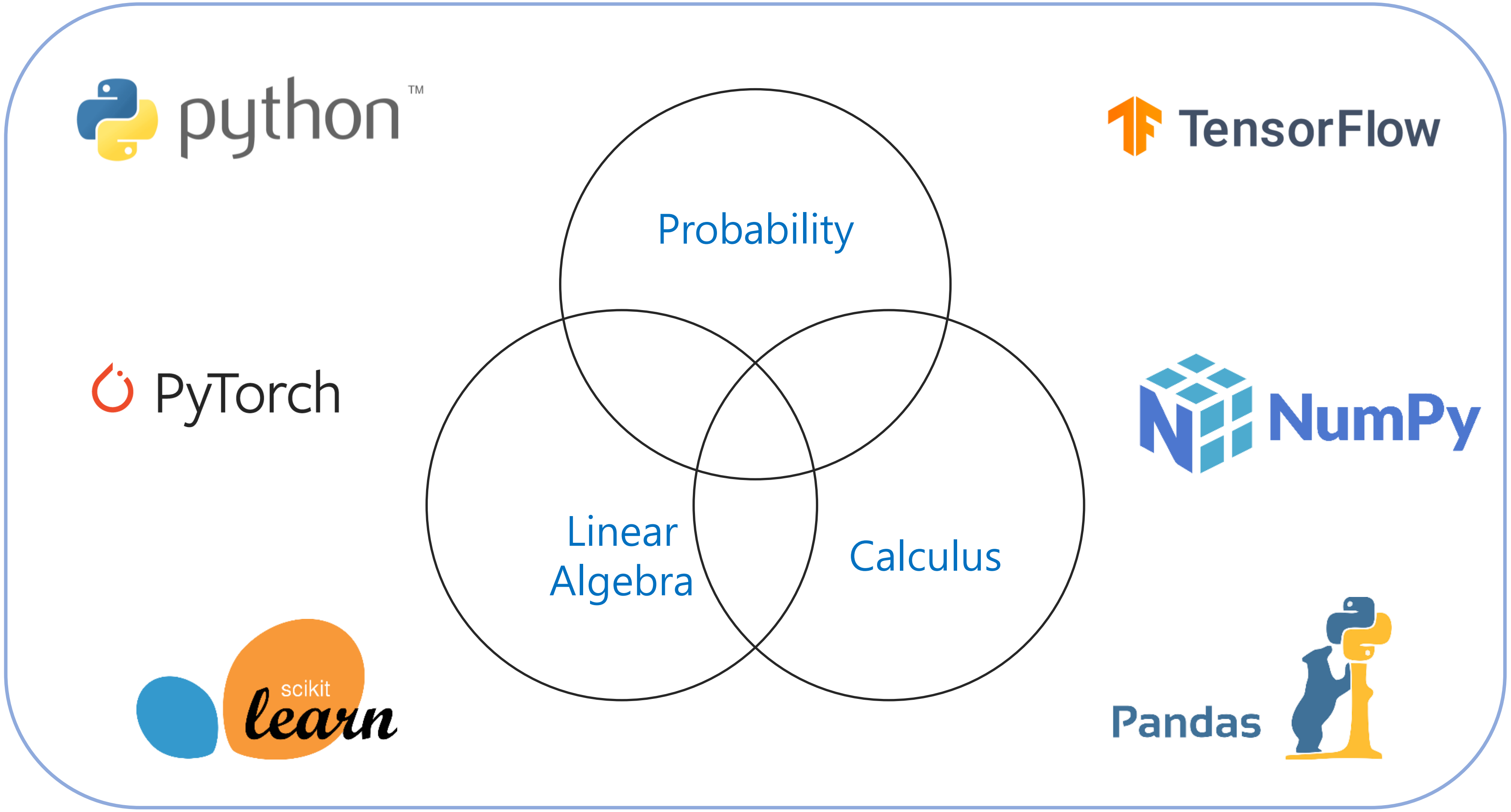
$$h_3^1 = w_{13}x_1 + w_{23}x_2 + w_{33}x_3 + \;\cdots\; + w_{n3}x_n + b_1$$

$$h_4^1 = w_{14}x_1 + w_{24}x_2 + w_{34}x_3 + \;\cdots\; + w_{n4}x_n + b_1$$

$$\vdots$$

$$h_m^1 = w_{1m}x_1 + w_{2m}x_2 + w_{3m}x_3 + \;\cdots\; + w_{nm}x_n + b_1$$

Machine Learning (Deep Learning)



1.1 Introduction to Systems of Linear Equations

Key Learning in Section 1.1

- **Recognize a linear equation in n variables.**
- **Find a parametric representation of a solution set.**
- **Determine whether a system of linear equations is consistent or inconsistent.**
- **Use back-substitution and Gaussian elimination to solve a system of linear equations.**

- a linear equation in *n* variables:

$$a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n=b$$

$a_1, a_2, a_3, \ldots, a_n, b$: real number

a_1: leading coefficient

x_1: leading variable

- Notes:
  - Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.
  - Variables appear only to the first power.

- Ex 1: (Linear or Nonlinear)

Linear $(a)\ 3x+2y=7$

$(b)\ \frac{1}{2}x+y-\pi z=\sqrt{2}$ Linear

Linear $(c)\ x_1-2x_2+10x_3+x_4=0$

$(d)\ (\sin\frac{\pi}{2})x_1-4x_2=e^2$ Linear

Exponential

Nonlinear $(e)\ xy+z=2$

not the first power

$(f)\ e^x-2y=4$ Nonlinear

Nonlinear $(g)\ \sin x_1+2x_2-3x_3=0$

trigonometric functions

$(h)\ \frac{1}{x}+\frac{1}{y}=4$ Nonlinear

not the first power

- **<u>a solution</u>** of a linear equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

$$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3,\ \cdots,\ x_n = s_n$$

such that $a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$

- Solution set:
  - the set of all solutions of a linear equation

- Ex 2： (Parametric representation of a solution set)

$$x_1 + 2x_2 = 4$$

a solution: (2, 1), i.e. $x_1 = 2, x_2 = 1$

If you solve for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2,$$

By letting $x_2 = t$ you can represent the solution set as

$$x_1 = 4 - 2t$$

And the solutions are $\{(4-2t, t) \mid t \in R\}$ or $\{(s, 2-\tfrac{1}{2}s) \mid s \in R\}$

- a system of m linear equations in n variables:
  - A set of m equations, which is linear in the same n variables

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

- Consistent:

  A system of linear equations has at least one solution.

- Inconsistent:

  A system of linear equations has no solution.

- Notes:

Every system of linear equations has either

(1) **exactly one** solution,

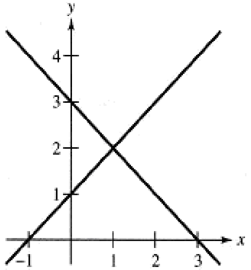
(2) **infinitely many** solutions, or

(3) **no** solution.

- Ex 4: (Solution of a system of linear equations)

(1) $$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$

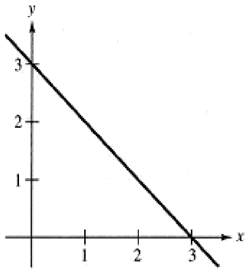
two intersecting lines



exactly one solution

(2) $$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

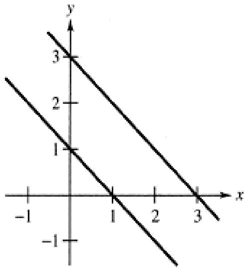
two coincident lines



infinitely many solutions

(3) $$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

two parallel lines



no solution

- Row-Echelon Form
  - The system on the right is clearly easier to solve.
  - To solve such a system, use back-substitution.

$$
\begin{aligned}
x - 2y + 3z &= 9 \\
-x + 3y &= 9 \\
2x - 5y + 5z &= 17
\end{aligned}
\qquad\qquad
\begin{aligned}
x - 2y + 3z &= 9 \\
y + 3z &= 5 \\
z &= 9
\end{aligned}
$$

- Ex 5: (Using back substitution to solve a system in row echelon form)

$$
\begin{aligned}
x - 2y &= 5 \qquad (1) \\
y &= -2 \qquad (2)
\end{aligned}
$$

Sol: By substituting $y = -2$ into (1), you obtain

$$
\begin{aligned}
x - 2(-2) &= 5 \\
x &= 1
\end{aligned}
$$

The system has exactly one solution: $x = 1,\ y = -2$

- Ex 6: (Using back substitution to solve a system in row echelon form)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \quad (1) \\ & & y & + & 3z & = & 5 & \quad (2) \\ & & & & z & = & 2 & \quad (3) \end{array}$$

Sol: Substitute $z = 2$ into (2)

$$\begin{array}{rcrcr} y & + & 3(2) & = & 5 \\ & & y & = & -1 \end{array}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{array}{rcrcrcr} x & - & 2(-1) & + & 3(2) & = & 9 \\ & & & & x & = & 1 \end{array}$$

The system has exactly one solution:

$$x = 1,\ y = -1,\ z = 2$$

- Equivalent:

  Two systems of linear equations are called **equivalent**

  if they have precisely the same solution set.

- Notes:

  Each of the following operations on a system of linear equations produces an equivalent system.

  (1) Interchange two equations.

  (2) Multiply an equation by a nonzero constant.

  (3) Add a multiple of an equation to another equation.

- Ex 7: Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \quad (1) \\ -x & + & 3y & & & = & -4 & \quad (2) \\ 2x & - & 5y & + & 5z & = & 17 & \quad (3) \end{array}$$

Sol: $(1)+(2) \to (2)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \quad (4) \\ 2x & - & 5y & + & 5z & = & 17 & \end{array}$$

$(1)\times(-2)+(3) \to (3)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \\ & & -y & - & z & = & -1 & \quad (5) \end{array}$$

$(4)+(5)\to(5)$

$$\begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \end{array} \qquad (6)$$

$(6)\times\frac{1}{2}\to(6)$

$$\begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$$

So the solution is $x=1,\ y=-1,\ z=2$ (<u>only one solution</u>)

- Ex 8: Solve a system of linear equations (inconsistent system)

$$\begin{array}{rcrcrcrl} x_1 & - & 3x_2 & + & x_3 & = & 1 & \qquad (1) \\ 2x_1 & - & x_2 & - & 2x_3 & = & 2 & \qquad (2) \\ x_1 & + & 2x_2 & - & 3x_3 & = & -1 & \qquad (3) \end{array}$$

Sol: $(1)\times(-2)+(2)\rightarrow(2)$

$(1)\times(-1)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcrl} x_1 & - & 3x_2 & + & x_3 & = & 1 & \\ & & 5x_2 & - & 4x_3 & = & 0 & \qquad (4) \\ & & 5x_2 & - & 4x_3 & = & -2 & \qquad (5) \end{array}$$

$$(4)\times(-1)+(5)\rightarrow(5)$$

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ & & 5x_2 & - & 4x_3 & = & 0 \\ & & & & \boxed{0} & \boxed{=} & \boxed{-2} \end{array} \quad \text{(a false statement)}$$

So the system has no solution (an inconsistent system).

- Ex 9: Solve a system of linear equations (infinitely many solutions)

$$\begin{array}{rcrcrcrl} & & x_2 & - & x_3 & = & 0 & \quad (1) \\ x_1 & & & - & 3x_3 & = & -1 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

Sol: $(1) \leftrightarrow (2)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & \quad (1) \\ & & x_2 & - & x_3 & = & 0 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

$(1)+(3) \to (3)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & \\ & & x_2 & - & x_3 & = & 0 & \\ & & 3x_2 & - & 3x_3 & = & 0 & \quad (4) \end{array}$$

$$\begin{array}{rcrcr} x_1 & & - \ 3x_3 & = & -1 \\ & x_2 & - \ \ x_3 & = & 0 \end{array}$$

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

let $x_3 = t$ Choose x_3 to be the free variable and represent it by *t*

then $x_1 = 3t - 1,$

$x_2 = t, \qquad t \in R$

$x_3 = t,$

So this system has infinitely many solutions.

Key Learning in Section 1.1

- **Recognize a linear equation in n variables.**

- **Find a parametric representation of a solution set.**

- **Determine whether a system of linear equations is consistent or inconsistent.**

- **Use back-substitution and Gaussian elimination to solve a system of linear equations.**

- **linear equation: 線性方程式**
- **system of linear equations: 線性方程式系統**
- **leading coefficient: 領先係數**
- **leading variable: 領先變數**
- **solution: 解**
- **solution set: 解集合**
- **parametric representation: 參數化表示**
- **consistent: 一致性 (有解)**
- **inconsistent: 非一致性 (無解、矛盾)**
- **equivalent: 等價**