

Linear Algebra

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3.1 The Determinant of a Matrix

▲ Key Learning in Section 3.1

- **Find the determinant of a 2×2 matrix.**
- **Find the minors and cofactors of a matrix.**
- **Use expansion by cofactors to find the determinant of a matrix.**
- **Find the determinant of a triangular matrix.**

▲ 3.1 The Determinant of a Matrix

- the determinant of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

▲ 3.1 The Determinant of a Matrix

- Ex. 1: (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$$

$$\begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2\left(\frac{3}{2}\right) = 0 - 3 = -3$$

- **Note:** The determinant of a matrix can be positive, zero, or negative.

▲ 3.1 The Determinant of a Matrix

- Minor of the entry a_{ij}

The determinant of the matrix determined by deleting the i th row and j th column of A

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- Cofactor of a_{ij} :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

▲ 3.1 The Determinant of a Matrix

- Ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

3.1 The Determinant of a Matrix

- **Notes:** Sign pattern for cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

- **Notes:**

Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs.

▲ 3.1 The Determinant of a Matrix

- Ex 2: Find all the minors and cofactors of A .

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Sol: (1) All the minors of A .

$$\Rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5, \quad M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2, \quad M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \quad M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3, \quad M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

Sol: (2) All the cofactors of A.

$$\because C_{ij} = (-1)^{i+j} M_{ij}$$

$$\Rightarrow C_{11} = + \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad C_{12} = - \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5, \quad C_{13} = + \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = + \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \quad C_{23} = - \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad C_{32} = - \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3, \quad C_{33} = + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

▲ 3.1 The Determinant of a Matrix

- Thm 3.1: (Expansion by cofactors)

Let A is a square matrix of order n .

Then the determinant of A is given by

$$(a) \quad \det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

(Cofactor expansion along the i -th row, $i = 1, 2, \dots, n$)

or

$$(b) \quad \det(A) = |A| = \sum_{i=1}^n a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

(Cofactor expansion along the j -th column, $j = 1, 2, \dots, n$)

▲ 3.1 The Determinant of a Matrix

- Ex: The determinant of a matrix of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

▲ 3.1 The Determinant of a Matrix

- Ex 3: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \xRightarrow{\text{Ex2}} C_{11} = -1, C_{12} = 5, C_{13} = 4 \\ C_{21} = -2, C_{22} = -4, C_{23} = 8 \\ C_{31} = 5, C_{32} = 3, C_{33} = -6 \end{array}$$

Sol:

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14 \end{aligned}$$

▲ 3.1 The Determinant of a Matrix

- Ex 5: (The determinant of a matrix of order 3)

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ -4 & 1 \end{vmatrix} = 7 \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (-1)(-5) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix} = -8$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (0)(7) + (2)(5) + (1)(-8) \\ &= 2 \end{aligned}$$

■ Notes:

The row (or column) containing the most zeros is the best choice for expansion by cofactors .

- Ex 4: (The determinant of a matrix of order 4)

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

▲ 3.1 The Determinant of a Matrix

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43})$$

$$= 3C_{13}$$

$$= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3 \left[(0)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right]$$

$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

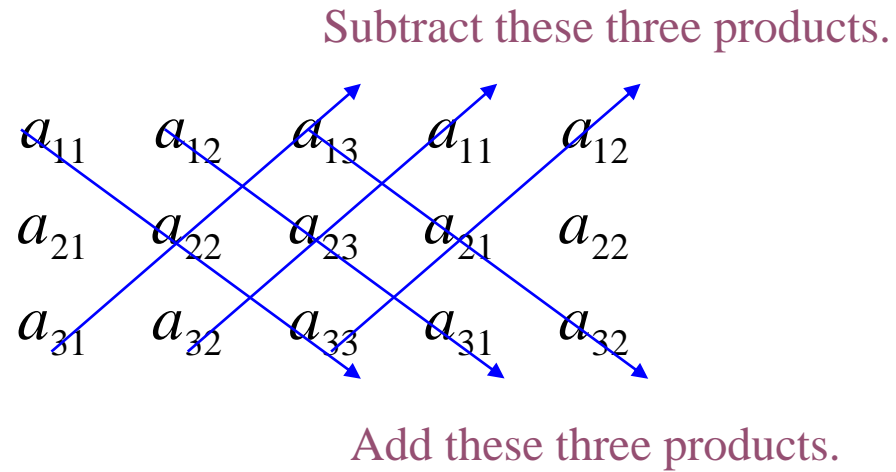
$$= (3)(13)$$

$$= 39$$

▲ 3.1 The Determinant of a Matrix

- The determinant of a matrix of order 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\Rightarrow \det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} \\ - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

▲ 3.1 The Determinant of a Matrix

- Ex 5:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$$

Diagram illustrating the calculation of the determinant of a 3x3 matrix A using Sarrus' rule. The matrix is:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$$

The diagram shows the expansion of the determinant using the first row. The terms are:

- Positive terms (downward arrows): $0 \cdot 16 = 0$, $2 \cdot (-12) = -24$, $1 \cdot 4 = 4$
- Negative terms (upward arrows): $1 \cdot 0 = 0$, $2 \cdot 3 = 6$, $1 \cdot 4 = 4$

The final calculation is:

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4) - 0 - 6 = 2$$

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4) - 0 - 6 = 2$$

▲ 3.1 The Determinant of a Matrix

- Upper triangular matrix:

All the entries below the main diagonal are zeros.

- Lower triangular matrix:

All the entries above the main diagonal are zeros.

- Diagonal matrix:

All the entries above and below the main diagonal are zeros.

- Note:

A matrix that is both upper and lower triangular is called diagonal.

▲ 3.1 The Determinant of a Matrix

■ Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

diagonal

- Thm 3.2: (Determinant of a Triangular Matrix)

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdots a_{nn}$$

- **Ex 6:** Find the determinants of the following triangular matrices.

$$(a) \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

$$(b) \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$(a) \quad |A| = (2)(-2)(1)(3) = -12$$

$$(b) \quad |B| = (-1)(3)(2)(4)(-2) = 48$$

▲ Key Learning in Section 3.1

- **Find the determinant of a 2×2 matrix.**
- **Find the minors and cofactors of a matrix.**
- **Use expansion by cofactors to find the determinant of a matrix.**
- **Find the determinant of a triangular matrix.**

▲ Keywords in Section 2.2

- **determinant** : 行列式
- **minor** : 子行列式
- **cofactor** : 餘因子
- **expansion by cofactors** : 餘因子展開
- **upper triangular matrix**: 上三角矩陣
- **lower triangular matrix**: 下三角矩陣
- **diagonal matrix**: 對角矩陣