112-1 Calculus Final Assignment 2023/12/29

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試著分析在期末考範圍中，積分有哪些題型，並且簡單的描述其解題的策略。

整理公式

Trigonometric Functions

$$
\begin{array}{ccc}
 & \sin \quad \cos & \\
\tan & 1 & \cot \\
 & \sec \quad \csc &
\end{array}
\quad\Bigg|\quad
\begin{aligned}
\csc &= \frac{1}{\sin} \\
\sec &= \frac{1}{\cos} \\
\cot &= \frac{1}{\tan}
\end{aligned}
\quad\Bigg|\quad
\begin{aligned}
\sin^2 &= 1-\cos^2 \\
\cos^2 &= 1-\sin^2 \\
\tan^2 &= \sec^2-1 \\
\csc^2 &= \cot^2+1 \\
\sec^2 &= \tan^2+1 \\
\cot^2 &= \csc^2-1
\end{aligned}
\quad\Bigg|\quad
\begin{aligned}
\sin^2+\cos^2 &= 1 \\
\tan^2+1 &= \sec^2 \\
\cot^2+1 &= \csc^2 \\
\sin^2 x &= \frac{1-\cos 2x}{2} \\
\cos^2 x &= \frac{1+\cos 2x}{2}
\end{aligned}
$$

$$
\begin{aligned}
\sin\left(\cos^{-1}x\right) &= \sqrt{1-x^2} & \sin\left(\sin^{-1}x\right) &= x \\
\cos\left(\sin^{-1}x\right) &= \sqrt{1-x^2} & \cos\left(\cos^{-1}x\right) &= x \\
\sec\left(\tan^{-1}x\right) &= \sqrt{1+x^2} & \sin^2\left(\sin^{-1}x\right) &= x^2 \\
\tan\left(\sec^{-1}x\right) &= \begin{cases} \sqrt{x^2-1} &, x \geqslant 1 \\ -\sqrt{x^2-1} &, x \leqslant -1 \end{cases} & \cos^2\left(\cos^{-1}x\right) &= x^2
\end{aligned}
$$

Standard derivatives

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x & \frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x & \frac{d}{dx}(\csc x) &= -\csc x \cot x
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\left(a^x\right) &= a^x \cdot \ln x & \frac{d}{dx}\left(x^x\right) &= x^x\left(x \cdot \frac{1}{x} + \ln x\right) = x^x(1+\ln x) \\
\frac{d}{dx}\left(e^x\right) &= e^x & \frac{d}{dx}\left(e^{f(x)}\right) &= e^{f(x)} \cdot \frac{d}{dx} f(x) \\
\frac{d}{dx}(\ln x) &= \frac{1}{x} & \frac{d}{dx}\ln(x+a) &= \frac{1}{x+a} \\
\frac{d}{dx}(\ln(ax+b)) &= \frac{a}{(ax+b)} & \frac{d}{dx}(\log_a x) &= \frac{1}{x \ln a} & \left(\log_a x = \frac{\ln x}{\ln a}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\left(\sin^{-1}x\right) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\left(\cos^{-1}x\right) &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\left(\tan^{-1}x\right) &= \frac{1}{1+x^2} & \frac{d}{dx}\left(\cot^{-1}x\right) &= -\frac{1}{1+x^2} \\
\frac{d}{dx}\left(\sec^{-1}x\right) &= \frac{1}{|x|\sqrt{x^2-1}} & \frac{d}{dx}\left(\csc^{-1}x\right) &= -\frac{1}{|x|\sqrt{x^2-1}}
\end{aligned}
$$

Standard Ingegral Forms

Constants, Powers	$1. \int k \cdot du = ku \cdot + C$	$2. \int u^r \cdot du = \begin{cases} \dfrac{u^{r+1}}{r+1} + C & , r \neq -1 \\ \ln\lvert u\rvert + C & , r = -1 \end{cases}$
Exponentials	$3. \int e^u \cdot du = e^u + C$	$4. \int a^u \cdot du = a^u \cdot \dfrac{1}{\ln a} + C,\ a \neq 1,\ a > 0$
Trigonometric Functions	$5. \int \sin u \cdot du = -\cos u + C$	$6. \int \cos u \cdot du = \sin u + C$
	$7. \int \sec^2 u \cdot du = \tan u + C$	$8. \int \csc^2 u \cdot du = -\cot u + C$ (*Remember minus sign*)
	$9. \int \sec u \tan u \cdot du = \sec u + C$	$10. \int \csc u \cot u \cdot du = \csc u + C$ (*Remember minus sign*)
	$11. \int \tan u \cdot du = -\ln\lvert\cos u\rvert + C$	$12. \int \cot u \cdot du = \ln\lvert\sin u\rvert + C \quad \left(\cot u = \dfrac{\cos u}{\sin u}\right)$
	$13. \int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$	$14. \int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
	$15. \int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left(\dfrac{\lvert u\rvert}{a}\right) + C$	$= \dfrac{1}{a}\cos^{-1}\left(\dfrac{a}{\lvert u\rvert}\right) + C$
Hyperbolic Functions	$16. \int \sinh u \cdot du = \cosh u + C$	$17. \int \cosh u \cdot du = \sinh u + C$

Disk Method

$$V = \pi \int_a^b f(x)^2 \cdot dx$$

Shell Method

$$V = 2\pi \int_a^b x \cdot f(x) \cdot dx$$

Arc Length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx \qquad (y = f(x))$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

Area of a Surface of Revolution

$$A = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2} \cdot dx$$

$$A = 2\pi \int_a^b g(x) \cdot \sqrt{[f'(t)]^2 + [g'(t)]^2} \cdot dt$$

積分題型：

- $\int \sin^n x\, dx$ *and* $\int \cos^n x\, dx$

  (n *Odd*)
  use $\sin^2 = (1 - \cos^2 x)$

  (n *Even*)
  use half – angle identities

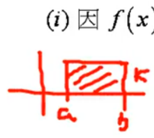
- 高斯函數的定積分

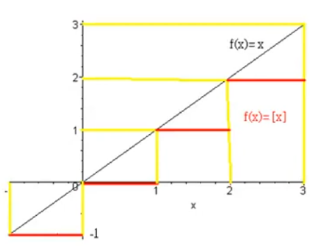
高斯函數的定積分 97/02/22

題目: 求 $(i)\ \int_{-1}^{3}[x]dx = ?$; $(ii)\ \int_{-1}^{2}[x^2]dx = ?$,其中[ ]爲高斯符號

解法:

(i) 因 $f(x) = [x] = \begin{cases} -1 & ,-1 \le x < 0 \\ 0 & ,0 \le x < 1 \\ 1 & ,1 \le x < 2 \\ 2 & ,2 \le x < 3 \\ 3 & ,x = 3 \end{cases}$





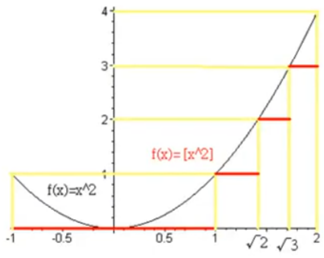
故 $\int_{-1}^{3}[x]dx = \int_{-1}^{0}[x]dx + \int_{0}^{1}[x]dx + \int_{1}^{2}[x]dx + \int_{2}^{3}[x]dx + \int_{3}^{3}[x]dx$

$= \int_{-1}^{0}(-1)dx + \int_{0}^{1}(0)dx + \int_{1}^{2}(1)dx + \int_{2}^{3}(2)dx + \int_{3}^{3}(3)dx$

$= (-1)(1) + (0)(1) + (1)(1) + (2)(1) + (3)(0) = -1 + 0 + 1 + 2 + 0 = 2$

(ii) 因 $-1 \le x \le 2 \Rightarrow 0 \le x^2 \le 4$，故$\left[x^2\right]$的可能出現值爲 $0,1,2,3,4$

又 $f(x)=\left[x^2\right]=\begin{cases} 0 & ,-1 \le x < 1 \\ 1 & ,1 \le x < \sqrt{2} \\ 2 & ,\sqrt{2} \le x < \sqrt{3} \\ 3 & ,\sqrt{3} \le x < 2 \\ 4 & ,x = 2 \end{cases}$



$$\begin{aligned} 故 \int_{-1}^{2}\left[x^2\right]dx &= \int_{-1}^{1}\left[x^2\right]dx+\int_{1}^{\sqrt{2}}\left[x^2\right]dx+\int_{\sqrt{2}}^{\sqrt{3}}\left[x^2\right]dx+\int_{\sqrt{3}}^{2}\left[x^2\right]dx+\int_{2}^{2}\left[x^2\right]dx \\ &= \int_{-1}^{1}(0)dx+\int_{1}^{\sqrt{2}}(1)dx+\int_{\sqrt{2}}^{\sqrt{3}}(2)dx+\int_{\sqrt{3}}^{2}(3)dx+\int_{2}^{2}(4)dx \\ &= (0)(2)+(1)\left(\sqrt{2}-1\right)+(2)\left(\sqrt{3}-\sqrt{2}\right)+(3)\left(2-\sqrt{3}\right)+(4)(0) \\ &= 5-\sqrt{2}-\sqrt{3} \end{aligned}$$

微分題型：

取 ln 再微分，把$1/y$移到右邊。

$$Find\ \frac{dy}{dx}\ by\ logarithmic\ differentiation$$

$$y=\frac{\left(x^2+3\right)^{\frac{2}{3}}(3x+2)^2}{\sqrt{x+1}}(P.330\ \#34)$$

$$\begin{aligned} \ln y &= \frac{2}{3}\ln\left(x^2+3\right)+2\ln(3x+2)-\frac{1}{2}\ln(x+1) \\ \frac{1}{y}\cdot\frac{dy}{dx} &= \frac{2}{3}\cdot\frac{2x}{x^2+3}+\frac{2\cdot 3}{3x+2}-\frac{1}{2(x+1)} \\ \frac{dy}{dx} &= \frac{\left(x^2+3\right)^{\frac{2}{3}}(3x+2)^2}{\sqrt{x+1}}\left[\frac{4x}{3\left(x^2+3\right)}+\frac{6}{3x+2}-\frac{1}{2(x+1)}\right] \end{aligned}$$

期末考練習卷

1. $Find\ \frac{dy}{dx}$

a) $y=\int_1^{x^2+x}\sqrt{2z+\sin z}\cdot dz$

$let\ u = x^2+x$

$du = 2x+1$

$$\begin{aligned} y' &= \frac{d}{du}\int_1^u\sqrt{2z+\sin z}\cdot dz\cdot\frac{du}{dx} \\ &= \sqrt{2u+\sin u}\cdot(2x+1) \\ &= \sqrt{2\left(x^2+x\right)+\int\left(x^2+x\right)}\cdot(2x+1) \end{aligned}$$

b) $y=\ln\left(x^2+3x+\pi\right)$

$$y'=\frac{1}{x^2+3x+\pi}\cdot(2x+3)$$

c) $y=\frac{\left(x^2+3\right)^{2/3}(3x+2)^2}{\sqrt{x+1}}$

$$\begin{aligned}
\ln y &= \frac{2}{3}\ln\left(x^2+3\right) + 2\ln(3x+2) - \frac{1}{2}\ln(x+1) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{2}{3}\cdot\frac{2x}{x^2+3} + \frac{2\cdot 3}{3x+2} - \frac{1}{2(x+1)} \\
\frac{dy}{dx} &= \frac{\left(x^2+3\right)^{\frac{2}{3}}(3x+2)^2}{\sqrt{x+1}}\left[\frac{4x}{3\left(x^2+3\right)} + \frac{6}{3x+2} - \frac{1}{2(x+1)}\right]
\end{aligned}$$

d) $y = e^{2x^2-x}$
$y' = e^{2x^2-x}\cdot(4x-1)$

f) $y = \cos^{-1}(1-3x)$