112-1 Calculus 2023-11-17 Homework

學號：412770116　　　　　　姓名：許嘉隆

期中考後作業 p.77 #8 #10 p.106 #67 p.114 #49 #52 p.117 #17 p.123 #16 p.129 #16

(p.77 #8)

1. $\lim\limits_{\theta \to 0} \dfrac{\tan 5\theta}{\sin 2\theta}$

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\tan 5\theta}{\sin 2\theta} &= \lim_{\theta \to 0} \frac{1}{\sin 2\theta} \cdot \frac{\sin 5\theta}{1} \cdot \frac{1}{\cos 5\theta} \cdot \frac{5\theta}{5\theta} \cdot \frac{2\theta}{2\theta} \\
&= \lim_{\theta \to 0} \frac{1}{\frac{\sin 2\theta}{2\theta}} \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{1}{\cos 5\theta} \cdot \frac{5\theta}{2\theta} \\
&= 1 \cdot 1 \cdot 1 \cdot \frac{5}{2} \\
&= \frac{5}{2} \qquad \blacksquare
\end{aligned}$$

(p.77 #10)

2. $\lim\limits_{t \to 0} \dfrac{\sin^2 3t}{2t}$

$$\begin{aligned}
\lim_{t \to 0} \frac{\sin^2 3t}{2t} &= \lim_{t \to 0} \frac{\sin 3t}{1} \cdot \frac{\sin 3t}{1} \cdot \frac{1}{2t} \cdot \frac{3t}{3t} \cdot \frac{3t}{3t} \\
&= \lim_{t \to 0} \frac{\sin 3t}{3t} \cdot \frac{\sin 3t}{3t} \cdot \frac{3t \cdot 3t}{3t} \\
&= 1 \cdot 1 \cdot 0 \\
&= 0 \qquad \blacksquare
\end{aligned}$$

(p.106 #67)

3. *Let* $f(x) = \begin{cases} mx + b & if\ x < 2 \\ x^2 & if\ x \geqslant 2 \end{cases}$

Determine m and b so that f is differentiable everywhere.

Solution :

Because f is differentiable everywhere, it is continuous everywhere, then

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (mx + b) = 2m + b = f(2) = 4$$

and $b = 4 - 2m$.

For f to be differentiable everywhere,

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \text{ must exist.}$$

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2^+} (x + 2) = 4$$

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{mx + b - 4}{x - 2} = \lim_{x \to 2^-} \frac{mx + 4 - 2m - 4}{x - 2} = \lim_{x \to 2^-} \frac{m(x-2)}{x - 2} = m$$

Therefore, $m = 4$ *and* $b = 4 - 2(m) = 4 - 8 = -4$ ■

(p.114 #49)

4. $y = x^2 - 2x + 2$, *find the tangent line at the point* $(1, 1)$.

Solution:

$y' = 2x - 2$

At $x = 1$: $m_{\tan} = 0$

Tangent line: $y = 1$ ■

(p.114 #52)

5. $y = \frac{1}{3}x^3 + x^2 - x$, *where the slope has* 1.

Solution:

$y' = x^2 + 2x - 1$

$m_{\tan} = y' = x^2 + 2x - 1 = 1$

$x^2 + 2x - 2 = 0$

$$
\begin{aligned}
x &= \frac{-2 \pm \sqrt{4-4(1)(-2)}}{2} \\
&= \frac{-2 \pm \sqrt{12}}{2} \\
&= -1 \pm \sqrt{3}
\end{aligned}
$$

The points $\left(-1+\sqrt{3},\ \frac{5}{3}-\sqrt{3}\right)$ *and* $\left(-1-\sqrt{3},\ \frac{5}{3}+\sqrt{3}\right)$ *have slope* 1. ■

(p.117 #17)

6.$y = \tan^2 x$

$$
\begin{aligned}
D_x \tan^2 x &= D_x (\tan x)(\tan x) \\
&= \left(\tan x \sec^2 x\right) + \left(\tan x \sec^2 x\right) \\
&= 2 \tan x \sec^2 x
\end{aligned}
$$ ■

(p.123 #16)

7.$y = \cos^3\left(\frac{x^2}{1-x}\right)$

Solution :

Let $y = u^3,\ u = \cos v,$ *and* $v = \frac{x^2}{1-x}$

$$
\begin{aligned}
D_x y &= D_u y \cdot D_v u \cdot D_x v \\
&= \left(3u^2\right)(-\sin v)\frac{(1-x)(2x)-\left(x^2\right)(-1)}{(1-x)^2} \\
&= -3\cos^2\left(\frac{x^2}{1-x}\right)\sin\left(\frac{x^2}{1-x}\right)\frac{(1-x)(2x)-\left(x^2\right)(-1)}{(1-x)^2} \\
&= \frac{-3\left(2x-x^2\right)}{(1-x)^2}\cos^2\left(\frac{x^2}{1-x}\right)\sin\left(\frac{x^2}{1-x}\right)
\end{aligned}
$$ ■

(p.129 #16)

8.$f(x) = \frac{(x+1)^2}{x-1}$

$$\begin{aligned}
f'(x) &= \frac{(x-1)\cdot 2(x+1)-(x+1)^2}{(x-1)^2} \\
&= \frac{x^2-2x-3}{(x-1)^2}
\end{aligned}$$

$$\begin{aligned}
f''(x) &= \frac{(x-1)^2(2x-2)-\left(x^2-2x-3\right)\cdot 2(x-1)}{(x-1)^4} \\
&= \frac{(x-1)(2x-2)-\left(x^2-2x-3\right)\cdot 2}{(x-1)^3} \\
&= \frac{2x^2-2x-2x+2-2x^2+4x+6}{(x-1)^3} \\
&= \frac{8}{(x-1)^3}
\end{aligned}$$

$$\begin{aligned}
f''(2) &= \frac{8}{1^3} \\
&= 8 \qquad \blacksquare
\end{aligned}$$