

Vignette of Variance Components Model

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This package contains three approaches for estimating variance components of linear model [1]: the parameter expanded EM (PX-EM) algorithm [2, 3], Minorization-Maximization (MM) algorithm [4] and the Method of Moments (MoM) [5].

Variance components model

Suppose we have dataset $\{\mathbf{X}, \mathbf{Z}, \mathbf{y}\}$ where $\mathbf{X} \in \mathbb{R}^{n \times p}$ is the design matrix whose columns are normalized with mean 0 and variance $1/p$, $\mathbf{y} \in \mathbb{R}^n$ is the response vector and $\mathbf{Z} \in \mathbb{R}^{n \times c}$ is the covariate matrix of fixed effects. The linear mixed model links \mathbf{y} with \mathbf{X} and \mathbf{Z} :

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\omega} + \mathbf{X}\boldsymbol{\beta} + \mathbf{e}, \quad \boldsymbol{\beta} \sim \mathcal{N}(0, \sigma_\beta^2 \mathbf{I}_p), \quad \mathbf{e} \sim \mathcal{N}(0, \sigma_e^2 \mathbf{I}_n), \quad (1)$$

where $\boldsymbol{\beta} \in \mathbb{R}^p$ is the random effect, $\boldsymbol{\omega} \in \mathbb{R}^c$ is the fixed effect and σ_β^2 and σ_e^2 are model parameters. This linear mixed model can also be re-written as a variance components model:

$$\mathbf{y} = \mathcal{N}(\mathbf{Z}\boldsymbol{\omega}, \sigma_\beta^2 \mathbf{K} + \sigma_e^2 \mathbf{I}_n), \quad (2)$$

where $\mathbf{K} = \mathbf{X}\mathbf{X}^T$. Since \mathbf{X} has been normalized with mean 0 and variance $1/p$, $\text{tr}(\mathbf{K}) = \text{tr}(\mathbf{I}_n) = n$. The goal is to estimate the variance components $\boldsymbol{\theta} = \{\sigma_\beta^2, \sigma_e^2\}$ [1]. This package provides three approaches to estimate the parameters: PX-EM algorithm, MM algorithm and the method of moments. The first two are based on maximum likelihood (MLE) approach and the third one adopts the moment matching approach.

PX-EM algorithm

The PX-EM algorithm is an extension of classical EM algorithm with faster speed [2, 3]. We first consider the parameter expanded version of (1):

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\omega} + \delta \mathbf{X}\boldsymbol{\beta} + \mathbf{e},$$

where $\delta \in \mathbb{R}^1$ is the expanded parameter. The complete-data log-likelihood is given as

$$\begin{aligned} \mathcal{L} &= \log \Pr(\mathbf{y}, \boldsymbol{\beta} | \boldsymbol{\theta}; \mathbf{Z}, \mathbf{X}) \\ &= -\frac{n}{2} \log(2\pi\sigma_e^2) - \frac{1}{2\sigma_e^2} \|\mathbf{y} - \mathbf{Z}\boldsymbol{\omega} - \delta \mathbf{X}\boldsymbol{\beta}\|^2 \\ &\quad - \frac{p}{2} \log(2\pi\sigma_\beta^2) - \frac{1}{2\sigma_\beta^2} \|\boldsymbol{\beta}\|^2, \end{aligned} \quad (3)$$

from which we can easily recognize that the terms involving $\boldsymbol{\beta}$ are of a quadratic form:

$$\boldsymbol{\beta}^T \left(-\frac{\delta^2}{2\sigma_e^2} \mathbf{X}^T \mathbf{X} - \frac{1}{2\sigma_\beta^2} \mathbf{I}_p \right) \boldsymbol{\beta} + \frac{\delta}{\sigma_e^2} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T \mathbf{X} \boldsymbol{\beta} + \text{Constant}.$$

Therefore, the posterior distribution of β is Gaussian $\mathcal{N}(\beta|\mu, \Sigma)$, where

$$\begin{aligned}\Sigma^{-1} &= \frac{\delta^2}{\sigma_e^2} \mathbf{X}^T \mathbf{X} + \frac{1}{\sigma_\beta^2} \mathbf{I}_p, \\ \mu &= \left(\frac{\delta^2}{\sigma_e^2} \mathbf{X}^T \mathbf{X} + \frac{1}{\sigma_\beta^2} \mathbf{I}_p \right)^{-1} \frac{\delta}{\sigma_e^2} \mathbf{X}^T (\mathbf{y} - \mathbf{Z}\omega).\end{aligned}$$

Now in the E-step, we evaluate the Q -function by taking the expectation of the complete-data log-likelihood with respect to the posterior $\mathcal{N}(\beta|\mu, \Sigma)$. Specifically, the quadratic terms involving β are evaluated as following:

$$\begin{aligned}\mathbb{E}[||\tilde{\mathbf{y}} - \delta \mathbf{X}\beta||^2] &= \mathbb{E}[\tilde{\mathbf{y}}^T \tilde{\mathbf{y}} - 2\delta \tilde{\mathbf{y}}^T \mathbf{X}\beta + \delta^2 \beta^T \mathbf{X}^T \mathbf{X}\beta] \\ &= \tilde{\mathbf{y}}^T \tilde{\mathbf{y}} - 2\delta \tilde{\mathbf{y}}^T \mathbf{X}\mu + \delta^2 \text{tr}(\mathbf{X}^T \mathbf{X}\Sigma), \\ \mathbb{E}[||\beta||^2] &= \mu^T \mu + \text{tr}(\Sigma),\end{aligned}$$

where $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{Z}\omega$. Then the Q -function given the current parameter estimates θ_{old} is obtained as:

$$\begin{aligned}\mathcal{Q}(\theta|\theta_{old}) &= -\frac{n}{2} \log(2\pi\sigma_e^2) - \frac{p}{2} \log(2\pi\sigma_\beta^2) \\ &\quad - \frac{1}{2\sigma_e^2} ||\mathbf{y} - \mathbf{Z}\omega - \delta \mathbf{X}\mu||^2 - \frac{1}{2\sigma_\beta^2} ||\mu||^2 \\ &\quad - \frac{1}{2\sigma_\beta^2} \mu^T \mu - \text{tr} \left(\left(\frac{\delta^2}{2\sigma_e^2} \mathbf{X}^T \mathbf{X} + \frac{1}{2\sigma_\beta^2} \mathbf{I}_p \right) \Sigma \right).\end{aligned}$$

In the M-step, the new estimates of parameter θ is obtained by setting the derivative of Q -function to be zero. The resulting updates are given as follows:

$$\begin{aligned}\delta &= \frac{(\mathbf{y} - \mathbf{Z}\omega)^T \mathbf{X}\mu}{\mu^T \mathbf{X}^T \mathbf{X}\mu + \text{tr}(\mathbf{X}^T \mathbf{X}\Sigma)}, \\ \omega &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (\mathbf{y} - \delta \mathbf{X}\mu), \\ \sigma_e^2 &= \frac{1}{n} [||\mathbf{y} - \mathbf{Z}\omega - \delta \mathbf{X}\mu||^2 + \delta^2 \text{tr}(\mathbf{X}^T \mathbf{X}\Sigma)], \\ \sigma_\beta^2 &= \frac{1}{p} [\mu^T \mu + \text{tr}(\Sigma)].\end{aligned}$$

This PX-EM algorithm is summarized in Algorithm 1. After convergence, the posterior mean and variance of $\mathcal{N}(\beta|\mu, \Sigma)$ can be evaluated given the obtained parameter estimates $\hat{\theta} = \{\hat{\sigma}_e^2, \hat{\sigma}_\beta^2\}$ and $\hat{\omega}$:

$$\begin{aligned}\Sigma^{-1} &= \frac{1}{\hat{\sigma}_e^2} \mathbf{X}^T \mathbf{X} + \frac{1}{\hat{\sigma}_\beta^2} \mathbf{I}_p, \\ \mu &= \left(\frac{1}{\hat{\sigma}_e^2} \mathbf{X}^T \mathbf{X} + \frac{1}{\hat{\sigma}_\beta^2} \mathbf{I}_p \right)^{-1} \frac{1}{\hat{\sigma}_e^2} \mathbf{X}^T (\mathbf{y} - \mathbf{Z}\hat{\omega}).\end{aligned}\tag{4}$$

Algorithm 1 PX-EM algorithm for model (1)

Initialization: Parameters are initialized by setting $\boldsymbol{\omega} = (\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T \mathbf{y}$, $\sigma_e^2 = \sigma_\beta^2 = \text{Var}(y - \mathbf{Z}\boldsymbol{\omega})$.

repeat

E-step: At the t -th iteration, evaluate the posterior $\mathcal{N}(\boldsymbol{\beta}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ given the current parameter estimates $\boldsymbol{\theta}^{(t)} = \{(\sigma_e^{(t)})^2, (\sigma_\beta^{(t)})^2\}$, $\boldsymbol{\omega}^{(t)}$ and set $\delta^{(t)} = 1$:

$$\boldsymbol{\Sigma}^{-1} = \frac{(\delta^{(t)})^2}{(\sigma_e^{(t)})^2} \mathbf{X}^T \mathbf{X} + \frac{1}{(\sigma_\beta^{(t)})^2} \mathbf{I}_p,$$
$$\boldsymbol{\mu} = \left(\frac{(\delta^{(t)})^2}{(\sigma_e^{(t)})^2} \mathbf{X}^T \mathbf{X} + \frac{1}{(\sigma_\beta^{(t)})^2} \mathbf{I}_p \right)^{-1} \frac{(\delta^{(t)})^2}{(\sigma_e^{(t)})^2} \mathbf{X}^T (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})$$

M-step: Update the model parameters by

$$\delta^{(t+1)} = \frac{(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T \mathbf{X}\boldsymbol{\mu}}{\boldsymbol{\mu}^T \mathbf{X}^T \mathbf{X}\boldsymbol{\mu} + \text{tr}(\mathbf{X}^T \mathbf{X}\boldsymbol{\Sigma})},$$
$$\boldsymbol{\omega}^{(t+1)} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (\mathbf{y} - \delta \mathbf{X}\boldsymbol{\mu}),$$
$$(\sigma_e^{(t+1)})^2 = \frac{1}{n_r} [||\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)} - \delta \mathbf{X}\boldsymbol{\mu}||^2 + \delta^2 \text{tr}(\mathbf{X}^T \mathbf{X}\boldsymbol{\Sigma})],$$
$$(\sigma_\beta^{(t+1)})^2 = \frac{1}{p} [\boldsymbol{\mu}^T \boldsymbol{\mu} + \text{tr}(\boldsymbol{\Sigma})].$$

Reduction-step: Rescale $(\sigma_e^{(t+1)})^2 = (\delta^{(t+1)})^2 (\sigma_e^{(t+1)})^2$ and reset $\delta^{(t+1)} = 1$.

until the incomplete-data log-likelihood stop increasing or maximum iteration reached

MM algorithm

Unlike the PX-EM algorithm, the MM algorithm maximize the incomplete-data log-likelihood by considering the variance components model (2) [4]. The incomplete-data log-likelihood is given as

$$\log \Pr(\mathbf{y}|\boldsymbol{\theta}; \mathbf{Z}, \mathbf{K}) = -\frac{1}{2} \log |\boldsymbol{\Omega}| - \frac{1}{2} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T \boldsymbol{\Omega} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}), \quad (5)$$

where $\boldsymbol{\Omega} = \sigma_\beta^2 \mathbf{K} + \sigma_e^2 \mathbf{I}_n$. The MM algorithm updates $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ alternatively by iteratively maximizing the lower bound of incomplete-data log-likelihood. Given $\boldsymbol{\theta}^{(t)}$, the updata of $\boldsymbol{\omega}$ is simply a weighted least square problem

$$\boldsymbol{\omega}^{(t)} = (\mathbf{Z}^T \boldsymbol{\Omega}^{(t)} \mathbf{Z})^{-1} \mathbf{Z}^T (\boldsymbol{\Omega}^{(t)})^{-1} \mathbf{y}. \quad (6)$$

The update of $\boldsymbol{\theta}$ given $\boldsymbol{\omega}$ depends on two minorizations. First,

$$-(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T (\boldsymbol{\Omega}^{(t)})^{-1} \left(\frac{(\sigma_\beta^{(t)})^4}{\sigma_\beta^2} \mathbf{K} + \frac{(\sigma_e^{(t)})^4}{\sigma_e^2} \right) (\boldsymbol{\Omega}^{(t)})^{-1} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}) \leq -\frac{1}{2} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T \boldsymbol{\Omega} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}), \quad (7)$$

which separates the variance components in the quadratic term of the likelihood (5). Second, the convexity of function $-\log |\boldsymbol{\Omega}|$ implies that

$$-\log |\boldsymbol{\Omega}^{(t)}| - \text{tr}((\boldsymbol{\Omega}^{(t)})^{-1} (\boldsymbol{\Omega} - \boldsymbol{\Omega}^{(t)})) \leq -\frac{1}{2} \log |\boldsymbol{\Omega}|, \quad (8)$$

which separates the variance components in the log determinant of the likelihood (5). Combining (7) and (8), the overall minorization is given as

$$\begin{aligned}
& \mathcal{G}(\boldsymbol{\theta}|\boldsymbol{\theta}_{old}) \\
&= -\log |\boldsymbol{\Omega}^{(t)}| - \frac{1}{2}\text{tr}((\boldsymbol{\Omega}^{(t)})^{-1}\boldsymbol{\Omega}) - \frac{1}{2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T(\boldsymbol{\Omega}^{(t)})^{-1}\left(\frac{(\sigma_\beta^{(t)})^4}{\sigma_\beta^2}\mathbf{K} + \frac{(\sigma_e^{(t)})^4}{\sigma_e^2}\right)(\boldsymbol{\Omega}^{(t)})^{-1}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)}) \\
&= -\log |\boldsymbol{\Omega}^{(t)}| - \frac{\sigma_\beta^2}{2}\text{tr}((\boldsymbol{\Omega}^{(t)})^{-1}\mathbf{K}) - \frac{1}{2}\frac{(\sigma_\beta^{(t)})^4}{\sigma_\beta^2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T(\boldsymbol{\Omega}^{(t)})^{-1}\mathbf{K}(\boldsymbol{\Omega}^{(t)})^{-1}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)}) \\
&\quad - \frac{\sigma_e^2}{2}\text{tr}((\boldsymbol{\Omega}^{(t)})^{-1}) - \frac{1}{2}\frac{(\sigma_e^{(t)})^4}{\sigma_e^2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T(\boldsymbol{\Omega}^{(t)})^{-2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)}).
\end{aligned} \tag{9}$$

By setting the derivative of \mathcal{G} -function to be zero, the resulting updates are given as follows:

$$\begin{aligned}
(\sigma_\beta^{(t+1)})^2 &= (\sigma_\beta^{(t)})^2 \sqrt{\frac{(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T(\boldsymbol{\Omega}^{(t)})^{-1}\mathbf{K}(\boldsymbol{\Omega}^{(t)})^{-1}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})}{\text{tr}((\boldsymbol{\Omega}^{(t)})^{-1}\mathbf{K})}} \\
(\sigma_e^{(t+1)})^2 &= (\sigma_e^{(t)})^2 \sqrt{\frac{(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T(\boldsymbol{\Omega}^{(t)})^{-2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})}{\text{tr}((\boldsymbol{\Omega}^{(t)})^{-1})}}.
\end{aligned}$$

The resulting algorithm is summarized in Algorithm 2.

Algorithm 2 MM algorithm for model (2)

Initialization: Parameters are initialized by setting $\boldsymbol{\omega} = (\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{y}$, $\sigma_e^2 = \sigma_\beta^2 = \text{Var}(y - \mathbf{Z}\boldsymbol{\omega})$.
repeat

$$\begin{aligned}
\boldsymbol{\Omega}^{(t)} &= (\sigma_\beta^{(t)})^2\mathbf{K} + (\sigma_e^{(t+1)})^2\mathbf{I}_n, \\
\boldsymbol{\omega}^{(t)} &= (\mathbf{Z}^T\boldsymbol{\Omega}^{(t)}\mathbf{Z})^{-1}\mathbf{Z}^T(\boldsymbol{\Omega}^{(t)})^{-1}\mathbf{y}, \\
(\sigma_\beta^{(t+1)})^2 &= (\sigma_\beta^{(t)})^2 \sqrt{\frac{(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T(\boldsymbol{\Omega}^{(t)})^{-1}\mathbf{K}(\boldsymbol{\Omega}^{(t)})^{-1}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})}{\text{tr}((\boldsymbol{\Omega}^{(t)})^{-1}\mathbf{K})}}, \\
(\sigma_e^{(t+1)})^2 &= (\sigma_e^{(t)})^2 \sqrt{\frac{(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})^T(\boldsymbol{\Omega}^{(t)})^{-2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega}^{(t)})}{\text{tr}((\boldsymbol{\Omega}^{(t)})^{-1})}}.
\end{aligned}$$

until the incomplete-data log-likelihood stop increasing or maximum iteration reached

For MLE methods including MM and PX-EM, the covariance matrix of variance components estimates are calculated from inverse of Fisher Information Matrix (FIM). $FIM = -\mathbb{E}\left[\frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\theta}^2}\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial \boldsymbol{\theta}^2} - \frac{1}{2}\log |\boldsymbol{\Omega}| - \frac{1}{2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T\boldsymbol{\Omega}^{-1}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})\right]$. The first derivatives are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \sigma_g^2} &= \frac{1}{2}\text{tr}\left[-\boldsymbol{\Omega}^{-1}\mathbf{K} + (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T\boldsymbol{\Omega}^{-1}\mathbf{K}\boldsymbol{\Omega}^{-1}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})\right], \\
\frac{\partial \mathcal{L}}{\partial \sigma_e^2} &= \frac{1}{2}\text{tr}\left[-\boldsymbol{\Omega}^{-1} + (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T\boldsymbol{\Omega}^{-2}(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})\right];
\end{aligned} \tag{10}$$

and the second derivatives are given as

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial(\sigma_g^2)^2} &= \frac{1}{2} \text{tr} [(\mathbf{\Omega}^{-1} \mathbf{K})^2 - 2(\mathbf{\Omega}^{-1} \mathbf{K})^2 \mathbf{\Omega}^{-1} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T], \\
\frac{\partial^2 \mathcal{L}}{\partial(\sigma_e^2)^2} &= \frac{1}{2} \text{tr} [\mathbf{\Omega}^{-2} - 2\mathbf{\Omega}^{-3} (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T], \\
\frac{\partial^2 \mathcal{L}}{\partial\sigma_g^2 \partial\sigma_e^2} &= \frac{1}{2} \text{tr} [\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-1} - (\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-2} + \mathbf{\Omega}^{-2} \mathbf{K} \mathbf{\Omega}^{-1}) (\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T].
\end{aligned} \tag{11}$$

Since the the only random variable is \mathbf{y} , and $\mathbb{E}[(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})(\mathbf{y} - \mathbf{Z}\boldsymbol{\omega})^T] = \mathbf{\Omega}$, the FIM is

$$FIM = -\frac{1}{2} \begin{bmatrix} \text{tr}[(\mathbf{\Omega}^{-1} \mathbf{K})^2] - 2\text{tr}[(\mathbf{\Omega}^{-1} \mathbf{K})^2] & \text{tr}(\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-1}) - 2\text{tr}(\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-1}) \\ \text{tr}(\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-1}) - 2\text{tr}(\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-1}) & \text{tr}[\mathbf{\Omega}^{-2}] - 2\text{tr}[\mathbf{\Omega}^{-2}] \end{bmatrix} \tag{12}$$

$$= \frac{1}{2} \begin{bmatrix} \text{tr}[(\mathbf{\Omega}^{-1} \mathbf{K})^2] & \text{tr}(\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-1}) \\ \text{tr}(\mathbf{\Omega}^{-1} \mathbf{K} \mathbf{\Omega}^{-1}) & \text{tr}[\mathbf{\Omega}^{-2}] \end{bmatrix}. \tag{13}$$

Inverting the FIM leads to the covariance matrix of $\hat{\boldsymbol{\theta}}$.

Method of Moments

While the MM algorithm and PX-EM algorithm adopts the MLE, MoM estimator is obtained by first multiplying Equation (2) by the projection matrix $\mathbf{M} = \mathbf{I}_n - \mathbf{Z}(\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T$ and then solving the following ordinary least squares (OLS) problem [5]:

$$\text{argmin}_{\sigma_\beta^2, \sigma_e^2} \|(\mathbf{M}\mathbf{y})^T(\mathbf{M}\mathbf{y}) - (\sigma_\beta^2 \mathbf{M}\mathbf{K}\mathbf{M} + \sigma_e^2 \mathbf{M})\|_F^2, \tag{14}$$

. Using the fact that $\|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}\mathbf{A}^T)}$, the OLS problem in (10) can be re-written as

$$\text{argmin}_{\sigma_\beta^2, \sigma_e^2} \text{tr}[(\mathbf{M}\mathbf{y})^T(\mathbf{M}\mathbf{y}) - (\sigma_\beta^2 \mathbf{M}\mathbf{K}\mathbf{M} + \sigma_e^2 \mathbf{M})(\mathbf{M}\mathbf{y})^T(\mathbf{M}\mathbf{y}) - (\sigma_\beta^2 \mathbf{M}\mathbf{K}\mathbf{M} + \sigma_e^2 \mathbf{M})^T],$$

which leads to the normal equation

$$\mathbf{S}\boldsymbol{\theta} = \mathbf{q}, \tag{15}$$

$$\text{with } \mathbf{S} = \begin{bmatrix} \text{tr}(\mathbf{M}\mathbf{K}\mathbf{M}\mathbf{K}) & \text{tr}(\mathbf{M}\mathbf{K}) \\ \text{tr}(\mathbf{M}\mathbf{K}) & n - c \end{bmatrix}, \quad \boldsymbol{\theta} = \begin{bmatrix} \sigma_\beta^2 \\ \sigma_e^2 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \mathbf{y}^T \mathbf{M}\mathbf{K}\mathbf{M}\mathbf{y} \\ \mathbf{y}^T \mathbf{M}\mathbf{y} \end{bmatrix}.$$

The MoM estimates of $\boldsymbol{\theta}$ is then given by $\hat{\boldsymbol{\theta}} = \mathbf{S}^{-1} \mathbf{q}$. The covariance matrix of MoM estimators are given by the sanwich estimator: $\boldsymbol{\Sigma}_\theta = \mathbb{E} \left[\frac{\partial B}{\partial \boldsymbol{\theta}} \right]^{-1} \text{Cov}(B) \mathbb{E} \left[\frac{\partial B}{\partial \boldsymbol{\theta}} \right]^{-1}$, where B is the normal equation $\mathbf{q} - \mathbf{S}\boldsymbol{\theta}$. Specifically,

$$\mathbb{E} \left[\frac{\partial B}{\partial \boldsymbol{\theta}} \right]^{-1} = \mathbf{S}^{-1}, \tag{16}$$

and

$$\text{Cov}(B) = \text{Cov} \left(\begin{bmatrix} \mathbf{y}^T \mathbf{M}\mathbf{K}\mathbf{M}\mathbf{y} \\ \mathbf{y}^T \mathbf{M}\mathbf{y} \end{bmatrix} \right) = \begin{bmatrix} \text{Var}(\mathbf{y}^T \mathbf{M}\mathbf{K}\mathbf{M}\mathbf{y}) & \text{Cov}(\mathbf{y}^T \mathbf{M}\mathbf{K}\mathbf{M}\mathbf{y}, \mathbf{y}^T \mathbf{M}\mathbf{y}) \\ \text{Cov}(\mathbf{y}^T \mathbf{M}\mathbf{K}\mathbf{M}\mathbf{y}, \mathbf{y}^T \mathbf{M}\mathbf{y}) & \text{Var}(\mathbf{y}^T \mathbf{M}\mathbf{y}) \end{bmatrix}, \tag{17}$$

where the elements are calculated by $\text{Var}(\mathbf{y}^T \mathbf{M}\mathbf{K}\mathbf{M}\mathbf{y}) = 2\text{tr}([\mathbf{M}\mathbf{K}\mathbf{M}\mathbf{\Omega}]^2)$, $\text{Var}(\mathbf{y}^T \mathbf{M}\mathbf{y}) = 2\text{tr}([\mathbf{M}\mathbf{\Omega}]^2)$, $\text{Cov}(\mathbf{y}^T \mathbf{M}\mathbf{K}\mathbf{M}\mathbf{y}, \mathbf{y}^T \mathbf{M}\mathbf{y}) = 2\text{tr}(\mathbf{M}\mathbf{K}\mathbf{M}\mathbf{\Omega}\mathbf{M}\mathbf{\Omega})$.

Example

```
library(VCM)
n <- 1000
d <- 1000

sb2 <- 0.1
se2 <- 1
X <- matrix(rnorm(n*d),n,d)
X <- scale(X)/sqrt(d)

w <- c(rnorm(d,0,sqrt(sb2)))
y0 <- X%*%w

y <- y0 + sqrt(se2)*rnorm(n)

fit_PXEM <- linRegPXEM(X=X,y=y,tol = 1e-6,maxIter =500,verbose=F)
fit_MM <- linRegMM(X=X,y=y,tol=1e-6,maxIter = 500,verbose=F)
fit_MoM <- linReg_MoM(X=X,y=y)

c(fit_PXEM$se2,fit_PXEM$sb2)

## [1] 0.98701371 0.06671157
c(fit_MM$se2,fit_MM$sb2)

## [1] 0.98685611 0.06688737
c(fit_MoM$se2,fit_MoM$sb2)

## [1] 1.00508465 0.04914072
```

References

- [1] Jiang, J., Li, C., Paul, D., Yang, C., & Zhao, H. (2016). On high-dimensional misspecified mixed model analysis in genome-wide association study. *The Annals of Statistics*, 44(5), 2127-2160.
- [2] Liu, C., Rubin, D. B., & Wu, Y. N. (1998). Parameter expansion to accelerate EM: the PX-EM algorithm. *Biometrika*, 85(4), 755-770.
- [3] Foulley, J. L., & Van Dyk, D. A. (2000). The PX-EM algorithm for fast stable fitting of Henderson's mixed model. *Genetics Selection Evolution*, 32(2), 143.
- [4] Zhou, H., Hu, L., Zhou, J., & Lange, K. (2018). MM algorithms for variance components models. *Journal of Computational and Graphical Statistics*, (just-accepted), 1-30.
- [5] Wu, Y., & Sankararaman, S. (2018). A scalable estimator of SNP heritability for biobank-scale data. *Bioinformatics*, 34(13), i187-i194.

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