18.650 Problem Set 5 Spring 2017
Statistics for Applications
Due Date: Wed 4/5/2017, prior to 4:00pm
Where: Problem Set Box (outside 4-174) (preferred)
or Electronically to Stellar website

Problems from John A. Rice, Third Edition. [*Chapter.Section.Problem*]

1. Problem 9.11.4

   Let X have one of the following distributions:

X	H_0	H_1
x_1	.2	.1
x_2	.3	.4
x_3	.3	.1
x_4	.2	.4

(a). Compare the likelihood ratio, Λ for each possible value X and order the x_i according to Λ.

X	H_0	H_A	Λ
x_1	.2	.1	2.
x_2	.3	.4	3/4
x_3	.3	.1	3
x_4	.2	.4	2/4

Sorting the x_i from highest to lowest Λ : x_3, x_1, x_2, x_4

(b). What is the likelihood ratio test of H_0 versus H_A at level $\alpha = 0.2$?

For a fixed constant λ_0, consider the likelihood ratio test with rejection region $R_{\lambda_0} = \{x : \Lambda(x) \leq \lambda_0\}$.

The signficance level α_0 of this likelihood ratio test is computed as:

$$\alpha_0 = P(X \in R_{\lambda_0} \mid H_0) = P(\Lambda(X) \leq \lambda_0 \mid H_0).$$

We rewrite the table, ordering the rows by decreasing Λ. We add a column corresponding to the significance levels of the tests for values of λ_0 corresponding to the distinct values of $\Lambda(x_i)$:

X	H_0	H_A	Λ	$P(\Lambda(X) \leq \Lambda(x_i) \mid H_0)$
x_3	.3	.1	3	1.0
x_1	.2	.1	2.	0.7
x_2	.3	.4	3/4	0.5
x_4	.2	.4	2/4	0.2

The test that rejects H_0 if $\Lambda(X) \leq \Lambda(x_4) = \lambda_0 = 2/4$ has level 0.2 and the test that rejects H_0 if $\Lambda(X) \leq \Lambda(x_2) = \lambda_0 = 3/4$ has level 0.5.

NOTE(!) In the text there is some inconsistency with defining the likelihood ratio test. On p. 332 of Rice, the Neyman-Pearson Lemma states that the rejection region for the likelhood ratio test with signficance level α is

$$R_c = \{x : \Lambda(x) < c\}$$

and has significance level $\alpha = P(R_c \mid H_0)$

On p. 341 of Rice, he states the that likelihood ratio of significance level α is given by the rejection region

$$R_{\lambda_0} = \{x : \Lambda(x) \leq \lambda_0\}$$

where the constant λ_0 is chosen so that

$$P(\Lambda \leq \lambda_0 \mid H_0) = \alpha.$$

When the distribution of the likelihood ratio is discrete, these definitions will have constants c and λ_0 hat difer for the same significance level. The rejection regions are equal, but have different specifications.

(c). If the prior probabilities are $P(H_0) = P(H_A)$ then the outcomes favoring H_0 are those for which the posterior probability ratio is greater than 1:

$$\begin{aligned}
1 &< \tfrac{P(H_0|X)}{P(H_A|X)} = \tfrac{P(H_0)P(X|H_0)}{P(H_A)P(X|H_A)} \\
&= (\tfrac{P(H_0)}{P(H_A)}) \cdot (\tfrac{P(X|H_0)}{P(X|H_A)}) \\
&= (\tfrac{P(H_0)}{P(H_A)}) \cdot \Lambda(X) \\
&= \Lambda(X)
\end{aligned}$$

For equal prior probabilities, the posterior odds is greater than 1 if the likeihood ratio is greater than 1. This is true for outcomes x_3 and x_1.

(d). What prior probabilities correspond to the decision rules with $\alpha = 0.2$ and $\alpha = 0.5$

For $\alpha = 0.2$, the rejection region of the test is $\{x_4\} = \{x : \Lambda < 3/4\}$.

For the posterior odds to be less than one, the prior odds must be less than 4/3, which corresponds to

$$P(H_0) = 1 - P(H_A) < 4/7.$$

For $\alpha = 0.5$, the rejection region of the test is $\{x_4, x_2\} = \{x : \Lambda < 2\}$.

For the posterior odds to be less than one, the prior odds must be less than 1/2, which corresponds to

$$P(H_0) = 1 - P(H_A) < 1/3.$$

2. Problem 9.11.5

(a). The significance level of a statistical test is equal to the probability that the null hypothesis is true.

False. In non-Bayesian hypothesis testing, there are no probabilities on the truth of the null (or alternative) hypothesis. The significance level is the conditional probability of rejecting the null hypothesis when the null hypothesis is true. The probability is computed under the conditional distribution of the data under the null hypothesis.

(b). If the significance level of a test is decreased, the power would be expected to increase.

False. If the significance level of a test is decreased, then the test rejects the null hypothesis with lower probability (less frequently) under the null hypothesis. The power of the test is 1 minus the probability of rejecting the null hypothesis evaluated under the conditional probability for a given alternate hypothesis. The power should decrease too since the test rejects less often if it has a lower level .

(c). If a test is rejecte at the significance level α, the probability that the null hypothesis is true equals α.

False. See (a) – we cannot evaluate the probability of the null hypothesis being true.

(d). The probability that the null hypothesis is falsely rejected is equal to the power of the test.

False. The power of the test is the probability that the null hypothesis is *correctly* rejected.

(e). A type I error occurs when the test statistic falls in the rejection region of the test.

False. When the test statistic falls in the rejection region of the test, the test procedure rejects the null hypothesis. This may lead to a correct decision of the null hypothesis is false, or to an incorrect decision (the type I error) if the null hypothesis is true. We do not know which.

(f). A type II error is more serious than a type I error.

False. Both error types are serious. In Neyman-Pearson testing, we limit the size of the type I error and find the test that minimizes the type II error subject to the constraint. By controlling the size of the type I error, it is given precedence in specifying tests.

(g). The power of a test is determined by the null distribution of the test statistic.

False. The power of a test is the conditional probability of correctly rejecting the null hypothesis when the alternate hypothesis is true. It is a function which is computed for every distribution in the alternate hypothesis.

(h). The likelihood ratio is a random variable.

True. The likelihood ratio is the ratio of the conditional pmf/pdf of the data (X) under H_0 and H_A. It is a function of the data which is a random variable.

3. Problem 9.11.23

   Suppose that a 99% confidence interval for the mean μ of a normal distribution is found to be

   $(-2.0, 3.0)$. Would a test of $H_0 : \mu = -3$ versus $H_A : \mu \neq -3$ be rejected at the 0.01 significane level?

   Yes, by the duality of confidence intervals and hypothesis tests (with two-sided alternatives).

   The confidence interval contains the true parameter with probability 0.99 (before observing the data). Accepting H_0 if and only if the confidence interval contains H_0 must have level $1 - 0.99$.

4. Problem 9.11.26

   (a). The generalized likelihood ratio statistic Λ is always less than or equal to 1.

   True. $\Lambda = \dfrac{\max_{\theta \in \Theta_0} Lik(\theta)}{\max_{\theta \in \Theta} Lik(\theta)}$

   where $\Theta_0 \subset \Theta$.

   (b). If the p-value is 0.03, the corresponding test will reject at the signficance level 0.02.

   False. Since the p-value is greater than 0.02, the data not *surprising enough* to reject the null hypothesis. The p-value is the lowest significance level of tests that reject the null hypothesis.

   (c). If a test rejects at significance level 0.06, then the p-value is less than or equal to 0.06.

   True. The p-value of the test statistic must be as low or lower than the significance level for the test to reject.

   (d). The p-value of a test is the probability that the null hypothesis is correct.

   False. The p-value is a conditional probability given the null hypothesis is true.

   (e). In testing a simple versus simple hyopthesis via the likleihood ratio, the p-value equals the likelihood ratio.

   False. The p-value is a probability computed assuming the null hypothesis is true. For the likelihood ratio test, it is the probability that the likelihood ratio (H_0 to H_1) is smaller than the observed value, computed under the conditional probability given H_0.

   (f). If a chi-square test statisti with 4 degrees of freedom has a value of 8.5, the p-value is less than 0.05

   False. Using R, the p value of the test statistic is

   $1 - pchisq(8.5, df = 4) = 0.07488723$

   The p-value is greater than 0.05.