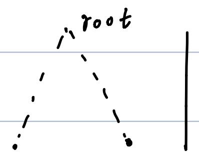
1. (1). assume: $\exists$ a root, make the height of the tree $< \lceil \frac{x}{2} \rceil$



$< \lceil \frac{x}{2} \rceil$ the longest simple path should be a path from the leaf node at left to the leaf node at right

when x is an odd number, height $< (x-1)/2 + 1$

the longest simple path $\leq 2(\text{height} - 1) + 1 < x$

when x is an even number, height $< \frac{x}{2}$

the longest simple path $\leq 2(\text{height} - 1) + 1 < x - 1$

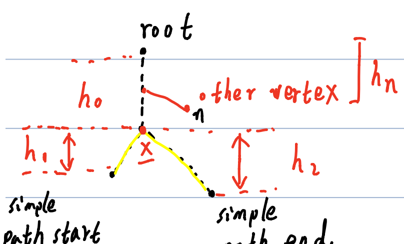
thus: the longest simple path $< x$

while the longest simple path is x.

thus: the height of the tree with arbitary root $\geq \lceil \frac{x}{2} \rceil$

(2) assume: $v \notin S$

then the tree has a structure below:



we assume $h_1 \leq h_2$

For: $n \notin S$ and $n \neq$ root

if $h_n - h_0 > h_1$

then the longest simple path is $n \sim$ simple path end.

the longest simple path $\geq h_n - h_0 + h_2 > h_1 + h_2$

thus: $h_n - h_0 \leq h_1$, $\rightarrow$ thus $h_n \leq h_1 + h_0 \leq h_2 + h_0$

thus: the height of the tree depend on simple path end.

$\boxed{\text{the height is } h_0 + h_2}$

however, when we take x as the root vertex the height is $\max\{h_0, h_1, h_2\}$

because, we assume $h_1 \le h_2$ the height is $\max\{h_0, h_2\}$

the original height is $h_0 + h_2$. because $h_0 \geq 1$

thus $h_0 + h_2 > \{h_0, h_2\}$

thus when $v \notin S$. the height is not minimized.

thus. when we choose v to minimize the height, $v \in S$.

(3). form (2) we know the root v must satisfy: $v \in S$.

when $v \in S$. the height is $\max\{h_1, h_2\}$

$h_1 + h_2 = x$. thus: the mini height is $\lceil \frac{x}{2} \rceil$

thus: the middle vertex of one of the longest simple paths is

always an answer to the problem.