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RESEARCH ARTICLE

PDE-Based Event-Triggered Containment Control of Multi-Agent Systems with Input Delay under Spatial Boundary Communication

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Summary

This paper studies the containment control problem for multi-agent systems with input delay under spatial boundary communication by employing an event-based approach. Firstly, the collective dynamics of multi-agent systems are described as parabolic partial differential equations (PDEs). Applying the integral transformation method developed in PDEs, the delayed parabolic PDEs are transformed into a series of new coupled PDE-PDE systems. Then, by taking the spatial boundary communication scheme into account and using the local boundary information, two boundary containment control protocols together with a dynamic event-triggered scheme (DETS) are designed, such that the states of all followers converge to a convex hull formed by multiple leader agents with and without input delay. The optimal protocols are given by minimizing the 2-norm of the designed control gain matrix, and the exclusion of the Zeno behavior is analysed. Finally, a numerical simulation example is provided to support the main results.

KEYWORDS:
Multi-agent systems, delayed parabolic PDEs, containment problem, spatial boundary communication

1 | INTRODUCTION

Multi-agent systems (MASs) have received much attention over the past few decades, since their collective behaviors can describe various engineering applications including distributed optimization, sensor networks, formation control, to name just a few. Generally, these collective behaviors are modelled as special differential inclusions [1], linear ordinary differential equations [2], or second-order differential operators [3]. However, as more and more spatio-temporal behaviors appear in 4,5, partial differential equations (PDEs), which have attracted ever-increasing attention (see 6,7 and references therein), have been a better choice to model MASs. There are numerous works on the control problems of PDEs [8,9,10,11,12]. For instance, the authors in 6 have modelled the MASs as high-order PDEs. Based on the PDE models, a boundary controller has been proposed for complicated formations of the MASs. In addition, other control problems have also been addressed for PDEs including dissipative boundary conditions [8], boundary control [9], stabilization [10,11], and output consensus problems [12].

Containment, as a kind of particular collective behavior, has been a hot topic recently, and various containment algorithms have been developed [13,14,15,16,17,18,19,20,21,22,23,24]. One typical representative of containment is that a portion of autonomous vehicles are driven into the required safe areas by designing a distributed control protocol using the neighbours' information of every vehicle. In 17, an active disturbance rejection containment controller has been designed for the nonaffine nonlinear MASs with uncertainties. The containment problem of general linear MASs with multiple leaders [18] have been also investigated. To tackle the containment problem with time-varying delays, a novel Lyapunov-Razumikhin function has been proposed for second-order

MASs in 19, where the multiple leaders and dynamic followers are both involved. Afterwards, when it comes to unknown MASs and heterogeneous MASs, the corresponding containment control protocols have been established by using the adaptive control technology and the output regulation error technology. However, until now, no PDE-based containment problem has been studied for MASs due to the complexity of PDE dynamics and unknowability of spatial boundary control scheme.

Meanwhile, there are many practical constraints in engineering systems including agent constraints and environment constraints, which may deteriorate system performances and even leads to system' instability. Noticeably, among environment constraints, communication constraints and time delay have been non-ignorable factors in the process of implementing control algorithms in digital platforms. To adapt to the limited communication environment, one effective strategy is introducing an event-triggered scheme (ETS) into information transmission from sensors to controllers. Several works have been published about the event-triggered control[25,26,27,28,29,30,32]. At first, ETSs have been formulated by comparing the norm of triggering errors and the norm of the last triggering states. Then, since the seminal work in 26, where an adaptive learning term is added as the new triggering threshold, more and more dynamic ETSs have been set up[27,28,29,30]. Note that the aforementioned ETSs aim at the control problems of finite-dimensional systems, and the infinite-dimensional case can be found in 32. On the other hand, numerous delays have been successfully dealt with for the leader-following MASs, in which the Lyapunov-Krasovskii functional is used[33,34]. However, the results on containment of PDE-based MASs with time delay and communication constraints have been scattered, which reflects containment algorithms need to be updated. Therefore, it is desirable to study the PDE-based MASs by taking ETSs and time delay into account.

Motivated by the above discussions, this paper develops two event-based boundary control protocols for the containment of a class of MASs with and without input delay, where agent dynamics can be captured by parabolic PDEs. Inspired by 9 with the input delay and 26 with the dynamic ETS, the MASs with input delay can be converted into a coupled PDE-PDE system, and the dynamic ETS can be constructed using the local boundary information. The optimization control problems are formulated by minimizing the control gain matrix. It is proven that the negative effect of input delay is counted by the designed event-based controllers, and no Zeno behavior is exhibited in implementing these controllers.

Contributions: The main contributions include three-folds. (a) A novel framework of containment problem for MASs is proposed, where PDE-based MASs and input delay are involved. Compared with the existing containment frameworks[17,18,19,20,21,22,23,24], the containment ODE problem is extended to more realistic and complicated PDE systems with input delay. (b) A modified containment error is employed for multiple leaders, and the optimal containment protocols can be obtained by solving some optimization problems. With comparison to the works related to PDE-based MASs, the requirement on the global information is relaxed. (c) Unlike the existing works[33,34], where the delay systems are viewed as an ODE system with disturbances, this paper treats the addressed delayed MASs as a coupled PDE-PDE system. It reveals a new perspective of the containment control problem for MASs.

The subsequent sections are arranged as follows. Section II states the containment problem of parabolic PDEs and some basic knowledge. Section III gives the design of the dynamic ETS and the boundary controller. Section IV analyses the convergence of the addressed MASs with and without input delay, and presents the optimal control protocols and the exclusion of Zeno bahavior. Section V provides a numerical simulation to testify the validity of main results. Section VI concludes the whole paper.

2 | PROBLEM FORMULATION

2.1 | Graph Theory

Consider a MAS composed of N followers and M leaders. The communication topology of these followers is projected as an undirected and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \cdots, N\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and $\mathcal{A} = (a_{ij})_{N\times N}$ are the node set, the edge set, and the adjacency symmetric matrix, respectively. Each node represents an agent, and each edge represents a communication link between any two nodes. If there exists an edge between node i and j, it means that these two nodes have access to each other's information and $a_{ij} > 0$. Otherwise, $a_{ij} = 0$. For every agent itself, there are no self-edges, that is, $a_{ii} = 0$ holds for $i \in \mathcal{V}$. In addition, the connected graph $\mathcal{G}$ implies that a path can always be found for any two nodes. The Laplacian matrix of the undirected and connected graph $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}]$ with $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$, where $\mathcal{L} = \mathcal{L}^T$, $\mathbf{1}^T\mathcal{L} = 0$, and other non-zero eigenvalues of $\mathcal{L}$ are positive scalars.

The leaders are indexed as a set $\Gamma = \{N+1, N+2, \cdots, N+M\}$. For $k \in \{N+1, N+2, \cdots, N+M\}$, the connection matrix of leader k is defined as $B_k = diag\{b_{k1}, \cdots, b_{kN}\}$, where b_{kl} is the connection weight between the leader k and the follower l for $l \in \mathcal{V}$. If the connection between the leader k and the follower l exists, $b_{kl} = 1$, otherwise, $b_{kl} = 0$.

2.2 | Problem Statement

For the addressed MAS, the dynamics of the $i-$th follower are governed by the following parabolic PDE:

$$\begin{cases} \partial_t x_i(z,t) = \partial_{zz} x_i(z,t), \quad t \in \mathbb{R}^+ \\ \partial_z x_i(0,t) = 0, \partial_z x_i(1,t) = -u_i(t-d) \\ x_i(z,0) = x_i^0(z) \\ x_i(t-d) = 0, \quad \text{if} \quad t-d<0 \end{cases} \tag{1}$$

where z is the spatial position satisfying $z \in [0,1]$, t is the time, and $x_i(z,t)$ is the state. The second equation in (1) is the boundary conditions, and $u_i(t)$ is boundary control input to be designed. The third equation in (1) is the known initial condition, and d is the boundary constant input delay.

For $k \in \Gamma$, the $k-$th leader is assumed to have the following dynamics:

$$\begin{cases} \partial_t s_k(z,t) = \partial_{zz} s_k(z,t), \quad t \in \mathbb{R}^+ \\ \partial_z s_k(0,t) = 0, \partial_z s_k(1,t) = 0 \\ s_k(z,0) = s_k^0(z) \end{cases} \tag{2}$$

where $s_k(z,t)$ is the state. In addition, the second equation and the third equation in (2) are the known boundary conditions and initial condition, respectively.

Let us define $e_i(z,t)$ as the error vector of the $i-$th follower, which can be expressed as

$$e_i(z,t) = \sum_{j=1}^{N} a_{ij}(x_i - x_j) + \sum_{k=N+1}^{N+M} b_{ki}(x_i - s_k). \tag{3}$$

Denote $x(z,t) = [x_1^T(z,t), \cdots, x_N^T(z,t)]^T$ as the state vector of N followers, $s(z,t) = [s_1^T(z,t), \cdots, s_M^T(z,t)]^T$ as the state vector of M leaders, $u(t) = [u_1^T(t), \cdots, u_n^T(t)]^T$ as the overall input vector, and $e(z,t) = [e_1^T(z,t), \cdots, e_N^T(z,t)]^T$ as the overall error vector, respectively. Thus, the overall error dynamic can be described as

$$e(z,t) = \sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k (x(z,t) - \mathbf{1}_N \otimes s_k(z,t)) \tag{4}$$

where $\bar{\mathcal{L}}_k = \mathcal{L}_k \otimes I_n$, and $\mathcal{L}_k = \frac{1}{M}\mathcal{L} + B_k$ for $k \in \Gamma$.

Combining (1), (2), and (3), the overall error system (4) can be rewritten as

$$\begin{cases} \partial_t e(z,t) = \partial_{zz} e(z,t) \\ \partial_z e_k(0,t) = 0, \partial_z e_k(1,t) = -\sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k u(t-d) \\ e(z,0) = \sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k (x_i^0(z) - \mathbf{1}_N \otimes s_k^0(z)). \end{cases} \tag{5}$$

Inspired by 18,19,20, where the input delay dynamics can be viewed as a transport PDE, the overall error system (5) can be converted into the following coupled PDE-PDE system:

$$\begin{cases} \partial_t e(z,t) = \partial_{zz} e(z,t), z \in (0,1) \\ \partial_z e(0,t) = 0, \partial_z e(1,t) = -\sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k v(1,t) \\ e(z,0) = \sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k (x^0(z) - \mathbf{1}_N \otimes s_k^0(z)) \\ \partial_t v(z,t) = \partial_z v(z,t), z \in [1, 1+d) \\ v(1+d,t) = u(t) \end{cases} \tag{6}$$

where $v(z,t) = u(t+z-1-d)$ is the state of the input delay.

Remark 1. In the existing containment works 19,20,21,22, the final positions of the followers are decided by the Laplacian matrix of all agents and the leaders' positions, which means that the controller design requires the prior knowledge of leaders' connections. To avoid this requirement, this paper employs a modified error, where the final positions of the followers can be obtained directly instead of partitioning the whole Laplacian matrix. Furthermore, the required control protocol is only related to the followers' topology.

Remark 2. Note that the control problem of PDE-based systems has been studied in 4,5,6,7,8,9,10,11,12, and two class of control methods have been developed. The first method is the domain control, which requires acturators and sensors to be deployed on a flat surface. This domain control is carried out by solving the liniar matrix inequalities. The second method is the boundary control, which means that the controller acts on the boundary of PDE systems. This bounday control usually is accomplished with the baskstepping approach. However, until now, most controllers of PDE-based MASs are implemented in a domain control manner, which brings some difficulties when poor physical conditions are avaiable. This motivates us to solve the control problem of PDE-based MASs by developing an event-triggered control scheme.

For the simplicity of further analysis, we denote $x(z,t) = x$, $s(z,t) = s$, $e(z,t) = e$, $e(1,t) = e(1)$, and $v(z,t) = v$.

Lemma 1. For any square integrable and absolutely continuous function $f(z)$ with $z \in [0,1]$, we have $||f(z)||_2^2 \leq 4\pi^{-2}||\partial_z f(z)||_2^2$ if $f(0) = 0$ or $f(1) = 0$ holds.

Definition 1. For a finite set of given points $Y = \{y_1, \cdots, y_q\}$, the minimal convex set is defined as $Co(Y) = \{\sum_{i=1}^{q} \alpha_i y_i | y_i \in Y, \alpha_i > 0, \sum_{i=1}^{q} \alpha_i = 1\}$.

Definition 2. The containment performance of the MAS (1) and (2), i.e. the convergence of all followers to a convex hull formed by multiple leaders, is satisfied if $\lim_{t\to\infty} \inf_{y\in Co(s_k(z,t)),k\in\Gamma} ||x_i(z,t) - y||_2 = 0$ holds.

In this paper, the objective is to design two event-triggered boundary control protocols by using the local agents' information based on the spatial boundary communication scheme, such that the addressed MAS without or with input delay can achieve containment in the sense of the norm $\Theta_1(t)$ or $\Theta_2(t)$, respectively. $\Theta_1(t)$ and $\Theta_2(t)$ are defined as $||e(.,t)||^2_{L^2(0,1)} + 2\sum_{i=1}^{N} \zeta_i(t)$ and $||e(.,t)||^2_{L^2(0,1)} + ||v(.,t)||^2_{L^2(1,1+d)} + 2\sum_{i=1}^{N} \zeta_i(t)$, respectively. The L^2 norm of the function $f(z)$ in the Soblov space is defined as $||f(z)||^2_{L^2(a,b)} = \int_a^b f^2(z)dz$. That is, the key points of the containment problem are states as follows. One is how to design the proper state boundary contrrollers with the spatial boundary communication scheme rather than the domain communication scheme, by utilizing the local information of agents, the interaction topology infromation and the known delay information. The other is how to construct the corresponding dynamic ETS to make the containment performance analysis be easier, by combing the static ETSs and the dynamic triggered thresholds.

3 | DYNAMIC EVENT-TRIGGERED CONTROLLER

3.1 | Dynamic Event-Triggered Scheme

Under the spatial communication scheme and by employing the available spatial state information $e(1)$, the control problem of the infinite-dimensional PDE system can be accomplished by a boundary controller. In the process of implementing the control law, the continuous controller updates are required. With the purpose of relaxing this requirement, we assume that its triggering sequence is $\Sigma_i = \{t_0^i, \cdots, t_k^i, \cdots\}$ and $t_0^i = 0$ for agent $i \in \mathcal{V}$. Then, the dynamic event-triggered scheme is constructed as follows:

$$\begin{cases} t_{k+1}^i = \inf\big\{t > t_k^i | \tilde{e}_i^T(1)\tilde{e}_i(1) - \alpha_{1i}e_i^T(1)e_i(1) \\ \qquad\qquad -\alpha_{2i}\zeta_i(t) \geq 0\big\} \\ \dot{\zeta}_i(t) = -\beta_i\zeta_i(t) + \sigma_i[\alpha_{1i}e_i^T(1)e_i(1) - \tilde{e}_i^T\tilde{e}_i], \zeta_i(0) > 0 \end{cases} \tag{7}$$

where $\tilde{e}_i(1) = \bar{e}_i(1) - e_i(1)$ is the event-triggered error with $\bar{e}_i(1)$ being the real state to be transmitted and $e_i(1)$ being the ideal state of agent i. t_k^i is the $k-$th event instant belonging to the triggering sequence Σ_i, which decides when to sample and deliver the real system state. α_{1i}, α_{2i}, β_i and σ_i are finite sets of positive scalars to be determined later. $\zeta_i(0)$ is the known initial state satisfying $\zeta_i(0) > 0$.

Remark 3. It has been proven in 27 that dynamic thresholds $\zeta_i(t)$ are always positive because of the strictly positive property of its initial value $\zeta_i(0)$. Besides, the value of $\zeta_i(t)$ is monotonously decreasing for $t \in \Sigma_i$, if the designed controller can drive the MAS (1) to achieve containment. In detail, when the consensus is ensured, we have $e_i(1, t_{k+1}^i) \leq e_i(1, t_k^i)$ and $\zeta_i(1, t_{k+1}^i) \leq \zeta_i(1, t_k^i)$ according to the definition of the Lyapunov energy function $\Theta_1(t)$ or $\Theta_2(t)$.

Remark 4. It is known to us that control design of the continuous MAS is on the basis of event-triggered scheme without Zeno phenomena. Traditionally, the exclusion of Zeno phenomena is achieved by adding some factors into the triggering function, such as, constants, exponential decay terms, noise terms, and so on. These factors play a significant role in refusing executing infinite events during a finite time. However, since the traditional factors can not adaptively change the triggering function, this paper adopts a dynamic adaptive learning learning factor $\zeta_i(t)$.

Remark 5. Compared with the existing dynamic ETSs 26,27,28,29,30,32, which have been applied into the finite-dimensional system, the developed dynamic ETS (7) not only covers the infinite-dimensional case, but also can adaptively adjust the controller strategy to optimiza the impact of ETS on system performance. Besides, it provides a new perspective about how to design the required boundary controller for MASs. In this case, the developed dynamic event-triggered controller not only solves the control problem of infinite-dimensional MASs with the boundary control scheme and less events than the existing event-triggered controllers, but also provides more freedom of the controller design in the Lyapunov stability analysis due to the injection of the dynamic thresholds $\zeta_i(t)$.

3.2 | Boundary Controller

In the ideal situation without any resource limitation, the state boundary controller for the couple PDE-PDE system (6) can be constructed as follows:

$$u(t) = (\mathcal{L} \otimes I_n + I_N \otimes H)e(1) \tag{8}$$

where $H = diag\{h_1, \cdots, h_n\}$ is the diagonal matrix to be determined later.

After the application of the constructed event-triggered scheme, the event-triggered boundary control protocol should be implemented as follows:

$$u(t) = (\mathcal{L} \otimes I_n + I_N \otimes H)\bar{e}(1). \tag{9}$$

Thus, the coupled PDE-PDE system (6) can be rewritten as the following compact form:

$$\begin{cases} \partial_t e(z,t) = \partial_{zz} e(z,t), z \in (0,1) \\ \partial_z e(0,t) = 0, \partial_z e(1,t) = -\sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k v(1,t) \\ e(z,0) = \sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k (x^0(z) - \mathbf{1}_N \otimes s_k^0(z)) \\ \partial_t v(z,t) = \partial_z v(z,t), z \in [1, 1+d) \\ v(1+d,t) = (\mathcal{L} \otimes I_n + I_N \otimes H)\bar{e}(1). \end{cases} \tag{10}$$

Remark 6. Note that the constant delay information can be treated as a first-order hyperbolic PDE, which has been proven to be effective in 9 . Then, the delayed MAS can be converted into a coupled complex system, where the coupling terms only act on the spatial boundary position. In this case, the performance analysis of the whole system will become easier than the conventional delayed system. Different from the conventional delay methods (such as constructing more complicated lyapunov function), this paper empoys the delay convertion method to make the control of infinite-time system with delay come true, relax the constraint of the upper bound of delay, and reduce the difficulties of performance analyses.

4 | MAIN RESULTS

In this section, we first analyze the convergence of the system (10) when neglecting the input delay. Then, we examine the effect of the time delay on the convergence analysis. In what follows, we present the Zeno behavior analysis under the constructed event-triggered scheme (7). Finally, we give the corresponding optimal protocols without and with input delay.

4.1 | Convergence Analysis without Input Delay

Theorem 1. Consider the MAS (1) and (2) with any initial states under the designed event-based boundary consensus control protocol (9) when neglecting the input delay. The MAS achieve containment, if there exists a diagonal matrix H with proper dimensions such that the following inequality holds

$$\mathcal{L} \otimes I_n + I_N \otimes H - F_1 > 0 \tag{11}$$

where $F_1 = diag\{\sigma_1\alpha_{11}, \cdots, \sigma_N\alpha_{1N}\}$.

Proof: Consider the following Lyapunov function along with state trajectories of the error system (6):

$$\begin{aligned} V_1(t) =& ||e(.,t)||^2_{L^2(0,1)} + 2\sum_{i=1}^{N} \zeta_i(t) \\ =& \int_0^1 e^T e dz + 2\sum_{i=1}^{N} \zeta_i(t). \end{aligned} \tag{12}$$

Calculating its time derivative with the system trajectories (6) and employing the integrating by parts leads to

$$\begin{aligned}\frac{d}{dt}\int_0^1 e^T e dz =& 2\int_0^1 e^T \partial_t e dz \\ =& 2e^T\partial_z e\Big|_{z=0}^{z=1} - 2\int_0^1 \partial_z e^T \partial_z e dz.\end{aligned} \tag{13}$$

Associated with the event inequality (7), the time derivative of $\sum_{i=1}^N \zeta_i(t)$ can be computed as

$$\begin{aligned}\sum_{i=1}^N \dot{\zeta}_i(t) =& \sum_{i=1}^N \big(-\beta_i\zeta_i(t) + \sigma_i[\alpha_{1i}e_i^T(1)e_i(1) - \tilde{e}_i^T\tilde{e}_i]\big) \\ =& -\sum_{i=1}^N \beta_i\zeta_i(t) + e^T(1)F_1 e(1) - \tilde{e}^T F_2\tilde{e}.\end{aligned} \tag{14}$$

where $F_2 = diag\{\sigma_1, \cdots, \sigma_N\}$.

Combining the event-triggered boundary controller (9), the inequality (11), and the condition (14), we have

$$\begin{aligned}\dot{V}_1(t) =& -2\sum_{i=1}^N \beta_i\zeta_i(t) + 2e^T(1)F_1 e(1) - 2\tilde{e}^T F_2\tilde{e} - 2e^T(1)(\mathcal{L}\otimes I_n + I_N\otimes H)e(1) - 2\int_0^1 \partial_z e^T\partial_z e dz \\ \leq& -2e^T(1)(\mathcal{L}\otimes I_n + I_N\otimes H - F_1)e(1) - 2\sum_{i=1}^N \beta_i\zeta_i(t) - 2\int_0^1 \partial_z e^T\partial_z e dz - 2\tilde{e}^T F_2\tilde{e} \\ \leq& 0\end{aligned} \tag{15}$$

which indicates that the MAS including (1) and (2) can achieve containment asymptotically under the designed event-triggered controller (9), when the input delay is neglected. ■

4.2 | Convergence Analysis with Input Delay

Theorem 2. For the MAS with input delay (1) and (2) under any initial states $\{x_i^0(t)\}_{i\in\mathcal{V}}$ and $\{s_k^0(t)\}_{k\in\Gamma}$, the designed event-based boundary consensus control protocol (7) and (9) can drive it to achieve containment shown in definition 2 in the sense of the norm $\Theta(t)$, if there exist two positive scalars a and γ, a quad $\{\alpha_{1i}, \alpha_{2i}, \beta_i, \sigma_i\}_{k\in\mathcal{V}}$, a diagonal matrix H with proper dimensions satisfying the following condition:

$$\Omega = \begin{bmatrix}\Omega_{11} & \Omega_{12} & 0 & 0 & 0 \\ * & \Omega_{22} & \Omega_{23} & 0 & \Omega_{25} \\ * & * & \Omega_{33} & 0 & 0 \\ * & * & * & \Omega_{44} & \Omega_{45} \\ * & * & * & * & \Omega_{55}\end{bmatrix} < 0 \tag{16}$$

where $\Omega_{11} = -\frac{\pi^2}{2}I + \gamma I$, $\Omega_{12} = \frac{\pi^2}{2}I$, $\Omega_{23} = -\sum_{N+1}^{N+M}\bar{\mathcal{L}}_k$, $\Omega_{22} = -\frac{\pi^2}{2}I + F_1$, $\Omega_{25} = (a+1+d)(\mathcal{L}\otimes I_n + I_N\otimes H)^T$, $\Omega_{33} = -2(a+1)I$, $\Omega_{44} = -F_2$, $\Omega_{45} = (a+1+d)(\mathcal{L}\otimes I_n + I_N\otimes H)^T$, and $\Omega_{55} = -\frac{a+1+d}{2}I$.

Proof: Consider the following Lyapunov function along with state trajectories of the error system (6):

$$\begin{aligned}V_2(t) =& ||e(.,t)||^2_{L^2(0,1)} + ||v(.,t)||^2_{L^2(1,1+d)} + 2\sum_{i=1}^N \zeta_i(t) \\ =& \int_0^1 e^T e dz + \int_1^{1+d}(a+z)v^T v dz + 2\sum_{i=1}^N \zeta_i(t).\end{aligned} \tag{17}$$

By integrating by parts and combining the error system (6), the time derivative of $\int_0^1 e^T e dz$ can be computed as

$$\frac{d}{dt}\int_0^1 e^T e dz = 2e^T \partial_z e\Big|_{z=0}^{z=1} - 2\int_0^1 \partial_z e^T \partial_z e dz. \tag{18}$$

From Lemma 1, the equation (18) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int_0^1 e^T e dz =& 2e^T \partial_z e\Big|_{z=0}^{z=1} - 2\int_0^1 \partial_z e^T \partial_z e dz \\ \leq& 2e^T \partial_z e\Big|_{z=0}^{z=1} - \frac{1}{2}\pi^2 \int_0^1 (e(z,t)-e(1,t))^T (e(z,t)-e(1,t)) dz.\end{aligned} \tag{19}$$

Similarly, the time derivative of $\int_1^{1+d}(1+z)v^T v dz$ can be derived as

$$\frac{d}{dt}\int_1^{1+d}(a+z)v^T v dz = \quad 2(a+z)v^T v\Big|_{z=1}^{z=1+d} - 2\int_1^{1+d} v^T v dz. \tag{20}$$

Substituting (14), (19), and (20) into (17), yields

$$\begin{aligned}\dot{V}_2(t) =& -\frac{\pi^2}{2}\int_0^1 (e(z,t)-e(1,t))^T (e(z,t)-e(1,t)) dz + 2e^T \partial_z e\Big|_{z=0}^{z=1} + 2(a+z)v^T v\Big|_{z=1}^{z=1+d} - 2\sum_{i=1}^{N} \beta_i \zeta_i(t) \\ & - 2\int_1^{1+d} v^T v dz + e^T(1) F_1 e(1) - \tilde{e}^T F_2 \tilde{e} \\ =& -\frac{\pi^2}{2}\int_0^1 (e-e(1))^T (e-e(1)) dz + 2(a+1+d)u^T(t)u(t) - 2(a+1)v^T(1)v(1) + e^T(1)F_1 e(1) - \tilde{e}^T F_2 \tilde{e} \\ & + \gamma\int_0^1 e^T e dz - 2e^T(1)\sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k v(1,t) - 2\int_1^{1+d} v^T v dz - 2\sum_{i=1}^{N}\beta_i \zeta_i(t) - \gamma\int_0^1 e^T e dz \\ \leq& \int_0^1 \xi^T \Omega \xi dz - \min\left\{2, \min\{2\beta_i | i \in \mathcal{V}, \gamma\}\right\} V(t)\end{aligned} \tag{21}$$

where $\xi = [e^T, e^T(1), v^T(1), \tilde{e}^T]^T$.

Since $\Omega < 0$ is ensured in Theorem 1, it follows that

$$\dot{V}_2(t) \leq -\min\left\{2, \min\{2\beta_i | i \in \mathcal{V}\}, \gamma\right\} V(t). \tag{22}$$

Thus, the containment performance given in definition 2 is guaranteed, which completes the proof. ■

Remark 7. Theorem 1 gives the sufficient condition (11) of the event-based boundary consensus control protocol for the non-delay case, and Theorem 2 (16) for the delay case. Inequality (11) presents the relationship among the interaction topology $\mathcal{L}$, the triggering parameters σ_i and α_i, and the controller gain matrix H. When the interaction topology is fixed, bigger triggering parameters usually requires bigger controller gain matrix. It can be seen that the required controller (9) is directly influenced by the constructed triggering condition (7). But, this influence is indirect, which is reflected in inequitility (16) for the delay case, owing to the presence of the free parameter α.

4.3 | Zeno Behavior Analysis

From the definition of $\tilde{e}_i(1)$ and the system (10), we have

$$\dot{\tilde{e}}_i(1) = -\dot{e}_i(1) = \sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k v_i(1,t). \tag{23}$$

Taking the time derivative of $||e_i(1)||$ yields

$$\begin{aligned}\frac{d||e_i(1)||}{dt} &= \frac{e_i^T(1)}{||e_i(1)||}\dot{e}_i(1) \\ &\leq ||\dot{e}_i(1)|| \leq ||\sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k|| \, ||v_i(1,t)||.\end{aligned} \tag{24}$$

When the containment of the system (10) is achieved, we have the deduction that the virtual boundary value $v_i(l,t)$ and the boundary state $e_i(1)$ are bounded. Then, $v_i(l) - e_i(1)$ is also bounded. Denote $\mathcal{L}^k = ||\sum_{k=N+1}^{N+M} \bar{\mathcal{L}}_k||$ and assume that the constant c is the upper bound of $v_i(l) - e_i(1)$, we have

$$\frac{d||\tilde{e}_i(1)||}{dt} \leq ||\mathcal{L}^k|| \, ||e_i(1)|| + c||\mathcal{L}^k||. \tag{25}$$

For a non-negative function $\phi = ||\mathcal{L}^k||\phi + c||\mathcal{L}^k||$, it follows that $\phi(t) \leq c(\mathrm{e}^{||\mathcal{L}^k||t} - 1)$. Similarly, during $t \in [t_k^i, t_{k+1}^i)$, the following inequality holds:

$$||\tilde{e}_i(1, t + t_k^i)| \leq c(\mathrm{e}^{||\mathcal{L}^k||t} - 1). \tag{26}$$

We can see that an event is not triggered for $t \in [t_k^i, t_{k+1}^i)$, if the following condition holds:

$$||\tilde{e}_i(1)| \leq \alpha_{2i}\zeta_i(t). \tag{27}$$

Associated the inequality (26) with the condition (27), the lower bound of $t_{k+1}^i - t_k^i$ must satisfy the following condition:

$$\alpha_{2i}\zeta_i(t) \leq c(\mathrm{e}^{||\mathcal{L}^k||(t_{k+1}^i - t_k^i)} - 1). \tag{28}$$

Thus, we have

$$t_{k+1}^i - t_k^i \geq \frac{1}{||\mathcal{L}^k||} \ln\left(1 + \frac{1}{c}\sqrt{\alpha_{2i}\zeta_i(t)}\right) > 0 \tag{29}$$

which indicates that the inter-event time interval is always positive. Therefore, there is no Zeno behavior for all agents.

4.4 | Optimal Control Protocols

Since the control input is only related to the designed controller gain, the following performance function is usually adopted to minimize the control effort for the overall system (6):

$$J = \int_{t=0}^{\infty} u^T(t)(I_N \otimes Q)u(t) \tag{30}$$

with Q being the specified positive definite matrix.

However, owing to the coupling property of the control input (9), this paper modifies the performance function by only taking the controller gain matrix H into account, which is described as follows:

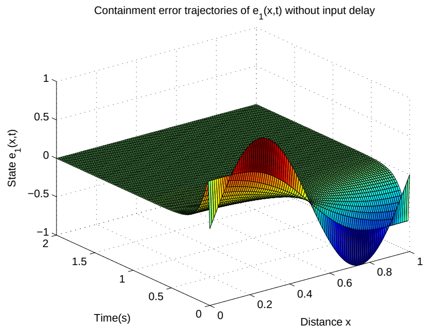
$$J = \int_{t=0}^{\infty} e^T(1)(I_N \otimes HQH)e(1). \tag{31}$$

As a result, for the non-input delay case, the optimal containment protocol for the MAS (1) should be obtained by solving the following optimization problem:

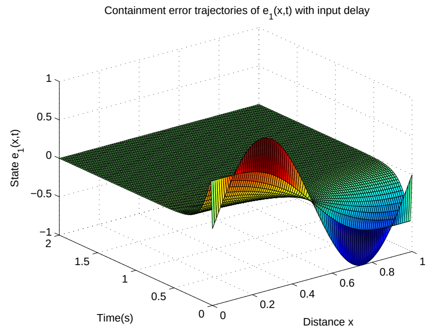
$$\begin{cases} \min Tr(HQH) \\ s.t. \quad \lambda I_N \otimes I_n + I_N \otimes H - F_1 > 0 \end{cases} \tag{32}$$

and for the input delay case

$$\begin{cases} \min Tr(HQH) \\ s.t. \quad \Omega < 0. \end{cases} \tag{33}$$



(a)



(b)

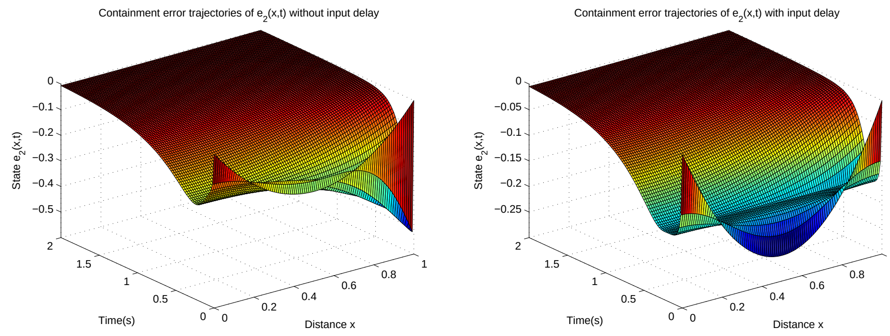
FIGURE 1 Containment error trajectories $e_1(z,t)$ with or without input delay.

Remark 8. It is noted that Theorem 1 and Theorem 2 reveal that the convergence of the MAS is related to the information including the triggering parameters σ_i, the Laplacian matrix $\mathcal{L}$, and the designed control gain matrix H, and the controller design depends on the positive and negative property of the term $u(t)$. That leads to a fact that the co-design of the event scheme (7) and the controller (8) requires the prior knowledge about the complete topology information. The problem is that the fact is not always practical when more and more agents are involved in the MAS (1). To overcome this problem, combining the properties of the Laplacian matrix $\mathcal{L}$ allows us to replace $\mathcal{L}$ in (11) with λ_2 and to replace $\mathcal{L}$ in (16) with λ_N, where λ_2 and λ_N are the smallest non-zero eigenvalue and the biggest eigenvalue, respectively.

5 | SIMULATION STUDIES

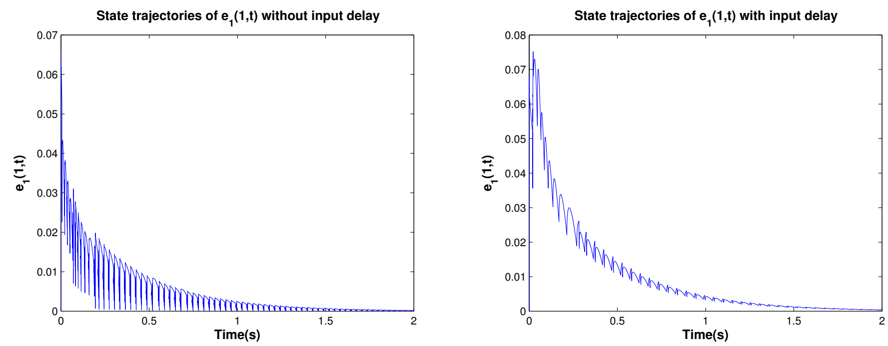
In this section, a numerical simulation is carried out with the proposed event-based boundary protocol (9) to prove the validity of the main results. Consider the overall error system (5) with a fixed topology, where two leaders and two followers are involved. The corresponding parameters are given as:
$e_1(z,0) = \sin(2\pi z)$, $e_2(z,0) = z(z-1)$, $\alpha_{1_i} = 0.01$, $\alpha_{2i} = 0.12$, $\beta_i = 1.1$, $\sigma_i = 9.6524$, $i = 1,2$, $d = 0.2s$,
$L = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

From Theorem 1 and Theorem 2, two event-based boundary protocols (9) with the corresponding dynamic event-triggered scheme (7) can be obtained. Then, the simulation results including two cases are shown in Fig. 1.a-Fig. 5.b, where the non-delay case and the delay case are given in the figures labelled a and the figures labelled b, respectively. The containment error trajectories plotted in Figs. 1-2, where $e_1(z,t)$ is shown in Fig. 1 and $e_2(z,t)$ is shown in Fig. 2, show that the design controller (9) is effective in controlling the containment problem.The boundary containment error trajectories are depicted in Figs. 3-4, where $e_1(1,t)$ is shown in Fig. 3 and $e_2(1,t)$ is shown in Fig. 4. As is seen from figs. 3-4, where the fluctuation value of boundary containment error trajectorie $e_2(1,t)$ is bigger in the delay case than the non-delay case, we can conclude that the derived protocol from (16) behaves better than the protocol from (11) in controlling the MAS (1).The triggering instants are displayed in Fig. 5 including Fig. 5.a and Fig. 5.b. Form these figures, it is shown that the input delay results in a bigger error and its negative effect to system performances can be compensated by the developed protocols. Meanwhile, in order to show the superiority of the proposed dynamic event-triggered scheme, comparisons have been added with the corrresponding static event-triggered scheme (that is, set $\alpha_{2i} = 0$). The triggering instants are depicted in Fig. 6 including Fig. 6.a and Fig. 6.b. The number of packages under static ETS and dynamic ETS are listed in Table I. It can be found easily that the proposed dynamic ETS behaves better than the corresponding static ETS in saving network bandwidth resources.



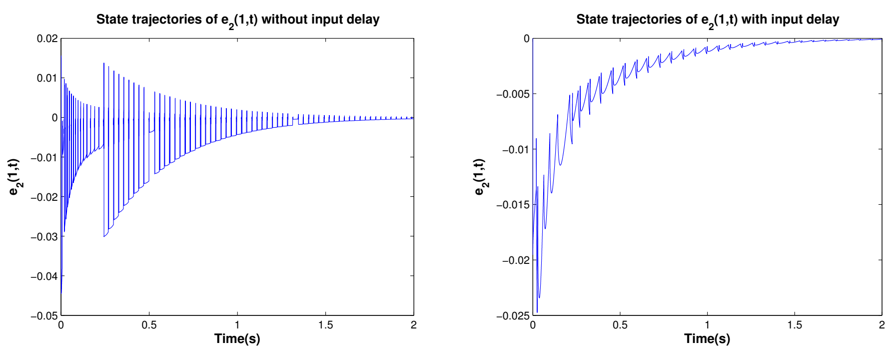
(a) (b)

FIGURE 2 Containment error trajectories $e_2(z, t)$ with or without input delay.



(a) (b)

FIGURE 3 Boundary containment error trajectories $e_1(1, t)$ with or without input delay.

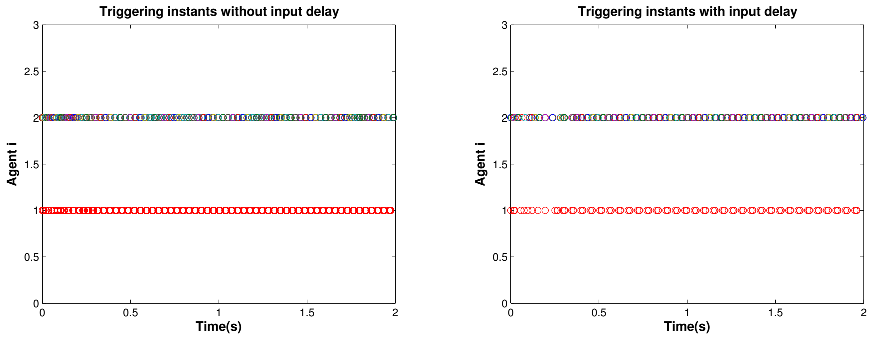


(a) (b)

FIGURE 4 Boundary containment error trajectories $e_2(1, t)$ with or without input delay.

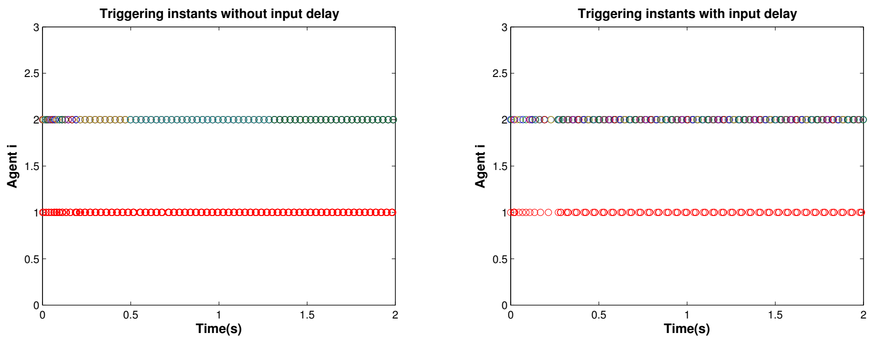
6 | CONCLUSION

This paper has investigated the containment control problem for a class of MASs modelled by parabolic PDEs through an event-triggered control approach. The input delay dynamics can be viewed as a hyperbolic PDE. By utilizing the spatial boundary information, two event-based boundary control protocols have been proposed to ensure the containment of the PDEs with



(a) (b)

FIGURE 5 Triggering instants under the proposed dynamic event-triggered scheme (7).



(a) (b)

FIGURE 6 Triggering instants under the cooresponding static event-triggered scheme (7 with $\alpha_{2i} = 0$).

TABLE 1 Triggering numbers for agents with different triggering schemes during $[0, 2]s$.

agents	static ETS without delay	dynamic ETS without delay	static ETS with delay	dynamic ETS with delay
agent 1	456	412	116	102
agent 2	499	401	324	298

and without input delay. For these two cases, the corresponding optimal consensus protocols can be obtained by solving two optimization problems, where the global topology information is not essential. The exclusion of the Zeno behavior is also provided. Finally, a simulation example is performed for the proposed control strategy.

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