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RESEARCH ARTICLE

# Asymptotical stability for fractional-order Hopfield neural networks with multiple time delays

Zichen Yao[1] | Zhanwen Yang*[1] | Yongqiang Fu[1] | Jiachen Li[2]

[1]School of Mathematics, Harbin Institute of Technology, Harbin, China
[2]School of Astronautics, Harbin Institute of Technology, Harbin, China

**Correspondence**
*Zhanwen Yang, School of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China.
Email: yangzhan_wen@hit.edu.cn

**Summary**

This paper is concerned with the asymptotical stability of fractional-order Hopfield neural networks with multiple delays. The problem is actually a generalization of stability for linear fractional-order delayed differential equations: ${}_0^C\mathrm{D}_t^{\alpha} X(t) = MX(t) + CX(t-\tau)$, which is widely studied when $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$. However, the stability is rarely known when $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \le \frac{\pi}{2}$. Hence, this work is mainly devoted to the stability analysis for $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \le \frac{\pi}{2}$. By virtue of the Laplace transform method and a decoupling technique for the characteristic equation, we propose a necessary and sufficient condition to ensure the stability, which improves the existing stability results for $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$. Afterward, by a linearization technique, a necessary and sufficient stability condition is also presented for fractional-order Hopfield neural networks with multiple delays. The conditions are established by delay-independent coefficient-type criteria. Finally, several numerical simulations are given to show the effectiveness of our results.

**KEYWORDS:**
Caputo's fractional derivative, Hopfield neural networks, time delays, asymptotical stability, nonlinear equations.

## 1 | INTRODUCTION

Artificial neural networks, or neural networks, are established by the association of neurons to simulate the neural system of human brain and form an artificial system. In recent decades, several kinds of neural networks have been investigated extensively such as Hopfield neural networks, Memristor-based neural networks, Competitive neural networks, Cohen-Grossberg neural networks and so on, see in 6,19,24,33. In 1982, Hopfield neural networks were first studied in 13. Since then, Hopfield neural networks have been applied in various optimization problems, associative memories, and engineering problems, considerable efforts have been put into the investigation of their analysis and synthesis[10,14,32]. As we all known, due to the finite switching speeds of the amplifiers, time delays are unavoidable in neural networks and they may disturb the dynamic systems, resulting in oscillations or instability. Therefore, qualitative analysis of neurodynamics, such as some fundamental properties of stability and oscillation, is indispensable for practical design of neural-network models and tools. Some stability criteria for Hopfield neural networks with delays have been proposed in 7,26. The general Hopfield neural networks with time delay can be expressed as follows:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t-\tau_{ij})), \quad i = 1, 2, \cdots, n,$$

where $n$ is the number of units in the neural networks, $x_i(t), i = 1, 2, \cdots, n$ represent the state variables at time $t$, $a_i, b_{ij}, c_{ij}, 1 \leq i, j \leq n$ present the self-regulating and synaptic connection weight of the neuron, $f, g$ denote the measures of response or activation to its incoming potentials, $\tau_{ij}, 1 \leq i, j \leq n$ represent time delays.

It is notable that the above mentioned papers about neural networks are on the basic of integer calculus. Fractional calculus, as a representative mathematical notion with a long history, is the elevation of integer calculus to derivation and integration of arbitrary non-integer order. Compared to the classical integer derivative, the fractional derivative is global and non-transient in nature, implying that all previous states have different effects on future states, which fits perfectly with the motion processes of neurons with memory and hereditary properties, see 21,22,30. Regarding the benefits, it has been successfully integrated into the modelling of neural networks with delay, especially in Hopfiled neural networks with delay. For example, in 1, based on fractional-order delayed Hopfield neural networks, the initially generated static substitution boxes are evolved to improve the nonlinearity and thus construct highly nonlinear key-dependent dynamic S-boxes to improve the security and robustness of the overall cryptosystem. Hence, we consider fractional-order Hopfield neural networks with time delays:

$$
{}_0^C\mathrm{D}_t^{\alpha} x_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t-\tau_{ij})), \quad i = 1, 2, \cdots, n, \tag{1}
$$

where $n$ is the number of units in the neural networks, $x_i(t), i = 1, 2, \cdots, n$ represent the state variables at time $t$, $a_i > 0, i = 1, 2, \cdots, n$ is the self-regulating parameters of the neuron, ${}_0^C\mathrm{D}_t^{\alpha}$ is the Caputo's fractional operator of order $\alpha (0 < \alpha < 1)$, $\tau_{ij}, 1 \leq i, j \leq n$ represent time delays, $f_j(x_j(t))$ and $g_j(x_j(t-\tau_{ij}))$ denote, respectively, the measures of response or activation to its incoming potentials of the unit $j$ at time $t$ and $t - \tau_{ij}$, $b_{ij}, c_{ij}$ are constants with $b_{ij}$ denoting the synaptic connection weight of the unit $j$ to unit $i$ at time $t$, and $c_{ij}$ denoting the synaptic connection weight of the unit $j$ to unit $i$ at time $t - \tau_{ij}$. Besides, we assume that $X^* = (x_1^*, x_2^*, \cdots, x_n^*)^\top$ is the equilibrium solution of (1) and the initial condition is $X_0 = \phi(s) = (\phi_1(s), \phi_2(s), \cdots, \phi_n(s))^\top \in \mathbf{C}([-\tau, 0], \mathbb{R}^n)$, where $\tau = \max_{1 \leq i,j \leq n} \{\tau_{ij}\}$.

Just like the integer-order Hopfield neural networks, qualitative properties of fractional-order Hopfield neural networks with delay have also been widely studied. Many interesting results in a great deal aspects of dynamical behaviours were obtained, see 6,28,29. For example, in 28, Wang et al. derived the stability conditions for two and three dimensional fractional-order delayed Hopfiled neural networks with ring structures. Especially, in 29, they also proved the existence and uniqueness of the equilibrium point for fractional-order Hopfield neural networks with time delay and provided the asymptotical stability conditions. The main result in 29 is the following theorem:

**Theorem 1.** Assume that the neuron activation functions $f_j, g_j$ are Lipschitz continuous, i.e. there exist positive constants $L_j, K_j, j = 1, \cdots, n$ such that

$$
\left| f_j(u) - f_j(v) \right| < L_j |u - v|, \left| g_j(u) - g_j(v) \right| < K_j |u - v|, u, v \in \mathbb{R}
$$

and

$$
\hat{K} < \gamma \sin \frac{\alpha \pi}{2}, \tag{2}
$$

where

$$
\hat{K} = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \left| c_{ji} \right| K_i \right), \gamma = \min_{1 \leq i \leq n} \left( a_i - \sum_{j=1}^{n} \left| b_{ji} \right| L_i \right).
$$

Then there exists a unique equilibrium solution for (1) and the equilibrium solution is asymptotically stable.

In fact, by denoting $A = \mathrm{diag}(a_1, a_2, \cdots, a_n), \overline{B} = (b_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$, $F(X(t)) = (f_1(x_1(t)), f_2(x_2(t), \cdots, f_n(x_n(t)))^\top$, and classifying $c_{ij} g_j(x_j(t-\tau_{ij}))$ as $G^k(X(t-\tau_k)) = (g_1^{(k)}(x(t-\tau_k)), g_2^{(k)}(x(t-\tau_k)), \cdots, g_n^{(k)}(x(t-\tau_k)))^\top, k = 1, 2, \cdots, m, 1 \leq m \leq n^2$ with regard to $\tau_{ij}$, system (1) can be rewritten in the following vector form:

$$
{}_0^C\mathrm{D}_t^{\alpha} X(t) = -AX(t) + \overline{B} F(X(t)) + \sum_{k=1}^{m} G^k(X(t-\tau_k)), \tag{3}
$$

which is a generalization of linear fractional-order differential equations with delay:

$$
{}_0^C\mathrm{D}_t^{\alpha} X(t) = MX(t) + CX(t-\tau). \tag{4}
$$

Naturally, the delay term in (4) can be seen as a disturbance. When the delay term degenerates, i.e. $C = 0$, Matignon[20] proved that the zero solution is asymptotically stable if and only if $|\mathrm{Arg}(\lambda_M)| > \frac{\alpha\pi}{2}$ ($\lambda_M$ denote the eigenvalues of matrix $M$). However, including Theorem 1, the existing stability results for (4) necessitate $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$, for example, fractional Lyapunov stability theorem[12,15], fractional Halanay inequality[27], fractional comparison theoem[29] and fractional Razumikhin-type stability theorem[2], which is violated by $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \leq \frac{\pi}{2}$ in fractional-order Hopfiled neural networks with multiple time delays (1) (see, for example, 29). Hence, the key question now is:

(**1**) What is the dynamic behaviour of the solution for (4) when $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \leq \frac{\pi}{2}$?

In fact, the imaginary part of $\lambda_M$ also affects the dynamic behaviour of solutions heavily, which was investigated in 31. Hence, we bring out the second question:

(**2**) Can the existing stability results be improved when $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$?

Hence, motivated from both mathematical theories and practical applications, we investigate the stability of fractional-order Hopfield neural networks with time delays (1). By virtue of linearization technique and Laplace transform, the local stability of (1) is expressed by the roots of the characteristic equation. Notwithstanding, it is very difficult to get the roots because of transcendence and singularity. In order to cope with the difficulties, a widely used method is the root locus curve method (or boundary curve method), see 11,16. However, the root locus curve is not available for our system due to the high dimension and multiple delays. Hence, we choose a decoupling technique (see 31) to approach the boundary of the stability region . Following such a path, a necessary and sufficient condition for the stability of fractional-order Hopfiled neural networks with multiple time delays (1) is attained in a coefficient-type criterion. The noteworthy contributions of this paper can be stated as follows.

- We give a boundary of the stability region for linear fractional-order differential equations with delay (4), which is an open problem for six years[3].
- A framework for stability analysis of fractional-order Hopfiled neural networks with multiple time delays (1) is established. We obtain a necessary and sufficient condition in a coefficient-type criterion, which is delay-independent.
- Compared with the existing results, such as 29, our results not only cover $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \leq \frac{\pi}{2}$ but also improve the results for $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$.

The rest of this article is set as follows. In section 2, we provide some preliminaries and model formulation. In section 3, we perform a framework of stability analysis for fractional-order Hopfield neural networks with multiple delays. In section 4, we present three illustrative examples to confirm our theoretical results. In section 5, conclusions and future work are provided.

## 2 | PRELIMINARIES AND MODEL FORMULATION

**Definition 1.** [17] The Caputo's fractional derivative of order $\alpha$ for a function $x(t) \in \mathbf{C}^n([0,\infty),\mathbb{R})$ is defined by

$${}_0^C\mathrm{D}_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $t \geq 0$ and $n$ is a positive integer such that $n-1 < \alpha < n$. Particularly, when $0 < \alpha < 1$,

$${}_0^C\mathrm{D}_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s)}{(t-s)^{\alpha}} \mathrm{d}s.$$

In this paper, Laplace transform is an important technique for the stability analysis, Thus, we give the Laplace transform of the Caputo's fractional derivative.

**Lemma 1.** [17] The Laplace transform of the Caputo's fractional derivative is

$$\mathfrak{L}[{}_0^C\mathrm{D}_t^\alpha x(t)](s) = s^\alpha \mathfrak{L}[x(t)](s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} x^{(k)}(0), \quad n-1 < \alpha < n, \tag{5}$$

where $\mathfrak{L}[x(t)](s)$ is the Laplace transform of $x(t)$, i.e., $\mathfrak{L}[x(t)](s) = \int_0^t e^{-st}x(t)\mathrm{d}t$. Particularly, when $0 < \alpha < 1$,

$$\mathfrak{L}[{}_0^C\mathrm{D}_t^\alpha x(t)](s) = s^\alpha \mathfrak{L}[x(t)](s) - s^{\alpha-1}x(0).$$

In addition, it is easy to obtain the Laplace transform for delay term $x(t-\tau)(\tau \in \mathbb{R}^+)$ as follows.

$$\mathfrak{L}[x(t-\tau)](s) = \int_0^{+\infty} x(t-\tau)e^{-st}\mathrm{d}t = e^{-s\tau}\mathfrak{L}[x(t)](s) + e^{-s\tau}\int_{-\tau}^{0} x(t)e^{-st}\mathrm{d}t. \tag{6}$$

In the sequel, we consider the stability problem of fractional-order Hopfiled neural networks with multiple delays (1). Just like the classical ordinary differential equations, the stability of equilibrium solutions for fractional-order differential equations can be investigated by linearization technique, see 8,9. Thus, we linearize system (3) around the equilibrium solution $X^*$ to get

$${}_0^C\mathrm{D}_t^\alpha X(t) = -AX(t) + \overline{B}HX(t) + C_1X(t-\tau_1) + \cdots + C_mX(t-\tau_m),$$

where $X(t-\tau_k) = (x_1(t-\tau_k), x_2(t-\tau_k), \cdots, x_n(t-\tau_k))^\top$. In addition,

$$H = \frac{\partial F}{\partial X}|_{X^*}, C_k = \frac{\partial G^k}{\partial X}|_{X^*}, k = 1, 2, \cdots, m. \tag{7}$$

Then, take $B = \overline{B}H$ to get

$${}_0^C\mathrm{D}_t^\alpha X(t) = -AX(t) + BX(t) + C_1X(t-\tau_1) + \cdots + C_mX(t-\tau_m). \tag{8}$$

*Remark 1.* Note that the non-delay term of (8) and (4) are the same if we denote $M = -A + B$. Nevertheless, because $A$ and $B$ present different effects of neurons, i.e., self-regulating and synaptic connection weight of neurons, we do not simplify the notation.

**Definition 2.** The zero solution of (8) is said to be stable if, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for any $\|X_0\| \leq \delta$, we have $\|X(t)\| < \varepsilon$, for all $t > 0$. The zero solution of equation (8) is said to be asymptotically stable if it is stable and there exists $\hat{\delta} > 0$, such that $\lim_{t\to+\infty} \|X(t)\| = 0$, whenever $\|X_0\| < \hat{\delta}$.

**Lemma 2.** [20] Consider the fractional-order differential equations (3) with $F(X) = 0, G^{(k)}(X) = 0, k = 1, 2, \cdots, m$. Then the zero solution of (3) is asymptotically stable if and only if $|\mathrm{Arg}(\lambda_A)| > \frac{\alpha\pi}{2}$, where $|\mathrm{Arg}(\lambda_A)|$ denote the minimum value of the absolute value of argument for matrix $A$. Besides, we regard the argument of 0 as $\mathrm{Arg}(0) = 0$.

## 3 | MAIN RESULTS

In this section, we will present a necessary and sufficient condition to guarantee the asymptotical stability of (8) and (1). At first, we present the stability region of (8) according to Definition 2:

$$S_{\alpha,\tau} = \left\{ (A, B, C_1, \cdots, C_m) \in \mathbb{R}^{n\times n} \times \cdots \times \mathbb{R}^{n\times n} | \lim_{t\to+\infty} \|X(t)\| = 0, \text{for all } X_0 \right\}.$$

Taking Laplace transform on both sides of (8) and by virtue of (5) and (6), we get

$$\begin{cases} s^\alpha Y_1(s) - s^{\alpha-1}\phi_1(0) = -a_1Y_1(s) + \sum\limits_{j=1}^{n} b_{1j}Y_j(s) + \sum\limits_{k=1}^{m}\sum\limits_{j=1}^{n} c_{1j}^{(k)}e^{-s\tau_k}\left(Y_j(s) + \int_{-\tau_k}^{0}\phi_j(t)\mathrm{d}t\right), \\ s^\alpha Y_2(s) - s^{\alpha-1}\phi_2(0) = -a_2Y_2(s) + \sum\limits_{j=1}^{n} b_{2j}Y_j(s) + \sum\limits_{k=1}^{m}\sum\limits_{j=1}^{n} c_{2j}^{(k)}e^{-s\tau_k}\left(Y_j(s) + \int_{-\tau_k}^{0}\phi_j(t)\mathrm{d}t\right), \\ \cdots \\ s^\alpha Y_n(s) - s^{\alpha-1}\phi_n(0) = -a_nY_n(s) + \sum\limits_{j=1}^{n} b_{nj}Y_j(s) + \sum\limits_{k=1}^{m}\sum\limits_{j=1}^{n} c_{nj}^{(k)}e^{-s\tau_k}\left(Y_j(s) + \int_{-\tau_k}^{0}\phi_j(t)\mathrm{d}t\right), \end{cases}$$

where $Y_i(s) = \mathfrak{L}[x_i(t)](s)$ is the Laplace transform of $x_i(t)(i = 1, 2, \cdots, n)$ and $C_k = (c_{ij}^{(k)})_{n\times n}(k = 1, 2, \cdots, m)$. Then, we can rewrite this equation as follows:

$$\Delta(s)\begin{pmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_n(s) \end{pmatrix} = \begin{pmatrix} d_1(s) \\ d_2(s) \\ \vdots \\ d_n(s) \end{pmatrix}, \tag{9}$$

in which

$$\begin{cases} d_1(s) = s^{\alpha-1}\phi_1(0) + \sum\limits_{k=1}^{m}\sum\limits_{j=1}^{n} c_{1j}^{(k)} e^{-s\tau_k} \int\limits_{-\tau_k}^{0} \phi_j(t)\mathrm{d}t, \\ d_2(s) = s^{\alpha-1}\phi_2(0) + \sum\limits_{k=1}^{m}\sum\limits_{j=1}^{n} c_{2j}^{(k)} e^{-s\tau_k} \int\limits_{-\tau_k}^{0} \phi_j(t)\mathrm{d}t, \\ \cdots \\ d_n(s) = s^{\alpha-1}\phi_n(0) + \sum\limits_{k=1}^{m}\sum\limits_{j=1}^{n} c_{nj}^{(k)} e^{-s\tau_k} \int\limits_{-\tau_k}^{0} \phi_j(t)\mathrm{d}t, \end{cases}$$

$$\Delta(s) = \begin{pmatrix} s^\alpha + a_1 - b_{11} - \sum\limits_{k=1}^{m} c_{11}^{(k)} e^{-s\tau_k} & -b_{12} - \sum\limits_{k=1}^{m} c_{12}^{(k)} e^{-s\tau_k} & \cdots & -b_{1n} - \sum\limits_{k=1}^{m} c_{1n}^{(k)} e^{-s\tau_k} \\ -b_{21} - \sum\limits_{k=1}^{m} c_{21}^{(k)} e^{-s\tau_k} & s^\alpha + a_2 - b_{22} - \sum\limits_{k=1}^{m} c_{22}^{(k)} e^{-s\tau_k} & \cdots & -b_{2n} - \sum\limits_{k=1}^{m} c_{2n}^{(k)} e^{-s\tau_k} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} - \sum\limits_{k=1}^{m} c_{n1}^{(k)} e^{-s\tau_k} & -b_{n2} - \sum\limits_{k=1}^{m} c_{n2}^{(k)} e^{-s\tau_k} & \cdots & s^\alpha + a_n - b_{nn} - \sum\limits_{k=1}^{m} c_{nn}^{(k)} e^{-s\tau_k} \end{pmatrix}.$$

We call $\Delta(s)$ the characteristic matrix of system (8) and $\det(\Delta(s)) = 0$ the characteristic equation of (8). In fact, the distribution of roots for characteristic equation $\det(\Delta(s)) = 0$ totally determines the stability of system (8). The characteristic equation can be written in a matrix form, i.e.,

$$\det(\Delta(s)) = \det\left(s^\alpha I + A - B - C_1 e^{-s\tau_1} - \cdots - C_m e^{-s\tau_m}\right) = 0, \tag{10}$$

where $I = \mathrm{diag}(1, 1, \cdots, 1)$ denotes the identity matrix.

**Theorem 2.** If all the roots of characteristic equation $\det(\Delta(s)) = 0$ have negative real parts, then the zero solution of system (8) is asymptotically stable.

*Proof.* If all the roots of characteristic equation $\det(\Delta(s)) = 0$ have negative real parts for $\alpha \in (0, 1)$, then $\Delta(s)$ is an invertible matrix. From equation (9), we get

$$Y(s) = \Delta(s)^{-1} D(s),$$

where $Y(s) = (Y_1(s), Y_2(s), \cdots, Y_n(s))^\top$, $D(s) = (d_1(s), d_2(s), \cdots, d_n(s))^\top$. According to the Final Value Theorem[31] of the Laplace transform, we have

$$\lim_{t\to\infty} X(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \Delta(s)^{-1} D(s)s = 0.$$

Hence, the zero solution of system (8) is asymptotically stable. □

By virtue of Theorem 2, the stability region $S_{\alpha,\tau}$ can be equivalently described as follows:

$$S_{\alpha,\tau} = \left\{(A, B, C_1, \cdots, C_m) \in \mathbb{R}^{n\times n} \times \cdots \times \mathbb{R}^{n\times n} \,|\, \det(\Delta(s)) = 0 \text{ implies that } \mathrm{Re}(s) < 0\right\}.$$

*Remark 2.* The characteristic equations are significant in stability analysis for both integer-order and fractional-order differential equation, with or without delays. As a generalization of classical ordinary differential equations, Theorem 2 has been used extensively in many models. For example, in 28 and 29, such approaches have been used in stability analysis for fractional-order Hopfield neural networks with delay. Although Theorem 2 is so elegant, it still has some disadvantages. One of them is that it is not an algebraic criterion, i.e., difficult to verify in practical applications. Hence, further analysis for various types of characteristic equations have been carried out in 3,4 and 31 to give a complete or relatively complete criterion for ensure the stability. However, as we known, there is not a relatively complete stability analysis for characteristic equation (3). This is mainly due to the fact that $\det(\Delta(s)) = 0$ is a transcendental equation and $\det(\Delta(s))$ is not analytic at point $s = 0$.

In order to cope with the two difficulties above, many scholars have applied the root locus curve method (or boundary curve method), see 11,16. It is, an effective method for exploiting the critical case to analyse the sign of real parts of the characteristic equation. However, the root locus curve is not available for (8) due to the high dimension of our system and multiple delays. Hence, following the path in 31, we utilize a decoupling method for the characteristic equation. Denote two $n$-dimensional systems as

$$\begin{cases} Z = s^{\alpha} I + A - B, \\ Z = C_1 e^{-s\tau_1} + C_2 e^{-s\tau_2} + \cdots + C_m e^{-s\tau_m} \end{cases}$$

and two regions as

$$\begin{cases} \mathcal{A} = \{Z = s^{\alpha} I + A - B, \mathrm{Re}(s) \geq 0\}, \\ \mathcal{B} = \left\{Z = C_1 e^{-s\tau_1} + C_2 e^{-s\tau_2} + \cdots + C_m e^{-s\tau_m}, \mathrm{Re}(s) \geq 0\right\}. \end{cases}$$

Therefore, $\mathcal{A} \cap \mathcal{B} = \emptyset$ implies that $\det(\Delta(s)) \neq 0$ for $\mathrm{Re}(s) \geq 0$, i.e.,

$$S_{\alpha,\tau} \supset \left\{(A, B, C_1, \cdots, C_m) \in \mathbb{R}^{n\times n} \times \cdots \times \mathbb{R}^{n\times n} | \ \mathcal{A} \cap \mathcal{B} = \emptyset\right\}.$$

*Remark 3.* We investigate the restricted region instead of $S_{\alpha,\tau}$ to express the stability of 1 by the intersection of sets $\mathcal{A}$ and $\mathcal{B}$. Without getting the exact roots of the characteristic equation, we only need to verify whether $\mathcal{A} \cap \mathcal{B} = \emptyset$, then we get whether the roots of (3) have only negative real parts. In fact, it is extremely hard to express the complete stability region $S_{\alpha,\tau}$, which has been an open problem for six years, as mentioned in 3. Hence, we intend to provide some conditions to give a critical boundary of $S_{\alpha,\tau}$ in the sense of $\mathcal{A} \cap \mathcal{B} = \emptyset$.

In the next, two conditions are offered to investigate the stability of (8).

**Condition 1** (**C1**). $|\mathrm{Arg}(\lambda_{-A+B})| > \frac{\alpha\pi}{2}$.

**Condition 2** (**C2**). There exists a norm such that

$$\sum_{k=1}^{m} \left\|C_k\right\| < \inf_{|\mathrm{Arg}(z)| \leq \frac{\alpha\pi}{2}} \left\|(zI + A - B)^{-1}\right\|^{-1}.$$

*Remark 4.* By virtue of Lemma 2, we know that if (**C1**) is satisfied, then the zero solution of the non-delay system is asymptotically stable. On the other hand, (**C2**) provides an algebraic criterion, which is derived by the decoupling technique of characteristic equation. In the sequel, several useful lemmas are proved.

**Lemma 3.** If (**C1**) and (**C2**) hold, then there exist no roots of characteristic equation $\det(\Delta(s)) = 0$ in a neighbourhood of $s = 0$.

*Proof.* According to (**C1**) and (**C2**), we have that $(A - B)$ is invertible and $\left\|(zI + A - B)^{-1}\right\| \sum_{k=1}^{m} \left\|C_k\right\| < 1$ for all $|\mathrm{Arg}(z)| \leq \frac{\alpha\pi}{2}$. Therefore, there exists a positive number $M_1$ such that

$$\frac{1}{\|A - B\|} \leq M_1.$$

What is more, there exists a positive number $R > 1$ such that

$$R \left\|(zI + A - B)^{-1}\right\| \sum_{k=1}^{m} \left\|C_k\right\| < 1 \tag{11}$$

holds for all $|\mathrm{Arg}(z)| \leq \frac{\alpha\pi}{2}$. Especially, (11) holds for $z = 0$. Hence, we denote

$$M_2 = \frac{1}{1 - R\left\|(A - B)^{-1}\right\| \sum_{k=1}^{m} \left\|C_k\right\|}$$

and

$$\delta = \min\left\{-\frac{1}{\tau} \ln R, (M_1 M_2)^{-\frac{1}{\alpha}}\right\}.$$

Suppose that there exists a root $s$ of the characteristic equation $\det(\Delta(s)) = 0$ with $|s| < \delta$ and denote the spectral radius of a matrix $P$ by $\rho(P)$. Thus,

$$\begin{aligned}
&\rho\left((A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right)\\
<&\left\|(A-B)^{-1}\right\|\left\|C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m}\right\|\\
\leq&\left\|(A-B)^{-1}\right\|\left(\left\|C_1\right\|e^{-s\tau_1}+\cdots+\left\|C_m\right\|e^{-s\tau_m}\right)\\
<&e^{s\tau}\left\|(A-B)^{-1}\right\|\sum_{k=1}^{m}\left\|C_k\right\|\\
<&1,
\end{aligned}$$

from which we can conclude that

$$\begin{aligned}
&\left\|\left[I+(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right]^{-1}\right\|\\
=&\left\|\left[I-\left(-(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right)\right]^{-1}\right\|\\
\leq&\frac{1}{1-\left\|(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right\|}\\
\leq&\frac{1}{1-e^{\delta\tau}\left\|(A-B)^{-1}\right\|\sum\limits_{k=1}^{m}\left\|C_k\right\|}\\
<&M_2.
\end{aligned}$$

Then, we have that $\left\|\left[I+(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right]^{-1}\right\|$ is uniformly bounded with regard to $s$ and further

$$\begin{aligned}
&\rho\left(z(A-B)^{-1}\left(I+(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right)^{-1}\right)\\
\leq&\left\|s^{\alpha}(A-B)^{-1}\left(I+(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right)^{-1}\right\|\\
\leq&M_1M_2|\delta|^{\alpha}\\
<&1.
\end{aligned}$$

Therefore,

$$Q_1 = s^{\alpha}(A-B)^{-1}\left(I+(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right)^{-1} - I$$

is invertible. At the same time, on the basis of (11), we get that

$$Q_2 = I+(A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})$$

is invertible too. Through simple calculation, we also conclude that

$$(A-B)Q_1Q_2 = s^{\alpha}I + A - B - C_1e^{-s\tau_1} - \cdots - C_me^{-s\tau_m}$$

is invertible, which contracts the fact that $\det(\Delta(s)) = 0$. Therefore, there are no roots of $\det(\Delta(s)) = 0$ in a neighbourhood of $s = 0$. □

**Lemma 4.** If (**C1**) and (**C2**) hold, then $\mathcal{A}\cap\mathcal{B}=\emptyset$.

*Proof.* We use reduction to absurdity. Assume that $\mathcal{A}\cap\mathcal{B}\neq\emptyset$, then there exist $s$ with $\mathrm{Re}(s)\geq 0$ and $z$ with $|\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}$ such that

$$(zI + A - B) - C_1e^{-s\tau_1} - \cdots - C_me^{-s\tau_m} = 0.$$

According to (**C1**), we have

$$(zI+A-B)^{-1}(zI+A-B)-(zI+A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})=0.$$

If we denote the spectral set of $P$ by $\sigma(P)$, then

$$1\in\sigma\left((zI+A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right),$$

from which it follows that

$$1\leq\rho\left((zI+A-B)^{-1}(C_1e^{-s\tau_1}+\cdots+C_me^{-s\tau_m})\right)\leq\left\|(zI+A-B)^{-1}\right\|\sum_{k=1}^{m}\left\|C_k\right\|.$$

This is contradictory to (**C2**). Thus, $\mathcal{A} \cap \mathcal{B} = \emptyset$. □

**Lemma 5.** There exists a group of $(A, B, C_1, \cdots, C_m) \in \mathbb{R}^{n\times n} \times \cdots \times \mathbb{R}^{n\times n}$ such that $\sum_{k=1}^{m} \|C_k\| = \inf_{|\mathrm{Arg}(z)|\leq \frac{\alpha\pi}{2}} \left\|(zI + A - B)^{-1}\right\|^{-1}$ and the zero solution is not asymptotically stable.

*Proof.* Choose that $\tau_1 = 1, \tau_2 = \cdots = \tau_m = 0, A = \mathrm{diag}(a, a, \cdots, a), B = \mathrm{diag}(b, b, \cdots, b), C_1 = \mathrm{diag}(c_1, c_1, \cdots, c_1), C_2 = \cdots = C_m = 0$, where

$$a = -\frac{\left(\pi - \frac{(\alpha-1)\pi}{2}\right)^{\alpha}}{2\cos\frac{\alpha\pi}{2}}, b = \frac{\left(\pi - \frac{(\alpha-1)\pi}{2}\right)^{\alpha}}{2\cos\frac{\alpha\pi}{2}}, c_1 = \left(\pi - \frac{(\alpha-1)\pi}{2}\right)^{\alpha} \tan\frac{\alpha\pi}{2}.$$

By simple calculation, we have that $|\mathrm{Arg}(\lambda_{-A+B})| > \frac{\alpha\pi}{2}$ and

$$c_1 = \sum_{k=1}^{m} \|C_k\| = \inf_{|\mathrm{Arg}(z)|\leq \frac{\alpha\pi}{2}} \left\|(zI + A - B)^{-1}\right\|^{-1} = (b - a)\sin\frac{\alpha\pi}{2}.$$

In addition,

$$\omega = \pi - \frac{(\alpha - 1)\pi}{2}$$

is the solution of

$$(\mathrm{i}\omega)^{\alpha} I = -A + B - C_1 e^{-\mathrm{i}\omega\tau_1},$$

which means that there exists a root with $\mathrm{Re}(s) = 0$. Thus, the zero solution of (8) is not asymptotically stable. □

**Theorem 3.** The zero solution of (8) is asymptotically stable if and only if (**C1**) and (**C2**) hold.

*Proof.* By virtue of Lemma 3 and 4, we have the sufficiency. Meanwhile, the necessity is a direct result of Lemma 5. □

*Remark 5.* Theorem 3 solves the stability problem for $n$-dimensional linear fractional-order differential equations with delay to a certain extent, which is an open problem since 2016 (see 3). It is evident that Theorem 3 successfully solves Question (**1**) and (**2**). On the one hand, Theorem 3 gives the stability conditions under the framework of $|\mathrm{Arg}(\lambda_M)| > \frac{\alpha\pi}{2}$, $(M = -A + B)$, especially for the gap $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \leq \frac{\pi}{2}$. On the other hand, the existing results such as Theorem 1 are improved in Theorem 3 owing to the consideration of the influence of imaginary part. We will illustrate these promotions in Example 1.

*Remark 6.* Theorem 3 provides us with a simple procedure for determining the stability of the fractional-order nonlinear systems with Caputo's derivative of order $0 < \alpha < 1$. If the coefficient matrix satisfies (**C2**), then it is unnecessary to get the exact solution. What is required is to calculate the eigenvalues of the matrix $(-A+B)$ and test their arguments. If $|\mathrm{Arg}(\lambda_{-A+B})| > \frac{\alpha\pi}{2}$, we conclude that the system is asymptotically stable. Besides, although multiple delays are considered, the criterion is delay independent.

So far, we have derived the stability criterion for linear fractional-order differential equations with delays. In the next, according to the linearization theory of fractional-order differential equations in 9, a similar stability criterion for (1) can be obtained immediately.

**Theorem 4.** The equilibrium solution of system (1) is locally asymptotically stable if and only if (**C1**) and (**C2**) hold, where the coefficients are given by (7).

*Remark 7.* In our work, we offers a necessary and sufficient condition for stability of (1) under $|\mathrm{Arg}(\lambda_M)| > \frac{\alpha\pi}{2}$. Compared with many related researches such as Lyapunov direct method[15] and comparison theorem[29], Theorem 4 is the most complete. In 28, it is noted that "it is difficult to get the stability conditions of the system under the assumption that the eigenvalues of $M$ satisfy $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \leq \frac{\pi}{2}$, because it is required to know the direction of the solution curve of system" in Remark 4. We give effective programmes for the stability analysis in the research gap $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \leq \frac{\pi}{2}$.

*Remark 8.* Although the discussions above are established in real domain, they can also be extended to complex domain, i.e., complex-valued fractional-order neural networks with delays. The proof is left for interested readers. We will give a complex-valued numerical example in Example 3.

In the sequel, we will give a more specific corollary, which are more convenient for calculating. We assume that $M = -A + B$ can be diagonalized, i.e., there exists an invertible matrix $P$ such that $P^{-1}MP = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. What is more, $\|\cdot\|_M$ denotes the norm induced by $M$. Then we have the following conclusion.

**Corollary 1.** Suppose that there exist eigenvalues of $M$ satisfy $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \leq \frac{\pi}{2}$, then the equilibrium solution of system (1) is locally asymptotically stable if

$$\sum_{k=1}^{m} \left\| C_k \right\|_M < |\lambda_M| \sin\left(|\mathrm{Arg}(\lambda_M)| - \frac{\alpha\pi}{2}\right)$$

where $|\lambda_M| = \min_{1\leq i\leq n}\{|\lambda_i|\}$.

*Proof.* Let $\|x\|_M = \|P^{-1}x\|_\infty$, then we have

$$\left\|(zI - M)^{-1}\right\|_M = \left\|(zI - \Lambda)^{-1}\right\|_\infty = \max_{1\leq i\leq n}\{|z - \lambda_i|\}.$$

As a consequence,

$$\begin{aligned}
&\inf_{|\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}} \left\|(zI - M)^{-1}\right\|^{-1} \\
=&\frac{1}{\sup\limits_{|\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}} \left\|(zI - M)^{-1}\right\|_M} \\
=&\frac{1}{\sup\limits_{|\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}} \max\limits_{1\leq i\leq n}\{|z - \lambda_i|\}} \\
=&\min_{\substack{1\leq i\leq n \\ |\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}}} \{|z - \lambda_i|\}.
\end{aligned}$$

For eigenvalues with $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_i)| \leq \frac{(\alpha+1)\pi}{2}$, we have

$$\begin{aligned}
|z - \lambda_i| &= |\lambda_i| \sin\left(|\mathrm{Arg}(\lambda_i)| - \frac{\alpha\pi}{2}\right) \\
&> |\lambda_M| \sin\left(|\mathrm{Arg}(\lambda_M)| - \frac{\alpha\pi}{2}\right) > \sum_{k=1}^{m} \left\| C_k \right\|_M .
\end{aligned}$$

For eigenvalues with $|\mathrm{Arg}(\lambda_i)| > \frac{(\alpha+1)\pi}{2}$, we get

$$|z - \lambda_i| = |\lambda_i| > |\lambda_M| > \sum_{k=1}^{m} \left\| C_k \right\|_M .$$

Then we yield

$$\sum_{k=1}^{m} \left\| C_k \right\|_M < \inf_{|\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}} \left\|(zI - M)^{-1}\right\|^{-1} = \min_{\substack{1\leq i\leq n \\ |\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}}} \{|z - \lambda_i|\}.$$

By applying Theorem 4, we complete the proof. □

**Corollary 2.** Suppose that all the eigenvalues of $M$ satisfy $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$, then the equilibrium solution of system (1) is locally asymptotically stable if all the eigenvalues of $M$ have imaginary parts satisfy

$$|\mathrm{Im}(\lambda_M)| > \frac{\sum\limits_{k=1}^{m} \left\| C_k \right\|_M}{\cos\left(\frac{\alpha\pi}{2}\right)} \tag{12}$$

where $|\mathrm{Im}(\lambda_M)| = \min_{1\leq i\leq n}\{|\mathrm{Im}(\lambda_i)|\}$.

*Proof.* Following the same approach of the proof in Corollary 1, we obtain

$$\inf_{|\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}} \left\|(zI - M)^{-1}\right\|^{-1} = \frac{1}{\sup\limits_{|\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}} \max\limits_{1\leq i\leq n}\{|z - \lambda_i|\}} = \min_{\substack{1\leq i\leq n \\ |\mathrm{Arg}(z)|\leq\frac{\alpha\pi}{2}}} \{|z - \lambda_i|\}.$$

Hence, by (12), we obtain

$$\begin{aligned}
|z - \lambda_i| = |\lambda_i| \sin\left(|\mathrm{Arg}(\lambda_i)| - \frac{\alpha\pi}{2}\right) &= |\lambda_i| \cos\left(\frac{\pi}{2} - |\mathrm{Arg}(\lambda_i)| + \frac{\alpha\pi}{2}\right) \\
> |\mathrm{Im}(\lambda_M)| \cos\left(\frac{\alpha\pi}{2}\right) &> \sum_{k=1}^{m} \left\| C_k \right\|_M
\end{aligned}$$

for eigenvalues with $\frac{\pi}{2} < |\mathrm{Arg}(\lambda_i)| \le \frac{(\alpha+1)\pi}{2}$ and

$$|z-\lambda_i| = |\lambda_i| > |\mathrm{Im}(\lambda_M)| > \frac{\sum_{k=1}^{m}\left\|C_k\right\|_M}{\cos\left(\frac{\alpha\pi}{2}\right)} \ge \sum_{k=1}^{m}\left\|C_k\right\|_M$$

for eigenvalues with $|\mathrm{Arg}(\lambda_i)| > \frac{(\alpha+1)\pi}{2}$. Hence, (**C2**) can be verified by

$$\sum_{k=1}^{m}\left\|C_k\right\|_M < \inf_{|\mathrm{Arg}(z)|\le\frac{\alpha\pi}{2}}\left\|(zI-M)^{-1}\right\|^{-1} = \min_{\substack{1\le i\le n\\ |\mathrm{Arg}(z)|\le\frac{\alpha\pi}{2}}}\{|z-\lambda_i|\}.$$

By applying Theorem 4, we complete the proof. □

*Remark 9.* We note that the questions (**1**) and (**2**) have been solved successfully by Corollary 1 and Corollary 2, respectively. Corollary 1 provides a coefficient-type criterion to ensure the stability for $M$ with $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \le \frac{\pi}{2}$. Corollary 2 shows the close relationship between $|\mathrm{Im}(\lambda_M)|$ and the stability for $M$ with $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$. Because both the imaginary parts and real parts are taken into account, our result is better than Theorem 2. We will illustrate this in the next section.

## 4 | NUMERICAL EXAMPLES

In this section, three numerical examples are given to illustrate the effectiveness of our main results. The utilized numerical method is Grünwald-Letnikov method[25].

**Example 1.** As the first example, we consider linear fractional-order differential equations with delay (4). Set $\alpha = 0.8, \tau = 1$, $C = \mathbf{diag}(c, c)$. The initial values are $x_1(t) = 1.6, x_2(t) = 1.3, t \in [-1, 0]$. In order to show the comprehensiveness of Theorem 3, the following parameters are considered, respectively:

$$(\mathbf{D1}):\ M = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}, C = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix};\ (\mathbf{D2}):\ M = \begin{pmatrix} -1 & \tan(0.1\pi) \\ -\tan(0.1\pi) & -1 \end{pmatrix}, C = \begin{pmatrix} 0.64 & 0 \\ 0 & 0.64 \end{pmatrix};$$

$$(\mathbf{D3}):\ M = \begin{pmatrix} -1 & \tan(0.1\pi) \\ -\tan(0.1\pi) & -1 \end{pmatrix}, C = \begin{pmatrix} 1.04 & 0 \\ 0 & 1.04 \end{pmatrix}.$$

On the one hand, we note that Theorem 3 is valid for both $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$ and $\frac{\alpha\pi}{2} < |\mathrm{Arg}(\lambda_M)| \le \frac{\pi}{2}$, while Theorem 1 only valid for $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$. Consider (4) with parameters (**D1**), whose eigenvalues are $\lambda = 1 \pm 4\mathrm{i}$ with

$$\frac{2\pi}{5} < |\mathrm{Arg}(\lambda_M)| \le \frac{\pi}{2}.$$

If we apply Theorem 1 for stability analysis, we have $L_1 = L_2 = 1 + \epsilon(\epsilon > 0)$ and

$$\gamma = 1 - 8(1+\epsilon) < 0, \sin\frac{\alpha\pi}{2} > 0, \hat{K} > 0.$$

Thus, (2) is never satisfied no matter how "small" the delay disturbance is, which is obviously unreasonable. More specific, Theorem 1 is invalid for parameters (**D1**). But we can still use Theorem 3, i.e., (**C1**) and (**C2**) can be verified by

$$|\mathrm{Arg}(\lambda_M)| > \frac{2\pi}{5}, 0.2 = \|C\|_2 < \inf_{|\mathrm{Arg}(z)|\le\frac{4\pi}{5}}\left\|(zI-M)^{-1}\right\|^{-1} \approx 0.285.$$

Therefore, the zero solution is asymptotically stable, which is depicted in Figure 1 . The figures illustrate the effectiveness of Theorem 3 perfectly.

On the other hand, we note that Theorem 3 performs better than Theorem 1 when $|\mathrm{Arg}(\lambda_M)| > \frac{\pi}{2}$. We set $M = \begin{pmatrix} -1 & \tan(0.1\pi) \\ -\tan(0.1\pi) & -1 \end{pmatrix}$, then we can calculate that Theorem 1 is satisfied if

$$c < \frac{1 - \tan 0.1\pi(1+\epsilon)}{1+\epsilon}\sin 0.4\pi \to 0.6420, \text{when } \epsilon \to 0.$$

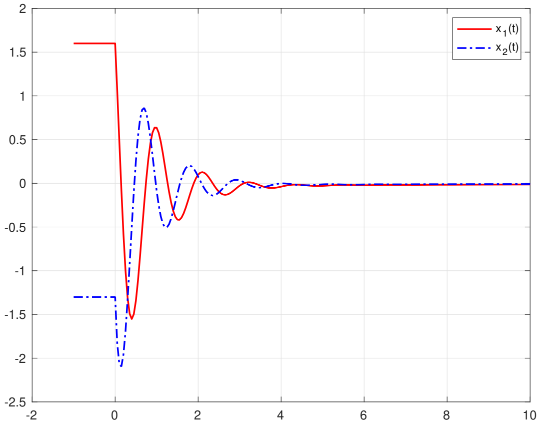


**FIGURE 1** The state variables for (4) with (**D1**).

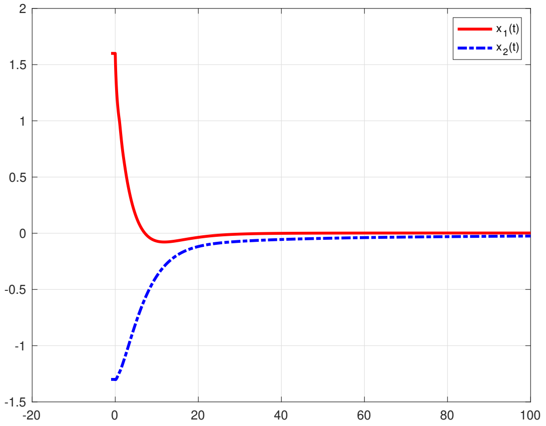


**FIGURE 2** The state variables for (4) with (**D2**).

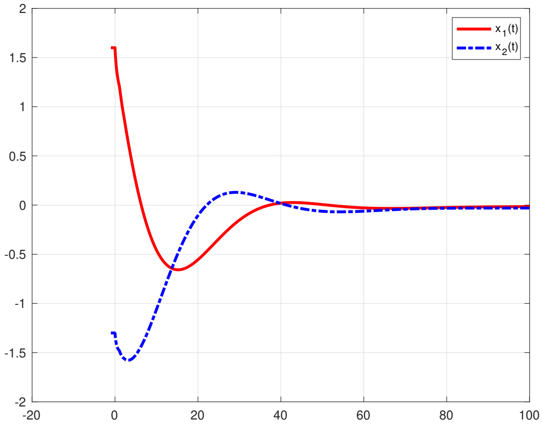


**FIGURE 3** The state variables for (4) with (**D3**).

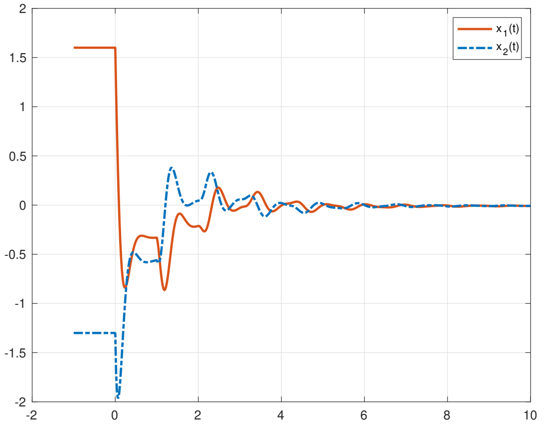


**FIGURE 4** The state variables for (**D4**) with $k = 7$.

But for Theorem 3, the condition is

$$c < \min_{\substack{1\le i\le n \\ |\mathrm{Arg}(z)|\le\frac{\alpha\pi}{2}}} \{|z-\lambda_i|\} \approx 1.0515.$$

Consider (4) with parameters (**D2**) and (**D3**), the state variables are depicted in Figs 2 and 3 , respectively. It is easy to see the convergence behavior of the two solutions, which show the wider applicability of Theorem 3 than the existing results.

Next, in order to show the influence of $\mathrm{Im}(\lambda_M)$ on the stability, we consider the following parameters:

$$(\mathbf{D4}) : \alpha = 0.8, \tau = 1, M = \begin{pmatrix} -1 & k \\ -k & -1 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is known that the eigenvalues are $\lambda = -1 \pm k\mathrm{i}$. By applying Corollary 2, the zero solution is asymptotically stable if the imaginary part

$$|\mathrm{Im}(\lambda_M)| = |k| > \frac{\|C\|_M}{\cos\left(\frac{\alpha\pi}{2}\right)} \approx 6.4721.$$

Hence, we choose $k = 7 > 6.4721$, the zero solution is stable, which is shown in Figure 4 .

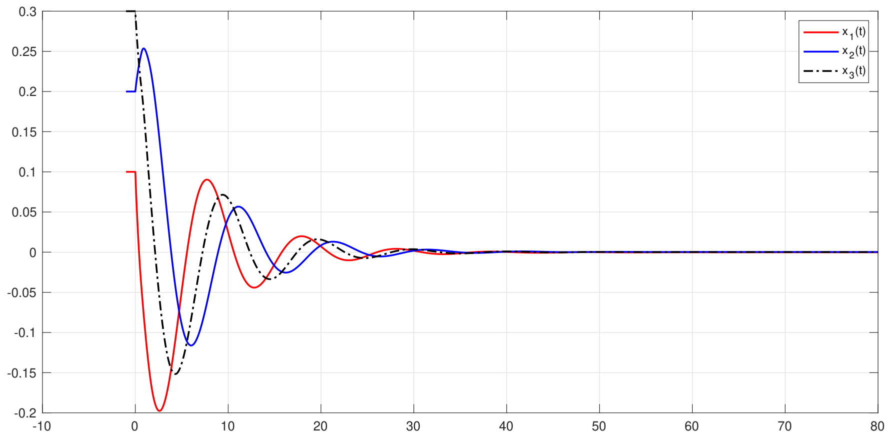


**FIGURE 5** The state variables for (13) with (**E1**).

**Example 2.** In the second example, let us consider the three-dimensional fractional-order Hopfield neural network with time delay in 28:

$$\begin{cases} {}_0^C\mathrm{D}_t^{\alpha} x_1(t) = -a_1 x_1(t) + b_{11}\sin(x_1(t)) + b_{12}\sin(x_2(t)) + c_{11}\tanh(x_1(t-\tau)) + c_{12}\tanh(x_2(t-\tau)), \\ {}_0^C\mathrm{D}_t^{\alpha} x_2(t) = -a_2 x_2(t) + b_{22}\sin(x_2(t)) + b_{23}\sin(x_3(t)) + c_{22}\tanh(x_2(t-\tau)) + c_{23}\tanh(x_3(t-\tau)), \\ {}_0^C\mathrm{D}_t^{\alpha} x_3(t) = -a_3 x_3(t) + b_{31}\sin(x_1(t)) + b_{33}\sin(x_3(t)) + c_{31}\tanh(x_1(t-\tau)) + c_{33}\tanh(x_3(t-\tau)), \\ x_1(t) = x_1(0), x_2(t) = x_2(0), x_3(t) = x_3(0), t \in [-\tau, 0]. \end{cases} \tag{13}$$

The network parameters of (13) are chosen as $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.5 \end{pmatrix}$. The initial value is $\left(x_1(0), x_2(0), x_3(0)\right) = (0.1, 0.2, 0.3)$. It is clear the $\left(x_1(t), x_2(t), x_3(t)\right) = (0,0,0)$ is the equilibrium solution of (13). Moreover, (**C1**) is satisfied since the eigenvalues of $(-A+B)$ are $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

In order to show the vital influence of coefficient matrices on the stability, the following parameters are considered, respectively:

$$(\mathbf{E1}):\ \tau = 0.7, C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};\ (\mathbf{E2}):\ \tau = 0.7, C = \begin{pmatrix} 0 & -3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix};\ (\mathbf{E3}):\ \tau = 1, C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consider parameters (**E1**), we can calculate that there exists matrix norm $\|\cdot\|_2$ such that $1 = \|C\|_2 < \inf_{|\mathrm{Arg}(z)| \le \frac{\alpha\pi}{2}} \left\|(zI + A - B)^{-1}\right\|^{-1} = 2$. By Theorem 3, the zero solution is asymptotically stable. Similarly, when the parameters are changed, it is not difficult to verify the satisfaction of (**C2**) for (**E3**) and the dissatisfaction for (**E2**).

Figs 5 , 6 , and 7 depict the state variables with above parameters, respectively. It is easy to see the convergence behavior of the solution from the figures. The different dynamic behaviors in Figure 5 and figure 6 clearly show the important influence of coefficient matrix $C$. Comparing Figure 5 with Figure 7 , we can see the delay independence of our criterion.

**Example 3.** As the third example, we consider the four-dimensional fractional-order Hopfield neural network with hub structure and two time delays in 23 as follows:

$$\begin{cases} {}_0^C\mathrm{D}_t^{\alpha} z_1(t) = -a_1 z_1(t) + \sum_{q=1}^{4} b_{1q} f_q(z_q(t)) + c_1 f_1(z_1(t-\tau_1)), \\ {}_0^C\mathrm{D}_t^{\alpha} z_p(t) = -a_p z_p(t) + b_{p1} f_1(z_1(t) + b_{pp} f_p(z_p(t)) + c_2 f_p(z_p(t-\tau_2)), p = 2,3,4. \end{cases} \tag{14}$$

where the nonlinear functions $f_1(x) = f_2(x) = \sin x$, $f_3(x) = f_4(x) = \tanh x$, delays $\tau_1 = 1, \tau_2 = 0.7$.

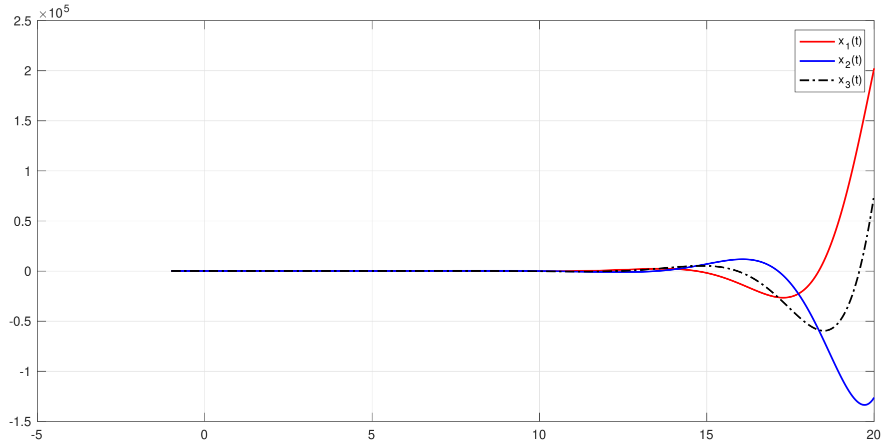


**FIGURE 6** The state variables for (13) with (**E2**).

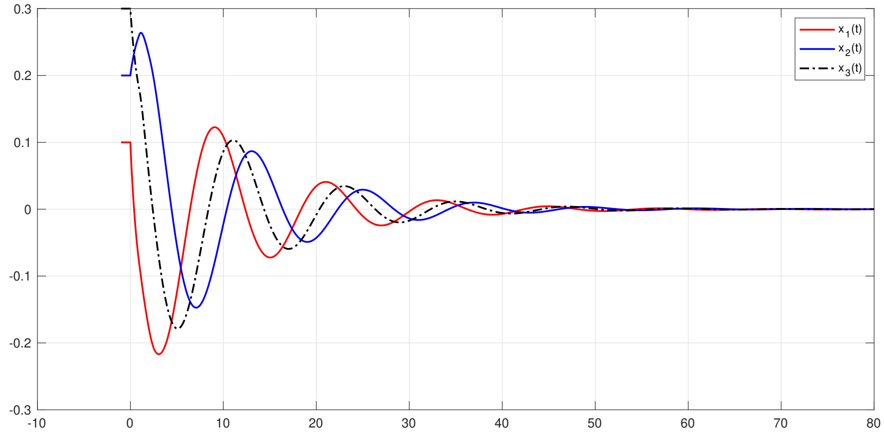


**FIGURE 7** The state variables for (13) with (**E3**).

In order to show the vital influence of fractional exponent on the stability, we keep

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 6 & 18 & 1 & 1 \\ -18 & 5 & 0 & 0 \\ 1 & 0 & -18 & 0 \\ 1 & 0 & 0 & -23 \end{pmatrix}, C_1 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2.4 & 0 & 0 \\ 0 & 0 & -2.3 & 0 \\ 0 & 0 & 0 & 2.2 \end{pmatrix}.$$

Then we set $\alpha = 0.75$, $\bar{\alpha} = 0.99$ and depict the state variables for initial values $z_1(t) = 1.6, z_2(t) = -1.3, z_3(t) = 4.5, z_4(t) = -3.8, t \in [-1, 0]$ in Figure 8 and 9 , which completely consistent with our results , i.e.,

$$1.79 \approx \inf_{|\mathrm{Arg}(z)| \le \frac{0.99\pi}{2}} \left\| (zI + A - B)^{-1} \right\|^{-1} < \|C_1\|_2 + \|C_2\|_2 = 4.4 < \inf_{|\mathrm{Arg}(z)| \le \frac{0.75\pi}{2}} \left\| (zI + A - B)^{-1} \right\|^{-1} \approx 5.92.$$

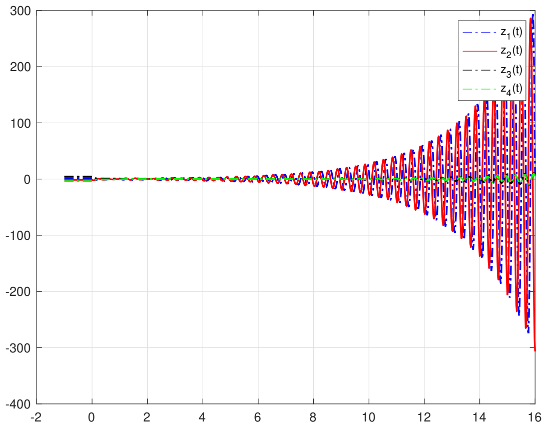


**FIGURE 8** The state variables for (14) with $\alpha = 0.99$.

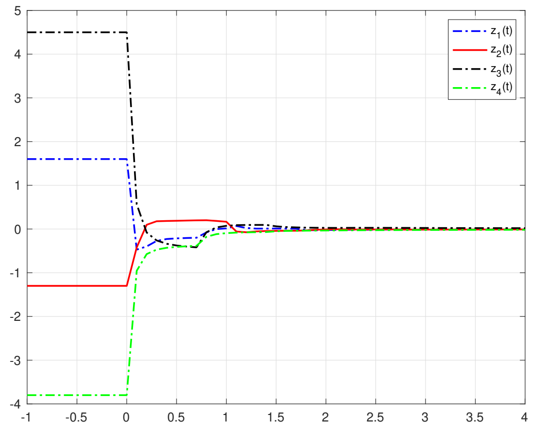


**FIGURE 9** The state variables for (14) with $\alpha = 0.75$.

Moreover, in Remark 8, we note that Theorem 4 is still valid for complex-valued system. Hence, we give a simple example, consider (14) with complex-valued parameters

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2.5 \end{pmatrix}, B = \begin{pmatrix} -1+0.5\mathrm{i} & -2+0.2\mathrm{i} & 3-0.3\mathrm{i} & 1-1.5\mathrm{i} \\ 0 & 2+0.2\mathrm{i} & 0 & 0 \\ 0 & 0 & 1+0.2\mathrm{i} & 0 \\ 0 & 0 & 0 & 0.5+0.2\mathrm{i} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -0.5\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.5+0.5\mathrm{i} & 0 & 0 \\ 0 & 0 & -0.5+0.5\mathrm{i} & 0 \\ 0 & 0 & 0 & -0.5+0.5\mathrm{i} \end{pmatrix}.$$

For $\alpha = 0.9, \tau_1 = 1, \tau_2 = 0.7$ and initial values $z_1(t) = 0.1+0.6\mathrm{i}, z_2(t) = 0.2+0.4\mathrm{i}, z_3(t) = 0.3+0.5\mathrm{i}, z_4(t) = 0.1+0.3\mathrm{i}, t \in [-1, 0]$, we can verify the conditions:

$$|\mathrm{Arg}(\lambda_{-A+B})| > \frac{0.9\pi}{2}, 1.21 \approx \|C_1\| + \|C_2\| < \inf_{|\mathrm{Arg}(z)| \le \frac{\alpha\pi}{2}} \left\|(zI + A - B)^{-1}\right\|^{-1} \approx 5.92.$$

Hence, the zero solution is asymptotically stable by Theorem 4, which is displayed in Figure 10 and 11 $(z_i(t) = x_i(t) + \mathrm{i}y_i(t))$. This illustrate the wide applicability of our results.

## 5 | CONCLUSIONS

In the present work, the asymptotical stability for fractional-order Hopfield neural network with multiple delays was investigated. A necessary and sufficient stability criterion is provided by the coefficient matrices and fractional exponent to ensure the stability. There are several possible research directions in the future such as fractional-order Competitive neural networks, Memristor-based neural networks and neural networks with complex-valued state. In addition, the stability for numerical solutions is also a direction in which we are interested in.

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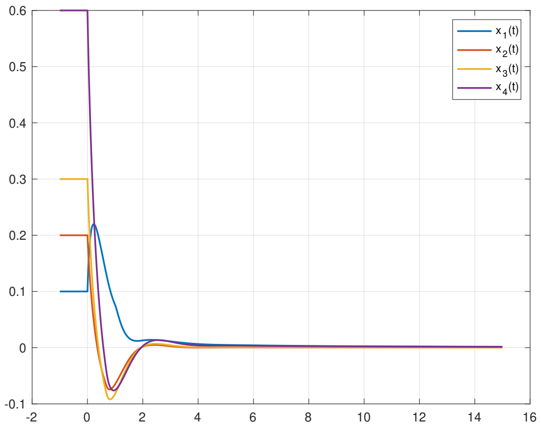


**FIGURE 10** The real parts $x_1(t), x_2(t), x_3(t), x_4(t)$ for (14).

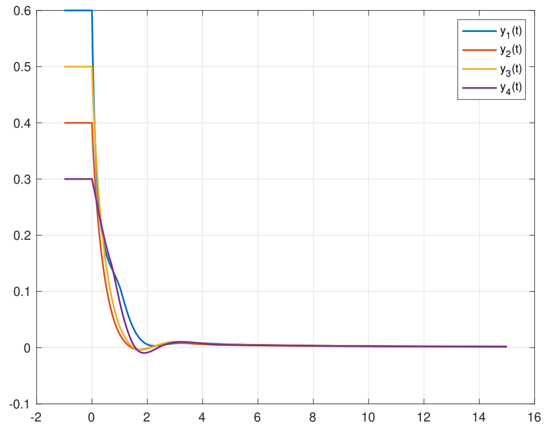


**FIGURE 11** The imaginary parts $y_1(t), y_2(t), y_3(t), y_4(t)$ for (14).

## DECLARATIONS OF INTEREST

The authors declare that they have no conflict of interest.

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