Control of Nonlinear Coupled Burgers' PDE-ODE Systems with Sensor Nonlinearities: A Mixed Control Approach

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Abstract: This paper studies the stability analysis and control design problems for a class of nonlinear coupled complex systems with sensor nonlinearities via a mixed control approach. The nonlinear coupled complex systems are modelled as a nonlinear finite-dimensional ordinary differential equation coupled with an infinite-dimensional viscid burgers' equation. Due to the existence of both time and space dimensional nonlinearities and based on the Lyapunov stability theory, a mixed control approach is proposed to produce a fuzzy-model-based controller tackling the time dimensional nonlinearity and a boundary controller eliminating the space dimensional nonlinearity, such that the addressed coupled complex systems achieve asymptotic stability. Sufficient conditions are obtained via the convex optimization technology. Different from the existing results on control of coupled PDE-ODE systems, a mixed control approach is firstly developed by combining the boundary control technology and the fuzzy-model-based control technology. Finally, the proposed mixed control approach is applied to a nonlinear hypersonic rocket car to testify its validity.

Key Words: Fuzzy control, viscid burgers' equation, coupled complex systems, infinite-dimensional nonlinearity.

1 Introduction

Recently, a coupled complex system, which is described as a partial differential equation (PDE) coupled with an ordinary differential equation (ODE), has been a significant active research topic for their widely engineering application potentials, including macroscopic traffic acceleration [1], deep drilling [2, 3], hetero-directional transport [4], wastewater treatment [5], etc. The stability analysis and robust control problem of such systems have received considerable attention [6–9]. For instance, the stability analysis problem of a coupled diffusion PDE-ODE system has been studied by resorting to the PDE backstepping method [6], in which the actuator of the PDE is in the boundary of spatial location and the ODE is influenced by the boundary controller. The stability analysis problem for coupled linear ODE-hyperbolic PDE systems has been investigated with two time scales by utilizing the singular perturbation method [8].

Meanwhile, during the past few decades, Takagi-Sugeno (T-S) fuzzy model has been successfully applied to deal with the time dimensional nonlinearity, where the nonliearity is modelled as a combination of some linear terms [10–12]. The authors in [10] has proven that industrial processes such as water cleaning process can be identified as a fuzzy model via fuzzy reasoning knowledge. By using T-S fuzzy modelling technique, an adaptive finite-time control law has been proposed in [11]. When the grades of membership suffer from uncertainties, the interval Type-2 (IT2) fuzzy systems have been considered with periodic sampling scheme [12].

The existing results on the stability analysis problem of nonlinear PDE systems, have been accomplished through the Lyapunov stability theory, the fuzzy control theory [13], and the Wirtinger's inequality [14, 15] and the Euclidean space [16]. Generally, the nonlinear coupled PDE-ODE systems are often confronted with three scenarios: (1). a H_∞ disturbance attention index is introduced when there exist nonlinearities, input constraints, and disturbances [17]; (2). a cost function is guaranteed when the nonlinear coupled PDE-ODE system is stabilized via fuzzy PDE/ODE control strategies, where the PDE subsystems are unstable different from the former scenario [18]; (3). the control force can be exploited by means of a planing observer when the states of such systems are not available [19]. In particular, the H_∞ control problem of the nonlinear hyperbolic PDE systems has been studied [20], where the reliable fuzzy controller can be gained by using the interior-point method.

Most of the aforementioned works only focus on tackling time dimensional nonlinearities by using the fuzzy-model-based control approach. When it comes to the nonlinear coupled burgers' PDE-ODE system, where both the time dimensional nonlinearity and the space dimensional nonlinearity are involved, however,it is difficult to carrying out a fuzzy-model-based control mechanism to study the stability analysis problem of the nonlinear coupled burgers' PDE-ODE system, due to the interaction of the above two nonlinearities. Besides, it has been found that sensor nonlinearities have an negative effect on the system stability in [25–27], where the controllers and filters have been developed by taking the sensor nonlinearities into account. However, there are few studies on the control problem of the coupled PDE-ODE systems with sensor nonlinearities. Therefore, it is desirable to propose a novel control approach to investigate the stability analysis problem of the nonlinear coupled burgers' PDE-ODE system with sensor nonlinearities.

Prompted by the above discussions, this paper considers the stability analysis and control design problems of a coupled complex system described as a coupled scalar burgers' PDE and nonlinear ODE by developing a mixed control approach. The remainder of this paper is arranged as follows. Section II transfers the addressed nonlinear coupled complex systems into fuzzy nonlinear coupled systems, and gives the control objective. Section III presents the corresponding fuzzy controller and states some preliminary knowledge. Section IV presents main results including stability analysis and controller design. Section V provides a simulation study to support the main results. Finally, Section VI concludes

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2 Problem Formulation

Consider the following nonlinear coupled complex system expressed as an n-dimensional nonlinear ODE coupled with a scalar burgers' PDE:

$$\dot{z}_1(t) = h(z_1(t)) + f(z_1(t)) \int_0^l z_2(x,t)dx + b(z_1(t))u_1(t), \tag{1}$$

$$\frac{\partial z_2(x,t)}{\partial t} = a\frac{\partial^2 z_2(x,t)}{\partial x^2} - z_2(x,t)\frac{\partial z_2(x,t)}{\partial x} + \varphi(z_1(t)) \tag{2}$$

with the following Robin boundary conditions

$$z_2(0,t) = 0, \left.\frac{\partial z_2(x,t)}{\partial x}\right|_{x=l} = u_2(t) \tag{3}$$

and the initial conditions

$$z_1(0) = z_{1_0}, z_2(x,0) = z_{2_0}(x) \tag{4}$$

where $z_1(t) \in \mathbb{R}^n$ and $u_1(t) \in \mathbb{R}^m$ are the state and the required control input of the ODE subsystem (1), respectively. $z_2(t) \in \mathbb{H} \triangleq \mathscr{L}_2([0,l],\mathbb{R})$ is the state of the PDE subsystem (2), where $\mathbb{H}$ stands for an infinite-dimensional Hilbert space of 1-dimensional scalar function. $u_2(t)$ is the boundary control input of the PDE subsystem (1) to be designed later. $x \in [0,l] \subset \mathbb{R}$ and $t \in [0,\infty)$ represent the spatial position and the time, respectively. $h(\cdot)$, $f(\cdot)$, $b(\cdot)$, and $\varphi(\cdot)$ are nonlinear locally Lipschitz continuous smooth functions satisfying $h(0) = 0$, $f(0) = 0$, $b(0) = 0$, and $\varphi(0) = 0$. a is a known positive diffusion coefficient, which characters the dissipativity of the PDE subsystem (2). z_{1_0} and $z_{2_0}(x)$ are the initial constant condition of ODE and known space-varying condition of PDE, respectively.

Remark 1. It can be obtained from (1) and (2) that the state variable $z_1(t)$ acts on the PDE dynamics via a transformation function $\varphi(\cdot)$, and in turn the state variable $z_2(x,t)$ influences the ODE dynamics through an integral operator, forming a coupled relationship between a finite dimensional function and an infinite dimensional function.

Remark 2. From [21–24], the structure of the burgers' equation (2) is similar to a 1-dimensional Navier-Stokes equation, which reflects the complex dynamics of many practical processes such as turbulent fluid motion and shock waves. On the other hand, the burgers' equation can be analysed by converting it into a linear parabolic equation by using the Hopf-Cole transformation, which will not lead to chaotic behaviours. That is why we study the coupled burgers' PDE-ODE systems via T-S fuzzy model, where the burgers' PDE subsystem is in a slow stable motion.

Note that the T-S fuzzy model, consisting of a family of linear local models, can approximate any smooth nonlinear functions with arbitrary accuracy in any convex compact region. Thus, by using the sector nonlinearity approach [10], the nonlinear coupled burgers' PDE-ODE systems can be represented as the following form with p fuzzy rules:

Plant rule i: IF $\theta_1(t)$ is $\mathscr{M}_1^i$ and $\theta_2(t)$ is $\mathscr{M}_2^i$ and $\cdots$ and $\theta_r(t)$ is $\mathscr{M}_r^i$, THEN

$$\begin{aligned}\dot{z}_1(t) &= A_i z_1(t) + F_i \int_0^l z_2(x,t)dx + B_i u_1(t)\\ \frac{\partial z_2(x,t)}{\partial t} &= a\frac{\partial^2 z_2(x,t)}{\partial x^2} - z_2(x,t)\frac{\partial z_2(x,t)}{\partial x} + \phi_i z_1(t)\end{aligned} \tag{5}$$

where $i \in \mathbb{S} \triangleq \{1,2,\cdots,p\}$. $\{\theta_i(t)|i \in \mathbb{S}\}$ are the premise variable set, and $\{\mathscr{M}_s^i|j = 1,\cdots,r\}$ refers to the fuzzy sets corresponding to the premise variable set. A_i, F_i, B_i, and ϕ_i are known local models' matrices with proper dimensions.

By utilizing the product inference engine, center-average defuzzifier and singleton fuzzifier, the overall fuzzy nonlinear coupled burgers' PDE-ODE system can be described as

$$\begin{aligned}\dot{z}_1(t) &= A(\mu)z_1(t) + F(\mu) \int_0^l z_2(x,t)dx + B(\mu)u_1(t)\\ \frac{\partial z_2(x,t)}{\partial t} &= a\frac{\partial^2 z_2(x,t)}{\partial x^2} - z_2(x,t)\frac{\partial z_2(x,t)}{\partial x} + \phi(\mu)z_1(t)\end{aligned} \tag{6}$$

with the following Robin boundary conditions

$$z_2(0,t) = 0, \left.\frac{\partial z_2(x,t)}{\partial x}\right|_{x=l} = u_2(t) \tag{7}$$

where

$$\begin{aligned}\mu_i[\theta(t)] &= \frac{\prod_{j=1}^r \mu_{ij}[\theta_i(t)]}{\sum_{i=1}^p \prod_{j=1}^r \mu_{ij}[\theta_i(t)]} \geq 0\\ A(\mu) &= \sum_{i=1}^p \mu_i A_i, F(\mu) = \sum_{i=1}^p \mu_i F_i, \mathscr{M}^i = \prod_{s=1}^r \mathscr{M}_s^i\\ B(\mu) &= \sum_{i=1}^p \mu_i B_i, \phi(\mu) = \sum_{i=1}^p \mu_i \phi_i, \sum_{i=1}^p \mu_i[\theta(t)] = 1\end{aligned} \tag{8}$$

with $\theta(t) = [\theta_1(t),\cdots,\theta_r(t)]^T$, and $\mu_{ij}[\theta(t)]$ being the normalized membership function.

The objective is to design a fuzzy control law involving the ODE states and the PDE states with the knowledge of membership functions and a boundary control law only acting on the spatial boundary position l, to ensure that the nonlinear coupled burgers' PDE-ODE system (1) achieves asymptotic stability with any initial conditions z_{1_0} and $z_{2_0}(x)$.

3 Mixed Controller Design

In this section, a fuzzy feedback control law is constructed based on the finite dimensional ODE states and the infinite dimensional PDE states, and a boundary controller is designed to eliminate the space dimensional nonlinearity according to the Lyapunov stability theory.

Given the fuzzy coupled burgers' PDE-ODE systems (6), the corresponding fuzzy controller is designed as

Controller rule i: IF $\theta_1(t)$ is $\mathscr{M}_1^i$ and $\theta_2(t)$ is $\mathscr{M}_2^i$ and $\cdots$ and $\theta_r(t)$ is $\mathscr{M}_r^i$, THEN

$$u_1(t) = K_i z_1(t) + G_i \int_0^l z_2(x,t)dx \tag{9}$$

where $K_i \in \mathbb{R}^{m\times n}$ and $G_i \in \mathbb{R}^{m\times 1}$ are controller gains to be designed. Similarly, the overall fuzzy controller is inferred as

$$u_1(t) = K(\mu)z_1(t) + G(\mu)\int_0^l z_2(x,t)dx \qquad (10)$$

where

$$K(\mu) = \sum_{i=1}^{p}\mu_i K_i, G(\mu) = \sum_{i=1}^{p}\mu_i G_i.$$

Traditionally, the boundary controller is formulated by using the spatial boundary information $z_2(l,t)$. However, owing to the injection of infinite-dimensional nonlinearities, a nonlinear boundary controller is constructed as

$$u_2(t) = cz_2^2(l,t) \qquad (11)$$

with c being the boundary controller gain to be determined later.

However, due to the unideal phenomena in the detecting process of sensor signal such as sensor nonlinearities, the above mixed controller composed of the fuzzy controller (9) and the boundary controller (10) is impossible in reality. Hence, taking the unideal phenomena into account yields the following mixed controller

$$\begin{aligned} &u_1(t) = K(\mu)\varsigma(z_1(t)) + G(\mu)\int_0^l \varsigma_1(z_2(x,t))dx \\ &u_2(t) = cz_2^2(l,t) \end{aligned} \qquad (12)$$

where $\varsigma(\cdot)$ and $\varsigma_1(\cdot)$ describe the sensor nonlinearities satisfying the following sector-bounded inequalities:

$$\Omega(a) = (\varsigma(a) - L_1 a)^T(\varsigma(a) - L_2 a) \le 0, L_1 \le L_2 \qquad (13)$$

and

$$\Omega_1(a) = (\varsigma_1(a) - s_1 a)^T(\varsigma_1(a) - s_2 a) \le 0, s_1 \le s_2 \qquad (14)$$

with L_1 and L_2 being the known diagonal matrices, and s_1 and s_2 being the known scalars.

Applying the mixed controller (12) into the fuzzy nonlinear coupled burgers' PDE-ODE system (6), the overall closed-loop system can be written as

$$\begin{aligned} \dot{z}_1(t) &= A(\mu)z_1(t) + B(\mu)K(\mu)\varsigma(z_1(t)) \\ &\quad + F(\mu)\int_0^l z_2(x,t)dx + B(\mu)G(\mu)\int_0^l \varsigma_1(z_2(x,t))dx \\ \frac{\partial z_2(x,t)}{\partial t} &= a\frac{\partial^2 z_2(x,t)}{\partial x^2} - z_2(x,t)\frac{\partial z_2(x,t)}{\partial x} + \phi(\mu)z_1(t) \end{aligned} \qquad (15)$$

with the following Robin boundary conditions

$$z_2(0,t) = 0, \left.\frac{\partial z_2(x,t)}{\partial x}\right|_{x=l} = c\varsigma^2(z_2(l,t)). \qquad (16)$$

Denote $\mu_i(\theta(t)) = \mu_i$, and the following definition and Lemmas are introduced for the convenience of subsequent stability analysis.

Definition 1. The overall fuzzy nonlinear coupled burgers' PDE-ODE system (6) is asymptotically stable if $\lim_{t\to\infty}||\Pi(t)|| = 0$ for any initial conditions z_{1_0} and $z_{2_0}(x)$, where $\Pi(t) = \int_0^l \| z_2(x,t) \|^2 dx + \| z_1(t) \|^2$.

Lemma 1 (Jensen's Inequality). For any vector function $z_2(x) \in \mathbb{H}$, any positive definite matrix M with proper dimensions, and any positive constant l, we have

$$\left(\int_0^l z_2(s)ds\right)^T M\left(\int_0^l z_2(s)ds\right) \le l\int_0^l z_2^T(s)Mz_2(s)ds. \qquad (17)$$

Lemma 2 (Wirtinger's Inequality). For any absolutely continuous scalar function $z_2(\cdot,t) \in \mathbb{H}$ with the initial condition $z_2(0,t) = 0$, the following inequality holds if its partial derivative $\frac{\partial z_2(x,t)}{\partial x}$ is square integrable

$$\| z_2(\cdot,t) \|_2^2 \le 4l^2\left|\left|\frac{\partial z_2(x,t)}{\partial x}\right|\right|_2^2. \qquad (18)$$

4 Main Results

4.1 Stability Analysis

Theorem 1: Consider the fuzzy nonlinear coupled burgers' PDE-ODE systems (6) with the robin boundary condition (16) and any initial conditions z_{1_0} and $z_{2_0}(x)$. The designed fuzzy controller (9), which guarantees the asymptotic stability of the overall closed-loop systems (15), can be derived if there exist scalars ρ_{1i}, ρ_{2i}, and $c = 1/3a$, positive definite symmetric matrix P, and matrices K_i and G_i with proper dimensions, and positive constants b, d, m_1, and m_2 for $i \in \mathbb{S}$ such that the following inequalities holds

$$\begin{aligned} &\Sigma_{ii} < 0, i \in \mathbb{S} \\ &\Sigma_{ij} + \Sigma_{ji} < 0, i < j \;\; s.t. \;\; \mu_i \cap \mu_j \neq \emptyset \end{aligned} \qquad (19)$$

where

$$\Sigma_{ij} = \begin{bmatrix} \Sigma_{ij11} & \Sigma_{ij12} & \Sigma_{ij13} & \Sigma_{ij14} \\ * & \Sigma_{ij22} & \mathbf{0} & \mathbf{0} \\ * & * & \Sigma_{ij33} & \Sigma_{ij34} \\ * & * & * & \Sigma_{ij44} \end{bmatrix},$$

$$\begin{aligned} &\Sigma_{ij11} = PA_i + A_i^T P - m_1 L_1 L_2, \Sigma_{ij22} = -m_1 I, \\ &\Sigma_{ij12} = PB_i K_j + m_1(L_1 + L_2)/2, \Sigma_{ij33} = -m_2 I, \\ &\Sigma_{ij13} = PB_i G_j, \Sigma_{ij14} = PF_i + b\phi(i), \\ &\Sigma_{ij34} = m_2(s_1 + s_2)/2, \Sigma_{ij44} = -ab/(2l^3) - m_2 s_1 s_2. \end{aligned}$$

Proof: For the overall closed-loop systems (15), consider the following Lyapunov function candidate

$$V(t) = \sum_{i=1}^{2} V_i(t, z_1(t), z_2(t)) \qquad (20)$$

where

$$V_1(t, z_1(t)) = z_1^T(t)Pz_1(t), V_2(t, z_2(t)) = b\int_0^l z_2^2(x,t)dx.$$

Taking the time derivative of $V_i(t), i = 1,2$ along the solution of (15) yields

$$\begin{aligned} \dot{V}_1(t) &= z_1^T(t)[PA(\mu) + A^T(\mu)P]z_1(t) \\ &\quad + 2z_1^T(t)PB(\mu)K(\mu)\varsigma(z_1(t)) \\ &\quad + 2z_1^T(t)PF(\mu)\int_0^l z_2(x,t)dx \\ &\quad + 2z_1^T(t)PB(\mu)G(\mu)\int_0^l \varsigma_1(z_2(x,t))dx, \end{aligned} \qquad (21)$$

and

$$\dot{V}_2(t) = 2ab\int_0^l z_2(x,t)\frac{\partial^2 z_2(x,t)}{\partial x^2}dx - 2b\int_0^l z_2^2(x,t)\frac{\partial z_2(x,t)}{\partial x}dx + 2b\int_0^l z_2(x,t)\phi(\mu)z_1(t)dx. \tag{22}$$

Combining Lemma 2, it follows that

$$-\int_0^l \left[\frac{\partial z_2(x,t)}{\partial x}\right]^2 dx \le -\frac{1}{4l^2}\int_0^l z_2^2(x,t)dx. \tag{23}$$

According to Lemma 1 with M being an unit matrix with proper dimensions, we have

$$-\int_0^l z_2^2(x,t)dx \le -\frac{1}{l}\left(\int_0^l z_2(x,t)ds\right)^T\left(\int_0^l z_2(x,t)ds\right). \tag{24}$$

From the boundary conditions (2) and the method of integration by parts, it implies that

$$\int_0^l z_2(x,t)\frac{\partial^2 z_2(x,t)}{\partial x^2}dx = z_2(x,t)\frac{\partial z_2(x,t)}{\partial x}\bigg|_{x=0}^{x=l} - \int_0^l \left[\frac{\partial z_2(x,t)}{\partial x}\right]^2 dx. \tag{25}$$

Substituting (23) and (24) into (25), it can be obtained that

$$\int_0^l z_2(x,t)\frac{\partial^2 z_2(x,t)}{\partial x^2}dx \le z_2(x,t)\frac{\partial z_2(x,t)}{\partial x}\bigg|_{x=0}^{x=l} - \frac{1}{4l^3}\left(\int_0^l z_2(x,t)ds\right)^T\left(\int_0^l z_2(x,t)ds\right). \tag{26}$$

Note that

$$-2b\int_0^l z_2^2(x,t)\frac{\partial z_2(x,t)}{\partial x}dx = -\frac{2b}{3}z_2^3(x,t)|_{x=0}^{x=l}. \tag{27}$$

Associated (13), (14), (21), (22), (26), (27) with the robin boundary conditions (16), it can be derived that

$$\begin{aligned}\dot{V}(t) \le& z_1^T(t)[PA(\mu)+A^T(\mu)P]z_1(t) \\ &+ 2z_1^T(t)PB(\mu)K(\mu)\varsigma(z_1(t)) \\ &- m_1\Omega(z_1(t)) - m_2\int_0^l \Omega_1(z_2(x,t))dx \\ &+ 2z_1^T(t)[PF(\mu)+b\phi(\mu)]\int_0^l z_2(x,t)dx \\ &+ 2z_1^T(t)PB(\mu)G(\mu)\int_0^l \varsigma_1(z_2(x,t))dx \\ &- \frac{ab}{2l^3}\left(\int_0^l z_2(x,t)ds\right)^T\left(\int_0^l z_2(x,t)ds\right)\end{aligned} \tag{28}$$

Denoting $\xi(x,t) = [z_1^T(t), \varsigma(z_1(t)), \int_0^l \varsigma_1(z_2(x,t))dx, \int_0^l z_2(x,t)dx]^T$ and applying the Schur complement, it follows from the inequality (13) that

$$\begin{aligned}\dot{V}(t) \le& 2\sum_{i=1}^p\sum_{j=1}^r \mu_i\mu_j z_1^T(t)\Sigma_{ij11}z_1(t) \\ &+ 2\sum_{i=1}^p\sum_{j=1}^r \mu_i\mu_j z_1^T(t)\Sigma_{ij12}\varsigma(z_1(t)) \\ &+ \sum_{i=1}^p\sum_{j=1}^r \mu_i\mu_j \varsigma^T(z_1(t))\Sigma_{ij22}\varsigma(z_1(t)) \\ &+ 2\sum_{i=1}^p\sum_{j=1}^r \mu_i\mu_j z_1^T(t)\Sigma_{ij14}\int_0^l z_2(x,t)dx \\ &- m_2\left(\int_0^l \varsigma_1(z_2(x,t))dx\right)^T\left(\int_0^l \varsigma_1(z_2(x,t))dx\right) \\ &+ 2\sum_{i=1}^p\sum_{j=1}^r \mu_i\mu_j\int_0^l \varsigma_1(z_2(x))dx\Sigma_{ij34}\int_0^l z_2(x)dx \\ &- (\frac{ab}{2l^3}+m_2s_1s_2)\left(\int_0^l z_2(x)dx\right)^T\left(\int_0^l z_2(x)dx\right) \\ \le& \sum_{i=1}^p\sum_{j=1}^r \mu_i\mu_j\xi^T(x,t)\Sigma_{ij}\xi(x,t).\end{aligned} \tag{29}$$

Based on Theorem 1, it can be deduced that $\dot{V}(t) < 0$, which indicates that the overall closed-loop coupled system (15) is asymptotically stable. This completes the proof.

4.2 Controller Design

With the purpose of solving (19), some transformations are performed to remove the nonlinear terms PB_iK_j and PB_iG_j.

Theorem 2: Consider the fuzzy nonlinear coupled burgers' PDE-ODE system (6) with the robin boundary condition (16) and any initial conditions z_{1_0} and $z_{2_0}(x)$. The designed fuzzy controller (9), which guarantees the asymptotic stability of the overall closed-loop system (15), can be derived if there exist positive definite matrix X with proper dimensions, scalars ρ_{1i}, ρ_{2i}, and $c = 1/3a$, matrices W_j and G_j with proper dimensions, and positive constants b, d, m_1,m_2, and m_3 for $i \in \mathbb{S}$ such that the following inequalities holds

$$\begin{aligned}&\bar{\Sigma}_{ii} < 0, i \in \mathbb{S}, \\ &\bar{\Sigma}_{ij} + \bar{\Sigma}_{ji} < 0, i < j \ \ s.t. \ \ \mu_i \cap \mu_j \ne \emptyset \\ &\begin{bmatrix} -m_3 I & X \\ * & -m_1 L_1 L_2 \end{bmatrix} \le 0\end{aligned} \tag{30}$$

where

$$\bar{\Sigma}_{ij} = \begin{bmatrix} \bar{\Sigma}_{ij11} & \bar{\Sigma}_{ij12} & \bar{\Sigma}_{ij13} & \bar{\Sigma}_{ij14} \\ * & \Sigma_{ij22} & \mathbf{0} & \mathbf{0} \\ * & * & \Sigma_{ij33} & \Sigma_{ij34} \\ * & * & * & \Sigma_{ij44} \end{bmatrix},$$

$$\begin{aligned}&\bar{\Sigma}_{ij11} = A_iX + XA_i^T - m_3I, \\ &\bar{\Sigma}_{ij12} = B_iK_j + m_1X(L_1+L_2)/2, \\ &\bar{\Sigma}_{ij13} = B_iG_j, \bar{\Sigma}_{ij14} = F_i + bX\phi(i).\end{aligned}$$

Proof: Let $X = P^{-1}$ and denote $\Delta = diag\{X, I, I, I\}$. Then, pre- and post-multiplying the inequality $\Sigma_{ij} < 0$ in

(19) by the matrix Δ and Δ for $i, j \in \mathbb{S}$, respectively, give

$$\Delta\Sigma_{ij}\Delta = \begin{bmatrix} \Xi & \bar{\Sigma}_{ij12} & \bar{\Sigma}_{ij13} & \bar{\Sigma}_{ij14} \\ * & \Sigma_{ij22} & \mathbf{0} & \mathbf{0} \\ * & * & \Sigma_{ij33} & \Sigma_{ij34} \\ * & * & * & \Sigma_{ij44} \end{bmatrix} < 0, i, j \in \mathbb{S}. \tag{31}$$

where $\Xi = A_i X + X A_i^T - m_1 X L_1 L_2 X$.

Note that if the inequalities $m_1 X L_1 L_2 X \le m_3 I$ and $\bar{\Sigma}_{ij} < 0$, we can obtain $\Delta\Sigma_{ij}\Delta < 0$. Hence, the original LMIs (19) can be transferred into LMIs (30). Under this case, the solution of the gain matrices K_j and G_j can be easily obtained by solving new LMIs (30). This completes the corresponding design and the proof.

Remark 3. Here, sensor nonlinearities are considered only in the fuzzy controller, not in the boundary controller. Why we do that is that sensor nonlinearities can be easily tackled by using the inequality technique. However, if the sensor nonlinearities appear in the boundary controller, the conventional controller in the finite-dimensional systems does not work and the adaptive backstepping method needs to be used. Future research will focus on the sensor nonlinearities in the boundary control problems.

5 Simulation Studies

In this section, different from the linear hypersonic rocket car model [9], a nonlinear hypersonic rocket car modelled as a nonlinear coupled PDE-ODE systems is presented to prove the effectiveness of the proposed scheme. The burgers' PDE subsystem represents the surface temperature distribution of the car with the injection of infinite-dimensional nonlinearities, and the nonlinear ODE stands for its motion. Thus, the nonlinear coupled PDE-ODE system can be derived as follows:

$$m_c \ddot{\omega}(t) = u(t) - \mu_a \dot{\omega}^2(t) - \mu_f \dot{\omega}(t) \tag{32}$$

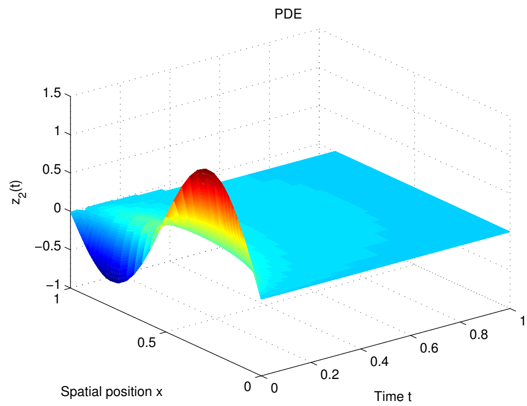
$$\frac{\partial z_2(x,t)}{\partial t} = a\frac{\partial^2 z_2(x,t)}{\partial x^2} - z_2(x,t)\frac{\partial z_2(x,t)}{\partial x} + \dot{\omega}^2(t) \tag{33}$$

where $\mu_f, \mu_a \ge 0$. m_c, $\omega(t)$, $\dot{\omega}(t)$, and $\ddot{\omega}(t)$ are the mass, the position, the velocity, and the acceleration of hypersonic rocket car, respectively. $u(t)$, $\mu_f \dot{\omega}(t)$, and $\mu_a \dot{\omega}^2(t)$ are the power force, the viscous friction force, and the aerodynamic drag, respectively. x, t, and $z_2(x,t)$ represent the surface position, the real time, and the temperature of a hypersonic rocket car, respectively. a is a nonnegative diffusion coefficient, and the boundary conditions and the initial conditions are in accordance with (16) and (4).

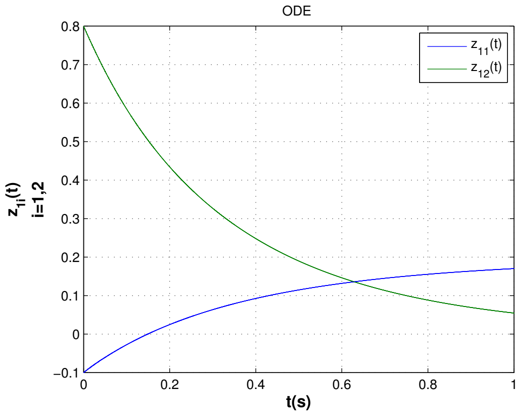
Denote $z_1(t) = [z_{11}(t) \;\; z_{12}(t)]^T = [\omega(t) \;\; \dot{\omega}(t)]^T$ as the ODE states, the whole state-space expression of nonlinear coupled PDE-ODE system can be expressed as the form (1) and (2) with the following parameters:

$$\begin{aligned} &H(z_1(t)) = [z_{12}(t) \quad -m_c^{-1}(\mu_a z_{12}^2(t) + \mu_f) z_{12}(t))]^T, \\ &F(z_1(t)) = [0.001 \;\; 0.001]^T, B(z_1(t)) = [0 \;\; m_c^{-1}]^T, \\ &\phi(z_1(t)) = z_{12}^2(t), z_1(0) = [0.8 \quad -0.1]^T, z_{2_0}(x) = 0, \\ &a = 0.3, l = 1, m_c = 0.1, \mu_a = 0.01, \mu_f = 0.025. \end{aligned} \tag{34}$$

The simulation results are shown in Figs. 1, where Fig. 1(a) presents the evolution of temperature profile and Fig. 1(b) displays the evolution of the position and the velocity. It can be found that the position of the nonlinear hypersonic rocket car will not equal to zero after some time, which means that the overall system (1) and (2) is unstable.



(a)



(b)

Fig. 1: (a) Temperature profile $z_2(x,t)$ and (b) State trajectories $z_1(x,t)$.

The corresponding T-S fuzzy-coupled model can be obtained by exploiting the local sector nonlinearity approach. First, let $\theta(t) = z_{12}(t)$ and $|z_{12}(t)| \le \kappa$, we have $\min\{\theta(t)\} = -\kappa$ and $\max\{\theta(t)\} = \kappa$. Then, we choose $\mu_1(\theta(t)) = \frac{\kappa+\theta(t)}{2\kappa}$ and $\mu_2(\theta(t)) = \frac{\kappa-\theta(t)}{2\kappa}$ as the membership functions. Second, define the fuzzy sets as "big" and "small". The exact T-S fuzzy-coupled model can be described as

Plant rule 1: IF $\theta(t)$ is "big", THEN

$$\begin{aligned} &\dot{z}_1(t) = A_1 z_1(t) + F_1 \int_0^l z_2(x,t)dx + B_1 u(t) \\ &\frac{\partial z_2(x,t)}{\partial t} = a\frac{\partial^2 z_2(x,t)}{\partial x^2} - z_2(x,t)\frac{\partial z_2(x,t)}{\partial x} + \phi_1 z_1(t) \end{aligned} \tag{35}$$

Plant rule 2: IF $\theta(t)$ is "small", THEN

$$\begin{aligned} &\dot{z}_1(t) = A_2 z_1(t) + F_2 \int_0^l z_2(x,t)dx + B_2 u(t) \\ &\frac{\partial z_2(x,t)}{\partial t} = a\frac{\partial^2 z_2(x,t)}{\partial x^2} - z_2(x,t)\frac{\partial z_2(x,t)}{\partial x} + \phi_2 z_1(t) \end{aligned} \tag{36}$$

where

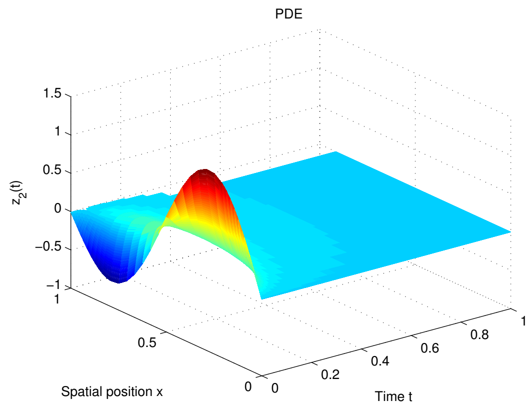
$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -m_c^{-1}(\mu_a\kappa + \mu_f) \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ m_c^{-1} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & m_c^{-1}(\mu_a\kappa - \mu_f) \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ m_c^{-1} \end{bmatrix},$$

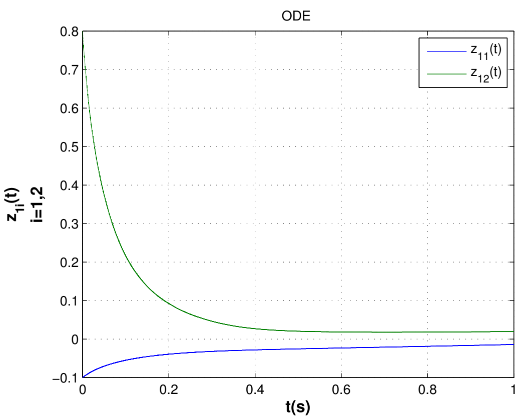
$$F_1 = F_2 = 0.001, \phi_1 = \begin{bmatrix} 0 & \kappa \end{bmatrix}, \phi_2 = \begin{bmatrix} 0 & -\kappa \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, L_2 = 0.8I, s_1 = 0.1, s_2 = 0.7.$$

Choosing $\kappa = 4$ and applying Algorithm 1 to the overall closed-loop system (15), the controller gain matrices and the parameters can be obtained, and the simulation results are displayed in Figs. 2. Fig. 2(a) gives an overview of the temperature profile $z_2(x,t)$ along the position x and the time t under the fuzzy controller (10), which exhibits less fluctuation than Fig. 1. Fig. 2(b) shows the state trajectories of $z_1(x,t)$ including the position and the velocity under the fuzzy controller (10), where $z_1(x,t)$ approaches zero asymptotically.



(a)



(b)

Fig. 2: Under the controller (10) and (15), (a) Temperature profile $z_2(x,t)$ and (b) State trajectories $z_1(x,t)$.

6 Conclusion

In this paper, the stability problem has been investigated for a class of nonlinear coupled PDE-ODE systems, with sensor nonlinearities. Firstly, fuzzy set theory has been adopted to counter the effects of the finite-dimensional nonlinearities, and the Wirtinger's inequality and Jensen's inequality have been used to compensate the infinite-dimensional nonlinearities. Secondly, the corresponding fuzzy controller has been designed to guarantee the asymptotic stability of the nonlinear coupled PDE-ODE systems. Finally, an simulation example of hypersonic rocket car has been provided to validate the effectiveness of the proposed controller.

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