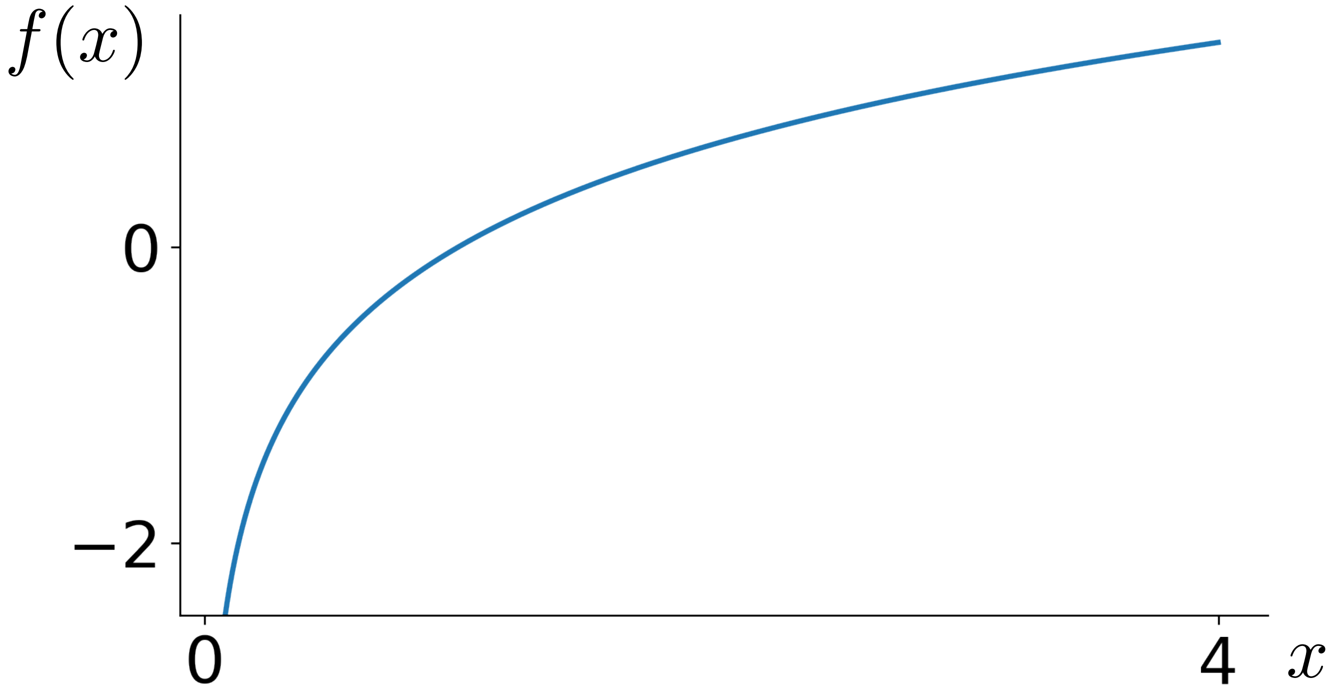
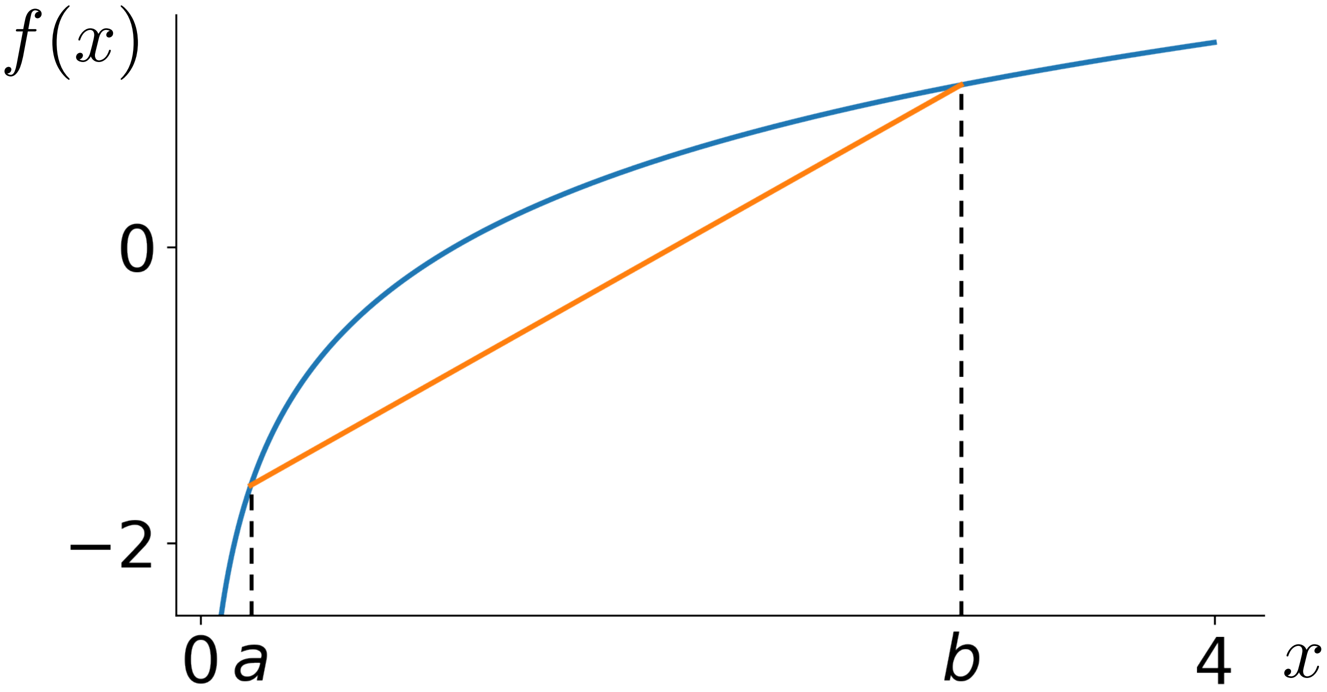
General form of Expectation Maximization

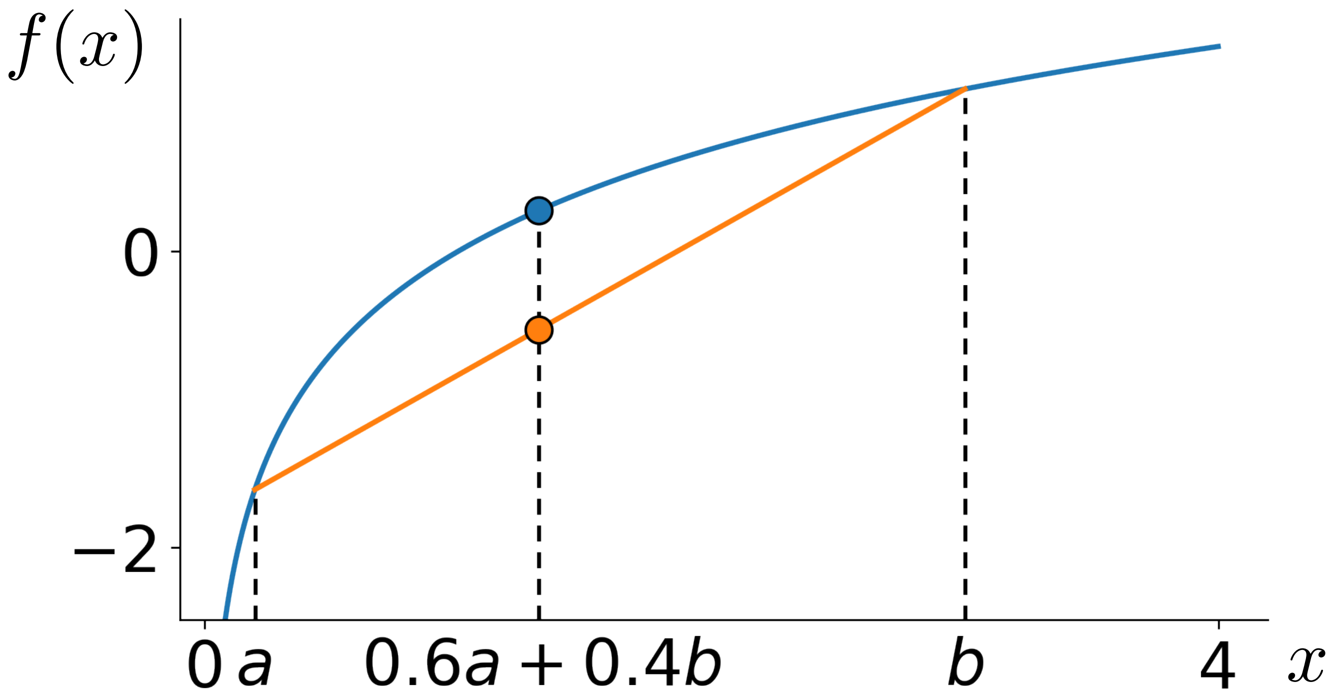
Concave functions



Concave functions



Concave functions



Def.: $f(x)$ is concave if

for any a, b, α : $f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b)$

$$0 \leq \alpha \leq 1$$

Jensen's inequality

$$f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b)$$

Jensen's inequality

If $\quad f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b)$

Then $\quad \alpha_1 + \alpha_2 + \alpha_3 = 1;\ \alpha_k \geq 0.$

$$f(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) \geq \alpha_1 f(a_1) + \alpha_2 f(a_2) + \alpha_3 f(a_3)$$

Jensen's inequality

If $\quad f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b)$

Then $\quad \alpha_1 + \alpha_2 + \alpha_3 = 1;\ \alpha_k \geq 0.$

$$f(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) \geq \alpha_1 f(a_1) + \alpha_2 f(a_2) + \alpha_3 f(a_3)$$

$$p(t = a_1) = \alpha_1,$$
$$p(t = a_2) = \alpha_2,$$
$$p(t = a_3) = \alpha_3$$

Jensen's inequality

If $\quad f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b)$

Then $\quad \alpha_1 + \alpha_2 + \alpha_3 = 1;\ \alpha_k \geq 0.$

$$f(\underbrace{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3}_{\mathbb{E}_{p(t)} t}) \geq \underbrace{\alpha_1 f(a_1) + \alpha_2 f(a_2) + \alpha_3 f(a_3)}_{\mathbb{E}_{p(t)} f(t)}$$

$$p(t = a_1) = \alpha_1,$$
$$p(t = a_2) = \alpha_2,$$
$$p(t = a_3) = \alpha_3$$

Jensen's inequality

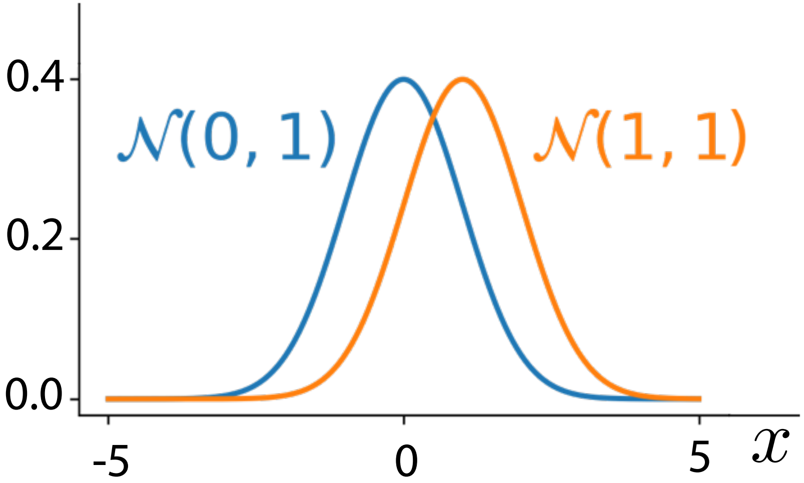
If $$f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b)$$

Then Jensen's inequality:

$$f\left(\mathbb{E}_{p(t)} t\right) \geq \mathbb{E}_{p(t)} f(t)$$

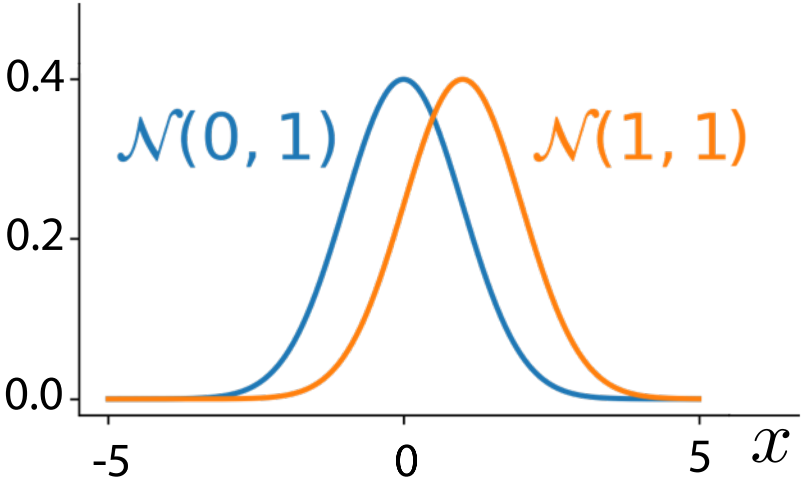
Kullback–Leibler divergence

Kullback–Leibler divergence



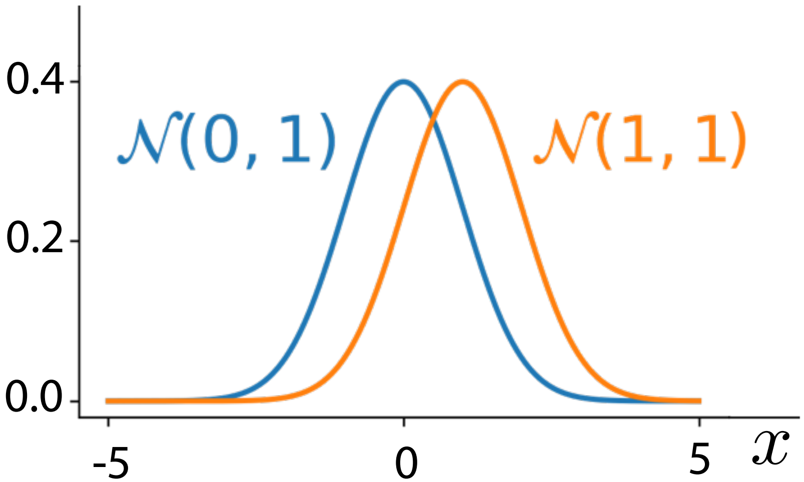
Kullback–Leibler divergence

Parameters difference: 1

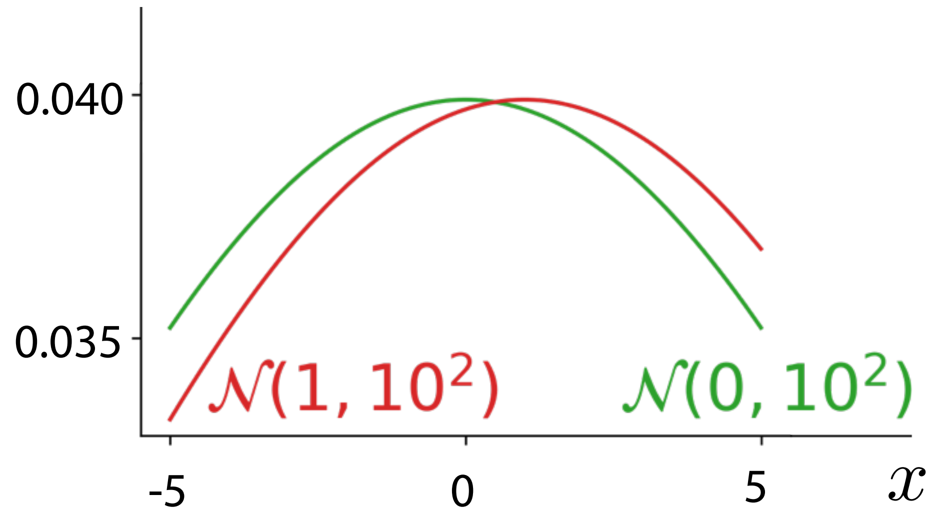


Kullback–Leibler divergence

Parameters difference: 1



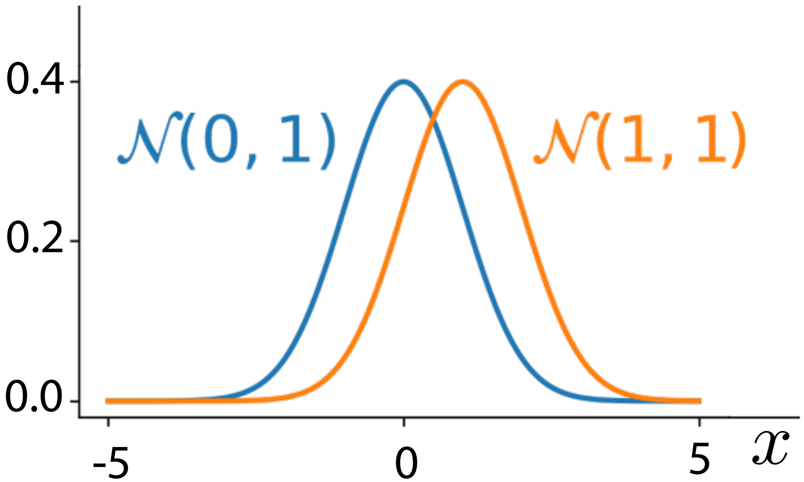
Parameters difference: 1



Kullback–Leibler divergence

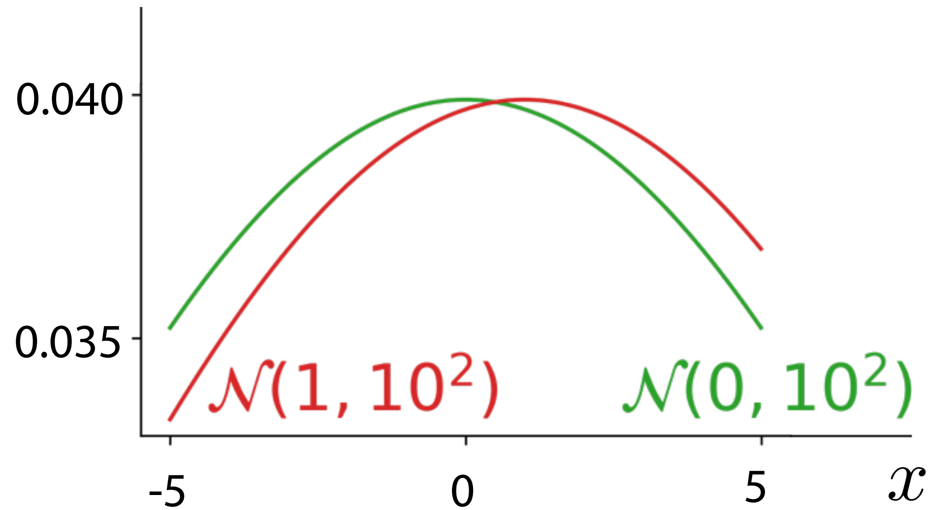
Parameters difference: 1

$$\mathcal{KL}(q_1 \parallel p_1) = 0.5$$



Parameters difference: 1

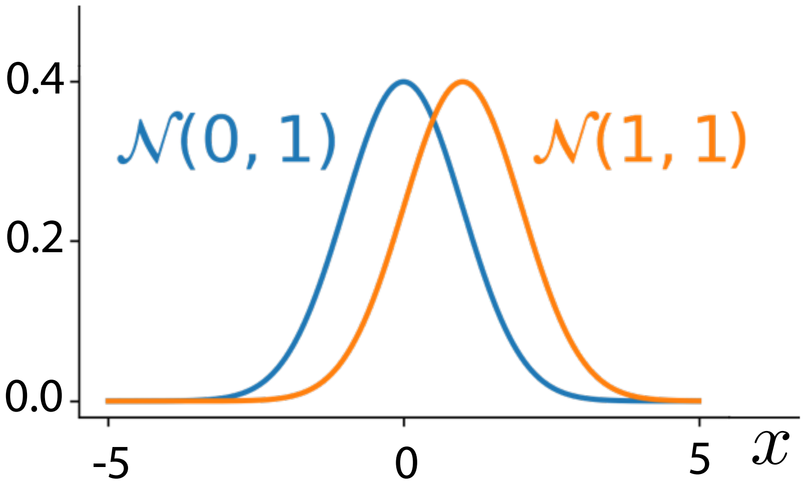
$$\mathcal{KL}(q_2 \parallel p_2) = 0.005$$



Kullback–Leibler divergence

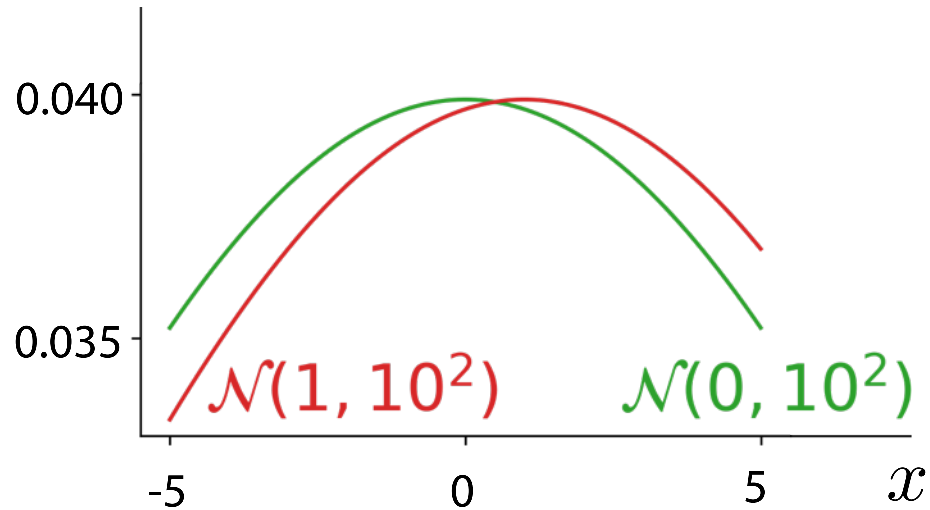
Parameters difference: 1

$$\mathcal{KL}(q_1 \parallel p_1) = 0.5$$



Parameters difference: 1

$$\mathcal{KL}(q_2 \parallel p_2) = 0.005$$



$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$
2. $\mathcal{KL}(q \parallel {\color{red}q}) = 0$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$
2. $\mathcal{KL}(q \parallel q) = 0$
3. $\mathcal{KL}(q \parallel p) \geq 0$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$
2. $\mathcal{KL}(q \parallel q) = 0$
3. $\mathcal{KL}(q \parallel p) \geq 0$

Proof: $-\mathcal{KL}(q \parallel p) = \mathbb{E}_q \left(-\log \frac{q}{p} \right)$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$
2. $\mathcal{KL}(q \parallel q) = 0$
3. $\mathcal{KL}(q \parallel p) \geq 0$

Proof: $-\mathcal{KL}(q \parallel p) = \mathbb{E}_q \left(-\log \frac{q}{p} \right) = \mathbb{E}_q \left(\log \frac{p}{q} \right)$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$
2. $\mathcal{KL}(q \parallel q) = 0$
3. $\mathcal{KL}(q \parallel p) \geq 0$

Proof: $$-\mathcal{KL}(q \parallel p) = \mathbb{E}_q \left(-\log \frac{q}{p} \right) = \mathbb{E}_q \left(\log \frac{p}{q} \right)$$

$$\leq \log(\mathbb{E}_q \frac{p}{q})$$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$
2. $\mathcal{KL}(q \parallel q) = 0$
3. $\mathcal{KL}(q \parallel p) \geq 0$

Proof: $$-\mathcal{KL}(q \parallel p) = \mathbb{E}_q \left(-\log \frac{q}{p} \right) = \mathbb{E}_q \left(\log \frac{p}{q} \right)$$

$$\leq \log(\mathbb{E}_q \frac{p}{q}) = \log \int q(x) \frac{p(x)}{q(x)} dx = 0$$

Kullback–Leibler divergence

$$\mathcal{KL}(q \parallel p) = \int q(x) \log \frac{q(x)}{p(x)} dx$$

Summary

A way to compare distributions
not a proper distance

1. $\mathcal{KL}(q \parallel p) \neq \mathcal{KL}(p \parallel q)$
2. $\mathcal{KL}(q \parallel q) = 0$
3. $\mathcal{KL}(q \parallel p) \geq 0$